

# From General Relativity to Quantum Gravity

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In general relativity (GR), spacetime geometry is no longer just a background arena but a physical and dynamical entity with its own degrees of freedom. We present an overview of approaches to quantum gravity in which this central feature of GR is at the forefront. However, the short distance dynamics in the quantum theory are quite different from those of GR and classical spacetimes and gravitons emerge only in a suitable limit. Our emphasis is on communicating the key strategies, the main results and open issues. In the spirit of this volume, we focus on a few avenues that have led to the most significant advances over the past 2-3 decades.<sup>1</sup>

PACS numbers: 04.60.Pp, 04.60.Ds, 04.60.Nc, 03.65.Sq

## I. INTRODUCTION

The necessity of reconciling general relativity (GR) with quantum physics was recognized by Einstein [1] already in 1916 when he wrote:

*“Nevertheless, due to the inner-atomic movement of electrons, atoms would have to radiate not only electro-magnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in Nature, it appears that quantum theory would have to modify not only Maxwellian electrodynamics, but also the new theory of gravitation.”*

Yet, almost a century later, we still do not have a satisfactory reconciliation. Why is the problem so difficult? An obvious response is that this is because there are no observations

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<sup>1</sup> To appear in *General Relativity and Gravitation: A Centennial Survey*, commissioned by the International Society for General Relativity and Gravitation and to be published by Cambridge University Press. Abhay Ashtekar served as the ‘coordinating author’ and combined the three contributions.

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to guide us. However, this cannot be the entire story because, if there are no observational constraints, one would expect an overabundance of theories, not scarcity!

The viewpoint in approaches discussed in this Chapter is that the primary obstacle is rather that, among fundamental forces of Nature, gravity is special: it is encoded in the very geometry of spacetime. This is a central feature of GR, a crystallization of the equivalence principle that lies at the heart of the theory. Therefore, one argues, it should be incorporated at a fundamental level in a viable quantum theory. The perturbative treatments which dominated the field since the 1960s ignored this aspect of gravity. They assumed that the underlying spacetime can be taken to be a continuum, endowed with a smooth background geometry, and the quantum gravitational field can be treated as any other quantum field on this background. But the resulting quantum GR turned out to be non-renormalizable; the strategy failed by its own criteria. The new strategy is to free oneself of the background spacetime that seemed indispensable for formulating and addressing physical questions; the goal is to lift this anchor and learn to sail the open seas. This task requires novel mathematical techniques and conceptual frameworks. From the perspective of this Chapter, we do not yet have a satisfactory quantum gravity theory primarily because serious attempts to meet these challenges squarely are relatively recent. However, as our overview will illustrate, the community *has* made notable advances towards this goal in recent years.

In this Chapter, we will focus on two main programs, each of which in turn has two related but distinct parts: i) *Loop Quantum Gravity* (LQG) whose Hamiltonian or canonical framework is well suited for cosmological issues, and whose Spinfoam or covariant framework is geared to address scattering theory [2–6]; and, ii) the *Asymptotic Safety* paradigm which includes the Effective Average Action Framework with its functional renormalization group equation in the continuum, and the Causal Dynamical Triangulation Approach in which one uses numerical simulations a la lattice gauge theory [7–10]. (String Theory is discussed in Chapter 12 and other approaches in the Introduction to Part IV.)

A common theme in these programs is that their starting point is the physical, dynamical spacetime geometry of GR. However, as will be clear from the detailed discussion, this does not imply a conventional quantization of GR. In LQG, for example, the fundamental quanta of geometry are one dimensional, polymer-like excitations over nothing, rather than gravitons, the wavy undulations over a continuum background. In particular, classical general relativity is recovered only in an appropriate coarse-grained limit. Another common theme is that these programs first focus on geometry and rely on non-perturbative effects—rather than specific matter couplings—to cure the ultraviolet difficulties of perturbative quantum GR. The viewpoint is that the short distance behavior of quantum geometry is qualitatively different from that suggested by the continuum picture and it would be more efficient to first develop a detailed understanding of quantum geometry in the Planck regime and then couple matter in a second stage. In the Asymptotic Safety scenario, for example, this strategy was successfully implemented first for pure gravity, and incorporating certain matter fields afterwards did not change the basic picture [11]. Finally, as in QCD, the first priority in these programs is to uncover and explore qualitatively new, non-perturbative features of quantum gravity by focusing on just one interaction, rather than on achieving unification. Such features have already emerged. Examples are: a quantum resolution of singularities of GR [5, 12], finiteness of microstates of black hole and cosmological horizons [2, 13], and effective dimension reduction in the Planck regime [7–10].

Although these programs share several common elements, there are also some key differences in the underlying viewpoints. Let us begin with Asymptotic Safety. Recall first

that, although GR is perturbatively non-renormalizable, there does exist a well-developed and powerful effective field theory [14] which, for example, has been applied with remarkable success to the long standing problem of equations of motion of compact binaries in GR [15]. However, this theory abandons the idea of handling the Planck regime and focuses on low energy processes. Asymptotic Safety can be thought of as a specific *ultraviolet (UV) completion* of this effective field theory using Wilson’s generalized notion of renormalization [16]. The idea is to avoid the notorious proliferation of undetermined couplings in the UV faced by perturbative GR by using a reliable strategy that has already been successfully tested in well understood, *perturbatively non-renormalizable* field theories where one can, so to say, ‘renormalize the non-renormalizable’ [17]. This success suggests that a state of ‘peaceful coexistence’ of perturbative divergences with Asymptotic Safety may be possible also for gravity [7–9, 18].

In LQG the guiding principle is rather different. The viewpoint is that, just as Riemannian geometry is essential to the formulation of general relativity, an appropriate *quantum* Riemannian geometry should underlie a viable theoretical account of space, time, and gravitation that does not disregard quantum theory. To meet this goal, a specific quantum theory of geometry was constructed in detail drawing motivation from geometric structures that underlie the phase space of GR [2–5]. In this and subsequent constructions one makes a heavy use of non-perturbative techniques that have already been successful in gauge theories but with a crucial twist: now there is no reference to a background metric. This requirement of background independence is surprisingly powerful and leads to a unique kinematical framework [19, 20] on which dynamics of the quantum theory is being built. While the Hamiltonian LQG has broad similarities with the older Wheeler-DeWitt (WDW) theory [21], the quantum nature of underlying geometry makes a key difference leading, for example, to a natural resolution of classical singularities in cosmological models [12]. Similarly, Spinfoams provide transition amplitudes that are UV finite to any order in a natural expansion. Furthermore, a *positive* cosmological constant provides a natural mechanism for regulating their infrared (IR) behavior [5, 6].

Thus, although both LQG and Asymptotic Safety programs have similar goals, the physical concepts and mathematical techniques used in subsequent analysis are quite different. In particular, because the quantum geometry underlying LQG is fundamentally discrete, the physical degrees of freedom terminate at the Planck scale, much like in string theory. In the Asymptotic Safety program, on the other hand, there is no kinematic reason that would prevent degrees of freedom at arbitrarily small scale. A first reading of the flow equations suggests that there are physical degrees of freedom at any scale, all the way to the infinitely small. However, it is the fixed point action that determines the physical degrees of freedom in this approach. Non-perturbative renormalizability indicates that these are fewer than what one would expect classically and the mean field considerations indicate that there are at most as many as in a theory in 2 space-time dimensions. A more thorough understanding of the fixed point is necessary to settle this important question in the Asymptotic Safety program.

This Chapter is organized as follows. Section II provides a broad brush overview of the two programs. Since this volume is likely to draw readership from diverse quarters, we have made a special attempt to make the sub-sections self-contained. Thus a reader interested only in Asymptotic Safety can skip sections II B and II C and a reader interested only in LQG can skip section II A without loss of continuity. Section III discusses illustrative applications to cosmology of the very early universe, black holes physics and scattering theory. While

advances over the past decade are encouraging, a large number of issues remain. These are discussed in section IV.

## II. FRAMEWORKS

This section is divided into three parts. The first summarizes the main ideas and results in the Asymptotic Safety program, the second, in Hamiltonian LQG and the third in Spinfoams.

### A. Asymptotic Safety

Since GR is not renormalizable in the standard perturbation theory, it is commonly argued that a satisfactory microscopic quantum theory of the gravitational interaction cannot be set up within the realm of quantum field theory without adding further symmetries, extra dimensions or new principles such as holography. In contrast, the Asymptotic Safety program [22] retains quantum field theory without such additions as the theoretical arena and instead abandons the traditional techniques of perturbative renormalization. Moreover, as we will see, in a certain sense it even abandons the standard notion of ‘quantization’ because its starting point is not a given classical model to be promoted to a quantum theory.

Rather, in its modern incarnation, this program may be thought of as a systematic *search strategy among theories that are already ‘quantum’*; it identifies the ‘islands’ of physically acceptable theories in the ‘sea’ of unacceptable ones plagued by short distance pathologies. Since the approach is based on Wilson’s generalized notion of renormalization [16] and the use of functional renormalization group (RG) equations, concepts from statistical field theory play an important role. They provide a unified framework for approaching the problem with both continuum and discrete methods. In this section we discuss two such complementary approaches within the Asymptotic Safety paradigm: the Effective Average Action (EAA) with its Functional renormalization group Equation (FRGE) [23], and Causal Dynamical Triangulations (CDT) [24].

#### 1. The Functional Renormalization Group

The goal of the Asymptotic Safety program consists in giving a mathematically precise meaning to, and actually computing functional integrals over ‘all’ spacetime metrics of the form  $\int \mathcal{D}\tilde{g}_{\mu\nu} \exp(iS[\tilde{g}_{\mu\nu}])$ , or

$$Z = \int \mathcal{D}\tilde{g}_{\mu\nu} e^{-S[\tilde{g}_{\mu\nu}]}, \quad (2.1)$$

from which all quantities of physical interest can be deduced then. Here  $S[\tilde{g}_{\mu\nu}]$  denotes the classical or, more appropriately, the bare action. It is required to be diffeomorphism invariant, but is kept completely arbitrary otherwise. In general it differs from the usual Einstein-Hilbert action. This generality is essential in the Asymptotic Safety scenario: the viewpoint is that the functional integral would exist only for a certain class of actions  $S$  and the task is to identify this class.

Following the approach proposed in [23] one attacks this problem in an indirect way: rather than dealing with the integral per se, one interprets it as the solution of a certain

differential equation, a functional renormalization group equation, or ‘FRGE’. The advantage is that, contrary to the functional integral, the FRGE is manifestly well defined. It can be seen as an ‘evolution equation’ in a mathematical sense, defining an infinite dimensional dynamical system in which the RG scale plays the role of time. Loosely speaking, this reformulation replaces the problem of defining functional integrals by the task of finding evolution histories of the dynamical system that extend to *infinitely late times*. According to the Asymptotic Safety conjecture the dynamical system possesses a fixed point which is approached at late times, yielding well defined, fully extended evolutions, which in turn tell us how to construct (or ‘renormalize’) the functional integral.

Let us start by explaining the passage from the functional integrals to the FRGE. Recall that in trying to put the integrals on a solid basis one is confronted with a number of obstacles:

(i) As in every field theory, difficulties arise since one tries to quantize *infinitely many degrees of freedom*. Therefore, at the intermediate steps of the construction one keeps only finitely many of them by introducing cutoffs at very small and very large distances,  $\Lambda^{-1}$  and  $k^{-1}$ , respectively. We shall specify their concrete implementation in a moment. The ultraviolet (UV) and infrared (IR) cutoff scales  $\Lambda$  and  $k$ , respectively, have the dimension of a mass, and the original system is recovered for  $\Lambda \rightarrow \infty$ ,  $k \rightarrow 0$ .

(ii) Conceptually, the most severe problem one encounters when quantizing the gravitational field, one which is not shared by any conventional matter field theory, is the requirement of *background independence*: no particular spacetime (such as Minkowski space, say) should be given a privileged status. Rather, the geometry of spacetime should be determined dynamically. In the approach to Asymptotic Safety along the lines of [23] this problem is dealt with by following the spirit of DeWitt’s background field method [25] and introducing a (classical, non-dynamical) background metric  $\bar{g}_{\mu\nu}$  which, however, is kept absolutely arbitrary. One then decomposes the integration variable as  $\tilde{g}_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}$ , and interprets  $\mathcal{D}\tilde{g}_{\mu\nu}$  as an integration over the nonlinear fluctuation,  $\mathcal{D}\tilde{h}_{\mu\nu}$ . In this way one arrives at a conceptually easier task, the quantization of the matter-like field  $\tilde{h}_{\mu\nu}$  in a generic, but classical background  $\bar{g}_{\mu\nu}$ . The availability of the background metric is crucial at various stages of the construction of an FRGE. However the final physical results do not depend on the choice of a specific background.

(iii) As in every gauge field theory, the *redundancy of gauge-equivalent field configurations* (diffeomorphic metrics) has to be carefully accounted for. Here we employ the Faddeev-Popov method and add a gauge fixing term  $S_{\text{gf}} \propto \int \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu$  to  $S$  where  $F_\mu \equiv F_\mu(\tilde{h}; \bar{g})$  is chosen such that the condition  $F_\mu = 0$  picks a single representative from each gauge orbit. The resulting volume element on orbit space, the Faddeev-Popov determinant, we express as a functional integral over Grassmannian ghost fields  $\tilde{C}^\mu$  and  $\tilde{\bar{C}}_\mu$ , governed by an action  $S_{\text{gh}}$ . In this way the original integral (2.1) gets replaced by  $\tilde{Z}[\bar{\Phi}] = \int \mathcal{D}\tilde{\Phi} \exp(-S_{\text{tot}}[\tilde{\Phi}, \bar{\Phi}])$ . Here the total bare action  $S_{\text{tot}} \equiv S + S_{\text{gf}} + S_{\text{gh}}$  depends on the dynamical fields  $\tilde{\Phi} \equiv (\tilde{h}_{\mu\nu}, \tilde{C}^\mu, \tilde{\bar{C}}_\mu)$ , the background fields  $\bar{\Phi} \equiv (\bar{g}_{\mu\nu})$ , and possibly also on (both dynamical and background) matter fields, which for simplicity are not included here.

Using the gauge fixed and regularized integral we can compute arbitrary ( $\bar{\Phi}$ -dependent!) expectation values  $\langle \mathcal{O}(\tilde{\Phi}) \rangle \equiv \tilde{Z}^{-1} \int \mathcal{D}\tilde{\Phi} \mathcal{O}(\tilde{\Phi}) e^{-S_{\text{tot}}[\tilde{\Phi}, \bar{\Phi}]}$ , for instance  $n$ -point functions where  $\mathcal{O}$  consists of strings  $\tilde{\Phi}(x_1)\tilde{\Phi}(x_2)\cdots\tilde{\Phi}(x_n)$ . For  $n = 1$  we use the notation  $\Phi \equiv \langle \tilde{\Phi} \rangle \equiv (h_{\mu\nu}, C^\mu, \bar{C}_\mu)$ , i.e. the elementary field expectation values are  $h_{\mu\nu} \equiv \langle \tilde{h}_{\mu\nu} \rangle$ ,  $C^\mu \equiv \langle \tilde{C}^\mu \rangle$  and  $\bar{C}_\mu \equiv \langle \tilde{\bar{C}}_\mu \rangle$ . Thus the full dynamical metric has the expectation value  $g_{\mu\nu} \equiv \langle \tilde{g}_{\mu\nu} \rangle = \bar{g}_{\mu\nu} + h_{\mu\nu}$ .

The dynamical laws which govern the expectation value  $\Phi(x)$  have an elegant description in terms of the *effective action*  $\Gamma$ . It is a functional depending on  $\Phi$  similar to the classical  $S[\Phi]$  to which it reduces in the classical limit. Requiring stationarity,  $S$  yields the classical field equation  $(\delta S/\delta\Phi)[\Phi_{\text{class}}] = 0$ , while  $\Gamma$  gives rise to a quantum mechanical analog satisfied by the expectation values, the *effective field equation*  $(\delta\Gamma/\delta\Phi)[\langle\tilde{\Phi}\rangle] = 0$ . If, as in the case at hand,  $\Gamma \equiv \Gamma[\Phi, \bar{\Phi}] \equiv \Gamma[h_{\mu\nu}, C^\mu, \bar{C}_\mu; \bar{g}_{\mu\nu}]$  depends also on background fields, the solutions to this equation inherit this dependence and so  $h_{\mu\nu} \equiv \langle\tilde{h}_{\mu\nu}\rangle$  functionally depends on  $\bar{g}_{\mu\nu}$ . Technically,  $\Gamma$  is obtained from a functional integral with  $S_{\text{tot}}$  replaced by  $S_{\text{tot}}^J \equiv S_{\text{tot}} - \int dx J(x)\tilde{\Phi}(x)$ . The new term couples the dynamical fields to an external, classical source,  $J(x)$ , and repeated functional differentiation  $(\delta/\delta J)^n$  of  $\ln \tilde{Z}[J, \bar{\Phi}]$  yields the  $n$ -point functions. In particular,  $\Phi = \delta \ln \tilde{Z}/\delta J$ . It is a standard result that  $\Gamma[\Phi, \bar{\Phi}]$  equals exactly the Legendre transform of  $\ln \tilde{Z}[J, \bar{\Phi}]$ , at fixed background fields  $\bar{\Phi}$ . The importance of  $\Gamma$  also resides in the fact that *it is the generating functional of special  $n$ -point functions from which all others can be easily reconstructed*. Therefore, finding  $\Gamma$  in some quantum field theory is often considered equivalent to completely ‘solving’ this theory.

To calculate  $\Gamma[\Phi, \bar{\Phi}]$  it is advantageous to employ a gauge breaking condition  $F_\mu$  which fixes a gauge belonging to the distinguished class of the so called *background gauges*. To see the benefit, recall that the original gauge transformations read  $\delta\tilde{g}_{\mu\nu} = \mathcal{L}_v\tilde{g}_{\mu\nu}$  where  $\mathcal{L}_v$  denotes the Lie derivative w.r.t. the vector field  $v$ . When we decompose  $\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu}$  we can distribute the gauge variation of  $\tilde{g}_{\mu\nu}$  in different ways over  $\bar{g}_{\mu\nu}$  and  $\tilde{h}_{\mu\nu}$ . In particular this gives rise to what is known as *quantum gauge transformations* ( $\delta^{\mathbf{Q}}\tilde{h}_{\mu\nu} = \mathcal{L}_v(\bar{g}_{\mu\nu} + \tilde{h}_{\mu\nu})$ ,  $\delta^{\mathbf{Q}}\bar{g}_{\mu\nu} = 0$ ) and *background gauge transformations* ( $\delta^{\mathbf{B}}\tilde{h}_{\mu\nu} = \mathcal{L}_v\tilde{h}_{\mu\nu}$ ,  $\delta^{\mathbf{B}}\bar{g}_{\mu\nu} = \mathcal{L}_v\bar{g}_{\mu\nu}$ ). Since the functional integral is defined by fixing an externally prescribed background metric,  $\bar{g}_{\mu\nu}$ , we must ensure invariance under the ‘ordinary’ or ‘true’ gauge transformations the Faddeev-Popov method deals with. Hence it is the  $\delta^{\mathbf{Q}}$ -invariance which needs to be gauge-fixed by the condition  $F_\mu = 0$ . Interestingly enough, there exist  $F_\mu$ ’s, a variant of the harmonic coordinate condition, for example, which indeed fix the  $\delta^{\mathbf{Q}}$ -transformations, but at the same time are *invariant under  $\delta^{\mathbf{B}}$ -transformations*:  $\delta^{\mathbf{B}}F_\mu = 0$ . They implement the background gauges, and from now on we assume that we employ one of those. Then, as a consequence, the effective action  $\Gamma[\Phi, \bar{\Phi}]$  *is invariant under background gauge transformations* which include the ghosts:  $\delta^{\mathbf{B}}\Gamma[\Phi, \bar{\Phi}] = 0$  for all  $\delta^{\mathbf{B}}\Phi = \mathcal{L}_v\Phi$ ,  $\delta^{\mathbf{B}}\bar{\Phi} = \mathcal{L}_v\bar{\Phi}$ . We emphasize that this property should not be confused with another notion of ‘gauge independence’ which the above  $\Gamma[\Phi, \bar{\Phi}]$  actually does *not* have: It is not independent of which particular  $F_\mu$  is picked from the class with  $\delta^{\mathbf{B}}F_\mu = 0$ . This  $F_\mu$ -dependence will disappear only at the level of observables.

Turning now to the concept of a *functional renormalization group equation* recall that the above definition of  $\Gamma$  is based on the functional integral regularized in the IR and UV, hence it depends on the corresponding cutoff scales:  $\Gamma \equiv \Gamma_{k,\Lambda}[\Phi, \bar{\Phi}]$ . It is this object for which we derive a FRGE, more precisely a closed evolution equation governing its dependence on the IR cutoff scale  $k$ . This is possible only if the IR regularization is implemented appropriately, as in the so called *effective average action* (EAA) [26].

The EAA is related to the modified integral,  $\int \mathcal{D}\tilde{\Phi} e^{-S_{\text{tot}}^J} e^{-\Delta S_k[\tilde{\Phi}, \bar{\Phi}]} \equiv Z_{k,\Lambda}[J, \bar{\Phi}]$  whose second exponential factor in the integrand, containing the *cutoff action*  $\Delta S_k$ , is designed to achieve the IR regularization. To see how this works, assume the integration variable  $\tilde{\Phi} = (\tilde{h}, \tilde{C}, \tilde{\bar{C}})$  is expanded in terms of eigenfunctions  $\varphi_p$  of the covariant tensor Laplacian related to the background metric,  $\bar{D}^2 \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$ . Writing  $-\bar{D}^2\varphi_p = p^2\varphi_p$  we have, symbolically,  $\tilde{\Phi}(x) = \sum_p \alpha_p\varphi_p(x)$ . The  $\alpha_p$ ’s are generalized Fourier coefficients, and so the functional

integration over  $\tilde{\Phi}$  amounts to integrating over all  $\alpha_p$ :

$$Z_{k,\Lambda}[J, \bar{\Phi}] = \prod_{p^2 \in [0, \Lambda^2]} \int_{-\infty}^{\infty} d\alpha_p \exp\left(-S_{\text{tot}}^{J'}[\{\alpha\}, \bar{\Phi}]\right) \quad (2.2)$$

Here  $S_{\text{tot}}^{J'}$  equals  $S_{\text{tot}}^J[\tilde{\Phi}, \bar{\Phi}] + \Delta S_k[\tilde{\Phi}, \bar{\Phi}]$  with the expansion for  $\tilde{\Phi}$  inserted. In (2.2) we implemented the UV regularization by retaining only eigenfunctions (or ‘modes’) corresponding to  $-\bar{D}^2$ -eigenvalues (or squared ‘momenta’) smaller than  $\Lambda^2$ . The IR contributions, i.e. those corresponding to eigenvalues between  $p^2 = 0$  and about  $p^2 = k^2$  are cut off smoothly instead, namely by a  $p^2$ -dependent suppression factor arising from  $\Delta S_k$ . To obtain a structurally simple FRGE,  $\Delta S_k$  should be chosen quadratic in the dynamical fields. Usually one sets  $\Delta S_k = \frac{1}{2} \int dx \tilde{\Phi} \mathcal{R}_k \tilde{\Phi}$  with an operator  $\mathcal{R}_k \propto k^2 R^{(0)}(-\bar{D}^2/k^2)$  containing a dimensionless function  $R^{(0)}$ . In the  $-\bar{D}^2$ -basis we have then  $\Delta S_k \propto k^2 \sum_p R^{(0)}(p^2/k^2) \alpha_p^2$  which shows that  $\Delta S_k$  represents a kind of  $p^2$ -dependent mass term: A mode with eigenvalue  $p^2$  acquires a (mass)<sup>2</sup> of the order  $k^2 R^{(0)}(p^2/k^2)$ . We require  $R^{(0)}(p^2/k^2)$  to have the qualitative properties of a smeared step function which, around  $p^2/k^2 \approx 1$ , drops smoothly from  $R^{(0)} = 1$  for  $p^2/k^2 \lesssim 1$  to  $R^{(0)} = 0$  for  $p^2/k^2 \gtrsim 1$ . This achieves precisely the desired IR regularization: In the product over  $p^2$  in (2.2),  $\Delta S_k$  equips all  $\int d\alpha_p$ -integrals pertaining to the *low momentum modes*, i.e. those with  $p^2 \in [0, k^2]$ , with a Gaussian suppression factor  $e^{-k^2 \alpha_p^2}$  since for such eigenvalues  $R^{(0)}(p^2/k^2) \approx 1$ . The *high momentum modes*, having  $p^2 \in [k^2, \Lambda^2]$ , yield  $R^{(0)}(p^2/k^2) \approx 0$  and so they remain unaffected by  $\Delta S_k$ . At least on a flat background, low (high) momentum modes  $\varphi_p(x)$  have long (short) wavelengths. Therefore, when one lowers  $k$  from  $k = \Lambda$  down to  $k = 0$  one ‘un-suppresses’ modes of increasingly long wavelengths, thus proceeding from the UV to the IR. (In FRGE jargon, this is called the ‘integrating out’ of the high momentum modes since in older approaches the low momentum modes were completely discarded, rather than just suppressed.) This process of encoding the contribution of an increasing number of modes in a scale dependent, or ‘running’ functional is precisely a *renormalization in the modern sense* due to Wilson [16].

The effective average action,  $\Gamma_{k,\Lambda}[\Phi, \bar{\Phi}]$ , is defined to be the Legendre transform of  $\ln Z_{k,\Lambda}[J, \bar{\Phi}]$  given by (2.2), with respect to  $J$ , for  $k$ ,  $\Lambda$ , and  $\bar{\Phi}$  fixed (and with  $\Delta S_k[\Phi, \bar{\Phi}]$  subtracted from the result of the transformation, which is not essential here). As for the  $\Phi$ ,  $\bar{\Phi}$ -arguments, we stress that the modes classified low or high momentum are only those of the *fluctuation* field,  $\tilde{\Phi}$ . The externally prescribed background and source fields  $\bar{\Phi}(x)$  and  $J(x)$ , which are also present under the integral defining  $Z_{k,\Lambda}[J, \bar{\Phi}]$ , have nonzero Fourier coefficients for all  $p^2 \in [0, \Lambda^2]$  in general, they may contain both high and low momentum components. As a consequence, the same is true for the  $\Phi$ -argument of the EAA, since  $J$  and  $\Phi = \delta \ln Z_{k,\Lambda} / \delta J$  are Legendre-conjugates of one another.

The EAA,  $\Gamma_{k,\Lambda}[\Phi, \bar{\Phi}]$ , has a number of important features not realized in other functional RG approaches:

- (i) Since no fluctuation modes are taken into account in the  $k = \Lambda \rightarrow \infty$  limit, the EAA approaches the bare (i.e., un-renormalized) action,  $\Gamma_{\Lambda,\Lambda} = S_{\text{tot}}$ . In the limit  $k \rightarrow 0$ , it yields the standard effective action (with an UV cutoff).
- (ii) It satisfies a closed FRGE, and can be computed by integrating this FRGE towards low  $k$ , with the initial condition  $\Gamma_{\Lambda,\Lambda} = S_{\text{tot}}$  at  $k = \Lambda$ .
- (iii) The functional  $\Gamma_{k,\Lambda}[\Phi, \bar{\Phi}]$  is invariant under background gauge transformations  $\delta^{\mathbf{B}}$  for all values of the cutoffs. This property is preserved by the FRGE: the RG evolution does not generate  $\delta^{\mathbf{B}}$ -noninvariant terms.

(iv) The FRGE continues to be well behaved when the UV cutoff is removed ( $\Lambda \rightarrow \infty$ ). Denoting solutions to the UV cutoff-free FRGE by  $\Gamma_k[\Phi, \bar{\Phi}]$ , it reads:

$$k\partial_k\Gamma_k[\Phi, \bar{\Phi}] = \frac{1}{2} \text{STr} \left[ \left( \Gamma_k^{(2)}[\Phi, \bar{\Phi}] + \mathcal{R}_k[\bar{\Phi}] \right)^{-1} k\partial_k\mathcal{R}_k[\bar{\Phi}] \right] \quad (2.3)$$

Here  $\text{STr}$  denotes the functional supertrace, and  $\Gamma_k^{(2)}$  stands for the matrix of second functional derivatives of  $\Gamma_k$  with respect to  $\Phi$  at fixed  $\bar{\Phi}$ . Since  $R^{(0)}$  is essentially a step function, the derivative  $\partial_k\mathcal{R}_k$  is nonzero only in a thin shell of momenta near  $p^2 = k^2$ , and so the supertrace on the RHS of (2.3) receives contributions only from such momenta. As a result, it is perfectly finite both in the IR and the UV, and this is why sending  $\Lambda \rightarrow \infty$  was unproblematic.

(v)  $\Gamma_k$  is closely related to a generating functional for field averages over finite domains of size  $k^{-1}$ ; hence the name EAA [26]. Thanks to this property, when treated as a *classical* action  $\Gamma_k$  can provide an effective field theory description of *quantum* physics involving typical momenta near  $k$ . This property has been exploited in numerous applications of the EAA to particle and condensed matter physics, but it plays no role in the present context. Rather, it is its interpolating property between  $S$  and  $\Gamma_{k=0}$  which is instrumental in the Asymptotic Safety program.

The arena in which the RG dynamics takes place is the infinite dimensional *theory space*,  $\mathcal{T}$ . It consists of all well behaved action functionals  $(\Phi, \bar{\Phi}) \mapsto A[\Phi, \bar{\Phi}]$  which depend on a given set of fields and are invariant under some symmetry group possibly. In metric gravity  $\mathcal{T}$  comprises arbitrary  $\delta^{\mathbf{B}}$  invariant functionals  $A[g_{\mu\nu}, \bar{g}_{\mu\nu}, C^\mu, \bar{C}_\mu]$ . The RHS of the FRGE (2.3) defines a vector field  $\beta$  on  $\mathcal{T}$ . Its natural orientation is such that  $\beta$  points from higher to lower momentum scales  $k$ , from the UV to the IR. (This is the direction of increasing ‘coarse-graining’ in which the microscopic dynamics is ‘averaged’ over increasingly large spacetime volumes.) The integral curves of this vector field,  $k \mapsto \Gamma_k$ , are the *RG trajectories*, and the pair  $(\mathcal{T}, \beta)$  is called the *RG flow*. It constitutes the dynamical system alluded to earlier.

One usually assumes that every  $A \in \mathcal{T}$  can be expanded as  $A[\Phi, \bar{\Phi}] = \sum_{\alpha=1}^{\infty} \bar{u}_\alpha P_\alpha[\Phi, \bar{\Phi}]$  where the set  $\{P_\alpha\}$  forms a basis of invariant functionals. Writing the RG trajectory correspondingly,  $\Gamma_k[\Phi, \bar{\Phi}] = \sum_{\alpha=1}^{\infty} \bar{u}_\alpha(k) P_\alpha[\Phi, \bar{\Phi}]$ , one encounters infinitely many *running coupling constants*,  $\bar{u}_\alpha(k)$ , whose  $k$ -dependence is governed by an infinite coupled system of differential equations:  $k\partial_k\bar{u}_\alpha(k) = \bar{\beta}_\alpha(\bar{u}_1, \bar{u}_2, \dots; k)$ . The dimensionful *beta functions*  $\bar{\beta}_\alpha$  arise by expanding the RHS of the FRGE:  $\frac{1}{2}\text{STr}[\dots] = \sum_{\alpha=1}^{\infty} \bar{\beta}_\alpha P_\alpha[\Phi, \bar{\Phi}]$ . The coefficients  $\bar{\beta}_\alpha$  are similar to the familiar beta functions of perturbative quantum field theory (where, however, only the finitely many beta functions of the relevant couplings are considered.)

Reexpressing the RG equations in terms of dimensionless couplings  $u_\alpha \equiv k^{-d_\alpha}\bar{u}_\alpha$  with  $d_\alpha$  the canonical mass dimension of  $\bar{u}_\alpha$ , the resulting *FRGE in component form* is autonomous, i.e. its  $\beta$ -functions have no explicit  $k$ -dependence:  $k\partial_k u_\alpha(k) = \beta_\alpha(u_1(k), u_2(k), \dots)$ . The coupling constants ( $u_\alpha \equiv u$ ) serve as local coordinates on  $\mathcal{T}$ , and the  $\beta_\alpha$ ’s are the components of the vector field  $\beta \equiv (\beta_\alpha(u))$ .

Later on *fixed points* of the RG flow will be of special interest. At a fixed point,  $\beta = 0$ , so its coordinates ( $u_\alpha^* = u^*$ ) satisfy the infinitely many conditions  $\beta_\alpha(u^*) = 0$ . The fixed point’s *UV critical hypersurface*,  $\mathcal{S}_{\text{UV}}$ , or synonymously its *unstable manifold* is defined to consist of all points in  $\mathcal{T}$  which are pulled into the fixed point under the inverse RG flow, i.e. for increasing scale  $k$ . Linearizing the flow about  $u^*$  one has  $k\partial_k u_\alpha(k) = \sum_\gamma B_{\alpha\gamma} (u_\gamma(k) - u_\gamma^*)$  with the *stability matrix*  $\mathbf{B} = (B_{\alpha\gamma})$ ,  $B_{\alpha\gamma} \equiv \partial_\gamma\beta_\alpha(u^*)$ . If the eigenvectors of  $\mathbf{B}$  form a basis, its solution reads  $u_\alpha(k) = u_\alpha^* + \sum_\alpha C_I V_\alpha^I (k_0/k)^{\theta_I}$ . Here the  $C_I$ ’s are constants

of integration and the  $V^I$ 's denote the right-eigenvectors of  $\mathbf{B}$  with eigenvalues  $-\theta_I$ , i. e.  $\sum_\gamma B_{\alpha\gamma} V_\gamma^I = -\theta_I V_\alpha^I$ . In general  $\mathbf{B}$  is not symmetric and the *critical exponents*  $\theta_I$  are complex. Along eigendirections with  $\text{Re } \theta_I > 0$  ( $\text{Re } \theta_I < 0$ ) deviations from  $u_\alpha^*$  grow (shrink) when  $k$  is lowered from the UV towards the IR; they are termed relevant (irrelevant).

A trajectory  $u_\alpha(k)$  within  $\mathcal{S}_{\text{UV}}$ , by definition, approaches  $u_\alpha(k \rightarrow \infty) = u_\alpha^*$  in the UV. For the constants  $C_I$  in its linearization this implies that  $C_I = 0$  for all  $I$  with  $\text{Re } \theta_I < 0$ . Hence the trajectories in  $\mathcal{S}_{\text{UV}}$  are labeled by the remaining  $C_I$ 's related to the critical exponents with  $\text{Re } \theta_I > 0$ . (For simplicity we assume all  $\text{Re } \theta_I$  nonzero.) As a consequence, *the dimensionality of the critical hypersurface,  $s \equiv \dim(\mathcal{S}_{\text{UV}})$ , equals the number of critical exponents with  $\text{Re } \theta_I > 0$ , i.e., the number of relevant directions.*

A fixed point is called *Gaussian* if it corresponds to a free field theory. Its critical exponents agree with the canonical mass dimension of the corresponding operators. A fixed point whose critical exponents differ from the canonical ones is referred to as nontrivial or as a *non-Gaussian fixed point* (NGFP).

## 2. Asymptotic Safety

The construction of a quantum field theory involves finding an RG trajectory which is infinitely extended in the sense that it is a curve, entirely within theory space, with well defined limits  $k \rightarrow 0$  and  $k \rightarrow \infty$ , respectively. *Asymptotic Safety is a proposal for ensuring the existence of the second limit.* Its crucial prerequisite is a nontrivial RG fixed point  $\Gamma_*$  on  $\mathcal{T}$ . Let us assume there is such a fixed point. Then it is sufficient to simply pick any of the trajectories within its hypersurface  $\mathcal{S}_{\text{UV}}$  to be sure that the trajectory has a singularity free ultraviolet behavior since it will always hit the fixed point for  $k \rightarrow \infty$ . There exists a  $\dim(\mathcal{S}_{\text{UV}})$ -parameter family of such trajectories.

Most probably an UV fixed point is not only sufficient but also necessary for an acceptable theory without divergences. Therefore, in the simplest case when there exists only one, the physically inequivalent asymptotically safe quantum theories one can construct are labeled by the  $\dim(\mathcal{S}_{\text{UV}})$  parameters characterizing trajectories inside  $\mathcal{S}_{\text{UV}}$ . Thus the degree of predictivity of asymptotically safe theories is essentially determined by the number of *relevant* eigendirections at  $\Gamma_*$ . If this is a finite number  $s \equiv \dim(\mathcal{S}_{\text{UV}})$ , it is sufficient to measure only  $s$  of the couplings  $\{u_\alpha(k)\}$  characterizing  $\Gamma_k$  in order to predict the infinitely many others. In particular, at  $k = 0$  the standard effective action  $\Gamma \equiv \Gamma_0$  is obtained which ‘knows’ all possible predictions.

The only input required for this construction is the theory space  $\mathcal{T}$ , that is the field contents and the symmetries. It fully determines the FRGE and its fixed point properties. Since  $\Gamma_{k \rightarrow \infty}$  is closely related to the bare action  $S$ , the Asymptotic Safety program essentially consists in *computing*  $S \sim \lim_{k \rightarrow \infty} \Gamma_k = \Gamma_*$  from the fixed point condition. In this sense the approach amounts to a selection process among quantum theories rather than the quantization of a classical system known beforehand. It has become customary to call *Quantum Einstein Gravity*, or QEG, any quantum field theory of metric-based gravity, regardless of its bare action, which is defined by a trajectory on the theory space  $\mathcal{T}_{\text{QEG}}$  of diffeomorphism invariant functionals  $A[g_{\mu\nu}, \bar{g}_{\mu\nu}, C^\mu, \bar{C}_\mu]$ .

A priori the functional integral over ‘all’ metrics is only formal and plagued by mathematical problems. Knowing  $\Gamma_*$  and the RG flow in its vicinity one can give a well defined meaning to it. The only extra ingredient that needs to be selected is an UV regularization

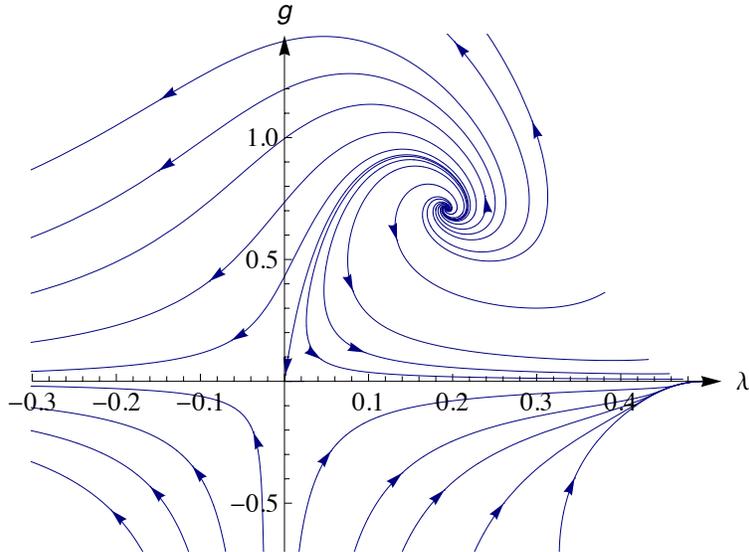


Figure 1: RG flow of the Einstein-Hilbert truncation on the  $(g, \lambda)$ -plane. The arrows point towards decreasing scales  $k$ . (First obtained in [28]).

for the integral. It is then possible to use the information encoded in the flow of  $\Gamma_k$  near  $\Gamma_*$  in order to determine how the (‘bare’) parameters, on which the integral depends, must be tuned in order to obtain a meaningful limit when the UV regulator is removed [27]. Thus the mathematical subtleties of the functional integral are overcome if the long time-behavior of the associated dynamical system on  $\mathcal{T}_{\text{QEG}}$  can be controlled, e.g. by means of a fixed point. For an evolution equation as complicated as the FRGE, on an infinite dimensional theory space, it is by no means clear from the outset that this is possible, i.e. that there exist RG trajectories that extend to infinite values of the evolution parameter. An essential part of the Asymptotic Safety program consists in demonstrating that this is indeed the case, for the concrete reason that the trajectory hits a fixed point in the long time-limit.

Practical computations require a nonperturbative approximation scheme. The method of choice consists in a *truncation of theory space*. One sets all but a certain subset of couplings  $u_\alpha$  to zero, and expands  $\Gamma_k$  in terms of the appropriately chosen reduced set  $\{P_\alpha, \alpha = 1, \dots, N\}$  where, as before,  $P_\alpha$  is a basis of invariant functionals in terms of which now only the actions in the truncated theory space can be expanded. Hence the FRGE boils down to a system of  $N$  coupled differential equations. This amounts to a severe restriction, of course, which needs to be justified a posteriori by systematically changing and enlarging the subset chosen. This difficulty is not specific to gravity; the same strategy is followed in FRGE-based investigations of matter field theories on flat space and in statistical physics.

At the time Weinberg conjectured the possibility of Asymptotic Safety, due to the lack of nonperturbative computational techniques, a NGFP was known to exist only for a single coupling, Newton’s constant, and only in  $d = 2 + \epsilon$  spacetime dimensions [22]. The situation changed when the EAA-based methods became available [23]. Starting from early work on the ‘Einstein-Hilbert truncation’ [23, 29] and a generalization with an additional  $R^2$ -term [30], a considerable number of truncations with increasingly large subsets  $\{P_\alpha\}$  were analyzed in the following decade [31]. Quite remarkably, they all agree in that *QEG indeed seems to possess a NGFP suitable for the Asymptotic Safety construction*. Although a complete

proof is not within reach, by now there is highly nontrivial evidence for a NGFP on the full (un-truncated) theory space, rendering QEG nonperturbatively renormalizable [7–9]. As a representative example, Fig. 1 shows the phase portrait of the Einstein-Hilbert truncation [23] based upon the running action  $\Gamma_k = \frac{1}{16\pi G_k} \int d^d x \sqrt{g} (-R(g) + 2\Lambda_k) + S_{\text{gf}} + S_{\text{gh}}$  which has  $N = 2$ . It involves the approximation of neglecting the  $k$ -dependence in the gauge fixing and ghost sectors which can be justified by BRST methods [23]. This ansatz contains a running Newton constant  $G_k$  and cosmological constant  $\Lambda_k$ , their dimensionless analogs being, in  $d$  spacetime dimensions,  $g(k) \equiv k^{d-2}G_k$  and  $\lambda(k) \equiv \Lambda_k/k^2$ , respectively. Their beta functions  $(\beta_g, \beta_\lambda) \equiv \beta$  have been computed for any  $d$  [23]. The first steps of the calculation are reminiscent of those in perturbatively quantized general relativity [32] but this is a coincidence due to the specific ansatz for  $\Gamma_k$ . Moreover,  $\beta_g$  and  $\beta_\lambda$  are quite different from beta-functions in perturbation theory. They sum up contributions from arbitrary orders of perturbation theory and, what is more important, they contain also information about the strong power law-type renormalization effects which are not seen usually in perturbative calculations employing dimensional regularization. This is important however, for instance in order to ‘tame’ the notorious quadratic (and higher) divergences due to the non-zero mass dimension of  $G$ .

Fig. 1 shows the flow diagram obtained by solving the coupled equations  $k\partial_k g(k) = \beta_g(g, \lambda)$  and  $k\partial_k \lambda(k) = \beta_\lambda(g, \lambda)$  for  $d = 4$  [28]. Besides a Gaussian fixed point at  $g_* = 0 = \lambda_*$  there is indeed a second, *non-Gaussian* fixed point at  $g_* > 0, \lambda_* > 0$ . Both of its critical exponents have a positive real part. Hence  $s \equiv \dim S_{\text{UV}} = 2$ . In sufficiently general truncations one finds that  $s < N$ , and the reduced dimensionality then allows us to predict  $N - s$  couplings after  $s$  couplings have been measured. The predictions are encoded in the way the  $s$ -dimensional hypersurface  $\mathcal{S}_{\text{UV}}$  is immersed into the (truncated or complete) theory space. There are general arguments suggesting that  $s$  should saturate at a small finite value when  $N$  is increased [22]. Concrete calculations confirmed this picture, first in  $d = 2 + \epsilon$  where the ‘ $\Lambda + R + R^2$  truncation’, having  $N = 3$ , yields a NGFP with  $s = 2$  [30]. So, given two input parameters, the third one is a prediction. One might for instance express the coefficient of the term  $\int \sqrt{g} R^2$  added to the Einstein-Hilbert action in terms of  $g$  and  $\lambda$ . In  $d = 4$ , all known truncations confirmed that the projection of the flow onto the  $g$ - $\lambda$ -plane has the same structure as in Fig. 1, with ‘perpendicular’ directions added. By now, there exist very impressive analyses of  $f(R)$  truncations, with  $f$  a polynomial of high degree. They do indeed display the expected stabilization of  $s$  at a small finite value when  $N$  is made large [38]. Furthermore, first explorations of infinite dimensional truncated theory spaces were performed [33] and truly functional flows in non-polynomial  $f(R)$  truncations are within reach now [34]. Trying to make the truncations more accurate it is not sufficient to generalize their  $g_{\mu\nu}$ -dependence only; at the same time we must also allow for a more general dependence of  $\Gamma_k$  on the background metric. The first results on such ‘bi-metric truncations’ which treat the  $g_{\mu\nu}$ - and  $\bar{g}_{\mu\nu}$ -dependence on a similar footing further support the viability of the Asymptotic Safety program [35]. The same is true for a different type of generalization, the inclusion of scale dependent surface terms into  $\Gamma_k$  for spacetimes with boundaries [36]. There is yet another important, but technically difficult generalization, namely *non-local* terms. They are particularly important in the infrared where they are expected to cure a problem of the Einstein-Hilbert truncation not visible in Fig. 1: singularities of the beta-functions at  $\lambda = 1/2$  which indicate that the truncation becomes insufficient in the IR. In [37] a simple but genuinely ‘functional’ flow of a non-local EAA was analyzed which turned out to possess an infrared fixed point, i.e. a non-local ‘fixed functional’.

Besides a better understanding of the RG flow in QEG, future work will also have to address the question of observables. The running couplings parameterizing the EAA have no direct physical significance in general. While under very special circumstances it might be possible to deduce observable effects directly from the  $k$ -dependence of certain couplings (by some kind of ‘RG improvement’), the general strategy is to first construct the functional integral, then find interesting observables in terms of the fundamental fields, and finally compute their expectation values. In this respect the status of observables within the Asymptotic Safety program is not different from any other functional integral based approach.

### 3. Causal Dynamical Triangulations

The partition functions of standard model-like quantum field theories, analytically continued to Euclidean space and discretized, have been extensively studied by Monte Carlo techniques. It is therefore natural to apply similar ideas to gravity and to attempt a definition of the formal functional integral (2.1) as the  $a \rightarrow 0$  limit of the partition function belonging to a suitably chosen statistical mechanics model, specified by a choice of dynamical variables, bare action  $S[g]$  and measure  $\mathcal{D}g$ . Here the discretization scale  $a$  is analogous to a lattice spacing. A priori the ‘lattice units’ defined by  $a$  are unphysical; they can be converted to physical lengths or masses only later when it comes to computing observables.

The limit  $a \rightarrow 0$  is to be taken indirectly, as follows. The statistical system has a chance of describing physics in the continuum if  $a$  can be made much smaller than any relevant physical length scale  $\ell$ , or more adequately, if all lengths  $\ell$  are much larger than  $a$ . In fact, in numerical simulations where  $a$  is necessarily nonzero ( $a = 1$ , say) the requirement  $(\ell/a) \gg 1$  is met if the free parameters of the statistical model (bare couplings) are tuned such that its correlation length diverges and  $\ell$ , in lattice units, becomes very large. Thus the *continuum limit*  $a/\ell \rightarrow 0$  amounts to  $\ell \rightarrow \infty$  with  $a$  fixed (rather than  $a \rightarrow 0$  and  $\ell$  fixed). As it is well known from the statistical physics of critical phenomena, for instance, the correlation length does indeed diverge at second order phase transition points. So the strategy will be to propose a plausible statistical model, compute numerically its partition function in dependence of the bare parameters, and search for points in parameter space where the correlation length diverges. If such a critical point exists one would use it to define a continuum theory and explore its properties.

The statistical systems underlying critical phenomena are conveniently analyzed in terms of their RG flow under successive ‘coarse graining’. While there is considerable ambiguity in how this is done concretely, it typically boils down to a space averaging of the degrees of freedom (block spin transformation, etc.) which, in a continuum language, amounts to a step-by-step integrating out of field modes with increasing wavelengths. In this setting systems at second order phase transition points, displaying no preferred length scale, are described by fixed points of the RG flow.

This observation brings us back to Asymptotic Safety: The discrete system describes a continuum theory when its bare parameters are tuned to their fixed point values. Then the partition function  $Z$  is a sum over contributions from fluctuations whose wavelengths, in physical units, range from zero to infinity. In EAA language this amounts to specifying a *complete* trajectory  $\Gamma_k$ , well behaved in particular in the UV since  $\lim_{k \rightarrow \infty} \Gamma_k = \Gamma_*$ . One can show that  $\Gamma_*$  is indeed very closely related to the RG fixed point of the statistical model, and that the large  $k$  behavior of  $\Gamma_k$ , with minimal additional input, can be mapped onto

the RG flow of the model near the second order phase transition point [27].

The CDT approach [10, 39, 40] is a specific proposal for a statistical system representing gravity. It sums over the class of piecewise linear 4-geometries which can be assembled from 4-dimensional simplicial building blocks (with link length  $a$ ) in such a way that the resulting spacetime is ‘causal’ in a certain technical sense. A priori the spacetimes  $\mathcal{M}$  summed over have Lorentzian signature. However, to ensure the existence of a generalized Wick rotation they are restricted to be globally hyperbolic which allows introducing a global proper-time foliation,  $\mathcal{M} = I \times \Sigma$ , where  $I$  denotes a ‘time’ interval and space is represented by 3-dimensional leaves  $\Sigma$  whose topology is not allowed to change in time. A choice extensively studied is  $\Sigma = S^3$  so that at each proper-time step  $t_n \in I$  the spatial geometry is represented by a triangulation of  $S^3$ . It is made up of equilateral spatial tetrahedra with positive squared side-length  $\ell_s^2 \equiv a^2 > 0$ . The number  $N_3(t)$  of tetrahedra, and the way they are glued together to form a piecewise flat 3-dimensional manifold will change in general when we go from  $t = t_n$  to the next time slice at  $t_{n+1}$ . In order to constitute a 4-dimensional triangulation, the 3-dimensional slices must be connected in a ‘causal’ way, preserving the  $S^3$ -topology at all intermediate times. (This ensures that a branching of the spatial universe into several disconnected pieces (baby universes) does not occur.) For the gluing of two consecutive time slices  $S^3(t_n)$  and  $S^3(t_{n+1})$  it is sufficient to introduce four types of 4-simplices, namely the so-called (4,1)-simplices, which have 4 of its vertices on  $S^3(t_n)$  and 1 on  $S^3(t_{n+1})$ , the (3,2)-simplices with 3 vertices on  $S^3(t_n)$  and 2 on  $S^3(t_{n+1})$ , as well as (1,4)- and (2,3)- simplices defined the other way around. The integration over spacetimes  $\mathcal{M}$  boils down to a sum over all possible ways to connect given triangulations of  $S^3(t_n)$  and  $S^3(t_{n+1})$  compatible with the topology  $I \times S^3$ , along with a summation over all 3-dimensional triangulations of  $S^3(t)$ , at all times  $t$ .

Denoting by  $\ell_t$  and  $\ell_s$  the length of the time-like and the space-like links, respectively, one has  $\ell_t^2 = -\alpha\ell_s^2$  where the constant  $\alpha$  is positive in the Lorentzian case, whence  $\ell_t^2 < 0$ . It was shown [39] that there exists a well defined rotation in the complex  $\alpha$  plane ( $\alpha \rightarrow -\alpha$ ) which, thanks to the restriction to a given foliation in the simplicial decomposition, connects the Lorentzian to the Euclidean signature, with  $\ell_t^2 = |\alpha|\ell_s^2 > 0$ . This turns oscillating exponentials  $e^{iS}$  into Boltzmann factors  $e^{-S}$ , so that the resulting partition function can be computed with Monte Carlo integration methods. It reads [39]:

$$Z(\kappa_0, \kappa_4, \Delta) = \sum_T \frac{1}{C_T} \exp\left(-S_{\text{Regge}}[T]\right) \quad (2.4)$$

The symmetry factor  $C_T$  equals the order of the automorphism group of the triangulation  $T$ , and  $S_{\text{Regge}}$  is the Regge-discretized Einstein-Hilbert action:  $S_{\text{Regge}} = -(\kappa_0 + 6\Delta)N_0 + \kappa_4(N_4^{(4,1)} + N_4^{(3,2)}) + \Delta(2N_4^{(4,1)} + N_4^{(3,2)})$ . Here  $N_4^{(4,1)}$  and  $N_4^{(3,2)}$  denote the number of (4,1)- and (3,2)-simplices in  $T$ , respectively, and  $N_0$  is the total number of vertices. The couplings  $\kappa_0$  and  $\kappa_4$  correspond to  $1/G$  and  $\Lambda/G$ , respectively, and  $\Delta$  parameterizes a possible asymmetry between  $\ell_t$  and  $\ell_s$ ; it is nonzero if  $|\alpha| \neq 1$ .

Extensive Monte Carlo simulations of the partition function (2.4) have been performed at a number of points in the space of bare couplings  $(\kappa_0, \kappa_4, \Delta)$ . Three different phases were discovered, and one of them seems indeed capable of representing continuum physics. A surface in parameter space,  $\kappa_4 = \kappa_4(\kappa_0, \Delta)$ , has been identified on which the 4-volume becomes large. This ‘infinite’ volume limit should however not be confused with the continuum limit. The crucial question is whether the latter can actually be realized à la Asymptotic Safety by tuning the remaining two parameters to a second order phase transition point. The answer

is not known yet, but this is the topic of very active current research.

The most important result of the CDT model is that it is able to describe the emergence of a classical 4-dimensional de Sitter universe with small superimposed quantum fluctuations. The calculation is carried out in the Euclidean signature, but thanks to the above  $\alpha$ -rotation it admits a Lorentzian interpretation. The reason this result is interesting is that it resolves a difficulty of previous attempts to address quantum gravity with dynamical triangulations: the 4-d Euclidean triangulation models without the ‘causality’ constraint produced only states with Hausdorff dimensions  $d_H = 2$  and  $d_H = \infty$ , respectively, contradicting the classical limit.

In these CDT simulations the link length  $a$  is still as large as about 2 Planck lengths so they do not yet probe the physics on sub-Planckian length scales [40]. Once simulations well beyond the Planck scale become feasible they should be able to make contact with the RG fixed point predicted by the EAA based calculations in the continuum. Indeed, it has been shown already [41, 42] that the CDT and EAA predictions for the running spectral dimension agree quite precisely in the semiclassical regime. It is also known how, at least in principle, the information about the  $k$ -dependence of the EAA can be used to predict the expected RG running of a statistical model near the continuum limit [27]. In this respect it should also be mentioned that while most EAA studies have been performed for *Euclidean signature*, they also apply to the *Lorentzian* case almost unchanged [43]. It will be very interesting to see whether future Monte Carlo results lead to the same picture of physics near the fixed point as the FRGE studies.

Finally, CDT breaks Lorentz invariance because the spatial and temporal cut-offs are independent. There is no general argument that Lorentz invariance must be restored in the continuum limit. If it is not restored, the classical limit of the continuum theory might not be general relativity, but rather something akin to the Hořava–Lifshitz theory, which is non-Lorentz invariant [44]. Hořava–Lifshitz theory is renormalizable, and if it were physically viable, it would represent a possible solution to the quantum gravity problem.

## B. Hamiltonian Theory and Quantum Geometry

The *Asymptotic Safety* program generalizes the procedures that have been successful in Minkowskian quantum field theories (MQFTs) by going beyond traditional perturbative treatments. An avenue that is even older is canonical quantization, pioneered by Dirac, Bergmann, Arnowitt, Deser, Misner and others. Over the past 2-3 decades, these ideas have inspired a new approach, known as *Loop Quantum Gravity* (LQG).

While the point of departure is again a Hamiltonian framework, as explained in section I, there is an important conceptual shift: the idea now is to construct a *quantum* theory of geometry and then use it to formulate quantum gravity systematically. This theory was constructed in detail in the 1990s. Since then, research in LQG has progressed along two parallel avenues. In the first, discussed in this sub-section, one continues the development of the canonical quantization program, now using quantum geometry to properly handle the field theoretical issues. In the second, discussed in the next sub-section, one develops a path integral framework and defines dynamics via transition amplitudes between quantum 3-geometries. In the final picture, the fundamental degrees of freedom are *quite different* from those that would result in a ‘direct’ quantization of GR –they are not metrics and extrinsic curvatures but chunks, or atoms, of space with quantum attributes. Classical geometries emerge only upon coarse graining of their coherent superpositions.

This subsection is divided into three parts. The first summarizes the Hamiltonian framework that provides the point of departure, the second explains the basic structure of quantum geometry and the third sketches the status of quantum dynamics in canonical LQG.

### 1. Connection Dynamics

The key idea underlying the Hamiltonian framework used in LQG is to cast GR in the language of gauge theories that successfully describe the electroweak and strong interactions. This requires a shift from metrics to connections; Wheeler’s ‘geometrodynamics’ [21] is replaced by a dynamical theory of spin-connections [45]. Once this is achieved, the phase space of general relativity becomes *the same* as that of gauge theories: All four fundamental forces of Nature are unified at a kinematical level. However, dynamics of GR has two distinguishing features. First, whereas the Hamiltonian of QED or QCD uses the flat background metric, the Hamiltonian constraints that generate dynamics of GR are built entirely from the spin connection and its conjugate momentum; the theory is manifestly *background independent*. Second, the gauge group now refers to rotations in the physical space rather than in an abstract, internal space. This is why in contrast to, say, QCD, *spacetime* geometry can now emerge from this gauge theory. As we will see, these two features have a powerful consequence: one is led to a *unique* quantum Riemannian geometry.

Fix a 3-manifold  $M$  which is to represent a Cauchy surface in spacetime. The gravitational phase space  $\Gamma$  is coordinatized by pairs  $(\underline{A}_a^j, E_j^a)$  of an  $SU(2)$  connection  $\underline{A}_a^j$  and its conjugate ‘electric field’  $E_j^a$  on  $M$ , where  $j$  refers to the Lie algebra  $\mathfrak{su}(2)$  of  $SU(2)$  and  $a$  to the tangent space of  $M$ . Thus, the fundamental Poisson brackets are:

$$\{\underline{A}_a^j(x), E_k^a\} = -i\kappa_N \delta_a^b \delta_k^j \delta^3(x, y) \quad (2.5)$$

where  $\kappa_N = 8\pi G_N$  is the gravitational coupling constant. As remarked above, although the phase space variables have the familiar Yang-Mills form, they also admit a natural interpretation in terms of spacetime geometry. To spell it out, let us first recall from Chapter 8 that the standard Cauchy data of GR consists of a pair,  $(q_{ab}, K_{ab})$ , representing the intrinsic positive definite metric  $q_{ab}$  and the extrinsic curvature  $K_{ab}$  on  $\Sigma$ . If we denote by  $e_j^a$  an orthonormal triad—a ‘square root’ of  $q^{ab}$ —then in the Lorentzian signature we have:

$$E_j^a = \sqrt{q} e_j^a \quad \text{and} \quad \underline{A}_a^j = \Gamma_a^j - \iota K_a^j \quad (2.6)$$

where  $q$  denotes the determinant of the metric  $q_{ab}$ ,  $\Gamma_a^j$  is the intrinsic spin connection on  $M$  defined by  $e_j^a$ ,  $K_a^j = K_{ab} e_j^b$  and  $\iota = 1$  in the Euclidean signature and  $\iota = i (\equiv \sqrt{-1})$  in the Lorentzian signature (used in most of this Chapter). The connection  $\underline{A}_a^j$  parallel transports left handed (or unprimed) spacetime spinors. In the final solution, its curvature  $\underline{F}_{ab}^j := 2\partial_{[a} \underline{A}_{b]}^j + \epsilon^{jkl} \underline{A}_{ak} \underline{A}_{bl}$  represents the (pull-back to  $M$  of the) self-dual part of the spacetime Weyl curvature.<sup>1</sup>

<sup>1</sup> For simplicity we assume that  $M$  is compact; in the asymptotically flat case, one has to specify appropriate boundary conditions at infinity and keep track of boundary terms. See. e.g., [2, 4, 45]. Since the electric field  $E_i^a$  is a density of weight 1, mathematically, it is often simpler to work with its dual  $\Sigma_{ab}^j := \eta_{abc} E_j^c$ , which is just a 2-form on  $M$ . Finally, as is standard in Yang-Mills theories, the internal indices  $j, k, \dots$  are raised and lowered using the Cartan-Killing metric on  $\mathfrak{su}(2)$ .

As is well-known, dynamics of GR is generated by a set of constraints. While they are rather complicated and non-polynomial functionals of the geometrodynamical ADM variables, they become low order polynomials in the connection variables. In absence of matter sources, they are [45]:

$$\mathcal{G}_j := \underline{D}_a E_j^a = 0, \quad \mathcal{D}_b := E_j^b \underline{F}_{ab}^j = 0, \quad \text{and} \quad \mathcal{H} := \epsilon^{jkl} \left( \underline{F}_{abl} - \Lambda \eta_{abc} E_l^c \right) E_j^a E_k^b = 0 \quad (2.7)$$

The first constraint is just the familiar Gauss law of Yang Mills theory, the second is the Diffeomorphism constraint of GR, and the third the Hamiltonian constraint. Interestingly, these are the simplest gauge invariant, local expressions one can construct from a connection and its conjugate electric field *without* reference to a background metric. Indeed, these are the *only* such expressions that are at most quartic in the canonical variables  $A_a^i, E_j^a$ . At first one might expect that it would be difficult to couple matter to gravity using these connection variables since they refer only to the self dual part of spacetime curvature. But this is not the case; one can couple spin zero, half and one fields keeping the simplicity [46, 47] and recently the framework has also been extended to include higher dimensions [48], and supersymmetry [49].

Since the constraints are polynomial in the connection variables, so are the equations of motion. Furthermore, the framework represents a small extension of GR: Since, in contrast to the ADM variables, none of the equations require us to invert  $E_j^a$ , they remain viable even when  $E_j^a$  become degenerate. At these phase space points one no longer has a (non-degenerate) spacetime metric but connection dynamics continues to remain meaningful. The standard causal structures have been extended to such configurations [50]. Finally, the connection dynamics framework provides a natural setting for proofs of the positive energy theorems a la Witten [51]; one can establish the positivity of the gravitational Hamiltonian not only on the constraint surface as in the original theorems but also in a neighborhood of the constraint surface, i.e., even ‘off-shell’ [52].

As we noted after Eq. (2.6), in the Lorentzian signature the connection  $\underline{A}_a^i$  is complex valued, or, equivalently, it a 1-form that takes values in the Lie algebra of  $\mathbb{C}SU(2)$ , the complexification of  $SU(2)$ . While this feature does not create any obstacle at the classical level, a key mathematical difficulty arises in the passage to quantum theory: Because  $\mathbb{C}SU(2)$  is *non-compact*, the space of connections  $\underline{A}_a^j$  is not known to carry diffeomorphism invariant measures that are necessary to construct a satisfactory Hilbert space of square integrable functions of connections. To bypass this difficulty, the main-stream strategy has been to replace the complex, left handed connections  $\underline{A}_a^j$  with *real*  $SU(2)$  connections  $A_a^j$ , obtained by replacing  $i$  in (2.5) and (2.6) by a real, non-zero parameter  $\gamma$  [53]. Then, both the phase space variables are real and the fundamental Poisson-brackets become

$$\{\underline{A}_a^j(x), E_k^a\} = \gamma \kappa_N \delta_a^b \delta_k^j \delta^3(x, y). \quad (2.8)$$

$\gamma$  is known as the Barbero-Immirzi parameter [54] and taken to be positive for definiteness. As we will see in section II B 2, one can introduce well-defined measures on the space  $\mathcal{A}$  of these real connections  $A_a^j$  and develop rigorous functional analysis to introduce the quantum Hilbert space and operators without any reference to a background geometry. This passage from left handed to real connections represents a systematic generalization of the Wick rotation one routinely performs to obtain well-defined measures in MQFTs. However, the rotation is now performed in the ‘internal space’ rather than spacetime. Indeed, the *spacetime* Wick rotation does not naturally extend to general curved spacetimes while this internal Wick rotation does and serves the desired purpose of taming the functional integrals.

However, the strategy has two limitations. First, the form of the constraints (and evolution equations) is now considerably more complicated. But thanks to several astute techniques introduced by Thiemann [4, 55], these complications can be handled in the canonical approach, and they are not directly relevant to spin foams. The second limitation is that, while the connection  $A_a^j$  is well-defined on  $M$  and continues to have a simple relation to the ADM variables, it does not have a natural 4-dimensional geometrical interpretation in solutions to the field equations [56]. Nonetheless, one *can* arrive at the canonical pair  $(A_a^j, E_j^a)$  by performing a Legendre transform of a 4-dimensionally covariant action  $S(e, \omega)$  that depends on a space-time co-tetrad  $e_\mu^I$  and a Lorentz connection  $\omega_\mu^{IJ}$  [57].

## 2. Quantum Riemannian Geometry

The first step in the passage to quantum theory is to select a preferred class of *elementary phase space functions* which are to be directly promoted to operators in the quantum theory without factor ordering ambiguities. In geometrodynamics, these are taken to be the positive definite 3-metric  $q_{ab}$  on  $M$  and its conjugate momentum,  $P^{ab} = \sqrt{q} (K^{ab} - Kq^{ab})$  (integrated against suitable test fields). In connection dynamics the choice is motivated by structures that naturally arise in gauge theories. Thus, the configuration variables are now the *Wilson lines, or holonomies*  $h_\ell$  which enable one to parallel transport left handed spinors along 1-dimensional (curves or) *links*  $\ell$  in  $M$ , and the conjugate momenta are the ‘*electric field fluxes*’  $E_{f,S}$  across 2-dimensional *surfaces*  $S$  (smeared with test fields  $f^i$  that take values in  $\text{su}(2)$ ) [2–4, 58–60]:

$$h_\ell := \mathcal{P} \exp \int_\ell A, \quad \text{and} \quad E_{f,S} := \int_S d^2 S_a f^i(x) E_i^a(x). \quad (2.9)$$

Note that the definitions do not require a background geometry; since  $A$  is an  $\text{su}(2)$ -valued 1-form, it can be naturally integrated along 1-dimensional links to yield  $h_\ell \in \text{SU}(2)$ , and since  $E$  is the Hodge-dual of a  $(\text{su}(2)$ -valued) 2-form the second integral is also well-defined without any background fields. However, the Poisson brackets between these variables fail to be well-defined if  $\ell$  and  $S$  are allowed to have an infinite number of intersections. Therefore they have to satisfy certain regularity conditions. Two natural strategies are to use piecewise linear links and 2-surfaces or piecewise analytic ones (more precisely, ‘semi-analytic’ in the sense of [2, 19, 61]). The first choice is well-adapted to the simplicial decompositions often used in Spinfoam models while the second is commonly used in canonical LQG.

Formal sums of products of these elementary operators  $\hat{h}_\ell$  and  $\hat{E}_{n,S}$  generate an abstract algebra  $\mathfrak{A}$ . This is the analog of the familiar Heisenberg algebra in quantum mechanics and one’s first task is to find its representations. The Hilbert space  $\mathcal{H}_{\text{grav}}^{\text{kin}}$  underlying the chosen representation would then serve as the space of kinematical quantum states, the quantum analog of the gravitational phase  $\mathbf{\Gamma}$  of GR, the arena to formulate dynamics.

In quantum mechanics, von-Neumann’s theorem guarantees that the Heisenberg algebra admits a unique representation satisfying certain regularity conditions (see, e.g., [62]). However, in MQFTs, because of the infinite number of degrees of freedom, this is not the case in general: The standard result on the uniqueness of the Fock vacuum assumes *free field dynamics* [63, 64]. What is the situation with the algebra  $\mathfrak{A}$  of LQG? Now, in addition to the standard regularity condition, we can and *have to* impose the strong requirement of background independence. A fundamental and surprising result due to Lewandowski, Okolow, Sahlmann, and Thiemann [19] and Fleishhack [20] is that the requirement is in

fact so strong that it suffices to single out a unique representation of  $\mathfrak{A}$ , without having to fix dynamics. Thus, *thanks to background independence, quantum kinematics is unique in LQG.*

This powerful result lies at the foundation of much of LQG because the unique representation it selects leads to the fundamental discreteness in quantum geometry. Therefore let us discuss the key features of this representation and compare and contrast it with representations used in MQFTs. The underlying Hilbert space  $\mathcal{H}_{\text{grav}}^{\text{kin}}$  is the space  $L^2(\mathcal{A}, d\mu_o)$  of square integrable functionals of (generalized) connections with respect to a regular, Borel measure  $\mu_o$ . As one would expect, the holonomy operators  $\hat{h}_\ell$  act by multiplication while their ‘momenta’  $\hat{E}_{f,S}$  act by differentiation. There is a state  $\Psi_o$  in  $\mathcal{H}_{\text{grav}}^{\text{kin}}$  which is cyclic in the sense that  $\mathcal{H}_{\text{grav}}^{\text{kin}}$  is generated by repeated actions of  $\hat{h}_\ell$  on  $\Psi_o$ . These properties are shared by MQFTs where the Fock space can also be represented as the space of square-integrable functionals over the space of (distribution-valued) fields on  $\mathbb{R}^3$  and the vacuum plays the role of  $\Psi_o$ . In these theories, the vacuum state is Poincaré invariant and this invariance implies that the Poincaré group is unitarily implemented in the quantum theory. In LQG, the state  $\Psi_o$  is invariant under the kinematical symmetry group  $\text{SU}(2)_{\text{loc}} \times \text{Diff}(M)$  of connection dynamics—the semi-direct product of the local  $\text{SU}(2)$  gauge transformations and diffeomorphisms of  $M$ —and this group is unitarily represented on  $\mathcal{H}_{\text{grav}}^{\text{kin}}$ . This fact provides a natural point of departure in the imposition of quantum constraints, discussed below.

However, the representation also has two unfamiliar features: i)  $\mathcal{H}_{\text{grav}}^{\text{kin}}$  is non-separable, and, ii) while the holonomies  $\hat{h}_\ell$  are well-defined operators on  $\mathcal{H}_{\text{grav}}^{\text{kin}}$ , the connection operators themselves do not exist (because  $\hat{h}_\ell$  fail to be continuous with respect to the links  $\ell$ ). These aspects of LQG kinematics have caused some unease among researchers outside LQG (see, e.g. [65]) because it is not widely appreciated that they are *not* peculiarities of LQG but follow, in essence, just from background independence. In particular, if one seeks a representation of the properly constructed kinematical algebra of geometrodynamics in which the cyclic state is invariant under the kinematical symmetry group  $\text{Diff}(M)$ , *one again finds that the representation inherits these two features* [66]. Intuition derived from the  $\text{Diff}(S^1)$  group used, e.g., in string theory does not carry over to higher dimensions in this respect.

Let us now discuss quantum states and operators in some detail. Recall that in MQFTs, while the characterization of the Fock space as the space of square integrable functionals of (generalized) fields is succinct, detailed calculations are most efficiently performed in a convenient basis that diagonalizes the number operators. The situation with  $\mathcal{H}_{\text{grav}}^{\text{kin}}$  is analogous. More precisely, it is convenient to decompose  $\mathcal{H}_{\text{grav}}^{\text{kin}}$  into orthogonal subspaces,  $\mathcal{H}_{\text{grav}}^{\text{kin}} = \bigoplus \mathcal{H}_\alpha$ , associated with graphs  $\alpha$  in  $M$  with a finite number of oriented links  $\ell$ . Next, if one labels each link  $\ell$  of  $\alpha$  with a non-trivial, irreducible representation  $j_\ell \neq 0$  of  $\text{SU}(2)$ , one obtains a further decomposition [67, 68]

$$\mathcal{H}_{\text{grav}}^{\text{kin}} = \bigoplus_{\alpha} \mathcal{H}_\alpha = \bigoplus_{\alpha, j_\ell} \mathcal{H}_{\alpha, j_\ell}. \quad (2.10)$$

If  $\alpha$  has  $L$  links,  $\mathcal{H}_{\alpha, j_\ell}$  is a *finite* dimensional Hilbert space which can be identified with the space of quantum states of a system of  $L$  spins. Therefore (2.10) is called a *spin-network decomposition* of  $\mathcal{H}_{\text{grav}}^{\text{kin}}$ . To make this relation explicit, note first that a (generalized) connection  $A$  assigns to each link  $\ell$  a holonomy  $h_\ell$  and elements  $\Psi$  of  $\mathcal{H}_{\text{grav}}^{\text{kin}}$  are functions of these (generalized) connections. States  $\Psi$  in  $\mathcal{H}_{\alpha, j_\ell}$  are of the form

$$\Psi(A) = \psi(h_{\ell_1}, \dots, h_{\ell_L}) \quad (2.11)$$

where  $\psi$  is a function of the  $L$   $SU(2)$  group-elements in its argument, which is square integrable with respect to the Haar measure on  $[SU(2)]^L$ . They know only about the action of the connection  $A$  pulled back to the  $L$  links of  $\alpha$ . Thus, by restricting attention to a single graph  $\alpha$ , one truncates the theory and focuses only on a finite number of degrees of freedom. The spirit is the same as in MQFTs. In any calculation with Feynman diagrams of a weakly coupled theory (such as low energy QED) one truncates the theory by allowing only a *finite* number of virtual particles. Similarly, in strongly coupled theories (such as low energy QCD) one truncates the theory by making a lattice approximation. In both cases, the full Hilbert space is recovered in the limit in which the degrees of freedom are allowed to go to infinity. In LQG this is achieved by taking a well-defined (projective) limit in the space of graphs [60]. Finally, the second equality in (2.10) is obtained by carrying out Fourier transforms (using the Peter-Weyl theorem) on  $[SU(2)]^L$ .

Such truncations are useful if the operators of interest leave the truncated Hilbert spaces invariant. This is indeed the case with geometric operators of LQG. As one would expect from the phase space description, these operators are constructed from  $\hat{E}_{f,S}$  since the electric field  $E_i^a$  also serves as the orthonormal triad in the classical theory. The action of  $\hat{E}_{f,S}$  on a state  $\Psi \in \mathcal{H}_\alpha$  is non-trivial only if the surface  $S$  intersects one or more links of the graph  $\alpha$  and then the action involves only group theory at the intersection [2, 4, 61]. This is just the structure one would expect from background independence! To construct geometric operators such as those corresponding to areas of 2-surfaces and volumes of 3-dimensional regions, one first expresses their classical expressions in terms of the ‘elementary’ phase space functions  $\hat{E}_{f,S}$  and then promotes the classical expression to a quantum operator. In the intermediate stages one has to introduce auxiliary structure but the procedure ensures that the final expressions are background independent [2–4].

Let us now consider the operator  $\widehat{\text{Ar}}_{S,\alpha}$  on  $\mathcal{H}_\alpha$ , representing the area of a 2-surface  $S$  (without boundary) [69–71] which has played a particularly important role in LQG. Let us first suppose that the surface  $S$  intersects  $\alpha$  only at a node  $n$ . Then, one can naturally define a *node-Laplacian operator*  $\Delta_{\alpha,S,n}$  whose action on  $\Psi$  of (2.11) is an appropriate sum of the Laplacians on the copies of  $SU(2)$  associated with links  $\ell_i$  that intersect  $S$  at  $n$  [2, 60, 71]. As one might expect,  $\Delta_{\alpha,S,n}$  is a negative definite self-adjoint operator on  $\mathcal{H}_\alpha$ . The final expression of the area operator  $\widehat{\text{Ar}}_{S,\alpha}$  is given by

$$\widehat{\text{Ar}}_{S,\alpha} = 4\pi \gamma \ell_{\text{Pl}}^2 \sqrt{\Delta_{\alpha,S,n}} \quad (2.12)$$

If there are multiple intersections  $n_i$  between  $\alpha$  and  $S$ ,  $\widehat{\text{Ar}}_{S,\alpha}$  is just the sum of these operators for each  $n_i$ . The non-trivial result is that operators defined on various  $\mathcal{H}_\alpha$  can be naturally *glued together* to obtain a self-adjoint operator  $\widehat{\text{Ar}}_S$  on the entire  $\mathcal{H}_{\text{grav}}^{\text{kin}}$ .

Properties of  $\widehat{\text{Ar}}_{S,\alpha}$  have been analyzed in detail. Its spectrum is discrete in the sense that all its eigenvectors are normalizable. In the special case when all intersections between  $\alpha$  and  $S$  are at bi-valent nodes at which ‘straight’ links pierce  $S$ , the expression of eigenvalues simplifies to a form that is useful in many applications [71, 72]:

$$a_S = 8\pi \gamma \ell_{\text{Pl}}^2 \sum_n \sqrt{j(j+1)}. \quad (2.13)$$

There is a smallest non-zero eigenvalue among these:

$$\Delta a_S = 4\pi \gamma \ell_{\text{Pl}}^2 \sqrt{3}. \quad (2.14)$$

This *area gap* plays an important role in the theory. The level spacing between consecutive eigenvalues is *not* uniform but decreases *exponentially* for large eigenvalues [71]. This implies that, although the eigenvalues are fundamentally discrete, the continuum approximation becomes excellent *very rapidly*.

For the volume and length operators, the strategy is the same and the background independence of LQG again fixes the precise form of the final expressions [2, 4, 69, 73]. However, the detailed procedure is technically more complicated. The length operator has not had significant applications. The volume operator has been investigated in greater detail because features prominently in the dynamical considerations of the canonical theory [4, 55, 74]. The problem of finding its spectrum has been cast in a form that makes it accessible to numerical studies [75]. Although the eigenvalues are discrete, there are indications that, in contrast to the area operator, the spectrum of the volume operator may not have an volume gap. This is but one indication that the quantum geometry has qualitatively different features from what one may naively expect from the classical Riemannian geometry or a naive discretization thereof.

Let us summarize. The kinematical framework of LQG is well developed, with full control on functional analysis. In particular, the infinite dimensional integrals are not formal symbols but performed with well defined measures [76, 77]. There are two key results that simplify the analysis: the uniqueness theorem [19] and the spin-network decomposition of the full Hilbert space [67, 68]. The natural truncation of the theory is achieved by restricting oneself to the Hilbert space  $\mathcal{H}_\alpha$  defined by a graph  $\alpha$ . Elements of these  $\mathcal{H}_\alpha$  describe elementary quanta of geometry; to obtain classical geometries one needs to coherently superpose a large number of them.

Perhaps the simplest way to visualize the elementary quanta is to introduce a simplicial decomposition  $\mathcal{S}$  of the 3-manifold  $M$  and consider a graph  $\alpha$  which is dual  $\mathcal{S}$ : Each cell in  $\mathcal{S}$  is a topological tetrahedron  $T_n$ , dual to a node  $n$  of  $\alpha$ ; each face  $F_\ell$  of  $\mathcal{S}$ , is dual to a link  $\ell$ . In Regge calculus, every  $T_n$  has the geometry of a tetrahedron in flat space and the curvature is encoded in the holonomies of the connection around ‘bones’ that lie at the intersection of any two faces of  $T_n$ . What is the situation in LQG? To bring out the similarities and contrasts, it is convenient to consider a basis  $\Psi_{\alpha, v_n, a_\ell}$  in  $\mathcal{H}_{\alpha, j_\ell}$  that simultaneously diagonalizes the volume operator associated with the tetrahedron  $T_n$ , and the area operators associated with the faces  $F_\ell$ , for all  $n, \ell$ . Each of these spin-network states describes a specific *elementary* quantum geometry. One can think of the node  $n$  as a ‘grain’ or a ‘quantum’ of space captured in the (topological) tetrahedron  $T_n$ . As in Regge calculus each  $T_n$  has a well defined volume  $v_n$  and each of its faces  $F_\ell$  has a well-defined area  $a_\ell$ . But now the  $v_n, a_\ell$  are *discrete*. More importantly, because the operators  $J_\ell^i$  do not commute,  $T_n$  no longer has the sharp geometry of a geometrical tetrahedron in the Euclidean space. In particular, operators describing angles between any two distinct faces  $F_\ell, F_{\ell'}$  of a  $T_n$  are *not* diagonal in the basis. Furthermore, although the area of any common face  $F$  of two adjacent tetrahedrons is unambiguous, in contrast with the Regge geometry, curvature now resides not just at the bones of tetrahedra but also along the faces; the geometry is ‘twisted’ in a precise sense [78]. These properties of the quantum geometry associated with the basis  $\Psi_{\alpha, v_n, a_\ell}$  are closely analogous to the properties of angular momentum captured by the basis  $|j, m\rangle$  in quantum mechanics: it too diagonalizes only some of the angular momentum operators, leaving values of other angular momentum observables fuzzy. Thus, each of the elementary cells in the simplicial decomposition is now a ‘tetrahedron’ in the same heuristic sense that the a spinning particle in quantum mechanics is a ‘rotating body’.

To conclude, we note that tri-valent spin-networks were introduced by Roger Penrose already in 1971 in a completely different approach to quantum gravity [79]. He expressed his general view of that construction as follows: “*I certainly do not want to suggest that the universe ‘is’ this picture . . . But it is not unlikely that essential features of the model I am describing could still have relevance in a more complete theory applicable to more realistic situations*”. In LQG one finds that the trivalent graphs  $\alpha_{\text{tri}}$  are indeed ‘too simple’ because all states in the  $\mathcal{H}_{\alpha_{\text{tri}}}$  have zero volume [80]. Also, we now have detailed geometric operators and find that the angles cannot be sharply specified. Nonetheless, Penrose’s overall vision is realized in a specific and precise way in the LQG quantum geometry.

### 3. Quantum Einstein’s equations

Recall from (2.7) that we have three sets of constraints. In the classical theory, the Gauss and the Diffeomorphism constraints generate kinematical symmetries while dynamics is encoded in the Hamiltonian constraint. In the quantum theory the physical Hilbert space  $\mathcal{H}_{\text{phy}}$  is to be constructed by imposing the quantum constraints  $\hat{C}\Psi_{\text{phy}} = 0$  a la Dirac. This requires one to solve two non-trivial technical problems: i) Introduce well-defined constraint operators  $\hat{C}$  on  $\mathcal{H}_{\text{kin}}$  starting from the classical constraint functions  $C$ ; and ii) Introduce the appropriate scalar product on the solutions  $\Psi_{\text{phy}}$  to obtain  $\mathcal{H}_{\text{phy}}$ . The second step is non-trivial already for systems with a finite number of degrees of freedom if the constraint operator  $\hat{C}$  has a continuous spectrum because then the kinematical norm of physical states  $\Psi_{\text{phy}}$  diverges. In geometrodynamics, the operators  $\hat{C}$  have been defined only formally and generally the issue of scalar product is not addressed. In LQG by contrast, the availability of a rigorous kinematical framework provides the necessary tools to address both these issues systematically.

For the kinematical constraints, both these steps have been carried out [70]. Since these constraints  $C_{\text{kin}}$  have a natural geometrical interpretation, the quantum operators  $\hat{C}_{\text{kin}}$  simply implement those geometrical transformations on the kinematical (spin-network) states  $\Psi_{\text{kin}}$  in  $\mathcal{H}_{\text{kin}}$ . The second task, that of introducing the appropriate scalar product, is carried out using a general strategy called *group averaging* [70, 81]. The detailed implementations of these ideas is straightforward for the Gauss constraint but there are important subtleties in the case of the diffeomorphism constraint [2, 4, 61, 70]. In particular, the strategy described here allows only the exponentiated version of the diffeomorphism constraint, i.e., *finite* diffeomorphisms, and one has to specify the precise class of diffeomorphisms that are allowed.<sup>2</sup> The Hilbert space  $\mathcal{H}^{\text{diff}}$  on which both the kinematical constraints are satisfied provides a completion of the Dirac quantization program.

For the Hamiltonian constraint  $C_H$ , on the other hand, the situation is still in flux. There is a non-trivial result due to Thiemann that one *can* regulate this constraint systematically on  $\mathcal{H}^{\text{diff}}$  [55]. By contrast, no such regularization is available for the WDW equation of geometrodynamics. But the procedure involves introduction of additional structures in the intermediate steps, whence the final result is ambiguous. Furthermore, the *physical*

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<sup>2</sup> With a natural choice of this class,  $\mathcal{H}^{\text{diff}}$  is *separable*, although  $\mathcal{H}_{\text{kin}}$  is not. This may seem surprising at first. But the situation is completely analogous to what happens already in the quantum theory of free Maxwell theory with the Gauss constraint if one does not wish to work with an indefinite metric [66, 82].

meaning of the additional structures has remained unclear. Finally, recall that in GR, the Poisson bracket between Hamiltonian constraints smeared with lapse functions  $N$  and  $M$  is the diffeomorphism constraint smeared with a ‘q-number’ shift field  $K^a = q^{ab}(ND_aM - MD_aN)$ . An important question is whether this Poisson bracket structure is reflected in the quantum theory. In these regularizations, in the quantum theory the commutator of the two Hamiltonian constraints vanishes and so does the diffeomorphism constraint on right hand side on the ‘kinematical habitat’ on which the calculation is carried out [83]. While this establishes consistency, one would hope for a better scheme in which neither side vanishes and the commutator structure captures the non-trivial, off-shell relation between the constraints.

Recently, a promising approach to this problem has been introduced by Laddha, Varadarajan and others [84, 85]. The first underlying idea is to take hints from earlier work on 1+1 dimensional parameterized field theories where well-understood, unconstrained field theories are recast in an extended setting with constraints. The constraints mimic those of GR in that they are again related to space-time diffeomorphisms [86]. One finds that the techniques used in LQG provide a natural avenue to implement quantum constraints in the parameterized form, leading to the correct final quantum theory. The second and deeper observation is motivated by the fact that in connection dynamics the diffeomorphism and the Hamiltonian constraints of (2.7) can be naturally combined in the spinorial setting as  $E^{aA}{}_B E^{bB}{}_C F^C{}_{abD} = 0$  where  $A \dots D$  are spinorial indices [47]. (The trace over  $A$  and  $D$  yields the Hamiltonian constraint while the trace-free part yields the diffeomorphism constraint). This unity suggests that although Hamiltonian constraint generates time evolution, this action could be recast in terms of geometric operations *within the 3-manifold  $M$* . This has been shown to be the case [85]: in the classical theory, time evolution can be re-expressed as the action of *diffeomorphism and Gauss constraints* smeared with certain ‘q-number’ smearing fields on  $M$ . As a consequence, an entirely new perspective emerges. These features do not carry over to geometrodynamics since there the two constraints cannot be naturally combined into a single one. In a simplified theory, where the gauge group  $SU(2)$  is replaced by the Abelian group  $U(1)^3$ , the program has been carried out and it has been shown that the algebra of constraints *closes off-shell non-trivially*. There is ongoing research to extend these results to the full theory using  $SU(2)$ .

Finally there has been considerable research on coupling matter fields to gravity, particularly those that can serve as physical clocks and rods [61, 87, 88]. The idea, as in geometrodynamics, is to use the matter fields to ‘deparameterize’ the constraints and study the ensuing *relational* dynamics. On the conceptual side, these ideas will play a key role in the physical interpretation of canonical LQG. On the technical side, it is rather surprising that certain matter fields do make the quantum constraints manageable enabling one to extract the notion of ‘evolution’ from the solution to quantum constraints. Once there is a fully satisfactory implementation of the Hamiltonian constraint, these ideas will play a key role in extracting physics from canonical LQG. A qualitative understanding has already begun to emerge because the strategy of [84] to better regularize the constraints and that of [87, 88] to deparameterize the theory using matter can be seen as generalizations to the full theory of the successful strategies used in loop quantum cosmology to first obtain and then interpret the quantum theory.

A complementary approach to dynamics is provided by Spinfoams, discussed in the next subsection.

### C. Covariant Loop Gravity: Spinfoams

The covariant or *Spinfoam* formulation of LQG is built again on the quantum theory of geometry discussed in section II B, but the now dynamics is specified by defining the transition amplitudes, order by order in a suitable expansion. This is akin to spirit used by Feynman to build QED directly in terms of the Feynman rules, which streamlined and simplified the theory.

As with Feynman diagrams, the amplitudes defined in this manner can also be seen as given by a sum over histories. The relevant histories, however, describe spacetime as the *evolution of individual quanta of geometry*, rather than of classical configurations. Thus, a 3-geometry is still represented by a spin-network, and a 4-geometry, by a history of spin-networks. These histories are called *Spinfoams* [3, 89, 90]. The sum over Spinfoams define transition amplitudes between quantum 3-geometries and, as discussed in section III C, the  $n$ -point functions of non-perturbative quantum gravity.

The main results of covariant LQG to date are the following:

- i) The amplitudes are *finite* at every order.
- ii) At each order, the amplitudes have a well-defined classical limit, related to a truncation of classical general relativity.
- iii) The theory has been extended to include fermions and Yang-Mills fields [91].

Regarding point (i), there are two potential sources of infinities in the theory. The ultraviolet (UV) divergence which corresponds to the conventional infinities of perturbative Feynman diagrams, and infrared (IR) diverges that can arise from the contributions of intermediate states with large-scale geometries. The UV divergences are naturally cured by the discreteness of the underlying quantum geometry itself. The IR divergences are cured by the presence of a positive cosmological constant  $\underline{\Lambda}$ . Therefore, interestingly, the structure of the theory is such that a cosmological constant *with a positive sign* naturally acts as a physical IR regulator. Regarding point (ii), recall that the classical limit of lattice QCD on a fixed triangulation is just the classical lattice theory. Similarly, the classical limit of covariant loop quantum gravity at a fixed order is related to Regge calculus on a finite triangulation in a precise sense. We will now discuss these issues in detail.

#### 1. Transition Amplitudes

For simplicity, we describe the case without fermions and Yang-Mills fields and with  $\underline{\Lambda} = 0$  (thus ignoring IR problems). We later discuss the necessary modifications to incorporate  $\underline{\Lambda} > 0$ .

In quantum theory, one calculates the transition amplitudes between initial and final states. In LQG these are states of quantum geometry and therefore belong to the Hilbert spaces  $\mathcal{H}_\alpha^{\text{diff}}$  labeled by (abstract) graphs  $\alpha$ . In the absence of an external time parameter, there is no distinction between the initial and the final states. Therefore, it is convenient to combine the two graphs that refer to the boundary states; we denote this total graph by  $\Gamma$ . Then, if  $\Gamma$  has  $N$  nodes  $n$  and  $L$  links  $l$ ,  $\mathcal{H}_\Gamma^{\text{diff}}$  is spanned by states  $\psi(U_l)$  which are in  $L^2(\text{SU}(2))^L$  and invariant under  $\text{SU}(2)$  gauge transformations at the nodes. Thus,  $\mathcal{H}_\Gamma^{\text{diff}}$  is the same as the Hilbert space of a  $\text{SU}(2)$  lattice Yang-Mills theory. The theory defines a transition amplitude for each of these states  $\Psi(U_l)$ .

To any given order, the transition amplitudes are labeled by a *2-complex*  $\mathcal{C}$ , a higher-dimensional analog of a graph: it is defined as a (combinatorial) set of *faces*  $f$  meeting at

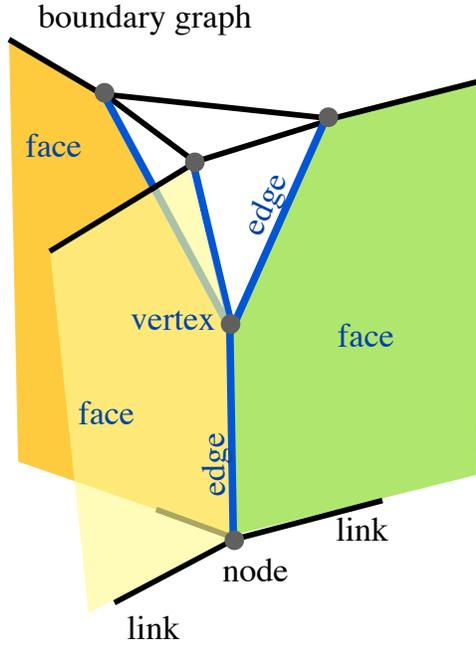


Figure 2: 2-complex terminology.

*edges*  $e$ , which in turn meet at *vertices*  $v$  (see Fig. 2). It can be regarded as a history of a spin-network in which each link  $l$  of the spin-network ‘evolves’ to form a face  $f$ , each node  $n$  ‘evolves’ to an edge  $e$ , and non-trivial dynamics occurs when a new vertex  $v$  appears. The number of vertices in  $\mathcal{C}$  defines the order in the expansion of the transition amplitude.

To grasp the interplay with spacetime geometry, it is useful to note that a triangulation  $\Delta$  of the four dimensional spacetime defines a dual 2-complex: each 4-simplex of  $\Delta$  corresponds to a vertex in  $\mathcal{C}$ , each tetrahedron of  $\Delta$  to an edge in  $\mathcal{C}$ , and each triangle in  $\Delta$  to a face of  $\mathcal{C}$ . For simplicity, we will only consider 2-complexes dual to triangulations. The boundary  $\Gamma = \partial\mathcal{C}$  of a 2-complex is a graph that the state  $\psi(U_l)$  refers to (see Fig. 2).

Let us fix a 2-complex with boundary graph  $\Gamma$  and  $v$  vertices. Then, the  $v$ th order term in the expansion of the transition amplitude associated with a state  $\psi \in \mathcal{H}_\Gamma^{\text{diff}}$  is defined as the scalar product in  $\mathcal{H}_\Gamma^{\text{diff}}$  of the state  $\psi$  with the function  $W_{\mathcal{C}}(U_l)$  defined as follows:

$$W_{\mathcal{C}}(U_l) = \int_{\text{SU}(2)} dh_{vf} \prod_f \delta(h_f) \prod_v A_v(h_{vf}). \quad (2.15)$$

This  $W$  is the key object in Spinfoams because, to order  $v$  in our expansion, quantum dynamics is encoded in  $W$ . The integral in (2.15) is over one  $\text{SU}(2)$  variable  $h_{vf}$  associated to each vertex-face pair, the delta distribution is over  $\text{SU}(2)$  and its argument  $h_f$  is the (oriented) product of the variables  $h_{vf}$  around a face. In the 4d Lorentzian theory, the ‘vertex amplitude’  $A_v$  has the form

$$A_v(h_{vf}) = \int_{\text{SL}(2,\mathbb{C})} dg_{ve} \prod_e \sum_j (2j+1) \text{Tr}_j [Y^\dagger g_e g_{e'} Y h_f]. \quad (2.16)$$

Here the  $\text{SL}(2, \mathbb{C})$  integration variables  $g_{ve}$  are associated to each edge emerging from  $v$  and the two edges  $e$  and  $e'$  in the trace are those bounding the face  $f$ .  $Y$  is a map from the spin  $j$

representation of  $SU(2)$  to the unitary representation of  $SL(2, \mathbb{C})$  with continuous quantum number  $\gamma(j+1)$  and discrete quantum number  $j$ , defined by<sup>3</sup>

$$Y : |j; m\rangle \mapsto |\gamma(j+1), \gamma; j, m\rangle \quad (2.17)$$

where, as before,  $\gamma$  is the Barbero-Immirzi parameter. These three equations completely define the theory. Quite remarkably, to order  $v$  in the vertex expansion, they encode the entire quantum dynamics.

This form of the amplitude is variously denoted EPRL, EPRL-FK, or EPRL-FK-KKL amplitude. It was derived in [92] building on results in [93, 94] and extended to arbitrary 2-complexes in [95], and forms the basis of the 4d Lorentzian theory so far. Variants could be interesting; some of these have been considered for the Euclidean theory [96].

Let us examine the structures of this amplitude. The appearance of  $SL(2, \mathbb{C})$  is not surprising: it reflects the local Lorentz invariance of GR. The appearance of the *unitary* (infinite dimensional) representations of this group should not be too surprising either, given that unitary representations of symmetry groups are ubiquitous in quantum gravity. Indeed, one may wonder why the mathematics of the infinite dimensional unitary representations of  $SL(2, \mathbb{C})$  has played such a small role in the attempts to construct a quantum theory of gravity so far. The map  $Y$  on the other hand is a new ingredient that constitutes the technical core of the Spinfoam model and deserves explanation. For this, let us first return to the classical theory discussed in section II B. In the space-time picture, the momentum conjugate to the  $SL(2, \mathbb{C})$  connection  $\omega$  is [57]

$$\pi_{IJ} = \frac{1}{4\kappa_N} (\epsilon_{IJKL} e^K \wedge e^L + \frac{1}{2\gamma\kappa_N} e_I \wedge e_J) \quad (2.18)$$

On a boundary of a spacetime region, the one-form normal to the boundary contracted with the tetrad gives a vector in the internal Minkowski space, which determine a preferred Lorentz frame. We can decompose  $\pi_{IJ}$  in this frame in the same manner in which the Maxwell field  $F_{IJ}$  decomposes in the electric and magnetic field. Simple algebra then shows that the electric and magnetic parts of  $\pi_{IJ}$ , denoted respectively  $\vec{K}$  and  $\vec{L}$  satisfy the algebraic equation

$$\vec{K} = \gamma \vec{L} \quad (2.19)$$

This is a key equation in covariant loop quantum gravity, called the *simplicity constraint*. To ensure a correct classical limit, this constraint has to be implemented in the quantum theory in an appropriate fashion. This is precisely what the map  $Y$  does: in the quantum theory (2.19) holds on the image of this map as a weak operator equation (i.e., for all matrix elements of the operators) [97].

## 2. Classical Limit

Spinfoam dynamics presented in the last subsection was arrived at from several independent considerations: the Hamiltonian LQG [89], the fact that GR can be regarded as a constrained BF theory [98], the Ponzano-Regge and Turiiev-Viro models [99–102] for quantum gravity in 3-dimensions and group field theory [3, 103]. Furthermore, the overall paradigm

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<sup>3</sup> The maps extends easily from functions on  $SU(2)$  to functions of  $SL(2, \mathbb{C})$ .

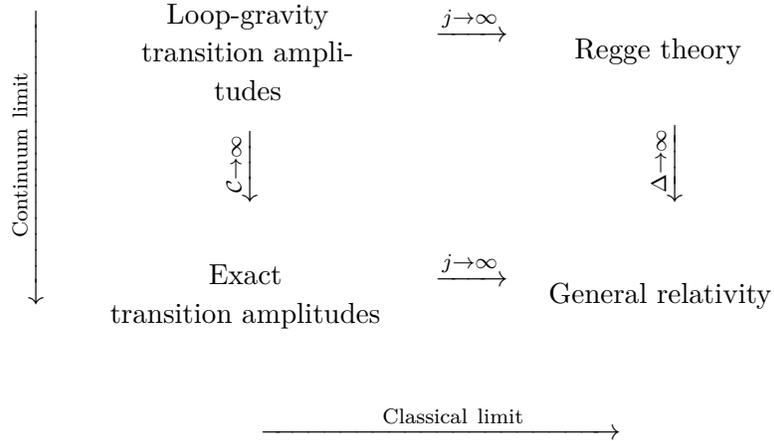


Table I: Relation between continuum limit and classical limit of the transition amplitudes.

underlying Spinfoams is borne out in symmetry reduced, cosmological models, where the transition amplitudes obtained by summing over quantum geometries have been shown to be finite and in agreement with the Hamiltonian theory [104]. While these considerations provide a reasonably strong motivation, one still needs direct evidence in favor of the specific proposal (2.15). Analysis of the classical limit provides a natural avenue to test its viability.

In any quantum theory, the classical limit is obtained in a regime where quantum numbers are large. Then the relevant actions are large compared to the Planck constant and the limit can be interpreted as  $\hbar \rightarrow 0$ . For example, for a particle with a Hamiltonian  $H$ , in the  $\hbar \rightarrow 0$  limit we have:

$$W(x, t; x', t') \sim \int [Dx] e^{\frac{i}{\hbar} S[x]} \sim A e^{\frac{i}{\hbar} S(x, t; x', t')}. \quad (2.20)$$

where the integration is over the paths from  $(x, t)$  to  $(x', t')$  and  $S(x, t; x', t')$  is the Hamilton function, namely the value of the action on the solution of the classical equations of motion that start at  $(x, t)$  and ends at  $(x', t')$ . In gravity the analogous procedure requires us to consider areas and volumes that are large compared to the Planck scale. Thus, to study the classical limit of Spinfoam dynamics, one can compare the large  $j$  limit of the transition amplitude (2.15) with the classical action. The asymptotic analysis of the vertex amplitude (2.16) is nontrivial, and has been carried out mainly by the Nottingham group [105]. For the simplest case where the 2-complex  $\mathcal{C}$  has only one vertex  $v$ , and the results can be summarized as follows. Recall that the amplitude is a function of the boundary quantum state  $\psi$  and quantum geometries are more general than Regge geometries. If  $\psi$  does not endow the 4-simplex  $\Delta$  dual to  $\mathcal{C}$  with a consistent classical geometry, the transition amplitude is suppressed exponentially. If it does, then the asymptotic form of the amplitude is given by

$$A_v \sim A \left( e^{\frac{i}{\hbar} (S_R + \frac{\pi}{4})} + e^{-\frac{i}{\hbar} (S_R + \frac{\pi}{4})} \right). \quad (2.21)$$

where  $S_R$  is the Regge action of  $\Delta$ . The presence of two terms in (2.21) is a consequence of the fact that, as we saw in section IIB, the starting point of the analysis is tetrad gravity and, when the tetrad changes orientation, the first order LQG action changes sign, while the Einstein-Hilbert action does not. Consequently, for each classical metric solution we have two tetrad solutions whose action is equal in magnitude but with opposite signs. The  $\frac{\pi}{4}$  is also well understood: it is the Maslov index that always appear in the semiclassical

limit when the two classical solutions sit on different branches of the solution space [106]. Therefore the result 2.21 has the following simple interpretation: the classical limit of the transition amplitude defined by a 2-complex  $\mathcal{C}$  dual to a spacetime triangulation  $\Delta$  is the Regge amplitude associated with that triangulation. This is precisely what one would have hoped. In this sense the proposal passes the viability criterion and it is reasonable to regard equations (2.15), (2.16) and (2.17) as providing a tentative definition of the dynamics of LQG.

This covariant formulation of LQG has some similarities with the path-integral approach based on Regge Calculus [107, 108] where one sums over configurations representing a Regge discretization of general relativity. This approach was introduced already in the 1980s and has evolved considerably since then [109]. In spite of the formal structural similarity, there is an important conceptual difference between the two approaches. In Regge calculus, the lengths of the individual links can be arbitrarily small. By contrast, the geometries that are summed over in Spinfoams represent histories of *quanta* of space, whence the areas of plaquettes cannot be arbitrarily small; they are bounded below by the area-gap of LQG. This fundamental discreteness naturally removes the UV divergences and introduces the Planck scale already in the permissible histories that are summed over. Consequently, the scaling structure of the theory with respect to the Regge Calculus is quite different.

We conclude this discussion by noting that the classical limit we have discussed here should not be confused with the continuum limit of the theory. The first is the standard  $\hbar \rightarrow 0$  limit while the second refers to refinement, i.e., adding more and more degrees of freedom. Recall that the classical limit of lattice QCD on a fixed lattice is of course a classical lattice theory. In LQG, the lattice is replaced by a triangulations, but with the crucial difference that its geometry is not pre-specified but constitutes the dynamical variable. Nonetheless, situation with respect to the classical limit is similar: in covariant loop quantum gravity, this limit is related in a precise way to Regge calculus on a finite triangulation. As well known, classical Regge calculus converges to full GR in the limit in which the triangulation is refined. The structure of the theory is therefore as in the Table I: To arrive at GR from Spinfoams, one can start from the upper left corner of the diagram, move first to the right and then down.

### 3. Cosmological Constant and IR Finiteness

As we noted above, the reason behind the UV finiteness of the Spinfoam amplitude (2.15) is intuitively simple: because of the discreteness of space at the Planck scale, there is an in-built and natural physical cut off preventing the standard quantum field theory divergences. In other words, there are no degrees of freedom at arbitrary small scales. Therefore the sum over intermediate states in a perturbation expansion does not include field configurations of arbitrary high momentum. From this perspective, the UV divergences of standard quantum field theory can be interpreted as pathologies introduced by the fact of neglecting the discrete nature of space.

However, the amplitude (2.15) can have IR divergences. This can happen every time the 2-complex has a *bubble*, i.e., a set of continuous faces with the topology of a two-sphere. These bubbles are the Spinfoam analog of loops in Feynman diagrams: The quantity circulating around a Feynman diagram loop is the momentum, and high momentum means UV; while the quantity circulating around a Spinfoam bubble is the area, and high area means IR. On a bubble, the sum over spins  $j$  in (2.15) can lead to divergent terms because  $j$  is unbounded

above. Geometrically, these divergences correspond to ‘spikes, representing large regions of spacetime bounded by small hypersurfaces.’

Remarkably, these divergences disappear naturally if there is a positive cosmological constant  $\underline{\Lambda}$  in the theory. Technically, the effect of a positive  $\underline{\Lambda}$  is to replace  $\text{SL}(2, \mathbb{C})$  with a quantum deformation of  $\text{SL}(2, \mathbb{C})$ . The mathematics for implementing this deformation has been developed [110, 111] and the Spinfoam amplitude with the cosmological constant has also been defined [112, 113]. The transition amplitudes of the theory with a quantum deformation of  $\text{SL}(2, \mathbb{C})$  are finite, and the classical limit of their vertex amplitude is still given by Eq. (2.21). But now the Regge action that appears in the classical limit has a cosmological constant  $\underline{\Lambda}$ , related to the deformation parameter  $q$  of the quantum group via [113, 114].

$$q = \exp \underline{\Lambda} \hbar G. \quad (2.22)$$

Therefore the full theory now depends on *two* dimensionless parameters:  $q$  or  $\underline{\Lambda} \hbar G$ , and the Barbero-Immirzi parameter  $\gamma$ . The bare cosmological constant enters the theory as a free parameter, therefore the theory does not prescribe its value. To explore various limiting regimes, one has to calculate the behavior of physical observables, keeping appropriate combinations of these constants fixed and let a complementary combination tend to the desired value.

#### 4. QED, QCD and LQG

Similarities between the Spinfoam model defined in the last three subsections and QCD on a fixed lattice are evident: In both cases, we have a discretization of the classical theory where the connection is replaced by group elements, and a quantum theory defined by an integral over configurations of an amplitude which is a product of local quantities. The use of a triangulation in Spinfoams instead of a square lattice simply reflects the fact that a square lattice is unnatural in absence of a flat metric. However, there is also a crucial difference. The Wilson QCD action depends on an external parameter, the lattice spacing  $a$ , while appropriate discretizations of the Einstein-Hilbert action, like the Regge action, do not. To recover the continuum theory, in QCD it is not sufficient to increase the total size of the lattice; it is also necessary to send  $a$  to zero. Equivalently, the lattice spacing  $a$  can be absorbed in the coupling constant  $\beta$  in front of the action and, in order to recover the continuum limit, it is necessary to tune  $\beta$  to its critical value,  $\beta = 0$ . In gravity, instead, the Regge action (or any other admissible discretization) does *not* include a lattice spacing  $a$  (nor, therefore, a coupling constant that needs to be tuned to a critical value as  $a \rightarrow 0$ ). The reason is simply that the lattice spacing  $a$  refers to a background geometry—the Yang-Mills theory depends on a fixed, externally given spacetime metric—while in gravity the geometry is included in the dynamical variables. It can be shown in general [115] that the discretization of a reparametrization invariant theory can be defined *without* a parameter that needs to be tuned to a critical value in the continuum limit. Accordingly, in a suitable discretization of general relativity the continuum limit can be defined just by making the triangulation (or the two complex) increasingly finer.<sup>4</sup> An alternative approach to the continuum limit is discussed in [116].

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<sup>4</sup> In concrete physical calculations, however, only finite triangulations suffice, as is generally the case in QCD. Similarly, in QED a finite number of Feynman graphs suffice.

Interestingly, there are also similarities between Spinfoams and perturbative QED. The nodes of the graph can be seen as quanta of space and the 2-complex can be read as a history of these quanta, showing where these quanta interact, join and split, just as real and virtual particles do in the Feynman graphs. The analogy is reinforced by the fact that the Spinfoam amplitude can actually be concretely obtained as a term in a Feynman expansion of a ‘group field theory’ (see for instance Chapter 9 of [3] and [96, 103]). The specific group field theory that gives the gravitational amplitude (2.15) has been derived (in the Euclidean context) in [117].

Thus, the Spinfoam paradigm shares some key features with QCD as well as QED, our two most successful, fundamental quantum theories. In addition, Spinfoams bring out a novel interplay between these theories and quantum gravity. A Feynman graph of QED is a history of quanta of a field while the lattice used in QCD is a collection of discrete chunks of spacetime. They are distinct and unrelated. But general relativity taught us that spacetime itself is a field—the gravitational field—and in LQG its discrete chunks are the quanta of this field. Therefore, once we recognize that the gravitational field is both dynamical and quantum, the quantum gravity analog of the lattice used in QCD can be seen as a Feynman graph of a quantum theory, representing the history of gravitational quanta. In this sense, the Feynman graphs of QED and lattices of QCD merge in LQG via Spinfoams.

### III. APPLICATIONS

Exploration of the physical consequences of Asymptotic Safety is still at its beginning. First investigations on both cosmological [118, 119] and black-hole spacetimes [120, 121] have been performed within asymptotically safe QEG. The main idea is to employ a method often used in particle physics that goes under the name *RG improvement*. Here, it amounts to replacing  $G$ ,  $\Lambda$  with  $G_k$ ,  $\Lambda_k$  and identifying  $k$  with an appropriately chosen dynamical or geometrical scale. Since this identification suffers from a certain degree of ambiguity, ultimately the method will have to be replaced by a more precise one. Nonetheless, these investigations have already provided a first idea of the QEG effects to be expected. Because the subject is still evolving, we will discuss these ideas in the ‘Outlook’ section IV B.

On the other hand, three applications of LQG have been investigated in detail over the last 10-15 years, resulting in thousands of publications whose results have been summarized in several detailed reviews (see, e.g., [5, 6, 12, 13, 122, 123]). In this section we will present some highlights of those developments. Even though LQG is still far from being a complete theory, advances could be made by using a *truncation strategy*: One first chooses the physical problem of interest, focuses just on that sector of the full theory which is relevant to the problem, and then uses LQG techniques to analyze it, making full use of the quantum geometry summarized in sections II B.

The section is divided into three parts. In the first we discuss the very early universe; in the second, quantum aspects of black holes, and in the third, the issue of defining  $n$ -point functions in a manifestly background independent theory.

#### A. The Very Early Universe

It is evident from Chapter 3 that there has been a huge leap in our understanding of the early universe over the past two decades. However, on the conceptual front a number

of issues have remained in the Planck era of the very early universe. Over the last decade these issues have been systematically addressed in Loop Quantum Cosmology (LQC).

In particular, the big bang singularity was resolved and cosmological perturbations are being analyzed following several approaches [12, 123–126]. For brevity, we will focus on one of these which provides an internally consistent paradigm starting from the Planck regime, with detailed predictions that are compatible with the WMAP and Planck data. In the first two parts of this subsection we summarize the main results and in the third we present a critical analysis of adequacy of the truncation strategy that underlies the discussion of quantum cosmology in *any* approach.

### 1. Singularity Resolution

Every expanding Friedman, Lemaitre, Robertson, Walker (FLRW) solution of GR, has a big bang singularity if matter satisfies the standard energy conditions. But scalar fields with potentials that feature in the inflationary scenarios violate these energy conditions. Therefore, initially there was a hope that the standard singularity theorems of GR [127] could be avoided in the inflationary context. However, this turned out not to be the case: Borde, Guth and Vilenkin [128] showed, *without any reference to energy conditions*, that if the expansion of a congruence of past directed time-like or null geodesics is negative (on an average), then they are necessarily past incomplete; the finite beginning represented by the big bang in GR is not avoided. But these arguments assume a smooth, classical geometry all the way back to the big bang which has no physical basis since quantum effects cannot be ignored in the Planck regime. Thus, although it is often heralded as reality, big bang is a prediction of classical gravity theories in a domain in which they are *not* applicable. A key result of LQC is that the quantum geometry effects in the Planck regime lead to a natural resolution of the big bang in a wide variety of cosmological models [12, 122].

To illustrate how this comes about, consider the simple example of the  $k = 0$  FLRW spacetimes with a massless scalar field  $\phi$  as a source. It is convenient to fix a fiducial cell  $\mathcal{C}$  in co-moving coordinates and plot these solutions directly in terms of physical variables of the problem, the scalar field and the volume  $v$  of  $\mathcal{C}$ . As the left panel of Fig. 3 suggests, one can regard the scalar field as a *relational* time variable, in term of which the volume  $v$  — and hence the curvature— ‘evolves’. In Bianchi models, the ‘evolving’ quantities would also include anisotropies and in, say, the Gowdy model, the inhomogeneities that encapsulating gravitational waves. Since the massless scalar field *does* satisfy all the energy conditions, all these solutions are singular. In the  $k=0$  FLRW case the universe either expands starting with the big bang or contracts into the big crunch singularity. Quantum cosmology was introduced in the 1970s in the hope that these classical singularities would be tamed by quantum effects [21]. However, in the WDW quantum geometrodynamics of the simple model under consideration unfortunately this hope is not realized [129, 130]. The ideas was revived some three decades later in a pioneering paper by Bojowald who showed that the situation is quite different in LQC [131]. Subsequent conceptual completions and technical improvements of this reasoning have provided a systematic understanding of how this comes about.

First, the uniqueness theorem of LQG kinematics descends to LQC [132] making LQC inequivalent to the WDW theory already at the kinematical level. The WDW differential equation turns out not to be well-defined on the LQC Hilbert space and one has to return to the classical Hamiltonian constraint and systematically construct the corresponding

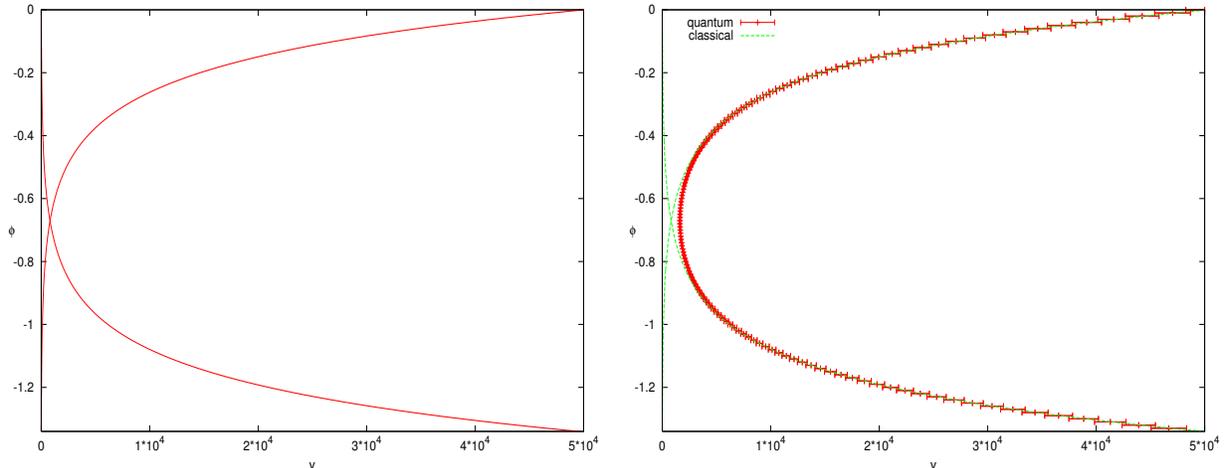


Figure 3: The FLRW model with a massless scalar field. *Left Panel:* In classical GR, there are two types of solutions; those which begin with a big bang expand forever and those that start out with zero energy density and contract to a big crunch singularity. *Right Panel:* In LQG, quantum geometry effects create a novel repulsive force which starts becoming significant when the energy density and curvature are  $\sim 10^{-3}$  in Planck units. The force grows with curvature and the big bang is replaced by a quantum bounce. From A. Ashtekar, *Gen. Rel. Grav.* 41, 707-741 (2006). With kind permission from Springer Science and Business Media.

quantum operator making full use of the underlying quantum geometry of LQG [133]. This construction is subtle and brings out a delicate interplay between the discreteness of *physical* areas and mathematics underlying the definition of the Hamiltonian constraint in the connection dynamics framework [129, 134]. A detailed analytical argument [135] has established that the density operator is well defined on the physical Hilbert space with an upper bound  $\rho_{\max} = 3/(8\pi\gamma^2 G(\Delta a_S))$  which is *directly* controlled by the area gap  $\Delta a_S$  of LQG. Thus, density and curvature cannot diverge in *any* physical state. Numerical simulations showed that this upper bound is in fact reached in states which are sharply peaked at late times and, not surprisingly, it is reached precisely at the bounce. Finally, the LQC singularity resolution has also been established in the consistent histories approach [136].

Details of LQC dynamics can be summarized as follows. Let us first consider classical solution depicted by an expanding trajectory (Fig. 3, left panel). Fix a point at a late time, consider a quantum state sharply peaked at that point and evolve it using the LQC Hamiltonian constraint. One then finds that the wave packet remains sharply peaked on the classical trajectory so long as the matter density or curvature is less than  $\sim 10^{-3}$  of the Planck scale. Thus, in this regime there is good agreement with GR. However, if we evolve the quantum state backwards towards the singularity, instead of following the classical trajectory into the singularity —as is the case in the WDW theory— the wave packet bounces. The expectation value of the energy density now starts decreasing and once it reaches a few thousandths of  $\rho_{\text{Pl}}$ , the peak of the wave packet again follows a classical trajectory along which the universe expands as we continue to move *backward* in time (Fig. 3, right panel). An important feature of LQC dynamics is that while the quantum geometry effects are strong enough to resolve the big bang, agreement with GR is recovered quickly, already when the curvature has fallen by a factor only of  $\sim 10^{-3}$  from the Planck scale. One

can modify Einstein’s equations by introducing some quantum gravity effects by hand and resolve the singularity. But such modifications generally lead to departures from GR already at the density of water! LQC naturally achieves the delicate balance: the UV pathology is tamed leaving GR in tact rather close to the Planck regime.

Although in the Planck regime the peak of the wave function deviates very substantially from the general relativistic trajectory, rather surprisingly it follows an *effective trajectory* with very small fluctuations (see Fig. 3). This effective trajectory was derived [137] using techniques from geometric quantum mechanics. The effective equations incorporate the leading corrections from quantum geometry. They modify the left hand side of Einstein’s equations. However, to facilitate comparison with the standard form of Einstein’s equations, one moves this correction to the right side through an algebraic manipulation. Then, one finds that the Friedmann equation  $(\dot{a}/a)^2 = (8\pi G \rho/3)$  is replaced by

$$\left(\frac{\dot{a}}{a}\right)^2 = (8\pi G \rho/3) \left(1 - \frac{\rho}{\rho_{\max}}\right). \quad (3.1)$$

At  $\rho = \rho_{\max}$ , the right side vanishes, whence  $\dot{a}$  vanishes and the universe bounces. This can occur because the LQC correction  $\rho/\rho_{\max}$  *naturally* comes with a *negative* sign which gives rise to an effective ‘repulsive force’. The occurrence of a negative sign is non-trivial: in the standard brane world scenario, for example, Friedmann equation is also receives a  $\rho/\rho_{\max}$  correction but it comes with a positive sign (unless one makes the brane tension negative by hand) whence the singularity is not resolved. Finally, there is an excellent match between analytical results within the quantum theory, numerical simulations and effective equations. In particular, the effective equations capture the leading LQC corrections to Einstein’s equations very efficiently.

This analysis has been extended to include the cosmological constant of either sign [138], the  $k=1$  FLRW models [134], the Bianchi I, II and IX models which include anisotropies [139, 140] and the Gowdy models which include inhomogeneities [141]. Furthermore the effective equations have been used to show that in LQC *all* curvature singularities—including, e.g., the big rip—are resolved in all FLRW models [142]. These results suggest that the quantum geometry effects of full LQG may well lead to a resolution of all space-like, strong curvature singularities of GR.

Finally, note that in all the models that have been studied in detail, singularity resolution occurs *generically* without any exotic matter or need to fine-tune initial conditions. Furthermore, one does not have to introduce a new boundary condition such as in the Hartle-Hawking proposal. Why then does the LQC singularity resolution not contradict the standard singularity theorems of Penrose, Hawking and others? These theorems are inapplicable because *the left hand side* of the classical Einstein’s equations is modified by the quantum geometry corrections of LQC. What about the more recent singularity theorems that Borde, Guth and Vilenkin [128] proved in the context of inflation? They do not refer to Einstein’s equations. But, motivated by the eternal inflationary scenario, they assume that the expansion is positive along any past geodesic. Because of the pre-big-bang contracting phase, this assumption is violated in the LQC effective theory.

## 2. Phenomenology: Implications of the Pre-inflationary Dynamics

The inflationary scenario has had an impressive success in accounting for the observed 1 part in  $10^5$  anisotropies in the CMB. Therefore, although many of the LQC results hold in

a broad class of early universe paradigms (see, e.g., [143]), for brevity and concreteness we will restrict ourselves to inflation here.

The resolution of the big bang singularity opens a natural avenue to extend this scenario to the Planck regime by systematically investigating the pre-inflationary dynamics. It is often argued that while this phase is conceptually important, it can not be relevant for observations because the near-exponential expansion during inflation would wash away all memory of prior dynamics. The reasoning is that modes seen in the CMB cannot be excited by the pre-inflationary dynamics because, when evolved back in time starting from the onset of the slow roll, their physical wave lengths  $\lambda_{\text{phy}}$  continue to remain within the Hubble radius  $\mathfrak{R}_{\text{H}}$  all the way to the big bang. However, this argument is flawed on two accounts. First, what matters to the dynamics of these modes is the curvature radius  $\mathfrak{R}_{\text{curv}} = \sqrt{6/R}$  determined by the Ricci scalar  $R$ , and not  $\mathfrak{R}_{\text{H}}$ , and the two scales are equal only during slow roll. Thus we should compare  $\lambda_{\text{phy}}$  with  $\mathfrak{R}_{\text{curv}}$  in the pre-inflationary epochs. The second and more important point is that the pre-inflationary evolution should not be computed using general relativity, as is done in the argument given above. One has to use an appropriate quantum gravity theory since the two evolutions can well be very different in the Planck epoch. Therefore, modes that are seen in the CMB could have  $\lambda_{\text{phy}} \gtrsim \mathfrak{R}_{\text{curv}}$  in the pre-inflationary phase. If this happens, these modes *would be* excited and the quantum state at the onset of the slow roll could be quite different from the Bunch Davies (BD) vacuum used at the onset of the slow roll.

Now, another common assumption was that even if there are such excitations over the BD vacuum at the onset of inflation, they would have no effect because they would be diluted away during inflation. However, this is not the case: stimulated emission compensates for expansion so the excitations persist at the end of inflation [144, 145]. Indeed, the difference from the standard prediction could well be so large that the resulting power spectrum is incompatible with the amplitude and the spectral index observed by WMAP. In this case, that particular quantum gravity scenario would be ruled out. On the other hand, the differences could be more subtle: the new power spectrum for scalar modes could be the same but there may be departures from the standard predictions that involve tensor modes or higher order correlation functions of scalar modes, changing the conclusions on non-Gaussianities. In this case, the quantum gravity theory would have interesting predictions for future observational missions [145]. Thus, pre-inflationary dynamics can provide an avenue to confront quantum gravity theories with observations.

To analyze what happens during the pre-inflationary phase, in LQC one proceeds as follows. Since in the inflationary paradigm it is adequate to consider just the FLRW geometries and first order scalar (or curvature) and tensor perturbations  $\mathcal{R}$ ,  $\mathcal{T}$ , one first truncates the full phase of GR to this sector, replaces the FLRW metrics with the quantum wave functions  $\Psi_o$  provided by LQC and investigates the dynamics of first order quantum perturbations  $\hat{\mathcal{R}}$ ,  $\hat{\mathcal{T}}$  on these *quantum* FLRW geometries [124, 146]. Since quantum perturbations now propagate on quantum geometries which are all regular, free of singularities, *the framework automatically encompasses the Planck regime*. What is then the status of the ‘trans-Planckian issues’ discussed in the context of inflation? A careful examination shows that they boil down to the following question: Is the LQC truncation scheme self-consistent? That is, is it consistent to ignore the back reaction and work just with first order quantum perturbations on quantum FLRW backgrounds? This central issue is extremely difficult to analyze in any approach to quantum gravity because it requires a careful treatment of regularization and renormalization of the stress energy tensor of quantum perturbations on

FLRW *quantum* geometries.

The LQC analysis was carried out in detail using the simplest  $\frac{1}{2}m^2\phi^2$  potential that is compatible with the current observations. It has revealed three interesting features [12, 123, 124]. First, there exist quantum states of the background FLRW geometry that remain sharply peaked on solutions to effective equations all the way from the bounce till the curvature has fallen by several orders of magnitude, when general relativity is an excellent approximation. Therefore, one can focus on effective dynamics and ask if these solutions would generically encounter the phase of slow roll inflation that is compatible with observations. It turns out that these solutions are completely determined by the value  $\phi_B$  of the inflaton at the bounce and it is constrained to lie in a finite interval,  $|\phi_B| \in [0, 7.47 \times 10^5]$ . *This is the parameter space of LQC.* For definiteness, let us suppose that the inflaton and its time derivative have the same sign at the bounce. Then, the detailed analysis shows that the dynamical trajectory *will* encounter an inflationary phase compatible with observations (within the WMAP error bars) provided  $\phi_B > 0.93$ , i.e., in almost the entire parameter space [147].

So we can choose an effective trajectory with  $\phi_B > 0.93$ , select a quantum state  $\Psi_o$  which is sharply peaked on it and consider quantum fields  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$  representing scalar and tensor perturbations on the quantum geometry  $\Psi_o$ . At first the problem of studying their dynamics seems intractable. However, the detailed investigation has brought out a second non-trivial and completely unforeseen feature: assuming that the back reaction can be neglected, dynamics of  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$  on quantum geometry  $\Psi_o$  is *identical* to that of quantum fields  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$  propagating on a smooth, classical FLRW metric  $\bar{g}$  constructed from  $\Psi_o$ . This construction is quite subtle and involves rather complicated combinations of the expectation values of various operators in the state  $\Psi_o$ . Thus, although the scalar and tensor modes  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$  propagate on the quantum geometry  $\Psi_o$ , their dynamics is sensitive to only those features of  $\Psi_o$  that are captured in  $\bar{g}$ . This  $\bar{g}$  is a ‘dressed’ effective metric: While the metric determined by the effective equations discussed above knows only about the expectation values,  $\bar{g}$  knows also about certain fluctuations, i.e., a finite number of ‘higher moments’ of  $\Psi_o$ . Physics behind this result can be intuitively understood in terms of a simple analogy: As light propagates in a medium, while there are many interactions between the Maxwell field and the atoms of the medium, the net effect can be neatly coded in just a few parameters such as the refractive index. In LQC, the result provides a powerful technical simplification because it enables one ‘lift’ various well-developed mathematical techniques from QFT on classical FLRW spacetimes to  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$  propagating on quantum geometries  $\Psi_o$ .

However, this analysis assumes that the back reaction can be neglected. One can always start by restricting oneself to states  $\psi$  for which this assumption holds at the bounce. But there is no guarantee that the condition will continue to be satisfied under evolution especially in the Planck regime immediately after the bounce. Does the energy density of the fields  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$  remain negligible all the way from the deep Planck regime of the bounce to the onset of slow roll, removed from the bounce by some 11 orders of magnitude in curvature? This issue can be settled only numerically. These simulations require great care because: i) the renormalization procedure subtracts two diverging terms whence even a tiny loss of precision can result in a significant error; ii) the simulation has to be carried over a very large number of time steps; and, iii) since the background density falls rapidly, even extremely small numerical errors (of the order of one part in  $10^{15}$ ) can be comparable to the background energy density. Simulations with all the due care have been performed to establish firm *upper bounds* on the energy density in perturbations. They showed that if

$\phi_B > 1.23$ , there is a natural choice of initial conditions for  $\psi$  at the bounce such that the back reaction can indeed be ignored from the bounce to the onset of inflation. Furthermore, there are analytical arguments to show that if a state  $\psi$  satisfies this condition, then all states in an open neighborhood do so. Any of these states provide a *self consistent* solution in which the initial truncation hypothesis is seen to be satisfied in the final solution. This is the third non-trivial result. Together, the three results establish that, LQC does provide a self-consistent extension of the standard inflationary scenario to the Planck regime for almost all of the LQC parameter space.

What are then the phenomenological predictions of these self-consistent solutions? The power spectrum and the spectral index have been calculated and, as in the standard inflationary calculations, they agree with observations within error bars. However, there is a small window in the LQC parameter space where certain LQC predictions differ from those of standard inflation. For example, the standard ‘consistency relation’  $r = -8n_t$  relating the ratio  $r$  of the tensor to scalar power spectra to the tensor spectral index is modified [124]. These deviations arise precisely by the mechanism we discussed above: the LQC effective dynamics of the FLRW background is qualitatively different from that of GR so that certain modes *can* have wave lengths  $\lambda$  larger than the curvature radius. Therefore, at the onset of inflation the LQC quantum state  $\psi$  of perturbations has excitations over the BD vacuum in these modes. This departure from the BD vacuum also has implications for the CMB and galaxy distribution [145] and observational tests for such effects have already been proposed [148]. A careful analysis of this window in the LQC parameter space is a focus of current research.

To summarize, LQC has led to a natural resolution of the initial singularity in cosmological models of direct physical interest via quantum geometry effects that replace the big bang with a big bounce [129, 131, 134]. Cosmological perturbations on these quantum geometries have been studied in detail [124–126]. There are natural choices of states at the bounce for which one obtains self consistent extensions of the inflationary scenario all the way to the Planck regime of the bounce. By combining these results with the very rich set of results on inflationary and post-inflationary dynamics, one obtains a coherent paradigm to account for large scale structure, starting right at the quantum bounce. Furthermore, in a small window of the parameter space, this analysis provides results that differ from standard inflation, thereby opening an avenue to extend the reach of observational cosmology to the Planck scale.

### 3. *Is Quantum Cosmology Justified?*

As emphasized in section IIC, in our most successful theories, such as QED and QCD, the *actual* calculations of physical effects have always involved truncations. The mini and midi superspace were introduced in the 1960s in the hope that this truncation would be sufficient to capture the salient quantum effects that tame cosmological singularities. Now that this hope is borne out, it is appropriate to reexamine the strategy and ask: Is this truncation where one ignores an infinite number of degrees of freedom not too severe?

The LQC strategy is guided by the following considerations. First, there is an analogy with Dirac’s solution to the Hydrogen atom problem. From the perspective of full QED, Dirac’s restriction to spherical symmetry is a drastic truncation because it removes all physical photons and ignores all but a finite number of degrees of freedom. But the results of this truncated theory are in excellent agreement with observations and we need quantum

corrections from QED only when the accuracy of experiments is at the level of the Lamb shift when the vacuum fluctuations of the photon field cannot be ignored. The viewpoint is that the situation is similar in cosmology: An analysis of the problem in the mini-superspace approximation *that appropriately takes into account quantum geometry effects from the full theory* should provide a good approximation to the predictions of the full theory. The second source of intuition is provided by the Belinskii Khalatnikov Lifshitz (BKL) conjecture in GR discussed in Chapter 9. It suggests that as one approaches a generic space-like singularity in GR, the local evolution is well approximated by the Bianchi I, II and IX models. Therefore the fate of singularities in Bianchi models is of special interest. A common concern is that even if the big bang is replaced by a big bounce in the isotropic case, typically this singularity resolution would not survive in Bianchi models (primarily because the anisotropic shear terms diverge as  $1/a^6$  where  $a$  is the scale factor). In LQC, by contrast, the big bang singularity is again resolved once the quantum geometry effects from full LQG are correctly incorporated [139, 140]. Furthermore, if one traces the Hamiltonian constraint of the Bianchi I model over anisotropies, one is led precisely to the FLRW hamiltonian constraint, bringing out robustness of the scheme. Finally, in the CDT simulations one finds that even when one allows all fluctuations in geometry keeping only the scale factor fixed, the behavior of the scale factor, including quantum fluctuations, is described accurately by a mini superspace model which assumes homogeneity and isotropy from the outset [10]. Putting together these diverse results, it is not unreasonable to hope that these models adequately capture the behavior of global observables (such as the scale factor and average matter density) that would be predicted by the full theory.

What about the truncation used in treating cosmological perturbations  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$ ? Full LQG *will* admit states in which there are huge quantum fluctuations in the Planck regime whose physics cannot be captured by states of the type  $\Psi_o \otimes \psi$  where  $\Psi_o$  is a state of the quantum FLRW geometry and  $\psi$  is the state of linear quantum perturbations  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$ . It is often implicitly assumed that *all* states of the full quantum gravity will have huge fluctuations. LQC has provided concrete evidence that this need not be the case: there do exist states of the type  $\Psi_o \otimes \psi$  for which truncation is *self consistent*. These states lead to an unforeseen, tame behavior in which  $\hat{\mathcal{R}}, \hat{\mathcal{T}}$  evolve as linear perturbations on a background quantum geometry  $\Psi_o$  carrying energy densities that are negligible compared to that in the background. The non-triviality lies in the fact that these self consistent, truncated solutions lead to the power spectrum and spectral index that are consistent with observations. Thus, the situation is similar to that in the standard  $\Lambda$ CDM model where it suffices to restrict oneself to the simplest cosmological solutions. The early universe appears to be simpler than what one would have a priori imagined!

## B. Black Holes

As is clear from Chapter 4, black holes (BHs) serve as powerful engines that drive the most energetic astrophysical phenomena. But, as discussed in Chapter 10, they have also driven developments in fundamental physics, particularly quantum gravity, raising deep conceptual questions about the statistical mechanical origin of the Bekenstein-Hawking entropy [149, 150] and a quantum gravity description of the BH evaporation process [150]. In this subsection we will provide a brief summary of developments in LQG in this area.

### 1. Quantum Horizon Geometry and Micro-canonical Entropy

In statistical mechanics, entropy is generally associated with systems in equilibrium. BHs in equilibrium were first modeled using event horizons of stationary space-times in GR. However, in statistical mechanics equilibrium refers only to the system under consideration, and not the entire universe. Therefore, about 15 years ago, a quasi-local framework was introduced through the notion of *isolated horizons* (IHs) to better model BHs which are themselves in equilibrium, allowing for dynamical processes in the exterior [151]. Event horizons of stationary space-times as well as the cosmological horizons in de Sitter space-time are special cases of IHs. Interestingly, the first law of BH thermodynamics naturally extends to IHs, with a further advantage that mass and angular momentum in the law now refer to the BH itself, defined at the IH, rather than to the ADM quantities defined at infinity which receive contributions also from the exterior region [152]. Its form is again similar to the first law of thermodynamics, suggesting that a multiple of the area  $a_\Delta$  of the IH  $\Delta$  should be interpreted as entropy  $S_\Delta$ . Hawking's analysis of quantum radiance provides the numerical value of the multiple, yielding the Bekenstein-Hawking formula, now for IHs:  $S_\Delta = a_\Delta/4G_N\hbar$ .

In LQG, one investigates the statistical mechanical origin of this entropy,  $S_\Delta$  [13]. The shift of focus to IHs has two advantages. First, one can consider realistic, astrophysical black holes: Not only does one not have to invoke 'charges' to make BHs near-extremal, but one can even allow for distortions in the horizon geometry that may be caused by matter rings or other black holes. Second, the cosmological horizons (for which thermodynamic considerations are known to hold) are automatically incorporated. The idea is to first investigate the quantum geometry of these IHs [153, 154] and then calculate the number of quantum microstates in the specified ensemble [13, 155]. This procedure provides a statistical mechanical derivation of entropy in terms of quantum geometry. We will now summarize these developments.

As before, one carries out a truncation of the theory that is motivated by the physical problem of interest. Thus, one begins with the phase space of GR in connection dynamics, now with a spatial 3-manifold  $M$  that is asymptotically flat *and* has an internal boundary  $S$ , the intersection of  $M$  with an IH 3-manifold  $\Delta$ . Detailed analysis shows that the total phase space can now be written as  $\Gamma = \Gamma_{\text{bulk}} \times \Gamma_S$  where  $\Gamma_S$  turns out to be the phase space of an U(1) Chern-Simons theory. The IH boundary condition relates the curvature  $F$  of the U(1) Chern-Simons connection to the pull-back  $\underline{\Sigma}$  of the 2-form  $\eta_{abc}E_i^c r_i$  where  $r^i$  is the unit internal vector normal to  $S$ :  $F = -(2\pi/a_\Delta) 8\pi G_N \gamma \underline{\Sigma}$ .

For this sector of GR, one has to extend the quantum geometry framework of section II B to allow for an inner boundary  $S$  corresponding to an IH. The bulk Hilbert space  $\mathcal{H}_{\text{bulk}}$  is again spanned by spin networks. However, the links of these spin-networks can now end on the boundary  $S$ , piercing it on a node (see Fig. 4). The surface Hilbert space  $\mathcal{H}_{\text{CS}}$  is now the Hilbert space of an U(1) Chern-Simons theory on the resulting punctured sphere, with the level (or, dimensionless coupling constant)  $k = a_\Delta/4\pi\gamma\ell_{\text{Pl}}^2$ . The total kinematical Hilbert space is now a tensor product  $\mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{bulk}} \otimes \mathcal{H}_{\text{CS}}$ . States in  $\mathcal{H}_{\text{kin}}$  are now subject to the *quantum horizon boundary condition* which is an operator equation:

$$(1 \otimes \hat{F})\Psi = -\frac{2\pi}{a_\Delta} 8\pi G_N \gamma (\underline{\Sigma} \otimes 1)\Psi \quad (3.2)$$

Note that solutions to (3.2) can exist only if  $\hat{F}$  on the surface Hilbert space  $\mathcal{H}_{\text{CS}}$  has the same eigenvalues as the triad operator  $\underline{\Sigma}$  on  $\mathcal{H}_{\text{bulk}}$ . This is a highly non-trivial condition since the

two operators have been defined *completely independently* on two *distinct* Hilbert spaces. However, the framework passes this severe test because the two operators share an infinite number of eigenvalues. Finally, the physical meaning of this condition is as follows: The intrinsic curvature of the IH can fluctuate and so can the bulk geometry in its neighborhood, but they have to fluctuate in tandem, satisfying (3.2).

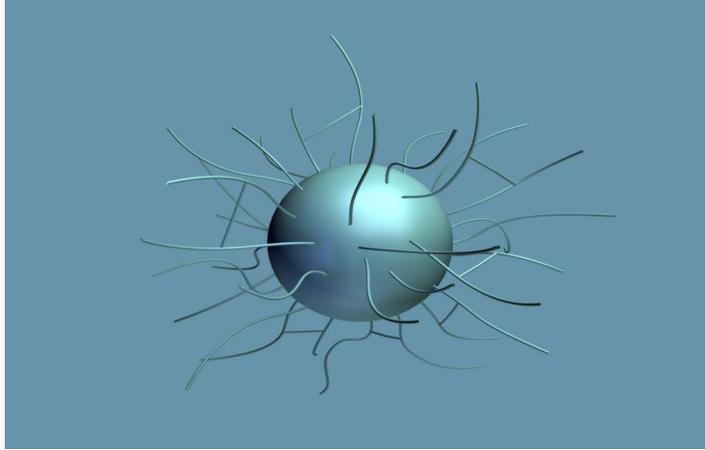


Figure 4: An artist's rendering of an isolated horizon punctured by spin-network links. Image credit: Alejandro Corichi.

Next, one has to impose quantum constraints. There are some interesting subtleties which lead to mapping class groups and quantum deformation of  $U(1)$  on  $S$  [153]. The final result is that what matters is only the number of punctures, not their location on  $S$ , *and* that each puncture has to be treated as ‘distinguishable.’ We emphasize that these are systematic implications of the constraint equations, and *not* additional inputs, as is sometimes thought. The net result is that, assuming the Hamiltonian constraint does admit a sufficient number solutions in the bulk, LQG provides a coherent description of quantum space-times with IHs [153]. This quantum geometry is depicted in Fig. 4. (For more detailed summaries, see e.g. [2, 13]).

To calculate entropy, one has to fix an ensemble and count the number of states compatible with the macroscopic parameters characterizing the ensemble. This is done via the notion of multipole moments that characterize the geometry of  $\Delta$  in a diffeomorphism invariant manner [154]. (In the simplest case when all moments except the mass monopole vanish, the IH is spherically symmetric with respect to *some*  $SO(3)$  action.) The ensemble is specified by requiring that all multipoles lie in a small interval around some pre-specified values. The idea is to calculate the number  $\mathcal{N}$  of quantum microstates of the horizon geometry that satisfy this constraint. Its logarithm gives the microcanonical entropy of the ensemble.

To determine  $\mathcal{N}$  one has to count specific types of finite sequences of half integers subject to certain constraints [13]. This problem has been investigated in detail in a series of mathematical papers by Barbero, Villaseñor and others that are of interest in their own right [155]. They combine known types of Diophantine equations with techniques involving generating functions and Laplace transforms. The final result is that the micro-canonical entropy is given by

$$S_{\text{micro}} = \frac{\gamma_o}{\gamma} \frac{a_\Delta}{4\ell_{\text{Pl}}^2} + O\left(\ln \frac{a_\Delta}{\ell_{\text{Pl}}^2}\right) \quad (3.3)$$

where  $\gamma_o \sim 0.2$  is a root of an algebraic equation<sup>5</sup> [13]. Thus, in the sector of the theory where the BI parameter is set to  $\gamma_o$ , one recovers the Bekenstein-Hawking result to the leading order with a logarithmic correction. The Barbero-Immirzi parameter is a quantization ambiguity in LQG, rather similar to the  $\theta$ -ambiguity in QCD [57]. In QCD, the value of this parameter is determined experimentally. In LQG, an experimental measurement of, say, the area gap would similarly determine  $\gamma$ . The LQG viewpoint is that while such measurements are completely out of reach of current technology, the Bekenstein-Hawking formula can be used as a theoretical constraint to determine  $\gamma$ . Note that once we set  $\gamma = \gamma_o$  to get agreement with this formula for one type of IH (say spherical ones), the agreement extends to all IHs.

## 2. Semiclassical Considerations and Dynamical Processes

The description of quantum horizons we just summarized has the advantage that it is fully background independent. But that very feature makes it difficult to relate it to the rich body of semi-classical results that have been derived in Kerr and Rindler space-times. Therefore, over the last three years, two independent avenues have been introduced to make closer contact with semi-classical results and study quantum dynamical processes. In this subsection we will briefly describe their current status.

In the first approach, developed by Ghosh, Perez and others, [156] one considers the near horizon geometry of Kerr space-times and asks the question: How would near horizon, stationary observers describe physics *within* LQG? Denote by  $\chi^a$  the Killing vector which is the null normal to the Kerr horizon  $\Delta$  and consider observers  $\mathcal{O}$  with 4-velocity  $u^a = \chi^a / \sqrt{\chi \cdot \chi}$ , at a fixed distance  $d \ll R_\Delta = \sqrt{(a_\Delta/4\pi)}$  from  $\Delta$ . Note that the observers  $\mathcal{O}$  are approximately at rest with respect to  $\Delta$  since their angular momentum is  $O(d/R_\Delta)$ . If one were to consider the Hamiltonian framework with a boundary at the location of the observers  $\iota$ , one would find that the Hamiltonian acquires, in addition to the ADM surface integral at infinity, a 2-surface integral  $H_{\mathcal{O}}$  at the inner boundary which, one argues, is given by  $H_{\mathcal{O}} = a_\Delta / (8\pi Gd)$ . In LQG, the corresponding operator is  $\hat{H}_{\mathcal{O}} = \widehat{\text{Ar}}_S / (8\pi Gd)$ , where  $\widehat{\text{Ar}}_S$  is the area operator of section II B, now associated with the intersection  $S$  of  $\Delta$  with a partial Cauchy surface  $M$  used in the Hamiltonian framework. Next, since the acceleration of  $u^a$  is given to the leading order by  $1/d$ , one assumes that observers  $\mathcal{O}$  would experience the Unruh temperature  $T_U = 1/(2\pi d)$  [157]. This is supported by two independent considerations: i) if one red-shifts  $T_U$  to infinity, one obtains the Hawking temperature  $T_H$ , and, ii) detectors carried by the observers  $\mathcal{O}$  coupled to  $\hat{H}_{\mathcal{O}}$  would read local temperature  $T_U$  [156].

Using these ingredients, one arrives at the following physical picture: the observers  $\mathcal{O}$  would describe the punctured quantum horizon  $\Delta$  as a grand canonical ensemble  $\tilde{\rho}_{\mathcal{O}}[\beta, \mu; \gamma]$  of *punctures*  $p$  endowed with spin labels  $j_p$ , at an inverse temperature  $\beta_U = (2\pi d)/\hbar$  and a chemical potential  $\mu$ . (The dependence on the Barbero-Immirzi parameter  $\gamma$  comes from  $\hat{H}_{\mathcal{O}}$ ). As usual, this is equivalent to a canonical ensemble  $\rho_{\mathcal{O}}[\beta_U; \gamma]$  in which  $-T \frac{\partial S}{\partial N} |_E$  equals the chemical potential  $\mu$  of the grand canonical ensemble. An explicit calculation of  $-T \frac{\partial S}{\partial N} |_E$  provides  $\mu$  as a function  $\mu(\gamma)$  of  $\gamma$ . Finally, recall from section II B that the level

<sup>5</sup> Because the surface states on  $\Delta$  are intertwined with the bulk spin network states, a priori one can give assign two meanings to the term ‘pure surface terms’ that are to be counted. They lead to two values  $\approx 0.27$  and  $0.24$  of  $\gamma$ . (See, e.g., [13].) In detailed LQG calculations this difference only changes numerical values by small amounts. But conceptually it is important to better understand and resolve this ambiguity.

spacing between eigenvalues of the area operator goes to zero exponentially for large areas. Hence the energy required to create a new puncture is arbitrarily small for large black holes. One therefore makes the final assumption that, as for photons, the physical value of the chemical potential  $\mu$  should be zero. This condition determines  $\gamma$  uniquely and the value is precisely the  $\gamma_o$  arrived at by state counting in the micro-canonical ensemble irrespective of the choice of  $d$  (which enters only in the local temperature that  $\mathcal{O}$  attribute to the BH). In the resulting canonical ensemble  $\rho_{\mathcal{O}}[\beta_U; \gamma_o]$  that the observers  $\mathcal{O}$  would use to describe the BH, the entropy is given by  $S = a_{\Delta}/4\ell_{\text{Pl}}^2$  to leading order, exactly as in the micro-canonical ensemble. Thus, these semi-classical considerations provide the same final result but with a novel description of the quantum horizon as a gas of punctures carrying spins. Therefore, this approach opens new avenues to describe dynamical processes, including the BH evaporation.

The second and complementary development is due to Gambini and Pullin [158] and follows a strategy that is analogous to the one used in LQC. It considers a different truncation of GR, that of spherically symmetric space-times. While this truncation was discussed in the LQG literature already in the 90s, the *global* structure of the quantum space-times — including both the asymptotic part and the portion that is classically inside the horizon— was analyzed relatively recently.

In this symmetry reduced model, it suffices to consider spin networks with graphs along just the radial line. However, the nodes now carry additional labels that encode information about the connection and geometry in the two transverse directions. While there are close similarities with LQC, there is a major difference: Since the 3-geometry is now inhomogeneous, we have infinitely many Hamiltonian as well as diffeomorphism constraints, smeared with radial lapse and shift fields [159]. Remarkably, it is possible to express solutions to the Hamiltonian constraints in a closed form as a linear combination of the spin networks [158]. The diffeomorphism constraint can then be solved as in section II C by group averaging [70]. In the resulting physical Hilbert space, the ADM mass is a Dirac observable as in the classical theory. Furthermore, as in LQC, by appropriately deparameterizing the theory, one can also express the metric as a parameterized Dirac observable. As one would expect from quantum geometry, the metric is an operator valued distribution concentrated at the nodes of the spin networks. There are semi-classical states which upon coarse graining on an appropriate scale —say, a thousand times the Planck length— yield smooth classical geometries. However, as in LQC, the quantum space-time is singularity-free and, as was anticipated by calculations within effective LQG equations for this model, the quantum space-time is ‘larger’ than that of classical GR. At a technical level, the fact that one can solve the infinite set of both Hamiltonian and diffeomorphism constraints is highly non-trivial.

As in LQC, it is now natural to investigate the behavior of test quantum fields on the quantum geometry of the symmetry reduced model. For this, one now truncates the theory allowing linear scalar fields on spherically symmetric space-times, again ignoring the back reaction in the first step. Then, as in LQC, the scalar field  $\hat{\Phi}$  now propagates on a quantum state  $\Psi_o$  of the background geometry that, on coarse graining, yields the classical Schwarzschild geometry of a large black hole. In the interaction picture, in the approximation in which the back reaction is ignored, the field  $\hat{\Phi}$  again propagates on an effective dressed quantum geometry. The main effect of the background quantum space time on quantum field theory is to replace the partial differential equation governing  $\hat{\Phi}$  with a difference equations. However, for frequencies (at infinity) which are significantly smaller than the Planck frequency, there is negligible difference from the thermal spectrum at infinity. This

is not surprising because the Hawking radiation is robust with respect to the near horizon microstructure of space-time [160]. But conceptually the underlying discreteness of quantum geometry does have one important effect: it removes the UV divergences encountered in the Boulware and Unruh vacua at the horizon [158].

This recent development has provided a coherent framework to describe Hawking radiation from first principles using the strategy of truncating LQG to the physical problem of interest. At a technical level, as we indicated, there is a close similarity with the framework used in LQC. On the physical side, on the other hand, there is a difference. Since the issue of the back reaction of *quantum* perturbations is significant only in the very early universe, in LQC one could analyze this issue systematically and show that the truncation used is physically self-consistent. For black holes, on the other hand, it is the back reaction that drives the evaporation process. Therefore, the truncation used so far for black holes is not adequate to systematically analyze the issue of information loss.

### C. $n$ -point Functions in a Diffeomorphism Invariant Theory

As Wightman emphasized already in the 1950s, in MQFTs the  $n$  point functions

$$W(x_1, \dots, x_n) = \langle 0 | \phi(x_n) \dots \phi(x_1) | 0 \rangle, \quad (3.4)$$

completely determine the theory [161]. In particular, one can calculate the scattering amplitudes from these distributions. However, since they make an explicit reference to the Minkowski metric, it is far from being a priori clear that these ideas can be extended in a meaningful manner to non-perturbative quantum gravity. Indeed, at first it may appear that, because manifolds do not admit non-trivial, diffeomorphism invariant  $n$ -point distributions, a background independent framework cannot lead to non-trivial  $n$ -point functions either. However, as we will see, this argument is too naive. The  $n$ -point functions refer to a state and in gravity that state can naturally encode information about a specific geometry which can then appear in the expressions of these distributions. In particular, LQG does lead to non-trivial  $n$ -point functions. Furthermore, to the leading order they have been shown [162–165] to agree in the appropriate sense with the  $n$ -point functions calculated in the effective low energy quantum general relativity [14] referred to in section II A. These calculations have created a bridge from the rather abstract and unfamiliar background independent framework of LQG to notions and techniques used in concrete calculations in familiar MQFTs.

To spell out the construction, let us first return to MQFTs and recall that an  $n$ -point function can be written as a path integral.

$$W(x_1, \dots, x_n) = \langle 0 | \phi(x_n), \dots, \phi(x_1) | 0 \rangle = \int D\phi \phi(x_n) \dots \phi(x_1) e^{i\mathcal{S}[\phi]} \quad (3.5)$$

For simplicity, consider the two-point function. We can organize the integration in (3.5) as follows. Select an arbitrary *compact* region  $\mathcal{R}$ , as in the Fig. 5, such that the points  $x, x'$  of interest lie on its boundary  $b$ . Denote by  $W(\varphi)$  the integral over fields  $\phi$  defined only on  $\mathcal{R}$  with the boundary value  $\varphi$  on  $b$ , and by  $\Psi_b[\varphi]$  the integral over fields defined only on the exterior region  $M - \mathcal{R}$ , again with the boundary value  $\varphi$  on  $b$  (and appropriate fall-off at infinity). Then, we have:

$$W(x, x') = \int D\varphi W[\varphi] \varphi(x) \varphi(x') \Psi_b[\varphi_b] \quad (3.6)$$

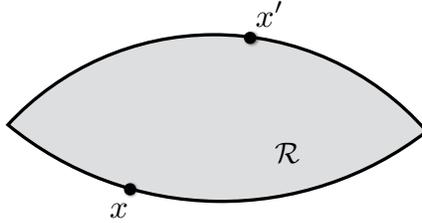


Figure 5: Geometrical interpretation of the transition amplitude.

From the perspective of the region  $\mathcal{R}$ , this expression can be interpreted as providing the 2-point function for the boundary state  $\Psi_b[\varphi]$ , the transition amplitude ( $\sim e^{iS_{\mathcal{R}}}$ ) being given by  $W[\varphi]$ .

In the form (3.6), the functional integral can be taken over to quantum gravity using Spinfoam transition amplitudes  $W_{\mathcal{C}}$  introduced in section II B. But there are crucial differences in the underlying structures that are needed in MQFTs versus LQG. In MQFTs, in addition to the value  $\varphi$  of the field on the boundary  $b$ , we must *also* use the background metric to fix the shape and geometry of the boundary  $b$ , and the spacetime distance between  $x'$  and  $x$ . Only then can we calculate the transition amplitude  $W[\varphi]$  associated with the region  $\mathcal{R}$ , and the boundary state  $\Psi_b[\varphi]$ :

$$W[\varphi] = \int D\phi e^{i \int_{\mathcal{R}} L[\phi]}, \quad \text{and} \quad \Psi_b[\varphi] = \int D\phi e^{i \int_{M-\mathcal{R}} L[\phi]}, \quad (3.7)$$

But in LQG, these expressions cannot and do not make reference to a background metric. As we saw in section II B, a signature of the non-triviality of the LQG construction is that the transition amplitude  $W[\varphi]$  is a function only of the field  $\varphi$ . It does not depend on any background structures; it refers only to the dynamical fields and the physical process of interest.

How can one then make contact with the effective field theory calculations which do refer to a background geometry? This is achieved through the state  $\Psi_b[\varphi]$ . Because of the functional integration involved in its definition,  $\Psi_b[\varphi]$  depends on the dynamics of the theory as well as the boundary conditions at infinity. But as in MQFTs, these ingredients are invisible to the calculation of the 2-point function (3.6); what matters directly is the state  $\Psi_b[\varphi]$  itself. But the key difference from MQFTs is that, because we are considering gravitational fields, *this state now encodes information about the geometry*. For appropriate boundary conditions at infinity,  $\Psi_b[\varphi]$  would be peaked at a certain classical (intrinsic and extrinsic) 3-geometry of  $b$ . Therefore, it assigns to  $b$  a certain shape and size and also selects a classical 4-geometry in  $\mathcal{R}$  that extremizes the classical action for the given boundary geometry  $\varphi$  at  $b$ . The relative position of the  $n$  arguments of the  $n$ -point function is well defined with respect to that 4-geometry. In particular, for comparison with the effective theory, one picks a region  $\mathcal{R}$  of Minkowski spacetime and approximates it with a triangulation that is sufficiently fine to capture the relevant dynamical scale of the phenomenon of interest. One then determines the intrinsic and extrinsic geometry of the boundary of this region, and picks a quantum state  $\Psi \in \mathcal{H}_{\Gamma}$  of gravity peaked on these data. Then the two points  $\vec{x}$  and  $\vec{x}'$  sit on nodes of the boundary graph. The operators  $E_{\ell}$ , associated to these nodes and links, provide the geometric interpretation of the quanta in the boundary state.

This construction resolves a longstanding confusion in quantum gravity, referred to in the very beginning of our discussion. Formally, if  $S[g]$  is a generally covariant action and  $D[g]$  a generally covariant measure, the distribution

$$W(x_1, \dots, x_n) = \int D[g] g(x_1) \dots g(x_n) e^{iS[g]} \quad (3.8)$$

would be trivial, i.e., independent of the position of the points  $(x_1, \dots, x_n)$  (as long as they do not overlap). Therefore (3.8) are not the physically interesting  $n$ -point functions in a generally covariant theory. In particular, they are completely unrelated to the  $n$  point functions used in the effective field theory: while the (3.8) refer only to the  $n$  points  $x_1 \dots x_n$  in the spacetime manifold, those in the effective theory depend on *physical distances* between these points computed in the background metric. In the LQG construction, the  $n$  point functions *do know* about the physical distances between  $x_n$  in the background geometry determined by the boundary state  $\Psi_b[\varphi]$ . However, under a spacetime diffeomorphism that sends  $\mathcal{R} \rightarrow \mathcal{R}'$ ,  $b \rightarrow b'$  and  $(x_1, \dots, x_n) \rightarrow (x'_1, \dots, x'_n)$  and  $\varphi \rightarrow \varphi'$  the boundary state also transforms covariantly,  $\Psi_b(\phi) \rightarrow \Psi_{b'}(\phi')$ , whence the geodesic distances between any two points  $x_i, x_j$  on the boundary and their images  $x'_i, x'_j$  are preserved. Therefore, the final results are diffeomorphism invariant.

Calculation of a  $n$ -point functions have been performed in this framework, using states and operators of the canonical theory and the transition amplitude (2.15) provided by Spinfoams [162–164]. The result is that, in a suitable semiclassical limit (to terms  $\mathcal{O}(\hbar)$ ) the two-point function *exactly matches* [163] with the one obtained from Lorentzian Regge calculus [100]. In turn, this limit is consistent with the effective field theory [14]. Thus, although the basic notions and techniques appear to be very different from those used in perturbative treatments the final results show that, as in the Asymptotic Safety scenario, a peaceful co-existence with low energy results is possible. A program to make a systematic connection with effective field theory has been initiated recently [165]. Some radiative corrections have also computed using more refined 2-complexes, containing bubbles [166]. Ideally, one would hope that the low energy behavior of the  $n$  point functions computed in the non-perturbative theory would agree with the effective theory, while at high energies the LQG calculations would provide an UV completion of the non-renormalizable perturbation theory. Whether this is the case is still an open question.

The conceptual non-triviality of results to date lies in the fact that they provide a streamlined approach to compare the non-perturbative theory with background dependent effective theories, without sacrificing the underlying diffeomorphism invariance. Reconciling the two had been a long standing open issue in quantum gravity.

## IV. DISCUSSION

In this section we will first summarize the main ideas and results, putting them in a broader context, and then discuss open issues that remain.

### A. Summary

Approaches discussed in this Chapter are rooted in well-established physics: principles of GR and QFT. The viewpoint is that ideas that have no observational support should not

constitute an integral part of the foundation of quantum gravity, even when they can lead to rich mathematical structures. In particular, these approaches do not rely on a negative cosmological constant, or extended objects or specific matter content involving towers of fields and particles.<sup>6</sup> The primary goal is not unification with other fundamental forces. Rather, the emphasis is on qualitatively new insights into *quantum* spacetimes that can emerge from non-perturbative techniques. In classical GR, the dynamical nature of geometry led to new phenomena —such as gravitational waves, black hole horizons and the big bang— that could not even be imagined before. As earlier Chapters in this volume vividly bring out, these notions have had deep impact on the subsequent developments in astrophysics, cosmology, computational physics, and geometric analysis. In these developments one can often use perturbative techniques but they have to be built around the novel non-linear configurations and use qualitatively new boundary conditions, dictated by full general relativity. The expectation is that the situation will be similar in the quantum domain with the new, unforeseen features of quantum geometry. Results to date, e.g. on UV finiteness and physics of the very early universe discussed in sections II and III provide concrete evidence in favor of this expectation.

We saw in section IIA that the new notion of Effective Average Action (EAA) in the continuum can be used to give a meaning to the basic functional integral by reformulating the problem as a question about solutions of a functional flow equation [23]. These renormalization group trajectories possess a well defined ultraviolet limit, allowing one to reconstruct the functional integral from them if they hit a non-Gaussian fixed point in this limit. We summarized the evidence for the existence of such a non-trivial fixed point [23, 28–30]. We then discussed CDT, a lattice approach based on statistical mechanics ideas [10, 39, 40, 167].

Over the last two decades, concrete progress has occurred by carrying out suitable, finite dimensional truncations of the infinite dimensional theory space  $\mathcal{T}$ . The initial truncation had only two coupling constants,  $G_N$  and  $\Lambda$ , corresponding to the Einstein-Hilbert and the cosmological constant terms. By now, the truncations have reached a mature level, allowing for *nine* different coupling constants in the gravitational sector. Not only has the non-trivial fixed point persisted but there is consistency with the two dimensional truncation in a precise sense. The analysis has also been extended beyond pure gravity and several matter couplings have been investigated in detail [11]. These results have provided highly non-trivial evidence in support of Asymptotic Safety.

The CDT approach has led to an unforeseen result that had eluded earlier lattice simulations: In the Euclidean signature, de Sitter spacetime with small fluctuations was shown to emerge from Monte Carlo simulations using the discretized Einstein-Hilbert action. The primary importance of this demonstration is not so much that it is the de Sitter spacetime that resulted but rather that the result has an interpretation as a 4-dimensional classical geometry in the first place. Indeed, previous dynamical triangulation simulations had led only to ‘crumpled’ or ‘polymer-like’ phases rather than the one corresponding to a smooth macroscopic geometry.

In LQG, the emphasis is again on non-perturbative methods. But while in the EAA framework one introduces a background metric  $\bar{g}_{ab}$  in the intermediate stages, splits the physical metric  $\tilde{g}_{\mu\nu}$  as  $\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$  and interprets  $\mathcal{D}\tilde{g}_{\mu\nu}$  as an integration over the nonlinear

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<sup>6</sup> In the same spirit, they do not *demand* supersymmetry nor higher dimensions, but the methods used in these approaches have been extended to incorporate these possibilities [11, 48, 49].

fluctuations,  $\mathcal{D}\tilde{h}_{\mu\nu}$ , the LQG framework is *manifestly* background independent. Quantum geometry, developed in the canonical framework [2–5] provides well-defined techniques to carry out path integrals directly in terms of Spinfoams [3, 5, 6] which represent physical, quantum spacetimes, without any split.

In section II C we discussed LQG dynamics via a Spinfoam model [92–95]. This model has drawn a great deal of attention because it represents a notable confluence of ideas from apparently distinct directions: Canonical LQG [2, 4], Regge calculus [100–102], topological field theories and group field theory [3, 96, 103]. The number of simplices that feature in the underlying 4-geometries provides a mathematically natural expansion parameter to calculate transition amplitudes. These amplitudes are UV finite to any order in this expansion [92–95] and, in presence of a positive cosmological constant, there is an elegant procedure involving a quantum deformation of  $SL(2, \mathbb{C})$  (the double cover of the local Lorentz group) that provides a natural infrared regulator [112, 113]. Finally, in III C we summarized the construction of  $n$ -point functions in the semi-classical limit of this model. The leading term in the 2-point function reproduces the low energy graviton propagator in a precise sense [162–164]. These developments have begun to create a bridge [165] from the background independent, non-perturbative framework of LQG to effective field theories that encompass low energy scattering processes in quantum gravity. Thus, LQG offers a well-defined set of fundamental equations describing quantum spacetime, free of UV and IR divergences in a natural expansion, with substantial evidence for GR to emerge in a suitable limit.

Over the past two decades, LQG has also been used to analyze long-standing issues which originally constituted the main motivation for quantum gravity. As in Asymptotic Safety, progress has occurred by truncating the full theory appropriately and analyzing the truncated sectors in detail. But now truncations are motivated directly by each physical problem under consideration. In the cosmological truncation, discussed in section III A, not only are the strong curvature singularities naturally resolved by the quantum geometry effects [12, 129, 131, 134, 139, 141, 142] but the standard paradigms have been extended all the way to the bounce by facing the Planck regime squarely, using quantum field theory on *quantum* cosmological space-times [124–126, 146]. Furthermore there is a small window in the parameter space where the theory can be confronted with future observations. In section III B, we summarized the current status of quantum black holes in LQG. Using the notion of isolated horizons and techniques from quantum geometry one can treat all black hole and cosmological horizons in one go, without having to restrict oneself to extremality [153, 154]. More recently, semi-classical considerations have brought the LQG description closer to the more familiar treatments in terms of energy and temperatures measured by suitable families of near-horizon observers [156]. Finally, there is now a novel approach to investigate the quantum evaporation process, using quantum field theory on *quantum* space-times describing black holes in LQG [158].

## B. Outlook

Every quantum gravity program faces two types of issues: i) those which are *internal* to any given program which must be resolved before one has a conceptually complete, coherent theory with the correct low energy limit in 4 spacetime dimensions; and, ii) those which are *common* to all programs, addressing the long standing physical questions. As our summary illustrates, concrete advances have occurred on both these fronts. However, a number of important challenges remain. In particular, so far *none* of the approaches to quantum

gravity satisfies the ‘internal’ criterion of completeness.

We will now illustrate these challenges and ensuing opportunities through examples. Strategies summarized here are necessarily provisional; our primary intent is only to provide a general idea of the directions that are being currently pursued.

- **Infrared Issues:** In the EAA approach of the Asymptotic Safety program there exist trajectories admitting the non-trivial UV fixed points which are known to reduce to GR in the low energy limit. In the CDT approach, because of the usual limitations on the size of lattices that can be handled in simulations, the smallest physical length  $a$  of the link is still about  $2\ell_{\text{Pl}}$ , and the infrared regime corresponds to  $\sim 20\ell_{\text{Pl}}$  [10]. The IR behavior is already illuminating in that not only does a classical de Sitter geometry with small quantum fluctuations arise in the Euclidean signature, but this occurs even for universes of radius  $\sim 20\ell_{\text{Pl}}$ .<sup>7</sup> An important open issue is whether other physically interesting spacetimes can be recovered from CDT.

In full LQG, as we saw in sections II C and III C, low energy limit is recovered in a certain well-defined sense. However, there is considerable room for improvement. In particular, although the boundary states currently used are well-motivated, being peaked on the metric as well as the extrinsic curvature of the boundary induced by the Minkowski metric, there is still considerable ambiguity in their choice which descends to the transition amplitude and  $n$ -point functions. Conceptually, this is not problem because both these quantities are, by definition, functions of the boundary state. But different boundary states would give rise to different sub-leading terms, making comparison with the effective theory ambiguous. A principle to select *canonical* states corresponding to Minkowski and de Sitter space-times is still lacking. A second important limitation is that most of the results on classical and semi-classical limits we summarized have been carried out using only one simplex. There is substantial ongoing work that considers refinements, allowing a large number of simplexes in the interior, keeping the boundary state peaked on the classical geometry of interest. These results will either firmly establish the infrared viability of the specific Spinfoam model that is currently used, or, suggest better alternatives.

- **Matter couplings:** In the cosmological truncation of LQG, matter fields have been incorporated and their effect has been analyzed in detail [12, 122, 123]. In full connection dynamics, matter couplings have been discussed exhaustively at the classical level [4, 46, 47] and the framework has also been extended to incorporate supersymmetry [49]. However, at the quantum level, so far only formal schemes have been laid out in the full theory [4, 91]. Interestingly, one can arrive at a unification which is ‘dual’ to the Kaluza-Klein scheme in the following sense: One can continue with 4 spacetime dimensions but enlarge the *internal* group to a product of the Lorentz group (associated with gravity) with groups associated with Yang Mills fields governing other interactions [168]. Whether these ideas are fully compatible with particle physics phenomenology is, however, still unclear. More generally, constructing a detailed quantum theory with matter coupling represents a challenging and fertile area for LQG research in coming years.

In the Asymptotic Safety program, by contrast, there is already very substantial work on incorporating matter. It has provided interesting constraints on the number of fermions and

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<sup>7</sup> Interestingly, the same scale arose completely independently in LQC: in the  $k = 1$  Lorentzian FLRW cosmology, for example, dynamics of the quantum wave functions is accurately described by GR once the radius of the universe exceeds  $8\ell_{\text{Pl}}$  even in the case when the universe grows to a radius only of  $23\ell_{\text{Pl}}$  before undergoing a recollapse a la GR [134].

gauge fields that can be accommodated within this scenario, constraints that are satisfied by the standard model of particle physics [11]. Furthermore, insights on the quantum nature of geometry provided by results to date are likely to have implications to the ultraviolet issues in field theories of other interactions as well. Indeed, there are already indications that the coupling to asymptotically safe gravity might cure certain notorious problems in the matter sector [169, 170], and it is conceivable that the coupled system is more predictive than the standard model of particle physics without gravity. There are for example scenarios in which the Higgs mass [171] or the fine structure constant [169] are computable quantities. These promising ideas are likely to be more fully developed in the coming decade.

- **New Physics:** In both approaches, there is a large number of avenues that will be pursued to explore new physics. We will present just a few illustrative examples.

In the Asymptotic Safety program one is naturally led to an EAA-based ‘quantum geometry’ of spacetime which goes beyond Riemannian geometry in a specific sense: in general, the metric is scale dependent. So a single (smooth) manifold is furnished not with just one metric, but rather a family,  $\{\langle g_{\mu\nu} \rangle_k, 0 \leq k < \infty\}$ , where  $\langle g_{\mu\nu} \rangle_k$  is a solution of the effective field equation following from  $\Gamma_k$ . This general framework [172] was used, for instance, to demonstrate that under certain conditions the EAA, while defined in the continuum, can give rise to a dynamically generated minimum length scale. It was also used to analyze the fractal-like properties of the ‘quantum spacetimes’ which follow from the EAA [29, 30]. In particular a running spectral dimension has been defined and computed [173, 174]. One finds that there is a dimensional reduction from 4 macroscopic to 2 microscopic dimensions.<sup>8</sup> As a consequence, the graviton propagator is modified near the UV fixed point [29, 118]. These novel features have interesting implications for the early universe and black holes [118–121] which provide interesting avenues for future research.

Applications of LQG discussed in section III also provide a number of interesting directions to explore new physics. First, as we discussed in section III A, there is a small window in the parameter space where LQC leads to new predictions [123, 124, 126, 148]. It needs to be analyzed in much greater detail, keeping in mind the planned astronomical surveys. While the a priori probability that this window is realized in Nature is small, if the initial observations were to favor it, it will be possible to use novel avenues to confront the theory with observations in detail, precisely because the window is small. On the conceptual front, there are a number of issue concerning the specification of initial conditions at the bounce. So far the focus has been on establishing the *existence* of initial conditions that lead to a self-consistent extension of standard inflation to the Planck regime. But the issue of uniqueness is quite open except for some preliminary ideas involving a quantum extension of Penrose’s Weyl curvature hypothesis [124]. These will be explored in detail in the coming years. If one uses inflation, observations inform us that the entire observable universe should originate from a ball of radius of less than  $10\ell_{\text{Pl}}$  at the bounce. But standard inflation does not explain why there was an extraordinary homogeneity at this scale. The repulsive force of LQC that dominates near the bounce provides a novel avenue to explore this issue. LQC models that have been analyzed in detail indicate that in the Planck regime the net effect of this repulsion is to dilute the wrinkles in the curvature and forcing homogeneity and isotropy at this scale. It is important to translate these physical ideas into detailed calculations also

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<sup>8</sup> It is interesting that this general phenomenon also occurs in LQG, where the 4-dimensional spacetime continuum arises from coarse graining of a 2 complex representing the evolution of the fundamental quanta of geometry.

because they imply that the repulsive force would wash away the memory of the pre-bounce phase as far as observations are concerned, making it natural to specify initial conditions at the bounce. Finally, in the self-consistent solutions, it has been possible to argue that while the LQG effects are critical for the background FLRW quantum geometry, they can be ignored for perturbations since the energy density in perturbations is so small. It is important to carry out detailed calculations to investigate new physics that may emerge in more general situations from a full LQG treatment of perturbations.

There are similar challenges and opportunities in the investigation of quantum properties of black hole and cosmological horizons. While there is a detailed understanding of the microscopic quantum geometry of horizons in equilibrium, [153], the relation between the number of these microstates and the more familiar semi-classical calculations of entropy [175] via path integrals has begun to receive attention only recently. This is a key open issue. More generally, the intriguing relation between the microscopic geometry of quantum horizons and the semi-classical ideas [156] discussed in section III B remains to be explored in detail. Finally, as we saw in section III B, recently, a new window has been opened to investigate the Hawking effect within LQG [158]. This important development offers many opportunities for detailed calculations that will lead us to a deeper understanding of the evaporation process.

- **Beyond Truncations:** Recall that in both approaches discussed in this Chapter, concrete progress could be made by studying the appropriate truncations of the full theory. As emphasized towards the end of sections II C and III A, this is the common situation in fundamental physics: all the concrete calculations in QED, QCD and scenarios of the early universe, for example, involve truncations. Nonetheless, from the conceptual viewpoint, a central question remains: Is there an underlying coherent theory without reference to truncations that is being approximated in these calculations?

In the Asymptotic Safety program, a conceptual framework to address this question is provided by the infinite dimensional theory space  $\mathcal{T}$ . At a fundamental level, one should find the renormalization group flows in  $\mathcal{T}$  and then investigate whether, in concrete physical problems, finite dimensional truncations carried out to date provide a trustable approximation. However, this lofty goal is far too ambitious for now. Progress is likely to occur by further enlarging the reach of truncations. In particular, a simplified version of the (particle physics) standard model in which the gauge fields are assumed to be Abelian has already been incorporated in the EAA program [11]. An important goal which may be within reach in the foreseeable future would be to extend these calculations to include the full standard model.

What is the situation with Spinfoams? Results to date have focused on finite simplicial decomposition of the spacetime manifold. The key open question is whether one should take a suitable limit by successively refining the decomposition in a *well-controlled fashion*, or whether one should sum these contributions, *appropriately avoiding the obvious redundancy*. Does the final transition amplitude remain finite in either case? In 3 spacetime dimensions, the refinement does converge and yields the correct result. Similarly, one can recast LQC in the Spinfoam framework and show that the sum converges and yields a result that agrees with the Hamiltonian theory [104]. These calculations are helpful but do not provide deep insight because these theories do not have local degrees of freedom. Therefore currently there is a great deal of activity in the full 4-dimensional theory. In particular, generalized renormalization group flows are being studied by Dittrich and others to constrain the refinement procedures and investigate the phase diagrams that result, and

group field theory is being used by Oriti and others to carry out the sum systematically. Interestingly, the two procedures have quite different conceptual underpinnings. In the first, the viewpoint is more akin to that in the study of condensed matter systems using statistical mechanics, where the atomic structure is fundamental and phonon fields are convenient tools to encode collective behavior of atoms. In LQG, the quanta of geometry play the role of atoms while continuum quantum fields are the rough analogs of phonons. In the second approach, quantum fields are more fundamental as in particle physics, and one uses well established methods with the goal of summing a perturbative expansion. But quantum gravity introduces a key difference: the quantum fields are now defined on a group manifold rather than spacetime. It is fortunate that the central issue of whether there is a coherent theory underlying Spinfoam truncations is being analyzed from very different, if not opposing, perspectives. Since this central issue is deep and difficult, it is essential to have variety.

**Acknowledgments:** We thank Alejandro Corichi for permission to use Fig.4. This work was supported in part by the NSF grant PHY-1205388 and the Eberly research funds of Penn state.

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