PROBLEM OF TIME IN QUANTUM GRAVITY

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Abstract

The Problem of Time occurs because the 'time' of GR and of ordinary Quantum Theory are mutually incompatible notions. This is problematic in trying to replace these two branches of physics with a single framework in situations in which the conditions of both apply, e.g. in black holes or in the very early universe. Emphasis in this Review is on the Problem of Time being multi-faceted and on the nature of each of the eight principal facets. Namely, the Frozen Formalism Problem, Configurational Relationalism Problem (formerly Sandwich Problem), Foliation Dependence Problem, Constraint Closure Problem (formerly Functional Evolution Problem), Multiple Choice Problem, Global Problem of Time, Problem of Beables (alias Problem of Observables) and Spacetime Reconstruction or Replacement Problem. Strategizing in this Review is not just centred about the Frozen Formalism Problem facet, but rather about each of the eight facets. Particular emphasis is placed upon A) relationalism as an underpinning of the facets and as a selector of particular strategies (especially a modification of Barbour relationalism, though with some consideration also of Rovelli relationalism). B) Classifying approaches by the full ordering in which they embrace constrain, quantize, find time/history and find observables, rather than only by partial orderings such as "Dirac-quantize". C) Foliation (in)dependence and Spacetime Reconstruction for a wide range of physical theories, strategizing centred about the Problem of Beables, the Patching Approach to the Global Problem of Time, and the role of the question-types considered in physics. D) The Halliwell- and Gambini–Porto–Pullin-type combined Strategies in the context of semiclassical quantum cosmology.

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1 Introduction

The notorious **Problem of Time (POT)** [6, 7, 11, 22, 46, 25, 26, 27, 66, 65, 88, 109, 118] occurs because the 'time' of GR and the 'time' of ordinary Quantum Theory are mutually incompatible notions. This incompatibility leads to a number of problems with trying to replace these two branches of physics with a single framework in situations in which the conditions of both apply, such as in black holes or in the very early universe. I begin with an outline of some relevant aspects of what time means in general [75, 118] (Sec 1.1) and for each of these theories (Sec 1.2-3).

1.1 Some properties often ascribed to time

i) **Time as an ordering**. Simultaneity enters here, i.e. a notion of present that separates future and past notions, which are perceived differently (e.g. one remembers the past but not the future). The structure of the present instant is one's theory's notion of space.
ii) **Causation**: that one phenomenon brings about another at a later value of 'the time'.
iii) **Temporal logic**: this extends more basic (atemporal) logic with "and then" and "at time t_1" constructs.
iv) In dynamics, one encounters the idea of **change in time** (so that time is a container): a parameter of choice with respect to which change is manifest. Newtonian absolute time is an example of this (and external to the system itself and continuous). Contrast the Leibniz–Mach view of time in Sec 1.3! One state of a system "**becoming**" another state of a system (see Sec 6.3) is another phrasing of this dynamical facet of time.
v) Mathematically, time is often taken to be modelled by the real line or an interval of this or a discrete approximation of this. Though time can easily be *position-dependent* in field theories: $\mathrm{t}(x^\mu)$ in place of t. [I use $\mathtt{t}$ to denote either, and follow suit with other quantities using these fonts for finite theories, field theories and portmanteaux of the two. This straight symbol for field-theoretic version and slanty symbol for finite theory version notation shall pervade this article.]
vi) Time is additionally habitually taken to be **monotonic** (rather than direction-reversing). This makes sense as regards possession of ordering and causation properties.
vii) There is to be freedom in prescribing a timefunction as to the **choice of calendar year zero** and **of tick-duration**, i.e. if $\mathtt{t}$ is a timefunction, so is $A + B\,\mathtt{t}$ for A, B constants.
viii) A good timefunction is **globally valid** [25, 41] both over time (antagonist to the half-finite and finite interval times unless there is a good physical reason for this) and over space (made necessary by field theories and, to a greater extent generic curved space/GR).
ix) It also makes sense for a time function to be **operationally meaningful** (computable from observable quantities – tangible and practically accessible).

Candidate times. Simply calling an entity a time or a clock does not make it one. One needs to check a fairly extensive property list before one can be satisfied that it is a) indeed a time or a clock and b) that it is useful for accurate work in comparing theory and experiment. However, what is a sufficient set of properties for a candidate time to be a time is disputed because GR and ordinary QM, so this a choice of a sufficiency of properties rather than just a check-list.

1.2 Time in QM and in GR

Newtonian Absolute Time is what ordinary QM is based upon. This is a *fixed* background parameter. Thus time comes to enter ordinary QM as a background parameter that is
1) used to mark the evolution of the system.
2) It is represented by an *anti*-hermitian operator (unlike the representation of other quantities).
3) The time–energy uncertainty relation $\Delta t^{\mathrm{Newton}} \Delta E \geq \hbar/2$ is also given an entirely different meaning to that of the other uncertainty relations. [I subsequently use $\hbar = c = G = 1$ units.]
4) There is unitary evolution in time, i.e. that probabilities always sum to one. This evolution is in accord with the theory's time-dependent wave equation (for now a time-dependent Schrödinger equation). The *scalar product* on the Hilbert space of states leads to conserved probability currents.
5) Moreover, there is a second dynamical process: collapse of the wavefunction that is held to occur in ordinary QM despite its not being described by the evolution equation of the theory.
Special Relativity (SR) brings in the further notions of
1) a proper time corresponding to each observer, and of
2) time as another coordinate on (for now, flat) spacetime, as opposed to the external absolute time of Newtonian physics.
3) However, time in SR is also external and absolute in the sense of there being (via the existence of Killing vectors) a presupposed set of privileged inertial frames (the quantum theory can be made independent of a choice of frame if it carries a unitary representation of the Poincaré group). In that sense in SR all one has done is trade one kind of absolute time for another [100] (the changes are in the notion of simultaneity [75] rather than in removal of the absolute; also the type of wave equation is now e.g. Klein–Gordon or Dirac). Thus the passage from Newtonian Mechanics to SR and thus from ordinary QM to relativistic QFT does *not* greatly affect the role of time.
It is only GR that frees one from absolute time, and so that the SR to GR conceptual leap is in many ways bigger than the one from Newtonian Mechanics to SR. Here,

1) The conventional **GR spacetime formulation** is in terms of (M, $\mathbf{g}$) for M the spacetime topological manifold and $\mathbf{g}$ the indefinite spacetime 4-metric that obeys the Einstein field equations (EFE's). [I usually restrict to the vacuum case for simplicity, but extending this article's study to include the ordinary matter fields is unproblematic.]
2) Time is a *general* spacetime coordinate in GR, clashing with ordinary QM's having held time to be a unique and sui generis extraneous quantity.
3) GR can furthermore be formulated as an evolution of spatial geometry – the spatial geometries themselves being the entities (configurations) that undergo the GR theory's dynamics: GR is a *geometrodynamics* (see Sec 2 for more).
4) Moreover, time as a general choice of coordinate is embodied geometrically in how GR spacetime is sliced into a **sequence of** (or **foliation by**) **spacelike hypersurfaces** corresponding to constant values of that time. There is one timefunction per choice of foliation; thus time in GR is said to be '**many-fingered**' (see Secs 2 and 5 for further detail). That space and time can be thought of as different and yet spacetime also carries insights probably accounts for why a number of POT facets are already present in classical GR (see Secs 3 and 5).
5) **The generic spacetime has no timelike Killing vector**, so much of the preceding QM structure ceases to have an analogue. This puts an end to how the passage from Newtonian to SR absolute structures is effectuated, since the latter's notion of absolute time is based on Minkowski spacetime's timelike Killing vector. Now instead one has to pass from having privileged frame classes to dealing with the spacetime diffeomorphisms (which are a much larger and less trivial group of transformations).
6) GR has the time-ordering property, and the notion of causality as an extension of SR but now with matter and gravitation influencing the larger-scale causal properties.
7) GR additionally has a time non-orientability notion [16], and a closed timelike curve notion (both of these are usually held to be undesirable features for a physical solution to possess).

External time is furthermore incompatible with describing truly *closed* systems, the ultimate of which is closed-universe Quantum Cosmology. In this regard, Page and Wootters [14] have given convincing arguments that such a system's only physical states are *eigenstates* of the Hamiltonian operator, whose time evolution is essentially trivial.

Returning to ordinary QM, the idea of events happening at a single time plays a crucial role in the technical and conceptual foundations of Quantum Theory, as follows.
1) *Measurements* made at a particular time are a fundamental ingredient of the conventional Copenhagen interpretation (which is anchored on the existence of a privileged time). Then in particular [26], an **observable** is a quantity whose value can be measured at a 'given time'. On the other hand, a 'history' has no direct physical meaning except in so far as it refers to the outcome of a sequence of time-ordered measurements. That histories are part QM-like in path integral form, but also capable of harbouring GR-like features, may bode well for (part of) a reconciliation of GR and QM. How QM should be *interpreted* for the universe as a whole is a recurring theme (see e.g. Secs 6.3–4).
2) In constructing a Hilbert space for a quantum theory, one is to select a complete set of observables that are required to *commute* at a fixed value of time – i.e. obey **equal-time commutation relations**. This again extends to SR by considering not one Newtonian time but the set of relativistic inertial frames; then one's QM can be made independent of frame choice via it carrying a unitary representation of the Poincaré group. For a relativistic QFT, the above equal-time commutation relations point is closely related to requiring *microcausality* [26]:

$$[\widehat{\phi}(X^\mu), \widehat{\phi}(Y^\mu)] = 0 \tag{1}$$

for relativistic quantum field operators $\widehat{\psi}$ and all spacelike-separated spacetime points X^μ and Y^μ. However, observables and some notion of time-equal commutation relations pose significant difficulties in the context of GR.

1.3 Background Independence, relational criteria and the POT

Our attitude is that Background Independence is philosophically-desirable and a lesson to be taken on board from GR, quite separately from any detail of the form of its field equations that relativize gravitation. 'Background Independence' is to be taken here in the sense of freedom from absolute structures, according to the following.

Temporal relationalism [13, 30, 101, 118] concerns time being not primary for the universe as a whole (Leibniz's view). This is to be implemented by reparametrization-invariance in the absense of extraneous time-like variables The conundrum of our apparent local experience of time is then to be along the lines of Mach's time is to be abstracted from change' (see Secs 2 and 3 for more detail of these last two sentences).
Configurational relationalism consists in regarding the configuration space Q of dynamical objects to possess a group G of transformations that are physically irrelevant [13, 59, 65, 124, 126]. A classic example are the translations and rotations with respect to absolute space, though the internal-space transformations of Gauge Theory are also of this nature [118].

Moreover, Background Independence already implies a number of POT facets at the classical level, and the rest at the quantum level. So whenever one adheres to Background Independence in the above sense, some particular form of the POT arises as a direct consequence, and then one must face it.

Note 1) Whilst Einstein was inspired by Mach, he did not implement Mach directly in setting up GR. However, direct implementation of Mach via temporal and configurational relationalism does give back ([59], Secs 3 and 7) the dynamical formulation of GR.
Note 2) Background independence criteria are held to be an important feature in e.g. geometrodynamics and Loop Quantum Gravity (LQG).
Note 3) Perturbative covariant quantization involves treating the metric as a small perturbation about a fixed background metric. This is neither background-independent nor successful on its own terms, at least in the many steps of this program that have been completed, due to nonrenormalizability, and non-unitarity in higher-curvature cases. Perturbative String Theory is another background-dependent approach. This amounts to taking time to be a fixed-background SR-like notion rather than a GR-like one in this formulation's deepest level. Here, GR's field equations are emergent, so that problems associated with them are not held to be fundamental but should be referred further along to background spacetime metric structure that the strings move in. Moreover, technical issues then drive one to seek nonperturbative background-independent strategies and then POT issues resurface among the various possibilities for the background-independent nature of M-theory.
Note 4) LQG has a greater degree of background independence than perturbative string theory: it is independent of background *metric* structure. [However the argument for this can be repeated at the level of background *topological structure*, for which one is presently very largely unprepared.]

1.4 Outline of the rest of this Review

Whilst I consider the Frozen Formalism Problem (FFP) in Sec 3 and a number of conceptualizations that have been suggested to solve it in Sec 6, unlike in previous POT reviews [11, 22, 27, 46, 65, 109], the present review concentrates on how there are other facets (Secs 4 and 5), and on freeing conceptual strategizing from dealing primarily with the FFP (Secs 7 to 13). The seven other facets are [25, 26] Configurational Relationalism, that generalizes the Best Matching Problem that itself generalizes the Thin Sandwich Problem, The Problem of Beables (alias Problem of Observables), the Foliation Dependence Problem, the Functional Evolution Problem, the Global POT, the Multiple Choice Problem and the Spacetime Reconstruction/Replacement Problem. Secs 6 to 13 additionally update the classic reviews [25, 26] due to the almost 20 years of work since. I use relational/Machian/background-independent criteria as a characterization/possible deeper explanation of POT facets, and then as a selection criterion on theories and on POT strategies. These select for semiclassical, many timeless approaches and histories approaches. I end by considering composite of these strategies in Sec 14: Halliwell-type [62, 64, 74, 99, 116, 125] and Gambini–Porto–Pullin-type [80, 90] approaches.

2 Dynamical formulation of GR

In dynamical/canonical approaches, the spacetime manifold M is 3 + 1 split (3 spatial dimensions and 1 time dimension) into $\Sigma \times \mathbb{R}$; this assumes that the spatial topology Σ does not change with coordinate time $t \in \mathbb{R}$ (or some subinterval thereof). This furthermore assumes that GR is globally hyperbolic [16], which amounts to determinability of GR evolution from GR initial data; this excludes e.g. time non-orientability and closed timelike curves. Some considerations focus around a single Σ embedded into GR spacetime (or without that assumption, usually leading to its deduction, c.f. Sec 5).

Some further considerations require a *foliation* $\mathcal{F} = \{\mathcal{L}_{\mathsf{A}}\}_{\mathsf{A} \in A}$, which is a decomposition of an m-dimensional manifold $\mathcal{M}$ into a disjoint union of connected p-dimensional subsets (the *leaves* $\mathcal{L}_{\mathsf{A}}$ of the foliation) such that the following holds. $m \in \mathcal{M}$ has a neighbourhood U in which coordinates $(x^1, ..., x^m)$: $U \to \mathbb{R}^m$ such that for each leaf $\mathcal{L}_{\mathsf{A}}$ the components of $U \bigcap \mathcal{L}_{\mathsf{A}}$ are described by x^{p+1} to x^m constant [see Fig 1a)]. The codimension of the foliation is then $c = m - p$. For 3 + 1 GR, $\mathcal{M} = $ M, so $m = 4$, the leaves $\mathcal{L}_{\mathsf{A}}$ are 3-space hypersurfaces so $p = 3$ and thus $c = 1$: the time dimension.

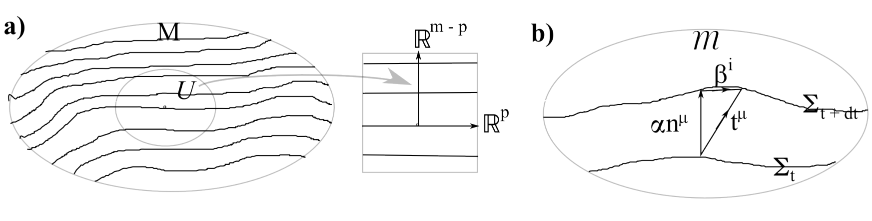

Figure 1: a) Arnowitt–Deser–Misner 3 + 1 split of $(\mathrm{M}, \mathbf{g})$. b) Picture supporting the text's definition of foliation.

Given 2 neighbouring hypersurfaces, one can {3 + 1}-decompose the spacetime metric [1] $\mathbf{g}$ [see Fig 1b] into

$$g_{\mu\nu} = \begin{pmatrix} \beta_k\beta^k - \alpha^2 & \beta_j \\ \beta_i & h_{ij} \end{pmatrix} \quad (2)$$

where α is the lapse (time elapsed), β^i is the shift (in spatial coordinates between the two slices) and $\mathbf{h}$ is the intrinsic metric on the original hypersurface Σ_t itself. The relation between the foliation vector n^μ and the above {3 + 1}-split quantities is $t^\mu = \alpha n^\mu + \beta^\mu$ and so $n^\mu = \alpha^{-1}[1, -\boldsymbol{\beta}]$.

Next,[1] the *extrinsic curvature* of a hypersurface is the rate of change of the normal along that hypersurface and thus of the bending of that hypersurface relative to its ambient space:

$$K_{\mu\nu} = h_\mu{}^\rho \nabla_\rho n_\nu = \pounds_n h_{\mu\nu}/2 \ , \tag{3}$$

or (since it is a hypersurface object so it can be thought of as a spatial tensor as well as a spacetime one),

$$K_{ij} = \{\dot{h}_{ij} - \pounds_\beta h_{ij}\}/2\alpha = \{\dot{h}_{ij} - 2D_{(i}\beta_{j)}\}/2\alpha \ . \tag{4}$$

As well as this being an important characterizer of hypersurfaces, it is relevant due to its close connection to the GR momenta,

$$\pi^{ij} = -\sqrt{h}\{K^{ij} - Kh^{ij}\} \tag{5}$$

for $K := K_{ij}h^{ij}$. Next, the GR constraints arise from the 3 + 1 split of the (here vacuum) EFE's, which have the given forms in terms of each of K_{ij} and of π^{ij},

$$\text{Hamiltonian constraint, } \mathcal{H} := 2G^{(4)}_{\perp\perp} = K^2 - K_{ij}K^{ij} + \text{Ric}(\mathbf{x};\mathbf{h}] = \mathcal{F}_{\mu\nu\rho\sigma}\pi^{\mu\nu}\pi^{\rho\sigma}/\sqrt{h} - \sqrt{h}\text{Ric}(\mathbf{x};\mathbf{h}] = 0 \ , \tag{6}$$

$$\text{momentum constraint, } \mathcal{H}_i := 2G^{(4)}_{i\perp} = 2\{D_jK^j{}_i - D_\mu K\} = -2D_j\pi^j{}_i = 0 \tag{7}$$

Note 1) The K_{ij}-forms of these serve to identify these as contractions of the Gauss–Codazzi equations for the embedding of spatial 3-slice into spacetime (a higher-d indefinite-signature generalization of Gauss' Theorema Egregium).
Note 2) The GR momentum constraint is straightforwardly interpretable in terms of the physicality residing solely in terms of the 3-geometry of space rather than in its coordinatization/point-identification. However, interpreting the GR Hamiltonian constraint is tougher; it leads to the FFP.
Note 3) On the other hand, the GR momentum constraint often becomes entwined in technical problems that afflict POT strategies. Some of these issues further involve the specific 3-diffeomorphism Diff(Σ) nature of the associated group of physically-irrelevant transformations [and by extension Superspace(Σ) = Riem(Σ)/Diff(Σ), for Riem(Σ) the configuration space of spatial 3-metrics on a fixed Σ]. There are various further ways in which the spacetime diffeomorphisms Diff(M) and their hypersurface-split version enter various POT Facets and Strategies; these are the subject of Secs 7 and 9.

Whilst the first two of these are just infinite-dimensional Lie groups, the third corresponds to the Dirac Algebra[2]. of constraints,

$$\{\mathcal{H}(\boldsymbol{\beta}'), \mathcal{H}(\boldsymbol{\beta})\} = \pounds_\beta \mathcal{H}(\boldsymbol{\beta}') \ , \tag{8}$$

$$\{\mathcal{H}(\alpha), \mathcal{H}(\boldsymbol{\beta})\} = \pounds_\beta \mathcal{H}(\alpha) \ , \tag{9}$$

$$\{\mathcal{H}(\alpha), \mathcal{H}(\alpha')\} = \mathcal{H}(\boldsymbol{\gamma}) \ \text{ for } \ \gamma^i := hh^{ij}\{\alpha\partial_j\alpha' - \alpha'\partial_j\alpha\} \ , \tag{10}$$

which, in its third bracket, possesses *structure functions*, so that this is far more complicated than a Lie algebra (it is a Lie algebroid [119]). (8) is the closure of the Lie algebra of 3-diffeomorphisms on the 3-surface, whilst (9) just means that $\mathcal{H}$ is a scalar density; neither of these have any dynamical content. It is the third bracket then that has dynamical content. The Dirac Algebra has remarkable properties at the classical level, as exposited in Sec 5. However, that the Dirac Algebra of the classical GR constraints is not a Lie algebra, which does limit many a quantization approach [15].

The Diff($\mathcal{M}$) that are not among the Diff(Σ) do cause a number of serious difficulties with quantization schemes [15]. The quantum GR Hamiltonian constraint is a Wheeler–DeWitt equation (WDE),[3]

$$\widehat{\mathcal{H}}\Psi := -\hbar^2 \text{‘}\{\triangle_{\boldsymbol{\mathcal{G}}} - \xi\,\text{Ric}_{\boldsymbol{\mathcal{G}}}(\mathbf{x};\mathbf{h}]\}\text{’}\,\Psi - \sqrt{h}\text{Ric}(\mathbf{x};\mathbf{h}]\Psi = 0 \ , \ \text{ for } \triangle_{\boldsymbol{\mathcal{G}}} := \frac{1}{\sqrt{\mathcal{G}}}\frac{\delta}{\delta h^{ij}}\left\{\sqrt{\mathcal{G}}\mathcal{F}^{ijkl}\frac{\delta\Psi}{\delta h^{kl}}\right\} \tag{11}$$

$$\widehat{\mathcal{H}}_i\Psi := -2D_jh_{ik}\delta\Psi/\delta h_{jk} = 0 \ . \tag{12}$$

In the **Ashtekar variables** formulation of GR [24], qualitative details of the POT facets remain largely unchanged. However, using these is relevant to some of the strategies subsequently discussed, so I provide an outline. Pass from h_{ij}, π^{ij} to a $SU(2)$ connection $\mathbb{A}_i{}^{\mathtt{AB}}$ and its conjugate momentum $\mathbb{E}^i{}_{\mathtt{AB}}$ [which is related to the 3-metric by $h_{ij} = -\text{tr}(\mathbb{E}_i\mathbb{E}_j)$].[4] This formulation's constraints are

$$\mathbb{D}_i\mathbb{E}^i{}_{\mathtt{AB}} \equiv \partial_i\mathbb{E}^i{}_{\mathtt{AB}} + |[\mathbb{E}_i, \mathbb{E}^i]|_{\mathtt{AB}} = 0 \ , \tag{13}$$

[1] The spatial topology Σ is taken to be compact without boundary. h_{ij} is a spatial 3-metric thereupon, with determinant h, covariant derivative D_μ, Ricci scalar Ric($\mathbf{x};\mathbf{h}$] and conjugate momentum π^{ij}. Λ is the cosmological constant. Here, the GR configuration space metric is $\mathcal{G}^{ijkl} = h^{ik}h^{jl} - h^{ij}h^{kl}$ is the undensitized inverse DeWitt supermetric with determinant $\mathcal{G}$ and inverse $\mathcal{F}_{ijkl}$ itself the undensitized DeWitt supermetric $h_{ik}h_{jl} - h_{ij}h_{kl}/2$. The Ricci scalar built out of this metric is $\text{Ric}_{\boldsymbol{\mathcal{G}}}(\mathbf{x};\mathbf{h}]$. In this article, $\lfloor\ \rceil$ is a portmanteau of 1) function dependence ( ) in finite theories and 2) functional dependence [ ] or mixed function-functional dependence (;] in field theories. $\pounds_V$ is the Lie derivative with respect to a vector field V^i. (4) superscripts denote spacetime tensors; $G^{(4)}_{\mu\nu}$ is the spacetime Einstein tensor.

[2] Computationally, this follows most succinctly from the Bianchi identity [16] using the first forms of eqs (6,7)

[3] The inverted commas indicate that the WDE has, in addition to the POT, various technical problems, including regularization problems (not at all straightforward in background-independent theories), what meaning to ascribe for functional differential equations, and operator-ordering issues (some of which are parametrized by the number ξ).

[4] The capital typewriter indices denote the Ashtekar variable use of spinorial $SU(2)$ indices. tr denotes the trace over these. $\mathbb{D}_i$ is the $SU(2)$ covariant derivative as defined in the first equality of (13). $|[\ ,\ ]|$ denotes the classical Yang–Mills-type commutator.

$$\mathrm{tr}(\mathbb{E}^i\mathbb{F}_{ij}) = 0\ , \tag{14}$$

$$\mathrm{tr}(\mathbb{E}^i\mathbb{E}^j\mathbb{F}_{ij}) = 0\ . \tag{15}$$

(13) arises because one is using a first-order formalism; in this particular case, it is an $SU(2)$ (Yang–Mills–)Gauss constraint. (14) and (15) are the polynomial forms taken by the GR momentum and Hamiltonian constraints respectively, where $\mathbb{F}_{ij}^{\mathsf{AB}} := 2\partial_{[i}\mathbb{A}_{j]}^{\mathsf{AB}} + |[\mathbb{A}_i, \mathbb{A}_j]|^{\mathsf{AB}}$ is the Yang–Mills field strength corresponding to $\mathbb{A}_i{}^{\mathsf{AB}}$. One can see that (14) is indeed associated with momentum since it is the condition for a vanishing (Yang–Mills–)Poynting vector. Again, the Hamiltonian constraint (15) lacks such a clear-cut interpretation. Finally, the Ashtekar variables formulation given above is of complex GR. But then one requires troublesome [27] *reality conditions* in order to recover real GR. Nowadays so as to avoid reality conditions, one usually prefers to work not with Ashtekar's original complex variables but with Barbero's real variables [35] that now depend on an Immirzi parameter, γ. **LQG** is then a QM scheme for this [66, 82]. Whilst the standard constraint algebra in this scheme is usually an enlarged but qualitatively-similar version of the Dirac Algebra, a **Master Constraint Approach** has also been proposed [82]. Here, the *master constraint* $\mathbb{M} := \mathcal{C}_{\mathsf{A}}\mathcal{K}^{\mathsf{AB}}\mathcal{C}_{\mathsf{B}}/2$ (for some arbitrary positive-definite array $\mathcal{K}^{\mathsf{AB}}$) is supposed to replace the set of constraints $\mathcal{C}_{\mathsf{A}}$. There are however a number of reservations with this (see each of [82, 109] and Sec 10), not least whether the arbitrariness in $\mathcal{K}^{\mathsf{AB}}$ will cause difficulties.

3 The Frozen Formalism Problem (FFP)

One notable facet of the POT shows up in attempting canonical quantization of GR (or many other gravitational theories that are likewise background-independent). It is due to GR's Hamiltonian constraint $\mathcal{H}$ being quadratic but not linear in the momenta. Because of this feature, I denote the general case of such a constraint by Quad. Then promoting a constraint with a momentum dependence of this kind to the quantum level gives a time-independent wave equation (11) that is schematically of the form

$$\widehat{\mathcal{H}}\Psi = 0\ , \tag{16}$$

in place of ordinary QM's time-dependent one,

$$i\partial\Psi/\partial t = \widehat{\mathsf{H}}\Psi \tag{17}$$

[or a functional derivative $\delta/\delta t(x^\mu)$ counterpart of this in the general GR case]. Here, H is a Hamiltonian, Ψ is the wavefunction of the universe and t is absolute Newtonian time [or a a local GR-type generalization $\mathtt{t}(x^\mu)$].

This suggests, in apparent contradiction with everyday experience, that nothing at all *happens* in the universe! Thus one is faced with having to explain the origin of the notions of time in the laws of physics that appear to apply in the universe; this paper reviews a number of strategies for such explanations. [Moreover timeless equations such as the WDE apply *to the universe as a whole*, whereas the more ordinary laws of physics apply to small subsystems *within* the universe, suggesting that this is an apparent, rather than actual, paradox.]

The usual form of the FFP is that variation with respect to the lapse implies the purely quadratic $\mathcal{H}$, which implies the FFP at the quantum level. That reparametrization invariance implies a purely quadratic $\mathcal{H}$ is also fairly commonly stated in the literature. However, the stance of Barbour and collaborators [2, 59] is that Leibnizian temporal relationalism is the starting point and that this is *implemented by* reparametrization invariance alongside an absense of auxiliary time-like variables such as Newtonian absolute time or the GR lapse. The subsequent product-type action, modellable in this sense by the Jacobi action of mechanics, $S = \sqrt{2}\int \mathrm{d}s\sqrt{E - V}$ (for $\mathrm{d}s$ the configuration space line element, V the potential energy and E the total energy), implies a purely quadratic constraint as a *primary* constraint which, as ever, then gives a FFP. One can then view Leibnizian temporal relationalism as *setting up* a **classical FFP**, which is resolved (at least in principle, see Sec 7 for complications) in a 'time is to be abstracted from change' Machian manner by the Jacobi–Barbour–Bertotti emergent time [13, 30, 65, 118],

$$t^{\mathrm{JBB}} = t^{\mathrm{JBB}}(0) + \int \mathrm{d}s/\sqrt{E - V}\ . \tag{18}$$

In the mechanics context, this arises as a simplifier of the relational approach's classical equations and amounts to a recovery of Newtonian time from relational first principles; while, in the context of GR, it amounts to recovery of proper time [and of cosmic time, in the case of (approximately) homogeneous cosmologies]. However, this fails to unfreeze the quantum FFP; thus something more is needed there.

An outline of the strategization about the FFP involves the longstanding philosophical fork between 'time is fundamental' and 'time should be eliminated from one's conceptualization of the world'. Approaches of the second sort are to reduce questions about 'being at a time' and 'becoming' to, merely, questions of 'being'. A finer classification [25, 26, 109] is into Tempus Ante Quantum, Tempus Post Quantum, Tempus Nihil Est [25, 26] and Non Tempus sed Historia [109]. [My separation out of the last of these from Kuchař and Isham's timeless approaches is due to its departure from conventional physics' dynamics and quantization of *configurations* and conjugates.] I denote these by, procedurally ordered from left to right, TQ, QT, Q and a new pair: HQ, QH ('Historia ante Quantum' and 'Historia post Quantum'). In fact, one has an octalemma, the final possible cases being H, T or none. There three correspond to opting out of quantizing GR, on grounds of regarding it as but an effective classical theory (much as one would usually not directly quantize the equations of classical fluid mechanics). In such a case, whatever structure eventually replaces GR as a fundamental theory might not possess GR-like features including Background Independence. I leave these Q-free options for someone else to review, concentrating on the first five above. One might also consider H and T as classical antecedents that are nevertheless interesting models in their own right (i.e. not abandon quantization of GR, but rather leave it instead for a future occasion).

Note 1) By aiming to involve a notion of time at all, TQ and QT strategies favour ordinary QM, whereas Tempus Nihil Est and Non Tempus sed Historia are more radical in this respect.
Note 2) In finer detail, *what* change is to be used in Mach's statement [126, 129]? Various suggestions are
i) **any change** (Rovelli [66, 96], see also [119, 73, 77]).
ii) **All change** (Barbour [95], following Leibniz in emphasizing the configuration of the universe as a whole, and holding this to be the only perfect clock).
iii) **A sufficient totality of local relevant change (STLRC)**, which is my position, in which all changes are given the opportunity to contribute but only those that do so locally significantly are kept in the actual time calculation, as exemplified by the actual calculations of the astronomers' *ephemeris time*. See [126, 129] for further detail of these, as well as further detail of how to classify POT strategies by this criterion. The above t^{JBB} *looks* to be of the 'all' type, but, upon detailed examination [118, 129], its practicalities are of the STLRC form.
Note 3) In these various approaches to timekeeping, one should be careful not to confuse a convenient reading-hand (usually small: a conventional clock-like device) with what system is actually used to maintain calibration of one's timestandards/clocks (often much larger, such as the Earth–Moon–Sun system).
Note 4) One might also expect it only to make sense for one subsystem to furnish a time for another if the two are coupled dynamically or by time-frame determination procedure, else how would the one subsystem know about the other one it is keeping time *for*.

4 The Problem of Time Possesses Other Facets

Over the past decade, it has become more common to suggest or imply that the POT *is* the Frozen Formalism Problem. However, a more long-standing point of view [22, 25, 26, 27] (and also argued in favour for in e.g. [46, 65, 109, 118] and the present article) is that the POT contains a number of further facets. Furthermore, it is not even a case of then having to address around eight facets in succession. For, as Kuchař found [27], these facets interfere with each other. I argue [109, 118] that this occurs because the facets arise from a common cause – the mismatch of the notions of time in GR and Quantum Theory – by which they bear conceptual and technical relations, making it advantageous, and likely necessary for genuine progress, to treat them as a coherent package rather than piecemeal as 'unrelated problems'.

I also note that, despite the facets' common origin, almost all of the strategizing toward overcoming the POT has started from how to address the FFP. This is largely how POT approaches have been classified and conceptualized about. Whilst it does make for particularly eye-catching conceptualization, I would argue, rather, that
1) each facet should directly receive conceptual and strategic consideration, for which this review is a first step.
2) That it is likely that only some *combined* strategy will stand much of a chance of overcoming most of, or even all of, the POT; this review's last section considers a few combined strategies.
Thus there are seven further facets to examine, alongside various of the inter-relations between them. The FFP itself has the following addendum.

Hilbert Space/Inner Product Problem, i.e. how to turn the space of solutions of the frozen equation in question into a Hilbert space. See Sec 6.2 for why this is a problem (i.e. why ordinary QM inspired guesses for this will not do for GR). It is a time problem due to the ties between inner products, conservation and unitary evolution.

5 Further facets of the Problem of Time

Best Matching Problem [13, 59]. If one's theory exhibits configurational relationalism, linear constraints $\mathcal{L}\mathrm{in}_{\mathsf{Z}}$ follow from variation with respect to the auxiliary G-variables [these constraints generalize (7) in the $G = \mathrm{Diff}(\Sigma)$ case of GR]. The Best Matching procedure is then whether one can solve the Lagrangian form of $\mathcal{L}\mathrm{in}_{\mathsf{Z}}\lfloor \mathsf{Q}^{\mathsf{A}}, \mathsf{P}_{\mathsf{A}} \rfloor = 0$ for the G-auxiliaries themselves; this is a particular form of reduction. One is to then substitute the answer back into the action. This is clearly a classical-level procedure, an indirect implementation of configurational relationalism that is a bringing into maximum congruence by keeping $\mathsf{Q}^{\mathsf{A}}_1$ fixed and shuffling $\mathsf{Q}^{\mathsf{A}}_2$ around with G-transformations until it is as close to $\mathsf{Q}^{\mathsf{A}}_1$ as possible. From another perspective, it is establishing a point identification map [21] between configurations. It becomes a Problem due to being an often-obstructed calculation. In particular this is so in the case of GR (or, more widely, theories with a Diff-type G) where it has been more usually called the **Sandwich Problem** [2, 3], in reference to the construction of the spacetime 'filling' between 2 given spatial hypersurfaces ('slices of bread'). See Sec 7 for further generalization of this facet. One relation of this to the POT is clear via the title of [2] is "*three-dimensional geometry as carrier of information about time*"; furthermore, it becomes entwined in some POT approaches that start at the pre-quantum level (see Sec 7).

Classically, there are no Closure, Foliation-Dependence or Spacetime Reconstruction Problems. These are all ensured by the nature of the Dirac Algebra. From (8-10), closure is clear. In classical GR, one can foliate spacetime in many ways, each corresponding to a different choice of timefunction. This is how time in classical GR comes to be 'many-fingered', with each finger 'pointing orthogonally' to each possible foliation. Classical GR then has the remarkable property of being refoliation-invariant [10], so that going between two given spatial geometries by means of different foliations in between produces the same region of spacetime and so the same answers to whatever physical questions can be posed therein. The 'relativity without relativity' (RWR) approach [59] then amounts to a classical-level spacetime reconstruction: here spacetime structure is not presupposed, only configurational relationalism and temporal relationalism, and then the Dirac procedure for constraint algebra consistency returns GR spacetime as one of very few consistent possibilities. [GR is further picked out among these by the demand of foliation independence and the demand for a finite nonzero propagation speed.] RWR includes how, for *simple* [78] matter, local SR is deduced rather than assumed [59]. Moreover, following on from Sec 1, RWR shows how GR can not only be cast in Machian form, but also can be derived from Machian principles that assume less structure (though consult [78] for an update of what assumptions this approach actually entails). Finally, RWR is an answer to Wheeler's question about why $\mathcal{H}$ takes the form that it does (see [10] for an earlier answer, though that did assume embeddability into spacetime and thus does not constitute a classical-level spacetime reconstruction). However, one then no longer knows any means of guaranteeing these three nice properties at the quantum level.

QM Foliation Dependence Problem. That this is obviously a time problem follows from each foliation by spacelike hypersurfaces being orthogonal to a GR timefunction: each slice can be interpreted as an instant of time for a cloud of observers distributed over the slice, and each foliation corresponds to the these moving in a particular way. As Kuchař states [25], *"When one starts with the same initial state Ψ_{in} on the initial hypersurface and develops it to the final hypersurface along two different routes $\Psi_{\text{fin}-1} \neq \Psi_{\text{fin}-2}$ on the final hypersurface. Such a situation certainly violates what one would expect of a relativistic theory."*

QM Constraint Closure Problem = Functional Evolution Problem Closure of a classical Poisson bracket algebra does not entail closure of the corresponding quantum commutator algebra, due to 1) these algebras not necessarily being isomorphic due to global effects [15]. 2) the existence of anomalies (by definition quantum obstructions arising in cases that did not possess classical obstructions); Dirac [5] said that avoiding this required luck. The Functional Evolution Problem is then a subset of the possibility of anomalies due to time/frame issues in GR or some toy model of or alternative thereto. [I say a subset since only some of the anomalies that one finds in physics are time- or frame-related.] Foliation-dependent anomalies clearly exhibit *two* Facets of the POT. [Non-closure is also entwined with the Operator Ordering Problem, since changing the ordering gives additional right-hand-side pieces not present in the classical Poisson brackets algebra.]

Global Problem of Time alias **Kuchař's Embarrassment of Poverty** [25]. This is a more basic problem, which is already present at the classical level (recall Sec 1), as an obstruction to making globally-valid gauge choices in parallel with the well-known Gribov effect of Yang–Mills theory, except that in the present situation, gauge choices involve choices of times and frames. Its subfacets are, in one sense, i) timefunctions may not be globally defined in space. ii) Timefunctions may not be globally defined in time itself. In another sense, at the classical level this might be seen as a need to use multiple coordinate patches held together by the meshing conditions of classical differential geometry. Whilst that is classically straightforward, it remains largely unclear how a 'meshing together' of quantum evolutions might be formulated.

Multiple Choice Problem alias **Kuchař's Embarrassment of Riches** [25]. This is the purely quantum-mechanical problem that different choices of time variable may give inequivalent quantum theories. There is a notion of time is among the coordinates in GR; in some approaches, it is among the observables. Foliation Dependence is one of the ways in which the Multiple Choice Problem can manifest itself. Moreover, the Multiple Choice Problem is known to occur even in some finite toy models [25]. Thus foliation issues are not the only source of the Multiple Choice Problem. For instance, another way the Multiple Choice Problem can manifest itself is as a subcase of how making different choices of sets of variables to quantize (as per the Groenewold–Van Hove phenomenon [45]) may give inequivalent quantum theories.

Problem of Beables. This is usually called the Problem of Observables [5, 25, 26, 66], but I follow Bell [9, 12] in in placing emphasis on conceiving in terms of beables, which carry no connotations of external observing, but rather simply of being, which make them more suitable for the quantum-cosmological setting. The Problem involves construction of a sufficient set of beables/observables for the physics of one's model, which are then involved in the model's notion of evolution.

Spacetime (Reconstruction or Replacement) Problem. 1) At the at the quantum level, fluctuations of the dynamical entities are unavoidable, i.e. here fluctuations of 3-geometry, and these are then too numerous to be embedded within a single spacetime (see e.g. [7]). Thus (something like) the superspace picture – considering the set of possible 3-geometries – might be expected to take over from the spacetime picture at the quantum level. It is then not clear what becomes of causality (or of locality, if one believes that the quantum replacement for spacetime is 'foamy' [7]); in particular, microcausality is violated in some such approaches [67].
2) **Recovering continuity and**, a forteriori, **something that looks like spacetime** (e.g. as regards dimensionality) is an issue in discrete or bottom-up approaches to Quantum Gravity. This is not a given, since some approaches give unclassical entities or too low a continuum dimension. See Sec 13 for a third subfacet.

6 More detail of a selection of Frozen Formalism Problem Strategies

Fig 1 further refines Sec 3's classification. This includes a dichotomy of Tempus Nihil Est between Rovelli's school [23, 61, 66, 73] and the somewhat broader and partly older schools of Hawking, Page, Wootters, Barbour, Gell-Mann, Hartle, Halliwell and the Author [18, 14, 40, 58, 28, 31, 47, 50, 64, 83, 99, 125]. This review mostly only covers the latter (though see Sec 8 and [109] for comments about the former). The present Sec very largely ignores configurational relationalism/the ensuing linear constraints, which situation is rectified in the subsequent Sec. Thus suitable toy models for the present section include **minisuperspace** (GR restricted to the homogeneous solutions, which also entails no possibility of structure formation) and Sec 3's relational JBB formulation of mechanics.

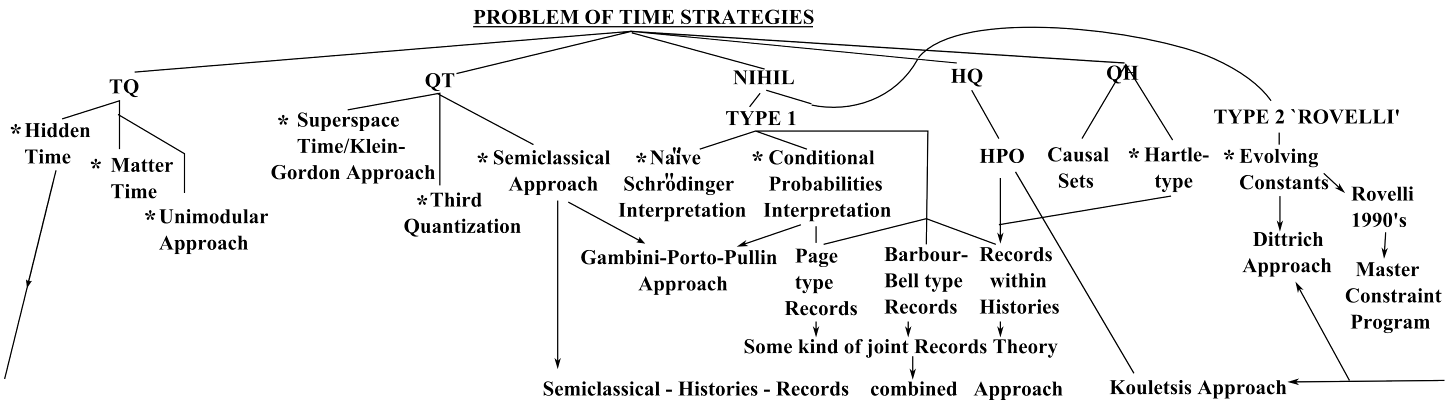

Figure 2: The various branches of strategy and their relations, in a diagram with 'cylindrical topology'. * indicates the 10 strategies covered by Kuchař and Isham's classic reviews [25, 26].

6.1 Tempus Ante Quantum (TQ)

Ante Postulate. There is a fundamental time to be found at the classical level for the full (i.e. untruncated) classical gravitational theory (possibly coupled to suitable matter fields). Candidate times of this type include the following.

1) Whilst $\mathtt{t}^{\mathrm{JBB}}$ is obtained in accordance with the Ante Postulate, it is not an unfreezing at the quantum level. Thus it is only a classical resolution of the FFP.

2) **Superspace time**. The space of spatial 3-geometries' DeWitt supermetric has signature $-+++++$. Perhaps one could then take the indefinite direction to be pick out a timefunction, in parallel to how the indefinite direction in Minkowski space does for such as Klein–Gordon theory. Schematically,

$$\triangle_{\mathcal{G}} \quad \text{is actually a} \quad \Box_{\mathcal{G}} \ . \tag{19}$$

This approach was traditionally billed as Tempus Post Quantum, through choosing to make this identification of a time after the quantization, but the approach is essentially unchanged if this identification is made priorly at the classical level. Regardless of when this identification is made, however, this approach fails by [11, 22] the GR potential not being as complicit as Klein–Gordon theory's simple mass term was (by not respecting the GR configuration space's conformal Killing vector).

3) The above is one example of a scale time. Scale or related quantities are often used as internal times, for instance the cosmological scalefactor a, or the local spatial volume element $\sqrt{h}$ that goes as a^3 in the (approximately) isotropic case.

Part-linear/parabolic implementation. While the above straightforward schemes fail, it may still be that a classical time exists but happens to be harder to find. We now consider starting one's scheme off by finding a way of solving $\mathcal{Q}$uad to obtain a part-linear/parabolic form, where I make a single-time versus time-and-frame formulation distinction

$$\mathtt{P}_{\mathtt{t}^{\mathrm{ante}}} + H^{\mathrm{true}}\lfloor \mathtt{t}^{\mathrm{ante}}, \mathtt{Q}^{\mathsf{A}}_{\mathrm{other}}, \mathtt{P}^{\mathrm{other}}_{\mathsf{A}} \rfloor = 0 \ \text{ or } \ \mathtt{P}^{\mathcal{X}}_{\mu} + H^{\mathrm{true}}_{\mu}\lfloor \mathcal{X}^{\mu}, \mathtt{Q}^{\mathsf{A}}_{\mathrm{other}}, \mathtt{P}^{\mathrm{other}}_{\mathsf{A}} \rfloor = 0 \ . \tag{20}$$

Here, $\mathtt{P}_{\mathtt{t}^{\mathrm{ante}}}$ is the momentum conjugate to a candidate classical time variable, $\mathtt{t}^{\mathrm{ante}}$ which is to play a role parallel to that of external classical time. Also, $\mathtt{P}^{\mu}_{\mathcal{X}}$ are the momenta conjugate to 4 candidate embedding variables $\mathcal{X}^{\mu}$, which form the 4-vector $[\mathtt{t}^{\mathrm{true}}, \mathrm{X}^i]$ X^i are 3 spatial frame variables. H^{true} is then the 'true Hamiltonian' for the system. Also, $H^{\mathrm{true}}_{\mu} = [H^{\mathrm{true}}, \Pi^{\mathrm{true}}_i]$, where Π^{true}_i is the true momentum flux constraint. By passing as soon as possible to having an object that plays such a role, TQ can be viewed as the most conservative family of strategies [26, 42]. Given such a parabolic form for $\mathcal{H}$, it becomes possible to apply a conceptually-standard quantization that yields the time-dependent Schrödinger equation,

$$i\mathtt{D}\Psi/\mathtt{D}\mathtt{t}^{\mathrm{ante}} = \widehat{H}_{\mathrm{true}}\lfloor \mathtt{t}^{\mathrm{ante}}, \mathtt{Q}^{\mathsf{A}}_{\mathrm{other}}, \mathtt{P}^{\mathrm{other}}_{\mathsf{A}} \rfloor \Psi \ \text{ or } \ i\mathtt{D}\Psi/\mathtt{D}\mathcal{X}^{\mu} = \widehat{H}^{\mathrm{true}}_{\mu}\lfloor \mathcal{X}^{\mu}, \mathtt{Q}^{\mathsf{A}}_{\mathrm{other}}, \mathtt{P}^{\mathrm{other}}_{\mathsf{A}} \rfloor \Psi \ , \tag{21}$$

($\mathtt{D}$ is the portmanteau of partial derivative ∂ for finite theories and functional derivative δ for field theories). The above unfreezing is accompanied, at least formally, by the obvious associated Schrödinger inner product. Single-time and time-and-frame part-linear forms then occur for parametrized nonrelativistic particle and parametrized field theory toy models respectively [25]. GR examples of part-linear forms are as follows.

4) **Hidden time Internal Schrödinger Approach** A first such suggestion is that there may be a *hidden* alias *internal* time [8, 11, 25, 26, 117] within one's gravitational theory itself. I.e. that the apparent frozenness is a formalism-dependent statement, to be removed via applying some canonical transformation. In the first part-linear form scheme, this sends GR's spatial 3-geometry configurations to 1 hidden time, so $\mathrm{t}^{\mathrm{ante}} = \mathrm{t}^{\mathrm{hidden}}$ plus 2 'true gravitational degrees of freedom' [which are the form the 'other variables' take, and are here 'physical' alias 'non-gauge']. The general canonical transformation is here

$$(\mathtt{Q}^{\mathsf{A}}, \mathtt{P}_{\mathsf{A}}) \longrightarrow (\mathtt{t}^{\mathrm{hidden}}, \mathtt{p}_{\mathtt{t}^{\mathrm{hidden}}}, \mathtt{Q}^{\mathsf{A}}_{\mathrm{true}}, \mathrm{P}^{\mathrm{true}}_{\mathsf{A}}) \quad \text{or} \quad (h_{ij}(x^k), \pi^{ij}(x^k)) \longrightarrow (\chi^\mu(x^k), \Pi_\mu(x^k), \mathrm{Q}^{\mathsf{A}}_{\mathrm{true}}(x^k), \mathrm{P}^{\mathrm{true}}_{\mathsf{A}}(x^k)) \ , \tag{22}$$

for $\mathrm{Q}^{\mathsf{A}}_{\mathrm{true}}(x^i)$ the true gravitational degrees of freedom and embedding variables χ^μ (a 4-component time-and-frame quantity). Thus one arrives at a hidden time-dependent Schrödinger equation of whichever of the two forms in (21), with the above significances attached to the variables (for $\mathrm{t}^{\mathrm{hidden}} = \mathrm{t}^{\mathrm{ante}}$ or $\chi^\mu = \mathcal{X}^\mu$).

A particular sub-example of this involves the canonical conjugate of $\sqrt{h}$, namely the *York time* ([8]). [This is proportional to the constant mean curvature (CMC) ($K =$ constant and thus $h_{ij}\pi^{ij}/\sqrt{h} =$ constant$'$.] In this shape-scale split formulation, the GR constraints decouple on CMC slices so that one can at least formally solve the momentum constraint prior to the Hamiltonian constraint. This approach's furtherly-reduced configuration space is the space of conformal 3-geometries: conformal superspace $\mathrm{CS}(\Sigma) := \mathrm{Superspace}(\Sigma)/\mathrm{Conf}(\Sigma)$, to which a solitary global spatial volume of the universe variable is usually adjoined; here, $\mathrm{Conf}(\Sigma)$ are the conformal transformations on Σ. This approach is in practice hampered by Example 1.2 of Sec 11's and by the Lichnerowicz–York equation (conformalized Hamiltonian constraint) not being explicitly solvable. Substantial progress with shape–scale split formulations in the case of Ashtekar variables has been but recent [123].

5) **Straightforward Matter Time**. Typically in minisuperspace for 1-component scalar matter, one simply isolates the corresponding momentum to play the role of the time part of the subsequent wave equation. This often taken to be 'the alternative' to scale time [e.g. this is often so in the **Loop Quantum Cosmology (LQC)** literature, which is a minisuperspace analogue but now based on a higher-order difference equation form of WDE]. However, the momenta conjugate to each of these represent 2 further possibilities, and then, if if canonical transformations are allowed, a whole further host of possibilities become apparent. Finally, using a matter scalar field as a time is often argued to be 'relational' [70], though this is clearly only the 'any change' form of the Machian recovery of time, so more scrutiny might be considered. [For each such model, does this candidate matter time possess the features expected of a time? If there is multi-component matter, why should one matter species be given the privilege of constituting the time? If one uses other times – *equally* relational in the above sense – does the Multiple Choice Problem appear?]

6) **Reference Matter Time**. a) One can look to attain the part-linear form i) of (22) is attained with $\mathrm{t}^{\mathrm{ante}} = \mathrm{t}^{\mathrm{matter}}$ and $\mathrm{Q}^{\Gamma}_{\mathrm{other}} = h_{ij}$, or (better) the 4-component version, by extending the geometrodynamical set of variables to include matter variables coupled to these, which then serve so as to label spacetime events [25]. b) One could additionally form a quadratic combination of constraints [38], resulting in strongly-vanishing Poisson brackets. Technical problems with these include involving a choice between time resolution and physicality of the matter, and the more general non-correspondence between reference matter and tangible, observed matter; see e.g. [25, 109, 132] for further details. [112] exemplifies newer such approaches using both further types/formulations of matter and the Ashtekar variables formulation of gravitation.

7) **Unimodular gravity implementation** One might also regard an undetermined cosmological constant Λ as a type of reference fluid [19]. Here, one does not consider the lapse to be a variable that is to be varied with respect to. Instead, $\mathcal{H}_i$ has $\mathcal{H}_{,i}$ as an integrability [c.f. (10)], leading to $\mathcal{H} + 2\Lambda = 0$ with $\mathcal{H}$ the vacuum expression and Λ now interpreted as a constant of integration. The unimodular approach has problems of its own as a POT resolution (one is in Sec 9; see [25, 26, 118] for further such).

Aside from the individual technical problems of each of the above approaches, finding or appending a time is contrary to inherent primary Leibniz timelessness and subsequent Machian time emergence. Furtherly, these times are particular special changes rather than allowing all changes to contribute in principle as per the STLRC interpretation of Machian time emergence. E.g. matter change has no opportunity to contribute to scale, York or unimodular times, whilst gravitational change has no opportunity to contribute to matter time. Finally, the link between such things as scale, the cosmological constant, conjugates to these or special, unobservable reference fluids, and what actually constitutes the reading-hand and calibration of clocks is largely unclear, conceptually or practically, both qualitatively and as regards how none of these things are known to the desired, and used, accuracy in timekeeping.

Machian Status Quo. This is based on *current knowledge* of $\mathtt{T}$ ... $\mathtt{Q}$ approaches that remain unfreezing at the quantum level and how these are all characterized as un-Machian and bereft of accuracy. Thus $\mathtt{Q}$ ... $\mathtt{T}$ *looks to be presently required for a satisfactory QM-level Machian time* in the STLRC sense.

6.2 Tempus Post Quantum (QT)

Post Postulate In strategies in which time is not always present at the fundamental level, time is nevertheless capable of emerging in the quantum regime. Because this is an emergence, it means that the Hilbert space structure of the final quantum theory is capable of being (largely) unrelated to that of the WDE-type quantum theory that one starts with. Such emergent strategies are of the following types.

1) A scheme based on Schrödinger inner product fails due to the indefiniteness of the WDE.
2) **Attempting a Klein-Gordon Interpretation based on superspace time** then fails just as for its previous oppositely-ordered case. Noting however the parallel with Klein–Gordon failing as a first-quantization leading to its reinterpretation as a second-quantized QFT, one might then try the following.
3) **Third Quantization**, i.e. that the solutions $\Psi[h]$ of the WDE might be turned into operators:

$$\widehat{\mathcal{H}}\widehat{\Psi}\psi = 0 \ . \tag{23}$$

However, this turns out not to shed much light on the POT [25], and, whilst Third Quantization recurs [76] in the Group Field Theory approach to Spin Foams (themselves described in Sec 6.4), its use there is not extended to furnish a POT strategy either.

4) **Semiclassical approach** [6, 17, 25, 26, 65, 79]. Perhaps one has slow, heavy 'h' variables that provide an approximate timestandard with respect to which the other fast, light 'l' degrees of freedom evolve [17, 25, 65]. In the Halliwell–Hawking [17] scheme for GR Quantum Cosmology, h is scale (and homogeneous matter modes) and the l-part are small inhomogeneities.[5] I.e., it is a perturbative midisuperspace model. The most usual Semiclassical Approach for a POT strategy involves making i) the Born–Oppenheimer (BO) ansatz $\Psi(\mathrm{h},\mathrm{l}) = \psi(\mathrm{h})|\chi(\mathrm{h},\mathrm{l})\rangle$,

ii) the WKB ansatz
$$\psi(\mathrm{h}) = \exp(i\,S(\mathrm{h})) \ ; \tag{24}$$

in each case one makes a number of associated approximations.
[Evoking semiclassicality only makes sense if one's goals are somewhat modest as compared to some more general goals in Quantum Gravity programs. Nevertheless this is still useful for some applications, including the foundations of practical Quantum Cosmology i.e. testing this robustness of calculations along the lines of Halliwell and Hawking's.]
iii) One forms the h-equation $\langle\chi|\widehat{\mathcal{H}}\Psi = 0$. Then, under a number of simplifications, this yields a Hamilton–Jacobi equation $\{\partial S/\partial \mathrm{h}\}^2 = 2\{E - V(\mathrm{h})\}$, where $V(\mathrm{h})$ is the h-part of the potential. One way of solving this involves doing so for an approximate emergent semiclassical time $t^{\mathrm{em(WKB)}}(\mathrm{h})$; this is a recovery of a 'time before quantization' timefunction, namely $\mathtt{t}^{\mathrm{JBB}}$; moreover, this now does have a QM-unfreezing status in this new emergence.
iv) One then forms the l-equation $\{1 - |\chi\rangle\langle\chi|\}\widehat{\mathcal{H}}\Psi = 0$. This fluctuation equation can be recast (modulo further approximations) into a $\mathtt{t}^{\mathrm{em(JBB)}}$-dependent Schrödinger equation for the l-degrees of freedom,

$$i\mathtt{D}|\chi\rangle/\mathtt{D}\mathtt{t}^{\mathrm{em}} = \widehat{\mathcal{H}}_{\mathrm{l}}|\chi\rangle \ \ , \tag{25}$$

the emergent time dependent left-hand side of which arises from the cross-term $\partial_{\mathrm{h}}|\chi\rangle\partial_{\mathrm{h}}\psi$. $\widehat{\mathcal{H}}_{\mathrm{l}}$ is the remaining surviving piece of $\widehat{\mathcal{H}}$, acting as a Hamiltonian for the l-subsystem. Note that the working leading to such a time-dependent wave equation ceases to function in the absence of making the WKB ansatz and approximation, which, additionally, in the quantum-cosmological context, is not known to be a particularly strongly supported ansatz and approximation to make. In the usual Semiclassical Approach, the time-provider to studied subsystem coupling feature is obligatory, else the emergent time derivative in eq (25) is built from a product containing a zero factor in it.

Note 1) The first approximation used here is rather un-Machian (in the STLRC sense) via deriving its change just from scale (plus possibly homogeneous isotropic matter modes, or, more widely, from the usually-small subset of h degrees of freedom). However, the second approximation remedies this by allowing anisotropic and inhomogeneous changes (or, more widely, whatever else the l-degrees of freedom may be) to contribute to the corrected emergent time. Contrast this with the criticism at the end of Sec 6.1. Finally, the detailed emergent time here is *not* exactly the same as the detailed classical $\mathtt{t}^{\mathrm{JBB}}$, as is to be expected from how *quantum* change corrections influence this new quantum-level version.
Note 2) **WKB Problem**. Making the WKB approximation requires justification (see e.g. [79] including for references to the original literature), without which there is a serious danger of 'merely passing the buck' from a time to a set of wavefronts very heavily implying a time. Without this ansatz, the above sketch of obtaining of an emergent-time-dependent Schrödinger equation for the l-subsystem in general fails.
Note 3) The qualitative types of often-omitted terms in semiclassical Quantum Cosmology are: non-adiabaticities, other (including higher) emergent time derivatives and averaged terms [79, 48, 121]. Including the last of these parallels the

[5] This is a quantum explanation for the origin of structure in the universe - the seeding of galaxies and of CMB inhomogeneities. The possibility of making a connection between quantum-cosmological perturbations and the observed universe is usually via some inflationary mechanism, and renders this semiclassical Quantum Cosmological regime particularly valuable among quantum-gravitational regimes from the perspective of future observations. This example only makes sense once linear constraints are included, i.e. it is a 'perturbations about minisuperspace' **midisuperspace** treatment.

use of Hartree–Fock self-consistent iterative schemes, though the system is now more complex via involving a chroniferous quantum-average-corrected Hamilton–Jacobi equation.
Note 4) The imprecision due to omitted terms means deviation from exact unitarity.

6.3 Tempus Nihil Est

Temporal and Atemporal questions. Questions are closely related to *propositional logic* (see [4, 33, 57, 118, 126] for applications of this to fundamental physics).
Questions of Being are covered by some atemporal logic. Among these, **Questions of Conditioned Being** play a particular role. These involve two properties within a single instant: given that an (approximate) (sub)configuration has property P_1, what is the probability that it also has property P_2?
Moreover, in the presence of a meaningful notion of time, one can additionally consider the following question-types to be primary.
I) **Questions of Being at a Particular Time** have the general form Prob(S_1 has property P_1 as the timefunction $\mathtt{t}$ takes a fixed value $\mathtt{t}_1$).
II) **Questions of Becoming**, on the other hand, furthermore involve a given particular (approximate) (sub)system state becoming some other (approximate) (sub)system state. As such, they are covered by a more complicated temporal logic.
Moreover, 1) is straightforward to eliminate and some schemes for eliminating 2) too have also been put forward. If these things are possible, one would then expect [83] the form of the remaining physics to strongly reflect the structure of atemporal logic (at the classical level, Boolean logic suffices, but this is not obeyed by quantum propositions in general, leading to suggestions for the use of 'quantum logic' [32, 33] or the 'intuitionistic logic' of Topos Theory [89]).

Nihil Postulate. One aims to supplant 'becoming' with 'being' at the primary level [40, 58, 31, 47, 28, 50, 62, 64, 83, 99]. In this sense, Timeless Approaches consider the instant/space as primary and spacetime/dynamics/history as secondary.
Adopting a Tempus Nihil Est approach allows one to avoid the difficult issue of trying to define time as outlined in Sec 1. However, one then has to face three other problems instead.

1) **Semblance of dynamics Problem** [31, 47, 64, 65]. How to explain the semblance of dynamics if the universe is timeless as a whole. Dynamics or history are perhaps now to be *apparent notions* to be constructed from one's instant, though possibly also Histories Theory could provide an explanation (at the cost of more structure being assumed), see Sec 6.4. Fig 2 gives a split into Type 1 and Type 2 'Rovelli' Tempus Nihil Est approaches. Type 1 includes various forms of timeless Records Theory have appeared, including Barbour's [31], Page's [40, 58] mine [83, 118, 129], and also Records Theory within Histories Theory (work of Halliwell [50] building on work of Gell-Mann and Hartle [28]); see below for more. Type 2 'Rovelli' is briefly covered in Sec 8; this is based on Rovelli's 'any change' relational approach to time.

2) **Nonstandard Interpretation of Quantum Theory**. Type 1 Tempus Nihil Est and Histories Theory both come to involve in general questions about the interpretation of Quantum Theory, in particular as regards whole-universe replacements for standard Quantum Theory's Copenhagen Interpretation. I note that some criticisms of Timeless Approaches [25, 46] are subject to wishing to preserve aspects of the Copenhagen Interpretation, which may not be appropriate for Quantum Cosmology and Quantum Gravity.

3) **WDE Dilemma**. Such approaches either1) invoke the WDE and so inherit some of its problems, or they do not, thus risking the alternative problem 2) of being incompatible with it, so that the action of the WDE operator kicks purported solutions out of the physical solution space [25].

Naïve Schrödinger interpretation (NSI) [18, 19]. This first example of a Nihil strategy concerns the 'being' probabilities for universe properties such as: what is the probability that the universe is large? One obtains these via consideration of the probability that the universe belongs to region R of the configuration space that corresponds to a quantification of a particular such property,

$$\mathrm{Prob}(\mathrm{R}) \propto \int_{\mathrm{R}} |\Psi|^2 \mathcal{D}\mathtt{Q} \ , \tag{26}$$

for $\mathcal{D}\mathtt{Q}$ the configuration space volume element. This approach is termed 'naïve' due to it not using any further features of the constraint equations; it also involves generally non-normalizable probabilities, which, however, support finite *ratios* of probabilities. Its implementation of propositions is Boolean via how the classical regions enter the integrals; this is problematic as per this SubSec's first paragraph.

Proposition–projector association. The aim here is to represent propositions at the quantum level by projectors (taken to include beyond [33] the usual context and interpretation that these are ascribed in ordinary Quantum Theory). This is in contradistinction to the above kind of attempts at representation via regions of integration.

In ordinary quantum theory, for state density matrix ρ and proposition P implemented by projector $\widehat{\mathrm{P}}$, $\mathrm{Prob}(P; \rho) = \mathrm{tr}(\widehat{\rho}\widehat{\mathrm{P}})$ with Gleason's theorem providing strong uniqueness criteria for this choice of object from the perspective of satisfying the basic axioms of probabilities (see e.g. [37]). The formula for conditional probability in ordinary Quantum Theory is then [37]

$$\text{Prob}(B \in b \text{ at } \mathtt{t} = \mathtt{t}_2 | A \in a \text{ at } \mathtt{t} = \mathtt{t}_1; \rho) = \frac{\text{Tr}\big(\text{P}_b^B(\mathtt{t}_2)\,\text{P}_a^A(\mathtt{t}_1)\,\rho\,\text{P}_a^A(\mathtt{t}_1)\big)}{\text{Tr}\big(\text{P}_a^A(\mathtt{t}_1)\,\rho\big)} \ . \tag{27}$$

N.B. that this is in the ordinary-QM 2-time context, i.e. to be interpreted as *subsequent* measurements. [P_a^A is the projection operator for an observable A an observable and a a subset of the values that this can take.]

Supplant at-a-time by value of a particular 'clock' subconfiguration. [Not all used of this necessarily carry 'good clock' connotations: in the literature, this is done for whichever of the 'any', 'all' and 'sufficient local' connotations.]

Conditional Probabilities Interpretation (CPI) [14]. This goes further than the NSI by addressing conditioned questions of 'being', and, moreover uses proposition–projector implementation in a *non*standard context. Namely, the conditional probability of finding B in the range b, given that A lies in a, and to allot to it the value

$$\text{Prob}(B \in b | A \in a; \rho) = \frac{\text{tr}\big(\text{P}_b^B\,\text{P}_a^A\,\rho\,\text{P}_a^A\big)}{\text{tr}\big(\text{P}_a^A\,\rho\big)} \ . \tag{28}$$

Note 1) it is these occurring within the one instant rather than ordered in time (one measurement and *then* another measurement) that places this construct outside the conventional formalism of Quantum Theory, for all that (28) superficially resembles (27).
Note 2) The CPI can additionally deal with the question of 'being at a time' by having the conditioning proposition refer to a 'clock' subsystem [14, 26].
Note 3) The CPI does implement propositions at the quantum level by use of projectors; its traditional development did not set up a scheme of logical propositions (it came historically before awareness of that began to enter the POT community via [33, 36]). However, this layer of structure can be added to the CPI by the first principles reasons argued above.
Note 4) **Supplanting becoming questions** in this kind of context has been considered e.g. in [40, 58, 47, 83]. In Page's particular version, one does not have a sequence of a set of events; rather the scheme is a single instant that contains memories or other evidence of 'past events'.
Note 5) Kuchař criticized the CPI in [25, 46] for its leading to incorrect forms for propagators. Page answered that this is a timeless conceptualization of the world, so it does not need 2-time entities such as propagators; see also Sec 14.

Records theory remains a heterogeneous subject, meaning that different people have postulated different axioms for it.
A) One can view the preceding extension as **Page's form of Records Theory**.
B) **Bell–Barbour Records** [12, 31, 47]. Take how α-particle tracks form in a bubble chamber as a "**time capsules**" paradigm for Records Theory; perhaps Quantum Cosmology can be studied analogously [31, 47, 64, 99], though here the paths are for the whole universe and in configuration space. Barbour's own approach has a number of additional elements [31, 47]; see [118] for some commentary and caution.
C) **Gell-Mann–Hartle–Halliwell Records** [28] and Halliwell [50, 62, 64] have found and studied records contained within Histories Theory (see the next SubSec).
D) My own axiomatization for Records Theory prior to the above 'semblance of dynamics' divergence is as follows [83].
Records Postulate 1. Records are information-containing subconfigurations of a single instant that are localized in both space and configuration space. Local in configuration space concerns the imperfection of knowledge in practise, i.e. a notion of coarse graining.

Records Postulate 2 Records can be tied to atemporal propositions, which, amount to a suitable logic. [The tying at the quantum level is preferably by the Projector–Proposition Association. The form of the logical structure remains open to debate. The notion of localization in configuration space may well furnish the graining/partial order/logical implication operation.] [89] may be viewed as basis for implementation of a Records theory, though this is a bridge in the process of being built [131].
Records Postulate 3. Records are furthermore required to contain useful information. I take this to mean information that is firstly and straightforwardly about correlations. Secondly, however, one would wish for such correlation information to form a basis for a semblance of dynamics or history.
Such a scheme therefore requires [83] suitable notions of locality in space and in configuration space, of propositional logic, of information, relative information and correlation.
Records Postulate 4 [129] Semblance of time to be abstracted from records is to be of STLRC form. This is not as strong a selector as one might suspect, in that many timeless schemes built on Rovelli- or Barbour-type notions of Machian emergent time can be converted to this framework. Indeed, Records 1 to 4 can be seen as a new body in which to cast various approaches - a second-generation CPI/Page Records approach can have logic considerations and seek for conditioning to be on one's best estimate at that point on time abstracted from the STLRC. This is laid out in more detail in [129].
Records Theory is, finally, the subsequent study of how dynamics (or history or science) is to be abstracted from correlations between such same-instant subconfiguration records.

6.4 Histories Theory

Perhaps instead it is the histories that are primary, a view brought to the GR context by Gell–Mann and, especially, Hartle [28, 34] and subsequently worked on by Isham, Linden, Savvidou and others [33, 36, 54, 67, 69, 86].

Histories postulate. Treat histories (rather than configurations) as one's primary dynamical entities.

Let us first assume that we treat histories at the classical level: Historia ante Quantum! (HQ) Then
1) histories are sequences of configuration instants at given label-times, $\mathtt{Q}^{\mathtt{A}}(\mathtt{t}_i)$ $i = 1$ to n (discrete model) or curves in configuration space parametrized by a continuous label-time.
2) One's taking the histories to be the dynamical objects means that one has to define history momenta and then history Poisson brackets (such approaches are then often referred to as 'histories brackets' approaches) and a histories quadratic constraint, $\mathrm{Quad}\lfloor \mathtt{Q}^{\mathtt{A}}(\mathtt{t}), \mathtt{P}_{\mathtt{A}}(\mathtt{t})\rfloor = 0$.
3) One also has notions of coarse- and fine-graining of histories (much as one does for records, and which in fact preceded that in the literature).
Let us now conceive of histories at the quantum level, whether because one is doing 'Historia Post Quantum' (QH) or one is promoting the classical part of 'Historia ante Quantum' to the quantum level.
0Q) Naively at the quantum level, one has a corresponding Feynman path integral structure ('sum over histories'). However, Histories Theory itself is usually taken to have more structure than that.
1Q) Individual histories are now built as strings of projectors $\mathrm{P}^{A_i}_{a_i}(\mathtt{t}_i)$, i = 1 to N at times $\mathtt{t}_i$ (Hartle-type approach [34]), or a continuous limit of a tensor product counterpart (Isham–Linden **Histories Projection Operator (HPO)** approach [33]). One distinction between these is that only the latter's products of projectors are themselves projectors and thus implementors of whole-histories propositions according to the very useful proposition–projector association. Questions about histories are then another simplified form of logical structure as compared to temporal logic [33, 102]. HPO is a QFT *in the (label) time direction*, even for 'conventionally finite' models.
2Q) In the HPO approach, which is usually taken to be HQ, there is a kinematical commutator algebra of histories, and a quantum histories quadratic constraint.
Regardless of whether it is HPO or of which H, Q ordering is used, the following additional layers of structure are considered.
3Q) There continue to be notions of coarse- and fine-graining at the quantum level.
4Q) Given a pair of histories γ, γ', the corresponding **decoherence functional** is

$$\mathrm{Dec}(c_{\gamma'}, c_\gamma) := \mathrm{tr}(c_{\gamma'}\rho c_\gamma) \ . \tag{29}$$

Note 1) In such as path integrals or composites thereof such as explicit computations of decoherence functionals, one is to use label time (or emergent time) when available (a few approaches use some kind of internal/matter time instead).
Note 2) 'Historia ante Quantum' (HQ) approaches can be seen as providing a second opportunity to a number of TQ approaches and group/geometrical quantization methods. E.g. new kinematical quantization, commutator and constraint bracket-algebras, and new possibilities for formulation of frame variables (such as Kouletsis' space maps [86]).
Note 3) Savvidou pointed out [52, 69] that this version of Histories Theory has a distinct structure for each of two conceptually distinct notions of time:
I) a kinematical notion of time that labels the histories as sequences of events (the 'labelling parameter of temporal logic' , taken by [69] to mean causal ordering, though see also [33].
II) A dynamical notion of time that is generated by the Hamiltonian.
Savvidou has argued that having these two distinct notions of time allows for such a Histories Theory to be canonical and covariant at once, which is of obvious interest in understanding, and reconciling various viewpoints in, Quantum Gravity.
Note 4) The records scheme sitting within a Histories Theory [28, 114, 50, 62, 64, 74, 99, 116] is independent of the Gell-Mann–Hartle versus Isham–Linden distinction because these involve the single-time histories, i.e. a single projector, and then the ordinary and tensor products of a single projector obviously coincide and indeed trivially constitute a projector. Thus one can apply the Projector–Proposition Association and nicely found a propositional logic structure on this.

It is convenient to end with some examples of (more or less) discrete approaches, some of which can be taken to be more structurally minimalist counterparts of Histories Theory.
1) **Spin foams** [66, 82, 104] can be viewed as path-integral counterparts of LQG. As such, one might further build these approaches up to possess further histories-theoretic structure. There is a Hartle-type spin-foam work presently nearing completion [130]. On the other hand, Savvidou has cast Barbero-real-variables GR in HPO form, and some elements of Histories Theory also enters into composite attempts at resolving the POT in Ashtekar Variables programs, see e.e. [82].)
2) The **Causal Sets Approach** [63] may be viewed as much like a Histories Theory in which less structure is assumed. Here, all that is kept at the primary level is the set of events and the causal ordering structure on these.
3) In juxtaposition and for later use, the **Causal Dynamical Triangulation (CDT)** approach [107] is a path integral/sum over histories approach but does not conventionally involve further histories-theoretic elements. It does also retain causal structure, and considers as primary a continuum limit rather than about the discrete structure used to build up to that. Thus it assumes somewhat more structure than in the Causal sets approach.]

7 Configurational relationalism generalization of Best Matching Problem

The most general conceptual entity here is indeed Sec 1's configurational relationalism; there is no strong reason why this needs to be addressed (just) at the level of the classical Lagrangian formalism that is the domain of the best-matching generalization of the thin-sandwich problem. Nontrivial configurational relationalism produces linear constraints, which need to be solved at *some* level; up until that level, one may well need to consider other indirectly-formulated G-invariant objects (notions of distance, of information, quantum operators...). All of these and best-matched actions themselves are objects of the following G-act, G-all form[6] e.g. the group action $\vec{G}_g$ followed by integration over the group G itself,

$$\mathrm{A} = \int_{g \in G} \mathcal{D}g \, \vec{G}_g \, A \, . \tag{30}$$

This is limited for full quantum gravity by not being more than formally implementable for the case of the 3-diffeomorphisms.

Thus one needs to order not just T and Q tuples, but also a R = (R, nothing) tuple. This generalizes 'reduced quantization' and 'Dirac quantization', which come from a context in which time-choosing was not necessary and simply mean R ... Q and Q ... R. Moreover, this is but one of several further procedural ordering ambiguities.[7] Classically, there are a priori 8 schemes. This is the same combinatorial 8 as in Sec 3. See Appendix A for the countings. Including the quantum also, there are now 27 combinations of strategies One can go a bit further with strategic diversity: can also consider 'χ or plain' versions of T and H (χ meaning time-and-frame constructions rather than the plain case's single-time constructions). However, the R ... χ ordering makes no sense and so is to be discarded. On the other hand, TR = RT since each of these procedures acts on separate parts of one's relational theory. Taking this further diversity and physical restriction into account too, there are 10 classical schemes (4 single, 4 frame and 2 Nihil). Allowing for the possibility of QM as well, there are now 35 schemes (16 single-time, 14 time-frame and 5 Nihil).

Example 1) Configurational relationalism is trivial in minisuperspace and whatever other Lin$_{\mathsf{Z}}$-bereft models.
Example 2) **relational particle mechanics (RPM's)** [13, 118] are useful models for configurational relationalism, midisuperspace nontrivialities and the Problem of Beables. The notation below is N particles in dimension d. $n = N - 1$ is used because of the triviality of removing translations. Rotations Rot(d) are here the analogues of Diff(Σ), with a linear zero total angular momentum constraint corresponding to the GR momentum constraint and relationalspace $\mathrm{R}(N,\, d) := \mathbb{R}^{nd}/\mathrm{Rot}(d)$ being the counterpart of superspace(Σ). Dilations Dil as analogues of Conf(Σ), corresponding to, in the pure-shape RPM case a $D = \sum_I Q^{aI} P_{aI} = 0$ constraint analogous to the GR maximum slicing condition, though in the scaled-RPM case D plays a similar role to the York time. [The I indexes particle labels.] Finally, preshape space is $\mathrm{P}(N,\, d) := \mathbb{R}^{nd}/\mathrm{Dil} = \mathbb{S}^{nd-1}$ and shape space is $\mathrm{S}(N,\, d) := \mathbb{R}^{nd}/\mathrm{Rot}(d) \circledS \mathrm{Dil}(d)$, which is the counterpart of GR's CS(Σ). I use S^{A} to denote shapes, σ for scale (the total moment of inertia or its square root, in parallel to $\sqrt{h} = a^3$ and a for cosmology), with corresponding shape momenta p_{S} and scale momentum p_σ. That particle models obviously have a notion of structure as well as of linear constraint, thus completing their midisuperspace-like features. RPM's serve as toy models for semiclassical, records and histories approaches.
Example 3) In geometrodynamics, resolving the Configurational Relationalism Problem at whatever level would entail working explicitly with the 3-geometries themselves.
Example 4) As regards LQG, using loops, holonomies or spin-nets involve having taken out the $SU(2)$, and the Diff(3) also in the case of knots. Note the LQG–RPM analogy: loops to preshapes and knots to shapes.

1) In the CT scheme, the **emergent JBB time** now takes the configurational relationalism entwined form

$$t^{\mathrm{JBB}} \underset{\text{of RPM action}}{\overset{\text{extremum } g \in \mathrm{G}}{=}} \int ||\mathrm{d}_g \mathbf{Q}||_{\mathbf{m}}/\sqrt{E - V} \text{ (RPM)} \quad \text{or} \quad \mathrm{t}^{\mathrm{JBB}} \underset{\text{of GR relational action}}{\overset{\text{extremum } g \in \mathrm{Diff}(\Sigma)}{=}} \int_\Sigma \mathrm{d}^3 x \sqrt{h} \int ||\mathrm{d}_g \mathbf{h}||_{\boldsymbol{\mathcal{G}}}/\sqrt{\mathrm{Ric}(\mathbf{x},\mathbf{h}]} \text{ (GR)} \, . \tag{31}$$

[Here, d_g is the differential corrected by the infinitesimal group action of G, and $\mathbf{m}$ is the mass matrix kinetic metric.] Further features of this candidate timefunction are discussed in [118, 129]. Moreover, the Best Matching Problem has been explicitly solved in 1- and 2-d RPM's [81, 118] (for GR, this gives the Thin Sandwich Problem). Unfortunately while this CT scheme yields a time function, it does not in any way unfreeze the GR Hamiltonian constraint or its energy constraint RPM analogue; thus a CTQ scheme based on it is not satisfactory. One might thus use an internal or matter time instead, or postpone enquiring about a time that is serviceable at the quantum level till one is at the quantum level.
2) Moreover, there is also a Direct Approach for these; this has no reduction step in use and yet the theory is nontrivial, via Kendall's procedure [51] for constructing the shape spaces for these cases at the metric level and my subsequent application of the Jacobi–Synge procedure that converts geometries to mechanical models [81, 118]. In this case the JBB time requires no group-extremization, being formally the same structure as in Sec 3.

[6] See [118] for a more detailed account of this concept and its scope. It is a group sum/group average/group extremization move.
[7] See the next Sec for another, whilst Q itself has a fair amount of internal structure [15, 126]: kinematical quantization, allotting a pre-Hilbert space, promoting constraints to operators and then solving them. Reduction itself can be split by solving for geometrically separable linear constraints at different points in the procedure. E.g. in RPM's, one can readily remove the translations at the start but quantize before constraining with respect to the rotations. Likewise, one might consider removing LQG's $SU(2)$ classically – passing to Wilson loops – prior to treating the Diff(3) constraint at the quantum level – passing to quantum states depending only on spin-knots.

3) One could solve the linear constraints at the Hamiltonian level instead, e.g. in geometrodynamics for the longitudinal potential W^k part of K_{ij}. This gives e.g. the alternative formal CTQ scheme in which a single scalar t is found by solving the Lichnerowicz–York equation with the value of W^k found from solving the GR momentum constraint substituted back in. This follows from York's [8] work, which was indeed built as an alternative to Wheeler's original sandwich conceptualization. However, Q steps involving solutions of the Lichnerowicz-York equation are particularly impassable. One might adhere to scale-and-frame auxiliary variables elimination from constraints (and CMC slicing condition) in Lagrangian form, as arising e.g. from the action in [68] or in connection with the distinct conformal thin sandwich reformulation [49] of the GR initial value problem itself. I note that neither of these have been investigated in depth as p.d.e. systems in this particular regard of classical reduction at the Lagrangian level/actually solving rather than just posing the Best Matching Problem. One could also follow solving the GR momentum constraint by finding a single 'time-map' Histories Theory (CHQ scheme).
4) Either of the preceding C-first moves could be followed up by a Tempus Nihil Est approach on the relational configuration space (CQ).
5) One could find a time-frame function(al) (internal or from matter) and then classically reduce away the frame so as to pass to a new single time function(al) (χCQ). However, as a second example of entwining of linear constraints [25], the Internal Time approach's evolutionary canonical transformation's generating function needs to be a function of the initial and final slices' metrics in the classical configuration representation, implying a need for the *prior* resolution of the Best Matching problem. One could instead find a classical histories theory with 'space-map' and 'time-map' [60, 86] , prior to reducing away the 'space-map' structure to pass to a new 'time-map' (χ_{H}CQ) The reduction here could e.g. be a 'Best Matching of histories': solve the Lagrangian form of the linear histories constraints $\mathrm{Lin}\lfloor \mathtt{Q}^{\mathsf{A}}, \mathtt{P}_{\mathsf{A}}\rfloor = 0$ for the histories auxiliary variables so as to remove these from the formalism. It could however also be a reduction at the Hamiltonian level.
6) One could face the linear constraints after quantizing, for instance in a classical internal or matter time-and-frame finding TQR scheme or in an emergent semiclassical time approach that has been enlarged to include h- and l-linear constraints (QRT scheme [65, 79].
7) A histories theory may still have linear constraints at this stage (QR orderings), in which case there is a nontrivial histories commutator constraint algebra. If this is an HQR ordering, the kinematical commutator algebra is obtained by selecting a subalgebra of the classical histories quantities, whose commutator suitably reflects global considerations, and the quantum histories constraints are some operator-ordering of their classical counterparts. [This will not always form the same constraint algebra that the histories Poisson brackets of the classical constraints did.]
8) One could consider Records Theory with linear constraints and G-act, G-all constructed G-invariant notions of distance and of information [118] and G-invariant operators at the QM level (QC), or a Histories Theory whose decoherence functional is constructed in such a fashion (QHC).
Thus, whilst a number of the above could be viewed as Best Matching Problem avoiding strategies, configurational relationalism itself is inevitable.

Kuchař 's Principle The reduced ordering R ... Q is the physical ordering. This is because, firstly, R ... Q and Q ... R do not always agree. Secondly, one would not expect that appending unphysical fields to the reduced description should change any of the physics of the of the true dynamical degrees of freedom. Thus, if they do differ, one should go with the reduced version. I finally note that this argument logically extends to its histories counterpart: H ... Q.

8 Strategizing about the Problem of Beables

I consider this to involve introduction of a fourth tuple O whose options are, in decreasing order of stringency, (D, K, P, –) standing for (Dirac, Kuchař, Partial, none).

Dirac observables/beables [6] alias **constants of the motion** alias **perennials** [27, 39, 53] are any function(al)s of the canonical variables $\mathrm{O} = \mathrm{D}\lfloor \mathtt{Q}^{\mathsf{A}}, \mathtt{P}_{\mathsf{A}}\rfloor$ such that, at the classical level, their Poisson brackets with all the constraint functions $\mathcal{C}_{\mathsf{X}}$ vanish (perhaps weakly [26]). I.e.

$$\{\mathcal{C}_{\mathsf{X}}, \mathrm{O}\} = 0 \ . \tag{32}$$

Thus, e.g. for geometrodynamics

$$\{\mathcal{H}_\mu, \mathrm{O}\} = 0 \ , \quad \text{for} \quad \mathcal{H}_\mu = [\mathcal{H}, \mathcal{H}_i] \ . \tag{33}$$

Justification of the name 'constants of the motion' conventionally follows from the total Hamiltonian being $H\lfloor \Lambda^{\mathsf{X}}\rfloor = \int_{\mathrm{S}} \mathrm{dS}\, \Lambda^{\mathsf{X}}\mathcal{C}_{\mathsf{X}}$ for multiplier coordinates Λ^{X}, so that (32) implies

$$\mathrm{dO}[\mathtt{Q}(\mathtt{t}), \mathtt{P}(\mathtt{t})]/\mathrm{d}\mathtt{t} = 0 \ . \tag{34}$$

True observables [23] alias **complete observables** [61] (which at least Thiemann [82] also calls evolving constant of the motion) are a similar notion, which, classically, involve operations on a system each of which produces a number that can be predicted if the state of the system is known.

Suppose one replaces (32) with split conditions

$$\{\mathcal{Q}\mathrm{uad}, \mathrm{O}\} = 0 \ , \quad \{\mathcal{L}\mathrm{in}_{\mathsf{Z}}, \mathrm{O}\} = 0 \quad \text{i.e.} \quad \{\mathcal{H}, O\} = 0 \ , \quad \{\mathcal{H}_i, O\} = 0 \text{ for geometrodynamics} \ . \tag{35}$$

Kuchař observables [27] are then as above except that only the second bracket of (35) need vanish. Kuchař then argued [27] for only the former needing to hold, in which case I denote the objects by $\mathrm{K}\lfloor \mathsf{Q}^{\mathsf{A}}, \mathsf{P}_{\mathsf{A}} \rfloor$. See also [27, 53, 86].

Note 1) It is clear that finding these is a timeless pursuit: it involves configuration space or at most phase space but not the Hamiltonian constraint and thus no dynamics. The downside now is that there is still a frozen Quad on the wavefunctions, so that one has to concoct some kind of Tempus Post Quantum or Tempus Nihil Est manoeuvre to deal with this.
Note 2) (A useful entwining with configurational relationalism) If configurational relationalism has by this stage been resolved, knowledge of the Kuchař observables is trivial. I.e. any R ... K ordered approaches.
Example 1) (RPM's). In the pure-shape case, any functional $\mathrm{F}[\mathrm{S}^{\mathsf{A}}, \mathrm{P}^{\mathrm{S}}_{\mathsf{A}}] = \mathrm{K}$, and in the scaled case, any functional $\mathrm{F}[\mathrm{S}^{\mathsf{A}}, \sigma, \mathrm{P}^{\mathrm{S}}_{\mathsf{A}}, \mathrm{P}_\sigma] = \mathrm{K}$; for 1- and 2-d RPM's a basis of shapes, a scale and the momenta conjugate to all of these are explicitly known [122]).
The next two examples are formal but not constructively known.
Example 2) 3-geometries and conjugates are Kuchař observables for geometrodynamics.
Example 3) knot quantities and conjugates are Kuchař observables for LQG.
Note 3) Formal G-act, G-all expressions for Kuchař observables are also available if configurational relationalism has not by this stage been solved.

The quantum counterpart of Dirac and Kuchař observables involves some operator form for the canonical variables and quantum commutators [,] in place of Poisson brackets. The operator-and-commutator counterparts of the Hamiltonian constraint constitute another 'Heisenberg' manifestation of the FFP. One's classical notion of observable is in each of the above cases to be replaced with the quantum one tied self-adjoint operators obeying a suitable commutator algebra in place of the classical Poisson algebra; this correspondence is however nontrivial (e.g. the two algebras may not be isomorphic) due to global considerations [15]. The Groenewold–Van Hove phenomenon is also an issue here if one tries to promote a classically-found set of observables to a QM set, i.e. in O ... Q schemes.

S-matrix quantities obey the first and not necessarily the second of (35), or possibly the QM counterpart of this statement. These do not carry background dependence connotations due to corresponding to scattering processes on configuration space rather than on space. Clearly then Kuchař AND S-matrix $\Rightarrow$ Dirac. Moreover, Halliwell [64] supplies a *specific construct* for a family of these, beginning from (for a simple particle mechanics model)

$$A(\mathbf{q}, \mathbf{q}_0, \mathbf{p}_0) = \int_{-\infty}^{+\infty} \mathrm{d}t\, \delta^{(k)}(\mathbf{q} - \mathbf{q}^{\mathrm{cl}}(t)) \ , \ C_{\mathrm{R}} = \theta\left(\int_{-\infty}^{\infty} \mathrm{d}t f_{\mathrm{R}}(\mathbf{q}(t)) - \epsilon \right) P(\mathbf{q}_{\mathrm{f}}, \mathbf{q}_0) \exp(iS(\mathbf{q}_{\mathrm{f}}, \mathbf{q}_0)) \ , \tag{36}$$

where the former is classical and the latter is a semiclassical QM *class function* (see [62] for more details of this variant and Sec 14 for discussion of more advanced variants).[8]

It is important to treat the whole path rather than segments of it, since the endpoints of segments contribute right-hand-side terms to the attempted commutation with H (Halliwell's specific context being that with no linear constraints, that is $H = \alpha\mathcal{H}$).

Note 4) A limitation until recently is that Halliwell's approach had only been done for cases with no linear constraints $\mathcal{L}\mathrm{in}_{\mathsf{Z}}$. However, I used the RPM arena for a first-such investigation [125], promoting the Kuchař resolution to the Problem of Beables to a Dirac resolution.

Partial observables/beables. The other type of timeless approaches (by Rovelli, Thiemann and Dittrich: see [66, 82, 73, 77, 109] for discussion and [6, 14] for some earlier roots) are often (but not always) used in further developments in LQG. These involve classical or QM operations on the system that produces a number that is possibly totally unpredictable even if the state is perfectly known (contrast with the definition of total/Dirac observables). The physics then lies in considering pairs of these objects which between them do encode some extractable purely physical information. This is also an 'any' move, in fact a second such: anything for time before, and anything to serve as partial observables. Here the problem of observables is held to have been a misunderstanding of the true nature of observables, which are in fact entities that are commonplace but meaningless other than *as regards correlations between more than one such*.
The P (Partial) option for observables can in principle be readily carried out at any stage relative to the other moves.

Master constraint program observables can be conceived of, obeying not (32) but $\{\mathrm{O}, \{\mathrm{O}, \mathbb{M}\}\} = 0$. The conceptual meaning of these candidate observables has not, as far as I am aware, been exposited.

Histories observables [86]. These are, conceptually, quantities that commute with the histories versions of the constraints. These are a distinct and meaningful concept in the Dirac and Kuchař cases of observables, whose definitions are constraint-dependent.

[8] Here, $\mathbf{q}^{\mathrm{cl}}(t)$ is the classical trajectory, $\mathbf{q}_0, \mathbf{p}_0$ is initial data, θ is the step function, f_{R} is the characteristic function of region R, ϵ is a small number, $S(\mathbf{q}_{\mathrm{f}}, \mathbf{q}_0)$ is the classical action between $\mathbf{q}_{\mathrm{f}}$ and $\mathbf{q}_0$; see [62] for the detailed form of the prefactor function P.

Note 6) The QORT classification, i.e. how to order Q, O, R, T tuples in dealing with the quantum version of a theory with Background Independence. The relation-free count of number of possible orderings of maps is by now over 100 at the classical level and several times that for programs including quantization. These large numbers of programs illustrate the value of restricting these via firstly the commutations of some of the maps and secondly by further principles as in Note 8).
Note 7) Using the QORT classification to properly identify strategies. As an example, recognizing Thiemann's argument for 'Dirac' rather than 'reduced' [72] to involve only subsets of these.[9] The argument is based on extra freedom in clock choices in the Dirac picture has its fluctuations suppressed in the reduced case, rendering the reduced case less physical. However, this argument specifically involves favouring TQR over TRQ rather than for Q ... R over R ... Q in general that standard useage of 'reduced quantization' would imply. Moreover, whilst the TQR scheme having more variables means that it has a larger variety of such clock variables, these extra degrees of freedom are clearly unphysical and thus fluctuations in them are physically irrelevant, thus forming an effective counterargument. This alongside Kuchař 's argument makes for a strong case for R ... Q approaches, contrary to much current literature. The general suggestion is that gauge theory is unfortunately no longer physically viable in the presence of a Quad that corresponds to the temporal relationalism aspect of background independence. Thus a leading challenge for quantum gravity would appear to be how to redo classical gauge theory in gauge-invariant terms and follow through on quantizing that. RPM's in 1- and 2-d have the good fortune of being tractable in this way, but, for now, the extension of that to field theoretical models remains a major obstacle.
Note 8) Furthermore, out of the remaining QORT's, I argue in favour of RQOT, RQTO, RQOH, RQHO and RQO using K or D for the O's and noting that all the T's and H's are single times and not time-frames, and basing my choice on Note 7) Kuchař's Principle and the Machian Status Quo. Thus, provided that T-H-Nihil combinations are not being sought (for these it is equitable to set up T and H and the same relative position to Q, O, R), RHQO, HRQO and χ_{H}RQO Histories Brackets approaches also fit these criteria.
Note 9) See Sec 11 for discussion of *localized* notions of observables/beables.

9 Strategizing about the Foliation Dependence Problem

In the continuum GR case, the n^μ of Sec 2 can furthermore be interpreted as the foliation 4-vector. One should also ask for analogues of spatial slices and then of foliations for discrete-type approaches to Quantum Gravity. In CDT, spatial triangulations are the instants, and in spin foams, they are spatial spin network states. In the Causal Sets program, this role is played by *antichains*; this makes good sense, since these are sets of causally *unrelated* points. Moreover, since points and causal relations are *all* of the structure in this approach, these antichains are just unstructured sets of points. However, they can be slightly thickened so as to have enough relations to be structured. One can then imagine in each of these discrete approaches for layered structures of each's means of modelling an instant, and then pose questions of 'foliation dependence' about this (or some suitable limit of this).

Attitude 1) Demand classical GR's foliation independence and refoliation invariance continue to hold at the quantum level. One then has to face that the commutator algebra at the quantum level is almost certainly distinct from classical GR's Dirac Algebra, (whilst not getting any simpler as regards having structure functions) and may contain foliation-dependent anomalies too.
Attitude 2) Background-dependent/privileged slicing alternatives are approaches with times with sufficient significance imposed on them that they cannot be traded for other times (these are more like the ordinary Quantum Theory notion of time than the conventional lore of time in GR. As the below examples show, the amount of theorizing involving attitude 2 has been on the increase; however, I first present a few cautions.

Note 1) One should disentangle Attitude 2) from how special highly-symmetric solutions *can* have geometrically-preferred foliations. However, GR is about generic solutions, and even perturbations about highly-symmetric solutions cease to have geometrically-privileged foliations to the perturbative order of precision [46]. Additionally, even highly-symmetric solutions admitting a privileged foliation in GR are *refoliable*, so the below problem with losing refoliation invariance does not apply.
Note 2) Kuchař also states that [25] *"The foliation fixing prevents one from asking what would happen if one attempted to measure the gravitational degrees of freedom on an arbitrary hypersurface. Such a solution amounts to conceding that one can quantize gravity only by giving up GR: to say that a quantum theory makes sense only when one fixes the foliation is essentially the same thing as saying that quantum gravity makes sense only in one coordinate system."*
Note 3) On the other hand, one cannot press too hard with envisaging different foliations as corresponding to various motions of a cloud of observers distributed throughout space, since that could also be modelled by multiple congruences of curves without reference to multiple foliations.

Example 2.1) Einstein–Aether theory [84] has privileged background structures due to a unit timelike vector field.
Example 2.2) Shape Dynamics in the sense of [113, 115, 124] has a fixed CMC foliation. Here, one "*trades refoliation invariance for a conformal symmetry*". [It is a fair point that not everybody working in this area is holding out for a fixed-foliation interpretation. The situation is an enlarged phase space 'linking theory', for which one gauge-fixing produces a

[9] [56] also talks of clock fluctuations in this way, motivated e.g. by the status quo approach to path integrals for constrained theories.

GR sector and another gauge-fixing produces a CMC-fixed sector. Then from the perspective of the linking theory, GR and CMC-fixed sector are gauge-related.]
Example 2.3) Hořava–Lifshitz theory (HLT) [92] has a privileged foliation (which has also been identified [98] as CMC). In this approach, one gives objective existence to the foliation and then have solely the foliation-preserving subset of the 4-diffeomorphisms, $\mathrm{Diff}_{\mathcal{F}}(\mathrm{M})$. There are in fact various versions of HLT, corresponding to whether $\alpha = \alpha(\mathrm{t}\text{ alone})$ or $\alpha(\mathrm{t}, \mathbf{x})$, crossed with what form the potential term is to take. The $\alpha = \alpha(\mathrm{t}\text{ alone})$ theories, by obvious parallel with Sec 3, avoid having a local $\mathcal{H}$ and thus FFP, though this is clearly at the expense of dropping many of the GR-like properties of time. The other theories produce a local $\mathcal{H}$ and have been found to require extension in the number of included terms [105] so as to remain viable.
Example 2.4) Some of the more successful forms of CDT have been found to involve preferred foliations.
Example 2.5) Soo and Yu [120] look to use a Master Constraint-type argument so as to also trade refoliation invariance for a privileged foliation theory.

Note 3) The above examples bear rich inter-relations. If Einstein–Aether Theory's unit timelike vector is restricted to be hypersurface-orthogonal, the IR limit of the extended $\alpha(\mathrm{t}, \mathbf{x})$ HLT is recovered [106]. Searching for a suitable classical limit, the abovementioned CDT's produced HLT spacetime, rather than GR spacetime [94]; [120] exposits links with HLT too. See the soon-forthcoming preprint [128] for further inter-relations between HLT and both the RWR and shape dynamics areas of Barbour's relational program.
Note 4) Fixed foliations are a type of background-dependence, which, from the relational perspective, is undesirable *from the outset*, for all that investigating the effects of making *just* this concession is indeed also of theoretical interest.

As regards Attitude 1), it can be used to provide a strong basic counting argument against the unimodular approach. For, there can only be 1 degree of freedom in a time t^{uni} that arises from the cosmological constant, and this cannot possibly index the plethora of foliations corresponding to GR's many-fingered time notion. The geometrical origin of this mismatch is that a cosmological time measures the 4-volume enclosed between two embeddings of the associated internal time functional t^{int}. However, given one of the embeddings the second is not uniquely determined by the value of t^{uni} (since pairs of embeddings that differ by a zero 4-volume are obviously possible due to the Lorentzian signature and cannot be distinguished in this way). Some constructive examples toward Attitude 1) are as follows.
Example 1.1) Kouletsis and Kuchař [60, 86] provided a means of including the set of foliations into an extension of the ADM phase space that is generally covariant. This amounts to extending phase space to include embeddings so as to take into account the discrepancy between the Diff(M) algebra and the Dirac Algebra (itself an older idea of Isham and Kuchař), here effectuated by construction of a time map and a space map. Thus this is a χ_{H} K... approach; it remains unstudied for GR past the classical level.
Example 1.2) In Savvidou's approach to HPO, she pointed out that the space of histories has implicit dependence on the foliation vector (the unit vector n^μ mentioned in Sec 2 as being orthogonal to a given hypersurface). With this view of HPO admitting 2 types of time transformation, the histories algebra turns out to now be foliation-dependent. However, the probabilities that are the actual physical quantities, are *not* foliation-dependent, so this approach avoids having a Foliation Dependence Problem at its end.
Example 1.3) Isham and Savvidou also considered the possibility of quantizing the classical foliation vector itself [54, 55]. Here,

$$\{\widehat{n}_\mu \Psi\} = n_\mu \Psi \ , \ \{\widehat{p}^{\mu\nu}\Psi\} = i\left\{n_\mu \frac{\partial}{\partial n_\nu} - n_\mu \frac{\partial}{\partial n_\nu}\right\}\Psi \ , \tag{37}$$

where the antisymmetric $p^{\mu\nu}$ is the conjugate of n^μ and satisfies the Lorentz algebra. They then apply a group-theoretic quantization to the configuration space of all foliation vectors for the Minkowski spacetime toy model of HPO.

10 Strategizing about the Functional Evolution Problem

Closure is complicated at the quantum level. Firstly, one has to bear in mind that the classical and quantum algebras are not in fact necessarily related due to global considerations [15].
Suppose then that we experience non-closure.
1) One could try to blame this on operator-ordering ambiguities, and continue to try to obtain closure.
2) Another counter sometimes available is to set some numerical factor of the offending term to zero, like how String Theory gets its particular dimensions.
3) Another possibility is to accept the QM-level loss of what had been a symmetry at the classical level.
4) Another possibility is that including additional matter fields could cancel off the anomaly hitherto found; some supergravity theories exhibit this feature.
5) A final possibility is to consider a new constraint algebra, in particular a simpler one, along the lines of e.g. the $\alpha(t \text{ alone})$ HLT or the Master Constraint Program. The latter cuts down the number of constraints so far that it would free the theory of all anomalies; however, that is in itself suspect since the same conceptual packaging of constraints into a Master Constraint would then appear to be applicable to flat-spacetime gauge theory, and yet these have not been declared to no longer have anomaly concerns.

On the other hand, other constraint algebra variants enlarge the algebra, e.g. more traditional forms of LQG algebra (anomaly analysis for which is in e.g. [44]), histories algebra [86] or a linking theory algebra [115].
Non-closure can be due to foliation-dependent terms, or, if scale is included among the physically irrelevant variables, then there can also be conformal anomalies; these are likely to plague Shape Dynamics.

I end by pointing out the concept of an '*observables anomaly*', i.e. finding that promoting a classical observable to a QM operator produces an observable that no longer commutes with the quantum constraints. This will tend to add to the futility of O ... Q schemes, since classical Problem of Beables solutions need not be straightforwardly promoteable to a QM Problem of Beables solution. Kuchař observables may well be largely exempt due to the nice properties of Lie groups under quantization schemes, but, in cases going beyond that (Dirac observables, the classical Dirac Algebra...) one may find one has to solve the QM Problem of Beables again. See the Halliwell combined scheme of Sec 14 for a good example, there are separate classical and QM implementations for Dirac observables, whilst the Kuchař observables translate over straightforwardly for the triangle RPM model [125].

11 Strategizing about the Global Problem of Time

Global Attitude 1) insist on only using globally-valid quantities for timefunctions and observables.

Example 1.1 Scale times are not useable globally-in-time in recollapsing universes due to non-monotonicity. Passing to dilational times (conjugate to scales) such as York time remedies this problem.
Example 1.2) The global POT then occurs e.g. in the separation into true and embedding (space frame and timefunction) variables in the internal time approach (the Torre Impasse [29]).
Example 1.3) Moreover the above example's split might be defineable in some case, but only for some sequence of slices that cannot be indefinitely continued as one progresses along a foliation that is to cover the entirety of that spacetime (e.g. not all spacetimes can be entirely foliated by CMC slices).
Example 1.4) Simpler models' timefunctions can eventually going astray for some other reason not involving foliations [after all, many simpler models do not have (meaningful) notions of foliation]. E.g. in the semiclassical approach, the allotment of h-l status may be only local in configuration space, with the approximate dynamical trajectory/wavepacket free to leave that region, and likewise the WKB regime itself may only be local, and all of this can already happen e.g. in RPM's for which the notion of foliation is moot.
Example 1.5) The global POT is partly avoidable by timeless approach, though elements of this construct including the semblance of dynamics itself may be local.

Global Attitude 2) allow for use of patching of various notions of time and of observables that are only locally valid [103, 110]. Whilst this was very naturally covered at the classical level by time being a coordinate and thus only in general locally valid and subject to the ordinary differential-geometric meshing condition, the problem now is how to mesh together different unitaty evolutions. Many of the other POT facets themselves have global issues (e.g. global observables, globally valid foliations, globally valid spacetime reconstructions).

Example 2.1) Bojowald et al's **fashionables** concept [110], or my **degradeables** parallel [118, 125]. [Fashionables are observables local in time and space, whereas degradeables are beables that are local in time and space. These are good words for local concepts, viz 'fashionable in Italy', 'fashionable in the 1960's', 'degradeable outside of the freezer' and 'degradeable within a year' all making good sense. Also, fashion is in the eye of the beholder – observer-tied, whereas degradeability is a mere matter of being, rather than of any observing.] These are **patching approaches**: observables/beables, and timefunctions, are held to only be valid on certain local patches. This holds for any of the Dirac, Kuchař and partial variants.
Example 2.2) Bojowald et al's patching approach uses a moments expansion as a bypass on the inner product problem. It is also semiclassical, in their sense that they neglect $O(\hbar^2)$ and moment-polynomials above some degree. Bojowald et al's specific fashionables implementation is based on time going complex around turning points within their notion of semiclassical regime. However, the geometrical interpretation of the transition between these fashionables is, for now, at least to me, unclear.
Example 2.3) At the level of Kuchař observables for RPM's, using fashionables/degradeables means that the functionals of the shapes (and scales) and their conjugates do not need to be valid over the whole of configuration space. Sec 8's examples can be upgraded likewise to be about functionals that are local in the corresponding configuration spaces. RPM's should also allow for Bojowald's working to be considered in cases which posses the major midisuperspace features.

12 Strategizing about the Multiple Choice Problem

Whilst patchings are multiple, and [110] claim that as an Multiple Choice Problem solver too, this does not address whole of this problem. Generally, the Multiple Choice Problem is multi-faceted – due to a heterogeneous collection of mathematical causes. Multiplicity of times is also inherent in Rovelli's 'any change' and 'partial observables' based relationalism, and in my STLRC (where one tests one's way among the many to find the locally-best).

1) Some cases of Multiple Choice Problem reflect foliation dependence. It has been suggested that one way out of the Multiple Choice Problem is to specify the lapse α and shift β^i a priori (e.g. in [43]). However, such amounts to yet another case of foliation fixing (so return to Sec 9).
2) It may be that some cases of Multiple Choice Problem could be due to context: different observers observing different subsystems that have different notions of time; this fits in well with the partial observables and patching paradigms.
3) Some cases of Multiple Choice Problem involve how choosing different times at the classical level for subsequent use ('promotion to operators')at the quantum level can lead to unitarily-inequivalent QM's, even if at the classical level they are canonically-related. This is due to the Groenewold–Van Hove phenomenon (which already holds for finite theories). This is a time problem if one's search for time takes one among the classical observables (i.e. O ... T ... Q approaches).
4) Possible problems with Patching are that the moments approach looks to rely on classical and quantum having the same bracket algebra (generally untrue [15]). The moments approach's use of all polynomials (up to a given degree in the approximate case detailed in [110]) may be at odds with the Groenewold–Van Hove phenomenon, and looks to indicate that subalgebra-selection reasons for the Multiple Choice Problem are not included, so that only a partial resolution of the Multiple Choice Problem is on offer. The Groenewold–Van Hove part of the Multiple Choice Problem is at least part-avoidable via the semiclassical approach being less reliant on subalgebra choices or unitary inequivalences.

13 Strategizing about the Spacetime Reconstruction Problem

A further Spacetime Reconstruction Problem Subfacet is that, in Internal Time Approaches, internal space or time coordinates are to be used in the conventional classical spacetime context; these need to be scalar field functions on the spacetime 4-manifold. In particular, functions of this form do not have any foliation dependence. However, the canonical approach to GR uses functionals of the canonical variables, and which there is no a priori reason for such to be scalar fields of this type. Thus one is faced with either finding functionals with this property (establishing Foliation Independence by construction and standard spacetime interpretation recovery), or with coming up with some new means of arriving at the standard spacetime meaning at the classical level.

Microcausality recovery is possible in the Savvidou [67] or Kouletsis [86] canonical-covariant formalisms (though this work has as yet very largely not been extended to the quantum level).

Programs are indeed often designed so that the continuum with spacetime properties recovery is the last facet to face.

Example 1) The status of the Spacetime Reconstruction Problem remains unclear for the Semiclassical Approach in detail [25, 26].
The next examples involve bottom-up approaches, i.e. less structure assumed.
Example 2) Getting back a semblance of dynamics or a notion of history from Timeless approaches counts as a type of spacetime reconstruction. This is in contradistinction to histories-assumed approaches.
Example 3) In LQG, semiclassical weave states have been considered by e.g. Ashtekar, Rovelli, Smolin, Arnsdorf and Bombelli; see [82] for a brief review and critique of these works. Subsequent LQG semiclassical reconstruction work has mostly involved instead (a proposed counterpart of the notion of) *coherent states* constructed by the complexifier method [82]; see [108] for a different take on such. In Lorentzian spin foams, semiclassical limits remain a largely open problem; e.g. [104] is a treatment of Lorentzian spin foams that also covers how the Regge action emerges as a semiclassical limit in the Euclidean case, whilst almost all the semiclassical treatment in the most review [127] remains Euclidean. The further LQC truncation [70] does possess solutions that look classical at later times (amidst larger numbers of solutions that do not, which are, for now, discarded due to not looking classical at later times, which is somewhat unsatisfactory in replacing predictivity by what ammounts to a *future* boundary condition). It also possesses further features of a semiclassical limit [71] (meaning a WKB regime with powers of both $\hbar$ and Immirzi's γ neglected, whilst still subject to open questions about correct expectation values of operators in semiclassical states).
Example 4) CDT succeeds in generating a classical regime [107], albeit this is HLT rather than GR, which is subject to the various foliation-related issues in Sec 9 as well as non-POT related advantages in ultra-violet behaviour. From that we may deduce that sometimes reconstructing spacetime is more straightforward if we forfeit its refoliation-invariance property (i.e. trading completion of one facet for taking the consequences of refusing to face another). See also [111] for a semiclassical analysis of CDT.
Example 5) Spacetime Reconstruction difficulties are to be expected from the Causal Sets Approach's insistence on very sparse structure; for recent advances here, see e.g. [93] for a recovery of a spacetime-like notion of topology, or [97] for a recovery of a metric notion.
Example 6) Euclidean counterparts of dynamical triangulation run into Wick rotation ambiguities (a similar situation to that observed in [20]), and [107] argue this to be evidence for the need to maintain the notion of causality.

14 Combined strategies for the POT

All of those considered here are semiclassical, which, whilst a limitation for other purposes, is argued to be valuable in the quantum-cosmological setting in footnote 5.

14.1 Halliwell-type combined strategies

Both histories and timeless approaches lie on the common ground of atemporal logic structures [36, 118], and there is a Records Theory within Histories Theory, as per Sec 6.4. The Semiclassical Approach and/or Histories Theory could support Records Theory by providing a mechanism for the semblance of dynamics Histories decohereing is a leading (but as yet *not fully established*) way by which the semiclassical regime's WKB approximation could be legitimately obtained in the first place. The elusive question of which degrees of freedom decohere which should be answerable through where in the universe the information is actually stored, i.e. where the records thus formed are [28, 64]; as Gell-Mann and Hartle say [28],

Records are "*somewhere in the universe where information is stored when histories decohere*". (38)

That emergent semiclassical time amounts to an approximate semiclassical recovery [79] of Barbour's classical emergent time [30], which is an encouraging result as regards making such a Semiclassical–Timeless Records combination.

More concretely, the semiclassical approach aids in the computation of timeless probabilities of histories entering given configuration space regions. This is by the WKB assumption giving a semiclassical flux into each region [64]; this approach does not, however, proceed via an emergent semiclassical time dependent Schrödinger equation such as in Sec 6.2. These timeless probabilities, and decoherence functionals, can be built from the class-function objects of eq (36 ii). These are are additionally S-matrix quantities, resolving part of the Problem of Beables, and a first simple model (RPM) with linear constraints included has also been resolved [125]. Finally these are indeed locally-interpretable concepts and so are furthermore compatible with the basic conceptual ethos of fashionables/degradeables, giving some hope of global POT resolution and partial Multiple Choice resolution, so this approach comes fairly close to resolving the POT in semiclassical regimes, and *very* close to resolving the POT for RPM's themselves.

Moreover, whilst conceptually illustrative, the class function (36 ii) itself has technical problems, in particular it suffers from the Quantum Zeno problem. This can be dealt with firstly by considering the following reformulation [74]:

$$\mathrm{P}(N\epsilon)...\mathrm{P}(2\epsilon)\mathrm{P}(\epsilon) = \exp(-iHt) \ , \tag{39}$$

as well as now conceptualizing in terms of probabilities of *never* entering the region. Next, one applies Halliwell's 'softening' (in the sense of scattering theory), passing to

$$\mathrm{P}(N\epsilon)...\mathrm{P}(2\epsilon)\mathrm{P}(\epsilon) = \exp(-i(H - iV_0P)t) \tag{40}$$

for $\epsilon V_0 \approx 1$. The class operator for not passing through region R is then

$$C_{\overline{\mathrm{R}}} \quad = \quad \lim_{t_1 \longrightarrow -\infty,\, t_2 \longrightarrow \infty} \exp(iHt_2)\exp(-i\{H - iV\}\{t_2 - t_1\})\exp(-iHt_1) \ . \tag{41}$$

This now does not suffer from the Quantum Zeno Problem whilst remaining an S-matrix quantity and thus commuting with Quad.

14.2 Gambini–Porto–Pullin-type combined strategies

This [80, 85] is built upon a different formulation of conditional probabilities objects:

$$\text{Prob(observable } O \text{ lies in interval } \Delta O \text{ provided that clock variable } \mathtt{t} \text{ lies in interval } \Delta\mathtt{t}) = \lim_{T \longrightarrow 0} \frac{\int_0^T \mathrm{d}\mathtt{t}\mathrm{Tr}(\mathrm{P}_{\Delta O}(\mathtt{t})\mathrm{P}_{\Delta\mathtt{t}}(\mathtt{t})\rho_0\mathrm{P}_{\Delta\mathtt{t}}(\mathtt{t}))}{\int_0^T \mathrm{d}\mathtt{t}\mathrm{Tr}(\rho_0\mathrm{P}_{\Delta\mathtt{t}}(\mathtt{t}))} \ . \tag{42}$$

The $\mathrm{P}(\mathtt{t})$'s here are Heisenberg time evolutions of projectors P: $\mathrm{P}(\mathtt{t}) = \exp(iH\mathtt{t})\,\mathrm{P}\exp(-iH\mathtt{t})$ These conditional probabilities allow for 'being at a time' to be incorporated. This approach is for now based on the 'any change' notion of relational clocks (that can be changed [129] to the STLRC notion). Moreover, these clocks are taken to be non-ideal at the QM level, giving rise to

1) a decoherence mechanism (distinct from that in Histories Theory; there is also a non-ideal rod source of decoherence postulated; the general form of the argument is in terms of imprecise knowledge due to imprecise clocks and rods).

2) A modified version of the Heisenberg equations of motion of the Lindblad type, taking the semiclassical form

$$i\hbar\frac{\partial\rho}{\partial\mathtt{t}} = [H, \rho] + D[\rho] \ . \tag{43}$$

Note 1) The form of the D-term here is $\sigma(\mathtt{t})[H, [H, \rho]]$ where $\sigma(\mathtt{t})$ is dominated by the rate of change of width of the probability distribution.

Note 2) By the presence of this 'emergent becoming' equation, this approach looks to be more promising in practise than Page's extended form of CPI. This approach also has the advantage over the original CPI of producing consistent propagators.

Note 3) By the presence of the D-term, the evolution is not unitary, which is all right insofar as it represents a system with imprecise knowledge (c.f. Sec 6.2).

Note 4) Dirac observables for this scheme are furthermore considered in [90].

Note 5) One advantage of this scheme over the Halliwell-type one is that it is based on the projector-proposition implementation rather than a classical regions implementation.

Note 6) Arce [121] has also provided a combined CPI–semiclassical scheme [for yet another notion of semiclassicality, that keeps some non-adiabatic terms and constituting an example of self-consistent approach, c.f. Sec 6.2].

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Appendix A: Counting numbers of maps

We want to know all possible compositions of maps where each map is used once or not at all. There are $K(n) := \sum_n P(n,r)$ of these for $P(n,r)$ the permutator, and this sum is computationally equal to $\text{int}(\text{e}\, n!)$ for $n \geq 1$. The first few values of this are $K(1) = 2$, $K(2) = 5$, $K(3) = 16$, $K(4) = 65$... Next, when one of the maps is chosen from a menu of two (T, H or nothing), the total number of maps is $D(n) := 2K(n) - K(n-1)$. The first few values of this are $D(2) = 8$, $D(3) = 27$, $D(4) = 114$... Further counting in this article just involve i) more/larger menus: choosing 1 or none from each but also having the courses in any order. ii) relation restrictions on the maps: some maps are disallowed to the right of others, and some adjacent pairs of maps are equivalent by commutation. These follow from the more detailed mathematical meaning of the maps in question.

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