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978-0-521-85623-2 - An Introduction to General Relativity and Cosmology

Jerzy Plebanski and Andrzej Krasinski

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