Time, TOEs, and UltraStructuralism

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1 Mathematical Structure and Reality

Mathematics, broadly speaking, is the science of patterns. Physics, broadly speaking, is the search for patterns *in the natural world*. Eugene Wigner's [12] famous problem concerning the "unreasonable effectiveness of mathematics in the natural sciences" constitutes an expression of puzzlement over the empirical success of physics, based as it is on mathematics.[1] Put this way, of course, the problem has a rather simple answer: mathematics (the science of patterns) is so effective because the natural world (the subject matter of physics) is itself patterned. The regularities of physics are instances of mathematical structures. For example, we can apply geometry to physical space because physical space has a structure that is (more or less) isomorphic to some mathematical structure. We might, in slightly different terms (and ignoring complications to do with representation), view our world as a *model* of the axioms of some systems of geometry (and of the axioms of quantum field theory, say—though this is debatable).

However, the problem is really an old one, and there are old solutions too. Pythagoras claimed that there was no distinction between the world of physics and the world of mathematics. Plato argued there was a very great difference: it amounted to concrete versus abstract, a distinction denied by Pythagoras. For Plato the concrete world instantiated (or 'partook of') the abstract forms (albeit imperfectly). This Platonic account is somewhat similar to the model-based view presented above. More recently structural realists have answered the question by

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[1] As he puts it: "the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena" ([12], p. 230).

arguing that science is about the discovery of structural aspects of the world, and these structural aspects are essentially mathematical.[2] Radical 'ontic' structural realists turn this in to an ontological claim: science is about structure and structure is all there is.

Max Tegmark [10] has recently extended this basic idea by combining it with something like David Lewis' extreme brand of modal realism [4]. Not only is there the structure we observe (which is mathematical), there exist mathematical structures of all possible types. We have here, then, an extreme case of the principle of plenitude. Why go to this extreme? To explain the nature and existence of the structure we observe. The laws (structure) of our universe are by no means necessary and so demand some explanation for why they are thus rather than so. Tegmark answers this question with absolute proliferation: our world is a mathematical structure in a multiverse of all possible structures. Interestingly, in this sense, though the structure we inhabit (and are, ourselves, part of) is itself contingent[3], the existence of the structure we inhabit is *necessary* since (being an instance of an eternal mathematical structure) it will be a possible world relative to all other possible worlds.

The maximal multiplicity of possible worlds, then, is utilized to ground a theory of everything that does not face problems of creation *ex nihilo*: mathematical structures are timeless, they are not the kinds of thing that can be created and destroyed. One can then, if one is so inclined, invoke an anthropic explanation of why we find ourselves in *this* particular mathematical structure. This is, admittedly, a hard view to swallow. However, if one wishes to explain why there is something rather than nothing (surely the ultimate explanandum), then I see no other alternative than to invoke eternally existing structures. If we are willing to accept this, then we are led to a belief in many types of (consistent) structure. If we are then further willing to view our world as one of these structures (i.e. literally a mathematical structure), then combined with the necessary anthropics, we have an explanation of what is often considered to be an insurmountable problem.

The major problematic step (by no means the only one), I take it, is that requiring belief that our world, including ourselves, is a mathematical structure. All aspects of reality in our world would have to be reconceptualized as such a structure. The common reaction to such a view bears similarities to Dr Johnson's attempted refuta-

[2]Not all structural realists would go this far. Some, for example, would prefer to say that structure is physical, and that there might be biological and social structures that are not necessarily mathematical.

[3]For example, one can conceive of the laws being different, and indeed, as in David Lewis' theory, the existence of a multiverse of the sort described can provide the machinery to ground such possibilities.

tion of Berkeley's idealism by the kicking of a rock. In this case, the objection is that the world does not seem to be anything like a mathematical structure: mathematical structures are abstract and physical reality is concrete (whereupon you are invited to notice that there are spatiotemporally located, impermanent solid objects). But we have no way of knowing what it is like to be a mathematical structure: it could, after all, be just like *this*! Moreover, given our present knowledge of spacetime (on which more later), the idea that spatiotemporal location is such that it can serve to play so crucial a role as demarcating abstract from concrete seems absurd: spacetime locations are dynamically determined. However, there are real problems in attempting to account for certain observed aspects of the world, time and change being cases in point. The world certainly appears to undergo change, and this, we usually (i.e. with our philosophy hats off) assume, must happen in time. How can time and change be part of a mathematical structure given that such structures are immutable and eternal?

In the remainder of this paper I argue that recent work on problems of time and change in classical and quantum gravity can be brought to bear on the matter resulting in a satisfactory resolution. To deploy a Wheelerism, they show how one can have time without time. We can give an account (or, at least the outline of an account) of our world, *qua* mathematical structure, that at a fundamental level does not contain time. This account makes use of structuralism in a direct way. Thus, the account I present can be usefully incorporated into Tegmark's theory—or what I shall call "ultrastructuralism"—in order to defuse a major potential problem.

2 Time and Symmetry

The universe, as a single object, is usually modeled as a four dimensional structure (a 4-geometry). This structure is naturally changeless: change happens *within* the universe, from one hypersurface (3-geometry) to another (with the time variable chosen arbitrarily)—at least, within ours it seems to. Of course, general relativity leads us to view spacetime geometry as part of a dynamical system, as something that satisfies equations of motion and evolves. But clearly the evolution here cannot be understood in a temporal sense, unless we have at our disposal some external time parameter in which to understand it. An alternative is to attempt to concoct some 'internal' parameter from the dynamical degrees of freedom that can then parametrize the evolution.

This, in a nutshell, is the problem of time in general relativity: spacetime geometry is a dynamical variable, but clearly the dynamics cannot be understood in the usual sense (that is, as involving an external time parameter). The problem is worse than this, however, and can trickle down from global (involving the universe as a whole and a timelessness that is fairly innocuous) to local (involving timelessness and changelessness of the states and observables *within* the universe). At the root of this problem is the symmetry group of general relativity, the group of four-dimensional diffeomorphisms of the spacetime manifold. Diffeomorphism invariance makes local observables (i.e. observables sitting at spacetime points or within regions of spactime) an impossibility, for the equations of motion (of generally relativistic theories) are invariant with respect to diffeomorphisms that shift the points and regions about. Since there clearly *are* (in some sense) local degrees of freedom, and these are what we observe (and what seem to evolve), we need some notion of observable that does not make reference to the spacetime manifold but that fits our experience. That is, we need a background independent notion of observable that does not utilize external spatial and temporal parameters for changes with respect to these will be symmetries of the theory.

A popular response—and one that has been mentioned in several of these FQXI essays—is to use physical degrees of freedom to define observables and evolution. This can be understood as one kind of implementation of the 'internalist' strategy mentioned above. The observables so 'localized' are relational in the sense that they are not defined on a background space but only relative to other dynamical entities (matter fields, spatial volume, etc.). Observables are not of the form $\mathcal{A}(x, t)$ (where x and t label an independent manifold) but $\mathcal{A}(\mathcal{B})$ (where $\mathcal{B}$ is another observable and neither $\mathcal{B}$ nor $\mathcal{A}$ is privileged in any sense). One can then consider the relative evolution of such observables, looking at the way in which changes in the value of one are correlated with changes in the value of the other. This approach can give us notions of time and change that emerge as a consequence of functional relations between elements of a mathematical structure. However, this is to oversimplify matters: in order to properly appreciate the nature of this problem, and the suggested resolution[4] I need to quickly cover the entangled concepts of gauge and constraints. I can then introduce Rovelli's partial observables framework, for constrained (gauge) systems, and show how it provides a structuralist response to the problem of time that can be utilized by the ultrastructuralist to explain time in an atemporal mathematical structure.[5]

[4]This view has been defended by a variety of authors; most notably Bryce DeWitt and Carlo Rovelli. Here I adopt Rovelli's 'partial observables' formalism [9]. See [8] for a general review of the problem of time and proposed solutions.

[5]I restrict the discussion to classical systems in order to make the presentation easier to follow. For the technically savvy, one can transform to the quantum case, roughly, by thinking of the functional relation or correlation $\mathcal{A}(\mathcal{B})$ as representing the expectation values of $\mathcal{A}$ relative to the eigenvalues of $\mathcal{B}$.

3 Constraints and Gauge

The problems of time and change sketched above are aspects of the fact that general relativity is a gauge theory—it's Hamiltonian formulation is characterized by the presence of constraints. We give a very rough and ready presentation of these ideas here—for more details (in the context of the problem of time), see [8].

The diffeomorphism symmetry mentioned above affects the dynamics so that a standard Hamiltonian or Lagrangian formulation of the theory is not possible. Respectively, the canonical variables, q and p, are not all independent (being required to satisfy identities known as constraints: $\phi(q,p)=0$) and the Euler-Lagrange equations are not all independent. These identities serve to 'constrain' the set of phase space points that represent genuine physical possibilities: only those points satisfying the constraints do so, and these form a subset in the full phase space known as the 'constraint surface'. This has a direct impact on the form of the observables. Since a pair of dynamical variables (not observables) that differ by a gauge transformation are indistinguishable, corresponding to one and the same physical state of affairs (the defining characteristic of a gauge transformation), the observables ought to register this fact too: that is, the observables of a gauge theory should be *insensitive* to differences amounting to a gauge transformation—as should the states in any quantization of such a theory: i.e. if $x \sim y$ then $\Psi(x) = \Psi(y)$.[6] Where '$\mathcal{A}$' is a dynamical variable, 'O' is the set of (genuine) observables, x, y are states (represented by points on the constraint surface), and '$\sim$' denotes gauge equivalence, we can express this as:

$$\mathcal{A} \in \mathsf{O} \iff (x \sim y) \supset (\mathcal{A}(x) = \mathcal{A}(y)) \tag{1}$$

Or, equivalently, we can say that the genuine observables are those dynamical variables that are constant on gauge orbits '$[x]$' (where $[x] = \{x : x \sim y\}$):

$$\forall [x]\ ,\ \mathcal{A} \in \mathsf{O} \iff \mathcal{A}[x] = \text{const.} \tag{2}$$

Most of the work done on finding the observables of general relativity is done using the $3+1$ projection of the spacetime Einstein equations. That is, the constraints are understood as conditions laid down on the initial data $\langle \Sigma, h, K \rangle$ when we project the spacetime solution onto a spacelike hypersurface Σ—here, h is a Riemannian metric on Σ and K is the extrinsic curvature on Σ; note that this formulation has since been

[6] It seems that Einstein was aware of this implication soon after completing his theory of general relativity, for he writes that "the connection between *quantities in equations* and *measurable quantities* is far more indirect than in the customary theories of old" ([3], p. 71).

superseded by a representation in terms of Wilson loops and their conjugate momenta (namely, fluxes). I won't go into the nitty gritty details here, but it turns out that the Hamiltonian of general relativity is a sum of constraints on this initial data (of the kind that generate gauge motions, namely 1st class)—hence, the dynamics is entirely generated by constraints and is therefore pure gauge. There is no evolution in time. This is the technical expression of the problems posed above.

This formulation allows us to connect the characterization of the observables up to the dynamics (generated by constraints, abbreviated to $\mathcal{H}_i$) more explicitly:

$$\mathcal{A} \in \mathsf{O} \iff \{\mathsf{O}, \mathcal{H}_i\} \approx 0 \ \ \forall i \tag{3}$$

In other words, the observables of the theory are those functions that have weakly vanishing (i.e. on the constraint surface) Poisson brackets with all of the (first-class) constraints. These are the gauge-invariant quantities: evolving with the constraints (the dynamics) does not generate a physically distinct state. A pressing problem in general relativity—especially pressing for quantum gravity—is to find suitable entities that satisfy this definition. There are at least two types that fit the bill: highly non-local quantities defined over the whole spacetime[7] and (differently) non-local, 'relational' quantities built out of correlations between field values. There seems to be some consensus forming, at least amongst 'canonical relativists', that the latter type are the most natural, and these will serve as the appropriate vehicle for defining time in an unchanging mathematical structure.

4 Partial Observables and Structural Correlations

John Earman calls quantities of the form $\mathcal{A}(\mathcal{B})$ "coincidence occurrences". As he explains, "a coincidence occurrence consists in the corealization of values of pairs of (non-gauge invariant) dynamical quantities" ([2], p. 16). Earman thinks that this new conception of physical quantities signals the necessity of a shift from the traditional 'subject-predicate'-based ontologies, such as substantivalism and relationalism. I think this is the right thing to say, and have argued this point elsewhere (see [6, 7, 8]). However, I spell it out rather differently, in terms of structuralism. Rovelli's framework of partial and complete observables—developed in [9] and beautifully explained in his submission to this competition—provides, I think, the perfect formal framework in which to make sense of the view.

[7] There is a proof (for the case of closed vacuum solutions of general relativity) that there can be no *local* observables at all [11], where 'local' here means that the observable is constructed as a spatial integral of local functions of the initial data and their derivatives.

A *partial* observable is a physical quantity to which we can associate a measurement leading to a number and a *complete* observable is defined as a quantity whose value (or probability distribution) can be predicted by the relevant theory. Partial observables are taken to coordinatize an extended configuration space $\mathcal{Q}$ and complete observables coordinatize an associated reduced phase space Γ_{red}. The "predictive content" of some dynamical theory is then given by the kernel of the map $f : \mathcal{Q} \times \Gamma_{red} \to \mathbb{R}^n$. This space gives the *kinematics* of a theory and the *dynamics* is given by the constraints, $\phi(q^a, p_a) = 0$, on the associated extended phase space $T^*\mathcal{Q}$. The content appears to be this: there are quantities that can be measured whose values are *not* predicted by the theory. Yet the theory *is* deterministic (modulo quantum theoretic probabilities) because it does predict correlations between partial observables. The dynamics is then spelt out in terms of *relations* between partial observables. Hence, the theory formulated in this way describes relative evolution of (non-gauge invariant) variables as functions of each other. No variable is privileged as *the independent one* (*cf.* [5], p. 5). The dynamics concerns the relations between elements of the space of partial observables, and though the individual elements do not have a well defined evolution, relations between them (i.e. correlations) do, and in such a way as to remain independent of coordinate space and time.

The interpretation here is as follows: $\phi = T$ is a partial observable parametrizing the ticks of a clock (laid out across a gauge orbit), and $f = a$ is another partial observable (also stretching out over a gauge orbit). Both are non-gauge invariant quantities. A gauge *invariant* quantity, a complete observable, can (here borrowing from [1]) be constructed from these partial observables as:

$$\mathcal{A}_{[f;T]}(\tau, x) = f(x') \tag{4}$$

These quantities encode correlations. They tell us what the value of a non-gauge invariant function f is when, under the gauge flow generated by the constraint, the non-gauge invariant function T takes on the value τ. This correlation is gauge invariant. These are the kinds of quantity that a background independent gauge theory like general relativity is all about. We don't talk about the value of the gravitational field *at a point of the manifold*, but where some other physical quantity (say, a value of the electromagnetic field) takes on a certain value. Once again, we find that Einstein was surprisingly modern-sounding on this point, writing that "the gravitational field at a *certain location* represents nothing 'physically real,' but the gravitational field together with other data does" ([3], p. 71). Likewise, the "other data" will represent nothing without yet more data (such as the gravitational field). The correlations are the fundamental things.

Let us return to the issue of structuralism. Epistemic structural realism argues that the best we can hope for is to get to know structural aspects of the world, since we only ever get to observe relational properties rather than intrinsic ones (in our experiments and so on). However, in a background independent gauge theory like general relativity we have seen that the physical observables just *are* relational quantities: this is all there is! In other words, there's nothing 'underneath' the relational properties (as encoded in the dynamical fields), so that these *exhaust* what there is, leading to an *ontological* structuralism.[8] This is why we face the problems regarding the 'subject-predicate'-style ontologies that Earman mentions: there *are* no independent subjects that are the 'bearers' of properties and the 'enterers' of relations. Hence, unless one can have objects without intrinsic properties (and I don't think this is a metaphysically healthy route to follow), structuralism is the alternative.

The position involves the idea that physical systems (which I take to be characterized by the values for their observables) are exhausted by extrinsic or relational properties: they have no intrinsic properties at all! This is a consequence of background independence coupled with gauge invariance. This leads to a rather odd picture in which objects and structure are deeply entangled in the sense that, inasmuch as there are objects, any properties they possess are structurally conferred: they have no reality outside the correlation. What this means is that the objects don't *ground* the structure; they are nothing independently of the structure, which takes the form of a (gauge-invariant) correlation between (non-gauge invariant) field values. With this view one can both evade the standard 'no relations without relata' objection and the problem of accounting for the appearance of time (in a timeless structure) in the same way.

5 Conclusion

The 'frozen' character of general relativity (and background independent theories) is usually considered to constitute a problem. However, the most obvious resolution of this problem, involving correlations, can (when appropriately interpreted) be shown to provide a natural explanation of the appearance of time in timeless mathematical

[8]Hence, we have here an empirical argument for ontic structural realism that evades the standard 'no relations without relata' objection. The relations are the correlations here (the gauge invariant, complete observables), and the 'relata' would be the non-gauge invariant, partial observables. But the partial observables being non-gauge invariant do not correspond to physical reality (at least not in any fundamental sense): only the complete observables do. We cannot *decompose* the correlations in an ontological sense, though we clearly can in a epistemic sense—indeed, the correlates constitute our 'access points' to the more fundamental correlations.

structures. With this problem resolved, ultrastructuralism (the view that all there are are timeless mathematical structures) is a position that ought to be taken seriously. There are serious problems remaining with this view, of course. Not least of these is the fact that for any given structure, there are lots of ways the structure is realizable. Now, Tegmark doesn't need a realization relation, and so can evade the problem. However, one needs to accept, on this ultrastructuralist view, that if our universe is a mathematical structure, then we have to accept the possibility that there is an *identical* structure with a radically different appearance. That, as it stands, is very hard to make sense of.

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