Lectures on loop gravity

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These are introductory lectures on loop quantum gravity. The theory is presented in self-contained form, without emphasis on its derivation from classical general relativity. Dynamics is given in the covariant form. The approximations needed to compute physical quantities are discussed. Some applications are described, including the recent derivation of de Sitter cosmology from full quantum gravity.

I. WHERE ARE WE IN QUANTUM GRAVITY?

What we know today with a good degree of confidence about the fundamental description of the physical world is summed up within three theories:

- Quantum mechanics,
- The standard model (with neutrino mass),
- General relativity (with cosmological constant).

Quantum mechanics still leaves us perplexed about its actual physical meaning, but its empirical adequacy gives no signs of failure. *The standard model* has always enjoyed negative press, but is among the most spectacularly predictive (if not *the* most predictive) physical theory ever. It is based on flat-space quantum field theory (QFT), which is an application of quantum mechanics to special-relativistic field theory that completely neglects the import of general relativity. *General relativity* (GR) is our present theory of space, time and gravitation; it has obtained spectacular empirical success, especially in the last decade; it completely neglects the import of quantum theory.

With the notable exception of the "dark matter" phenomenology, the description of the world provided by these theories appears to be in full accord with *all* present observations, and capable to account for all of them, at least in principle. All other ideas studied today in fundamental theoretical physics, including loops and strings, are for the moment –at best– tentative and speculative.

Standard QFT and GR are grounded on mutually contradictory assumptions. Concretely, there are physical situations in which the theories mentioned do not have predictive power. For instance, we can use the framework of effective QFT to compute the gravitational scattering amplitude of two elementary particles; but if the center-of-mass energy is approximately equal to the impact parameter in Planck units, then the effective QFT based on GR breaks down, and we do not have any prediction at all. For the same reason, these theories do not provide a viable framework for early cosmology, for predicting what happens at the end of a a black hole evaporation, or to describe the quantum structure of spacetime at very small scale.

Loop quantum gravity (LQG), or loop gravity, is a tentative predictive theory for these phenomena, based on the idea of writing a version of QFT that does not disregard the import of GR and –the other way around– a theoretical account of space, time and gravitation that does not disregard the import of quantum theory.

LQG is it is still incomplete in some respects, but it is already a rather mature theory, with a clear definition of kinematics and dynamics, where some concrete physical calculations can be performed.

The theory is built on the idea of taking the import of quantum mechanics and especially that of GR seriously. GR has proven spectacularly effective for describing relativistic gravitation, and this has been achieved by modifying in depth our understanding of space and time. LQG is the effort to bring such general relativistic understanding of space and time into QFT. The physical assumptions on which LQG is based are thus grounded on well-established physics: GR and quantum mechanics (in a slightly generalized form, in order to take into account the way temporal evolution is dealt with in the general relativistic context).

The theory utilizes the Ashtekar's formulation of GR and its variants, and can be "derived" in a number of ways. The three major ones (see Section IV for more details) are:

- Canonical quantization of GR,
- Covariant quantization of GR on a lattice,
- Formal quantization of geometrical "shapes".

There is a surprising convergence towards the same formalism, from these very different techniques and philosophies. This convergence is a remarkable hint in favor of the idea that LQG is a natural formalism for general relativistic QFT.

In these lectures I do not follow any of these paths leading to LQG. Rather, I give a self-contained presentation of the theory, and discuss its relation with classical GR a posteriori. This is like presenting QED by giving its Fock space, operators and Feynman rules. Section II describes states and operators. Section III the transition amplitudes. Section V some applications. A list of problems is given in Appendix A. Mathematical recalls, advanced comments and pointers to alternative formulations are in smaller characters.

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II. STATES AND OPERATORS

A. Elementary math: $SU(2)$

" It is the mark of the educated man to look for precision in each class of things just so far as the nature of the subject admits."
Aristotle, Nicomachean Ethics, I,3.

LQG uses heavily the group $SU(2)$, its representation theory and the Hilbert spaces of the square-integrable functions over the group. Here is a recall of some elementary facts about these.

$SU(2)$ is the group of 2×2 unitary matrices h with unit determinant. A basis in its algebra is provided by the three matrices $\tau_i = -\frac{i}{2}\sigma_i,\ i=1,2,3$, where $\sigma_i = \left\{ \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} & -i \\ i & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \right\}$ are the Pauli matrices. Every $h \in SU(2)$ can be written as

$$h = e^{\alpha^i\tau_i} = \cos\Big(\frac{\alpha}{2}\Big)\, 1\!\!1 + i \sin\Big(\frac{\alpha}{2}\Big)\, \frac{\alpha^i}{\alpha}\, \sigma_i. \qquad (1)$$

where $\alpha \equiv \sqrt{\alpha_i\alpha_i} < 2\pi$ is the rotation angle of the $SO(3)$ rotation corresponding to the $SU(2)$ element h. The group manifold of $SU(2)$ is the three sphere $(x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2 = 1$, where $x^0 = \cos\big(\frac{\alpha}{2}\big)$ and $x^i = \sin\big(\frac{\alpha}{2}\big)\frac{\alpha^i}{\alpha}$. The standard Haar measure on $SU(2)$ can be written as the invariant measure on this sphere: $dh = d^4x\ \delta(|x|^2-1)$.

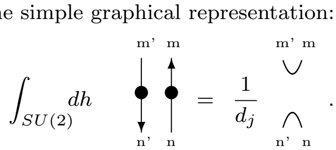
The irreducible unitary representations of $SU(2)$ are labelled by a half-integer $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ called "spin". The representation space $\mathcal{H}_j$ has dimension $d_j = 2j+1$. The standard basis that diagonalizes τ_3 is denoted $v^m,\ m=-j,...,+j$. The representation matrices are the Wigner matrices $D^j(h)^m{}_n$. The spin-j character is defined as $\chi^j(h) = tr[D^j(h)]$. A key property of these matrices is to be orthogonal in the Haar measure

$$\int_{SU(2)} dh\ \overline{D^{j'}(h)^{m'}{}_{n'}}\ \ D^j(h)^m{}_n = \frac{1}{d_j}\, \delta^{jj'}\delta^{mm'}\delta_{nn'}. \qquad (2)$$

Using the fact that the Wigner matrices are unitary, this can also be written in the more useful form

$$\int_{SU(2)} dh\ \ D^{j'}(h^{-1})_{n'}{}^{m'}\ \ D^j(h)^m{}_n = \frac{1}{d_j}\, \delta^{jj'}\delta^{mm'}\delta_{nn'}. \qquad (3)$$

which admits the simple graphical representation:



(4)

Spaces of functions on the group play an important role in LQG: in particular, the space $L_2[SU(2)]$ of the functions square-integrable in the Haar measure. Because of the orthogonality (2), the Wigner matrices form a basis in this space. Writing, in Dirac notation, $D^j(h)^m{}_n = \langle h|j,m,n\rangle$ and $\langle\psi|\phi\rangle = \int dh \overline{\psi(h)}\phi(h)$, equation (2) reads

$$\langle j',m',n'|j,m,n\rangle = \delta^{jj'}\delta^{mm'}\delta_{nn'}\ \frac{1}{2j+1}. \qquad (5)$$

This is the content of the Peter-Weyl theorem, which plays a major role below. This can equally be expressed as follows. Since $D^j : \mathcal{H}_j \to \mathcal{H}_j$, we can write $D^j \in (\mathcal{H}_j^* \otimes \mathcal{H}_j)$ and the Peter-Weyl theorem can be expressed in the useful notation

$$L_2[SU(2)] = \oplus_j \left(\mathcal{H}_j^* \otimes \mathcal{H}_j\right). \qquad (6)$$

Some operators are naturally defined on $L_2[SU(2)]$. The (matrix elements of the) $SU(2)$ group element h act as multiplicatif operators. The (hermitian) left and right invariant vector fields $\vec{L} = \{L_i\}$ and $\vec{R} = \{R_i\}$ are defined by

$$L_i\psi(h) \equiv i\frac{d}{dt}\,\psi(he^{t\tau_i})\Big|_{t=0}, \qquad R_i\psi(h) \equiv i\frac{d}{dt}\,\psi(e^{t\tau_i}h)\Big|_{t=0}. \qquad (7)$$

Acting on the Wigner matrices, they give

$$\vec{L}\ D^j(h) = i\ D^j(h)\ \vec{J}^j, \qquad \vec{R}\ D^j(h) = i\ \vec{J}^j\ D^j(h), \qquad (8)$$

where $\vec{J}^j$ are the (anti-hermitian) generators in the representation j. The Casimir operator $L^2 := L_iL_i$ acts on the individual $\mathcal{H}_j$ in the Peter-Weyl decomposition (because it acts only on the m indices and not on the j indices) and is diagonal in the spins

$$L^2\ D^j(h) = j(j+1)\ D^j(h). \qquad (9)$$

Given k spins $j_1, ..., j_k$, the tensor product $\mathcal{H}_{j_1}\otimes ... \otimes \mathcal{H}_{j_k}$ is the space of the tensors $i^{m_1...m_k}$ with indices in different representations. This tensor product can be decomposed into irreducibles, as in standard angular momentum theory. In particular, its invariant subspace is formed by the invariant tensors, satisfying $D^{j_1}(h)^{m_1}{}_{n_1}...D^{j_k}(h)^{m_k}{}_{n_k} i^{n_1...n_k} = i^{m_1...m_k}$. These are called *intertwiners* and the linear space they span

$$\mathcal{K}_{j_1...j_k} = \mathrm{Inv}[\mathcal{H}_{j_1}\otimes ... \otimes \mathcal{H}_{j_k}] \qquad (10)$$

is called "intertwiner space". Examples of invariant tensors are the fully antisymmetric tensor $i_{ijk} = \epsilon_{ijk}$ in $\mathcal{K}_{111} = \mathcal{H}_1\otimes\mathcal{H}_1\otimes\mathcal{H}_1$, the tensor $i_i{}^A{}_B = \sigma_i{}^A{}_B$ formed by the components of the Pauli matrices in $\mathcal{K}_{1\frac{1}{2}\frac{1}{2}} = \mathcal{H}_1\otimes\mathcal{H}_{\frac{1}{2}}\otimes\mathcal{H}^*_{\frac{1}{2}}$, and the two tensors $i_{ijkl} = \delta_{ij}\delta_{kl}$ and $i'_{ijkl} = \delta_{ik}\delta_{jl}$ in $\mathcal{K}_{1111} = \mathcal{H}_1\otimes\mathcal{H}_1\otimes\mathcal{H}_1\otimes\mathcal{H}_1$. Since an $SU(2)$ representation appears at most once in the tensor product of two others, it is easy to see that $\mathcal{K}_{j_1j_2j_3}$ is always 1-dimensional, and therefore i_{ijk} and $i_i{}^A{}_B$ are unique (up to scaling); while $\mathcal{K}_{1111}$ is 2-dimensional.

B. Elementary math: graphs

Graphs play a role in the following. Roughly, the adjacency relations (who is next to who) between the elementary quanta of space is described by graphs.

A graph Γ is a combinatorial object. It is defined as a triple $\Gamma = (\mathcal{L}, \mathcal{N}, \partial)$, where $\mathcal{L}$ is a finite set of L elements l, which we call "links", $\mathcal{N}$ is a finite set of N elements n, which we call "nodes", and the boundary relation $\partial = (s,t)$ is an (ordered) couple of functions $s : \mathcal{L} \to \mathcal{N}$ called "source" and $t : \mathcal{L} \to \mathcal{N}$, called "target".

The simplest way of visualizing a graph is of course to imagine the nodes as points and the links as (oriented) lines that join these points. Each link goes from its source to its target.

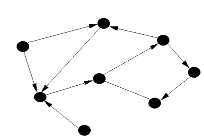


FIG. 1: Picturing of an graph with $N = 8$ and $L = 10$.

It is convenient to define also a "reversed" link l^{-1} for each l, where $s(l) = t(l^{-1})$ and $t(l) = s(l^{-1})$. We use the notation $l \in n$ to indicate that l is a link or a reversed link with $s(l) = n$. Thus the set of $l \in n$ is the set of the oriented links bounded by the node n, all considered with outgoing orientation.

An automorphism of a graph is a map from Γ to itself which preserves ∂. Given a graph, its automorphisms form a discrete group.

We say that a graph Γ' is a subgraph of a graph Γ, we write $\Gamma' \leq \Gamma$ and we say that Γ "contains" Γ', if there exists a map sending the links and the nodes of Γ' into the links and the nodes of Γ, preserving the boundary relations. Of course if Γ contains Γ' there may be more than one of the maps. That is, it might be possible to "place" Γ' into Γ in different manners.

The relation $\leq$ equips the set of all graphs with a partial order. This order admits an upper bound, namely given any two graphs

Γ and Γ' we can always find a third graph Γ'' which contains both Γ and Γ'. This implies that we can define the limit $\Gamma \to \infty$ of any quantity f_Γ that depends on graphs. It is exist, we say that

$$f_\infty = \lim_{\Gamma\to\infty} f_\Gamma \tag{11}$$

if for any ϵ there is a Γ' such that $|f_\infty - f_\Gamma| < \epsilon$ for all $\Gamma > \Gamma'$.

With these preliminaries, we are ready to define states and observables of quantum gravity. The kinematics of any quantum theory is given by a Hilbert space $\mathcal{H}$ carrying an algebra of operators A that have a physical interpretation in terms of observables quantities of the system considered. Let us start with $\mathcal{H}$.

C. Hilbert space

Let me start by recalling the structure of the Hilbert spaces used in QED and in QCD. A key step in constructing any interactive QFT is always a *finite* truncation of the dynamical degrees of freedom. In weakly coupled theories, such as low-energy QED or high-energy QCD, the truncation is provided by the particle structure of the free field, which allows us to consider virtual processes involving *finitely many* particles, described by Feynman diagrams. In strongly coupled theories, such as QCD, we can resort to a non-perturbative truncation, such as a *finite* lattice approximation.[1] The full theory is then formally obtained as a limit where all degrees of freedom are recovered.

In the first case, we can start by defining the single-particle Hilbert space $\mathcal{H}_1$. For a massive scalar theory, for instance, this can be taken to be the space $\mathcal{H}_1 = L_2[M]$ of the square-integrable functions on the Lorentz hyperboloid M. The n-particle Hilbert space is

$$\mathcal{H}_n = L_2[M^n]/\sim \tag{12}$$

where the factorization is by the equivalence relation determined by the action of the permutation group, which symmetrizes the states. The Hilbert space

$$\mathcal{H}_N = \bigoplus_{n=0}^{N} \mathcal{H}_n \tag{13}$$

contains all states up to N particles, and is actually sufficient for all calculations in perturbation theory. $\mathcal{H}_N$ is naturally a subspace of $\mathcal{H}_{N'}$ for $N < N'$, and the full Fock space is the limit

$$\mathcal{H}_{\text{Fock}} = \lim_{N\to\infty} \mathcal{H}_N. \tag{14}$$

In the second case, namely in lattice gauge theory, the canonical theory is defined on a lattice Γ with, say, L links l and N nodes n. The variables are group elements $U_l \in G$, where G is the gauge group, associated to the links, and the non-gauge-invariant Hilbert space is [1]

$$\tilde{\mathcal{H}}_\Gamma = L_2[G^L]. \tag{15}$$

Gauge transformations act on the states $\psi(h_l) \in \tilde{\mathcal{H}}_\Gamma$ at the nodes as

$$\psi(h_l) \to \psi(g_{\,s(l)}\, h_l\, g_{t(l)}^{-1}), \qquad g_n \in G \tag{16}$$

and the space of gauge invariant states is the physical Hilbert space

$$\mathcal{H}_\Gamma = L_2[G^L/G^N] \tag{17}$$

formed by the states satisfying (16). Again the full theory is obtained by appropriately taking L and N to infinity.

The Hilbert space of LQG has aspects in common with both these constructions. Let me now define it in three steps:

(i) For each graph Γ, consider a "graph space"

$$\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N] \tag{18}$$

which is precisely the Hilbert space (17) of an $SU(2)$ lattice gauge theory, over a graph which is not necessarily a cubic lattice. As we shall see, the local $SU(2)$ gauge is related to the freedom of rotating a 3d reference frame in space.

(ii) If Γ is a subgraph of Γ' then $\mathcal{H}_\Gamma$ can be naturally identified with a subspace of $\mathcal{H}_{\Gamma'}$ (the subspace formed by the states $\psi(h_l) \in \mathcal{H}_{\Gamma'}$ depending on h_l only if l is in the subgraph Γ). Define an equivalence relation $\sim$ as follows: two states are equivalent if they can be related (possibly indirectly) by this identification, or if they are mapped into each other by the group of the automorphisms of Γ. Let

$$\tilde{\mathcal{H}}_\Gamma = \mathcal{H}_\Gamma/\sim\,. \tag{19}$$

(iii) The full Hilbert space of quantum gravity is finally defined as

$$\mathcal{H} = \lim_{\Gamma\to\infty} \tilde{\mathcal{H}}_\Gamma. \tag{20}$$

It is separable.

This completes the construction of the Hilbert space of the theory.

$\mathcal{H}$ has aspects in common with Fock space as well as with the state space of lattice gauge theory. As we shall see, states in $\mathcal{H}_\Gamma$ can be viewed as formed by N quanta, where N is the number of nodes of the graph. Thus, each node of the graph is like a particle in QED, namely a quantum of electromagnetic field. Here, each node represents a quantum of gravitational field.

[1] In either case, the relevant effect of the remaining degrees of freedom can be subsumed under a dressing of the coupling constants, if a criterion of renormalizability or criticality is met.

But there is a key difference. QED Fock quanta carry quantum numbers coding where they are located in the background space-manifold. Here, since in general relativity the gravitational field is also physical space, individual quanta of gravity are also quanta of space. Therefore they do not carry information about their localization in space, but only information about the *relative* location with respect to one another. This information is coded by the graph structure. Thus, the quanta of gravity form themselves the texture of of physical space. Therefore the graphs in (20) can *also* be seen as a generalization of the lattices of lattice QCD.

This convergence between the perturbative-QED picture and the lattice-QCD picture follows directly from the key physics of general relativity: the fact that the gravitational field is physical space itself. Indeed, the lattice sites of lattice QCD are small regions of space; according to general relativity, these are excitations of the gravitational field, therefore they are themselves quanta of a (generally covariant) quantum field theory. An N-quanta state of gravity has therefore the same structure as a Yang-Mills state on a lattice with N sites. This convergence between the perturbative-QED and the lattice-QCD pictures is a beautiful feature of loop gravity.

Another similarity appears in the factorization by graph automorphisms, which is analogous to the symmetrization of individual particle states defining the Fock n-particle states.

Comments. This is the "combinatorial $\mathcal{H}$". An alternative studied in the literature is to consider embedded graphs in a fixed three-manifold Σ –namely collections of lines l embedded in Σ that meet only at their end points n– and to define Γ as an equivalence class of such embedded graphs under diffeomorphisms of Σ. This choice defines the "Diff $\mathcal{H}$". A third alternative is to do the same but using *extended* diffeomorphisms [2]. This choice defines the "Extended-Diff $\mathcal{H}$". With these definitions a graph is characterized also by its knotting and linking. (If Σ is chosen with non-trivial topology, by the homotopy class of the graph as well). In addition, with the first of these alternatives graphs are characterized by moduli parameters at the nodes as well (extended diffeos factor away these moduli [2]). The space Diff $\mathcal{H}$ is non-separable, leading to a number of complications in the construction of the theory. The combinatorial $\mathcal{H}$ considered here and the extended-Diff $\mathcal{H}$ are separable.

Neither knotting or linking, nor the moduli, have found a physical meaning so far, hence I tentatively prefer the combinatorial definition. But there are also ideas and interesting attempts to interpret knotting and linking as matter degrees of freedom [3–5]. If it worked it would be very remarkable success, but it is a long shot.

Another option is to restrict the theory to graphs Γ where all nodes are four valent. (The valence of a node n is the number of links for which n is the source plus the number of links for which it is the target.) I do not take this option here, although several results in the literature refer to the theory restricted in this manner.

D. Operators

The fact that $\mathcal{H}$ can be interpreted as a space describing quanta of space follows from its structure, as revealed by a crucial theorem due to Roger Penrose. Indeed, each Hilbert space $\mathcal{H}_\Gamma$ has a natural interpretation as a space of quantum metrics, early recognized by Penrose. Let's see how this happens.

The momentum operator on the Hilbert space of a particle $L_2[R]$ is the derivative operator $\vec{p} = -i\nabla = -i\frac{d}{d\vec{x}}$. The corresponding natural "momentum" operator on $L_2[SU2]$ is the derivative operator (7). There is one of these for each link, call it $\vec{L}_l$.

As in lattige gauge theory, operators are defined on the individual spaces $\mathcal{H}_\Gamma$, not on $\mathcal{H}$. Later I explain how these operators are used in computing observable quantities.

Because of the gauge invariance (16), we have

$$C_n = \sum_{l\in n} \vec{L}_l = 0. \tag{21}$$

at each node n. The operator $\vec{L}_l$ is not gauge invariant, namely it is not defined on gauge invariant functions. But is easy to write a gauge invariant operator:

$$G_{ll'} = \vec{L}_l \cdot \vec{L}_{l'} \tag{22}$$

where $s(l) = s(l') = n$. For reasons that will be clear in a moment, call this operator the "metric operator". In particular, denote the diagonal entries of $G_{ll'}$ as

$$A_l^2 = G_{ll}. \tag{23}$$

The operator $G_{ll'}$ coincides with Penrose's metric operator [6]. Penrose spin-geometry theorem states that the operator $G_{ll'}$ can be interpreted as defining angles in three dimensional space, at each node [6–8]. The theorem states that these angles obey the dependency relations expected of angles in three dimensional space.

I give here here in more detail a sharper version of Penrose's original spin-geometry theorem, based on a result by Minkowski. Consider the classical limit of the Hilbert space $\mathcal{H}_\Gamma$, that is, consider classical quantities $\vec{L}_l$ satisfying (21). Minkowski's theorem [9] states that whenever there are F non-coplanar 3-vectors $\vec{L}_l$ satisfying the condition (21), there exists a convex polyhedron in $\mathbb{R}^3$, whose faces have outward normals paralel to $\vec{L}_l$ and areas A_l. The resulting polyhedron is unique, up to rotation and translation. It follows that if we write $G_{ll'} = A_l A_{l'} \cos(\theta_{ll'})$, then the quantities $\theta_{ll'}$ satisfy all the relations satisfied by the angles normal to the faces of the polyhedra. The operators $G_{ll'}$ fully capture the "shape" of the polyhedron, namely its (flat) metric geometry, up to rotations. In other words, in the classical limit the states in the Hilbert space $\mathcal{H}_\Gamma$ describe a collection of flat polyhedra with different shapes, one per each node of Γ. The quantum operators A_l can be interpreted as giving the areas of these faces and the quantum operators $G_{ll'}$ as the (cosine of the) angles between two faces (multiplied by the areas). See Figure 2.

What has a collection of polyhedra to do with the gravitational field, which is classically described by a continuous metric? The answer is suggested by Regge gravity: a collection of flat polyhedra glued to one another defines a (non-differentiable) metric, where the curvature

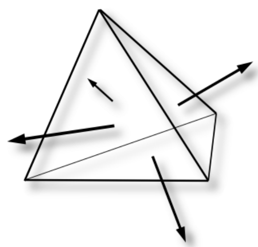


FIG. 2: Normals (here arrows) to the faces, and proportional to the area of the face, satisfy (21) and uniquely detrmine the polyhedron (here a tetrahedron).

is concentrated on the edges of the polyhedra. Thus, a collection of glued polyhedra provides a discretized geometry, and therefore a gravitational field up to a finite truncation of its degrees of freedom.

Thus, the Hilbert space $\mathcal{H}_\Gamma$ describes a truncation of the degrees of freedom of GR, like the N-particle Hilbert space of QED, or the Hilbert space of lattice QCD, describe truncations of the degrees of freedom of a Yang-Mils field.

Using standard geometrical relations, we can write the volume of these polyhedra in terms of the $\vec{L}_l$ operators. For instance, for a 4-valent node n, bounding the links $l_1, ..., l_4$ the volume operator V_n is given by the expression for the volume of a tetrahedron

$$V_n^2 = \frac{2}{9}\ |\vec{L}_{l_1} \cdot (\vec{L}_{l_2} \times \vec{L}_{l_3})|; \tag{24}$$

gauge invariance (16) at the node ensures that this definition does not depend on which triple of links is chosen.[2]

As pointed out by Thomas Thiemann and Cecilia Flori [10], the definition of the vertex operator for higher valent nodes given in the literature, is not entirely satisfactory. A good definition of the operator for general n-valent nodes (which reduces to the one on $(n-1)$-valent nodes when one of the links has zero spin) is still missing. This does not affect what follows.

Notice that the volume operator V_n acts precisely on the node space $\mathcal{H}_n$, which, I recall is the space of the intertwiners between the representations associated to the node n. It is therefore convenient to choose at each node n a basis of intertwiners $\mathtt{v}_n$ that diagonalizes the volume operator, and label it with the corresponding eigenvalue $\mathtt{v}_n$. I use the same notation $\mathtt{v}_n$ for the intertwiner and for its eigenvalue.

Finally, the *holonomy* operator is the multiplicative operator h_l associated to each link l. The operators $\vec{L}_l$ and h_l form a closed algebra and are the basic operators in terms of which all other observables are built, like the creation and annihilation operators in QFT.

E. Spin network basis

While the full set of $SU(2)$ invariant operators $G_{ll'}$ do not commute, the Area and Volume operators A_l and V_n commute. In fact, they form a complete set of commuting observables in $\mathcal{H}_\Gamma$, in the sense of Dirac (up to possible accidental degeneracies in the spectrum of V_n). We call the orthonormal basis that diagonalizes these operators the *spin-network* basis.

This basis has a well defined physical and geometrical interpretation [11–13]. The basis can be obtained via the Peter-Weyl theorem and it is defined by

$$\psi_{\Gamma,j_l,\mathtt{v}_n}(h_l) = \left\langle\ \otimes_l\ d_{j_l}\ D^{j_l}(h_l)\ \middle|\ \otimes_n\ \mathtt{v}_n\ \right\rangle_\Gamma \tag{25}$$

where $D^{j_l}(h_l)$ is the Wigner matrix in the spin-j representation and $\langle\cdot|\cdot\rangle_\Gamma$ indicates the pattern of index contraction between the indices of the matrix elements and those of the intertwiners given by the structure of the graph.[3]

Since the Area is the $SU(2)$ Casimir, while the volume is diagonal in the intertwiner spaces, the spin j_l is easily recognized as the Area quantum number and $\mathtt{v}_n$ as the Volume quantum number.

More in detail, the Peter-Weyl theorem states that $L_2[SU(2)^L]$ can be decomposed into irreducible representations

$$L_2[SU(2)^L] = \bigoplus_{j_l} \bigotimes_l (\mathcal{H}^*_{j_l} \otimes \mathcal{H}_{j_l}). \tag{27}$$

Here $\mathcal{H}_j$ is the Hilbert space of the spin-j representation of $SU(2)$, namely a $2j+1$ dimensional space, with a basis $|j,m\rangle, m = -j, ..., j$ that diagonalizes L^3. The star indicates the adjoint representation, but since the representations of $SU(2)$ are equivalent to their adjoint, we can forget about the star.[4] For each link l, the two factors in the r.h.s. of (27) are naturally associated to the two nodes $s(l)$ and $t(l)$ that bound l, because under (16) they transform under the action of $g_{s(l)}$ and $g_{t(l)}$, respectively. We can hence rewrite the last equation as

$$L_2[SU(2)^L] = \bigoplus_{j_l} \bigotimes_n \mathcal{H}_n \tag{28}$$

where the node Hilbert space $\mathcal{H}_n$ associated to a node n includes all the irreducible $\mathcal{H}_j$ that transform with g_n under (16), that is[5]

$$\mathcal{H}_n = \bigotimes_{l\in n} \mathcal{H}_{j_l}. \tag{29}$$

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[2] The factor $\frac{2}{9}$ is easy to compute from elementary geometry. Take a tetrahedron having three sides equal to the three orthogonal basis vectors in R^3. It has three faces orthogonal to one another and with area $\frac{1}{2}$. Therefore the triple product gives $\left(\frac{1}{2}\right)^3$. While the volume is $V = \frac{1}{3}Ah = \frac{1}{6}$.

[3] Both tensor products live in

$$H_\Gamma \subset L_2[(SU2)^L] = \bigoplus_{j_l} \bigotimes_l \mathcal{V}_{j_l} \otimes \mathcal{V}_{j_l} = \bigoplus_{j_l} \bigotimes_n \bigotimes_{e\in\partial n} \mathcal{V}_{j_l}. \tag{26}$$

where $\mathcal{V}_j$ is the $SU2$ spin-j representation space, here identified with its dual.

[4] The star does not regard the Hilbert space itself: it specifies the way it transforms under $SU(2)$.

[5] More precisely, $\mathcal{H}_n = (\bigotimes_{l\in s(n)} \mathcal{H}^*_l) \otimes (\bigotimes_{l\in t(n)n} \mathcal{H}_l)$ where $s(t)$ and $t(n)$ are the sets of the links for which n is, respectively, a source or a target.

The $SU(2)$ invariant part of this space

$$\mathcal{K}_n = \text{Inv}_{SU(2)}[\mathcal{H}_n]. \tag{30}$$

under the diagonal action of $SU(2)$ is the intertwiner space of the node n. The volume operator V_n acts on this space.

The Hilbert space $\mathcal{K}_\Gamma$ is the subspace of $\mathcal{H}_\Gamma$ formed by the gauge invariant states. Thus clearly

$$\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N] = \bigoplus_{j_l} \bigotimes_n \mathcal{K}_n. \tag{31}$$

I denote $P_{SU(2)} : \mathcal{H}_\Gamma \to \mathcal{K}_\Gamma$ the othogonal projector on the gauge invariant states. It can be written explicitly in the form

$$P_{SU(2)}\psi(h_l) = \int_{SU(2)^N} dg_n \; \psi(g_{s(l)} h_l g^{-1}_{t(l)}). \tag{32}$$

Notice also that the states where $j_l = 0$ for some l are precisely the states that belong also to the Hilbert space $\mathcal{H}_{\Gamma'}$ where Γ' is the subgraph of Γ obtained erasing those l's. It is therefore convenient to define the subspace $\mathcal{H}^*_\Gamma$ of $\mathcal{H}_\Gamma$, spanned by the spinnetwork states with all j_l nonvanishing. By doing so, we can rewrite (20) as

$$\mathcal{H} = \bigoplus_\Gamma \; \mathcal{H}^*_\Gamma \tag{33}$$

without then having to bother to factor the equivalence between spaces with different graph.

Concluding, a basis in $\mathcal{H}$ is labelled by three sets of "quantum numbers": an abstract graph Γ; a coloring j_l of the links of the graph with irreducible representations of $SU(2)$ different from the trivial one ($j = \frac{1}{2}, 1, \frac{3}{2}, ...$); and a coloring of each node of Γ with an element $\mathtt{v}_n$ in an orthonormal basis in the intertwiner space $\mathcal{H}_n$. The states $|\Gamma, j_l, \mathtt{v}_n\rangle$ labelled by these quantum numbers are called "spin network states" [11].

F. Physical picture

Spin network states are eigenstates of the area and volume operators. A spin network state has a simple geometrical interpretation. It represents a "granular" space where each node n represents a "grain" or "quantum" of space [14]. These quanta of space do not have a precise shape because the operators that decide their geometry do not commute. Classically, each node represents a polyhedron, thanks to Minkowski's theorem, but the polyhedra picture holds only in the classical limit and cannot be taken literally in the quantum theory. In the quantum regime, the operators $G_{ll'}$ do not commute among themselves, and therefore there is no sharp polyhedral geometry at the quantum level. In other words, these are "polyhedra" in the same sense in which a particle with spin is a "rotating body".

The spectrum of A_l is easy to find out, since A_l^2 is simply the Casimir of one of the $SU(2)$ groups. Therefore the area eigenvalues are

$$a_j = \sqrt{j(j+1)} \tag{34}$$

where $j \in \mathbb{N}/2$. Notice that the spectrum is discrete and it has a minimum step between zero and the lowest non-vanishing eigenvalue

$$a_{\frac{1}{2}} = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)} = \frac{\sqrt{3}}{2}. \tag{35}$$

The volume of each grain n is $\mathtt{v}_n$. Volume eigenvalues are not as easy to compute as area eigenvalues. They can be computed numerically for arbitrary intertwiners spaces (the problem is just to diagonalize a matrix) and there are elegant semiclassical techniques that give excellent results.

Two grains n and n' are adjacent if there is a link l connecting the two, and in this case the area of the elementary surface separating the two grains is determined by the spin of the link joining n and n'. Physical space is "weaved up" [15] by this net of atoms of space.

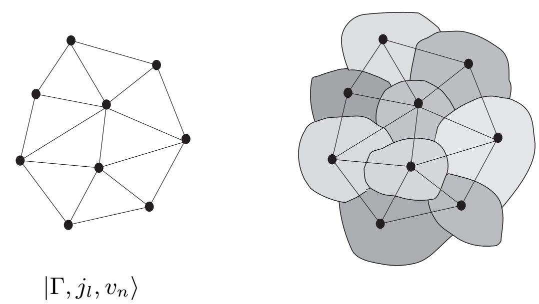


FIG. 3: "Granular" space. A node n determines a "grain" or "chunk" of space.

The geometry represented by a state $|\Gamma, j_l, \mathtt{v}_n\rangle$ is a *quantum* geometry for three distinct reasons.

i. It is discrete. The relevant quantum discreteness is not the fact that the continuous geometry has been discretized —this is just a truncation of the degrees of freedom of the theory. It is the fact that area and volume are quantized and their spectrum turns out to be discrete. It is the same for the electromagnetic field. The relevant quantum discreteness is not that there are discrete modes for the field in a box: it is that the energy of these modes is quantized.

ii. The components of the Penrose metric operator do not commute. Therefore the spin network basis diagonalizes only a subset of the geometrical observables, precisely like the $|j, m\rangle$ basis of a particle with spin. Angles between the faces of the polyhedra are quantum spread in this basis.

iii. A generic state of the geometry is not a spin network state: it is a linear superposition of spin networks. In particular, the *extrinsic* curvature of the

3-geometry[6], which, as we shall see later on, is captured by the group elements h_l, is completely quantum spread in the spin network basis. It is possible to construct coherent states in H_Γ that are peaked on a given intrinsic as well as extrinsic geometry, and minimize the quantum spread of both. A technology for defining these semiclassical states in H_Γ has been developed by a number of authors, yielding beautiful mathematical developments that unfortunately I do not have space to cover here. See for instance [16–20].

G. The Planck scale

So far, I have not mentioned units and physical dimensions. The gravitational field $g_{\mu\nu}$ has the dimensions of an area.[7] The dimension of the Ashtekar's electric field E (the densitized inverse triad), is also an area. The geometrical interpretation described above depends on a unit of length $L_{\rm loop}$, which characterizes the theory. For instance, in, say, centimeters, the minimum area eigenvalue $a_{\frac{1}{2}}$ will have the value

$$a_{\frac{1}{2}} = \frac{\sqrt{3}}{2}\, L^2_{\rm loop}. \tag{36}$$

and the metric operator will be defined by

$$G_{ll'} = L^4_{\rm loop}\, \vec{L}_l \cdot \vec{L}_{l'} \tag{37}$$

What is the value of $L_{\rm loop}$? The Hilbert space and the operator algebra described here can be derived from a canonical quantization of GR. In this case $\vec{L}_l$ is easily identified with the flux of the Ashtekar electric field, or the densitized triad, across the polyhedra faces, and canonical quantization fixes the multiplicative to be

$$L^2_{\rm loop} = 8\pi\gamma\ \hbar G \tag{38}$$

where γ, the Immirzi-Barbero parameter is a positive real number, G is the Newton constant. This relation may be affected by radiative corrections (the Newton constant may run between Planck scale and the infrared), therefore it is more prudent to keep $L^2_{\rm loop}$ as a free parameter in the theory for the moment. It is the parameter that fixes the scale at which geometry is quantized.

[6] In canonical general relativity the extrinsic curvature of a space-like surface is the quantity canonically conjugate to the intrinsic geometry of the surface.

[7] This follows from $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ and the fact that it is rather unreasonable to assign dimensions to the coordinates of a general covariant theory: coordinates are functions on spacetime, that can be arbitrarily *nonlinearly* transformed. For instance, they are often angles.

H. Boundary states

The states in $\mathcal{H}$ can be viewed as describing quantum space at some given coordinate time. A more useful interpretation, however, and the one I adopt here, is to take them to describe the quantum space *surrounding a given 4-dimensional finite region $\mathcal{R}$ of spacetime.* This second interpretation is more covariant and will be used below to define the dynamics. That is, a state in $\mathcal{H}$ is not interpreted as "state at some time", but rather as a "boundary state". See Figure 4.

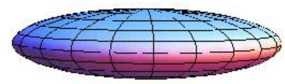


FIG. 4: The state described by a spin network can be taken to give the geometry of the three dimensional hypersurface surrounding a finite 4d spacetime region.

In the non-general-relativistic limit, therefore, $\mathcal{H}$ must be identified with the tensor product $\mathcal{H}^*_{fin} \otimes \mathcal{H}_{init}$ of the initial and final state spaces of conventional quantum theory.

This is the quantum geometry at the basis of loop gravity. Let me now move to the transition amplitudes between quantum states of geometry.

III. TRANSITION AMPLITUDES

A. Elementary math: $SL(2,\mathbb{C})$

I start with a few notions about $SL(2,\mathbb{C})$, the (double cover of the) Lorentz group $SO(3,1)$. $SL(2,\mathbb{C})$ is a six dimensional group. I denote by ψ the spinors of the fundamental representation defined on $\mathbb{C}^2$ by the 2×2 complex matrices with unit determinant. By v the vectors in the 4d real representation defined on Minkowski space by the Lorentz transformations. And by J the antisymmetric tensors in the adjoint representation (as the electromagnetic field).

It is convenient to study $SL(2,\mathbb{C})$ by choosing a "rotation" subgroup $H = SU(2) \subset SL(2,\mathbb{C})$. Choosing an $SU(2)$ subgroup in $SL(2,\mathbb{C})$ is like choosing a Lorentz frame in special relativity. In the vector representation H leaves a timelike vector t invariant, and we can choose Minkowski cordinates where, say $t = (1,0,0,0)$. Then we can distinguish the time space components of any vector $v = v^0 t + \vec{v}$, where $\vec{v} = (0, v^i), i = 1,2,3$ is orthogonal to t. In the fundamental representation a choice of H is equivalent to the choice of a scalar product. H is given by the matrices unitary with respect to this scalar product. A change of Lorentz frame is equivalent to rotation of the scalar product in $\mathbb{C}^2$. Given a scalar product $\langle\psi|\phi\rangle = g_{AB}\overline{\psi^A}\phi^B$, we can choose a basis in $\mathbb{C}^2$ where $g = \mathbb{1}$. The relation between the choice of basis in $\mathbb{C}^2$ and in Minkowski space is given by the Clebsch-Gordan map $v \to v^0\mathbb{1} + v^i\sigma_i$.

In the adjoint representation, a basis in the $SL(2,\mathbb{C})$ algebra is formed by the generators L_i of the $SU(2)$ rotations and the corresponding boosts generators K_i. Any group element can be written in the form

$$g = e^{\alpha^i\tau_i + i\beta^i\tau_i} \tag{39}$$

The left invariant vector fields are then given by

$$L_i\psi(h) \equiv i\frac{d}{dt}\,\psi(he^{t\tau_i})\Big|_{t=0}, \quad K_i\psi(h) \equiv i\frac{d}{dt}\,\psi(he^{it\tau_i})\Big|_{t=0}. \quad (40)$$

The two Casimirs of the group are $\vec{L}\cdot\vec{K}$ and $|\vec{L}|^2+|\vec{K}|^2$.

The finite-dimensional representations of $SL(2,\mathbb{C})$ are non-unitary. For instance the Minkowski "scalar product" $x\cdot y = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3$ is not a scalar product, because it is not positive definite. LQG uses instead *unitary* representations of $SL(2,\mathbb{C})$, which are infinite dimensional. These can be studied for instance in [21] or [22]. Roberto Pereira's thesis [23] is also very useful for this. Here I give the essential information about these representations.

Unitary representations of $SL(2,\mathbb{C})$ are labeled by a spin k and a positive real number p. The representation spaces are denoted $\mathcal{H}_{p,k}$. The two Casimirs take the values $2pk$ and p^2+k^2, respectively, on $\mathcal{H}_{p,k}$. Each $\mathcal{H}_{p,k}$ can be decomposed into irreducibles of the $SU(2)$ subgroup as follows

$$\mathcal{H}_{p,k} = \oplus_{j=k}^{\infty}\,\mathcal{H}^j_{p,k} \quad (41)$$

where $\mathcal{H}^j_{p,k}$ is a (2j+1) dimensional space that carries the spin-j representation of $SU(2)$. A useful basis in $\mathcal{H}_{p,k}$ is obtained diagonalizing the total spin and the third component of the spin of the $SU(2)$ subgroup. States in this basis can be written as $|p,k;j,m\rangle$. The representation matrices $D^{pk}(g)$ in this basis have the form $D^{pk}(g)^{jm}{}_{j'm'}$. The representation matrices $D^{pk}(g)$ span a linear space of functions on $SL(2,C)$. As elements of this space, they can be written in Dirac notation as

$$\langle g|p,k;(j,m),(j'm')\rangle = D^{pk}(g)^{jm}{}_{j'm'}. \quad (42)$$

$\chi^{p,k}(g) = tr[D^{pk}(g)]$ is the $SL(2,\mathbb{C})$ character in the (p,k) unitary representation. (This is generally a distribution on the group, since the representation spaces are infinite dimensional.)

Of particular importance in LQG is a subspace of this space of functions. This is the space spanned by the subspaces $\mathcal{H}^j_{p,k}$ where

$$p=\gamma j, \qquad k=j; \quad (43)$$

that is, the space

$$\mathcal{H}_\gamma = \oplus_j\left((\mathcal{H}^j_{\gamma j,j})^*\otimes\mathcal{H}^j_{\gamma j,j}\right). \quad (44)$$

In other words, this is the space of functions on $SL(2,\mathbb{C})$ of the form

$$\psi(g) = \sum_{j,mn} c_{j,mm'}\;D^{\gamma j,j}(g)^{jm}{}_{jm'}. \quad (45)$$

equivalently

$$|\psi\rangle = \sum_{j,mn} c_{j,mm'}\;|\gamma j,j;(j,m),(jm')\rangle. \quad (46)$$

The space $\mathcal{H}_\gamma$ has two remarkable properties

- It is naturally isomorphic to $L_2[SU(2)]$. The map

$$Y_\gamma : |j,m,n\rangle \mapsto |\gamma j,j;(j,m),(j,n)\rangle \quad (47)$$

sends $L_2[SU(2)]$ onto $\mathcal{H}_\gamma$. This maps is clearly $SU(2)$ invariant.

- For all $\psi,\phi\in H_\gamma$, we have [24–26]

$$\langle\psi|\vec{K}+\gamma\vec{L}|\phi\rangle \sim 0. \quad (48)$$

where $\sim$ means that the relation holds in the large j limit (which, as we shall see later on is the semiclassical limit of the quantum theory). This relation has an important meaning in quantum gravity, because, as we shall see in Section IV it is precisely the "simplicty condition" that reduce BF theory to GR — see equation (70).

The space $\mathcal{H}_\gamma$ has a natural Hilbert space structure, inherited from (47). Notice, however that it is *not* a subspace of the Hilbert space of square integrable functions $L_2[SL(2,\mathbb{C})]$. This is because it is formed by a discrete linear combination of functions with a sharp value of p, which is a continuous label. In other words, it is like a space of linear combinations of delta functions.[8]

Consider a graph Γ with and a function $\psi(h_l)$ of $SU(2)$ group elements on its links. The map Y_γ extends immediately (by tensoring it), and sends $\psi(h_l)$ in a function of $Y_\gamma\psi(g_l)$ of $SL(2,\mathbb{C})$ elements. This is not $SL(2,\mathbb{C})$ invariant at the nodes, but we can make it gauge invariant by integrating over a gauge action of $SL(2,\mathbb{C})$. That is, denote

$$P_{SL(2,\mathbb{C})}\psi(g_l) = \int_{SL(2,\mathbb{C})^N} dg'_n\;\psi(g_{s(l)}g_l g^{-1}_{t(l)}). \quad (49)$$

where the prime on dg_n indicates that one of the edge integrals is dropped (it is redundant). Thus, the linear map

$$f_\gamma := P_{SL(2,\mathbb{C})}\circ Y_\gamma \quad (50)$$

sends $SU(2)$ spin networks into $SL(2,\mathbb{C})$ spin networks. In particular, $(f_\gamma\psi)(\mathbb{1}_l)$ is a linear functional on the space of $SU(2)$ spin networks. By linearity

$$(f_\gamma\psi)(\mathbb{1}_l) = \int dh_l\;\psi(h_l)A_\gamma(h_l) \quad (51)$$

An explicit calculation shows that this can be written in the form

$$A_v(h_l) = \int_{SL(2,\mathbb{C})} dg'_n\;\prod_l K(h_l, g_{s_l}g^{-1}_{t_l}) \quad (52)$$

where l are the links and n the nodes of Γ and the kernel K is

$$K(h,g) = \sum_j\int_{SU(2)} dk\;d_j\;\chi^j(hk)\;\chi^{\gamma j,j}(kg). \quad (53)$$

We use this below.

B. Elementary math: 2-complexes

A (combinatorial) two-complex $\mathcal{C} = (\mathcal{F},\mathcal{E},\mathcal{V},\partial)$ is defined by a finite set $\mathcal{F}$ of F elements f called "faces", a finite set $\mathcal{E}$ of E elements e called "edges", a finite set $\mathcal{V}$ of V elements v called "vertices", and a boundary relation ∂ that associates to each edge an ordered couple of vertices $\partial e = (s_e,t_e)$) and to each face a cyclic sequence of edges. A cyclic sequences of edges is a sequence of n_f edges or reversed edges e such that $t_{e_n} = s_{e_{n+1}}$ with $n_f+1=1$. We define $\epsilon_{ef} = \pm 1$ if the edge e appears as e or as e^{-1} in the sequence. We write

$$\mathcal{V} = \sum_f\;n_f. \quad (54)$$

For each vertex v of the two-complex, we call Γ_v the graph formed by intersection of the two-complex with a small sphere surrounding it.

The boundary $\Gamma = \partial\mathcal{C}$ of a two-complex $\mathcal{C}$ is a (possibly disconnected) graph Γ, whose links l are edges of $\mathcal{C}$ bounding a single face and whose nodes n are vertices of $\mathcal{C}$ bounding (links and) a single internal edge.

The same definitions of automorphisms and partial order I gave for the graphs hold for the two-complexes. In particular, it makes sense to define the limit

$$f_\infty = \lim_{\mathcal{C}\to\infty} f_{\mathcal{C}} \quad (55)$$

[8] The relation 48 is exactly true (not just in the large j limit) if we replace $p=\gamma j$ by $p=\gamma(j+1)$. This alternative has been considered in the literature, but it seems to lead to problems in relating the dynamics of graphs to that on subgraphs. Sergei Alexandrov has noticed that H_γ is not the only subspace with these properties. It is the first of a family of spaces $\mathcal{H}^r_\Gamma$, defined by $p=\gamma(j+r)(j+r+1)/j,\;k=j+r,\;\;\tilde{j}=j+r$, with any integer r. The parameter r determines a different ordering of the constraints, and I do not consider it here. On this, see [26].

of a function that depends on a complex.

A two-complex can be visualized as a set of polygons f meeting along edges e in turn joining at vertices v (see 5).

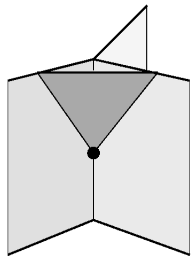


FIG. 5: A two-complex with one internal vertex.

Recall that in 3d, the physical interpretation of the graphs turned out to be as dual to cellular decompositions of 3d space. The same will be true for two-complexes in 4d. *Vertices* can be thought as dual to 4d cells in spacetime. *Edges* are dual to 3d cell bounding the 4d cells. Importantly, *faces* are dual to the surfaces to which we have assigned areas in 3d.

Notice that boundary relations then hold correctly: An edge hitting the boundary of the two-complex is simply a 3d cell that happens to sit on the boundary of a 4d cellular decomposition of spacetime. A face hitting the boundary of the two-complex is a 2d surface that sits on the boundary. Thus, for instance a 2d surface in spacetime is represented by a *link* l in the 3d graph, but it is represented by a *face* f in the 4d complex. If the surface is on the boundary, l is bounds f.

We now all the ingredients for defining the transition amplitudes of quantum gravity.

C. Transition amplitudes

Each two-complex $\mathcal{C}$ defines a (truncated) transition amplitude $Z_{\mathcal{C}}(h_l)$ for the states in $\mathcal{H}_{\partial\mathcal{C}}$. Recalling the discussion in Section II C on the convergence between the QED and the QCD picture, the truncation can be thought in two equivalent ways. Either as a background independent analog of the truncation provided by a finite 4d lattice in QCD, or as a truncation in the order of the Feynman diagrams, in a background-independent analog to the Feynman perturbative expansion. The two pictures turn out to converge because physically a spacetime lattice is nothing else that a "history" of space quanta, in the same sense in which a Feynman graph is a "history" of field quanta. In the first case, the two-complex can be thought as a discretization of spacetime. In the second, as a particular Feynman history of quanta of space. Let us now define the transition amplitude.

The transition amplitudes of LQG are defined by

$$Z_{\mathcal{C}}(h_l) = \int_{SU(2)} dh_{vf} \; \prod_f \delta(h_f) \; \prod_v A_v(h_{vf}). \tag{56}$$

Here $h_f = \prod_{v\in f} h_{vf}$ is the oriented product of the group elements around the face f and the vertex amplitude is

$$A_v(h_l) = \int_{SL(2,\mathbb{C})} dg'_n \; \prod_l K(h_l, g_{s_l} g_{t_l}^{-1}) \tag{57}$$

where l are the links and n the nodes of Γ_v (which are associated to the the faces and, respectively, the edges adjacent to v). Finally, the kernel K is

$$K(h,g) = \sum_j \int_{SU(2)} dk \; d_j \; \chi^j(hk) \; \chi^{\gamma j, j}(kg). \tag{58}$$

where, I recall, $d_j = 2j+1$, $\chi^j(h)$ is the spin-j character of $SU(2)$ and $\chi^{\rho,n}(g)$ is the character of $SL(2,\mathbb{C})$ in the (ρ, n) unitary representation. γ is a dimensionless parameter that characterizes the quantum theory. This is the theory.

The expressions above define truncations of the full transition amplitudes. Let me define also the full physical transition amplitude as

$$W(h_l) = \lim_{\mathcal{C}\to\infty} Z_{\mathcal{C}}(h_l). \tag{59}$$

In a general-covariant quantum theory, the dynamics can be given by associating an amplitude to each boundary state [27, 28]. This is determined by the linear functional W on $\mathcal{H}$. The modulus square

$$P(\psi) = |\langle W|\psi\rangle|^2 \tag{60}$$

determines (with suitable normalization) the probability associated to the process defined by the boundary state ψ.

This vertex amplitude (57-58) has been found independently by different research groups [29–34], following quite distinct research logics; the different vertices have only later been recognized as the same. (The presentation I have given here does not follow any of the original derivations.) Notice that (57) and (58) are precisely equations (52) and (53). Therefore the vertex amplitude for a spin-network state ψ on Γ_v can be also written in the compact form

$$A_v(\psi) = (f_\gamma \psi)(\mathbb{1}). \tag{61}$$

where f_γ is given in (50). In other words, the theory is determined by the imbedding Y_γ of $SU(2)$ functions into $SL(2,\mathbb{C})$ functions, defined in section III A, see equation (47). This is discussed more in detail below.

D. Properties and comments

The form of W is largely determined by general principles: Feynman's superposition principle, locality, and local Lorentz invariance [35].

1. *Superposition principle.* Following Feynman, we expect that the amplitude $\langle W\,|\,\psi\rangle$ can be expanded in a sum over "histories". The integral in (56) is like a truncated version of a Feynman path integral, analogous to the integration over the group elements in lattice QCD. The integration variables are precisely the $SU(2)$ group elements that form

a basis in the Hilbert space of the theory. As mentioned, a given two-complex can also be viewed as a history of quanta of spaces. The integration in (56) is then analog to the momenta integration in Feynman diagrams.

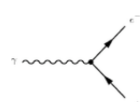
2. *Locality.* We expect the amplitude $W(\sigma)$ of a single history to be built in terms of *products* of elementary amplitudes associated to local *elementary* processes.[9] This is the case for the integrand of (56), which is a product of local face and vertex amplitudes. The face amplitude $\delta(h_f)$ simply glues the different vertices amplitude. The non trivial part of the amplitude is thus in the vertex amplitudes A_v.

3. *Local Lorentz invariance.* Classical GR has a local Lorentz invariance, and we expect the individual spinfoam vertex to be Lorentz invariant in an appropriate sense. If spinfoams states were $SL(2,\mathbb{C})$ spin networks $\Psi(g_l), g_l \in SL(2,\mathbb{C})$, gauge invariance could be easily implemented by projecting on locally Lorentz invariant states with $P_{SL(2,\mathbb{C})}$. But the Hilbert space $\mathcal{H}_\Gamma$ has no hint of $SL(2,\mathbb{C})$. So, to implement local Lorentz invariance, there should be a map from $\mathcal{H}_\Gamma$ to a Lorentz covariant language that characterizes the vertex. How? Well, I have just constructed such a map in the previous section: it is the map Y_γ, which depends only on a single parameter γ. The vertex amplitude is then simply obtained from Y_γ and $P_{SL(2,\mathbb{C})}$, as expressed by (61).

Quite astonishingly, the simple and natural vertex amplitude (61) appears to yield the Einstein equations in the large distance classical limit, as I will show below. A natural group structure based on $SU(2) \subset SL(2,\mathbb{C})$ appears to code the Einstein equations.

The skepticism prompted by this surprising claim may be tempered by two considerations. The first is that the same happens in QED. The simple vertex amplitude

$$\langle W|\psi^A_{e_1}(p_1), \psi^B_{e_2}(p_2), \psi^\gamma_\mu(k)\rangle \;=\; \tag{62}$$



$$= \; e\;\gamma^{AB}_\mu\;\delta(p_1+p_2+k)$$

yields the full complexity of the interacting Dirac-Maxwell equations. In other words, QED, with its fantastic phenomenology and its 12 decimal digits accurate predictions, is little more than momentum conservation plus the Dirac matrices γ^{AB}_μ, which, like f_γ, are essentially Clebsch-Gordan coefficients.

The second consideration is that, as will discuss later on, general relativity is BF theory plus the simplicity constraints. BF theory means flat curvature. Hence in a sense GR is flat curvature plus simplicity conditions (see (70) below). The map f_γ implements the simplicity conditions, since it maps the states to the space where the simplicity conditions (48) hold; while the evaluation on $G_l = \mathbb{1}$ codes (local) flatness.

The last observation does not imply that the theory describes flat geometries, for the same reason for which Regge calculus describes curved geometries using *flat* 4-simplices. In fact, there is a derivation of the vertex (61) which is precisely based on Regge calculus, and a single vertex is interpreted as a flat 4-simplex [31, 33]. In this derivation one only considers 4-valent nodes and 5-valent vertices. The resulting expression naturally generalized to an arbitrary number of nodes and vertices, and therefore defines the dynamics in full LQG. The existence of this generalization was clearly emphasized in [34].

The vertex amplitude (61) gives the probability amplitude for a single spacetime process, where n grains of space are transformed into one another. It has the same *crossing* property as standard QFT vertices. That is, it describes different processes, obtained by splitting differently the boundary nodes into "in" and "out" ones. For instance if $n = 5$ (this is the case corresponding to a 4-simplex in the triangulation picture), the vertex (61) gives the amplitude for a single grain of space splitting into four grains of space; or for two grains scattering into three, and so on. See Figure 6.

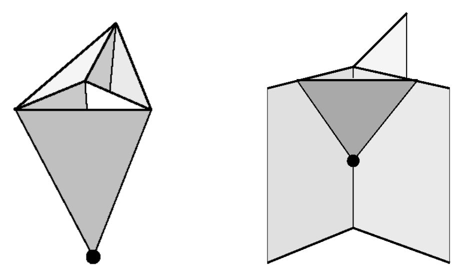


FIG. 6: Transition from nothing to four quanta of space and from a single quantum of space to three.

More precisely, the vertex $\langle W_v|\psi\rangle$ gives an amplitude associated to the spacetime process defined by a finite region of spacetime, bounded by a 3d region described by the state ψ: there is no distinction between "in" and "out" states.

The expressions (56-58) define the quantum field theory of (pure) gravity. What remains to do is to extract physics from this theory, and show that it gives general relativity in some limit. Before discussing how to extract physical predictions from the formalism above, however, let us pause to illustrate how the above construction is heuristically related to classical general relativity.

[9] Notice that this is true for the Feynman integral amplitudes, which are exponential of integrals, namely limits of exponentials of sums, which is to say (limits of) products of (exponentials of) terms which are local in spacetime, as well as for the amplitudes of the QED perturbation expansion, which are products of vertex amplitudes and propagators. In particular, in QED the QED vertex is the elementary dynamical process that gives an amplitude to the boundary Hilbert space of the states of two electrons and one photon.

IV. DERIVATIONS

I have presented the theory without *deriving* it from classical general relativity. There are a number of distinct derivations that converge to the theory. In this section, I sketch some basic ideas in these derivations. A word of caution is however needed.

Quantum-gravity research has often focused on setting up and following "quantization paths" from classical general relativity to a quantum theory. These are very useful to provide heuristic indications for constructing the quantum theory, but they are neither sufficient nor necessary for taking us to quantum gravity. If there was a straightforward quantization route, the quantum theory of gravity would have been found long ago. Any generalization requires a certain amount of guesswork. The "quantization paths" sketched below must be seen as nothing more than heuristics, which have given suggestions useful for construction of the theory, and shed light on aspects of the definitions.

The theory itself should not be evaluated on the basis of whether or not quantization procedures have been "properly followed" in setting it up. It must be judged on the basis of two criteria. The first is whether it provides a coherent scheme consistent with what we know about Nature, namely with quantum mechanics and, in an appropriate limit, with classical general relativity. The second is to predict new physics that agrees with future empirical observations. This is all we demand to a quantum theory of gravity.

Since for the moment we do not have so many useful empirical observations, it might sound that the considerations above give us far to much freedom. How then to choose between different quantum gravity theories, or different ways of constructing the theory? This question is asked often. I think it is a misleading question, for the following reason. At present, we do not have several consistent, complete and predictive theories of quantum gravity. In fact, we are near to have none at all. Most of the quantum gravity approaches lead to very incomplete theories where predictions are impossible. Therefore the scientifically sound problem, today, is whether *any* complete and consistent quantum theory of gravity can be set up at all. If we can solve *this* problem, it is already a great success, after decades of search. The issue of checking whether this is the *right* theory, namely the theory that agrees with experiments, comes after. And if history is any guide, solving a problem of this kind has almost always immediately lead to the right solution: Maxwell found one of the possible ways of combining electricity and magnetism, and it was the right one, Einstein found one of the possible ways of writing a relativistic field theory of gravity and it was the right one, and so on. The scientifically sound problem today, therefore, is whether *a* complete, consistent, predictive theory of quantum gravity exists. With this in mind, let us see what is the formal relation between classical general relativity and the quantum theory constructed above. Accordingly, this section is mostly sketchy, and relies on pointers to existing literature.

A. Dynamics

General relativity can be presented as the field theory for the field $g_{\mu\nu}(x)$ defined by the classical equations of motion that follow for instance from the action

$$S[g]=\int dx\ \sqrt{g}\ (R-2\Lambda). \tag{63}$$

In this section I neglect the cosmological constant Λ. The metric field $g_{\mu\nu}(x)$ cannot be the fundamental field, because it does not allow fermion coupling. A better presentation of the gravitational field, compatible with the physical existence of fermions, is the tetrad formulation, where the gravitational field is represented by the field $e(x)=e_\mu(x)dx^\mu$, where $e_\mu(x)=(e^I_\mu(x), I=0,1,2,3)$ is a vector in Minkowski space. The relation with the metric is well known: $g_{\mu\nu}=\eta_{IJ}e^I_\mu e^J_\nu$. It convenient to treat the theory in the so called first-order formalism, in which the connection is treated as an independent variable. Therefore we introduce a (a priori) independent $SL(2,\mathbb{C})$ connection field $\omega(x)=\omega_\mu(x)dx^\mu$, where $\omega_\mu(x)=(\omega^{IJ}_\mu(x))$ is an antisymmetric Minkowski tensor, namely an element in the adjoint representation of $SL(2,\mathbb{C})$. Another element in the adjoint representation is the Plebanski two form

$$\Sigma\equiv e\wedge e, \tag{64}$$

which is important in what follows. It is a simple exercise to rewrite (63) in terms of these quantity. This gives

$$S[e,\omega]=\int (e\wedge e)^*\wedge F[\omega] \tag{65}$$

where F is the curvature of ω, the star indicates the Hodge dual in the Minkowski indices, that is $(e\wedge e)^*_{IJ}\equiv\frac{1}{2}\ \epsilon_{IJKL}\ e^K\wedge e^L$, and a trace in the adjoint representation is understood (that is $\Sigma F\equiv\Sigma_{IJ}F^{IJ}$) Now, recall that in QCD we add to the classical action $\int F^*\wedge F$ a parity violating θ-term $\theta\int F\wedge F$ that does not affect the equations of motion, but which has an effect in the quantum theory. The same can be done in general relativity, giving the Holst action

$$S[e,\omega]=\int [(e\wedge e)^*+\frac{1}{\gamma}(e\wedge e)]\wedge F[\omega]. \tag{66}$$

This action is equivalent to (63) in the sense that all solutions of the Einstein equations of motion are also solutions of this action. Therefore this action is equally empirically validated as standard general relativity.

We are interested in the quantum states of this theory. Quantum states live at fixed time, or, in a more covariant language, on a 3d surface bounding a spacetime region. Let us therefore consider a region of spacetime with a boundary Σ, and let's study the canonical formalism associated to this boundary. The momentum conjugate to $\omega|_\Sigma$, namely to the restriction of ω to Σ, is immediately read out of the action:

$$\pi = \left((e\wedge e)^* + \frac{1}{\gamma}(e\wedge e)\right)\Big|_\Sigma. \tag{67}$$

π lives in the adjoint representation of $SL(2,\mathbb{C})$. In the quantization of any theory with an internal gauge, the momentum conjugate to the connection is the generator of the local gauge transformations, thus we identify π with the $SL(2,\mathbb{C})$ generator in the quantum theory.

It is convenient to partially gauge fix the internal $SL(2,\mathbb{C})$ symmetry. Choose (continuously) a scalar field $n = (n_I)$ with (timelike) values in Minkowski space at every point of the boundary, and gauge fix e by requiring that $ne|_\Sigma = n_I e^I_i dx^i = 0$, where x^i are coordinates on the boundary. Define its electric and magnetic components with respect to the gauge defined by n, that is

$$\vec{K} := n\pi, \quad \vec{L} := -n\pi^*. \tag{68}$$

where the arrow reminds us that these are in fact 3d quantities like the electric and magnetic field. In the time gauge, using $ne = 0$, this gives immediately

$$\vec{K} = n(e\wedge e)^*|_\Sigma, \quad \vec{L} = -\frac{1}{\gamma}n(e\wedge e)^*|_\Sigma, \tag{69}$$

that is

$$\vec{K} + \gamma\vec{L} = 0, \tag{70}$$

which is called the "linear simplicity constraint". $\vec{L}$, formed by the space-space components of π, generates rotations, while $\vec{K}$, formed by the time-space components of π, generates boosts. Therefore the states of the quantum theory will have to satisfy the constraint equation (70). Compare this equation with equation (48), and recall that the space $\mathcal{H}_\gamma$ is the subspace of the space of the $SL(2,\mathbb{C})$ representations where this equation holds. It is clear that the restriction of the $SL(2,\mathbb{C})$ representations defined by $\mathcal{H}_\gamma$ is precisely an implementation of the general relativity constraint (70).

Now, consider a theory defined by the action

$$S[B,\omega] = \int B\wedge F[\omega]. \tag{71}$$

where B is an arbitrary two-form with values in the adjoint representation. This is the same theory as general relativity by *wihout* the condition that B has the form $B = (e\wedge e)^* + \frac{1}{\gamma}(e\wedge e)$ for some e, or, equivalently, without the condition that in the time gauge *simplicity* holds. This theory, which is called BF theory, is well understood both in the classical and quantum domains. The equations of motion give $F = 0$, therefore the connection is flat. Therefore the dynamics of general relativity can be thought as the combination of two ingredients: an $SL(2,\mathbb{C})$ theory of a flat connection, but with the additional constraint (70). Compare now these observations with the definition of the vertex amplitude (61) that defines the LQG dynamics. The vertex is obtained by mapping $SU(2)$ spin networks into $SL(2,\mathbb{C})$ spin networks with the map Y_γ, and the evaluating these networks at the identity. The first step maps $SU(2)$ spin networks precisely in the subspace of $SL(2,\mathbb{C})$ spin networks where the simplicity constraint holds. The evaluation at the identity is like the request that the connection is flat. In fact, it can be easily shown that if we replace Y_Γ with the identity, we obtain one of the well known quantizations of BF theory. This is discussed in many review papers and I will not insist on this here. See for instance [36] and [37].

The considerations above do not represent a derivation of the LQG dynamics from classical general relativity. But they show that the basic ingredients on which the LQG dynamics is defined are precisely the basic ingredients of the general relativity dynamics, when expressed in the tetrad-connection form.

B. Kinematics

The consideration above illustrate the formal relation between the *dynamics* of LQG and that of classical general relativity. Let me now step back and discuss the logic behind the *kinematics* of LQG. Consider again the three-dimensional spacelike slice Σ of spacetime (that is, "space") and consider the restriction of Σ to this slice. Fix the time gauge. The "Ashtekar-Barbero" connection is the field

$$A \equiv n(\omega^* + \gamma\omega)|_\Sigma. \tag{72}$$

It transforms as an $SU(2)$ connection under the transformations generated by $\vec{L}$ and it is a simple calculation to show that $\vec{L}$ and A are conjugate variables. Given a two-dimensional surface l in space and a one-dimensional line σ , let

$$\vec{L}_l = \int_l \vec{L} \tag{73}$$

be the flux of $\vec{L}$ across the surface and

$$U_\sigma = \mathcal{P}\ e^{\int_\sigma A} \tag{74}$$

the parallel transport operator for A, which is an element of $SU(2)$. The two quantities $\vec{L}_l$ and U_σ are the basis of the canonical loop quantization of general relativity [37–39]. Their Poisson algebra can be represented by operators acting on a space $\mathcal{S}$ of functionals $\psi[A]$ of the connection. The space $\mathcal{S}$ is formed by (limits of sums of products of) functionals that depend on the value of A on graphs.

The key gauge invariance is 3d coordinate transformations, which plays three major roles. First, it is the main hypothesis for a class of theorems stating that the resulting representation is essentially unique [40, 41]. Second, it "washes away" the location of the graph Γ in Σ, so that all the Hilbert subspaces associated to distinct but topologically equivalent graphs in Σ end up identified [42, 43]. Depending on the particular class of coordinate transformations one allows in the classical theory, one ends up with the different versions of the Hilbert space mentioned above. Third, this gauge invariance resolves the difficulties that have plagued the previous attempts to use a basis of loop states in continuous gauge theories.

The other gauge invariance of the canonical theory is formed by the local $SU(2)$ transformations, which gives rise to (16).

The natural definition of the dynamics in the hamiltonian framework is in terms of a hamiltonian constraint operator [39]. A spinfoam expansion for the transition amplitudes can in principle be derived from the canonical formalism [44], but for the moment we do not know how to derive explicitly the amplitude given above from a well defined Hamiltonian constraint.

The purely canonical formulation of the dynamics defined by the Hamiltonian operator is now being developed by an active research program [45], which mostly uses the idea of gauge-fixing diffeormorphisms with matter fields.

C. Covariant lattice quantization

A different possibility to build the quantum theory is to discretize general relativity on a 4d lattice with a boundary, and study the resulting Hilbert space of the lattice theory. This is close in spirit to lattice gauge theory. The difference is diffeomorphism invariance: in general relativity the lattice is a "coordinate" lattice, and coordinates are gauges. Thus for instance there is no analog of the QCD lattice spacing a. More precisely, the physical dimensions (lengths, areas, volumes) of the cells of the lattice are not fixed, as in lattice gauge theory, but are determined by the discretized field variables themselves.

The (double covering of the) local gauge group of the covariant theory is $SL(2,\mathbb{C})$ and the boundary space that one obtains on the boundary of the lattice theory is

$$\mathcal{H}_\Gamma^{SL(2,\mathbb{C})} \;=\; L_2[SL(2,\mathbb{C})^L/SL(2,\mathbb{C})^N]. \qquad (75)$$

where Γ is the two-skeleton of the boundary of the lattice. The states in this Hilbert space $\psi(g_l), g_l \in SL(2,\mathbb{C})$, can be seen as wave functions of the holonomies $H_l = \mathcal{P}\exp\int_l \omega$ of the spin connection ω, along the links l. The corresponding generators J of the Lorentz group must therefore represent the conjugate momentum of ω.

Notice that $SL(2,\mathbb{C})$ has a natural complex structure and we can define the complex variables $\Pi = K + iL$ and $\overline{\Pi} = K - iL$. Then (70) can be interpreted as a reality condition. A technique for implementing reality conditions in the quantum theory is to choose a scalar product appropriately. This is because reality conditions depend on complex conjugation, and this is realized by the adjoint operation in the quantum theory. But the adjoint operation does not depend on the linear structure: it depends on the scalar product. Therefore we can implement the reality conditions by choosing the scalar product appropriately (see e.g. [46]). Viceversa, we can view the quantum mechanical scalar product as *determined* by the reality conditions.

For instance, in the usual Schrödinger representation the condition that x and p be real translates into the requirement that the linear operators x and $p = -i\frac{d}{dx}$ (which form a linear representation of the poisson algebra of the observables x and p) be self adjoint. The L_2 scalar product is the unique one making them so, and thus satisfying the reality conditions.

If we apply this idea here, we have to find a scalar product on a space of lineal functionals on $SL(2,\mathbb{C})$, such that (70) holds. The solution is clear: this is the scalar product implicity defined by the map (47). This reduces the space (75) to one isomorphic to (18).

D. Polyhedral quantum geometry

The idea of polyhedral quantum geometry is to describe "chunks" of quantum space by quantizing the space $\tilde{S}$ of the "shapes" of the geometry of solids figures (tetrahedra, or more general polyhedra) [23, 47–49]. This space can be given a rather natural symplectic structure as follows. Take a flat tetrahedron, for simplicity. Its shape can be coordinatized by the four normals $\vec{L}_l, l = 1,2,3,4$ to its faces, normalized so that $|\vec{L}_l| = a_l$ is the area of the face l. A natural $SO(3)$ invariant symplectic structure on $\tilde{S}$ is $\omega = \sum_l \epsilon_{ijk}\, L^i_l\, dL^j_l \wedge dL^k_l$, or, equivalently, by the Poisson brackets

$$\{L^i_l, L^j_{l'}\} = \delta_{ll'}\, \epsilon^{ij}{}_k\; L^k. \qquad (76)$$

A quantum representation of this Poisson algebra is precisely defined by the generators of $SU(2)$ on the space $\tilde{\mathcal{H}}_n$ given in (29) (for a 4-valent node n). The operator corresponding to the area $a_l = |\vec{L}_l|$ is the Casimir of the representation j_l, therefore the space "quantizes" the space of the shapes of the tetrahedron with areas $j_l(j_l+1)$. Furthermore, the normals of a tetrahedron satisfy

$$\vec{C} := \sum_l \vec{L}_l = 0. \qquad (77)$$

The Hamiltonian flow of $\vec{C}$, generates the rotations of the tetrahedron in R^3. By imposing equation (77) and factoring out the orbits of this flow, the space $\tilde{S}$ reduces to a space S which is still symplectic. In the same manner, imposing the operator equation (77) strongly on $\tilde{\mathcal{H}}_n$ gives the space $\mathcal{H}_n$ given in (30).

The construction generalizes to polyhedra with more than 4 faces. Then the shape of an ensemble of such polyhedra, with the same area and opposing normals on the shared faces[10], is quantized precisely by the Hilbert space $\mathcal{H}$ defined above.

What is the relation with gravity? The central physical idea of general relativity is of course the identification of gravitational field and metric geometry. Consider a polyhedron given on a (say, piecewise linear) manifold. A metric geometry is assigned by giving the value of a metric, or a triad field $e^i = e^i_a dx^a$, namely the gravitational field. Consider the quantity

$$L^i_l = \epsilon_{ijk} \int_l e^j \wedge e^k. \tag{78}$$

Observe that on the one hand this is precisely L_l defined in (73), namely the flux of the densitized inverse triad, or the flux of the Ashtekar's Electric field E^{ia} across the face l of the polyhedron:

$$E^i_l = \int_l n_a E^{ai} \,, \tag{79}$$

where n_a is the normal to the face; on the other hand, in locally flat coordinates it is the normalized normal $\vec{n}_l$ to the face l, multiplied by the area:

$$E^i_l = \int_l n_a E^{ai} = \int_l n^i = n^i_l a_l = L^i_l. \tag{80}$$

Therefore the quantized normals $\vec{L}_l$ of simplicial quantum geometry can be interpreted as the quantum operator giving the flux of the Ashtekar electric field, and we recover again the full kinematics of the previous section.

The spinfoam formalism is natural from this point of view, and the first spinfoam amplitude, called the Barrett-Crane amplitude was first formalized in this context [50]. The Barrett-Crane model was then improved to give the amplitude defined in these lectures.

After this parenthesis, let us return to the physics of LQG.

V. EXTRACTING PHYSICS

The predictions of the theory are in its transition amplitudes. Given a boundary state, the formalism presented above defines transition amplitudes, namely associates probabilities to boundary states (processes). We are particularly interested in processes involving (background) semiclassical geometries. Since the formalism is background independent, the information about the background over which we are computing amplitude must be fed into the calculation. This can only be done with the choice of the boundary state.

Consider a three-dimensional surface Σ with the topology of a three sphere. Let (q, k) be the three-metric and the extrinsic curvature of Σ. The *classical* Einstein equations determine uniquely whether or not (q, k) are physical: that is, whether or not there exist a Ricci-flat spacetime $\mathcal{M}$ (a solution of the Einstein equation) which is bounded by (Σ, q, k).

This is the generalization to general covariant field theory of the following formulation of dynamics. Calling q_o, p_o the coordinate and momenta at time t=0 and q_t, p_t at a final time t, dynamics is fully captured by the conditions the quadruplet (q_o, p_o, q_t, p_t) must satisfy in order to bound a physical trajectory. For a free particle, for instance, these are $p_t = p_o = m(q_t - q_o)/t$.

The *quantum* theory assigns an amplitude to any semiclassical boundary state peaked on a given boundary geometry (q, k), and for classical GR to be recovered this amplitude must (in the semiclassical regime) be suppressed if (Σ, q, k) does not bound a solution of the Einstein equations.[11]

Now, consider a (normalized) semiclassical boundary state $\psi_{(q,k)}$ that approximates the classical geometry (q, k) (these states are discussed in detail in the following section). If q, k is a solution of the Einstein equations, we must expect that, within the given approximation

$$P(\psi_{(q,k)}) = |\langle W|\psi_{(q,k)}\rangle|^2 \sim 1. \tag{81}$$

Next, if we modify the state $\psi_{(q,k)}$ with field operators $E_1, ..., E_n$, then the amplitude

$$W_{(q,k)}(E_1, ..., E_n) = \langle W|E_1...E_n|\psi_{(q,k)}\rangle. \tag{82}$$

can be interpreted as a scattering amplitude between the n "particles" (quanta) created by the field operators E over the spacetime $\mathcal{M}$. (The possibility of using the notion of "particle" in this context is discussed in detail in [51].) Since we know how to write the gravitational field operator (the triad), we can in principle compute graviton n-point functions in this way. To use this strategy, we must learn how to write semiclassical states in $\mathcal{H}_\Gamma$; this is the topic of the next section.

A. Coherent states and holomorphic representation

The relation between quantum states and the classical theory is clarified by the construction of coherent states. These are particularly valuable in the present context, where the relation with the classical theory is more indirect than usual. Various classes of coherent states have been studied. Here I describe the "holomorphic" coherent states, developed by a number of people

[10] The area and the normals match, but not the rest of the geometry of the face, in general. Thus, we have "twisted geometries", in the sense of Freidel and Speziale.

[11] Cfr: in the non-relativistic theory, if $\psi_{q,p}$ is a coherent state peaked on q, p, then $\langle W|\psi_{q_0,p_0} \otimes \overline{\psi_{q_t,p_t}}\rangle \equiv \langle \psi_{q_t,p_t}|e^{-iHt}|\psi_{q_0,p_0}\rangle$ is suppressed unless the conditions mentioned above are satisfied.

[16, 52–55] and recently discussed in detail by Bianchi-Magliaro-Perini [19], as well as the "semi coherent" states of Livine-Speziale (LS) [30].

Holomorphic states are labelled by an element H_l of $SL(2,\mathbb{C})$ for each link l. They are a special case of Thiemann's complexifier coherent states [16, 54, 55]. They are defined by

$$\psi_{H_l}(U_l) = \int_{SU(2)^N} dg_n \bigotimes_{l\in\Gamma} K_t(g_{s(l)} H_l g^{-1} U_l^{-1}). \quad (83)$$

Here t is a positive real number and K_t is the (analytic continuation to $SL(2,\mathbb{C})$ of) the heat kernel on $SU(2)$, which can be written explicitly as

$$K_t(g) = \sum_j (2j+1) e^{-j(j+1)t}\ \mathrm{Tr}[D^j(g)] \quad (84)$$

where D^j is the (Wigner) representation matrix of the representation j.

The $SL(2,\mathbb{C})$ labels H_l can be given two related interpretations. First, we can decompose each $SL(2,\mathbb{C})$ label in the form

$$H_l = e^{2itE_l}\ U_l \quad (85)$$

where $U_l \in SU(2)$ and $E_l \in su(2)$. Then it is not hard to show that U_l and E_l are the expectation values of the operators U_l and L_l on the state ψ_{H_l}

$$\frac{\langle\psi_{H_l}\,|\,U_l\,|\,\psi_{H_l}\rangle}{\langle\psi_{H_l}\,|\,\psi_{H_l}\rangle} = U_l\ , \quad \frac{\langle\psi_{H_l}\,|\,L_l\,|\,\psi_{H_l}\rangle}{\langle\psi_{H_l}\,|\,\psi_{H_l}\rangle} = E_l^i\ , \quad (86)$$

and the corresponding spread is small.[12]

Alternatively, we can decompose each $SL(2,\mathbb{C})$ label in the form

$$H_l = n_{s,l}\ e^{-i(\xi_l + i\eta_l)\frac{\sigma_3}{2}}\ n_{t,l}^{-1}. \quad (87)$$

where $n \in SU(2)$. Let $\vec{z} = (0,0,1)$ and $\vec{n} = D^1(n)\vec{z}$. Freidel and Speziale discuss a compelling geometrical interpretation for the $(\vec{n}_s, \vec{n}_t, \xi, \eta)$ labels defined on of each link by (87) [56] (see also [57–59]). For appropriate four-valent states representing a Regge 3-geometry with intrinsic and extrinsic curvature, the vectors $\vec{n}_s, \vec{n}_t$ are the $3d$ normals to the triangles of the tetrahedra bounded by the triangle; ξ is the extrinsic curvature at the triangle and η is the area of the triangle divided by $8\pi\gamma G\hbar$. For general states, the interpretation extends to a simple generalization of Regge geometries, that Freidel and Speziale have baptized "twisted geometries".

[12] Restoring physical units, $\Delta U_l \sim \sqrt{t}$ and $\Delta E_l \sim 8\pi\gamma\hbar G\sqrt{1/t}$. If we fix a length scale $L \gg \sqrt{\hbar G}$ and choose $t = \hbar G/L^2 \ll 1$, we have then $\Delta U_l \sim \sqrt{\hbar G}/L$ and $\Delta E_l \sim \sqrt{\hbar G}\,L$, which shows that both spreads go to zero with $\hbar$.

Freidel and Speziale give a slightly different definition of coherent states [56]. The two definitions converge for large spins, but differ at low spins. It would be good to clarify their respective properties, in view of the possible applications in scattering theory (see below).

Of great use are also the Livine-Speziale (LS) "semi-coherent" states. They are defined as follows. The conventional magnetic basis $|j,m\rangle$ with $m = -j,...,j$, in H_j diagonalizes L^3. Its highest spin state $|j,j\rangle := |j, m=j\rangle$ is a semiclassical state peaked around the classical configuration $\vec{L} = j\vec{z}$ of the (non commuting) angular momentum operators. If we rotate this state, we obtain a state peaked around any configuration $\vec{L} = j\vec{n}$. The state

$$|j,n\rangle = D^j(n)|j,j\rangle = \sum_m D^j_{jm}(n)|j,m\rangle, \quad (88)$$

is a semiclassical state peaked on $\vec{L} = j\vec{n} = jD^1(n)\vec{z}$.

The states (88) are generally denoted as

$$|j,\vec{n}\rangle := |j,n\rangle. \quad (89)$$

where $\vec{n} = D^j(n)\vec{z}$. I find this notation confusing. The problem is of course that there are many different n (many rotations) that yield the same $\vec{n}$, therefore the state $|j,\vec{n}\rangle$ is not defined by this equation. The common solution is to choose a "phase convention" that fixes a preferred rotation $\hat{n}$ for each $\vec{n}$. For instance, one may require that $D^1(\hat{n})$ leave $\vec{z}\times\vec{n}$ invariant. I would find it clearer, even after such a phase convention has been chosen, to still add a label to the notation (89), say for every rotation n_ϕ that leaves $\vec{n}$ invariant,

$$|j,\vec{n},\phi\rangle := |j, n_\phi \hat{n}\rangle = e^{ij\phi}\ |j,\hat{n}\rangle. \quad (90)$$

The reason is that this phase has a physical interpretation: it codes the extrinsic curvature at the face.

LS states are states in $\mathcal{H}_n$, where n is v-valent (unfortunate notation: here n indicates a node, not an $SU(2)$ element as above), labelled by a unit vector $\vec{n}_l$ for each link l in n, defined by

$$|j_l,\vec{n}_l\rangle = \int_{SU(2)} dg \bigotimes_{l\in n} D^{j_l}(g)|j_l,\vec{n}_l\rangle. \quad (91)$$

The integration projects the state on $\mathcal{H}_n$. These states are not fully coherent: they are eigenstates of the area, and the observable conjugated to the area (which is related to the extrinsic curvature) is fully spread.

Remarkably, in [19] it is shown that for large η_l the holomorphic states are essentially LS states which are also wave packets on the spins. That is

$$\langle j_l,\vec{n}_l|\psi_{H_l}\rangle \sim \prod_l e^{-\frac{(j_l-j_l^0)^2}{2\sigma_l}}\ e^{i\xi_l j_l} \quad (92)$$

where $\vec{n}$ and $\vec{\tilde{n}}$ are identified with the $\vec{n}$ in $s(l)$ and $t(l)$ respectively and where $2j_l + 1 = \eta_l/t_l$ and $\sigma_l = 1/(2t_l)$. Thus, the different coherent states that have been used in the covariant and the canonical literature, and which were long thought to be unrelated, are in fact essentially the same thing.

In summary, the Hilbert space $\mathcal{H}_\Gamma$ contains an (overcomplete) basis of "wave packets" $\psi_{H_l} = \psi_{\vec{n}_l,\vec{n}'_l,\xi_l,\eta_l}$, with a nice interpretation as discrete classical geometries with intrinsic and extrinsic curvature.

These states define a natural holomorphic representation of $\mathcal{H}_\Gamma$ [53, 60]. In this representation, states are represented by holomorphic functions on $SL(2,\mathbb{C})^L$

$$\psi(H_l) = \langle\psi_{H_l}\,|\,\psi\rangle. \quad (93)$$

The vertex amplitude takes a more manageable form when written in terms of coherent states. First, it is easy to show that in terms of LS states, it reads

$$A_v(j_l, \vec{n}_l, \vec{n}'_l) = \int d\tilde{g}_n \bigotimes_l \langle \vec{n}_l | g_{s(l)} g^{-1}_{t(l)} | \vec{n}'_l \rangle_{(\gamma j, j)} \quad (94)$$

The scalar product is taken in the irreducible $SL(2,\mathbb{C})$ representation $H_{(\gamma j,j)}$ and $|\vec{n}_l\rangle$ is the coherent state $|j, \vec{n}_l\rangle$ sitting in the lowest spin subspace of this representation.

Second, the form of the vertex in the holomorphic basis defined by the coherent states (83) can be obtained by combining the definition (57-58) of the vertex and the definition (83) of the coherent states. This gives [60]

$$\begin{aligned} A_v(H_l) &\equiv \langle W_v \,|\, \psi_{H_l} \rangle \quad (95) \\ &= \int_{SL(2,\mathbb{C})^N} d\tilde{g}_n \ \prod_l P(H_l \,,\, g_{s(l)} g^{-1}_{t(l)}) \end{aligned}$$

where

$$P(H,g) = \sum_j (2j+1)\, e^{-j(j+1)t} \ \mathrm{Tr}\big[D^{(j)}(H) Y^\dagger_\gamma D^{(\gamma j,j)}(g) Y_\gamma\big]. \quad (96)$$

Here $D^{(j)}$ is the analytic continuation of the Wigner matrix from $SU(2)$ to $SL(2,\mathbb{C})$ and Y_γ is defined in (47). Eq. (96) gives the "holomorphic" form of the vertex amplitude.

B. The euclidean theory

Before describing how to use the above definition of the dynamics, it is useful to introduce also "euclidean quantum gravity", which is the model theory obtained from the one above by replacing $SL(2,\mathbb{C})$ with $SO(4)$. The representations of $SO(4)$ are labelled by two spins (j^+, j^-). The theory is the same as above with the only difference that (43) is replaced by

$$j^\pm = \frac{|1 \pm \gamma|}{2} \quad (97)$$

and f_γ maps H_j into the lowest spin component of $H_{j^\pm}$ if $\gamma > 1$, but to the highest spin component of $H_{j^\pm}$ if $\gamma < 1$ (the case $\gamma = 1$ is ill defined.) All the rest goes through as above. The vertex amplitude can be written in the simpler form

$$A_v(j_l, \vec{n}_l, \vec{n}'_l) = \int dg^\pm_n \bigotimes_l \prod_{i=\pm} \langle \vec{n}_l | g^i_{s(l)} (g^i_{t(l)})^{-1} | \vec{n}'_l \rangle^{2j^i} \quad (98)$$

where now the integration is over $SU(2)^N \times SU(2)^N \sim SO(4)^N$ and the scalar product is in the fundamental representation of $SU(2)$.

C. Expansions

There is no physics without approximations. We need some appropriate way to compute approximate transition amplitudes, as we do for instance order by order in perturbative QED. What approximations can be effective in the background-independent context of quantum gravity? Three expansions naturally present themselves. (See also [61].)

1. Graph expansion

Consider the component $\mathcal{H}_\Gamma$ of $\mathcal{H}$. Notice that because of the equivalence relation defined in Section II, all the states that have support on graphs smaller than (subgraphs of) Γ are already contained in $\mathcal{H}_\Gamma$, provided that we include also the $j = 0$ representations. Therefore if we truncate the theory to a single Hilbert space $\mathcal{H}_\Gamma$ for a given fixed Γ, what we lose are only states that need a "larger" graphs to be defined. Let us therefore consider the truncation of the theory to a given graph.[13]

What kind of truncation is this? It is a truncation of the degrees of freedom of general relativity down to a finite number; which can be interpreted as describing the lowest modes on a mode expansion of the gravitational field on a compact space. Strictly speaking this is neither an ultraviolet nor an infrared truncation, because the whole physical space can still be large or small. What are lost are not wavelengths shorter than a given length, but rather wavelengths k times shorter than the full size of physical space, for some integer k.

It is reasonable to expect this truncation to define a viable approximation for all gravitational phenomena such that the ratio between the largest and the smallest relevant wavelengths in the boundary state is not large. The approximation can then be improved by taking a larger graph.

Notice that the graph expansion resolves the apparent problem that the operators of the theory are defined on $\mathcal{H}_\Gamma$ rather than on $\mathcal{H}$, since all calculation in this approximation can be performed on a single graph.

2. Vertex expansion

A second natural expansion of presents itself: the expansion in the number N of vertices of σ.

In which regime is this expansion useful? We have a hint from the Regge interpretation of the vertex amplitude: if we derive this amplitude from a Regge discretization of GR, a single vertex corresponds to a *flat* 4-simplex. It is therefore natural to expect that cutting the theory to small N defines an approximation valid around flat space, and where relevant wavelengths are not much shorter than the bounded scattering region $\mathcal{R}$.

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[13] The analog in QFT is to truncate the theory to the sector of Fock space with a number of particles less than a finite fixed maximum number. It is important to stress that that virtually *all* calculations in perturbative QED are performed within this truncation.

Notice the similarity of this expansion with the standard perturbation expansion of QED. In both cases, we describe a quantum field in terms of interactions of a finite number of its "quanta". In the case of QED, these are the photons. In the case of LQG, these are the "quanta of space", or "chunks of space", described in Section II F. In the QED case, individual photons can have small or large energy; in the quantum gravity case, the quanta of space can have small or large volume. In the case of QED, one should be careful not to take the photon picture too literally when looking at the semiclassical limit of the theory. For instance, the Feynman graph for the Coulomb scattering of two electrons is given in Figure 7. But Figure 7 does *not* provide a viable picture of the continuous electric field in the scattering region. Similarly, if we compute a transition amplitude between geometries at first order in the vertex expansion, we should not mistake the corresponding spinfoam for a faithful geometrical picture of the gravitational field in the corresponding classical spacetime.

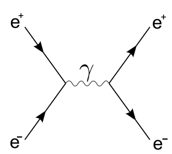

FIG. 7: Electromagnetic scattering of two electrons at first relevant order in QED.

An important observation regards radiative corrections. The QED perturbative expansion is viable because the effect of all the radiative corrections due to the higher frequency modes can be absorbed into the renormalization of a few parameters. Does the same happen in LQG? For the moment, this is not known. Preliminary calculations are encouraging: they indicate *finite* radiative corrections of the vertex and *logarithmic* correction for the "self energy" [62][14].

Potential divergences in the theory are infrared, not ultraviolet, because there is no short-scale geometry (sub Planckian geometry) in the theory.

3. Large distance expansion

Finally, a useful approximation can be taken by choosing the boundary to be large. This means that the boundary state must be peaked on a boundary geometry which is large compared with the Planck length. In particular, we can chose holomorphic boundary states ψ_{H_l} where $\eta_l \gg 1$ in each H_l. Recall that η determines the area of the faces in Planck units (see equation (87) and the discusson that follows it). Therefore large η means that the states we consider describe results of measurements of geometrical quantities that are large compared to the Planck area.

The analysis of the vertex (94) as well as that of its euclidean analog (98) in this limit has been carried out in great detail for the 5-valent vertex, by the Nottingham group [23, 49, 64, 65]. The remarkable result of this analysis is that in this limit the vertex behaves as

$$W_v \sim e^{iS_{Regge}} \tag{99}$$

where S_{Regge} is a function of the boundary variables given by the Regge action, under the identifications of these with variables describing a Regge geometry. The Regge action codes the Einstein equations' dynamics. This is an indication that the vertex can yield general relativity in the large distance limit.

In fact, what is shown in [64] is that $W_v \sim e^{iS_{Regge}} + e^{-iS_{Regge}}$. Concern has been raised by the fact that two terms appear in this sum. This concern is excessive. In the holomorphic representation only one of the terms in survives [60]. This is because of the ubiquitous mechanism of phase cancellations between propagator and boundary state in quantum mechanics. See [66] for a discussion of this mechanism. Therefore the existence of different terms in (99) does not affect the classical limit of the theory. On the other hand, I think that the amplitude of the theory *should* include different terms. This appears clearly in the three dimensional Ponzano Regge theory [67] as well as in low dimensional models [68], and can be viewed as related to the fact that the classical dynamics does not distinguish propagation "ahead in (proper) time" or "backward in (proper) time", in a theory where coordinate time is an unphysical parameter.[15]

D. What has already been computed

Using the approximations discussed above, a few transition amplitudes have already been computed in the (Euclidean) theory.

1. n-point functions

The two point function of general relativity over a flat spacetime has been computed in [69], following the earlier

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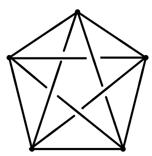
[14] When [62] was written, the choice of the face amplitude was still very unclear. Remarkably, later independent arguments were found, selecting the most favorable case ($d_j \sim j$) [63] .

[15] It is sometime argued that the presence of the two terms follows from the fact that one has failed to select the "positive energy" solutions in the course of the quantization. However, such choice makes sense only in the context of the specific strategy for quantization which consists in considering *complex* solutions of the classical equations and then *discarding* solutions with "negative energy". This strategy is not available here, because of the absence of a preferred time, or a preferred energy. But there are other quantization strategies that are available: we quantify the *real* solution space and keep *all* solutions. In other words, the physical scalar product is determined by all *real* solutions of the Wheeler DeWitt equation with the proper symplectic structure, not by a "positive energy sector" of the complex solutions.

attempts in [66, 70–73] and using the Euclidean theory rather than the Lorentzian one (that is, using (98) instead than (94)), and has been shown to converge to the free graviton propagator of quantum gravity in the large distance limit.

The calculation is to first order in the vertex expansion, on the complete graph with five nodes Γ_5, and to first order in the large-distance expansion. The boundary state ψ_L has been chosen as the coherent state determined by the (intrinsic and extrinsic) geometry of the boundary of a *regular* 4-simplex[16] of size L.

$\Gamma_5 =$  . (100)

The quantity computed is

$$W^{abcd}_{mn} = \langle W|\vec{L}_{na}\cdot\vec{L}_{nb}\ \vec{L}_{mc}\cdot\vec{L}_{md}|\psi_L\rangle \qquad (101)$$
$$-\langle W|\vec{L}_{na}\cdot\vec{L}_{nb}|\psi_L\rangle\langle W\vec{L}_{mc}\cdot\vec{L}^j_{md}|\psi_L\rangle.$$

where $m, n, a, b... = 1, ..., 5$ label the nodes of Γ_5. The resulting expression can be compared with the corresponding quantity

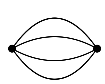
$$W^{abcd}(x_m, x_n) = \langle 0|g^{ab}(x_n)g^{cd}(x_m)|0\rangle_{\text{connected}} \qquad (102)$$

in conventional QFT, where $g^{ab}(x)$ is the gravitational field operator.

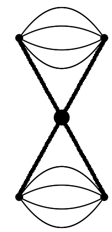
n-point functions in gravity can be computed order by order in this way.

2. Cosmology

The transition amplitude between two homogeneous and isotropic coherent states in quantum cosmology has been computed in [74]. The calculation is: (i) in the approximation where the theory is truncated on the graph formed by two copies of the graph Δ_2^*; Δ_2^* is the "dipole" graph formed by two nodes connected by four links [75]

$\Delta_2^* =$  .

and (ii) at first order in the large distance expansion and in the vertex expansion. The spinfoam considered has therefore the form



Homogenous isotropic states depend on two variables, p and c, at each Δ_2^*. These represent (the square of) the radius a and the extrinsic curvature of closed universe (or $\dot{a}$). They enter the definition of the holomorphic coherent states via a complex combination $z \sim c + ip$. To build these coherent states, let us introduce a regular (symmetric) triangulation of a 3-sphere, formed by two regular tetrahedra joined along all their faces. Let $\vec{n}_l,\ l = 1, ..., 4$ be four unit vectors in R^3 normal to the faces of a regular tetrahedron, that is, such that $\vec{n}_l\cdot\vec{n}_{l'} = -\frac{1}{3}$. The isotropic homogeneous states of a 3-sphere in this approximation are then the states

$$|z\rangle = |H_l(z)\rangle \qquad (103)$$

where $H_l(z)$ are given by

$$H_l(z) = e^{z\vec{n}_l\cdot\vec{\tau}}. \qquad (104)$$

where $z = c + itp$ We are interested in the transition amplitude between two such states, that is

$$W(z, z') = \langle W|(|z\rangle \otimes |z'\rangle). \qquad (105)$$

Using (B7), we obtain [74]

$$W(z, z') = \int_{SL(2,\mathbb{C})^4} dg_1...dg_4 \qquad (106)$$
$$\times \prod_{l=1,4}\sum_{j_l} d_{j_l} e^{-tj_l(j_l+1)} P_j(H_l(z), g_1g_2^{-1})$$
$$\times \prod_{l=1,4}\sum_{j_l} d_{j_l} e^{-tj_l(j_l+1)} P_j(H_l(z'), g_3g_4^{-1}).$$

That is

$$W(z, z') = W(z)W(z') \qquad (107)$$

where

$$W(z) = \int_{SL(2,\mathbb{C})} dg \prod_{l=1,4}\sum_{j_l} d_{j_l} e^{-tj_l(j_l+1)} P_j(H_l(z), g) \quad (108)$$

We are interested in computing this for large spaces, namely when the imaginary part of z is large. Consider the form (104) of H_l and $D^{(j)}(H_l)$ in this limit. Orienting the z-axis along n_l, we have

$$D^{(j)}(H_l) = D^{(j)}(e^{z\tau_3})^m{}_n = \delta^m{}_n e^{imz}. \qquad (109)$$

[16] This approximates flat space. More precisely, the quantity computed can be interpreted as the graviton two-point function *under the condition* that the state of the gravitational field is described by this spin network at large wavelength. In other words, what is assumed is not the full state of the gravitational field, but only the value of some of its variables. Intuitively, this can be seen as the translation into the theory of a *finite* number of macroscopic geometrical measurements that measure flat space.

When the imaginary part of z is large, only the highest magnetic quantum number $m = j$ survives, therefore in this limit

$$D^{(j)}(H_l) = \delta^m{}_j \delta^j{}_n e^{ijz} = e^{ijz} P_{n_l} \tag{110}$$

where P_{n_l} is the projector on the highest magnetic number eigenstate in the direction n_l. Observe now that the sum over j is (a discretization of) a gaussian integral in j_l, peaked on a large value $j^* \sim tp$. One can show that the rest of the expression contributes only polynomially, giving finally [74]

$$W(z) \sim z e^{-\frac{z^2}{2t}}. \tag{111}$$

Finally, restoring $\hbar \neq 1$ for clarity, the transition amplitude in this approximation is

$$W(z, z') \sim z z' e^{-\frac{z^2 + z'^2}{2t\hbar}}. \tag{112}$$

This amplitude reproduces the correct Friedmann dynamics in the sense that it satisfies a quantum constraint equation which reduces to the (appropriate limit of the) Friedmann hamiltonian in the classical limit [74]. Once appropriately normalized by the norm of the boundary coherent state, it can be shown to be peaked on the classical solutions of the theory Friedmann dynamics in the large scale limit we are considering.

The result can be improved by adding a cosmological constant to the theory [76]. To do this, we use the spin-intertwiner basis version of the amplitude described in Appendix B, and modify it as follows

$$Z_{\mathcal{C}} = \sum_{j_f, \mathtt{v}_e} \prod_f (2j+1) \prod_e e^{i\lambda \mathtt{v}_e} \prod_v A_v(j_f, \mathtt{v}_e). \tag{113}$$

This resulting amplitude turns out to be

$$W(z) = \sum_j (2j+1) \frac{N_o}{j^3} \; e^{-2t\hbar j(j+1) - izj - i\lambda \mathtt{v}_o j^{\frac{3}{2}}} \tag{114}$$

There are many ways of analyzing this amplitude. The simplest is to plot its modulus square: see figure 8. This shows a linear relation between a and $\dot{a}$, which is readily integrated giving

$$a(t) = e^{\sqrt{\frac{\Lambda}{3}} t} \tag{115}$$

where Λ is a constant fixed by the parameter λ in (113). See [76] for more details. Equation (115) is the DeSitter solution of the Einstein equations. This supports the conjecture that the dynamics of LQG gives non trivial solutions of the Einstein equations in the classical limit.

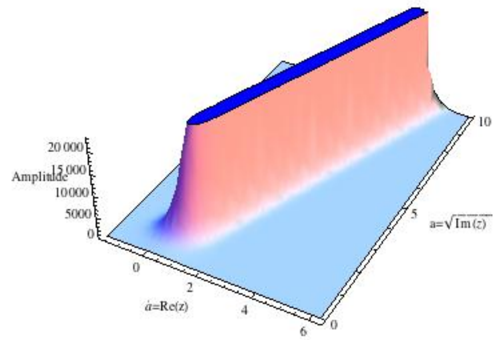


FIG. 8: Numerical analysis of the transition amplitude (114). The only free parameter is λv_o, here is set equal to .3. The computation has been done truncating the sum over j up to a maximum value $j_{max} = 200$. This choice is compatible to maximal scale factor plotted.

VI. CONCLUSION

These lectures are far from covering the full spectrum of the research in Loop Quantum Gravity. I have focused on the covariant formulation of the dynamics, at the expenses of the ongoing research in the canonical language. A formulation I haven't covered is group field theory [77, 78], which is the language in terms of which current research on the scaling of the theory is formulated [79–81]. I haven't covered the applications of the theory to black hole physics [82–85], and to hamiltonian loop quantum cosmology [86, 87], which are by far the most interesting applications of LQG. In particular, loop quantum cosmology is the most likely window for observations. Also, I have not covered several recent development, such as the manifest Lorentz invariant formulation of the theory [88], the coupling to fermions and Yang-Mills fields [89, 90], and to a cosmological constant [91, 92], using a quantum group. For a wide angle review, complementary to these lectures, on various aspects of the theory, including historical, and a more comprehensive bibliography and tentative overall evaluation, see [93].

In conclusion, the theory looks simple and beautiful to me, both in its kinematical and its dynamical parts. Some preliminary physical calculations have been performed and the results are encouraging. The theory is moving ahead fast. But it is far from being complete, and we do not yet know if it really works, and there is still very much to do.

My greatest wish is that one of the students studying these lectures will be able to complete the theory and show it wrong, or, more hopefully, show it right.

Marseille, February 17th, 2011

A warm thank to Leonard Cottrell.

Appendix A: Open problems

The theory is far from being complete. Here are some of the open problems that require further investigation.

1. Compute the propagator (101) in the Lorentzian theory, extending the euclidean result of [69].

2. Compute the three point function and compare it with the vertex amplitude of conventional perturbative quantum gravity on Minkowski space.

3. Compute the next vertex order of the two point function, for $N = 2$.

4. Compute the next graph order of the two point function, for $\Gamma > \Gamma_5$.

5. Understand the normalization factors in these terms, and their relative weight. Find out under which conditions the expansion is viable.

6. Study the radiative corrections in (56) and their possible (infrared) divergences, following the preliminary investigations in [62]. In particular, the sum can be split into a sum over two-complexes and a sum over labelings (spin and intertwiners) for a given two-complex. The potential divergences of the second are associated to "bubbles" (nontrivial elements of the second homotopy class) in the two-complex. Classify them and study how do deal with these.

7. Use the analysis of the these radiative corrections to study the scaling of the theory.

8. In particular, how does G scale?

9. Study the quantum corrections that this theory adds to the tree-level n-point functions of classical general relativity. Can any of these be connected to potentially observable phenomena?

10. Is there any reason for a breaking or a deformation of local Lorentz invariance, that could lead to observable phenomena such as γ ray bursts energy-dependent time of arrival delays, in this theory?

11. Compute the cosmological transition amplitude in the Lorentzian theory, extending the euclidean result of [74]. Compare with canonical Loop Quantum Cosmology [87, 94].

12. The possibility of introducing a spinfoam-like expansion starting from Loop Quantum Cosmology has been considered by Ashtekar, Campiglia and Henderson [95–99]. Can the convergence between the two approaches be completed?

13. Find a simple group field theory [61] whose expansion gives (56).

14. Find the relation between this formalism and the way dynamics can be treated in the canonical theory. Formally, if H is the Hamiltonian constraint, we expect something like the main equation
$$HW = 0 \tag{A1}$$
or $WP = 0$ where the operator P is given by $\langle W|\overline{\psi}\otimes\phi\rangle = \langle\psi|P|\phi\rangle$, since P is formally a projector on the solutions of the Wheeler de Witt equation
$$H\psi = 0. \tag{A2}$$
Can we construct the Hamiltonian operator in canonical LQG such that this is realized?

15. Is the node expansion related to the amount of boundary data available? How?

16. Where is the cosmological constant in the theory? It is tempting to simply replace (61) with a corresponding quantum group expression
$$\langle W_v|\psi\rangle = Ev_q(f\psi). \tag{A3}$$
where Ev_q is the quantum evaluation in $SL(2,\mathbb{C})_q$. Does this give a viable theory? Does this give a finite theory? (Since the first version of these lectures, a solution of this problem has appeared: [91, 92].)

17. How to couple fermions and YM fields to this formulation? The kinematics described above generalizes very easily to include fermions (at the nodes) and Yang Mills fields (on the links). Can we use the simple group theoretical argument that has selected the gravitational vertex also for coupling these matter fields? (Since the first version of these lectures, a first solution of this problem has appeared: [89, 90].)

Appendix B: Alternative forms of the amplitudes

I list here various form of the transition amplitudes that have been used in the literature and can be useful.

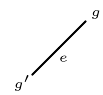
1. Single equation

The three equations (56), (57), (58), can be written compactly in a single equation in the form
$$Z_{\mathcal{C}}(h_l) = \int_{(SL2C)^{2(E-L)-V}} dg'_{ve} \int_{(SU2)^{\mathcal{V}-L}} dh_{ef} \sum_{j_f} N^{-1}_{\{j_f\}} \prod_f d_{j_f}\, \chi^{\gamma j_f, j_f}\Big(\prod_{e\in\partial f} g_{ef}^{\epsilon_{lf}}\Big) \prod_{e\in\partial f} \chi^{j_f}(h_{ef}) \tag{B1}$$
See [100] for details and the next subsection for more on the definition of each term.

2. Feynman rules

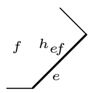
A more detailed description of this expression is given by the following Feynman rules. $Z_{\mathcal{C}}(h_l)$ is defined as the integral obtained associating:

1. Two group integrations to each internal edge (or one to each adjacent couple {internal edge, vertex})



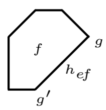
$$\longmapsto \quad \int_{SL2C} dg_{es_e} \int_{SL2C} dg_{et_e} \tag{B2}$$

2. A group integration to each couple of adjacent {face, internal edge}



$$\longmapsto \quad \int_{SU2} dh_{ef}\, \chi^{j_f}(h_{ef}) \tag{B3}$$

3. A sum to each face f



$$\longmapsto \quad \sum_{j_f} d_{j_f}\, \chi^{\gamma j_f, j_f}\Big(\prod_{e\in\partial f} g_{ef}^{\epsilon_{lf}}\Big). \tag{B4}$$

where $g_{ef} := g_{es_e} h_{ef} g_{et_e}^{-1}$ for internal edges, and $g_{ef} = h_l \in SU2$ for boundary edges. γ is a fixed real parameter, the Barbero-Immirzi parameter.

4. At each vertex, one of the integrals $\int_{SL2C} dg_{ev}$ in (B2) (which is redundant) is dropped. (This is the meaning of the prime on dg_{ev})

5. For each coloring j_f, divide the local amplitude by the combinatorial factor N_{j_j}. This factor has be taken to be unity in the text. If we choose it to be the number of automorphisms of $\mathcal{C}$ that preserves j_f, we have an interesting consequence:

   The *limit* (59) can be equivalently [101] expressed as an infinite *sum* over transitions

$$W(h_l) = \sum_{\mathcal{C}} Z^*_{\mathcal{C}}(h_l). \tag{B5}$$

   where $Z^*_{\mathcal{C}}(h_l)$ is defined by the same expression as $Z_{\mathcal{C}}(h_l)$, but with the sum over spins going from $\frac{1}{2}$ to ∞ rather than from 0 to ∞. That is, dropping trivial representations. That is, the full theory can be equivalently recovered by taking the "infinite refinement limit" or by "summing" over two-complexes.

3. Using Y explicitly

The kernel (58) can be written in the form

$$K(h,g) = \sum_j (2j+1)\, \mathrm{Tr}\big[D^{(j)}(h) Y_\gamma^\dagger D^{(\gamma j,j)}(g) Y_\gamma\big]. \tag{B6}$$

That it, using the explicit form of the Y_γ map,

$$P(U,g) = \sum_{jmn} (2j+1)\, D^{(j)}(U)_m{}^n D^{(\gamma j,j)}(g)_{jn}{}^{jm}. \tag{B7}$$

4. Spin-intertwiner basis

In the spin intertwiner basis, the amplitude reads

$$Z(j_l, v_n) = \sum_{j_f, v_e} \prod_f d_{j_f} \prod_v A_v(j_f, v_e) \tag{B8}$$

The euclidean vertex amplitude is

$$\begin{aligned} A(j_e, v_n) \;&=\; \sum_{v_n^+ v_n^-} 15j\big(\tfrac{(1+\gamma)j_l}{2}; v_n^+\big)\, 15j\big(\tfrac{|1-\gamma|j_l}{2}; v_n^-\big) \\ &\quad \bigotimes_a f^{i_n}_{i_n^+ i_n^-}(j_l) \end{aligned} \tag{B9}$$

where the $15j$ are the standard $SU(2)$ Wigner symbols, and the "fusion coefficients" are

$$f^v_{v^+v^-} := v^{m_1\ldots m_n}\; v^+_{q_1^+\ldots q_n^+}\; v^-_{q_1^-\ldots q_n^-}\; \bigotimes_i c^{q_i^+ q_i^-}_{m_i}. \tag{B10}$$

$c^{q_i^+ q_i^-}_{m_i}$. being the Clebsch-Gordan coefficients.

The vertex amplitude was first constructed in this language.

In the lorentzian theory:

$$\begin{aligned} A(j_l, i_n) \;&=\; \sum_{k_n} \int dp_n (k_n^2 + p_n^2) \Bigg(\bigotimes_a f^{i_n}_{k_n p_n}(j_l)\Bigg) \\ &\quad 15j_{SL(2,\mathbb{C})}\,((j_l, j_l\gamma); (k_n, p_n)) \end{aligned} \tag{B11}$$

where we are now using the 15j of $SL(2,\mathbb{C})$ and

$$f^v_{kp} := v^{m_1\ldots m_4}\; \bar{C}^{kp}_{(j_1,m_1)\ldots(j_4,m_4)}, \tag{B12}$$

where $j_1 \ldots j_n$ are the representations meeting at the node.

5. Alternatives

Alternative choices for the face amplitude have been considered in the literature. In the Euclidean case, where $SL(2,\mathbb{C})$ is replaced by $SO(4)$, there is a natural alternative which is the dimension of the $SO(4)$ irreducible into which the representation j is mapped by Y_γ. This choice is incompatible with the natural composition properties of the spin foam amplitude [63].

A simple modification of the theory is to multiply the vertex by a constant λ. This comes naturally if one derives the transition amplitudes from a group field theory [61]: then λ is the coupling constant in front of the group-field-theory interaction term. The physical interpretation of the constant λ is debated [61, 96].

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