

# Symplectic vs pseudo-Euclidean space-time with extra dimensions\*

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## Abstract

It is conjectured that the symplectic structure of space-time is superior to the metric one. Instead of the commonly adopted pseudo-orthogonal groups  $SO(1, d - 1)$ ,  $d \geq 4$ , the complex symplectic ones  $Sp(2l, C)$ ,  $l \geq 1$  are proposed as the local structure groups of the extended space-time. A discrete series of the metric space-times of the particular dimensionalities  $d = 4l^2$  and signatures, with  $l(2l - 1)$  time and  $l(2l + 1)$  spatial directions, defined over the set of the Hermitian second-rank spin-tensors is proposed as an alternative to the pseudo-Euclidean space-times with extra dimensions. The one-dimensional time-like direction remaining invariant under fixed boosts makes it possible the non-relativistic causality description despite the presence of extra times.

## 1 Space-time: symplectic vs pseudo-Euclidean

The space-time we live in is generally adopted to be (locally) the Minkowski one. Nevertheless, the spinor calculus in this space-time heavily relies on the isomorphism of the noncompact groups  $SO(1, 3) \simeq SL(2, C)/Z_2$ , as well as that  $SO(3) \simeq SU(2)/Z_2$  for their maximal compact subgroups. In fact, the whole relativistic field theory in four space-time dimensions can equivalently be formulated in the framework of representations of the complex unimodular group  $SL(2, C)$  alone (and in a sense it is even preferable [1]). For this reason, the space-time structure group with spinors as defining representations, i.e. the complex symplectic group  $Sp(2, C) \simeq SL(2, C)$ , is conceptually more appropriate than the pseudo-orthogonal one  $SO(1, 3)$  with vectors as defining representation. In the former approach, to a space-time point there corresponds a Hermitian spin-tensor of the second rank rather than a four-vector.

Then in searching for the space-times with extra dimensions it is natural to look for the extensions in the symplectic framework with the structure group  $Sp(2l, C)$ ,  $l > 1$  instead of  $SO(1, d - 1)$ ,  $d > 4$ . The symplectic series of the groups is peculiar quantum-mechanically for it retains the bilinear spinor product at any  $l > 1$ . Two alternative ways of the space-time extension can be pictured schematically as follows:

$$\begin{array}{ccc} SO(1, 3) & \simeq & Sp(2, C) \\ \downarrow & & \downarrow \\ SO(1, d - 1) & \not\simeq & Sp(2l, C). \end{array}$$

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The first, commonly adopted way of extension, corresponds to the pseudo-orthogonal structure groups while the second one relies on the complex symplectic groups. The scheme shows that the isomorphism of the two types of groups, valid at  $d = 4$  and  $l = 1$ , is no longer fulfilled at  $d > 4$  and  $l > 1$ . In the first way of extension the local metric properties of the space-times, i.e. their dimensionalities and signatures, are put in from the very beginning. In the second way, these properties are not to be considered as the primary ones but, instead, they should emerge as a manifestation of the inherent symplectic structure. In what follows, we develop the symplectic framework in general and elaborate somewhat the ordinary and next-to-ordinary space-time cases with  $l = 1, 2$ , respectively.<sup>1</sup>

## 2 Symplectic space-time

Let  $\psi_A$  and  $\bar{\psi}^{\bar{A}} \equiv (\psi_A)^*$ , as well as their respective duals  $\psi^A$  and  $\bar{\psi}_{\bar{A}} \equiv (\psi^A)^*$ ,  $A, \bar{A} = 1, \dots, 2l$  are the spinor representations of  $Sp(2l, C)$ . There exist the invariant spin-tensors  $\epsilon_{AB} = -\epsilon_{BA}$  and  $\epsilon^{AB} = -\epsilon^{BA}$  such that  $\epsilon_{AC}\epsilon^{CB} = \delta_A^B$ , with  $\delta_A^B$  being the Kroneker symbol (and similarly for  $\epsilon_{\bar{A}\bar{B}} \equiv (\epsilon^{BA})^*$  and  $\epsilon^{\bar{A}\bar{B}} \equiv (\epsilon_{BA})^*$ ). Owing to these tensors the spinor indices of the upper and lower positions are pairwise equivalent ( $\psi_A \equiv \epsilon_{AB}\psi^B$  and  $\bar{\psi}_{\bar{A}} \equiv \epsilon_{\bar{A}\bar{B}}\bar{\psi}^{\bar{B}}$ ), so that there are left just two inequivalent spinor representations (generically,  $\psi$  and  $\bar{\psi}$ ). They are the spinors of the first and the second kind, respectively.

Let us put in correspondence to an event point  $P$  a second rank  $2l \times 2l$  spin-tensor  $X_A^{\bar{B}}(P)$ , which is Hermitian, i.e., fulfil the restriction

$$X_A^{\bar{B}} = (X_B^{\bar{A}})^* \equiv \bar{X}^{\bar{B}}{}_A,$$

or in other terms  $X^{A\bar{B}} = (X_{B\bar{A}})^*$ . Now, one can define the quadratic scalar product

$$(X, X) \equiv \text{tr } X\bar{X} = X_A^{\bar{B}}\bar{X}_{\bar{B}}^A = -X_{A\bar{B}}(X_{B\bar{A}})^*,$$

with  $(X, X)$  being real though not sign definite. Under arbitrary  $S \in Sp(2l, C)$  one has in short notations:

$$\begin{aligned} X &\rightarrow SX S^\dagger, \\ \bar{X} &\rightarrow S^{\dagger-1}\bar{X}S^{-1}, \end{aligned}$$

so that  $X\bar{X} \rightarrow SX\bar{X}S^{-1}$  and hence  $(X, X)$  is invariant. At  $l > 1$ , the quadratic invariant above is just the lowest order one in a series of independent invariants  $\text{tr}(X\bar{X})^k$ ,  $k = 1, \dots, l$ , the highest order one with  $k = l$  being equivalent to  $\det X$ .

**Definition:** *the Hermitian spin-tensor set  $\{X\}$  equipped with the structure group  $Sp(2l, C)$  and the interval between points  $X_1$  and  $X_2$  equal to  $(X_1 - X_2, X_1 - X_2)$  constitutes the flat symplectic space-time.*

The noncompact transformations from the  $Sp(2l, C)$  are counterparts of the Lorentz boosts in the ordinary space-time, while transformations from the compact subgroup  $Sp(2l) = Sp(2l, C) \cap SU(2l)$  correspond to rotations. With account for translations  $X_A^{\bar{B}} \rightarrow X_A^{\bar{B}} + \Xi_A^{\bar{B}}$ , where  $\Xi_A^{\bar{B}}$  is an arbitrary constant Hermitian spin-tensor, the

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<sup>1</sup>For more detail see [2].

whole theory in the flat symplectic space-time should be covariant relative to the inhomogeneous symplectic group.

With restriction by the the maximal compact subgroup  $Sp(2l)$ , the indices of the first and the second kinds in the same position are indistinguishable relative to their transformation properties ( $\psi_A \sim \bar{\psi}_{\bar{A}}, \psi^A \sim \bar{\psi}^{\bar{A}}$ ). One can temporarily label  $X_{A\bar{B}}$  in this case as  $X_{XY}$ , where  $X, Y, \dots = 1, \dots, 2l$  generically mean spinor indices irrespective of their kind. Hence, one can reduce the tensor  $X_{XY}$  into two irreducible parts, symmetric and antisymmetric ones:  $X_{XY} = \sum_{\pm} (X_{\pm})_{XY}$ , where  $(X_{\pm})_{XY} = \pm (X_{\pm})_{YX}$  have the dimensionalities  $d_{\pm} = l(2l \pm 1)$ , respectively. One gets the following decomposition

$$(X, X) = \sum_{\pm} (\mp 1) (X_{\pm})_{XY} [(X_{\pm})_{XY}]^*.$$

At  $l > 1$ , one can further reduce the antisymmetric spin-tensor  $X_-$  into the trace relative to  $\epsilon$  and a traceless part. Therefore the whole extended space-time can be decomposed relative to the rotational subgroup into three irreducible subspaces of the dimensionalities 1,  $(l-1)(2l+1)$  and  $l(2l+1)$ , respectively. The first two subspaces correspond to time-like directions, the one-dimensional rotationally invariant and non-invariant ones, while the third subspace corresponds to the spatial extra dimensions. In the ordinary space-time which corresponds to  $l = 1$  the second subspace is empty.

The particular decomposition of  $X$  into two parts  $X_{\pm}$  is noncovariant with respect to the whole  $Sp(2l, C)$  and depends on the boosts. Nevertheless, the decomposition being valid at any boost, the number of the positive and negative components in  $(X, X)$  is invariant under the whole  $Sp(2l, C)$ . In other words, the metric tensor of the flat symplectic space-time

$$\eta_d = (\underbrace{+1, \dots}_{d_-}; \underbrace{-1, \dots}_{d_+})$$

is invariant. Hence, at  $l > 1$  the structure group  $Sp(2l, C)$  of the  $2l$ -th rank and the  $2l(2l+1)$ -th order, acting on the Hermitian second-rank spin-tensors with  $d = 4l^2$  components, is a subgroup of the embedding pseudo-orthogonal group  $SO(d_-, d_+)$ , of the rank  $2l^2$  and the order  $2l^2(4l^2 - 1)$ , acting on the pseudo-Euclidean space of the dimensionality  $d = 4l^2$ . What distinguishes  $Sp(2l, C)$  from  $SO(d_-, d_+)$ , is the total set of independent invariants  $\text{tr}(X\bar{X})^k$ ,  $k = 1, \dots, l$ . The isomorphism between the groups is valid only at  $l = 1$ , i.e., for the ordinary space-time  $d = 4$  where there is just one independent invariant  $1/2 \text{tr} X\bar{X} = \det X$ .

In the symplectic approach we consider, neither the discrete set of dimensionalities,  $d = 4l^2$ , of the extended space-time, nor its signature, nor the existence of the rotationally invariant one-dimensional time subspace are postulated from the beginning. Rather, these properties are the attributes of the underlying symplectic structure. In particular, the latter one seems to provide at the fundamental level the simple rationale for the four-dimensionality of the ordinary space-time, as well as for its signature  $(+ - - -)$ . Namely, the last properties just reflect the existence of one antisymmetric and three symmetric second-rank Hermitian spin-tensors at  $l = 1$ . The set of such tensors, in its turn, is the lowest admissible Hermitian space to accommodate the symplectic structure. On the other hand, right the one-dimensionality of time is what allows the events to be ordered at any fixed boosts and hence insures the causal description. Therefore the latter one may ultimately be attributed to the underlying symplectic structure, too. At  $l > 1$ , because of the one-dimensional time being mixed via boosts with the extra times, the causality is expected to be violated at large boosts.

### 3 C, P, T symmetries

Let us charge double the spinor space, i.e., for each  $\psi_A$  and  $(\psi_A)^\dagger \equiv \bar{\psi}^{\bar{A}}$  introduce two copies  $\psi_A^\pm$  and  $(\psi_A^\pm)^\dagger \equiv (\bar{\psi}^\mp)^{\bar{A}}$ , with  $\pm$  being the ‘‘charge’’ sign. We use here a dagger sign for complex conjugation to show that the Grassmann fields should undergo the change of the order in their products. In analogy to the ordinary case of  $SL(2, C)$  [3], one can define the following discrete symmetries:

$$\begin{aligned} C &: \psi_A^\pm \rightarrow \psi_A^\mp, \\ P &: \psi_A^\pm \rightarrow (\psi_A^\mp)^\dagger \equiv (\bar{\psi}^\pm)^{\bar{A}}, \\ T &: \psi_A^\pm \rightarrow (\psi_A^\pm)^\dagger \equiv (\bar{\psi}^\mp)^{\bar{A}}, \end{aligned}$$

and hence  $CPT: \psi_A^\pm \rightarrow \psi_A^\pm$  (all up to the phase factors). Under validity of the  $CPT$  invariance, only two of the discrete operations are independent ones. Without charge doubling, just one independent combination  $CP \equiv T: \psi_A \rightarrow \bar{\psi}^{\bar{A}}$  survives.

Now, let us introduce the Hermitian spin-tensor current  $J = J^\dagger$  as follows

$$J_A^{\bar{B}} \equiv \sum_{\pm} (\pm 1) \psi_A^\pm (\psi_B^\pm)^\dagger = \sum_{\pm} (\pm 1) \psi_A^\pm (\bar{\psi}^\mp)^{\bar{B}}.$$

( $\psi$ 's are the Grassmann fields). Under the discrete operations above the current  $J_A^{\bar{B}}$  transforms as follows

$$\begin{aligned} C &: J_A^{\bar{B}} \rightarrow -J_A^{\bar{B}}, \\ P &: J_A^{\bar{B}} \rightarrow -J_B^{\bar{A}}, \\ T &: J_A^{\bar{B}} \rightarrow J_B^{\bar{A}}. \end{aligned}$$

Fixing boosts and decomposing current  $J_{A\bar{B}}$  into the symmetric and antisymmetric parts,  $J_{XY} = \sum_{\pm} (J_{\pm})_{XY}$ , one gets

$$\begin{aligned} C &: (J_{\pm})_{XY} \rightarrow -(J_{\pm})_{XY}, \\ P &: (J_{\pm})_{XY} \rightarrow \mp (J_{\pm})^{XY}, \\ T &: (J_{\pm})_{XY} \rightarrow \pm (J_{\pm})^{XY}. \end{aligned}$$

This is in complete agreement with the signature association for the symmetric (anti-symmetric) part of the Hermitian spin-tensor  $X$  as the extended spatial (time) components.

### 4 $1 = 1$ space-time

The noncompact group  $Sp(2l, C)$  has  $2l(2l+1)$  generators  $M_{AB} = (L_{AB}, K_{AB})$ ,  $A, B = 1, \dots, 2l$ , with  $M_{AB} = M_{BA}$ . The generators  $L_{AB}$  are Hermitian and correspond to the extended rotations, whereas those  $K_{AB}$  are anti-Hermitian and correspond to the extended boosts. In the space of the first-kind spinors  $\psi_A$  these generators can be represented as  $(\sigma_{AB}, i\sigma_{AB})$  with  $(\sigma_{AB})_{CD} = 1/2(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC})$ , so that  $\sigma_{AB} = \sigma_{BA}$  and  $(\sigma_{AB})_{CD} = (\sigma_{AB})_{DC}$ ,  $(\sigma_{AB})_C^C = 0$ . Similar expressions hold true in the space of the second-kind spinors  $\bar{\psi}_{\bar{A}}$ . In these terms, a canonical formalism can be developed at arbitrary  $l \geq 1$ .

However, in the simplest case  $l = 1$ , corresponding to the ordinary four-dimensional space-time, there exists the isomorphism  $SO(3, C) \simeq Sp(2, C)/Z_2$ . Due to this property, the structure of  $Sp(2, C)$  can be brought to the form more familiar physically, though equivalent mathematically. We use here the complex group  $SO(3, C)$  instead of the real one  $SO(1, 3)$  to clarify the close similarity with the subsequent case  $l = 2$  where there is no real structure group. Because of the complexity of  $SO(3, C)$  one should distinguish vectors and their complex conjugate, the latter ones being omitted for simplicity in what follows. The same remains true for the  $SO(5, C)$  case corresponding to  $l = 2$ .

Let us introduce for the  $SO(3, C)$  group the double set of the Pauli matrices,  $(\sigma_i)_{A\bar{B}}$  and  $(\bar{\sigma}_i)_{\bar{A}B}$ ,  $i = 1, 2, 3$ . They should satisfy the anticommutation relations:  $\sigma_i\bar{\sigma}_j + \sigma_j\bar{\sigma}_i = 2\delta_{ij}\sigma_0$  and  $\bar{\sigma}_i\sigma_j + \bar{\sigma}_j\sigma_i = 2\delta_{ij}\bar{\sigma}_0$ , where  $(\sigma_0)_{A\bar{B}} \equiv \delta_{A\bar{B}}$ ,  $(\bar{\sigma}_0)_{\bar{A}B} \equiv \delta_{\bar{A}B}$  are the Kronecker symbols and  $\delta_{ij}$  is the metric tensor of  $SO(3, C)$ . Among these matrices,  $\sigma_0$  and  $\bar{\sigma}_0$  are the only independent ones which can be chosen antisymmetric:  $(\sigma_0)_{AB} \equiv \epsilon_{AB}$  and  $(\bar{\sigma}_0)_{\bar{A}\bar{B}} \equiv \epsilon_{\bar{A}\bar{B}}$ . On the other hand, with respect to the maximal compact subgroup  $SO(3)$ , all the matrices  $\sigma_i, \bar{\sigma}_i$  can be chosen both Hermitian and symmetric as  $(\sigma_i)_{X^Y} = [(\sigma_i)_{Y^X}]^*$  and  $(\sigma_i)_{XY} = (\sigma_i)_{YX}$  (and the same for  $\bar{\sigma}_i$ ). The matrices  $\sigma_{ij} \equiv -i/2(\sigma_i\bar{\sigma}_j - \sigma_j\bar{\sigma}_i)$  satisfying  $\sigma_{ij} = -\sigma_{ji}$  and  $(\sigma_{ij})_{AB} = (\sigma_{ij})_{BA}$  (and similarly for  $(\bar{\sigma}_{ij})_{\bar{A}\bar{B}} \equiv i/2(\bar{\sigma}_i\sigma_j - \bar{\sigma}_j\sigma_i)_{\bar{A}\bar{B}}$ ) are not linearly independent from  $\sigma_i$ . They can be brought to the form  $(\sigma_{ij})_{XY} = \epsilon_{ijk}(\sigma_k)_{XY}$ , with  $\epsilon_{ijk}$  being the Levi-Civita  $SO(3, C)$  symbol.

The matrices  $(\sigma_{ij}, i\sigma_{ij})$  can be identified as the generators  $M_{ij} = (L_{ij}, K_{ij})$  of the noncompact  $SO(3, C)$  group in the space of the first-kind spinors. Respectively, in the space of the second-kind spinors they are  $(-\bar{\sigma}_{ij}, i\bar{\sigma}_{ij})$ . The generators  $L_{ij}$  of the maximal compact subgroup  $SO(3) \simeq Sp(2)/Z_2$  correspond to rotations, while those  $K_{ij}$  of the noncompact transformations describe Lorentz boosts. Relative to the maximal compact subgroup  $SO(3)$  one has  $\bar{\sigma}_0 = \sigma_0$ ,  $\bar{\sigma}_i = \sigma_i$  and  $\bar{\sigma}_{ij} = -\sigma_{ij}$ . When restricted by  $SO(3)$ , the Hermitian second-rank spin-tensor may be decomposed in the complete set of the Hermitian matrices  $(\sigma_0, \sigma_{ij})$  with the real coefficients:  $X = 1/\sqrt{2}(x_0\sigma_0 + 1/2x_{ij}\sigma_{ij})$ , so that  $(X, X) = x_0^2 - 1/2x_{ij}^2$ . Identifying  $x_{ij} \equiv \epsilon_{ijk}x_k$  one gets as usually  $(X, X) = x_0^2 - x_i^2$ . Both the time and spatial representations being irreducible under  $SO(3)$ , there takes place the usual decomposition  $\underline{4} = \underline{1} \oplus \underline{3}$  relative to the embedding  $SO(3, C) \supset SO(3)$ .

## 5 1 = 2 space-time

This case corresponds to the next-to-ordinary space-time symplectic extension. There takes place the isomorphism  $SO(5, C) \simeq Sp(4, C)/Z_2$ . Cases  $l = 1, 2$  are the only ones when the structure of the symplectic group gets simplified in terms of the complex orthogonal groups. Relative to the maximal compact subgroup  $SO(5)$ , the double set of Clifford matrices  $(\Sigma_I)_{A\bar{B}}$  and  $(\bar{\Sigma}_I)_{\bar{A}B}$ ,  $I = 1, \dots, 5$  can be chosen as Hermitian  $(\Sigma_I)_{X^Y} = [(\Sigma_I)_{Y^X}]^*$  and antisymmetric  $(\Sigma_I)_{XY} = -(\Sigma_I)_{YX}$  (and similarly for  $\bar{\Sigma}_I$ ), like  $(\Sigma_0)_{AB} = \epsilon_{AB}$  and  $(\bar{\Sigma}_0)_{\bar{A}\bar{B}} = \epsilon_{\bar{A}\bar{B}}$ . One can also require that  $(\Sigma_I)_{X^X} = 0$  and  $(\bar{\Sigma}_I)_{X^X} = 0$ . Thus under restriction by  $SO(5)$ , six matrices  $\Sigma_0, \Sigma_I$  provide the complete independent set for the antisymmetric matrices in the four-dimensional spinor space. After introducing matrices  $\Sigma_{IJ} = -i/2(\Sigma_I\bar{\Sigma}_J - \Sigma_J\bar{\Sigma}_I)$ , so that  $\Sigma_{IJ} = -\Sigma_{JI}$ , one gets the symmetry condition for them:  $(\Sigma_{IJ})_{AB} = (\Sigma_{IJ})_{BA}$  (and similarly for

$(\bar{\Sigma}_{IJ})_{\bar{A}\bar{B}} = i/2(\bar{\Sigma}_I\Sigma_J - \bar{\Sigma}_J\Sigma_I)_{\bar{A}\bar{B}}$ . Therefore ten matrices  $\Sigma_{IJ}$  make up the complete set for the symmetric matrices in the spinor space. The matrices  $(\Sigma_{IJ}, i\Sigma_{IJ})$  represent the  $SO(5, C)$  generators  $M_{IJ} = (L_{IJ}, K_{IJ})$  in the space of spinors of the first kind, whereas matrices  $(-\bar{\Sigma}_{IJ}, i\bar{\Sigma}_{IJ})$  do the job in the space of the second kind spinors.

With respect to  $SO(5)$  the Hermitian second-rank spin-tensor  $X$  may be decomposed in the complete set of matrices  $\Sigma_0, \Sigma_I$  and  $\Sigma_{IJ}$  with the real coefficients:  $X = 1/2(x_0\Sigma_0 + x_I\Sigma_I + 1/2x_{IJ}\Sigma_{IJ})$ . In these terms one gets

$$(X, X) = x_0^2 + x_I^2 - \frac{1}{2}x_{IJ}^2.$$

There is one more independent invariant combination of  $x_0, x_I$  and  $x_{IJ}$  originating from the invariant  $\text{tr}(X\bar{X})^2 \sim \det X$ . Relative to the embedding  $SO(5, C) \supset SO(5)$  one has the following decomposition in the irreducible representations:

$$\underline{16} = \underline{1} \oplus \underline{5} \oplus \underline{10}.$$

Under the discrete transformations one gets

$$\begin{aligned} P & : x_0 \rightarrow x_0, x_I \rightarrow x_I, x_{IJ} \rightarrow -x_{IJ}, \\ T & : x_0 \rightarrow -x_0, x_I \rightarrow -x_I, x_{IJ} \rightarrow x_{IJ}. \end{aligned}$$

From the point of view of  $SO(5)$ ,  $x_I$  is the axial vector whereas  $x_{IJ}$  is the pseudo-tensor.

The rank of the algebra  $C_2$  being  $l = 2$ , an arbitrary irreducible representation of the noncompact group  $Sp(4, C)$  is uniquely characterized by two complex Casimir operators  $I_2$  and  $I_4$  of the second and the fourth order, respectively, i.e. by four real quantum numbers. Otherwise, an irreducible representation of  $Sp(4, C)$  can be described by the mixed spin-tensor  $\Psi_{A_1 \dots}^{\bar{B}_1 \dots}$  of a proper rank. This spin-tensor should be traceless in any pair of the indices of the same kind, and its symmetry in each kind of indices should correspond to a two-row Young scheme. In fact, antisymmetry is possible in no more than pairs of indices of the same kind. Therefore an irreducible representation of  $Sp(4, C)$  may unambiguously be characterized by a set of four integers  $(r_1, r_2; \bar{r}_1, \bar{r}_2)$ ,  $r_1 \geq r_2 \geq 0$  and  $\bar{r}_1 \geq \bar{r}_2 \geq 0$ . Here  $r_{1,2}$  (respectively,  $\bar{r}_{1,2}$ ) are the numbers of boxes in the first or second rows of the proper Young scheme. The rank of the maximal compact subgroup  $SO(5) \simeq Sp(4)/Z_2$  (the rotational group) being equal to  $l = 2$ , a state in a representation is additionally characterized under fixed boosts by two additive quantum numbers, namely, the eigenvalues of the mutually commuting momentum components of  $L_{IJ}$  in two different planes, say,  $L_{12}$  and  $L_{45}$ .

## 6 $\Delta l = 1$ reduction

The ultimate unit of dimensionality in the symplectic approach is the discrete number  $l = 1, 2, \dots$  corresponding to the dimensionality  $2l$  of the spinor space. The dimensionality  $d = 4l^2$  of the space-time appears just as a derivative quantity. In reality, the extended space-time with  $l > 1$  should supposedly compactify to the ordinary one with  $l = 1$  by means of the symplectic gravity. Let us consider the next-to-ordinary space-time case with  $l = 2$ . Three generic inequivalent types of the spinor decomposition relative to the embedding  $Sp(4, C) \supset Sp(2, C)$  are conceivable: (i)  $\underline{4} = \underline{2} \oplus \underline{2}$ , (ii)  $\underline{4} = \underline{2} \oplus \bar{\underline{2}}$  and (iii)  $\underline{4} = \underline{2} \oplus \underline{1} \oplus \underline{1}$ .

(i) Chiral spinor doubling

$$\underline{4} = \underline{2} \oplus \underline{2}$$

results in the decomposition of the Hermitian second-rank spin-tensor  $\underline{16} \sim \underline{4} \times \overline{4}$  as

$$\underline{16} = 4 \cdot \underline{4},$$

i.e., in a collection of four four-vectors (more precisely, of three vectors and an axial one). As for matter fermions, the number of the two-component fermions after compactification is twice that of the number of the four-component fermions prior compactification. If a kind of the family structure reproduces itself during the compactification, it is imperative that there should be at least two copies of the fermions in the extended space-time, with at least four copies of them in the ordinary space-time. For phenomenological reasons, the fermions in excess of three families should acquire rather large effective Yukawa couplings as a manifestation of curling-up of the space-time extra dimensions. This is not in principle impossible because the two-component fermions distinguish extra dimensions. Due to possible appearance of the additional moderately heavy vector bosons, the compactification scale  $\Lambda$  could in principle be both moderate and high without conflict with the standard model consistency. On the other hand, the extra time-like dimensions violate causality and the proper compactification scale  $\Lambda$  in the pseudo-orthogonal case should be not less than the Planck scale [4]. Nevertheless, one may hope that the latter restriction could be abandoned in the symplectic approach due to validity of the non-relativistic causality. It is to be valid at small boosts or gravitational fields, so that the compactification scale  $\Lambda$  could possibly be admitted to be not very high. For this reason, the given compactification scenario could still survive at any  $\Lambda$ .

(ii) Vector-like spinor doubling

$$\underline{4} = \underline{2} \oplus \overline{2}$$

results in the decomposition

$$\underline{16} = 2 \cdot \underline{4} \oplus (\underline{3} + \text{h.c.}) \oplus 2 \cdot \underline{1}.$$

In the traditional four-vector notations one has  $X \sim (x_\mu^{(1,2)}, x_{[\mu\nu]}, x^{(1,2)})$ ,  $\mu, \nu = 0, \dots, 3$ , with tensor  $x_{[\mu\nu]}$  being antisymmetric and all the components  $x$  being real. After compactification there should emerge the pairs of the ordinary and mirror matter fermions. For phenomenological reasons, one should require the mirror fermions to have masses supposedly of the order of the compactification scale  $\Lambda$ . Modulo reservations for the preceding case, this compactification scenario could be valid at any  $\Lambda$ , too.

(iii) Spinor-scalar decomposition

$$\underline{4} = \underline{2} \oplus \underline{1} \oplus \underline{1}$$

results in

$$\underline{16} = \underline{4} \oplus (2 \cdot \underline{2} + \text{h.c.}) \oplus 4 \cdot \underline{1},$$

or in the mixed four-vector and spinor notations  $X \sim (x_\mu, x_A^{(1,2)}, x^{(1,2,3,4)})$ ,  $A = 1, 2$ . There would take place the violation of the spin-statistics connection for matter fields in the four-dimensional space-time if this connection fulfilled in the extended space-time. The scale of this violation should be determined by the compactification scale  $\Lambda$  which, in contrast with the two preceding cases, have safely to be high enough for not to violate causality within the experimental precision.

## 7 Gauge interactions

Let  $D_A^{\bar{B}} \equiv \partial_A^{\bar{B}} + igG_A^{\bar{B}}$  be the generic covariant derivative, with  $g$  being the gauge coupling, the Hermitian spin-tensor  $G_A^{\bar{B}}$  being the gauge fields and  $\partial_A^{\bar{B}} \equiv \partial/\partial X^A_{\bar{B}}$  being the ordinary derivative. Now let us introduce the strength tensor

$$\begin{aligned} F_{\{A_1 A_2\}}^{[\bar{B}_1 \bar{B}_2]} &\equiv \frac{1}{ig} D_{\{A_1}^{\bar{B}_1} D_{A_2\}}^{\bar{B}_2]} \\ &= \frac{1}{4ig} \left( D_{A_1}^{\bar{B}_1} D_{A_2}^{\bar{B}_2} - D_{A_2}^{\bar{B}_2} D_{A_1}^{\bar{B}_1} + D_{A_2}^{\bar{B}_1} D_{A_1}^{\bar{B}_2} - D_{A_1}^{\bar{B}_2} D_{A_2}^{\bar{B}_1} \right) \end{aligned}$$

and similarly for  $\bar{F}_{[A_1 A_2]}^{\{\bar{B}_1 \bar{B}_2\}} \equiv (F_{\{B_2 B_1\}}^{[\bar{A}_2 \bar{A}_1]})^*$ , where  $\{\dots\}$  and  $[\dots]$  mean the symmetrization and antisymmetrization, respectively. One gets

$$F_{\{A_1 A_2\}}^{[\bar{B}_1 \bar{B}_2]} = \partial_{\{A_1}^{\bar{B}_1} G_{A_2\}}^{\bar{B}_2]} + ig G_{\{A_1}^{\bar{B}_1} G_{A_2\}}^{\bar{B}_2]}$$

and similarly for  $\bar{F}_{[A_1 A_2]}^{\{\bar{B}_1 \bar{B}_2\}}$ . These tensors are gauge invariant. The total number of the real components in the tensor  $F_{\{A_1 A_2\}}^{[\bar{B}_1 \bar{B}_2]}$  precisely coincides with the number of components of the antisymmetric second-rank tensor  $F_{[\alpha\beta]}$ ,  $\alpha, \beta = 0, 1, \dots, 4l^2 - 1$  defined in the pseudo-Euclidean space of the  $d = 4l^2$  dimensions. But in the symplectic case, tensor  $F$  is reducible and splits into a trace relative to  $\epsilon$  and a traceless part,  $F = F^{(0)} + F^{(1)}$ , where  $F^{(0)[\bar{B}_1 \bar{B}_2]}_{\{A_1 A_2\}} \equiv F^{(0)}_{\{A_1 A_2\}} \epsilon^{\bar{B}_1 \bar{B}_2}$  and  $F^{(1)[\bar{B}_1 \bar{B}_2]}_{\{A_1 A_2\}} \epsilon_{\bar{B}_1 \bar{B}_2} = 0$  (and similarly for  $\bar{F}_{[A_1 A_2]}^{\{\bar{B}_1 \bar{B}_2\}}$ ).

For an unbroken gauge theory with fermions, the generic gauge, fermion and mass terms of the Lagrangian  $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_M$  are, respectively,

$$\begin{aligned} \mathcal{L}_G &= \sum_{s=0,1} (c_s + i\theta_s) F^{(s)} F^{(s)} + \text{h.c.}, \\ \mathcal{L}_F &= \frac{i}{2} \sum_{\pm} (\psi^\pm)^\dagger \vec{D} \psi^\pm, \\ \mathcal{L}_M &= \psi^+ m_0 \psi^- + \sum_{\pm} \psi^\pm m_\pm \psi^\pm + \text{h.c.}, \end{aligned}$$

where  $F^{(s)} F^{(s)} \equiv F^{(s)[\bar{B}_1 \bar{B}_2]}_{\{A_1 A_2\}} F^{(s)\{A_2 A_1\}}_{[\bar{B}_2 \bar{B}_1]}$ . In the Lagrangians above,  $\psi^\pm$  are the charged conjugate fermions,  $m_0$  is the generic Dirac mass,  $m_\pm$  are Majorana masses,  $c_s$  and  $\theta_s$  are the real gauge parameters. One of the parameters  $c_s$ , supposedly  $c_0 \neq 0$ , can be normalized at will. The Lagrangian results in the following generalization of the Dirac equation

$$iD^C_{\bar{B}} \psi_C^\pm = m_0^\dagger \bar{\psi}_B^\pm + \sum_{\pm} m_\pm^\dagger \bar{\psi}_B^\mp$$



and the pair of Maxwell equations ( $c_0 \equiv 1$  and  $c_1 = \theta_1 = 0$ , for simplicity)

$$\begin{aligned} (1 + i\theta_0)D^{C\bar{B}}F^{(0)}_{\{CA\}} - \text{h.c.} &= 0, \\ (1 + i\theta_0)D^{C\bar{B}}F^{(0)}_{\{CA\}} + \text{h.c.} &= 2gJ_A^{\bar{B}}, \end{aligned}$$

with  $J_A^{\bar{B}} \equiv \sum_{\pm} (\pm 1)\psi_A^{\pm}(\psi_B^{\pm})^\dagger$  being the fermion Hermitian current.

Tensors  $F^{(s)}$ ,  $s = 1, 2$  are non-Hermitian, but under restriction by the maximal compact subgroup  $Sp(2l)$  they split into a pair of the Hermitian ones  $E^{(s)}$  and  $B^{(s)}$  as  $F^{(s)} = E^{(s)} + iB^{(s)}$ . Introducing the duality transformation  $F^{(s)} \rightarrow \tilde{F}^{(s)} \equiv -iF^{(s)}$ , so that  $\tilde{E}^{(s)} = B^{(s)}$  and  $\tilde{B}^{(s)} = -E^{(s)}$ , one gets  $\mathcal{R}eF^{(s)}F^{(s)} = E^{(s)2} - B^{(s)2}$  and  $\mathcal{I}mF^{(s)}F^{(s)} = \mathcal{R}e\tilde{F}^{(s)}F^{(s)} = 2E^{(s)}B^{(s)}$ . Though the splitting into  $E^{(s)}$  and  $B^{(s)}$  is noncovariant with respect to the whole  $Sp(2l, C)$ , the duality transformation is covariant. Tensors  $E^{(s)}$  and  $B^{(s)}$  are the counterparts of the ordinary electric and magnetic strengths, and  $\theta_0$  is the counterpart of the ordinary  $T$ -violating  $\theta$ -parameter for the  $l = 1$  case. Thus,  $\theta_1$  is an additional  $T$ -violating parameter at  $l > 1$ . Note that in the framework of symplectic extension the electric and magnetic strengths stay on equal footing. This is to be contrasted with the pseudo-orthogonal extension where these strengths have unequal number of components at  $d \neq 4$ . The electric-magnetic duality for the gauge fields (in the Euclidean space) play an important role for the study of the topological structure of the gauge vacuum in four space-time dimensions. Therefore the similar study might be applicable to the extended symplectic space-times with arbitrary  $l > 1$ .

## 8 Gravity

The field equations above are valid in the flat extended space-time or, otherwise, refer to the inertial local frames. To go beyond, one can introduce the local vielbeins  $e_{MA}^{\bar{B}}(X)$ , such that  $e_{MA}^{\bar{B}} = (e_{MB}^{\bar{A}})^*$ , the real world coordinates  $x_M \equiv e_M^A X_A^{\bar{B}}$ , as well as the generally covariant derivative  $\nabla_M(e)$ , with  $M = 0, 1, \dots, 4l^2 - 1$  being the world vector index. Now, the Lagrangian can be adapted to the  $d = 4l^2$  dimensional curved space-time equipped with the pseudo-Riemannian structure, i.e., the real symmetric metrics  $g_{MN}(x) = e_M^A e_{NA}^{\bar{B}}$ . One can also supplement gauge equations by the generalized gravity equations in line with [5]. But in the case at hand, the group of equivalence of the local vielbeins (structure group) is required to be only the symplectic group  $Sp(2l, C)$  rather than the whole pseudo-orthogonal group  $SO(d_-, d_+)$ . The former one leaves more independent components in the local symplectic vielbeins compared to the pseudo-Riemannian vielbeins and thus to the metrics. Hence the symplectic gravity is not in general equivalent to the metric one. The curvature tensor in the symplectic case, like the gauge strength one, splits additionally into irreducible parts which can a priori enter the gravity Lagrangian with the independent coefficients. The ultimate reason for this is that the space-time is meant to be in the symplectic approach not a fundamental entity. Therefore gravity as a generally covariant theory of the space-time deformations have to be just as an effective theory. The latter admits the existence of a number of free parameters, the choice of which should eventually be clarified by an underlying theory.

## 9 Conclusion

The hypothesis that the symplectic structure of space-time is superior to the metric one provides, in particular, the rationale for the four-dimensionality and the  $1+3$  signature of the ordinary space-time. When looking for the space-times with extra dimensions, the hypothesis predicts the discrete series of the metric space-times of the peculiar dimensionalities and signatures, both with the spatial and time extra dimensions. One of the time-like directions remaining rotationally invariant under fixed boosts makes it possible the approximate causality description despite the presence of extra times. The extended symplectic space-times provide a viable alternative to the pseudo-orthogonal ones. But beyond the physical adequacy of the extended space-times as such, by generalizing from the basic case  $l = 1$  to its counterpart for general  $l > 1$ , a deeper insight into the nature of the four-dimensional space-time itself may be gained.

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