

New Variables for Classical and Quantum Gravity in all Dimensions I. Hamiltonian Analysis

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May 19, 2011

Abstract

Loop Quantum Gravity heavily relies on a connection formulation of General Relativity such that 1. the connection Poisson commutes with itself and 2. the corresponding gauge group is compact. This can be achieved starting from the Palatini or Holst action when imposing the time gauge. Unfortunately, this method is restricted to $D + 1 = 4$ spacetime dimensions. However, interesting String theories and Supergravity theories require higher dimensions and it would therefore be desirable to have higher dimensional Supergravity loop quantisations at one's disposal in order to compare these approaches.

In this series of papers we take first steps towards this goal. The present first paper develops a classical canonical platform for a higher dimensional connection formulation of the purely gravitational sector. The new ingredient is a different extension of the ADM phase space than the one used in LQG which does not require the time gauge and which generalises to any dimension $D > 1$. The result is a Yang – Mills theory phase space subject to Gauß, spatial diffeomorphism and Hamiltonian constraint as well as one additional constraint, called the simplicity constraint. The structure group can be chosen to be $SO(1, D)$ or $SO(D+1)$ and the latter choice is preferred for purposes of quantisation.

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1 Introduction

The quantisation of General Relativity remains one of the most important open problems of contemporary physics. Early attempts to quantise the Hamiltonian formulation given by Arnowitt, Deser and Misner [1] have failed due to non-renormalisability [2, 3] among other problems. Supergravity in various dimensions entered the picture as a way to resolve these problems, however, not all could be addressed [4, 5, 6]. Meanwhile, Superstring theory [7, 8] and M-Theory [9, 10] have been proposed as theories of quantum gravity. They constrain the spacetime dimension to $D + 1 = 10$ (Superstring theory) or $D + 1 = 11$ (M-Theory) and symmetry arguments suggest that the respective Supergravities are their low energy limits [7, 8]. It is therefore interesting to (loop)-quantise these Supergravities as a new approach to quantising the low-energy limit of Superstring theory or M-theory.

However, the programme of loop quantisation (see e.g. [12] and references therein) requires the theory to be formulated in terms of a gauge theory. The reason for that is that only for theories based on connections and conjugate momenta background independent Hilbert space representations have been found so far, which also support the constraints of the theory as densely defined and closable operators. Of course, a connection formulation is also forced on us if we want to treat fermionic matter as well. A connection formulation for gravity in $D + 1 > 4$ that can be satisfactorily quantised, even in the vacuum case, has not been given so far. For the case $D + 1 = 4$, it was only in 1986 that Ashtekar discovered his new variables for General Relativity [13]. The most important property of these variables is that the connection A used has a canonically conjugate momentum E such that (A, E) have standard canonical brackets, in particular the connection Poisson commutes with itself. This is not trivial. Indeed, the naive connection that one would expect from the first order Palatini formulation does not have this crucial property, because the canonical formulation of Palatini gravity suffers from second class constraints and the Palatini connection then has non trivial corresponding Dirac brackets. This prohibited so far to find Hilbert space representations, in particular those of LQG type in which the connection is represented as a multiplication operator, for these Palatini connection formulations. The Ashtekar connection does not suffer from this problem because it is the

self-dual part of the Palatini connection (or spin connection in the absence of torsion terms). Unfortunately, for the only physically interesting case of Lorentzian signature this Ashtekar connection takes values in the non compact $SL(2, C)$ rather than a compact group and again it is very difficult to find Hilbert space representations of gauge theories with non compact structure groups.

As observed by Barbero [14], a possible strategy to deal with this non compactness problem is to use the time gauge and to gauge fix the boost part of $SO(1, 3)$. The resulting connection, which can be seen as the self dual part of the spin connection for Euclidean signature, is then an $SU(2)$ connection. The price to pay is that the Hamiltonian constraint for Lorentzian signature in terms of these variables is more complicated than in terms of the complex valued ones. However, this does not pose any problems in its quantisation [15]. Using these variables (which also allow a one parameter freedom related to the Immirzi parameter [16]) a rigorous quantisation of General Relativity with a unique Hilbert space representation could be derived [17, 18, 19, 20]. A different way to arrive at the same formulation is to start from the geometrodynamics phase space coordinatised by the ADM variables (three metric and extrinsic curvature) and to expand it by introducing (densitised) triads E and conjugate momenta K (basically the extrinsic curvature contracted with the triad). The connection is then the triad spin connection Γ plus this conjugate momentum, that is, $A = \Gamma + \gamma K$ where γ is the real valued Immirzi parameter. The first miracle that happens in 3 spatial dimensions is that this is at all possible: While K transforms in the defining representation of $SO(3)$, Γ transforms in the adjoint representation of $SO(3)$. But for the case of $SO(3)$, these are isomorphic and enable to define the object A . The second miracle that happens in 3 spatial dimensions is that this connection is Poisson self commuting which is entirely non trivial. Notice that in three spatial dimensions, the expansion of the phase space alters the number of configuration degrees of freedom from six per spatial point (described by the three metric tensor) to nine (described by the co-triad). To get back to the original ADM phase space, one therefore has to add three constraints and these turn out to comprise precisely an $SU(2)$ Gauß constraints just as in Yang Mills theory.

It is clear that this strategy can work only in $D = 3$ spatial dimensions: A metric in D spatial dimensions has $D(D + 1)/2$ configuration degrees of freedom per spatial point while a D -bein has D^2 . We therefore need $D^2 - D(D + 1)/2 = D(D - 1)/2$ constraints which is precisely the dimensionality of $SO(D)$. However, an $SO(D)$ connection has $D^2(D - 1)/2$ degrees of freedom. Requiring that connection and triad have equal amount of degrees of freedom leads to the unique solution $D = 3$. Thus in higher dimensions we need a generalisation of the procedure that works in $D = 3$. Attempts to construct a higher dimensional connection formulation have been undertaken, but few results are available [21, 22, 23]. Han et al. [24] have shown that the higher dimensional Palatini action leads to geometrodynamics when the time gauge is imposed before the canonical analysis.

In this paper, we will derive a connection formulation for higher dimensional General Relativity by using a different extension of the ADM phase space than the one employed in [13, 25] and which generalises to arbitrary spacetime dimension $D + 1$ for $D > 1$. It is based in part on Peldan's seminal work [26] on the possibility of using higher dimensional gauge groups for gravity as well as on his concept of a hybrid spin connection which naturally appears in the connection formulation of $2 + 1$ gravity [27]. More precisely, the idea is the following: If one starts from the Palatini formulation in $D + 1$ spacetime dimensions, then the natural gauge group to consider is $SO(1, D)$ or $SO(D + 1)$ respectively for Lorentzian or Euclidean gravity respectively. Both groups have dimension $D(D + 1)/2$. This motivates to look for a connection formulation of the Hamiltonian framework with a connection A_{aIJ} , $a = 1, \dots, D$; $I, J = 0, \dots, D$. Such a connection has $D^2(D + 1)/2$ degrees of freedom. The corresponding Gauß constraint removes $D(D + 1)/2$ degrees of freedom, leaving us with $(D - 1)D(D + 1)/2$ degrees of freedom.

However, a metric in D spatial dimensions has only $D(D+1)/2$ degrees of freedom, which means that we need $D^2(D-1)/2 - D$ additional constraints which together with the ADM constraints and the Gauß constraint form a first class system. To discover this constraint, we need an object that transforms in the defining representation of the gauge group. It is given by the “square root” of the spatial metric $q_{ab} = \eta_{IJ} e_a^I e_b^J$ where η has Lorentzian or Euclidean signature respectively. Since the D internal vectors e_a^I are linearly independent, we can complete them to a uniquely defined $(D+1)$ -bein by the unit vector e_0^I where $\eta_{IJ} e_a^I e_0^J = 0$. Now the momentum π^{aIJ} conjugate to A_{aIJ} is supposed to be entirely determined by e_a^I , that is, $\pi^{aIJ} \propto \sqrt{\det(q)} q^{ab} e_0^I e_b^J$. In other words, π is “simple” and we call these constraints therefore simplicity constraints. Since e_a^I has $D(D+1)$ degrees of freedom while π^{aIJ} has $D^2(D+1)/2$ these present precisely the required $D^2(D-1)/2 - D$ constraints. Furthermore, from e_a^I one can construct the hybrid spin connection Γ_{aIJ} which kills e_a^I and the idea, as for Ashtekar’s variables, is that $A - \Gamma$ is related to the extrinsic curvature. In order to show that the symplectic reduction of this extension of the ADM phase is given by the ADM phase space, similar to what happens in case of Ashtekar’s variables, we need that Γ is integrable at least modulo the simplicity constraints which we show to be the case.

It should be stressed that even in $D+1 = 4$ this extension of the ADM phase space is different from the one employed in LQG: In LQG the Ashtekar-Barbero connection is given by $A_{ajk}^{\text{LQG}} - \Gamma_{ajk} \propto \epsilon_{jkl} K_a^l$, $i, j, k = 1, \dots, D$, while in our case in the time gauge $e_0^I = \delta_0^I$ we have $A_{ajk}^{\text{NEW}} - \Gamma_{ajk}$ is pure gauge. Here Γ_{ajk} is the spin connection of the corresponding triad. Thus, in the new formulation the information about the extrinsic curvature sits in the A_{a0j} component which is absent in the LQG formulation. We also emphasise that it is possible to have gauge group $\text{SO}(D+1)$ even for the Lorentzian ADM phase space. While a Lagrangian formulation is only available when spacetime and internal signature match as we will see in a companion paper [28], this opens the possibility to quantise gravity in $D+1$ spacetime dimensions using LQG methods albeit with structure group $\text{SO}(D+1)$ and additional (simplicity) constraints [29, 30].

This paper is organised as follows:

In section 2, we will define the required kinematical structure of a $(D+1)$ -dimensional connection formulation of General Relativity. We will study in detail the properties of the simplicity constraint and the hybrid spin connection.

In section 3, we will postulate an extension of the ADM phase space in terms of a connection and its conjugate momentum subject to the corresponding Gauß constraint and the simplicity constraint discussed before. We will then prove that the symplectic reduction of this extension with respect to both constraints recovers the ADM phase space. There is a one parameter freedom in this extension, similar to but different from the Immirzi parameter of standard LQG [16].

In section 4, we express the spatial diffeomorphism constraint and the Hamiltonian constraint in terms of the new variables and prove that the full set of four types of constraints, namely Gauß, simplicity, spatial diffeomorphism and Hamiltonian constraints, is of first class. This can be done for either choice of $\text{SO}(1, D)$ or $\text{SO}(D+1)$ independently of the spacetime signature. Similar to the situation with standard LQG, the Hamiltonian simplifies when spacetime signature and internal signature match if one chooses unit Immirzi like parameter. There is an additional correction term present which accounts for the removal of the pure gauge degrees of freedom affected by the gauge transformations generated by the simplicity constraint.

In section 5, we conclude and discuss future research directions partly already addressed in our companion papers [28, 29, 30, 31, 32, 33].

The further organisation of this series is as follows:

In paper [28], we supplement the present paper by a Lagrangian, namely Palatini, formulation in the case that internal and spacetime signature match. As is well known, the canonical treatment of Palatini gravity, while leading to a connection formulation, is plagued by second class constraints and a non trivial Dirac bracket prohibiting a connection representation in the quantum theory [34]. This is in apparent contradiction to the first class structure found in the present paper. The link between the two approaches is through the machinery of gauge unfixing [35, 36, 37, 38], which transforms a second class system into an equivalent first class system subject to a modification of the Hamiltonian which in our case is precisely the additional correction term in the Hamiltonian constraint found in this paper.

In paper [29] we quantise the constraints found in this paper using the standard machinery developed for the $D = 3$ case. In paper [31], we consider coupling to fermionic matter and its quantisation where we have to solve the problem of how to switch from Lorentzian to Euclidean signature Clifford algebras. In paper [30], we quantise the simplicity constraint and show that in $D + 1 = 4$ the resulting Hilbert space and the representation of Gauß and simplicity invariant observables coincides with the standard LQG representation. In paper [32], we consider the classical canonical formulation of higher dimensional Supergravity theories, in particular the Rarita-Schwinger fields in terms of new canonical variables and formulate the corresponding quantum theory. Finally, we treat the p-form sector of Supergravity theories in paper [33].

2 Kinematical Structure of $(D + 1)$ -dimensional Canonical Gravity

This section is subdivided into three parts. In the first part we show that simple dimensional counting and natural considerations lead to a unique candidate connection formulation that works in any spacetime dimension $D + 1$ and has underlying structure group $SO(D + 1)$ or $SO(1, D)$ respectively. We also identify the simplicity constraints additional to the Gauß constraint that such a formulation requires and show that while there is no D -bein and no spin connection in such a formulation, there is a generalised D -bein and a hybrid connection. The latter is required in order to express the ADM variables in terms of the connection and its conjugate momentum. In the second part we formulate an equivalent expression for the simplicity constraint and discuss its properties and some subtleties. Finally, in the third part we prove a key property of the hybrid connection, namely its integrability modulo simplicity constraints. This will be key to proving in the next section that the symplectic reduction of the extended phase space by Gauß and simplicity constraints recovers the ADM phase space.

2.1 Preliminaries

As is well known (see e.g. [12] and references therein), the ADM Hamiltonian formulation of vacuum $D + 1$ General Relativity is based on a phase space coordinatised by a canonical pair (q_{ab}, P^{ab}) with non trivial Poisson brackets (we set the gravitational constant to unity for convenience)

$$\{q_{ab}(x), P^{cd}(y)\} = \delta_{(a}^c \delta_{b)}^d \delta^{(D)}(x - y), \quad (2.1)$$

where $a, b, c, \dots \in \{1, \dots, D\}$ and x, y, \dots are coordinates on a D -dimensional manifold σ . The images of σ under one parameter families of embeddings of σ into a $(D + 1)$ -dimensional manifold M constitute a foliation of M . Here q_{ab} is a metric on σ of Euclidean signature. The phase space

defined by (2.1) is subject to spatial diffeomorphism constraints

$$\mathcal{H}_a = -2q_{ac} D_b P^{bc} \quad (2.2)$$

and Hamiltonian constraint

$$\mathcal{H} = -\frac{s}{\sqrt{\det(q)}} [q_{ac}q_{bd} - \frac{1}{D-1}q_{ab}q_{cd}] P^{ab}P^{cd} - \sqrt{\det(q)}R^{(D)}, \quad (2.3)$$

where $R^{(D)}$ is the Ricci scalar of q_{ab} and D_a denotes the torsion free covariant derivative annihilating q_{ab} . Here s is the signature of the spacetime geometry. Expression (2.3) is problematic for $D = 1$ and in what follows we restrict to $D > 1$.

Similar to the formulation of standard LQG in $D + 1 = 4$, we would like to arrive at a connection formulation of this system which then can be quantised using standard LQG techniques. This requires the corresponding structure group to be compact. Let us recall and sketch how this is done for $D = 3$, see [12] for all the details:

Following Peldan [26], the idea is to extend the ADM phase space by additional degrees of freedom and then to impose additional first class constraints in such a way that the symplectic reduction of the extended system with respect to these constraints coincides with the original ADM phase space. In practical terms, this means that one considers a connection A_a^α , i.e. a Lie algebra valued one form with a Lie algebra of dimension N and a conjugate momentum π_α^a which is a Lie algebra valued vector density. Here $\alpha, \beta, \dots = 1, \dots, N$. Such a Yang-Mills phase space is subject to a Gauß constraint

$$G_\alpha = \mathcal{D}_a \pi_\alpha^a = \partial_a \pi_\alpha^a + f_{\alpha\beta}{}^\gamma A_a^\beta \pi_\gamma^a, \quad (2.4)$$

where $f_{\alpha\beta}{}^\gamma$ denote the structure constants of the corresponding gauge group. The requirement is then that there is a reduction $(A, \pi) \mapsto q_{ab} := q_{ab}[A, \pi]$, $P^{ab} := P^{ab}[A, \pi]$ such that the Poisson brackets of the ADM phase space are reproduced modulo the Gauß constraint and possible additional first class constraints that maybe necessary in order that the correct dimensionality of the reduced phase space is achieved.

The question is of course which group should be chosen depending on D and how to express q_{ab}, P^{ab} in terms of A_a^α, π_α^a . Furthermore, one may ask whether the Gauß constraint is sufficient in order to reduce to the correct number of degrees of freedom or whether there should be additional constraints. Consider first the case that the Gauß constraint is sufficient. Then the extended phase space has DN configuration degrees of freedom of which the Gauß constraint removes N . This has to agree with the dimension of the ADM configuration degrees of freedom which in D spatial dimensions is $D(D+1)/2$. It follows $N(D-1) = D(D+1)/2$. Next we need to relate $(A_a^\alpha, \pi_\alpha^a)$ to (q_{ab}, P^{ab}) . There may be many possibilities for doing so but here we will follow a strategy that is similar to the strategy of standard LQG. We consider some representation ρ of the corresponding Lie group G of dimension $M \geq D$ and introduce generalised D -beins e_a^I , $I, J, K, \dots = 1, \dots, M$ taking values in this representation with $q_{ab} = e_a^I \eta_{IJ} e_b^J$. The requirement $M \geq D$ is needed in order that q_{ab} can be chosen to be non degenerate and we furthermore require that it is positive definite. Here η is a G -invariant tensor, i.e. $\rho(g)_K^I \eta_{IJ} \rho(g)_L^J = \eta_{KL}$. The existence of such a tensor already severely restricts the possible choices of G and typically G is simply defined in this way whence ρ will typically be the defining representation of G . We extend the covariant derivative D_a to ρ valued objects by asking that D_a annihilates the co- D -bein

$$D_a e_b^I = \partial_a e_b^I - \Gamma_{ab}^c e_c^I + \Gamma_a^\alpha [X_\alpha^\rho]^I{}_J e_b^J = 0, \quad (2.5)$$

with the Levi-Civita connection Γ_{ab}^c . This equation defines the hybrid (or generalised) spin connection Γ_a^α . Here the X_α^ρ denote the generators of the Lie algebra of G in the representation ρ .

The idea is now that $\tilde{K}_a^b := [A_a^\alpha - \Gamma_a^\alpha] \pi_\alpha^b$ is the expression for the ADM extrinsic curvature $\sqrt{\det(q)} K_a^b$, $P_a^b = -s \sqrt{\det(q)} [K_a^b - \delta_a^b K_c^c]$, in terms of the new variables. However, there are several caveats. First of all, it is not clear that (2.5) has a non-trivial solution: These are $D^2 M$ equations for DN coefficients Γ_a^α and thus the system (2.5) could be overdetermined. Secondly, even if a solution exists, Γ_a^α will be a function of e_a^I while we need to express it in terms of the momentum π_α^a conjugate to A_a^α . If there is no other constraint than the Gauß constraint, then π_α^a itself must be already determined in terms of e_a^I which implies that $M = N$: The representation ρ has the same dimension as the adjoint representation of the Lie group. If one scans the classical Lie groups, then the only case where the defining representation and the adjoint representation have the same dimension (and are in fact isomorphic) is $SO(3)$ or $SO(1, 2)$ respectively, whence $N = 3$. In this case, the equation $N(D - 1) = D(D + 1)/2$ has the solutions $D = 2$ and $D = 3$ which can be shown to be the only solutions to this equation on the positive integers.

In order to go beyond $D = 3$, we therefore need more constraints. We consider now the case of the choice $G = SO(M + 1)$ or $G = SO(1, M)$ which is motivated by the fact that these Lie groups underly the Palatini formulation of General Relativity in $M + 1$ spacetime dimensions. Following Peldan's programme, other choices may be leading, conceivably, to canonical formulations of GUT theories. We will leave the investigation of such possibilities for future research. For this choice we obtain $N = M(M + 1)/2$ and thus (2.5) presents $D^2(M + 1)$ equations for $DM(M + 1)/2$ coefficients. Explicitly

$$\partial_a e_b^I - \Gamma_{ab}^c e_c^I + \Gamma_a^{IJ} e_{bJ} = 0, \quad (2.6)$$

where all internal indices are moved with η . Since $\Gamma_{a(IJ)} = 0$ we obtain the consistency condition

$$e_{(cI} \partial_a e_{b)}^I - \Gamma_{(c|a|b)} = 0, \quad (2.7)$$

where $q_{ab} = e_a^I e_{bI}$ was used. It is not difficult to see that (2.7) is in fact identically satisfied. Therefore the $D^2(M + 1)$ equations (2.6) are not all independent, there are $D^2(D + 1)/2$ identities (2.7) among them, reducing the number of independent equations to $D^2[M + 1 - \frac{1}{2}(D + 1)]$ for $DM(M + 1)/2$ coefficients Γ_{aIJ} . Equating the number of independent equations to the number of equations yields a quadratic equation for M with the two possible roots $M = D$ and $M = D - 1$. In the second case e_a^I is an ordinary D -bein and Γ_{aIJ} its ordinary spin connection. In the former case we obtain the hybrid spin connection mentioned before.

Let us discuss the cases $SO(D)$ and $SO(D + 1)$ separately (the discussion is analogous for $SO(1, D - 1)$ and $SO(1, D)$ except that $SO(1, D - 1)$ does not allow for a positive definite D metric and therefore must be excluded anyway). In the case of $SO(D)$ we have $D^2(D - 1)/2$ configuration degrees of freedom and $D(D - 1)/2$ Gauß constraints. In order to match the number of ADM degrees of freedom, we therefore need $S = D^2(D - 1)/2 - D(D - 1)/2 - D(D + 1)/2 = D^2(D - 3)/2$ additional constraints. These must be imposed on the momentum π^{aIJ} conjugate to A_{aIJ} and require that π^{aIJ} is already determined by e_a^I . Now e_a^I has D^2 degrees of freedom while π^{aIJ} has $D^2(D - 1)/2$ so that exactly S degrees of freedom are superfluous. However, there is no way to build an object π^{aIJ} with $\pi^{a(IJ)} = 0$ from e_a^I : In order to match the density weight we can consider $E^{aI} = \sqrt{\det(q)} q^{ab} e_b^I$, but we cannot algebraically build another object v^I from e_a^I without tensor index in order to define $\pi^{aIJ} = v^{[I} E^{a]J}$. The only solution is that there are no superfluous degrees of freedom, which leads back to $D = 3$. Now consider $SO(D + 1)$. In this case we have $D^2(D + 1)/2$ configuration degrees of freedom and $D(D + 1)/2$ Gauß constraints requiring $S = D^2(D + 1)/2 - D(D + 1)/2 - D(D + 1)/2 = D^2(D - 1)/2 - D$ additional constraints. The number of superfluous degrees of freedom in π^{aIJ} as compared to e_a^I is now also precisely $S = D^2(D + 1)/2 - D(D + 1)$. In contrast to the previous case, however, now it is possible to construct an object without tensor indices: If we assume that the D internal

vectors e_a^I , $a = 1, \dots, D$ are linearly independent then we construct the common normal

$$n_I := \frac{1}{D!} \frac{1}{\sqrt{\det(q)}} \epsilon^{a_1 \dots a_D} \epsilon_{IJ_1 \dots J_D} e_{a_1}^{J_1} \dots e_{a_D}^{J_D}, \quad (2.8)$$

which satisfies $e_a^I n_I = 0$, $n_I n^I = \zeta$ where $\zeta = 1$ for $\text{SO}(D+1)$ and $\zeta = -1$ for $\text{SO}(1, D)$. Notice that n_I is uniquely (up to sign) determined by e_a^I . We may now require that

$$\pi^{aIJ} = 2\sqrt{\det(q)} q^{ab} n^{[I} e_b^{J]} =: 2n^{[I} E^{a|J]}. \quad (2.9)$$

These are the searched for constraints on π^{aIJ} and constitutes our candidate connection formulation for General Relativity in arbitrary spacetime dimensions $D+1 \geq 3$. Since they require π to come from a generalised D -bein, we call them *simplicity constraints*. Notice that $D^2(D-1)/2 - D = 0$ for $D = 2$. Indeed, $2+1$ gravity is naturally defined as an $\text{SO}(1, 2)$ or $\text{SO}(3)$ gauge theory.

2.2 Properties of the Simplicity Constraints

The form of the constraint (2.9) is not yet satisfactory because the constraint should be formulated purely in terms of π^{aIJ} . The same requirement applies to the hybrid connection to which we will turn in the next subsection.

Given π^{aIJ} and any unit vector n_I we may define $E^{aI}[\pi, n] := -\zeta \pi^{aIJ} n_J$. This object then automatically satisfies $E^{aI} n_I = 0$. Furthermore we may define the transversal projector

$$\bar{\eta}_J^I[n] := \delta_J^I - \zeta n^I n_J \Rightarrow \bar{\eta}_J^I n^J = 0 \quad (2.10)$$

and define

$$\bar{\pi}^{aIJ} := \bar{\eta}_K^I[n] \bar{\eta}_L^J[n] \pi^{aKL}. \quad (2.11)$$

In what follows, all tensors with purely transversal components will carry an overbar. We obtain the decomposition

$$\pi^{aIJ} = \bar{\pi}^{aIJ} + 2n^{[I} E^{a|J]}. \quad (2.12)$$

It appears that the simplicity constraint now is equivalent to $\bar{\pi}^{aIJ} = 0$. However, there are two subtleties: First, at this point n^I is an extra structure next to π^{aIJ} which is required to define (2.11). Therefore the decomposition (2.12) is not intrinsic and n^I appears as an extra degree of freedom. It is therefore necessary to give an intrinsic definition of n^I . Next, suppose that we have achieved to do so, then $\bar{\pi}^{aIJ}$ constitute $D^2(D-1)/2$ degrees of freedom rather than the required $D^2(D-1)/2 - D$ while due to $E_I^a n^I = 0$ the E_I^a constitute only D^2 degrees of freedom rather than $D(D+1)$.

To remove these subtleties, it is cleaner to adopt the following point of view: We consider $D+1$ vector densities E_I^a to begin with such that the corresponding $D(D+1)$ -matrix has maximal rank. From these we can construct the densitised inverse metric

$$Q^{ab} := E_I^a E_J^b \eta^{IJ}, \quad (2.13)$$

which we require to have Euclidean signature as well as their common normal

$$n_I[E] := \frac{1}{D! \sqrt{\det(Q)}} \epsilon_{a_1 \dots a_D} \epsilon_{IJ_1 \dots J_D} E^{a_1 J_1} \dots E^{a_D J_D}, \quad (2.14)$$

which is now considered as a function of E . Notice that $n_I n^I = \zeta$. Therefore also $\bar{\eta}_J^I = \bar{\eta}_J^I[E]$ is a function of E . We can again apply the decomposition (2.12) and now have cleanly deposited

the searched for degrees of freedom into E_I^a . However, while n^I is now intrinsically defined via E_I^a , the constraints $\bar{\pi}^{aIJ} = 0$ are still D too many. We should remove D additional degrees of freedom from $\bar{\pi}^{aIJ}$. To do so we impose a tracefree condition. Consider the object

$$E_a^I := Q_{ab}E^{bI}, \quad Q_{ac}Q^{cb} := \delta_a^b. \quad (2.15)$$

It follows easily from the definitions that

$$E_a^I E_I^b = \delta_a^b, \quad E_a^I E_J^a = \bar{\eta}_J^I. \quad (2.16)$$

Consider the tracefree, transverse projector

$$P_{bKL}^{aIJ}[E] := \delta_b^a \bar{\eta}_{[K}^I \bar{\eta}_{L]}^J - \frac{2}{D-1} E^{a[I} E_{b[K} \bar{\eta}_{L]}^J]. \quad (2.17)$$

Then for any tensor π^{aIJ} we have with $\bar{\pi}_T^{aIJ} = P_{bKL}^{aIJ} \pi^{aIJ}$ that

$$\bar{\pi}^J := E_{aI} \bar{\pi}_T^{aIJ} = 0 \quad (2.18)$$

and $\bar{\pi}_T^{aIJ} n_I = 0$. Notice that $\bar{\pi}_T^{aIJ}$ has only $D^2(D-1)/2 - D$ degrees of freedom independent of E_I^a .

We therefore consider in what follows tensors π^{aIJ} of the following form

$$\pi^{aIJ}[E, \bar{S}_T] := \bar{S}_T^{aIJ} + 2n^{[I}[E] E^{a]J}], \quad (2.19)$$

where \bar{S}_T and E are considered as independent parameters for π . Notice that \bar{S}_T can be constructed as $P \cdot S$ from an arbitrary tensor S^{aIJ} . Such tensors can be intrinsically described as follows:

Given π , there exists a normal $n_I[\pi]$ such that the following holds: Define $E_I^a[\pi, n] = -\zeta \pi^{aIJ} n_J$ and $\bar{\pi}^{aIJ}[\pi, n]$ as above. Then automatically

$$\bar{\pi}^J[\pi, n] := \bar{\pi}^{aIJ}[\pi, n] Q_{ab}[\pi, n] E_I^b[\pi, n] = 0. \quad (2.20)$$

This is a set of D independent (since automatically $\bar{\pi}^I n_I = 0$ whatever n^I is), non-linear equations for the D independent (due to the normalisation $n_I n^I = \zeta$) components of n^I . In the appendix, we study this non trivial system of equations further and show that it can possibly be solved by fixed point methods. At present we do not know whether at least tensors π^{aIJ} subject to the condition that $\zeta \pi^{aIJ} \pi_{IJ}^b / 2$ is positive definite always allow for such a solution n^I , however, we know that the number of possible solutions is always finite because we can transform (2.20) into a system of polynomial equations. In what follows, we will assume that the solution $n^I[\pi]$ is in fact unique by suitably restricting the set of allowed tensors π^{aIJ} . This could imply that the set of such tensors no longer has the structure of a vector space which however does not pose any problems for what follows.

On the other hand, we can prove the following for general π^{aIJ} :

Theorem.

Let $D \geq 3$ and¹

$$S_{\overline{M}}^{ab} := \frac{1}{4} \epsilon_{IJKL\overline{M}} \pi^{aIJ} \pi^{bKL}, \quad (2.21)$$

where \overline{M} is any totally skew $(D-3)$ -tuple of indices in $\{0, 1, \dots, D\}$. Then

$$S_{\overline{M}}^{ab} = 0 \quad \forall \overline{M}, a, b \quad \Leftrightarrow \quad P_{bKL}^{aIJ}[\pi, n] \pi^{bKL} = 0 \quad (2.22)$$

for any unit vector n where $P_{bKL}^{aIJ}[\pi, n] := [P_{bKL}^{aIJ}[E]]_{E=E[\pi, n]}$ and $E^{aI}[\pi, n] = -\zeta \pi^{aIJ} n_J$ and where $P[E]$ is defined in (2.17). Here we assume that $Q^{ab}[\pi, n] := \pi^{aIK} \pi^{bJL} \eta_{IJ} n_K n_L$ is non degenerate for any (timelike for $\zeta = -1$) vector n_I .

This result implies that although $S_{\overline{M}}^{ab}$ are $D(D+1)/2 \binom{D+1}{4}$ equations which exceeds $D^2(D-1)/2 - D$ for $D > 3$ only $D^2(D-1)/2$ of them are independent. The constraint $S_{\overline{M}}^{ab} = 0$ does not fix n^I and makes no statement about the trace part $\bar{\pi}^J[\pi, n] = \bar{\pi}^{aIJ}[\pi, n] E_{aI}[\pi, n]$. Given that the theorem holds for any n it is natural to fix n such that the trace part vanishes simultaneously as otherwise we would have only that $\bar{\pi}^{aIJ} = 2E^{a[I} \bar{\pi}^{J]}/(D-1)$ and not $\bar{\pi}^{aIJ} = 0$ or $\pi^{aIJ} = 2n^{[I} E^{a|J]}$ on the constraint surface of the simplicity constraint.

Proof.

Obviously

$$S_{\overline{M}}^{ab} = 0 \quad \Leftrightarrow \quad \epsilon^{IJKL\overline{M}} S_{\overline{M}}^{ab} = \frac{\zeta}{4} 4! (D-3)! \pi^{aIJ} \pi^{bKL} = 0. \quad (2.23)$$

Given π , consider any unit vector n and decompose as in (2.12)

$$\pi^{aIJ} = \bar{\pi}^{aIJ}[\pi, n] + 2n^{[I} E^{a|J]}[\pi, n]. \quad (2.24)$$

Inserting into (2.24), we obtain

$$\pi^{a[IJ} \pi^{bKL]} = \bar{\pi}^{a[IJ} \bar{\pi}^{bKL]} + 4n^{[I} E^{(a|J} \bar{\pi}^{b)KL]} = 0. \quad (2.25)$$

Contracting with n_I yields

$$E^{(a|J} \bar{\pi}^{b)KL]} = 0. \quad (2.26)$$

Contracting further with E_{aJ} yields

$$(D-1) [\bar{\pi}^{bKL} - \frac{2}{D-1} E^{b[K} \bar{\pi}^{aJ|L]} E_{aJ}] = (D-1) P_{aIJ}^{bKL}[\pi, n] \pi^{aIJ} = 0. \quad (2.27)$$

We conclude $\pi^{aIJ} = 2v^{[I} E^{a|J]}$, $v^I = (n^I - \frac{1}{D-1} \bar{\pi}^{bJI} E_{bJ})$ and inserting back into (2.23) we see that it is identically satisfied. \square

The theorem therefore says that on the constraint surface $\pi^{aIJ} = 2v^{[I} E^{a|J]}$ for some vector v which is not necessarily normalised and not necessarily normal to E^{aI} but such that E^{aI}, v^I constitute $D+1$ linearly independent internal vectors. We can however draw, for $\zeta = -1$, some additional conclusion from the requirement that $Q^{ab} = \pi^{aIJ} \pi_{IJ}^b/(2\zeta)$ should have Euclidean signature. First of all, v^I cannot be null since otherwise $Q^{ab} \propto (E_I^a v^I)(E_J^b v^J)$ would be degenerate. If v^I would be spacelike then consider $\tilde{E}_I^a = E_I^a - E_J^a v^J v_I/(v^K v_K)$. It follows $\pi^{aIJ} = 2v^{[I} \tilde{E}^{a|J]}$ and $Q^{ab} \propto \tilde{E}^{aI} \tilde{E}_I^b$. Since v^I, \tilde{E}_I^a constitutes a $(D+1)$ -bein and v^I is spacelike while η is Lorentzian, also Q^{ab} would need to be Lorentzian. Hence v^I must in fact be timelike for $\zeta = -1$.

¹For $D = 2$ no simplicity constraints are needed since $D^2(D-1)/2 - D = 0$.

We may therefore absorb for either signature the normalisation of v into E_I^a and define $n_I := v_I/\sqrt{\zeta v_J v^J}$ as well as $\tilde{E}^{aI} = \sqrt{\zeta v_K v^K} E^{aJ} \tilde{\eta}_{J}^I$. Then $2v^{[I} E^{a|J]} = 2n^{[I} \tilde{E}^{a|J]}$ with $\tilde{E}^{aI} n_I = 0$, $n^I n_I = \zeta$.

Therefore, the constraint surface defined via (2.21) is the same as the one given by $\bar{\pi}_T^{aIJ}$ above, where we assumed that π is of the form (2.19) and constitutes the unique decomposition of π^{aIJ} with no trace part. In what follows, we will use the simplicity constraint in the form (2.21). However, it will be convenient to have the presentation (2.19) at one's disposal when we work off the constraint surface.

Notice that the proof given above also in the case $D = 3$ does not allow for a ‘‘topological sector’’ $\pi^{aIJ} = \epsilon^{IJKL} n^K E^{aL}$ or ‘‘degenerate sector’’ due to the non degeneracy assumption. This assumption is dropped in the alternative proof in [28] which is based on [39] which is why the topological sector does appear there.

2.3 Integrability of the Hybrid Connection Modulo Simplicity Constraint

The hybrid connection is defined via (2.6) on the constraint surface $S_{\bar{M}}^{ab} = 0$. We want to define an extension off the constraint surface such that the resulting expression is integrable, i.e. is the functional derivative $\Gamma_{aIJ} = \delta F/\delta \pi^{aIJ}$ of a generating functional $F = F[\pi]$. To that end, we need the explicit expression of Γ_{aIJ} in terms of e_a^I .

To begin with, we notice that $D_a n^I = 0$. To see this we consider its $D + 1$ independent components $n_I D_a n^I = \frac{1}{2} D_a (n^I n_I) = 0$ and $e_b^I D_a n^I = -n^I D_a e_b^I = 0$. We decompose

$$\Gamma_{aIJ} = \bar{\Gamma}_{aIJ} + 2n_{[I} \bar{\Gamma}_{a|J]}, \quad \bar{\Gamma}_{aI} = -\zeta \Gamma_{aIJ} n^J \quad (2.28)$$

and further

$$\bar{\Gamma}_{aIJ} = \bar{\Gamma}_{abc} e_I^b e_J^c, \quad \bar{\Gamma}_{aI} = \bar{\Gamma}_{ab} e_I^b, \quad (2.29)$$

with $e_I^b = q^{ab} e_{bI}$, $q^{ac} q_{cb} = \delta_b^a$, $q_{ab} = e_a^I e_{bI}$. We find

$$\bar{\Gamma}_{ab} = -\zeta n_I \partial_a e_b^I, \quad \bar{\Gamma}_{abc} = \Gamma_{bac} - e_{bI} \partial_a e_c^I, \quad (2.30)$$

where $\Gamma_{bac} = q_{bd} \Gamma_{ac}^d$ is the Levi-Civita connection. Combining these formulae, we obtain

$$\begin{aligned} \Gamma_{aIJ}[E] &= -[\eta_{K[I} + \zeta n_K n_{|I]} e_{|J]}^b \partial_a e_b^K + \Gamma_{ac}^b e_{b[I} e_{|J]}^c \\ &= \zeta n_{[I} \partial_a n_{|J]} + e_{b[I} \partial_a e_{|J]}^b + \Gamma_{ac}^b e_{b[I} e_{|J]}^c, \end{aligned} \quad (2.31)$$

where we used here and will also use frequently later $n_K \partial_a E^{bK} = -E^{bK} \partial_a n_K$, $n^K \partial_a n_K = 0$ and $n_{[I} \tilde{\eta}_{J]}^K = n_{[I} \eta_{J]}^K$.

To write Γ_{aIJ} in terms of π^{aIJ} , we notice the following weak identities modulo the simplicity

constraint, that is $\pi^{aIJ} \approx 2n^{[I} E^{a|J]}$,

$$\begin{aligned}
\pi^{aIJ} \pi_{IJ}^b &\approx 4n^{[I} E^{a|J]} n_{[I} E_{J]}^b = 2\zeta E^{aI} E_I^b = 2\zeta Q^{ab}, \\
Q_{ab} \pi^{aKI} \pi_{KI}^b &\approx [n^K E^{aI} - n^I E^{aK}] [n_K E_{aJ} - n_J E_{aK}] \\
&= Dn^I n_J + \zeta \bar{\eta}_J^I = (D-1)n^I n_J + \zeta \eta_J^I, \\
E^{a[I} n^{J]} &= -\zeta \pi^{a[I]L} n^{J]} n_L, \\
Q_{bd} \pi^{dK} \pi_{K[J}^c &\approx [n^K E_{b[I} - E_b^K n_{[I} [E_{J]}^c n_K - n_J E_K^c] = \zeta E_{b[I} E_{J]}^c = \zeta e_{b[I} e_{J]}^c, \\
Q_{bc} \pi^{bK} \pi_{[I} \partial_a \pi_{K]}^c &\approx [n^K E_{c[I} - E_c^K n_{[I} \partial_a [E_{J]}^c n_K - n_J E_K^c] \\
&= -n^K E_{c[I} [n_{J]} (\partial_a E_K^c) - (\partial_a E_{J]}^c) n_K] \\
&\quad + E_c^K n_{[I} [(\partial_a n_{J]} E_K^c - E_{J]}^c (\partial_a n_K)] \\
&= (D-1)n_{[I} (\partial_a n_{J]} + E_{c[I} n_{J]} E_K^c (\partial_a n^K) + \zeta E_{c[I} (\partial_a E_{J]}^c) \\
&= (D-2)n_{[I} (\partial_a n_{J]} + \zeta E_{c[I} (\partial_a E_{J]}^c) \\
&= (D-2)n_{[I} (\partial_a n_{J]} + \zeta e_{c[I} (\partial_a e_{J]}^c), \\
\bar{\eta}_I^K \bar{\eta}_J^L Q_{bd} \pi^{dM} \pi_{[K} \partial_a \pi_{M|L]}^c &\approx \zeta \bar{\eta}_{[I}^K \bar{\eta}_{J]}^L e_{b[K} \partial_a e_{L]}^b \\
&= \zeta e_{b[I} \partial_a e_{J]}^b - n_{[I} \partial_a n_{J]}. \tag{2.32}
\end{aligned}$$

Consider the quantities

$$T_{aIJ} := \pi_{bK[I} \partial_a \pi^{bK} \]_J, \quad T_{bIJ}^c := \pi_{bK[I} \pi^{cK} \]_J, \tag{2.33}$$

where $\pi_{aIJ} = Q_{ab} \pi_{IJ}^b$. Then

$$(D-1)n_{[I} \partial_a n_{J]} = T_{aIJ} - \bar{T}_{aIJ}, \quad (D-1)\zeta e_{b[I} \partial_a e_{J]}^b = T_{aIJ} + (D-2)\bar{T}_{aIJ}. \tag{2.34}$$

Inserting (2.33) and (2.34) into (2.31) then leads to the explicit expression

$$\Gamma_{aIJ}[\pi] = \frac{2\zeta}{D-1} T_{aIJ} + \frac{\zeta(D-3)}{D-1} \bar{T}_{aIJ} + \zeta \Gamma_{ac}^b T_{bIJ}^c. \tag{2.35}$$

Together with $Q^{ab} =: \det(q) q^{ab}$ which expresses Γ_{ac}^b in terms of $Q^{ab} = \pi^{aIJ} \pi_{IJ}^b / (2\zeta)$, this determines Γ_{aIJ} completely in terms of π^{aIJ} if we simply replace the \approx signs in (2.32) by $=$ signs and take the left hand sides as definitions for the right hand sides.

It transpires that Γ_{aIJ} is a rational, homogeneous function of π and its first derivatives which vanishes at $\pi = 0$. Therefore, if $\Gamma_{aIJ}[\pi]$ has a generating functional, then it is given by²

$$F'[\pi] = \int d^D x \pi^{aIJ} \Gamma_{aIJ}[\pi]. \tag{2.36}$$

Variation of F' with respect to π^{aIJ} yields

$$\begin{aligned}
\delta F' &= \int d^D x (\delta \pi^{aIJ} \Gamma_{aIJ}[\pi] + \pi^{aIJ} \delta \Gamma_{aIJ}[\pi]) \\
&= \int d^D x (\delta \pi^{aIJ} \Gamma_{aIJ}[\pi] + \pi^{aIJ} [\delta \Gamma_{aIJ}[E] + \delta S'_{aIJ}]) \\
&= \delta \left[\int d^D x \pi^{aIJ} S'_{aIJ} \right] + \int d^D x \left(\delta \pi^{aIJ} \Gamma_{aIJ}[\pi] + 2n^{[I} E^{a|J]} \delta \Gamma_{aIJ}[E] \right) \\
&\quad + \int d^D x (S^{aIJ} \delta \Gamma_{aIJ}[E] - \delta \pi^{aIJ} S'_{aIJ}), \tag{2.37}
\end{aligned}$$

²If a one form Γ_M is exact, i.e. has potential U with $\Gamma_M = U_{,M}$ then $U(\pi) - U(\pi_0) = \int_{\gamma_{\pi_0, \pi}} \Gamma$ for any path $\gamma_{\pi_0, \pi}$ between π_0 and π . If Γ is defined at $\pi_0 = 0$ to vanish then choosing the straight path $t \mapsto t\pi$ yields $U(\pi) = \text{const.} + \int_0^1 dt t^\pi \Gamma_M(t\pi)$.

where $S^{aIJ} := \pi^{aIJ} - 2n^{[I}E^{a|J]}$ and $S'_{aIJ} := \Gamma_{aIJ}[\pi] - \Gamma_{aIJ}[E]$ both vanish on the constraint surface of the simplicity constraint. We see that F' itself cannot be a generating functional but rather

$$F = F' - \int d^D x \pi^{aIJ} S'_{aIJ}, \quad (2.38)$$

i.e. F' has to be corrected by a term that vanishes on the constraint surface of the simplicity constraint, however, its variation does not necessarily vanish on that constraint surface. It follows that $\delta F/\delta\pi^{aIJ} = \Gamma_{aIJ} + \tilde{S}_{aIJ}$ for some \tilde{S}_{aIJ} which vanishes on the constraint surface of the simplicity constraint provided that

$$\int d^D x n^{[I}E^{a|J]} \delta\Gamma_{aIJ}[E] = \int d^D x \sqrt{\det(q)} n^{[I}e^{a|J]} \delta\Gamma_{aIJ}[E] = 0. \quad (2.39)$$

This is the key identity that one has to prove. It is the counterpart to the key identity that is responsible for the fact that the Ashtekar connection is Poisson commuting in $D + 1 = 4$. The reason for the correction $F' \rightarrow F$ is that $\Gamma_{aIJ}[\pi]$ is not strictly integrable but only modulo terms that vanish on the constraint surface of the simplicity constraint.

We proceed with the proof of (2.39). It is easiest to use (2.28) – (2.30). We have, using $n_K \delta n^K = 0$, $n_K \delta e_b^K = -e_b^K \delta n_K$ and that $\bar{\Gamma}_{a(bc)} = 0$,

$$\begin{aligned} n^{[I}e^{a|J]} \delta(2n_{[I}\bar{\Gamma}_{a|J]}) &= 2n^{[I}e^{a|J]} [n_I(\delta\bar{\Gamma}_{aJ}) + \bar{\Gamma}_{aJ}\delta n_I] \\ &= \zeta e^{aI}(\delta\bar{\Gamma}_{aI}) = -e^{aI} \delta(n_J(\partial_a e_b^J) e_I^b) \\ &= e^{aI} \delta(e_b^J(\partial_a n_J) e_I^b) = e^{aI} \delta(\bar{\eta}_I^J \partial_a n_J) \\ &= e^{aI} \nabla_a(\delta n_I), \\ n^{[I}e^{a|J]} \delta\bar{\Gamma}_{aIJ} &= n^I e^{aJ} \delta(\bar{\Gamma}_{abc} e_I^b e_J^c) \\ &= n^I e^{aJ} \bar{\Gamma}_{abc} e_J^c (\delta e_I^b) = q^{ac} \bar{\Gamma}_{acb} e_I^b (\delta n^I) \\ &= -[e_J^a(\partial_a e_b^J) - \Gamma_{ab}^a] e_I^b (\delta n^I) \\ &= -[e_J^a[\partial_a(e_I^b e_b^J) - e_b^J(\partial_a e_I^b) - \Gamma_{ab}^a e_I^b] (\delta n^I) \\ &= -[e_J^a \partial_a(\bar{\eta}_I^J) - (\nabla_a e_I^a)] (\delta n^I) = [\nabla_a e_I^a] (\delta n^I), \end{aligned} \quad (2.40)$$

where ∇_a is the torsion free covariant differential annihilating q_{ab} (it acts only on tensor indices, not on internal ones). We conclude

$$\int d^D x n^{[I}E^{a|J]} \delta\Gamma_{aIJ}[E] = \int d^D x \sqrt{\det(q)} \nabla_a [e_I^a \delta n^I] = \int d^D x \partial_a (E_I^a \delta n^I) = 0 \quad (2.41)$$

for suitable boundary conditions on E_I^a and its variations³.

We therefore have established:

Theorem.

There exists a functional $F[\pi]$ such that for δn^I vanishing sufficiently fast at spatial infinity, we have

$$\delta F[\pi]/\delta\pi^{aIJ}(x) = \Gamma_{aIJ}[\pi; x] + S_{aIJ}[\pi; x], \quad (2.42)$$

where S_{aIJ} vanishes on the constraint surface of the simplicity constraint, depending at most on its first partial derivatives and $\Gamma_{aIJ}[\pi]$ is the hybrid connection (2.35).

³For instance one could impose that n_I deviates from a constant by a function of rapid decrease at spatial infinity.

3 New Variables and Equivalence with ADM Formulation

We consider an $G = \text{SO}(D+1)$ or $G = \text{SO}(1, D)$ canonical gauge theory over σ with connection A_{aIJ} and conjugate momentum π^{aIJ} . These variables are subject to the canonical brackets

$$\{A_{aIJ}(x), \pi^{bKL}(y)\} = 2\beta\delta_a^b\delta_{[I}^K\delta_{J]}^L\delta^{(D)}(x-y), \quad \{A_{aIJ}(x), A_{bKL}(y)\} = \{\pi^{aIJ}(x), \pi^{bKL}(y)\} = 0, \quad (3.1)$$

as well as to the Gauß constraint

$$G^{IJ} := \mathcal{D}_a\pi^{aIJ} = \partial_a\pi^{aIJ} + 2A_a^{\quad [I} \pi^{a|K|J]} \quad (3.2)$$

and the simplicity constraint

$$S_M^{ab} = \frac{1}{4}\epsilon_{IJKL}\overline{M}^{\quad ab}\pi^{aIJ}\pi^{bKL}. \quad (3.3)$$

Internal indices as before are moved by the internal metric η which is just the Euclidean metric for $\text{SO}(D+1)$ ($\zeta = 1$) and the Minkowski metric for $\text{SO}(1, D)$ ($\zeta = -1$). We have for $g \in \text{SO}(\zeta, D)$ that $g^{IJ}g^{KL}\eta_{KL} = \eta^{IJ}$, $\det((g^{IJ})) = 1$. The covariant differential \mathcal{D}_a of A acts only on internal indices. This does not affect the tensorial character of (3.2) because π^{aIJ} is a Lie algebra valued vector density of weight one and (3.2) is its covariant divergence which is independent of the Levi-Civita connection. The real parameter $\beta \neq 0$ in (3.1) is similar to, but structurally different from the Immirzi parameter in $D = 3$.

Let $\Gamma_{aIJ}[\pi]$ be the hybrid connection (2.35) constructed from π . We define a map from this Yang-Mills theory phase space with coordinates (A_{aIJ}, π^{aIJ}) to the coordinates (q_{ab}, P^{ab}) of the ADM phase space by the following formulas

$$\begin{aligned} \det(q)q^{ab} &:= \frac{1}{2\zeta}\pi^{aIJ}\pi^{bIJ}, \\ P^{ab} &:= \frac{1}{4\beta}\left(q^{a[c} [A_{cIJ} - \Gamma_{cIJ}]\pi^{b]IJ} + q^{b[c} [A_{cIJ} - \Gamma_{cIJ}]\pi^{a]IJ}\right) \\ &= \frac{1}{2\beta}q^{d(a} [A_{cIJ} - \Gamma_{cIJ}]\pi^{b]IJ}\delta_d^{c]}. \end{aligned} \quad (3.4)$$

The central result of this section is:

Theorem.

- i. Gauß and simplicity constraints obey a first class constraint algebra.*
- ii. The symplectic reduction of the Yang-Mills phase space defined above with respect to Gauß and simplicity constraints coincides with the ADM phase space. More in detail, the functions $q_{ab}[\pi]$, $P^{ab}[A, \pi]$ defined in (3.4) are Dirac observables with respect to Gauß and simplicity constraints and obey the standard Poisson brackets*

$$\{q_{ab}(x), P^{cd}(y)\} = \delta_{(a}^c\delta_{b)}^d\delta^{(D)}(x-y), \quad \{q_{ab}(x), q_{cd}(y)\} = \{P^{ab}(x), P^{cd}(y)\} = 0 \quad (3.5)$$

on the constraint surface defined by simplicity and Gauß constraints.

Proof.

i.

Since S_M^{ab} only depends on π^{aIJ} , it Poisson commutes with itself. The Gauß constraint of course generates G gauge transformations under which A transforms as a connection and π as a section in an associated vector bundle under the adjoint representation of G . The Poisson algebra of the smeared Gauß constraints is therefore (anti-)isomorphic with the Lie algebra of G

$$\{G[f], G[f']\} = -\beta G[[f, f']], \quad G[f] := \int d^Dx \frac{1}{2}f_{IJ}G^{IJ}, \quad [f, f']_{IJ} = 2f_{[I} \quad^K f'_{|K|J]}. \quad (3.6)$$

Under finite Gauß transformations we have

$$\pi^{aIJ} \mapsto [g\pi^a g^{-1}]^{IJ}. \quad (3.7)$$

Since $G = \text{SO}(\zeta, D)$ is unimodular we obtain

$$S_{\overline{M}}^{ab} \mapsto \zeta g_{\overline{M}} \overline{N} S_{\overline{N}}^{ab}, \quad g_{\overline{M}} \overline{N} = \prod_{i=1}^{D-3} g_{M_i}^{N_i}. \quad (3.8)$$

It follows the first class structure $\{G, G\} \propto G$, $\{G, S\} \propto S$, $\{S, S\} = 0$.

ii.

The hybrid spin connection $\Gamma_{aIJ}[E]$ is a G connection by construction. Its extension $\Gamma_{aIJ}[\pi]$ off the simplicity constraint surface therefore transforms as a G connection modulo the simplicity constraint. Since both π^{aIJ} , $K_{aIJ} := \frac{1}{\beta}(A_{aIJ} - \Gamma_{aIJ})$ transform in the adjoint representation of G it is clear that $Q^{ab} \propto \text{Tr}(\pi^a \pi^b)$, $K_a^b \propto \text{Tr}(K_a \pi^b)$ are in fact Gauß invariant, possibly modulo the simplicity constraint, and thus are q_{ab} , P^{ab} . Since $S_{\overline{M}}^{ab}$ and q_{ab} are both constructed from π^{aIJ} alone it is clear that they strictly Poisson commute. As for P^{ab} we notice that it is a linear combination of the objects

$$K_a^b := -\frac{s}{4\beta} [A_{aIJ} - \Gamma_{aIJ}] \pi^{bIJ}, \quad (3.9)$$

with coefficients that depend only on q_{ab} . It is therefore sufficient to show that $\{K_a^b, S_{\overline{M}}^{cd}\} \approx 0$. We compute with the smeared simplicity constraint and using that $\Gamma_{aIJ}[\pi]$ depends only on π^{aIJ}

$$\begin{aligned} \{K_a^b(x), S[f]\} &= \int d^D y f_{cd}^{\overline{M}}(y) \{K_a^b(x), S_{\overline{M}}^{cd}(y)\} \\ &= -\frac{s}{16\beta} \int d^D y f_{cd}^{\overline{M}}(y) \pi^{bIJ}(x) \epsilon_{ABCD\overline{M}} \{A_{aIJ}(x), \pi^{cAB}(y) \pi^{dCD}(y)\} \\ &= -s f_{cd}^{\overline{M}}(x) \delta_a^{(c} S_{\overline{M}}^{d)b}(x). \end{aligned} \quad (3.10)$$

It follows that P^{ab} Poisson commutes with the simplicity constraint on its constraint surface.

It remains to verify the ADM Poisson brackets.

Since $q_{ab}(x)$ depends only on $\pi^{aIJ}(x)$ we have trivially $\{q_{ab}(x), q_{cd}(y)\} = 0$. Next, using $q^{ab} = Q^{ab} / \det(q)$, $\det(q) = [\det(Q)]^{1/(D-1)}$, we find

$$\begin{aligned} \{q_{ab}(x), P^{cd}(y)\} &= 2s q_{ae}(x) q_{bf}(x) \{q^{ef}(x), (q^{g(c} K_h^{[d] \delta_g^h]}) (y)\} \\ &= -\frac{1}{2\beta} [q_{ae} q_{bf}](x) [q^{g(c} \pi^{[d] IJ} \delta_g^h] (y) \times \\ &\quad \left[\frac{1}{\det(q)} \{Q^{ef}(x), A_{hIJ}(y)\} - \frac{1}{D-1} Q_{mn}(x) q^{ef}(x) \{Q^{mn}(x), A_{hIJ}(y)\} \right] \\ &= -\frac{1}{2\beta} \delta^{(D)}(x-y) q_{ae} q_{bf} q^{g(c} \pi^{[d] IJ} \delta_g^h] \times \\ &\quad \frac{1}{2\zeta} \left[-\frac{4\beta}{\det(q)} \delta_h^{(e} \pi_{IJ}^{f)} + \frac{4\beta}{D-1} Q_{mn}(x) q^{ef} \delta_h^m \pi_{IJ}^n \right] \\ &= \delta^{(D)}(x-y) q_{ae} q_{bf} q^{g(c} \left[q^{[d]|e|} \delta_h^{[f|} + q^{[d]|f|} \delta_h^{[e|} - \frac{2}{D-1} q_{mn} q^{[ef|} q^{[d]|n|} \delta_h^{[m|} \right] \delta_g^h] \\ &= \delta^{(D)}(x-y) \delta_{(a}^c \delta_{b)}^d \end{aligned} \quad (3.11)$$

by carefully contracting all indices and keeping track of the (anti)symmetrisations. The last bracket is the most complicated. We write

$$P^{ab} = P^{abeIJ} K_{eIJ}, \quad P^{abeIJ} = \frac{1}{2} q^{g(a} \pi^{[b]IJ} \delta_g^e] \quad (3.12)$$

and compute

$$\begin{aligned} \{P^{ab}(x), P^{cd}(y)\} &= P^{abeIJ}(x) \{K_{eIJ}(x), P^{cdfKL}(y)\} K_{fKL}(y) \\ &\quad - P^{cdfKL}(y) \{K_{fKL}(y), P^{abeIJ}(x)\} K_{eIJ}(x) \\ &\quad + P^{abeIJ}(x) P^{cdfKL}(y) \{K_{eIJ}(x), K_{fKL}(y)\} \\ &= \frac{1}{2} \{P^{ab}(x), q^{h(c}(y)\} \pi^{[d]KL} \delta_h^f] K_{fKL}(y) - \frac{1}{2} \{P^{cd}(y), q^{g(a}(x)\} \pi^{[b]IJ} \delta_g^e] K_{eIJ}(x) \\ &\quad + \frac{1}{2\beta} P^{abeIJ}(x) q^{h(c}(y)\} \{A_{eIJ}(x), \pi^{[d]KL}(y)\} \delta_h^f] K_{fKL}(y) \\ &\quad - \frac{1}{2\beta} P^{cdfKL}(y) q^{g(a}(x)\} \{A_{fKL}(y), \pi^{[b]IJ}(x)\} \delta_g^e] K_{eIJ}(x) \\ &\quad - \frac{1}{\beta^2} P^{abeIJ}(x) P^{cdfKL}(y) [\{A_{eIJ}(x), \Gamma_{fKL}(y)\} - \{A_{fKL}(y), \Gamma_{eIJ}(x)\}] \\ &= 2s \left[q^{h(a} q^{b)(c} K_f^{[d]} \delta_h^f] - q^{g(c} q^{d)(a} K_e^{[b]} \delta_g^e] \right] \delta^{(D)}(x-y) \\ &\quad - 2s \left[q^{h(c} \delta_e^{[d]} \delta_h^f] q^{g(a} K_f^{[b]} \delta_g^e] - q^{g(a} \delta_f^{[b]} \delta_g^e] q^{h(c} K_e^{[d]} \delta_h^f] \right] \delta^{(D)}(x-y) \\ &\quad - \frac{1}{\beta^2} P^{abeIJ}(x) P^{cdfKL}(y) [\{A_{eIJ}(x), \Gamma_{fKL}(y)\} - \{A_{fKL}(y), \Gamma_{eIJ}(x)\}]. \end{aligned} \quad (3.13)$$

By carefully carrying out the contractions, it is not difficult to see that the first two square brackets in the last equality are each proportional to

$$q^{ad} K^{[bc]} + q^{bd} K^{[ac]} + q^{ac} K^{[bd]} + q^{bc} K^{[ad]}, \quad K^{ab} := q^{ac} K_c^b. \quad (3.14)$$

We claim that $K^{[ab]}$ is constrained to vanish by the Gauß constraint. To see this, let \mathcal{D}'_a be the covariant differential of A which acts also on tensor indices and let D_a the covariant differential that kills the generalised D -bein. Then the Gauß constraint reads

$$G^{IJ} = \mathcal{D}_a \pi^{aIJ} = \mathcal{D}'_a \pi^{aIJ} = [\mathcal{D}'_a - D_a] \pi^{aIJ} + D_a \pi^{aIJ} \approx [(A - \Gamma)_a, \pi^a]^{IJ} = \beta [K_a, \pi^a]^{IJ}, \quad (3.15)$$

where we used that on the constraint surface of the simplicity constraint we have $D_a \pi^{bKL} = 2D_a n^{[K} E^{a]L} = 0$. With the convention $K_{aI} := -\zeta K_{aIJ} n^J$ we obtain for the Gauß constraint

$$\begin{aligned} G_{IJ} &= 2\beta K_{aL[I} \pi^a_{J]}{}^L \approx 2\beta K_{aL[I} (n_{J]} E^{aL} - E_{J]}^a n^L) \\ &= -2\zeta \beta K_{a[I} E_{J]}^a + 2\beta K_{[I} n_{J]} =: \bar{G}_{IJ} + 2n_{[I} G_{J]}, \end{aligned} \quad (3.16)$$

where $K_I = E^{aL} K_{aLI}$ is the trace part of K_{aIJ} . It follows that $\bar{K}_I = 0$ and $K_{a[I} E_{J]}^a = 0$ on the Gauß constraint surface. Now

$$\begin{aligned} K^{[ab]} E_{aI} E_{bJ} &\approx -\frac{s\zeta}{2} q^{c[a} K_{cL} E^{b]L} E_{aI} E_{bJ} = -\frac{s\zeta}{2} q^{ca} K_{cL} E^{bL} E_{a[I} E_{bJ]} \\ &= -\frac{s\zeta}{2 \det(q)} K_{cL} \bar{\eta}_{[J}^L E_{I]}^c = \frac{s\zeta}{2 \det(q)} K_{a[I} E_{J]}^a. \end{aligned} \quad (3.17)$$

Therefore $K^{[ab]} = [K^{[cd]} E_{cI} E_{cJ}] E^{aI} E^{bJ}$ vanishes on the Gauß constraint surface.

We now turn to the last square bracket in (3.13). It is a linear combination, with coefficients M depending on q_{ab} , of expressions of the form

$$M_f^{abe}(x)M_h^{cdg}(y)\pi^{fIJ}(x)\pi^{hKL}(y) [\{A_{eIJ}(x), \Gamma_{gKL}(y)\} - \{A_{gKL}(y), \Gamma_{eIJ}(x)\}]. \quad (3.18)$$

We now invoke the key result of the previous section and write $\Gamma_{aIJ} = \delta F / \delta \pi^{aIJ} + S_{aIJ}$ where S_{aIJ} vanishes on the constraint surface of the simplicity constraint and depends at most on its first partial derivatives. It is therefore given by an expression of the form

$$S_{gKL} = \lambda_{gKLgmn}^{\overline{M}} S_M^{mn} + \mu_{gKLmn}^{\overline{M}p} \partial_p S_M^{mn} \quad (3.19)$$

for certain coefficients λ, μ . First of all, we notice that due to the commutativity of partial functional derivatives

$$\{A_{eIJ}(x), \delta F / \delta \pi^{gKL}(y)\} - \{A_{gKL}(y), \delta F / \delta \pi^{eIJ}(x)\} = 0. \quad (3.20)$$

Next, due to the derivatives involved, the Poisson bracket is not ultralocal, however, what we intend to prove is that $\{P[f], P[f']\} \approx 0$ with the smeared functions $P[f] = \int d^D x f_{ab} P^{ab}$. Let $M_f^e = f_{ab} M_f^{abe}$, $M_h'^g = M_h^{cdg} f'_{cd}$, then the contribution from S_{gKL} in the first term of (3.18) becomes after smearing

$$\begin{aligned} &\approx \int d^D x \int d^D y M_f^e(x) \pi^{fIJ}(x) \left(M_h'^g \pi^{hKL} \lambda_{gKLmn}^{\overline{M}} - [M_h'^g \pi^{hKL} \mu_{gKLmn}^{\overline{M}p}]_p \right) \{A_{eIJ}(x), S_M^{mn}(y)\} \\ &= 4\beta \int d^D x M_f^e \pi^{fIJ}(x) \left(M_h'^g \pi^{hKL} \lambda_{gKLmn}^{\overline{M}} - [M_h'^g \pi^{hKL} \mu_{gKLmn}^{\overline{M}p}]_p \right) \delta_e^{(m} S_M^{n)f}(x) \\ &\approx 0, \end{aligned} \quad (3.21)$$

where (3.10) was used. The calculation for the second term is similar. In conclusion, $\{P^{ab}(x), P^{cd}(y)\}$ vanishes on the joint constraint surface of the Gauß and the simplicity constraint. \square

4 ADM Constraints in Terms of the New Variables

It remains to express the ADM constraints in terms of the new variables. Of course we could just substitute for the expressions (3.4), however, this is not the most convenient form for the ADM constraints because they involve the hybrid connection which is a complicated expression in terms of π . We will therefore adopt the strategy familiar from $D + 1 = 4$ and invoke the curvature F of A . In the end, we will arrive at expressions $\mathcal{H}_a, \mathcal{H}$ for spatial diffeomorphism and Hamiltonian constraint which differ from their counterparts $\mathcal{H}'_a, \mathcal{H}'$, obtained by naive substitution of q_{ab}, P^{ab} by (3.4) in (2.2), (2.3), by terms proportional to Gauß and simplicity constraints. This guarantees that the algebra of Gauß, simplicity, spatial diffeomorphism and Hamiltonian constraints is of first class:

To see this, let us write $\mathcal{H}_a = \mathcal{H}'_a + Z_a$, $\mathcal{H} = \mathcal{H}' + Z$ where Z_a, Z vanish on the constraint surface of the simplicity and Gauß constraint. We have seen already that $\{S, S\} = 0$, $\{G, S\} \propto S$, $\{G, G\} \propto G$. We also have shown that (3.4) are weak Dirac observables with respect to S and invariant under G . Since $\mathcal{H}'_a, \mathcal{H}'$ are defined in terms of (3.4) it follows that $\{S, \mathcal{H}'_a\} \propto S$, $\{S, \mathcal{H}'\} \propto S$. Altogether therefore $\{S, \mathcal{H}_a\}$, $\{S, \mathcal{H}\}$, $\{G, \mathcal{H}_a\}$, $\{G, \mathcal{H}\} \propto S, G$ thus S, G form an ideal. Next we have $\{\mathcal{H}'_a, \mathcal{H}'_b\} \propto \mathcal{H}'_c, S, G$, $\{\mathcal{H}'_a, \mathcal{H}'\} \propto \mathcal{H}', S, G$, $\{\mathcal{H}', \mathcal{H}'\} \propto \mathcal{H}'_a, S, G$ because the algebra of the variables (3.4) is the same as that of the ADM variables modulo S, G terms and therefore the algebra of the ADM constraints is reproduced modulo S, G terms. Together with what was already said, this implies that $\mathcal{H}_a, \mathcal{H}$ reproduce the ADM algebra of constraints modulo S, G terms.

We begin by deriving the relation between the hybrid curvature

$$R_{abIJ} = 2\partial_{[a}\Gamma_{b]IJ} + \Gamma_{aIK}\Gamma_b{}^K{}_J - \Gamma_{aJK}\Gamma_b{}^K{}_I \quad (4.1)$$

and the Riemann curvature of q_{ab} . Let ∇_a be the covariant derivative compatible with q_{ab} . Then we have by definition $D_a e_b^I = \nabla_a e_b^I + \Gamma_a{}^I{}_J e_b^J = 0$. Expanding out $D = \nabla + \Gamma$ in the commutator relation $[D_a, D_b]e_c^I = 0$ and using $[\nabla_a, \nabla_b]e_c^I = R_{abc}{}^d e_d^I$ we find

$$R_{abc}{}^d e_d^I + R_{ab}{}^I{}_J e_c^J = 0 \Rightarrow R_{abcd} = R_{abIJ} e_c^I e_d^J. \quad (4.2)$$

This relation looks familiar from the spin connection, but we stress Γ_{aIJ} is not the spin connection because e_a^I is not a D -bein. We obtain modulo S for the Ricci scalar

$$R_{abIJ} \pi^{aIK} \pi^b{}_{KJ} \approx R_{abIJ} [n^I E^{aK} - n^K E^{aI}] [n_K E^{bJ} - n^J E_K^b] = -\zeta \det(q) R. \quad (4.3)$$

Next, using (4.2)

$$R_{abIJ} \pi^{bIJ} \approx 2R_{abIJ} n^I E^{bJ} = 2q^{bc} \sqrt{\det(q)} R_{abIJ} n^I e_c^J = -2q^{bc} \sqrt{\det(q)} R_{abc}{}^d e_{dI} n^I = 0, \quad (4.4)$$

which is the analog of the algebraic Bianchi identity.

We now expand the curvature

$$F_{abIJ} := 2\partial_{[a}A_{b]IJ} + A_{aIK} A_b{}^K{}_J - A_{aJK} A_b{}^K{}_I \quad (4.5)$$

of $A = \Gamma + \beta K$ in terms of Γ, K and obtain

$$F_{abIJ} = R_{abIJ} + 2\beta D_{[a}K_{b]IJ} + 2\beta^2 K_{[aIK} K_{b]}{}^K{}_J, \quad (4.6)$$

where torsion freeness of $\nabla = D - \Gamma$ was employed. Contracting (4.6) with π^{bIJ} we find using (4.4)

$$F_{abIJ} \pi^{bIJ} \approx 2\beta (D_{[a}K_{b]IJ}) \pi^{bIJ} - \beta^2 \text{Tr}([K_a, K_b] \pi^b). \quad (4.7)$$

The second term is proportional to the Gauß constraint because $\text{Tr}([K_a, K_b] \pi^b) = \text{Tr}(K_a [K_b, \pi^b])$ and remembering (3.15). In the first term we notice that $D_a \pi^{bIJ} \approx 0$ so that

$$F_{abIJ} \pi^{bIJ} \approx -8s\beta D_{[a}K_{b]}^b = 4s\beta D_b [K_a{}^b - \delta_a^b K_c{}^c] = -4\beta D_b P_a{}^b = 2\beta \mathcal{H}_a \quad (4.8)$$

is proportional to the spatial diffeomorphism constraint modulo S, G .

Next, using (4.3)

$$F_{abIJ} \pi^{aIK} \pi^b{}_{KJ} \approx -\zeta \det(q) R + 2\beta D_a \text{Tr}(K_b [\pi^a, \pi^b]) - \beta^2 \text{Tr}([K_a, K_b] \pi^a \pi^b). \quad (4.9)$$

The second term is again proportional to the Gauß constraint, since $\text{Tr}(K_b [\pi^a, \pi^b]) = -\text{Tr}(\pi^a [K_b, \pi^b])$. So far all the steps were similar to the 3+1 situation. The difference comes in when looking at the third term in (4.9)

$$\begin{aligned} -\text{Tr}([K_a, K_b] \pi^a \pi^b) &\approx [K_{aIK} K_b{}^K{}_J - K_{bIK} K_a{}^K{}_J] [n^I E^{aL} - n^L E^{aI}] [n^L E^{bJ} - n^J E_L^b] \\ &= -\zeta [-(K_{aIK} E^{aI})(K_{bJ}{}^K E^{bJ}) + (K_{bIK} E^{aI})(K_{aJ}{}^K E^{bJ})]. \end{aligned} \quad (4.10)$$

By the Gauß constraint (3.16) we have $K_I = K_{aJI} E^{aJ} = \zeta [K_J n^J] n_I$ and $K_J n^J \approx -K_{aIJ} \pi^{aIJ} / 2 = 2s K_a^a$. Thus the first term in (4.10) is given by $4[K_a^a]^2$. However, the second term cannot be

written in terms of K_a^b . To explore the structure of the disturbing term we notice that from $\bar{K}_I = 0$ we have the decomposition

$$K_{aIJ} = \bar{K}_{aIJ}^T + 2n_{[I}K_{a|J]}, \quad K_{aI} = -\zeta K_{aIJ}n^J. \quad (4.11)$$

Hence

$$\begin{aligned} & -\zeta(K_{bIK}E^{aI})(K_{aJ}{}^K E^{bJ}) = -\zeta(\bar{K}_{bIK}^T E^{aI} - K_{bI}E^{aI}n_K)(\bar{K}_{aJ}{}^K E^{bJ} - K_{aJ}E^{bJ}n^K) \\ = & -\zeta(\bar{K}_{bIK}^T E^{aI})(\bar{K}_{aJ}{}^K E^{bJ}) - 4K_a^b K_b^a, \end{aligned} \quad (4.12)$$

where $K_{aI}E^{bI} = -\zeta K_{aIJ}E^{bI}n^J \approx K_{aIJ}\pi^{bIJ}/(2\zeta)$ was used. Altogether

$$- \text{Tr}([K_a, K_b]\pi^a \pi^b) = -4[K_a^b K_b^a - (K_c^c)^2] - \zeta(\bar{K}_{bIK}^T E^{aI})(\bar{K}_{aJ}{}^K E^{bJ}). \quad (4.13)$$

The first term in (4.13) has the structure that appears in the Hamiltonian constraint and can be written in terms of P^{ab}, q_{ab} , however, the second term does not appear in the Hamiltonian constraint and must be removed. Also notice that the Ricci term has sign $-\zeta$ while the first term has negative sign. If we are interested in Lorentzian Gravity then the relative sign between these two terms should be negative which is not the case for the choice of a compact gauge group $\zeta = 1$. Therefore the expression (4.9) fails to yield the Hamiltonian constraint for several reasons.

To assemble the Hamiltonian constraint without making use of Γ , the idea is to consider covariant derivatives which give access to A . Using suitable algebraic combinations then yields the desired expressions. To that end, let again \mathcal{D}_a be the covariant differential of A acting only on internal indices and let \mathcal{D}'_a be its extension by the Levi-Civita connection. Consider

$$D_b{}^a := \pi^{aK}{}_J (\mathcal{D}_b \pi^{cJL}) \pi_{cKL} = \pi^{aK}{}_J (\mathcal{D}'_b \pi^{cJL}) \pi_{cKL} - 2\pi^{aK}{}_J \pi_{cKL} \Gamma_{bd}^{[c} \pi^{d]JL}. \quad (4.14)$$

The second term equals modulo S

$$- 2[n^K E_J^a - n_J E^{aK}][n_K E_{cL} - n_L E_{cK}]\Gamma_{bd}^{[c} \pi^{d]JL} = -2\zeta E_J^a E_{cL} \Gamma_{bd}^{[c} \pi^{d]JL} \approx 0 \quad (4.15)$$

and thus vanishes modulo S . Writing $\mathcal{D}'_a = [\mathcal{D}'_a - D_a] + D_a$ and noticing $D_a \pi^{cJL} \approx 0$ we obtain

$$\begin{aligned} \pi^{aK}{}_J (\mathcal{D}_b \pi^{cJL}) \pi_{cKL} & \approx \zeta \beta E_J^a E_{cL} [K_b{}^J{}_M \pi^{cML} + K_b{}^L{}_M \pi^{cJM}] \\ & \approx \zeta \beta E_J^a E_{cL} [K_b{}^J{}_M E^{cL} n^M - K_b{}^L{}_M E^{cJ} n^M] \\ & = -\beta(D-1)E^{aJ} K_{bJ} = 2s\zeta\beta(D-1)K_b{}^a. \end{aligned} \quad (4.16)$$

It follows that

$$\frac{1}{(D-1)^2} [D_b{}^a D_a{}^b - (D_c{}^c)^2] \approx 4\beta^2 [K_b{}^a K_a{}^b - (K_c^c)^2] \quad (4.17)$$

and thus linear combinations of (4.9) and (4.17) can be used in order to produce the correct factor in front of the term quadratic in the extrinsic curvature.

In analogy to (4.14), consider

$$D^{aIJ} := \pi^{b[I}{}_K \mathcal{D}_b \pi^{a|K|J]} = \pi^{b[I}{}_K \mathcal{D}'_b \pi^{a|K|J]} - 2\pi^{b[I}{}_K \Gamma_{bc}^{[a} \pi^{c|K|J]}. \quad (4.18)$$

The second term equals modulo S

$$2\zeta E^{b[I} \Gamma_{bc}^{[a} E^{c]J]} = (-\zeta \Gamma_{bc}^c E^{b[I}]) E^{aJ]} \quad (4.19)$$

and thus is pure trace. Since we intend to cancel \bar{K}_{aIJ}^T we therefore consider instead of (4.18) its transverse tracefree projection

$$\bar{D}_T^{aIJ} := [P_{TT} \cdot D]^{aIJ}, \quad [P_{TT}]_{bKL}^{aIJ} = \delta_b^a \bar{\eta}_{[K}^I \bar{\eta}_{L]}^J - \frac{2}{D-1} E^{a[I} \bar{\eta}_{[K}^J] E_{bL]}, \quad (4.20)$$

under which (4.19) drops out. The projector P_{TT} can be expressed purely in terms of π^{aIJ} using (2.32) and

$$E^{a[I} \bar{\eta}_{[K}^J] E_{bL]} \approx -\zeta \left(\pi^{aM[I} \bar{\eta}_{[K}^J] \pi_{bL]M} + \delta_b^a n^{[I} \delta_{[K}^J] n_{L]} \right). \quad (4.21)$$

We continue using again $D_a \pi \approx 0$

$$\begin{aligned} \bar{D}_T^{aIJ} &\approx \beta P_{TT} \left(\pi^{b[I|K|} \left[K_{bKL} \pi^{a|L|J]} + K_b{}^J{}_{L} \pi^a{}_{K}{}^L \right] \right) \\ &\approx -\beta \zeta P_{TT} \left(E^{b[I} K_b{}^J{}_{L} E^{aL]} \right) = -\beta \zeta E^{b[I} \bar{K}_{bT}^{J]L} E_L^a. \end{aligned} \quad (4.22)$$

Notice that the last line is indeed tracefree and transverse. We write (4.22) as

$$\bar{D}_T^{aIJ} = \frac{\beta}{4} F^{aIJ,bKL} \bar{K}_{bKL}^T, \quad F^{aIJ,bKL} = 4\zeta E^{b[I} \bar{\eta}^{J][L} E^{aK]}. \quad (4.23)$$

The tensor $F^{aIJ,bKL}$ can be seen as bilinear form on transverse tensors of type \bar{K}_{aIJ} and has the following inverse

$$(F^{-1})_{aIJ,bKL} = \frac{\zeta}{4} [Q_{ab} \bar{\eta}_{[K|[I} \bar{\eta}_{J]|L]} - 2E_{b[I} \bar{\eta}_{J][K} E_{aL]}], \quad (4.24)$$

that is $[F \cdot F^{-1}]_{bKL}^{aIJ} = \delta_b^a \bar{\eta}_{[K}^I \bar{\eta}_{L]}^J$. Using (2.32) and

$$E_{aI} E_{bJ} \approx \zeta [\pi_{aIM} \pi_{bJ}{}^M - \zeta Q_{ab} n_I n_J], \quad (4.25)$$

F^{-1} is completely expressed in terms of π^{aIJ} . The quadratic combination of \bar{K}^T to be removed from (4.13) can now be compactly written as

$$\begin{aligned} E^{bI} \bar{K}_{bJM}^T E^{aJ} \bar{K}_{aI}^T{}^M &= E^{b[I} \bar{\eta}^{N][M} E^{aJ]} \bar{K}_{bJM}^T \bar{K}_{aIN}^T \\ &= \frac{\zeta}{4} F^{aIN,bJM} \bar{K}_{aIN}^T \bar{K}_{bJM}^T = 4 \frac{\zeta}{\beta^2} (F^{-1})_{aIJ,bKL} \bar{D}_T^{aIJ} \bar{D}_T^{bKL}. \end{aligned} \quad (4.26)$$

We now have all the pieces we need. The appropriate Hamiltonian constraint for spacetime signature s is displayed in (2.3). We find

$$\begin{aligned} \sqrt{\det(q)} \mathcal{H} &= \zeta \left(F_{abIJ} \pi^{aIK} \pi^b{}_{K}{}^J + 4 \bar{D}_T^{aIJ} (F^{-1})_{aIJ,bKL} \bar{D}_T^{bKL} + \frac{1}{(D-1)^2} [D_b{}^a D_a{}^b - (D_c{}^c)^2] \right) \\ &\quad - s \frac{1}{\beta^2} \frac{1}{(D-1)^2} [D_b{}^a D_a{}^b - (D_c{}^c)^2]. \end{aligned} \quad (4.27)$$

This expression simplifies for $s = \zeta$ and $\beta = 1$ in which case the terms quadratic in $D_b{}^a$ precisely cancel. This is again similar to the situation in 3+1 dimensions. This special situation can also be obtained more directly starting from the Palatini formulation as we will see in [28].

5 Conclusion

In this paper, we succeeded to construct a Hamiltonian connection formulation of General Relativity in all spacetime dimensions $D+1 \geq 3$ based on the gauge group $\text{SO}(D+1)$ or $\text{SO}(1, D)$. In addition to the usual Gauß, spatial diffeomorphism and Hamiltonian constraints, there are simplicity constraints that dictate that the momentum conjugate to the connection is determined by a generalised D -bein. The theory can be constructed for all four possible combinations of the internal (ζ) and spacetime (s) signature. This is especially attractive with an eye towards quantisation because unique [19] background independent representations of spatially diffeomorphism invariant theories of connections with compact structure group exist in any dimension and have been studied in great detail (see e.g. [12] and references therein).

The techniques for quantising Gauß, spatial diffeomorphism and Hamiltonian constraint that have been developed in $3+1$ dimensions generalise to arbitrary dimensions as we will show in [29]. The simplicity constraint provides a challenge. A similar kind of constraint plays a prominent role in Spin Foam models [40, 41, 42] and various proposals for its quantisation have been made. The problem is that the quantum simplicity constraints tend to be anomalous. This is due to the fact that the classically commuting π^{aIJ} become non commuting operators in the quantum theory. In [30] we propose some new strategies for how to make progress on this issue. Eventually, the solution of the simplicity constraint will consist in a restriction on the set of labels for spin network functions.

The application of interest of the present work is of course in higher dimensional Supergravity theories. Here we have to face two new technical challenges: For Lorentzian Supergravity the action is formulated in terms of a Lorentzian internal metric which would naturally imply the choice $\text{SO}(1, D)$. Hence, in order to keep $\text{SO}(D+1)$ we must carefully disassemble the $\text{SO}(1, D)$ Clifford algebra and reassemble it into an $\text{SO}(D+1)$ Clifford algebra which turns out to be possible. The second challenge is that higher dimensional Supergravity theories depend next to the Rarita-Schwinger field also on higher p -form fields for which background independent Hilbert space representations first need to be developed.

While many interesting technical issues are not settled by our analysis, the present work and its continuation in the companion papers hopefully contribute to the development of a non perturbative definition of quantum (Super)gravity in any dimension.

Acknowledgements

NB and AT thank the German National Merit Foundation for financial support. The part of the research performed at the Perimeter Institute for Theoretical Physics was supported in part by funds from the Government of Canada through NSERC and from the Province of Ontario through MEDT.

A Independent Set of Simplicity Constraints

The result that one would like to prove is as follows:

Theorem.

Let π^{aIJ} be a tensor antisymmetric in I, J and $a = 1, \dots, D$; $I, J = 1, \dots, D + 1$ subject to the condition that for any non zero vector the D vectors π^a ; $\pi^{aI} := -\zeta \pi^{aI} n^J$ are linearly independent. Then it is possible to construct a tensor $E_I^a = E_I^a[\pi]$ with the following properties:

Let $n^I = n^I[\pi]$ be the unique normal satisfying $E_I^a n^I = 0$, $n^I n^J \eta_{IJ} = \zeta$ where ζ corresponds to the signature of η . Let $\bar{\pi}^{aIJ} = \bar{\eta}_K^I \bar{\eta}_L^J \pi^{aKL}$ with the transversal projectors $\bar{\eta}_J^I = \delta_J^I - \zeta n^I n_J$. Then $\bar{\pi}^{aIJ} = \bar{\pi}_T^{aIJ}$ is automatically tracefree with respect to E , that is, $\bar{\pi}^{aIJ} E_{aI} = 0$ where E_{aI} is uniquely defined by $E^{aI} E_{aJ} = \bar{\eta}_J^I$, $E^{aI} E_{bI} = \delta_b^a$. Furthermore $\pi^{aIJ} = \bar{\pi}^{aIJ} + 2n^{[I} E^{a|J]}$.

In what follows we describe some ideas towards a possible proof.

Given π^{aIJ} , let n^I be any unit vector to begin with and construct $\bar{\eta}_{IJ}$ as above. Define $E^{aI}[\pi, n] := -\zeta \pi^{aI} n^J$. Notice that automatically $E^{aI} n_I = 0$. Then we obtain the decomposition

$$\pi^{aIJ} = \bar{\pi}^{aIJ} + 2n^{[I} E^{a|J]}. \quad (\text{A.1})$$

It is interesting to note that the non zero vector (due to the assumed linear independence)

$$N_I[\pi, n] := \epsilon_{IJ_1 \dots J_D} \epsilon_{a_1 \dots a_D} E^{a_1 J_1}[\pi, n] \dots E^{a_D J_D}[\pi, n] \quad (\text{A.2})$$

coincides up to normalisation with n_I whatever π is. Furthermore

$$\eta^{IJ} N_I N_J = \zeta [D!]^2 \det(Q); \quad Q^{ab} := \eta^{IJ} E_I^a E_J^b, \quad (\text{A.3})$$

where $\zeta = \pm 1$ if η has Euclidean or Lorentzian signature respectively. In particular, we verify that for $\zeta = -1$, the vector N_I is timelike, null or spacelike if and only if n_I is.

The tensor $E_{aI}[\pi, n]$ is given, up to normalisation, by

$$E_{aI} \propto -\epsilon_{IJ_1 \dots J_D} \epsilon_{a a_2 \dots a_D} n^{J_1} E^{a_2 J_2} \dots E^{a_D J_D}. \quad (\text{A.4})$$

The condition that $\bar{\pi}^{aIJ}$ be tracefree with respect to E becomes

$$\bar{\pi}^{aIJ} E_{aI} = [\pi^{aIJ} - 2n^{[I} E^{a|J]}] E_{aI} = \pi^{aIJ} E_{aI} + D n^J = 0. \quad (\text{A.5})$$

We can reformulate this as the condition that $\pi^{aIJ} E_{aI}$ is longitudinal, the coefficient of proportionality then follows from the normalisations. We define the tensor

$$\kappa_{IJ_1 \dots J_D}[\pi] := \epsilon_{IK_1 \dots K_D} \epsilon_{a_1 \dots a_D} \pi^{a_1 K_1} n^{J_1} \dots \pi^{a_D K_D} n^{J_D}, \quad (\text{A.6})$$

which is totally symmetric in J_1, \dots, J_D and only depends on π , not on n . In terms of this tensor the tracefree condition becomes

$$\pi^{aIJ} E_{aI} \propto \kappa_{IJ_2 \dots J_D} n^I n^{J_2} \dots n^{J_D} \stackrel{!}{\propto} n^J. \quad (\text{A.7})$$

Using the normalisation condition we can write this as the equality

$$\kappa_{J_1 I J_3 \dots J_{D+1}} n^{J_1} n^{J_3} \dots n^{J_{D+1}} = \zeta n_I \kappa_{J_1 \dots J_{D+1}} n^{J_1} \dots n^{J_{D+1}}. \quad (\text{A.8})$$

This is a system of D independent non-polynomial equations of order $D+2$ for the D independent unknowns n^I , $I = 1, \dots, D$; $n^{D+1} = \pm \sqrt{1 - \zeta \sum_{I=1}^D (n^I)^2}$. We can turn this into an equivalent system of $D + 1$ homogeneous, polynomial equations of order $D + 2$ by

$$\kappa_{J_1 I J_3 \dots J_{D+1}} n_{J_2} n^{J_1} \dots n^{J_{D+1}} = n_I \kappa_{J_1 \dots J_{D+1}} n^{J_1} \dots n^{J_{D+1}}, \quad (\text{A.9})$$

which leaves the normalisation of n^I undetermined.

Up to this point, n was just an extra structure independent of and next to π . The idea is now to fix n^I in terms of π by solving the system (A.8). Having determined $n^I = n^I[\pi]$ would then yield the desired tensor $E^{aI}[\pi] := E^{aI}[\pi, n[\pi]]$. However, it is far from clear whether a solution exists, nor that it is unique, although the number of independent equations matches with the number of degrees of freedom to be fixed. Being polynomial, it is clear that *complex* solutions of (A.9) exist, but the system of equations is far too complex in order to see whether *real* solutions exist. Hence to secure at least existence, we must resort to different methods.

Since the polynomial formulation (A.9) is of no help, we stick with (A.8). We write it in the form

$$n^I = \zeta \frac{f^I(n)}{n_J f^J(n)}, \quad f_I(n) = \kappa_{J_1 \dots J_{D+1}} n^{J_1} n^{J_2} \dots n^{J_{D+1}}. \quad (\text{A.10})$$

This equation takes the form of a *fixed point equation* $x = f(x)$. In order to apply established theorems, (A.10) is not useful because fixed point theorems typically are for compact sets and the right hand side of (A.10) has not manifestly bounded range, especially for signature $\zeta = -1$.

Consider instead the function

$$F^I(n) := \|n\| [2 - \|n\|] \frac{f^I(n)}{\|f(n)\|}. \quad (\text{A.11})$$

Notice that we use the *Euclidean* metric in both numerator and denominator also for the case $\zeta = -1$, i.e. $\|n\|^2 := \delta_{IJ} n^I n^J$. Let us now restrict n^I to the compact (closed and bounded) and convex⁴ $(D+1)$ -ball

$$B_{D+1} := \{n \in \mathbb{R}^{D+1}; \delta_{IJ} n^I n^J \leq 1\}. \quad (\text{A.12})$$

The map (A.11) is a continuous map from B_{D+1} to itself. To see this, notice that $\|f(n)\| \neq 0$ except at $\|n\| = 0$. This follows from the identity

$$f_I(n) n^I = N_I(n) n^I, \quad (\text{A.13})$$

which for $\zeta = 1$ and for $\zeta = -1$ and n not null shows that $\|f\| \neq 0$ unless $\|n\| = 0$. For $\zeta = -1$ and n null we have in fact $f_I = \gamma n_I$ with $\gamma \neq 0$. To see this, notice that the span of the E^{aI} , $a = 1, \dots, D$ contains n^I . Introduce some basis of the orthogonal complement, say b_α^I , $\alpha = 2, \dots, D$ and let $b_1^I = n^I$ so that $\eta_{IJ} b_\alpha^I b_\beta^J = \delta_{\alpha\beta}$, $\alpha, \beta = 2, \dots, D$ and $b_1^I b_{\alpha I} = 0$. Then we have an expansion $E^{aI} = r^{a\alpha} b_\alpha^I$ with $\det(r) \neq 0$ due to linear independence by assumption. Next, there exists $\gamma \neq 0$ such that

$$\epsilon_{IJ_1 \dots J_D} b_{\alpha_1}^{J_1} \dots b_{\alpha_D}^{J_D} = \gamma n_I \epsilon_{\alpha_1 \dots \alpha_D}, \quad (\text{A.14})$$

since the b_α are linearly independent and the left hand side of (A.14) is orthogonal to all of

⁴Suppose that $\|u\|, \|v\| \leq 1$ then

$$\|su + (1-s)v\|^2 = s^2\|u\|^2 + (1-s)^2\|v\|^2 + 2s(1-s) \langle u, v \rangle \leq s^2\|u\|^2 + (1-s)^2\|v\|^2 + 2s(1-s)\|u\| \|v\| \leq 1$$

for any $s \in [0, 1]$ due to the Cauchy Schwarz inequality.

them, hence it must be null. We can therefore compute

$$\begin{aligned}
f_I &= \delta_1^{\alpha_1} b_{\alpha_1}^{J_1} \epsilon_{J_1 K_1 \dots K_D} \epsilon_{a_1 \dots a_D} \pi^{a_1 K_1} {}_I E^{a_2 K_2} \dots E^{a_D K_D} \\
&= -\delta_1^{\alpha_1} [\epsilon_{J K_1 \dots K_D} b_{\alpha_1}^{K_1} \dots b_{\alpha_D}^{K_D}] \epsilon_{a_1 \dots a_D} \pi^{a_1 J} {}_I r^{a_2 \alpha_2} \dots r^{a_D \alpha_D} \\
&= -\gamma \epsilon_{1 \alpha_2 \dots \alpha_D} n_J \epsilon_{a_1 \dots a_D} \pi^{a_1 J} {}_I r^{a_2 \alpha_2} \dots r^{a_D \alpha_D} \\
&= \gamma \epsilon_{1 \alpha_2 \dots \alpha_D} \epsilon_{a_1 \dots a_D} E_I^{a_1} r^{a_2 \alpha_2} \dots r^{a_D \alpha_D} \\
&= \gamma \epsilon_{1 \alpha_2 \dots \alpha_D} b_{\beta I} \epsilon_{a_1 \dots a_D} r^{a_1 \beta} r^{a_2 \alpha_2} \dots r^{a_D \alpha_D} \\
&= \gamma \epsilon_{1 \alpha_2 \dots \alpha_D} b_{\beta I} \epsilon_{\beta \alpha_2 \dots \alpha_{D-1}} \det(r) \\
&= \gamma [(D-1)!] \det(r) n_I.
\end{aligned} \tag{A.15}$$

It follows that (A.11) is everywhere well defined except possibly at $\|n\| = 0$ where the fraction $f(n)/\|f(n)\|$ is ill defined. However, due to the prefactor $\|n\|$ we see that $F(n) := 0$ at $\|n\| = 0$ is a continuous extension. Next

$$\|F(n)\| = \|n\|(2 - \|n\|) = 1 - [1 - \|n\|]^2 \in [0, 1], \tag{A.16}$$

hence F maps B_{D+1} to itself. By the *Brouwer Fixed Point Theorem* [43] applicable to compact convex subsets of Euclidean space, it has a fixed point, that is, the equation $n^I = F^I(n)$ has at least one solution, a fixed point $n^I = n_*^I[\pi]$. Unfortunately, this is not very helpful because $n = 0$ is a trivial fixed point and the Brouwer fixed point theorem does not tell us anything about the number of fixed points, hence it could be that $n = 0$ is the only one. However, notice that $\|F(n)\| \geq \|n\|$ and $\|F(n)\| = \|n\| \Leftrightarrow \|n\| = 1$. This suggests that if a fixed point can be found by iteration $n_{k+1} := F(n_k)$ then it will lie on the sphere S^D . Indeed for $\|n\| = 1$ the map F maps S^D to itself.

In order to see whether a fixed point can be obtained by using iteration methods we estimate for $n_1, n_2 \in S^D$

$$\begin{aligned}
\|F(n_1) - F(n_2)\|^2 &= 2[1 - \frac{\langle f_1, f_2 \rangle}{\|f_1\| \|f_2\|}]^2 \\
&= \frac{1}{\|f_1\| \|f_2\|} [\|f_1 - f_2\|^2 - [\|f_1\| - \|f_2\|]^2] \\
&\leq \frac{\|f_1 - f_2\|^2}{\|f_1\| \|f_2\|}.
\end{aligned} \tag{A.17}$$

Now

$$D_{IJ} := \partial f_I / \partial n_J = [\kappa_{J I J_2 \dots J_D} + (D-1) \kappa_{I J J_2 \dots J_D}] n^{J_2} \dots n^{J_D}. \tag{A.18}$$

It follows $D_{IJ} n^J = D f_I$. For n_2 sufficiently close to n_1 , we obtain with the Cauchy Schwarz inequality

$$\|f_2 - f_1\|^2 \approx \|D(n_1)[n_2 - n_1]\|^2 \leq \text{Tr}(D^T(n_1)D(n_1)) \|n_2 - n_1\|^2 \tag{A.19}$$

and for $\|n\| = 1$ again due to the Cauchy Schwarz inequality

$$\text{Tr}(D^T D) = \sum_I \{ [\sum_J [D_{IJ}]^2] \|n\|^2 \} \geq \sum_I [\sum_J D_{IJ} n^J]^2 \geq D^2 \|f\|^2. \tag{A.20}$$

Thus the right hand side of (A.17) is given by $q(n_1, n_2) \|n_2 - n_1\|^2$, where $q(n, n) \geq D^2$. Hence F fails to be a contraction map and we cannot invoke techniques familiar from the *Banach Fixed Point Theorem* [43] in order to prove existence of a fixed point as this would need $\sup_{n_1, n_2} q(n_1, n_2) < 1$. Either sharper bounds are needed or we have to use a different iteration function (recall that fixed point equations can be written in many different but equivalent

ways and for some of them the iteration map maybe contractible, for others not). Notice that as long as we are only interested in obtaining $f(n) \propto n$ we may rescale f by a sufficiently large constant such that f itself becomes a contraction map and maps B_{D+1} to itself. This is possible because f and the matrix defined by $f(n_2) - f(n_1) = D(n_2, n_1) \cdot (n_2 - n_1)$ are continuous maps on the compact sets B_{D+1} and $B_{D+1} \times B_{D+1}$ respectively and thus are uniformly bounded. However, due to (A.20) the required constant would turn f into a strictly norm decreasing map and thus can only have $n = 0$ as a fixed point.

Remark:

In contrast to D odd, for D even there are two natural vectors that one construct purely from π namely

$$u_I = \kappa_{IJ_1 \dots J_D} \eta^{J_1 J_2} \dots \eta^{J_{D-1} J_D}, \quad v_I = \kappa_{J_1 I J_2 \dots J_D} \eta^{J_1 J_2} \dots \eta^{J_{D-1} J_D}. \quad (\text{A.21})$$

These are the only independent contractions that exist because $\kappa_{IJ_1 \dots J_D}$ is completely symmetric in its J indices. It is natural to assume that the fixed point vector is a linear combination of u, v and indeed in $D = 2$ it is easy to see that $n \propto u$. For $D \geq 4$ we were not able to verify this by direct calculation or any other means due to the complexity of the fixed point equation.

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