Towards Loop Quantum Supergravity (LQSG) II. p-Form Sector

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Abstract

In our companion paper, we focussed on the quantisation of the Rarita-Schwinger sector of Supergravity theories in various dimensions by using an extension of Loop Quantum Gravity to all spacetime dimensions. In this paper, we extend this analysis by considering the quantisation of additional bosonic fields necessary to obtain a complete SUSY multiplet next to graviton and gravitino in various dimensions. As a generic example, we study concretely the quantisation of the 3-index photon of $11d$ SUGRA, but our methods easily extend to more general p-form fields.

Due to the presence of a Chern-Simons term for the 3-index photon, which is due to local SUSY, the theory is self-interacting and its quantisation far from straightforward. Nevertheless, we show that a reduced phase space quantisation with respect to the 3-index photon Gauß constraint is possible. Specifically, the Weyl algebra of observables, which deviates from the usual CCR Weyl algebras by an interesting twist contribution proportional to the level of the Chern-Simons theory, admits a background independent state of the Narnhofer-Thirring type.

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1 Introduction

In our companion papers [1, 2, 3, 4, 5], we studied the canonical formulation of General Relativity (gravitons) coupled to standard matter in terms of connection variables for a compact gauge group without second class constraints in order that Loop Quantum Gravity (LQG) quantisation methods, so far formulated only in three and four spacetime dimensions [6, 7], apply.

The actual motivation for doing this comes from Supergravity and String theory [8, 9]: String theory is considered as a candidate for a UV completion of General Relativity, which in its present formulation requires extra dimensions and supersymmetry. Supergravity is considered as the low energy effective field theory limit of String theory. One may therefore call String theory a top − bottom approach. In this series of papers we take first steps towards a bottom − top approach in that we try to canonically quantise the Supergravity theories by LQG methods. While String theory in its present form needs a background dependent and perturbative quantum formulation, the LQG quantum formulation is by design background independent and non perturbative. On the other hand, quantum String theory is much richer above the low energy field theory limit, containing an infinite tower of higher excitation modes of the string, which come into play only when approaching the Planck scale and which are necessary in order to find a theory which is finite at least order by order in perturbation theory.

The quantisation of Supergravity is therefore the ideal arena in which to compare these two complementary approaches to quantum gravity, which was not possible so far. At least at low energies, that is, in the semiclassical limit, the two theories should agree with each other, as otherwise they would quantise two different classical theories. Evidently, this opens the very exciting possibility of cross fertilisation between the two approaches, which we are going to address in future publications.

The new field content of Supergravity theories as compared to standard matter Lagrangians are 1. Majorana (or Majorana-Weyl) spinor fields of spin $1/2, 3/2$ including the Rarita-Schwinger field (gravitino) and 2. additional bosonic fields that appear in order to obtain a complete supersymmetry multiplet in the dimension and the amount $\mathcal{N}$ of supersymmetry charges under consideration. The treatment of the Rarita-Schwinger sector and its embedding in the framework of [1, 2, 3, 4, 5] was accomplished in [10]. In this paper, we complete the quantisation of the extra matter content of many Supergravity theories by considering the quantisation of the additional bosonic fields, in particular, p-form fields. Specifically, for reasons of concreteness, we quantise the 3-index photon of $11d$ Supergravity but it will transpire that the methods employed generalise to arbitrary p.

What makes the quantisation possible is that the Gauß constraints of the 3-index photon form an Abelian ideal in the constraint algebra. If this ideal (or subalgebra) would be non − Abelian, then our methods would be insufficient and we most probably would have to use methods from higher gauge theory [11, 12, 13, 14, 15] such as p-groups, p-holonomies etc., a subject which at the moment is not yet sufficiently developed from the mathematical perspective (see [16] for the state of the art of the subject). Despite the Abelian character of this additional Gauß constraint, the quantisation of the theory is not straightforward and cannot be performed in complete analogy to the treatment of the Abelian Gauß constraint of standard 1-form matter [17]. This is due to a Chern-Simons term in the Supergravity action, whose presence is dictated by supersymmetry and which makes the theory in fact self-interacting, that is, the Hamiltonian is a fourth order polynomial in the 3-connection and its conjugate momentum just like in Yang-Mills theory. In particular, while one can define a holonomy flux algebra as for Abelian Maxwell-theory, the Ashtekar-Isham-Lewandowski representation [18, 19] is inadequate because the Abelian gauge group does not preserve the holonomy flux algebra.

A solution to the problem lies in performing a reduced phase space quantisation in terms of a twisted holonomy flux algebra, which is in fact Gauß invariant. We were not able to find a background independent representation of the corresponding Heisenberg algebra, which also differs by a twist from the usual one, however, one succeeds when formulating the quantum theory in terms of the corresponding Weyl elements. The resulting Weyl algebra is not of standard form and to the best of our knowledge it has not been quantised before. We show that it admits a state of the Narnhofer-Thirring type [20] whence the Hilbert space representation follows by the GNS construction. The Hamiltonian (constraint) can be straightforwardly expressed in terms of the Weyl elements, in fact it is quadratic in terms of the classical observables, that is, the generators of the Heisenberg algebra.

This paper's architecture is as follows:

In section 2, we sketch the Hamiltonian analysis of the 3-index photon in a self-contained fashion for the benefit of the reader and in order to settle our notation. We also describe in detail why one cannot straightforwardly apply methods from LQG as mentioned above.
In section 3, we display the reduced phase space quantisation solution in terms of the twisted holonomy flux algebra.
Finally, in section 4, we summarise and conclude.

2 Classical Hamiltonian Analysis of the 3-Index-Photon Action

The Hamiltonian analysis of the full $11d$ SUGRA Lagrangian has been performed in [21]. We will review the analysis of the contribution of the 3-index-photon 3-form $A_{\mu\nu\rho} = A_{[\mu\nu\rho]}$ to the $11d$ SUGRA Lagrangian with Chern-Simons term. This part of the Lagrangian is given up to a numerical constant by

$$\mathcal{L}_{\rm C} = -\frac{1}{2}|g|^{1/2} F_{\mu_1..\mu_4}\ F^{\mu_1..\mu_4} - \alpha|g|^{1/2} F_{\mu_1..\mu_4}\ J^{\mu_1..\mu_4} - \frac{c}{2}|g|^{1/2} F_{\mu_1..\mu_4}\ F_{\nu_1..\nu_4}\ A_{\rho_1..\rho_3}\epsilon^{\mu_1..\mu_4\nu_1..\nu_4\rho_1..\rho 3}. \tag{2.1}$$

Here, $F = dA,\ F_{\mu_1..\mu_4} = \partial_{[\mu_1} A_{\mu_2..\mu_3]}$ is the curvature of the 3-index-photon and indices are moved with the spacetime metric $g^{\mu\nu}$. Furthermore, J is a totally skew tensor current bilinear in the graviton field not containing derivatives, whose explicit form does not need to concern us here, except that it does not depend on any other fields. Finally, c, α are positive numerical constants whose value is fixed by the requirement of local supersymmetry [22]. The number c could be called the level of the Chern-Simons theory in analogy to $D + 1 = 3$.

We proceed to the $D+1$ split of this Lagrangian in a coordinate system with coordinates $t, x^a;\ a = 1, .., D$ adapted to a foliation of the spacetime manifold. The result of a tedious calculation is given by

$$\begin{aligned}
F_{\mu_1..\mu_4}\ F^{\mu_1..\mu_4} &= 4F_{ta_1..a_3}\ F^{ta_1..a_3} + F_{a_1..a_4}F^{a_1..a_4},\\
F_{ta_1..a_3}\ F^{ta_1..a_3} &= G^{a_1..a_3,b_1..b_3}\ F_{ta_1..a_3}\ F_{tb_1..b_3} - M^{a_1..a_3,b_1..b_4}\ F_{ta_1..a_3}\ F_{b_1..b_4},\\
G^{a_1..a_3,b_1..b_3} &= g^{tt}g^{a_1b_1}g^{a_2b_2}g^{a_3b_3} - 3g^{ta_1}g^{tb_1}g^{a_2b_2}g^{a_3b_3},\\
M^{a_1..a_3,b_1..b_4} &= g^{a_1b_2}g^{a_2b_2}g^{a_3b_3}g^{tb_4},\\
F_{a_1..a_4}\ F^{a_1..a_4} &= V_1 - 4M^{a_1..a_3,b_1..b_4}F_{ta_1..a_3}F_{b_1..b_4},\\
V_1 &= g^{a_1b_1}..g^{a_4b_4}\ F_{a_1..a_4}\ F_{b_1..b_4},\\
F_{\mu_1..\mu_4}\ F_{\nu_1..\nu_4}\ A_{\rho_1..\rho_3}\epsilon^{\mu_1..\mu_4\nu_1..\nu_4\rho_1..\rho_3} &= 8\epsilon^{a_1..a_3b_1..b_4c_1..c_3}F_{ta_1..a_3}F_{b_1..b_4}A_{c_1..c_3}\\
&\quad +3\epsilon^{a_1..a_4b_1..b_4c_1c_2}F_{a_1..a_4}F_{b_1..b_4}A_{tc_1c_2},\\
J^{\mu_1..\mu_4}F_{\mu_1..\mu_4} &= 4j^{a_1..a_3}F_{ta_1..a_3} + V_2,\\
V_2 &= J^{a_1..a_4}F_{a_1..a_4},
\end{aligned} \tag{2.2}$$

where we used $\epsilon^{a_1..a_D} = \epsilon^{ta_1..a_D}$ and defined $j^{a_1..a_3} := J^{ta_1..a_3}$. The potential terms V_1, V_2 only depend on the spatial components of the curvature and do not contain time derivatives.

Using

$$F_{ta_1..a_3} = \frac{1}{4}[\dot{A}_{a_1..a_3} - 3\partial_{[a_1}A_{a_2a_3]t}], \tag{2.3}$$

we may perform the Legendre transform. The momentum conjugate to A reads

$$\begin{aligned}
\pi^{a_1..a_3} &= \frac{\partial\mathcal{L}}{\partial\dot{A}_{a_1..a_3}} \\
&= -|g|^{1/2}[G^{a_1..a_3,b_1..b_3}F_{tb_1..b_3} - M^{a_1..a_3,b_1..b_4}F_{b_1..b_4} + \alpha j^{a_1..a_3}] - c\ \epsilon^{a_1..a_3b_1..b_4c_1..c_3}F_{b_1..b_4}A_{c_1..c_3}.
\end{aligned} \tag{2.4}$$

We may solve (2.4) for $F_{ta_1a_2a_3}$

$$\begin{aligned}
F_{ta_1..a_3} &= -|g|^{1/2}G_{a_1..a_3,b_1..b_3}\ [\pi^{b_1..b_3} + B^{b_1..b_3} + \alpha|g|^{1/2}j^{b_1..b_3}],\\
B^{a_1..a_3} &= c\epsilon^{a_1..a_3b_1..b_4c_1..c_3}F_{b_1..b_4}A_{c_1..c_3} - |g|^{1/2}\ M^{a_1..a_3,b_1..b_4}F_{b_1..b_4},
\end{aligned} \tag{2.5}$$

where

$$G_{a_1..a_3,c_1..c_3}G^{c_1..c_3,b_1..b_3} = \delta^{b_1}_{[a_1}\delta^{b_2}_{a_2}\delta^{b_3}_{a_3]} \tag{2.6}$$

defines the inverse of G.
Inverting (2.3) for $\dot{A}$ and using (2.4) and (2.2) we obtain for the Hamiltonian after a longer calculation

$$\begin{aligned} H &= \int d^{10}x \, \Big\{ \dot{A}_{a_1..a_3} \pi^{a_1..a_3} - \mathcal{L} \Big\} \\ &= -\int d^{10}x \Big\{ 3A_{ta_1a_2} \, G_C^{a_1a_2} + 2|g|^{-1/2} G_{a_1..a_3,b_1..b_3} [\pi + B + \alpha j]^{a_1..a_3} \, [\pi + B + \alpha j]^{b_1..b_3} \\ &\qquad + |g|^{1/2} [V_1/2 + \alpha V_2] \Big\}, \\ G_C^{a_1a_2} &:= \partial_{a_3} \pi^{a_1..a_3} - \frac{c}{2} \epsilon^{a_1a_2b_1..b_4c_1..c_4} F_{b_1..b_4} F_{c_1..c_4}, \end{aligned} \tag{2.7}$$

where an integration by parts has been performed in order to isolate the Lagrange multiplier $A_{ta_1a_2}$. Using the ADM frame metric components

$$g^{tt} = -1/N^2, \; g^{ta} = N^a/N^2, \; g^{ab} = q^{ab} - N^a N^b/N^2; \;\; g_{tt} = -N^2 + q_{ab} N^a N^b, \; g_{ta} = q_{ab} N^b, \; g_{ab} = q_{ab}, \tag{2.8}$$

with q_{ab} the induced metric on the spatial slices and lapse respectively shift functions N, N^a we can easily decompose the piece of H independent of the 3-index Gauß constraint $G_C^{a_1a_2}$ into the contributions $N^a \mathcal{H}_{Ca} + N \mathcal{H}_C$ to the spatial diffeomorphism constraint and Hamiltonian constraint, however, we will not need this at this point.
We will drop the subscript C in what follows, since in this paper we are only interested in the p-form sector. We smear the Gauß constraint with a 2-form Λ, that is

$$G[\Lambda] := \int d^{10}x \, \Lambda_{ab} \, G^{ab} \tag{2.9}$$

and study the gauge transformation behaviour of the canonical pair (A_{abc}, π^{abc}) with non-vanishing Poisson brackets

$$\{\pi^{a_1..a_3}(x), A_{b_1..b_3}(y)\} = \delta^{(10)}(x,y) \; \delta^{a_1}_{[b_1} \delta^{a_2}_{b_2} \delta^{a_3}_{b_3]}. \tag{2.10}$$

We find

$$\begin{aligned} \{G[\Lambda], A_{a_1..a_3}\} &= -\partial_{[a_1} \Lambda_{a_2a_3]}, \\ \{G[\Lambda], \pi^{a_1..a_3}\} &= c \, \epsilon^{a_1..a_3b_1..b_3c_1..c_4} \partial_{[b_1} \Lambda_{b_2b_3]} F_{c_1..c_4}. \end{aligned} \tag{2.11}$$

These equations can be written more compactly in differential form language, in terms of which they are easier to memorise. Introducing the dual 7-pseudo-form[1]

$$(*\pi)_{a_1..a_7} := \frac{1}{3! \, 7!} \epsilon_{b_1..b_3a_1..a_7} \pi^{b_1..b_3} \tag{2.12}$$

we may write (2.11) as

$$\delta_\Lambda A = -d\Lambda, \quad \delta_\Lambda \; *\pi = c \, (d\Lambda) \wedge F. \tag{2.13}$$

Since the right hand side of (2.13) is closed, in fact exact, it would seem that the observables of the theory can be coordinatised by integrals of A and $*\pi$ respectively over closed 3-submanifolds or 7-submanifolds respectively.

The $G(\Lambda)$ generate an Abelian ideal in the constraint algebra since

$$\{G(\Lambda), G(\Lambda')\} = 0, \quad \{G(\Lambda), H(x)\} = 0, \tag{2.14}$$

where $H(x)$ is the integrand of H in (2.7) and since the only π or A dependent contributions to the Hamiltonian and spatial diffeomorphism constraints are contained in $H(x)$.

We see that due to the non vanishing Chern-Simons constant c, the transformation behaviour of $*\pi$ differs from the transformation behaviour with respect to the higher dimensional analog of the usual

[1] Notice that π^{abc} is a tensor density of type T_0^3 and density weight one.

Maxwell type of Gauß law, which would be just the divergence term $\partial_{a_1}\pi^{a_1..a_3}$. In particular, π^{abc} itself is *not* gauge invariant. This "twisted" Gauß constraint (2.7) can be written in the form

$$G^{a_1a_2} := \partial_{a_3}[\pi^{a_1..a_3} - \frac{c}{2}\epsilon^{a_1..a_3b_1..b_3c_1..c_4}A_{b_1..b_3}F_{c_1..c_4}] =: \partial_{a_3}\pi'^{a_1..a_3}, \tag{2.15}$$

which suggests to introduce a new momentum π'. Unfortunately, this does not work because $*(\pi'-\pi) = A\wedge F$ does not have a generating functional K with $\delta K/\delta A = A\wedge F$, since the only possible candidate $K = \int A\wedge A\wedge F \equiv 0$ identically vanishes in the dimensions considered here. Since this is not the case, the Poisson brackets of π' with itself do not vanish and neither is π' gauge invariant as we will see below, so that there is no advantage of working with π' as compared to π.

The presence of the twist term in the Gauß constraint leads to the following difficulty when trying to quantise the theory on the usual LQG type kinematical Hilbert space:
Such a Hilbert space would roughly be generated by a holonomy flux algebra constructed from holonomies

$$A(e) = \exp(i\int_e A), \quad \pi(S) = \int_S *\pi, \tag{2.16}$$

where e and S are oriented 3-dimensional and 7-dimensional submanifolds respectively, which we call "edges" and surfaces in what follows. One could then study the GNS Hilbert space representation generated by the LQG type of positive linear functional

$$\omega(f\pi(S_1)..\pi(S_n)) = 0, \quad \omega(f) = \mu[f], \tag{2.17}$$

where μ is an LQG type measure on a space of generalised connections $\overline{\mathcal{A}}$. One can define it abstractly by requiring that the charge network functions

$$T_{\gamma,n} = \prod_{e\in\gamma} A(e)^{n_e}, \; n_e \in \mathbb{Z} \tag{2.18}$$

form an orthonormal basis in the corresponding $\mathcal{H} = L_2(\overline{\mathcal{A}},\mu)$, see [7] for details. Here, a graph γ is a collection of edges which are disjoint up to intersections in "vertices", which are oriented 2-manifolds. The possible intersection structure of these cobordisms should be tamed by requiring that all submanifolds are semi-analytic.
Up to here everything is in full analogy with LQG. The problem is now to isolate the Gauß invariant subspace of the Hilbert space: While the connection transforms as in a theory with untwisted Gauß constraint, it appears that we can solve it by requiring that charges add up to zero at vertices. However, this does not work because while such a vector is annihilated by the divergence term in G^{ab}, it is not by the second term $\propto A\wedge F$. Even more disastrous, the term $A\wedge F$ does not exist in this representation which is strongly discontinuous in the holonomies so that operators A, F do not exist. Finally, although π is not Gauß invariant, it leaves this would be gauge invariant subspace invariant, which reveals that this subspace is not the kernel of the twisted Gauß constraint.

We therefore must be more sophisticated. Since the A dependent terms in G cannot be quantised on the kinematical Hilbert space, we must exponentiate it:
Consider the Hamiltonian flow of $G[\Lambda]$

$$\exp(\{G[\Lambda],\cdot\})A = A - d\Lambda, \quad \exp(\{G[\Lambda],\cdot\}) * \pi = *\pi + c(d\Lambda)\wedge F, \tag{2.19}$$

which is a Poisson automorphism α_Λ (canonical transformation) and one would like to secure that an implementation of the corresponding automorphism group $\alpha_\Lambda \circ \alpha_{\Lambda'} = \alpha_{\Lambda+\Lambda'}$ by unitary operators $U(\Lambda)$ exists. The $U(\Lambda)$ would correspond to the desired exponentiation of the Gauß constraint. One way of securing this is by looking for an invariant state $\omega = \omega\circ\alpha_\Lambda$ on the holonomy - flux algebra (see [23]) for the details for this construction). This would then open the possibility that the Gauß constraint can be solved by group averaging methods. The first problem is that the automorphisms do not preserve the holonomy flux algebra because there appears an F on the right hand side of (2.19) which should appear exponentiated in order that the algebra closes. This forces us to pass to exponentiated fluxes, that is, to the corresponding Weyl algebra defined by exponentials of π, A. This algebra is now preserved by the automorphisms, as one can see by an appeal to the Baker-Campbell-Hausdorff formula. However, we now see that the state (2.17) is not invariant, because

$$\omega(e^{i\pi(S)}) = 1, \quad \omega(\alpha_\Lambda(e^{i\pi(S)})) = \omega(e^{i[\pi(S)+c\int_S d\Lambda\wedge F]}) = 0 \tag{2.20}$$

for suitable choices of Λ. Therefore, α_Λ is not implemented by conjugation by unitary operators of the GNS Hilbert space and solving the constraint by group averaging methods becomes non trivial if not impossible. Even if we could somehow construct the Gauß invariant Hilbert space, the observables $A(e), \exp(i\pi(S))$ with $\partial e = \partial S = \emptyset$, which leave the physical Hilbert space invariant, are insufficient to approximate (for small e, S) the π dependent terms appearing in the Hamiltonian (2.7), as one can check explicitly.

3 Reduced Phase Space Quantisation

In the previous section, we established that a quantisation in strict analogy to the procedure followed in LQG does not work. While a rigorous kinematical Hilbert space can be constructed, the Dirac operator constraint method of looking for the kernel of the Gauß constraint is problematic. As an alternative, a reduced phase space quantisation suggests itself. This has a chance to work due to the observation (2.14) which demonstrates that $H(x)$ only depends on observables. Indeed, $H(x)$ depends, except for G^{ab} which is a trivial observable since it is constrained to vanish, only on the combination $\pi + B + \alpha j$. Obviously j trivially Poisson commutes with G. Unpacking B from (2.5), we see that $\pi + B$ is a linear combination (with only metric dependent coefficients) of F and

$$P^{abc} := \pi^{abc} + c\epsilon^{abcd_1..d_4e_1..e_3} F_{d_1..d_4} A_{e_1..e_3} \quad \Leftrightarrow \quad *P = *\pi + c\; A \wedge F, \tag{3.1}$$

which suggests that $\{G(\Lambda), P^{abc}(x)\} = 0$ because F is already invariant. This indeed can be verified using (2.13)

$$\delta_\Lambda * P = \delta_\Lambda * \pi + c\delta_\Lambda A \wedge F = 0. \tag{3.2}$$

Our classical observables therefore are coordinatised by the 4-form and 7-form $F = dA$ and $*P = *\pi + cA \wedge F$ respectively. Since F is exact, it is determined entirely by a 3-form modulo an exact form, which in turn is parametrised by a 2-form. This 2-form worth of gauge freedom matches the number of Gauß constraints which can be read as a condition on π. Thus, on the constraint surface, the number of degrees of freedom contained in F and P match.

We compute the observable algebra. Let f be a 3-form and h a 6-form with dual $*h$ (a totally skew 4-times contravariant tensor pseudo density) and smear the observables with these

$$P[f] := \int d^{10}x\; f_{a_1..a_3}\; P^{a_1..a_3} = \int f \wedge *P, \quad F[h] := \int d^{10}x\; (*h)^{a_1..a_4} F_{a_1..a_4} = \int h \wedge F. \tag{3.3}$$

Then, we find after a short computation

$$\{F[h], F[h']\} = 0, \quad \{P[f], F[h]\} = \int h \wedge df, \quad \{P[f], P[f']\} = -3c\, F[f \wedge f']. \tag{3.4}$$

Thus, the observable algebra closes but P is not conjugate to F.

The form of the observable algebra (3.4) reveals the following:
Typically, background independent representations tend to be discontinuous in at least one of the configuration or the momentum variable. For instance, in LQG electric fluxes exist in non exponentiated form, but connections do not. Let us assume that we find such a representation in which $F[h]$ does not exist so that we have to consider instead its exponential (Weyl element). Then (3.4) tells us that in such a representation automatically also $P[f]$ cannot be defined, because if it could, then its commutator would exist, which however is proportional to some F which is a contradiction. Hence, either both F, P exist or only both of their corresponding Weyl elements.

We did not manage to find a representation in which the Weyl elements

$$W[h, f] := \exp\left(i(F[h] + P[f])\right) \tag{3.5}$$

are strongly continuous operators in both f, h. However, we did find one in which they are discontinuous in both h, f. This representation was studied in the context of QED in [20] and was applied to an LQG type of quantisation of the closed bosonic string in [24]. Before we define it, we must first define the Weyl algebra generated by the Weyl elements (3.5). The *-relations are obvious,

$$W[h, f]^* = W[-h, -f]. \tag{3.6}$$

However, the product relations are very interesting and non trivial, because they require the generalisation of the Baker-Campbell-Hausdorff formula [25, 26, 27, 28, 29, 30] to higher commutators [31]. Suppose that X, Y are operators on some Hilbert space such that the triple commutators $[X,[X,Y]]$ and $[Y,[Y,X]]$ commute with both X and Y. This formally applies to our case with $X = F[h]+P[f]$, $Y = F[h']+P[f']$, which obey the canonical commutation relations (we set $\hbar = 1$ for simplicity)

$$[X,Y] := i\{X,Y\} = i\left\{[\int(h' \wedge df - h \wedge df')]\,\mathbb{1} - 3cF[f \wedge f']\right\}. \tag{3.7}$$

From this follows for the triple commutators

$$\begin{aligned}
[X,[X,Y]] &= -3c(i)^2\,\{P[f], F[f\wedge f']\} = 3c\int f \wedge f' \wedge df\,\mathbb{1}, \\
[Y,[Y,X]] &= 3c(i)^2\,\{P[f'], F[f\wedge f']\} = -3c\int f \wedge f' \wedge df'\,\mathbb{1},
\end{aligned} \tag{3.8}$$

which thus are in the centre of the algebra.

The BCH formula for the case of all triple commutators commuting with X, Y reads

$$e^X\, e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]+[Y,[Y,X]])}, \tag{3.9}$$

which can also be proved using elementary methods. From this it is easy to derive the also useful Zassenhaus formula [31]

$$e^{X+Y} = e^X\, e^Y\, e^{-\frac{1}{2}[X,Y]}\, e^{-\frac{1}{6}([X,[X,Y]]+2[Y,[X,Y]])}. \tag{3.10}$$

Putting all these together, we obtain the Weyl relations

$$W[h,f]\,W[h',f'] = W[h+h'+\frac{3c}{2}f\wedge f', f+f']\ \ \exp\left(\frac{i}{4}\int[2(h\wedge df' - h'\wedge df) - cf\wedge f'\wedge d(f-f')]\right). \tag{3.11}$$

Hence also the Weyl relations get twisted as compared to the situation with $c = 0$. Notice that the first term in the phase is antisymmetric under the exchange $(h,f) \leftrightarrow (h',f')$, while the second is symmetric.

In order to obtain a representation of this *-algebra $\mathfrak{A}$ generated by the Weyl elements, it is sufficient to find a positive linear functional. We consider the Narnhofer-Thirring type of functional

$$\omega(W(h,f)) = \begin{cases} 1 & h = f = 0 \\ 0 & \text{else} \end{cases} \tag{3.12}$$

and show that it is positive definite on $\mathfrak{A}$. Let

$$a := \sum_{k=1}^{N} c_k\, W[z_k] \tag{3.13}$$

be a general element in $\mathfrak{A}$, where $N \in \mathbb{N}$, $c_k \in \mathbb{C}$ and the $z_k = (h_k, f_k)$ are arbitrary, where without loss of generality $z_k \neq z_l$ for $k \neq l$. We have

$$\begin{aligned}
\omega(a^*a) &= \sum_{k,l=1}^{N} \bar{c}_k\, c_l\, \omega(W[-z_k]\,W[z_l]) \\
&= \sum_{k,l=1}^{N} \bar{c}_k\, c_l\, \omega(W[z_{kl}])\exp(i\alpha_{kl}), \\
z_{kl} &= (-h_k + h_l - \frac{3}{2}cf_k\wedge f_l,\ \ -f_k + f_l), \\
\alpha_{kl} &= \frac{1}{4}\int[2(-h_k\wedge df_l + h_l\wedge df_k) - cf_k\wedge f_l\wedge d(f_k+f_l)].
\end{aligned} \tag{3.14}$$

For $k=l$, we have $z_{kl}=\alpha_{kl}=0$ because f_k, f_l are 3-forms. For $k\neq l$, we must have either $f_k\neq f_l$ or $h_k\neq h_l$ or both. If $f_k\neq f_l$, then obviously $z_{kl}\neq 0$. If $f_k=f_l$, then necessarily $h_k\neq h_l$ and $z_{kl}=(-h_k+h_l,0)\neq 0$. By definition (3.12) then

$$\omega(a^*a)=\sum_{k=1}^{N}|c_k|^2\geq 0;\;\; \omega(a^*a)=0 \;\;\Leftrightarrow\;\; a=0 \tag{3.15}$$

is positive definite. Thus, the left ideal $\mathfrak{I}=\{a\in\mathfrak{A};\;\omega(a^*a)=0\}=\{0\}$ is trivial and the Hilbert space representation is given by the GNS data [23]:
The cyclic vector is $\Omega=\mathbb{1}$, the Hilbert space $\mathcal{H}$ is the Cauchy completion of $\mathfrak{A}$ in the scalar product $<a,b>:=\omega(a^*b)$ and the representation is simply $\pi(a)b:=ab$ on the common dense domain $\mathcal{D}=\mathfrak{A}$. The representation is evidently strongly discontinuous in both h,f and while cyclic, it is not irreducible. Equivalently, ω is not a pure state [32, 33].

The question left open to answer is whether the algebra and the state ω are still well defined when restricting the smearing functions (h,f) to the form factors of 4-surfaces and 7-surfaces respectively. The bearing of this question is that in the Hamiltonian constraint the functions F and $*P$ appear in such a way, that in a discretisation of it, which results from replacing the integral by Riemann sums in the spirit of [34], these functions are naturally smeared over 4-surfaces and 7-surfaces respectively. They could thus be approximated by Weyl elements.
To answer this question, let S_4, S_7 be general 4 and 7 surfaces respectively. Consider the distributional forms ("form factors")

$$\begin{aligned} h^{S_4}_{a_1..a_6}(x) &:= \int_{S_4}\epsilon_{a_1..a_6b_1..b_4}dy^{b_1}\wedge dy^{b_4}\;\delta(x,y),\\ f^{S_7}_{a_1..a_3}(x) &:= \int_{S_7}\epsilon_{a_1..a_3b_1..b_7}dy^{b_1}\wedge dy^{b_7}\;\delta(x,y). \end{aligned}\tag{3.16}$$

Then

$$F[h^{S_4}]=\int_{S_4}F,\quad P[f^{S_7}]=\int_{S_7}*P. \tag{3.17}$$

Thus, the natural integrals of F,P over surfaces can be reexpressed in terms of distributional 6 forms and 4-forms respectively. It remains to check whether the exterior derivative and product combinations of these distributional forms appearing in the multiple Poisson brackets of (3.17) and in the Weyl relations remain meaningful. Three types of exterior derivative and product expressions appear. The first is, using formally Stokes theorem

$$\int h^{S_4}\wedge df^{S_4}=\int_{S_4}df^{S_7}=\int_{\partial S_4}f^{S_7}=\int_{\partial S_4}dx^{a_1}\wedge..\wedge dx^{a_3}\epsilon_{a_1..a_3b_1..b_7}\int_{S_7}dy^{b_1}\wedge..\wedge dy^{b_7}\;\delta(x,y)=:\sigma(\partial S_4,S_7). \tag{3.18}$$

The integral is supported on $\partial S_4\cap S_7$ and we can decompose this set into components (submanifolds) which are 0,1,2,3-dimensional. The number of these components will be finite if the surfaces are semianalytic. We define the *intersection number* $\sigma(\partial S_4,S_7)$ to be zero for the 1,2,3-dimensional components and by (3.18) for the isolated intersection points, which then takes the values ± 1. This can be justified by the same regularisation as in LQG for the holonomy flux algebra [7].

The second type of integral is given by $F[f^{S_7}\wedge f^{S'_7}]$. The support of the integral will be on $S^{S_7}\cap S^{S'_7}$ and in $D=10$ dimensions this will decompose into components that are at least 4-dimensional. By the same regularisation as in [7], one can remove the higher dimensional components and thus keep only the 4-dimensional ones. In what follows, we thus assume that $S_4:=S_7\cap S'_7$ is a single 4-dimensional component, otherwise the non vanishing contributions are over a sum of those. We have

$$\begin{aligned} F[f^{S_7}\wedge f^{S'_7}] &= \int_{S_7}F\wedge f^{S'_7} \\ &= \int_{S_7}dx^{a_1}\wedge..\wedge dx^{a_4}\wedge dx^{b_1}\wedge..\wedge dx^{b_3}\epsilon_{b_1..b_3c_1..c_7}\int_{S'_7}dy^{c_1}\wedge..\wedge dy^{c_7}\;\delta(x,y)\;F_{a_1..a_4}(x). \end{aligned}\tag{3.19}$$

By assumption, we have embeddings

$$X_{S_7}:\ U\to S_7;\quad Y_{S_7'}:\ V\to S_7';\quad Z_{S_4}:\ W\to S_4, \tag{3.20}$$

with open subsets U,V of $\mathbb{R}^7$ and an open subset W of $\mathbb{R}^4$ respectively, whose coordinates will be denoted by u,v,w respectively. The condition $X_{S_7}(u)=Y_{S_7'}(v)=Z_{S_4}(w)$ is solved by solving u,v for w, which leads to $u=u(w),\ v=v(w)$. Since the integrals are reparametrisation invariant, in the neighbourhood of S_4 on both S_7 and S_7' therefore we may use adapted coordinates so that $w^I=u^I=v^I,\ I=1,..,4$ on S_4 and $u^I,v^I,\ I=5,..,7$ denote the transversal coordinates, which take the value 0 on S_4. In this parametrisation both U,V are of the form $U=W\times U',\ V=W\times V'$ for some 3-dimensional subsets U',V' of $\mathbb{R}^3$. It follows $Z(w)=X(w,0)=Y(w,0)$ in this parametrisation. The δ distribution is then supported on $u^I=v^I,\ I=1,..,4$ and $u^I=v^I=0,\ \ I=5,..,7$ and we have in the neighbourhood of S_4

$$X^a(u)-Y^a(v)=-\sum_{I=1}^{4}Y_I^a(u,0)\left[u^I-v^I\right]+\sum_{I=5}^{7}\left[X_I^a(u,0)u^I-Y_I^a(u,0)v^I\right]. \tag{3.21}$$

We can now solve the δ distribution in (3.19) by performing the integral over $u^5,..,u^7,v^1,..,v^7$ and find with the notation $X_I^a=\partial X_{S_7}^a(u)/\partial u^I$ and $Y_I^a=\partial Y_{S_7'}^a(v)/\partial v^I$ etc.

$$\begin{aligned}
F[f^{S_7}\wedge f^{S_7'}] &= \int_U d^7u\,\epsilon^{I_1..I_7}\left[X_{I_1}^{a_1}..X_{I_4}^{a_4}X_{I_5}^{b_1}..X_{I_7}^{b_3}\right](u)\epsilon_{b_1..b_{10}}\int_V d^7v\,\epsilon^{J_1..J_7}\left[Y_{J_1}^{b_4}..Y_{J_7}^{b_{10}}\right](v)\ \times\\
&\quad\ \delta\left(X(u),Y(v)\right)F_{a_1..a_4}(X(u))\\
&= -\int_W d^4w\,\epsilon^{I_1..I_7}\left[Z_{I_1}^{a_1}..Z_{I_4}^{a_4}\right](w)\,\epsilon^{J_1..J_7}\,F_{a_1..a_4}(Z(w))\,\epsilon_{I_5..I_7J_1..J_7}\ \times\\
&\qquad\qquad \left[\mathrm{sgn}\left(\det\left(\frac{\partial(X(u)-Y(v))}{\partial(u^5,..,u^7,v^1,..,v^7)}\right)_{v^I=u^I=w^I;I=1,..,4;v^I=u^I=0;I=5,..,7}\right)\right]\\
&=: -3!\,7!\tilde\sigma(S_7,S_7')F[h^{S_4}],
\end{aligned} \tag{3.22}$$

where the $10d$ antisymmetric symbol is in terms of the coordinates $u^5,..,u^7,v^1,..,v^7$ and in the last step we noticed that the range of $I_1..I_4$ is restricted to 1..4. Also, we assumed that the sign function under the integral is constant and equal to $\tilde\sigma(S_7,S_7')$ on S_4 (which defines this function), otherwise we must decompose S_4 further. Under this assumption, we conclude the form factor identity

$$f_{S_7}\wedge f_{S_7'}=-3!\ 7!\ \tilde\sigma(S_7,S_7')h^{S_7\cap S_7'}. \tag{3.23}$$

Finally, we consider the integral of the third type, which now combining (3.18) and (3.24) is easily calculated

$$\int f_{S_7}\wedge f_{S_7'}\wedge df_{S_7}=-3!\ 7!\ \tilde\sigma(S_7,S_7')\int h^{S_7\cap S_7'}\wedge df^{S_7}=-3!\ 7!\ \tilde\sigma(S_7,S_7')\sigma(\partial(S_7\cap S_7'),S_7)=0, \tag{3.24}$$

because $\partial(S_7\cap S_7')\subset S_7$ for which σ vanishes by definition.

In order to make this restricted Weyl algebra close, we now have to decide whether the form factors should only be added with integer valued coefficients [17] or with real valued ones [35, 36, 37]. In the latter case we do not need to do anything and the restricted Weyl algebra already closes. In the former case we must replace the form factors f^{S_7} by $\frac{1}{\sqrt{3!\ 7!\ 3c/2}}f^{S_7}$, such that in the simplest situation we have

$$W[S_4,S_7]\ W[S_4',S_7']=W[S_4+S_4'-\tilde\sigma(S_7,S_7')S_7\cap S_7',S_7+S_7']\ \ \exp\left(\frac{i}{2}\left[\sigma(\partial S_4,S_7')-\sigma(\partial S_4',S_7)\right]\right), \tag{3.25}$$

from which the general case can be easily deduced.

We conclude that the restricted Weyl algebra is well defined in either case. Thus, wherever P or F appear in the Hamiltonian constraint, we follow the general regularisation procedure outlined in [34], which employs a combination of spatial diffeomorphism invariance and an infinite refinement limit of a Riemann sum approximation of the Hamiltonian constraint in terms of $P[S_7]$ and $F[S_4]=A[\partial S_4]$, which we approximate for instance by $\sin(P[S_7]),\ \sin(F[S_4])$ similar as in LQG. The details are obvious and are left to the interested reader.

4 Conclusions

Supergravity theories typically need additional bosonic fields next to the graviton, in order to obtain a SUSY multiplet (representation) containing the gravitino. In this paper, we focussed on $11d$ SUGRA for reasons of concreteness (and its relevance for lower dimensional SUGRA theories), which contains the 3-index photon in the bosonic sector. However, our analysis is easily generalised to arbitrary p-form fields. Without the Chern-Simons term in the action (i.e. $c = 0$) the analysis would be straightforward and in complete analogy to the background independent treatment of Maxwell theory in $D + 1 = 4$ dimensions [17]. In particular, the Hamiltonian constraint would be quadratic in the 3-form field and its conjugate momentum, which thus would reduce to a free field theory when switching off gravity. However, with the Chern-Simons term ($c \neq 0$) the Hamiltonian constraint becomes in fact quartic in the connection and thus becomes self-interacting even when switching off gravity, just like in non Abelian Yang Mills theories.

It is therefore the more astonishing that we can quantise the resulting *-algebra of observables (with respect to the 3-index-Gauß constraint) rigorously, even though the theory is self-interacting. In fact, in terms of the observables, the Hamiltonian constraint is a quadratic polynomial, however, the price to pay is that the observable algebra is non standard. Yet, the resulting Weyl algebra can be computed in closed form and we found at least one non trivial and background independent representation thereof, which nicely fits with the background independent quantisation of the gravitational degrees of freedom in the contribution to the Hamiltonian constraint depending on the 3-index-photon.

There are many open questions arising from the present study. One of them concerns the reducibility of the GNS representation found, which involves a mixed state. It would be nice to have control over the superselection sectors of the theory and, in particular, to analyse whether the cyclic GNS vector is not already cyclic for the Abelian subalgebra generated by the $W[h, 0]$. Next, it is worthwhile to study the question whether this algebra admits regular representations for both P and F, because then the GNS Hilbert space would admit a measure theoretic interpretation as an L_2 space. Finally, it is certainly necessary to work out the cobordism theory of relevance when restricting the Weyl algebra to distributional 4-form and 7-form factors as smearing functions which is only sketched in this paper. We plan to revisit these questions in future publications.

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References

[1] N. Bodendorfer, T. Thiemann, and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions I. Hamiltonian Analysis," `arXiv preprint`.

[2] N. Bodendorfer, T. Thiemann, and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions II. Lagrangian Analysis," `arXiv preprint`.

[3] N. Bodendorfer, T. Thiemann, and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions III. Quantum Theory," `arXiv preprint`.

[4] N. Bodendorfer, T. Thiemann, and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions IV. Matter Coupling," `arXiv preprint`.

[5] N. Bodendorfer, T. Thiemann, and A. Thurn, "On the Implementation of the Canonical Quantum Simplicity Constraint," `arXiv preprint`.

[6] C. Rovelli, *Quantum Gravity.* Cambridge University Press, Cambridge, 2004.

[7] T. Thiemann, *Modern Canonical Quantum General Relativity.* Cambridge University Press, Cambridge, 2007.

[8] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory, Vol. 1: Introduction.* Cambridge University Press, Cambridge, 1988.

[9] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory, Vol. 2: Loop Amplitudes, Anomalies and Phenomenology.* Cambridge University Press, Cambridge, 1988.

[10] N. Bodendorfer, T. Thiemann, and A. Thurn, "Towards Loop Quantum Supergravity (LQSG) I. Rarita-Schwinger Sector," `arXiv preprint`.

[11] J. C. Baez and J. Huerta, "An Invitation to Higher Gauge Theory," `arXiv:1003.4485 [hep-th]`.

[12] J. C. Baez, A. Baratin, L. Freidel, and D. K. Wise, "Infinite-Dimensional Representations of 2-Groups," `arXiv:0812.4969 [math.QA]`.

[13] L. Crane and M. D. Sheppeard, "2-categorical Poincare Representations and State Sum Applications," `arXiv:math/0306440`.

[14] L. Crane and D. N. Yetter, "Measurable Categories and 2-Groups," *Applied Categorical Structures* **13** (2005) 501–516, `arXiv:math/0305176 [math.QA]`.

[15] D. N. Yetter, "Measurable Categories," *Applied Categorical Structures* **13** (2005) 469–500, `arXiv:math/0309185 [math.CT]`.

[16] J. C. Baez and U. Schreiber, "Higher Gauge Theory," in *Categories in Algebra, Geometry and Mathematical Physics* (D. A., ed.), (Providence, Rhode Island), pp. 7–30, AMS, 2007. `arXiv:math/0511710 [math.DG]`.

[17] T. Thiemann, "Quantum spin dynamics (QSD) V: Quantum Gravity as the Natural Regulator of Matter Quantum Field Theories," *Classical and Quantum Gravity* **15** (1998), no. 5 1281–1314, `arXiv:gr-qc/9705019`.

[18] A. Ashtekar and C. J. Isham, "Representations of the holonomy algebras of gravity and non-Abelian gauge theories," *Classical and Quantum Gravity* **9** (1992) 1433–1468, `arXiv:hep-th/9202053`.

[19] A. Ashtekar and J. Lewandowski, "Representation Theory of Analytic Holonomy C* Algebras," in *Knots and Quantum Gravity* (J. Baez, ed.), (Oxford), Oxford University Press, 1994. `arXiv:gr-qc/9311010`.

[20] N. Narnhofer and W. Thirring, "Covariant QED Without Indefinite Metric," *Reviews in Mathematical Physics* **4** (1992) 197–211.

[21] A. Diaz, "Hamiltonian formulation of eleven-dimensional supergravity," *Physical Review D* **33** (1986) 2801–2808.

[22] P. van Nieuwenhuizen, "Supergravity," *Physics Reports* **68** (1981) 189–398.

[23] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*. Springer, 2 ed., 1996.

[24] T. Thiemann, "The LQG string - loop quantum gravity quantization of string theory: I. Flat target space," *Classical and Quantum Gravity* **23** (2006) 1923–1970, `arXiv:hep-th/0401172`.

[25] J. Campbell, "On a Law of Combination of Operators bearing on the Theory of Continuous Transformation Groups," *Proc. London Math. Soc.* **s1-28** (1896), no. 1 381–390.

[26] J. Campbell, "On a Law of Combination of Operators (Second Paper)," *Proc. London Math. Soc.* **s1-29** (1897), no. 1 14–32.

[27] H. Baker, "Further Applications of Metrix Notation to Integration Problems," *Proc. London Math. Soc.* **s1-34** (1901), no. 1 347–360.

[28] H. Baker, "On the Integration of Linear Differential Equations," *Proc. London Math. Soc.* **s1-35** (1902), no. 1 333–378.

[29] H. Baker, "Alternants and Continuous Groups," *Proc. London Math. Soc.* **s2-3** (1905), no. 1 24–47.

[30] F. Hausdorff, "Die symbolische Exponentialformel in der Gruppentheorie," *Ber. Verh. Saechs. Akad. Wiss. Leipzig* **58** (1906) 19–48.

[31] W. Magnus, "On the exponential solution of differential equations for a linear operator," *Comm. Pure Appl. Math.* **7** (1954) 649–673.

[32] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1: C*- and W*-Algebras. Symmetry Groups. Decomposition of States*. Springer, 1987.

[33] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2: Equilibrium States. Models in Quantum Statistical Mechanics*. Springer, 2003.

[34] T. Thiemann, "Quantum spin dynamics (QSD)," *Classical and Quantum Gravity* **15** (1996) 839–873, `arXiv:gr-qc/9606089`.

[35] A. Ashtekar, J. Lewandowski, and H. Sahlmann, "Polymer and Fock representations for a scalar field," *Classical and Quantum Gravity* **20** (2003) L11–L21, `arXiv:gr-qc/0211012`.

[36] W. Kamiński, J. Lewandowski, and M. Bobieński, "Background independent quantizations - the scalar field: I," *Classical and Quantum Gravity* **23** (2006) 2761–2770, `arXiv:gr-qc/0508091`.

[37] W. Kamiński, J. Lewandowski, and A. Okołów, "Background independent quantizations - the scalar field: II," *Classical and Quantum Gravity* **23** (2006) 5547–5585, `arXiv:gr-qc/0604112`.