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NAMBU-GOTO ACTION AND QUBIT THEORY

IN ANY SIGNATURE AND IN HIGHER DIMENSIONS

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Abstract

We perform an extension of the relation between the Nambu-Goto action and qubit theory. Of course, the Cayley hyperdeterminant is the key mathematical tool in such generalization. Using the Wick rotation we find that in four dimensions such a relation can be established no only in (2+2)-dimensions but also in any signature. We generalize our result to a curved space-time of $(2^{2n}+2^{2n})$-dimensions and $(2^{2n+1}+2^{2n+1})$-dimensions.

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Some years ago, Duff [1] discovers hidden new symmetries in the Nambu-Goto action [2]-[3]. It turns out that the key mathematical tool in such a discovery is the Cayley hyperdeterminant [4]. In this pioneer work, however, the target space-time turns out to have an associated $(2+2)$-signature, corresponding to two time and two space dimensions. It was proved in Ref. [5]-[6] that the Duff's formalism can also be generalized to $(4+4)$-dimensions and $(8+8)$-dimensions. Here, we shall prove that if one introduces a Wick rotations for various coordinates then one can actually extend the Duff's procedure to any signature in 4-dimensions. Moreover, we also prove that our method can be extended to curved space-time in $(2^{2n}+2^{2n})$-dimensions and $(2^{2n+1}+2^{2n+1})$-dimensions.

There are a number of physical reasons to be interested on these developments, but perhaps the most important is that eventually our work may be useful on a possible generalization of the remarkable correspondence between black-holes and quantum information theory (see Refs. [7]-[10] and references therein).

Let us start recalling the Duff's approach on the relation between the Nambu-Goto action and the $(2+2)$-signature. Consider the Nambu-Goto action [2]-[3],

$$S = \int d\xi^2 \sqrt{\epsilon \det(\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu})}. \tag{1}$$

Here, the space-time coordinates x^μ are real function of two parameters $(\tau,\sigma) = \xi^a$ and $\eta_{\mu\nu}$ is a flat metric, determining the signature of the target space-time. Moreover, the parameter ϵ takes the values $+1$ or -1, depending whether the signature of $\eta_{\mu\nu}$ is Euclidean or Lorenziana, respectively.

It turns out that by introducing the world-sheet metric g^{ab} one can prove that (1) is equivalent to the action [11] (see also Ref. [12] and references therein)

$$S = \int d\xi^2 \sqrt{-\epsilon \det g} g^{ab} \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu}, \tag{2}$$

which is, of course, the Polyakov action (see Ref. [12] and references therein). In fact, from the expression

$$\partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} - \frac{1}{2} g_{ab} g^{cd} \partial_c x^\mu \partial_d x^\nu \eta_{\mu\nu} = 0, \tag{3}$$

obtained by varying the action (2) with respect to g^{ab}, it is straightforward to show that from (2) one obtains (1) and *vise versa*. Hence, the actions (1) and (2) are equivalents.

It is convenient to define the induced world-sheet metric

$$h_{ab} \equiv \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu}. \tag{4}$$

Using this definition, the Nambu-Goto action (1) becomes

$$S = \int d\xi^2 \sqrt{\epsilon \det(h_{ab})}. \tag{5}$$

It is not difficult to see that in $(2+2)$-dimensions the expression (4) can be written as

$$h_{ab} = \partial_a x^{ij} \partial_b x^{kl} \varepsilon_{ik} \varepsilon_{jl}, \tag{6}$$

where x^{ij} denotes a the 2×2- matrix

$$x^{ij} = \begin{pmatrix} x^1 + x^3 & x^2 + x^4 \\ -x^2 + x^4 & x^1 - x^3 \end{pmatrix}. \tag{7}$$

It is important to observe that (7) corresponds to the set $M(2,R)$ of any 2×2-matrix. In fact, by introducing the fundamental base matrices

$$\begin{gathered} \delta^{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \varepsilon^{ij} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \eta^{ij} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \lambda^{ij} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{gathered} \tag{8}$$

one observes that (7) can be rewritten as the linear combination

$$x^{ij} = x^1 \delta^{ij} + x^2 \varepsilon^{ij} + x^3 \eta^{ij} + x^4 \lambda^{ij}. \tag{9}$$

Let us now introduce the expression

$$h = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} h_{ac} h_{bd}. \tag{10}$$

If one uses (4) one gets

$$h = \det(h_{ab}). \tag{11}$$

However, if one considers (6) one obtains

$$h = \mathcal{D}et(h_{ab}), \tag{12}$$

where $\mathcal{D}et(h_{ab})$ denotes the Cayley hyperdeterminant of h_{ab}, namely

$$\mathcal{D}et(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns} \partial_a x^{ij} \partial_c x^{kl} \partial_b x^{mn} \partial_d x^{rs}. \tag{13}$$

Of course, (11) and (12) imply that

$$\det(h_{ab}) = \mathcal{D}et(h_{ab}). \tag{14}$$

In turn, (14) means that in $(2+2)$-dimensions the Nambu-Goto action (5) can also be written as

$$S = \int d\xi^2 \sqrt{\mathcal{D}et(h_{ab})}. \tag{15}$$

Note that, since in this case one is considering the $(2+2)$-signature one must set $\epsilon = +1$ in (5).

In $(4+4)$-dimensions the key formula (6) can be generalized as

$$h_{ab} = \partial_a x^{ijm} \partial_b x^{kls} \varepsilon_{ik} \varepsilon_{jl} \eta_{ms}. \tag{16}$$

While in $(8+8)$-dimensions one has

$$h_{ab} = \partial_a x^{ijmn} \partial_b x^{klsr} \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{ms} \varepsilon_{nr}. \tag{17}$$

(see Refs. [5] and [6] for details). So by considering the real variables $x^{i_1 \cdots i_n}$ and properly considering the matrices ε_{ij} and η_{ij} the previous formalism can be generalized to higher dimensions. Of course, in such cases the Cayley hyperdeterminant $\mathcal{D}et(h_{ab})$ must be modified accordingly.

Observing (7) one wonders whether one can consider in (6) other signatures in 4-dimensions besides the (2+2)-signature. It is not difficult to see that using the Wick rotation in any of the coordinates x^1, x^2, x^3 or x^4 one can modify the signature. For instance, one can achieve the $(1+3)$-signature if one uses the prescription $x^2 \to ix^2$ in (6). This method lead us inevitable to generalize our method to a complex structure. One simple introduce the complex matrix

$$z^{ij} = z^1 \delta^{ij} + z^2 \varepsilon^{ij} + z^3 \eta^{ij} + z^4 \lambda^{ij}, \tag{18}$$

where the variables z^1, z^2, z^3 and z^4 are complex numbers. The expression (6) is generalized accordingly as [13]

$$h_{ab} = \partial_a z^{ij} \partial_b z^{kl} \varepsilon_{ik} \varepsilon_{jl}. \tag{19}$$

Thus, in this case, the Cayley hyperdeterminant becomes

$$\mathcal{D}et(h_{ab}) = \frac{1}{2!} \varepsilon^{ab} \varepsilon^{cd} \varepsilon_{ik} \varepsilon_{jl} \varepsilon_{mr} \varepsilon_{ns} \partial_a z^{ij} \partial_b z^{kl} \partial_a z^{mn} \partial_b z^{rs} \tag{20}$$

and consequently the Nambu-Goto action must be written using (20). Of course, the Nambu-Goto action, or the Polyakov action, must be real and therefore one must choose any of the coordinates z^1, z^2, z^3 and z^4 in (20) either as pure real or pure imaginary.

Similarly, the generalization to a complex structure can be made by introducing the complex variables $z^{i_1...i_n}$ and writing

$$\begin{array}{c}\mathcal{D}et(h_{ab})=\frac{1}{2!}\varepsilon^{ab}\varepsilon^{cd}\varepsilon_{i_1j_1}...\varepsilon_{i_{n-1}j_{n-1}}\eta_{i_nj_n}\varepsilon_{k_1l_1}...\varepsilon_{k_{n-1}l_{n-1}}\eta_{k_nl_n}\cdot\\ \cdot\partial_a z^{i_1...i_n}\partial_c z^{j_1...j_n}\partial_b z^{k_1...k_n}\partial_d z^{l_1...l_n}\end{array}\tag{21}$$

or

$$\begin{array}{c}\mathcal{D}et(h_{ab})=\frac{1}{2!}\varepsilon^{ab}\varepsilon^{cd}\varepsilon_{i_1j_1}...\varepsilon_{i_nj_n}\varepsilon_{k_1l_1}...\varepsilon_{k_nl_n}\cdot\\ \cdot\partial_a z^{i_1...i_n}\partial_c z^{j_1...j_n}\partial_b z^{k_1...k_n}\partial_d z^{l_1...l_n},\end{array}\tag{22}$$

depending whether the signature is $(2^{2n}+2^{2n})$ or $(2^{2n+1}+2^{2n+1})$, respectively.

One can further generalize our procedure by considering a target curved space-time. For this purpose let us introduce the curved space-time metric

$$g_{\mu\nu}=e^A_\mu e^B_\nu\eta_{AB}.\tag{23}$$

Here, e^A_μ denotes a vielbein field and η_{AB} is a flat metric. The Polyakov action in a curved target space-time becomes

$$S=\int d\xi^2\sqrt{-\epsilon\det g}g^{ab}\partial_a x^\mu\partial_b x^\nu g_{\mu\nu}.\tag{24}$$

Using (23), one sees that this action can be written as

$$S=\int d\xi^2\sqrt{-\epsilon\det g}g^{ab}(\partial_a x^\mu e^A_\mu)(\partial_b x^\nu e^B_\nu)\eta_{AB}.\tag{25}$$

So, by defining the quantity

$$E^A_a\equiv\partial_a x^\mu e^A_\mu,\tag{26}$$

the action in (25) reads as

$$S=\int d\xi^2\sqrt{-\epsilon\det g}g^{ab}E^A_aE^B_b\eta_{AB}.\tag{27}$$

Hence, in a target space-time of $(2+2)$-dimensions one can write (27) in the form

$$S=\int d\xi^2\sqrt{-\epsilon\det g}g^{ab}E^{ij}_aE^{kl}_b\varepsilon_{ik}\varepsilon_{jl},\tag{28}$$

where

$$E^{ij}_a\equiv\partial_a x^\mu e^{ij}_\mu.\tag{29}$$

Here, we considered the fact that one can always write

$$e_{\mu}^{ij} = e_{\mu}^{1}\delta^{ij} + e_{\mu}^{2}\varepsilon^{ij} + e_{\mu}^{3}\eta^{ij} + e_{\mu}^{4}\lambda^{ij}. \tag{30}$$

Observe that in this development one can consider a generalization of (4) namely

$$h_{ab} = E_{a}^{A}E_{b}^{B}\eta_{AB} \tag{31}$$

and therefore in $(2+2)$-dimensions this expression becomes

$$h_{ab} = E_{a}^{ij}E_{b}^{kl}\varepsilon_{ik}\varepsilon_{jl}, \tag{32}$$

while in $(4+4)$-dimensions and $(8+8)$-dimensions one obtains

$$h_{ab} = E_{a}^{ijm}E_{b}^{klr}\varepsilon_{ik}\varepsilon_{jl}\eta_{mr} \tag{33}$$

and

$$h_{ab} = E_{a}^{ijmn}E_{b}^{klrs}\varepsilon_{ik}\varepsilon_{jl}\varepsilon_{mr}\varepsilon_{ns}, \tag{34}$$

respectively.

At this stage, it is evident that if one wants to generalize the procedure to any signature in a curved space-time one simply substitute in the action (27) either

$$h_{ab} = \mathcal{E}_{a}^{i_1...i_n}\mathcal{E}_{b}^{j_1...j_n}\varepsilon_{ik}...\varepsilon_{i_{n-1}j_{n-1}}\eta_{i_n j_n} \tag{35}$$

or

$$h_{ab} = \mathcal{E}_{a}^{i_1...i_n}\mathcal{E}_{b}^{j_1...j_n}\varepsilon_{ik}...\varepsilon_{i_{n-1}j_{n-1}}\varepsilon_{i_n j_n}, \tag{36}$$

depending whether the signature is $(2^{2n}+2^{2n})$ or $(2^{2n+1}+2^{2n+1})$, respectively. Here, we used the prescription $E_{a}^{i_1...i_n} \rightarrow \mathcal{E}_{a}^{i_1...i_n}$, with $\mathcal{E}_{a}^{i_1...i_n}$ a complex function.

In order to include p-branes in our formalism, one notes that the expression (35) and (36) can still be used. In such a case, one allows the indice a in (35) and (36) to run from 0 to p. Braking such kind of indices as $a = (\hat{a}_1, \hat{a}_2)$ for a 3-brane, as $a = (\hat{a}_1, \hat{a}_2, \hat{a}_3)$, for a 5-brane and so on one observes that (35) and (36) can be written as

$$h_{\hat{a}_1...\hat{a}_2\hat{b}_1...\hat{b}_2} = \mathcal{E}_{\hat{a}_1...\hat{a}_2}^{i_1...i_p}\mathcal{E}_{\hat{b}_1...\hat{b}_2}^{j_1...j_p}\varepsilon_{ik}...\varepsilon_{i_{p-1}j_{p-1}}\eta_{i_p j_p} \tag{37}$$

or

$$h_{\hat{a}_1...\hat{a}_2\hat{b}_1...\hat{b}_2} = \mathcal{E}_{\hat{a}_1...\hat{a}_2}^{i_1...i_p}\mathcal{E}_{\hat{b}_1...\hat{b}_2}^{j_1...j_p}\varepsilon_{ik}...\varepsilon_{i_{p-1}j_{p-1}}\varepsilon_{i_p j_p}, \tag{38}$$

respectively. The analogue of Cayley hyperdeterminant in this case will be

$$\begin{array}{c}\hat{\mathcal{D}}et(h_{\hat{a}_1...\hat{a}_2\hat{b}_1...\hat{b}_2})=\\ =\varepsilon^{\hat{a}_1\hat{b}_1}...\varepsilon^{\hat{a}_p\hat{b}_p}\mathcal{E}^{i_1...i_p}_{\hat{a}_1...\hat{a}_2}\mathcal{E}^{j_1...j_p}_{\hat{b}_1...\hat{b}_2}\varepsilon_{ik}...\varepsilon_{i_{p-1}j_{p-1}}\varepsilon_{i_pj_p}\end{array}\tag{39}$$

and therefore the corresponding Nambu-Goto action becomes

$$S=\int d\xi^{p+1}\sqrt{\epsilon\hat{\mathcal{D}}et(h_{\hat{a}_1...\hat{a}_2\hat{b}_1...\hat{b}_2})}.\tag{40}$$

Summarizing, we have generalized the Duff's procedure concerning the combination of the Nambu-Goto action and the Cayley hyperdeterminant in target space-time of (2+2)-dimensions. Such a generalization first corresponds to a curved worlds with $(2^{2n}+2^{2n})$-signature or $(2^{2n+1}+2^{2n+1})$-signature. Using complex structure we may be able to extend the procedure to any signature. Further, we generalize the method to p-branes.

It turns out that these generalization may be useful in a number of physical scenario beyond string theory and p-branes. In fact, since the quantity $z^{j_1...j_n}$ can be identified with a n-qubit one may be interested in the route leading to oriented matroid theory [14] (see also Ref. [15]-[16]). In this direction, using the phirotope concept (see Ref. [17] and references therein), which is a complex generalization of the concept of chirotope in oriented matroid theory, a link between super p-branes and qubit theory has already been established [17]. Thus, it may be interesting for further developments to explore the connection between the results of the present work and supersymmetry *via* the Grassmann-Plücker relations (see Refs. [8]-[9] and references therein). It is worth mentioning that such relations are natural mathematical notions in information theory linked to n-qubit entanglement. Indeed, in such a case, the Hilbert space can be broken in the form $C^{2n}=C^L\otimes C^l$ with $L=2n-1$ and $l=2$. This allows a geometric interpretation in terms of the complex Grassmannian variety $Gr(L,l)$ of 2-planes in C^{2n} *via* the Plücker embedding. In this context, the Plücker coordinates of Grassmannians $Gr(L,l)$ are natural invariants of the theory (see Ref. [9] for details). However, it has been mentioned in Ref. [18], and proved in Refs. [19] and [20], that for normalized qubits the complex 1-qubit, 2-qubit and the 3-qubit are deeply related to division algebras via the Hopf maps, $S^3\xrightarrow{S^1}S^2$, $S^7\xrightarrow{S^3}S^4$ and $S^{15}\xrightarrow{S^7}S^8$, respectively. In order to clarify the possible application of these observations in the context of our formalism let us consider the general complex state $\mid\psi>\in C^{2n}$,

$$\mid\psi>=\sum_{i_1i_2...i_n=0}^{1}z^{i_1i_2...i_n}|i_1i_2...i_n>,\tag{41}$$

where $|i_1 i_2 ... i_n >= |i_1 > \otimes |i_2 > \otimes ... \otimes |i_n >$ correspond to a standard basis of the n-qubit. It is interesting to make the following observations. First, let us denote a n-rebit system (real n-qubit) by $x^{i_1 i_2 \cdots i_n}$. So, one finds that a 3-rebit and 4-rebit have 8 and 16 real degrees of freedom, respectively. Thus, one learns that the 4-rebit can be associated with the 16 degrees of freedom of a 3-qubit. It turns out that this is the kind of embedding discussed in Ref. [9]. In this context, one sees that in the Nambu-Goto context one may consider the 16-dimensions target space-time as the maximum dimension required by division algebras via the Hopf map $S^{15} \xrightarrow{S^7} S^8$.

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