A note on Lorentz-like transformations and superluminal motion

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Abstract

In this extended note a critical discussion of an extension of the Lorentz transformations for velocities faster than the speed of light given recently by Hill and Cox [1] is provided. The presented approach reveals the connection between faster-than-light speeds and the issue of isotropy of space. It is shown if the relative speed between the two inertial frames v is greater than the speed of light, the condition of isotropy of space cannot be retained. It further specifies the respective transformations applying to $-\infty < v < -c$ and $c < v < +\infty$. It is proved that such Lorentz-like transformations are improper transformations since the Jacobian is negative. As a consequence, the wave operator, the light-cone and the volume element are not invariant under such Lorentz-like transformations. Also it is shown that such Lorentz-like transformations are not new and already known in the literature.

Keywords: Lorentz transformation; special relativity; superluminal.

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1 Introduction

The early study of superluminal motion (motion faster than the speed of light) may be dated back to Heaviside [2] and Sommerfeld [3] (see also [4]). Recently, there has been a renewed interest in superluminal propagation in physics (e.g. [5, 6, 7, 8, 1]). In this connection, Hill and Cox [1] have introduced two possible "new" transformations to extend Lorentz transformations for inertial observers moving faster than the speed of light with respect to each other. Unlike the usual Lorentz transformations, the one parameter family of transformations derived by Hill and Cox [1] for the regime $c < v < +\infty$ do not form a one parameter group of transformations, since neither the identity characterized by $v = 0$ nor the inverse characterized by $-v$ are in the regime $c < v < +\infty$, and the extensions to negative v need to be obtained from other possible approaches.

In this work, we show that generalized Lorentz transformations can be derived formally from the connection between faster-than-light speeds and the issue of isotropy of space. On the other hand, we point out that the "Hill-Cox transformations" are not new since such transformations, sometimes called "Goldoni transformations", were earlier given in the 1970s and 1980s. In addition, we show that such generalized Lorentz transformations violate the condition of isotropy of space. For simplicity, most of the mathematical discussion will be restricted to the case of $(1+1)$-dimensions (for $(3+1)$-dimensions, one has to add: $y' = y$, $z' = z$ if the relative velocity is parallel to the x-axis). The modest goal of the present paper is not to derive a theory of superluminal motion but rather to discuss existing and recently published generalized Lorentz transformations.

2 Generalized Lorentz transformations

It is assumed that we have two inertial frames, S and S', moving with relative velocity v in the x-direction. Both inertial frames come with Cartesian coordinates: (x, t) for S and (x', t') for S'. The most general possible relationship should be of the form: $x' = f(x, t)$, and $t' = g(x, t)$ for some functions f and g. Since S and S' are both inertial frames, the map $(x, t) \to (x', t')$ must map straight lines to straight lines, that is, such maps are linear. f and g thus should take the form

$$\left\{ \begin{array}{c} x' \\ t' \end{array} \right\} = \left[\begin{array}{cc} a_1 & a_2 \\ a_3 & a_4 \end{array} \right] \left\{ \begin{array}{c} x \\ t \end{array} \right\} , \tag{1}$$

with a_i, $i = 1, 2, 3, 4$ each being a function of v, $a_i = a_i(v)$. Assume that S' is traveling at velocity v relative to S, which means that an observer sitting at $x' = 0$ of S' moves along the trajectory $x = vt$ in S, that is, $x = vt$ must map to $x' = 0$. There is actually one further assumption that $x' = 0$ coincides with $x = 0$ at $t = 0$. Together with the requirement of linearity, this restricts the coefficients a_1 and a_2 to be of the form,

$$x' = \gamma(v)(x - vt) , \tag{2}$$

with a coefficient $\gamma(v)$.

2.1 Case A: $\gamma(v)$ is an even function of v, i.e. $\gamma(v) = \gamma(-v)$

From the perspective of S', relative to which the frame S moves backwards with velocity $-v$, the argument that led us to Eq. (2) now leads to

$$x = \gamma(v)(x' + vt')\,. \tag{3}$$

Since $\gamma(v) = \gamma(-v)$, the $\gamma(v)$ appearing in Eq. (3) is the same as that appearing in Eq. (2).

Assume that the speed of light is equal to c in both S and S', which is nothing but the postulate of constancy of the speed of light. In S, a light ray has trajectory $x = ct$, while, in S', it has trajectory $x' = ct'$. Substituting these trajectories into Eqs. (2) and (3) gives the following equations:

$$ct' = \gamma(v)(c - v)t\,, \tag{4}$$

$$ct = \gamma(v)(c + v)t'\,, \tag{5}$$

which give

$$\gamma^2(v)\big(1 - v^2/c^2\big) = 1\,. \tag{6}$$

Eq. (6) shows that $|v| < |c|$, which means that (under the assumptions of homogeneity of space-time, linearity of inertial transformation, and invariance of the speed of light) assuming $\gamma(v)$ is an even function of v leads to the result that the relative speed between the two inertial frames is less than the speed of light.

Substituting the expression for x' in Eq. (2) into Eq. (3) and using Eq. (6), we get

$$t' = \gamma(v)\left(t - \frac{v}{c^2}\,x\right), \tag{7}$$

with the inverse transformation

$$t = \gamma(v)\left(t' + \frac{v}{c^2}\,x'\right). \tag{8}$$

Eqs. (4) and (5) show that $\gamma(v)$ is positive when $-c < v < c$, and from Eq. (6), we obtain the usual Lorentz factor

$$\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}\,, \qquad -c < v < c\,. \tag{9}$$

Eqs. (2), (7) and (9) are the usual Lorentz transformation connecting two subluminal frames. Standard Lorentz transformation is characterized by its invariant action of the quadratic form: $x'^2 - c^2t'^2 = x^2 - c^2t^2$ and its Jacobian is: $J = 1$. Thus, the standard Lorentz transformation is a proper transformation.

Assume a particle moves with constant velocity u' in frame S' which, in turn, moves with constant velocity v ($|v| < c$) with respect to frame S. The velocity u of the particle as seen in S is just $u = x/t = \gamma(v)(x' + vt')/[\gamma(v)(t' + vx'/c^2)]$. Substituting $x' = u't'$ into the expression for u gives $u = (u' + v)/(1 + uv'/c^2)$.

2.2 Case B: $\gamma(v)$ is an odd function of v, i.e. $\gamma(v) = -\gamma(-v)$

The same argument that led us to Eq. (2) now leads to

$$x = -\gamma(v)(x' + vt')\,. \tag{10}$$

Substituting the two trajectories $x = ct$ and $x' = ct'$ into Eqs. (2) and (10), we obtain

$$ct' = \gamma(v)(c - v)t\,, \tag{11}$$
$$ct = -\gamma(v)(c + v)t'\,, \tag{12}$$

which give

$$\gamma^2(v)\big(v^2/c^2 - 1\big) = 1\,. \tag{13}$$

Eq. (13) shows that $|v| > |c|$, indicating that (under the assumptions of homogeneity of space-time, linearity of inertial transformation, and invariance of the speed of light) assuming $\gamma(v)$ is an odd function of v leads to the result that the relative speed between the two inertial frames is greater than the speed of light.

Substituting the expression for x' in Eq. (2) into Eq. (10) and using Eq. (13), we obtain

$$t' = \gamma(v)\Big(t - \frac{v}{c^2}\,x\Big)\,, \tag{14}$$

with the inverse transformation

$$t = -\gamma(v)\Big(t' + \frac{v}{c^2}\,x'\Big)\,. \tag{15}$$

Eq. (11) shows that $\gamma(v)$ is positive when $-\infty < v < -c$, and Eq. (12) shows that $\gamma(v)$ is negative when $c < v < +\infty$, and combining with Eq. (13), we obtain the generalized Lorentz factor for superluminal motion

$$\gamma(v) = \begin{cases} \dfrac{1}{\sqrt{v^2/c^2 - 1}}\,, & -\infty < v < -c \\ -\dfrac{1}{\sqrt{v^2/c^2 - 1}}\,, & c < v < +\infty\,. \end{cases} \tag{16}$$

It is important to mention that this Lorentz factor is real-valued. Eqs. (2), (14) and (16) are the generalized Lorentz transformations towards superluminal motion.

Assume a particle moves with constant velocity u' in frame S' which, in turn, moves with constant velocity v ($|v| > c$) with respect to frame S. The velocity u of the particle as seen in S is just $u = x/t = -\gamma(v)(x' + vt')/[-\gamma(v)(t' + vx'/c^2)]$. Substituting $x' = u't'$ into the expression for u gives $u = (u' + v)/(1 + uv'/c^2)$, which is the same as in Case A.

Now some historical notes are in order to clarify the state of affairs. From the historical point of view, it seems that Synge [9] was the first who derived a real-valued Lorentz factor like the γ in (16a) for a point moving faster than the speed of light. Goldoni [10, 11] derived such generalized Lorentz transformations with positive Lorentz factor (16a). Lord and Shankara [12] and Sutherland and Shepanski [13] derived such generalized Lorentz transformations with negative Lorentz factor (16b). Many years later, Hill and Cox [1] derived their two generalized Lorentz transformations which are the two transformations given earlier by Goldoni [10, 11], Lord and Shankara [12] and Sutherland and Shepanski [13] and they agree with Eqs. (2), (14) and (16). It seems that Hill and Cox [1] have overlooked the original generalized Lorentz transformations given earlier by Goldoni [10, 11], Lord and Shankara [12] and Sutherland and Shepanski [13]. Thus, the "Hill-Cox transformations" are not at all new. Moreover, Hill and Cox [14] continue to ignore the scientific literature in the re-publication of their results. In addition, their statement that "these independent derivations (Hill and Cox [1] and Vieira [8]), obtained from entirely distinct perspectives, mean that there is now some commonality of agreement in the basic equations underlying superluminal motion" is of questionable veracity.

In addition, it is interesting to note that if c plays the role of the speed of sound, the Lorentz-like transformations with the Lorentz factor (16a) are in formal agreement with the form of the Lorentz-like transformations for supersonic motion in aerodynamics (e.g. [15, 16]). As it seems Miles [15] was the first who derived a Lorentz-like transformation with a real valued Lorentz-factor for the transformation of the supersonic wing equations using the theory of aerodynamics.

The corresponding inverse transformations are given by Eqs. (10), (15) and (16). The transformations (2), (14) and (16) extend the Lorentz transformations towards the regime $-\infty < v < -c$ and $c < v < +\infty$ in addition to the "classical" regime $-c < v < c$. Important enough to mention that for $c < v < +\infty$ as well as for $-\infty < v < -c$, Eq. (16) leads to real values and not to imaginary values of the Lorentz factor γ as often claimed in standard books on special relativity (e.g. [17, 18, 19, 20]). Fig. 1 plots the curve for γ as a function of v/c. The Lorentz factor γ has thus been discussed in three regimes: the regime I shown in the present paper is $-\infty < v < -c$, the regime of special relativity, II, is $-c < v < c$, and the regime discussed by Hill and Cox [1], III, lies in $c < v < +\infty$. Thus, Goldoni [10, 11] derived extended Lorentz transformations for the regime I, whereas Lord and Shankara [12] and Sutherland and Shepanski [13] derived them for the regime III. It can be seen that γ is singular at $v = -c$ and at $v = c$. At $v = -c$, γ possesses a singularity of $+\infty$, while γ is discontinuous at $v = c$ since it jumps from $+\infty$ to $-\infty$. Hence, the region II, the domain of special relativity, is clearly separated from the other two regions. In Fig. 1, it can be seen that $\gamma(v)$ is in the region II an even function and in the regions I and III an odd function of v. It can be clearly seen in Fig. 1 that the region II is "shielded" from regions I and III by two singularities at $v/c = -1$ and $v/c = +1$.

The limit of infinitely large velocity of Eqs. (14) and (15) is now studied. For the regime I the limit $v \longrightarrow -\infty$ in Eqs. (14) and (15) gives

$$c\,t' = x\,, \qquad c\,t = x' \tag{17}$$

and for the regime III the limit $v \longrightarrow +\infty$, Eqs. (14) and (15) are reduced to

$$c\,t' = x\,, \qquad c\,t = x'\,. \tag{18}$$

Also Eqs. (2) and (10) give this limit. Thus, the limit of infinitely large velocity gives in the regimes I and III the exchange of space and time coordinates in $(1+1)$-dimensions. In addition, we want to mention that when $|v| > c$, the new coordinates x' and t' cannot longer be identified with a spatial and a temporal one, because they become timelike and spacelike, respectively. Thus, the new variables x' and t', although perfectly viable mathematical labels in spacetime, lack a proper chronological interpretation, which is

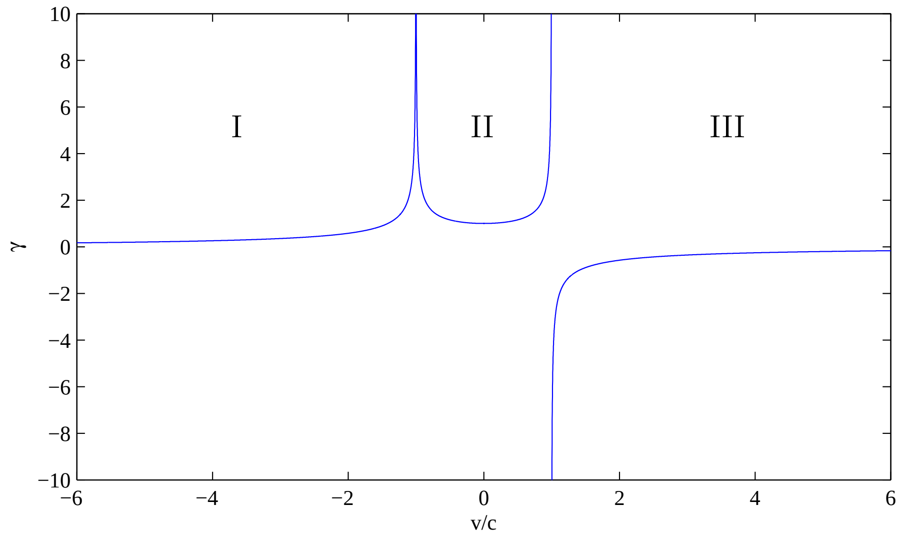

Figure 1: The curve for the Lorentz factor γ as a function of v/c.

what one wants from the coordinates in special relativity. This is particularly evident in the limit $|v| \longrightarrow \infty$, where equations (17) and (18) imply that t' is just a relabeling for x, and x' is just a relabeling for t.

In $(3+1)$-dimensions the Lorentz-like transformations transform the wave operator

$$\partial_{xx} + \partial_{yy} + \partial_{zz} - \frac{1}{c^2}\,\partial_{tt} \qquad \text{to} \qquad -\,\partial_{x'x'} + \partial_{y'y'} + \partial_{z'z'} + \frac{1}{c^2}\,\partial_{t't'} \tag{19}$$

and the light-cone (equation of the wave front)

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \qquad \text{to} \qquad -\,x'^2 + y'^2 + z'^2 + c^2t'^2 = 0\,. \tag{20}$$

Thus, the wave operator and the light-cone are not invariant under the Lorentz-like transformations for superluminal motion. The Lorentz-like transformations change the signature of the metric from $(+++-)$ to $(-+++)$. This shows that the rotational symmetry in (x, y, z)-space (or isotropy of space) is lost for superluminal motion. This gives rise to a "Mach cone" in space-time which divides space-time into two disconnected regions. But this is not surprising since it is known from a Mach cone for supersonic speed in acoustics. It can be seen that the surface to which Eq. (20b) is asymptotic, namely $-x'^2 + y'^2 + z'^2 = 0$, is a Mach cone.

In $(1+1)$-dimensions the Lorentz-like transformations transform the wave operator

$$\partial_{xx} - \frac{1}{c^2}\,\partial_{tt} \qquad \text{to} \qquad -\,\partial_{x'x'} + \frac{1}{c^2}\,\partial_{t't'} \tag{21}$$

and the light-cone

$$x^2 - c^2t^2 = 0 \qquad \text{to} \qquad -\,x'^2 + c^2t'^2 = 0\,, \tag{22}$$

which looks like a formal exchange of space and time coordinates in $(1+1)$-dimensions. The signature of the metric is changed from $(+-)$ to $(-+)$ under the Lorentz-like transformations. Accordingly, in the wave operator the x-axis becomes timelike and the t-axis spacelike. The Jacobian of the Lorentz-like transformations is: $J = -1$. Therefore, the Lorentz-like transformations are improper transformations even in $(1+1)$-dimensions. As a consequence the volume element is not invariant under Lorentz-like transformations: $\mathrm{d}x'\mathrm{d}t' = J\,\mathrm{d}x\,\mathrm{d}t = -\mathrm{d}x\,\mathrm{d}t$.

3 Conclusion and Discussion

In the standard derivations of Lorentz transformations, the Lorentz factor γ is assumed to be an even function of v (see, e.g., [17, 18, 21]). Following the definition in d'Inverno [20] and Levy-Leblond [21], this assumption is based on isotropy of space, i.e. space is non-directional, so that both orientations of the space axis are physically equivalent. The condition of isotropy of space is fulfilled only if γ is an even function of v. The case that γ is an odd function of v is not in agreement with the principle of relativity and with the condition of isotropy of space (see also [8, 22]). For special relativity, isotropy of space may be a sensible feature since both v and $-v$ lie in the same regime II, but for the case of speeds greater than the speed of light, the velocity $-v$ lies in the entirely different regime I, and involves passing through the speed of light twice from the regime I to the regime III. In this note, it has been shown that, under the assumptions of homogeneity of space-time, linearity of inertial transformation, and invariance of the speed of light, the assumption of isotropy of space leads to the result that the relative speed between the two inertial frames is less than the speed of light, and the violation of isotropy of space results in a relative speed greater than the speed of light. From this connection between faster-than-light speeds and the issue of the violation of isotropy of space, the (real-valued) expressions for the Lorentz factor γ in both the regimes I and III may be formally obtained. However, in the $(3+1)$-dimensional theory of special relativity Lorentz-like transformations (e.g. "Hill-Cox transformations") are not consistent because isotropy of space and rotational symmetry in space are broken. In general, the Lorentz-like transformations (like "Hill-Cox transformations") are improper transformations due to $J = -1$. Thus, the wave operator, the light-cone and the volume element are not invariant under such Lorentz-like transformations. We conclude that the "Hill-Cox transformations" are not new and they do not generalize Einstein's theory of special relativity in $(3+1)$-dimensions as claimed by Hill and Cox [1, 14].

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