

Ultraviolet and Infrared Divergences in Superstring Theory

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Abstract

Superstring theory is known to be free from ultraviolet divergences but suffers from the usual infrared divergences that occur in quantum field theories. After briefly reviewing the origin of ultraviolet finiteness of superstring theory we describe recent progress towards the understanding of infrared divergences in superstring theory.

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Quantum field theory is the standard theoretical tool for studying the physics of elementary particles. The most commonly used approach for studying quantum field theories is perturbation theory, where we take all the interaction effects to be small and carry out a Taylor series expansion in the coupling constants – the parameters that label the interaction strengths – of various physical quantities like the scattering amplitudes. The coefficients of the Taylor series expansion are given by sum of *Feynman diagrams*, each of which represents an integral over certain number of *loop momenta*. For a quantum field theory in d space-time dimensions a typical integral takes the form

$$\int d^d \ell_1 \cdots d^d \ell_g \prod_{j=1}^r (k_j^2 + m_j^2)^{-1} \mathcal{N} \quad (1)$$

where each ℓ_i is a d -dimensional vector labelling loop momenta, each k_j is a d -dimensional vector given by appropriate linear combination of the ℓ_i 's and the momenta p_1, \cdots, p_n carried by the incoming and outgoing particles whose scattering amplitude we are trying to calculate, m_j denotes the mass of one of the particles in the theory and the *numerator factor* \mathcal{N} is a polynomial in the loop momenta $\{\ell_i\}$ and the external momenta $\{p_k\}$. The components of the vector k_j (and similarly for ℓ_j) are labelled as $(k_j^0, \cdots, k_j^{d-1})$, and $k_j^2 \equiv -(k_j^0)^2 + (k_j^1)^2 + \cdots + (k_j^{d-1})^2$. The number of ℓ_i 's and k_j 's, the expressions for the k_j 's in terms of the ℓ_i 's and the p_k 's, which mass m_j to use in a given factor in (1) and the precise expression for the numerator factor \mathcal{N} are all fixed by the Feynman rules for a given Feynman diagram. Therefore all one needs to do to compute the scattering amplitude is to carry out the integrals of the form given in (1) and add the contributions from different diagrams. An expression containing integration over g loop momenta, like the one appearing in (1), is usually referred to as a g -loop contribution to the amplitude.

This has been an enormously successful program and lies at the heart of most of what we know about elementary particles. However, quantum field theories have inherent divergences – infinities encountered in the evaluation of (1) – which need to be dealt with before we can make concrete predictions. These divergences can be broadly classified into two kinds – *ultraviolet* and *infrared*. The ultraviolet divergences come from the region of integration where one or more of the ℓ_i 's in (1) become large. The infrared divergences arise from the vanishing of one or more factors of $(k_j^2 + m_j^2)$. It turns out that the ultraviolet divergences are unphysical, and can be removed in a class of quantum field theories called renormalizable quantum field theories. For describing the theory of elementary particles we use this kind of quantum field theories. On the other hand the infrared divergences have physical origin, in that their appearance signals that we are not asking the right question. Once the right question is asked, these divergences automatically disappear. Typical examples of infrared divergences

are *tadpole divergences* which arise when we incorrectly identify the ground state of the system about which we carry out the perturbation expansion, and *mass renormalization divergences* which arise when we use the wrong mass of the external states for computing the scattering amplitudes. These divergences typically arise when the ground state and/or the masses of the particles change after taking into account the effect of interactions, and we are not careful in taking into account the effect of these changes in our calculation.

In order to characterize these divergences it is useful to use the so called *Schwinger parameter* representation for the *propagator* factors $(k_j^2 + m_j^2)^{-1}$. For each propagator we introduce a real parameter s_j and write

$$(k_j^2 + m_j^2)^{-1} = \int_0^\infty ds_j \exp[-s_j(k_j^2 + m_j^2)]. \quad (2)$$

With the help of this identity, (1) can be written as

$$\int_0^\infty ds_1 \cdots \int_0^\infty ds_r \int d^d \ell_1 \cdots d^d \ell_g \exp \left[- \sum_j s_j (k_j^2 + m_j^2) \right] \mathcal{N}. \quad (3)$$

Since each k_j is a linear combination of the ℓ_i 's, the exponent is quadratic function of the ℓ_i 's for fixed s_j . Since the numerator factor \mathcal{N} is polynomial in ℓ_j , we can now explicitly perform the integration over the ℓ_j 's using the standard rules of Gaussian integration over multiple variables. Special care is needed to treat the integration over the ℓ_j^0 's; due to the fact that $(k_j^0)^2$ appears with a negative coefficient in the expression for k_j^2 , the coefficients of $(\ell_i^0)^2$ in the argument of the exponential in (3) is positive and the ℓ_i^0 integrals are *a priori* divergent. This is circumvented by the standard procedure of analytically continuing these integrals so that the ℓ_i^0 integrals run along the imaginary axis. Once this is done, one can carry out the integration over the ℓ_i 's without encountering any divergence, and express (3) as

$$\int_0^\infty ds_1 \cdots \int_0^\infty ds_r F(\{s_i\}), \quad (4)$$

for some function F of the s_j 's. It is easy to verify that the ultraviolet divergences of the original integral, coming from the region of large ℓ_j , now will appear as a divergence in the integral (4) from the region where a subset of the s_j 's go to zero. On the other hand infrared divergences of the original integral will appear in (4) from the region where one or more s_j 's become large.

In superstring theory (which for our discussion will stand for four different varieties of string theory named as *SO(32) heterotic*, *E₈ × E₈ heterotic*, *type IIA* and *type IIB* string theories) we replace the notion that the elementary building blocks of matter are point particles by the notion that they are strings – one dimensional extended objects. The main motivation for

superstring theory stems from the fact that this theory automatically incorporates gravity in its framework. This is to be contrasted with quantum field theory, which has great difficulty in incorporating gravity in its fold. The naive quantum field theory that one gets by applying the usual rules of quantum field theory to general theory of relativity – the theory developed by Einstein a hundred years ago – leads to a non-renormalizable theory and has uncontrolled ultraviolet divergences.

Superstring theory comes with its own prescription for computing scattering amplitudes which seems to differ from the sum over Feynman diagram expansion that emerges from a quantum field theory. The intrinsic difference arises from the fact that whereas particle trajectories are described by curves in space-time, the trajectory of a string is described by a surface in space-time – often referred to as the *world-sheet*. This intuitive picture allows us to represent the string scattering amplitudes as integrals over the *moduli space* of two dimensional Riemann surfaces – the moduli space being a space whose different points label different Riemann surfaces. In particular the g -loop, n -point scattering amplitude in superstring theory is given by an expression of the form:

$$\int_{\mathcal{M}_{g,n}} \prod_{i=1}^{6g-6+2n} dm_i f(\{m_j\}). \quad (5)$$

Here $\mathcal{M}_{g,n}$ denotes a $6g - 6 + 2n$ dimensional moduli space of *genus* g Riemann surface with n marked points – the genus of a Riemann surface being the number of handles that the Riemann surface has. The integrand $f(\{m_j\})$ is given by the correlation function of certain operators in a two dimensional conformal field theory inserted at the marked points on the Riemann surface. Which conformal field theory to use is determined by the specific background around which we study the superstring theory, whereas which operators in the conformal field theory we should use for our calculation is determined by the states whose scattering amplitude we want to compute.

This way of computing scattering amplitude *a priori* looks very different from the Feynman diagram expressions that we obtain from quantum field theories. However a closer examination reveals that in appropriate limit, the integral over moduli space that comes from superstring theory begins to resemble the integrals over the Schwinger parameters s_j in (4) that appear in quantum field theories. Since the divergences in quantum field theory arise from the integration over the parameters s_j , we see that to examine the fate of those divergences in superstring theory we have to examine possible divergences arising from the integral over the moduli spaces of Riemann surfaces.

Now the divergences from the integral over the moduli spaces of Riemann surfaces typically arise from the regions where the Riemann surface becomes singular. Study of singular

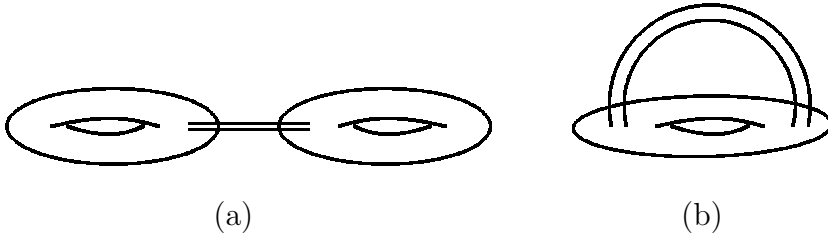


Figure 1: Degeneration of Riemann surfaces.

Riemann surfaces is a well-developed subject, and all such singularities are known to occur from *degenerations* of Riemann surfaces, where the Riemann surface either becomes a pair of Riemann surfaces connected by an infinitely narrow tube (Fig. 1(a)), or develops an infinitely narrow handle connecting two points on a single Riemann surface (Fig. 1(b)). By using the relation between the parameters m_j labelling the moduli space of Riemann surfaces and the parameters s_j appearing in the expression for an amplitude (4) in quantum field theory, one can show that all such singularities in the moduli space of Riemann surfaces can be interpreted as the region where one or more of the s_j 's become infinite. Therefore we conclude that all divergences in superstring theory can be interpreted as infrared divergences.

There is however a caveat. Even though the divergences coming from singularities of the moduli space of Riemann surfaces can be interpreted as infrared divergences, one may wonder whether there can be divergences from the regular regions in the interior of the moduli space. This would happen if the correlation function $f(\{m_j\})$ in the conformal field theory that we have to compute blows up at some regular point in the interior of the moduli space. One does not expect this to happen for a unitary conformal field theory, but it turns out that such a singularity could arise from the correlation function in the non-unitary conformal field theory of *superconformal ghost fields* – the fields which arise in the process of gauge fixing the supersymmetry transformation in the superstring world-sheet theory [1]. Physically these *spurious singularities* reflect the breakdown of the gauge fixing procedure. Recently a completely systematic procedure for avoiding these singularities has been developed [2, 3]. Furthermore one finds that while the procedure itself is not unique, different ways of avoiding the singularities lead to the same result for the scattering amplitudes. This establishes that there are no divergences coming from the interior of the moduli space, and the only possible divergences that arise in superstring theory are infrared divergences. An alternative approach to this problem, based on integration over the moduli space of *super Riemann surfaces*, has also been developed [4].

Therefore the relevant question is: how can we deal with the infrared divergences of superstring theory? As we have already mentioned earlier, in a quantum field theory, infrared

divergences have physical origin; they reflect that we are asking the wrong questions. Experience with quantum field theory also teaches us how to ask the right question and remove the infrared divergences. For example, quantum field theories have a systematic procedure for taking into account possible changes in the ground state and/or masses of elementary particles due to interaction, – this is lacking in the conventional approach to superstring perturbation theory. Therefore, if we had a quantum field theory whose Feynman diagrams reproduced the scattering amplitudes of superstring theory, then we would automatically know how to ask the right questions and avoid the infrared divergences in superstring theory.

This is another area where there has been progress in recent years. It turns out that it is indeed possible to write down a quantum field theory whose Feynman rules reproduce the amplitudes of the form (5) that come from superstring theory [5]. This quantum field theory – known as superstring field theory – is somewhat unusual, involving infinite number of fields and non-local interaction terms. Nevertheless it has the structure inherent to a quantum field theory that allows us to remove the infrared divergences exactly as we would do in a conventional quantum field theory.

To summarize, we now have a formulation of superstring theory which gives results free from all divergences, infrared and ultraviolet, when we ask the right questions. The scattering amplitudes computed from this formulation satisfy many of the desired properties *e.g.* *Ward identities* associated with general coordinate transformations and other gauge symmetries [6]. Work is in progress towards proving other desired properties of the scattering amplitude – *e.g.* unitarity (conservation of probability) – using this approach. We hope to make progress on this front in the near future.

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