

# Casting Loop Quantum Cosmology in the Spin Foam Paradigm

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The goal of spin foam models is to provide a viable path integral formulation of quantum gravity. Because of background independence, their underlying framework has certain novel features that are not shared by path integral formulations of familiar field theories in Minkowski space. As a simple viability test, these features were recently examined through the lens of loop quantum cosmology (LQC). Results of that analysis, reported in a brief communication [1], turned out to provide concrete arguments in support of the spin foam paradigm. We now present detailed proofs of those results. Since the quantum theory of LQC models is well understood, this analysis also serves to shed new light on some long standing issues in the spin foam and group field theory literature. In particular, it suggests an intriguing possibility for addressing the question of why the cosmological constant is positive and small.

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## I. INTRODUCTION

Four different avenues to quantum gravity have been used to arrive at spin-foam models (SFMs). The fact that ideas from seemingly unrelated directions converge to the same type of structures and models has provided a strong impetus to the spin foam program over the years [2].

The first avenue is the Hamiltonian approach to loop quantum gravity (LQG) [3–5]. By mimicking the procedure that led Feynman [6] to a sum over histories formulation of quantum mechanics, Rovelli and Reisenberger [7] proposed a space-time formulation of LQG. This work launched the spin-foam program. The second route stems from the fact that the starting point in canonical LQG is a rewriting of classical general relativity that emphasizes connections over metrics [8]. Therefore in the passage to quantum theory it is natural to begin with the path integral formulation of appropriate gauge theories. A particularly natural candidate is the topological B-F theory [9] because in 3 space-time dimensions it is equivalent to Einstein gravity, and in higher dimensions general relativity can be regarded as a constrained BF theory [10]. The well-controlled path integral formulation of the BF theory provided the second avenue and led to the SFM of Barret and Crane [11]. The third route comes from the Ponzano-Regge model of 3-dimensional gravity [12] that inspired Regge calculus in higher dimensions [13–15]. Here one begins with a simplicial decomposition of the space-time manifold, describes its discrete Riemannian geometry using edge lengths and deficit angles and constructs a path integral in terms of them. If one uses holonomies and

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discrete areas of loop quantum gravity in place of edge lengths, one is again led to a spin foam. These three routes are inspired by various aspects of general relativity. The fourth avenue starts from approaches to quantum gravity in which gravity is to emerge from a more fundamental theory based on abstract structures that, to begin with, have nothing to do with space-time geometry. Examples are matrix models for 2-dimensional gravity and their extension to 3-dimensions —the Boulatov model [16]— where the basic object is a field on a group manifold rather than a matrix. The Boulatov model was further generalized to a group field theory (GFT) tailored to 4-dimensional gravity [4, 17, 18]. The perturbative expansion of this GFT turned out to be very closely related to the vertex expansions in SFMs. Thus the SFMs lie at a junction where four apparently distinct paths to quantum gravity meet. Through contributions of many researchers it has now become an active research area (see, e.g., [4, 10, 19]).

Let us begin with the first path and examine SFMs from the perspective of LQG. Recall that spin network states are used in LQG to construct a convenient orthonormal basis in the kinematical Hilbert space. A key challenge is to extract physical states from them by imposing constraints. Formally this can be accomplished by the group averaging procedure which also provides the physical inner product between the resulting states [20, 21]. From the LQG perspective, the primary goal of SFMs is to construct a path integral that leads to this physical Hilbert space.

Heuristically, the main idea behind this construction can be summarized as follows. Consider a 4-manifold  $M$  bounded by two 3-surfaces,  $S_1$  and  $S_2$ , and a simplicial decomposition thereof. One can think of  $S_1$  as an ‘initial’ surface and  $S_2$  as a ‘final’ surface. One can fix a spin network on each of these surfaces to specify an ‘initial’ and a ‘final’ state of the quantum 3-geometry. A quantum 4-geometry interpolating between the two can be constructed by considering the dual triangulation of  $M$  and coloring its surfaces with half integers  $j$  and edges with suitable intertwiners. The idea is to obtain the physical inner product between the two states by summing first over all the colorings for a given triangulation, and then over triangulations keeping the boundary states fixed. The second sum is often referred to as the *vertex expansion* because the  $M$ -th term in the series corresponds to a dual triangulation with  $M$  vertices. Since each triangulation with a coloring specifies a quantum geometry, the sum is regarded as a path integral over physically appropriate 4-geometries. In ordinary quantum mechanics and Minkowskian field theories where we have a fixed background geometry, such a path integral provides the (dynamically determined) transition amplitude for the first state, specified at initial time, to evolve to the second state at the final time. In the background independent context of quantum gravity, one does not have access to a time variable and dynamics is encoded in constraints. Therefore the notion of a transition in a pre-specified time interval is not meaningful a priori. Rather, the sum over histories now provides the physical inner product between solutions to the quantum constraints, extracted from the two spin network states.

Over the last two years there have been significant advances in SFMs. While the structure of the path integral is well-motivated by the interplay between general relativity and the BF theory, its precise definition requires a key new ingredient —the vertex amplitude. The first proposal for the vertex amplitude was made over ten years ago [11]. But it turned out to have important limitations [22, 23]. New proposals have now been put forward [24–27] and, for the physically interesting regime of the Barbero-Immirzi parameter, they agree. Furthermore, one can regard these SFMs as providing an independent derivation of the kinematics underlying LQG. The detailed agreement between LQG and the new SFMs

[28, 29] is a striking development. There are also a number of results indicating that one does recover general relativity in the appropriate limit [32, 33]. Finally, the vertex amplitude is severely constrained by several general requirements which the new proposals meet.

However, so far, the vertex amplitude has not been systematically derived following procedures used in well-understood field theories, or, starting from a well-understood Hamiltonian dynamics. Therefore, although the convergence of ideas from several different directions is impressive, a number of issues still remain. In particular, the convergence is not quite as seamless as one would like; some rough edges still remain because of unresolved tensions.

For example, the final vertex expansion is a discrete sum, in which each term is itself a sum over colorings for a fixed triangulation. A priori it is somewhat surprising that the final answer can be written as a *discrete* sum. Would one not have to take some sort of a continuum limit at the end? One does this in the standard Regge approach [30] which, as we indicated above, is closely related to SFMs. Another route to SFMs emphasizes and exploits the close resemblance to gauge theories. In non-topological gauge theories one also has to take a continuum limit. Why not in SFMs? Is there perhaps a fundamental difference because, while the standard path integral treatment of gauge theories is rooted in the smooth Minkowskian geometry, SFMs must face the Planck scale discreteness squarely?

A second potential tension stems from the fact that the construction of the physical inner product mimics that of the transition amplitude in Minkowskian quantum field theories. As noted above, in a background independent theory, there is no a priori notion of time evolution and dynamics is encoded in constraints. However, sometimes it *is* possible to ‘deparameterize’ the theory and solve the Hamiltonian constraint by introducing an emergent or relational time a la Leibnitz. What would then be the interpretation of the spin-foam path integral? Would it yield both the physical inner product *and* the transition amplitude? Or, is there another irreconcilable difference from the framework used Minkowskian field theories?

There is also a tension between SFMs and GFTs. Although fields in GFTs live on an abstract manifold constructed from a Lie group, as in familiar field theories the action has a free part and an interaction term. The interaction term has a coupling constant,  $\lambda$ , as coefficient. One can therefore carry out a Feynman expansion and express the partition function, propagators, etc, as a perturbation series in  $\lambda$ . If one sets  $\lambda = 1$ , the resulting series can be identified with the vertex expansion of SFMs. But if one adopts the viewpoint that the GFT is fundamental and regards gravity as an emergent phenomenon, one is led to allow  $\lambda$  to run under the renormalization group flow. What then is the meaning of setting  $\lambda = 1$ ? Or, do other values of  $\lambda$  have a role in SFMs that has simply remained unnoticed thus far? Alternatively, one can put the burden on GFTs. They appear to be efficient and useful calculational schemes. But if they are to have a direct physical significance on their own, what then would the *gravitational* meaning of  $\lambda$  be?

Such questions are conceptually and technically difficult. However, they are important precisely because SFMs appear to lie at a junction of several cross-roads and the recent advances bring out their great potential. Loop quantum cosmology (LQC) provides a physically interesting yet technically simple context to explore such issues. In LQC the principles of LQG are applied to simple cosmological models which have a high degree of symmetry. Thanks to this symmetry, it has been possible to construct and analyze in detail quantum theories in a number of cases [34–47]. Furthermore, LQC shares many of the conceptual problems of LQG and SFMs. Therefore it provides a fertile ground to test various ideas and conjectures in the full theory. In the Hamiltonian context, LQC has served this role

successfully (for a recent review, see [48]). The goal of this paper is to first cast LQC in the spin foam paradigm and then use the results to shed light on the paradigm itself.

In LQC one can arrive at a sum over histories starting from a fully controlled Hamiltonian theory. We will find that this sum bears out the ideas and conjectures that drive the spin foam paradigm. Specifically, we will show that: i) the physical inner product in the timeless framework equals the transition amplitude in the theory that is deparameterized using relational time; ii) this quantity admits a vertex expansion a la SFMs in which the  $M$ -th term refers just to  $M$  volume transitions, without any reference to the time at which the transition takes place; iii) the exact physical inner product is obtained by summing over just the discrete geometries; no ‘continuum limit’ is involved; and, iv) the vertex expansion can be interpreted as a perturbative expansion in the spirit of GFT, where, moreover, the GFT coupling constant  $\lambda$  is closely related to the cosmological constant  $\Lambda$ . These results were reported in the brief communication [1]. Here we provide the detailed arguments and proofs. Because the Hilbert space theory is fully under control in this example, we will be able to avoid formal manipulations and pin-point the one technical assumption that is necessary to obtain the desired vertex expansion (see discussion at the end of section III A). In addition, this analysis will shed light on some long standing issues in SFMs such as the role of orientation in the spin foam histories [49], the somewhat puzzling fact that spin foam amplitudes are real rather than complex [31], and the emergence of the cosine  $\cos S_{\text{EH}}$  of the Einstein action —rather than  $e^{iS_{\text{EH}}}$ — in the classical limit [32, 33].

The paper is organized as follows. In section II we summarize the salient features of LQC that are needed to arrive at a sum over histories formulation. Section III establishes the main results in the timeless framework, generally used in SFMs. In particular, we show that the physical inner product can be expressed as a vertex expansion. In section IV we introduce a deparameterization using the relational time of LQC and obtain an equivalent but *distinct* vertex expansion, more directly related to the transition amplitude. The existence of distinct vertex expansions which sum to the same result suggests the possibility that there may well be distinct but physically equivalent vertex amplitudes in SFMs, each leading to a perturbative expansion that is tailored to a specific aspect of the physical problem. To avoid repetition, we adopted a strategy that is opposite of that used in [1]: here we provide detailed derivations in the timeless framework (section III) and leave out the details while discussing analogous results in the deparameterized picture (section IV). Section V summarizes the main results and discusses some generalizations and open issues. A number of technical issues are discussed in three Appendices.

## II. LQC: A BRIEF OVERVIEW

We will focus on the simplest LQC model that has been analyzed in detail [34–36, 39]: the  $k=0$ ,  $\Lambda=0$  Friedmann model with a massless scalar field as a source. However, it should not be difficult to extend this analysis to allow for a non-zero cosmological constant [40, 41] or anisotropies [43, 44] or to the spatially compact  $k=1$  case [37].

In the FRW models, one begins by fixing a (spatial) manifold  $S$ , topologically  $\mathbb{R}^3$ , Cartesian coordinates  $x^i$  thereon, and a fiducial metric  $q_{ab}^o$  given by  $q_{ab}^o dx^a dx^b = dx_1^2 + dx_2^2 + dx_3^2$ . The physical 3-metric  $q_{ab}$  is then determined by a scale factor  $a$ ;  $q_{ab} = a^2 q_{ab}^o$ . For the Hamiltonian analysis one fixes a cubical fiducial cell  $\mathcal{V}$  whose volume with respect to  $q_{ab}^o$  is  $V_o$  so that its physical volume is  $V = a^3 V_o$ . The quantity  $\nu$  defined by  $V = 2\pi\gamma\ell_{\text{Pl}}^2 |\nu|$  turns out to be a convenient configuration variable, where  $\gamma$  is the Barbero-Immirzi parameter of

LQG [39].<sup>1</sup>

The kinematical Hilbert space is a tensor product  $\mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{kin}}^{\text{grav}} \otimes \mathcal{H}_{\text{kin}}^{\text{matt}}$  of the gravitational and matter Hilbert spaces. Elements  $\Psi(\nu)$  of  $\mathcal{H}_{\text{kin}}^{\text{grav}}$  are functions of  $\nu$  with support on a countable number of points and with finite norm  $\|\Psi\|^2 := \sum_{\nu} |\Psi(\nu)|^2$ . The matter Hilbert space is the standard one:  $\mathcal{H}_{\text{kin}}^{\text{matt}} = L^2(\mathbb{R}, d\phi)$ .<sup>2</sup> Thus, the kinematic quantum states of the model are functions  $\Psi(\nu, \phi)$  with finite norm  $\|\Psi\|^2 := \sum_{\nu} \int d\phi |\Psi(\nu, \phi)|^2$ . A (generalized) orthonormal basis in  $\mathcal{H}_{\text{kin}}$  is given by  $|\nu, \phi\rangle$  with

$$\langle \nu', \phi' | \nu, \phi \rangle = \delta_{\nu'\nu} \delta(\phi', \phi). \quad (2.1)$$

To obtain the physical Hilbert space, one first notes that the quantum constraint can be written as

$$-C\Psi(\nu, \phi) \equiv \partial_{\phi}^2 \Psi(\nu, \phi) + \Theta\Psi(\nu, \phi) = 0 \quad (2.2)$$

where  $\Theta$  is a positive and self-adjoint operator on  $\mathcal{H}_{\text{kin}}^{\text{grav}}$  [50]. More explicitly,  $\Theta$  is a second order difference operator

$$\begin{aligned} (\Theta\Psi)(\nu) := & -\frac{3\pi G}{4\ell_o^2} \left[ \sqrt{|\nu(\nu + 4\ell_o)|} (\nu + 2\ell_o) \Psi(\nu + 4\ell_o) - 2\nu^2 \Psi(\nu) \right. \\ & \left. + \sqrt{|\nu(\nu - 4\ell_o)|} (\nu - 2\ell_o) \Psi(\nu - 4\ell_o) \right], \end{aligned} \quad (2.3)$$

where  $\ell_o$  is related to the ‘area gap’  $\Delta = 4\sqrt{3}\pi\gamma \ell_{\text{Pl}}^2$  via  $\ell_o^2 = \Delta$ . The form of  $\Theta$  shows that the space of solutions to the quantum constraint can be naturally decomposed into sectors in which the wave functions have support on specific ‘ $\nu$ -lattices’ [35]. For definiteness, we will restrict ourselves to the lattice  $\nu = 4n\ell_o$  where  $n$  is an integer. Details of the expression of  $\Theta$  will not be needed in most of our analysis.

The scalar field  $\phi$  is monotonic on all classical solutions (also in the cases when  $k=1$ , and  $\Lambda \neq 0$ ) and therefore serves as a relational time variable, a la Leibnitz, in the classical theory. This interpretation carries over to the quantum theory. For, the form of the quantum constraint (2.2) is similar to that of the Klein-Gordon equation,  $\phi$  playing the role of time and  $-\Theta$  of the spatial Laplacian (or, the elliptic operator generalizing the Laplacian if we are in a general static space-time). Therefore, in LQC, one can use  $\phi$  as an internal time variable with respect to which physical quantities such as the density, scalar curvature, anisotropies in the Bianchi models [43, 44], and infinitely many modes of gravitational waves in the Gowdy models [45–47], evolve.

In the spin foam literature, by contrast, one does not have access to such a preferred time and therefore one chooses to work with the timeless formalism. Therefore let us first forgo the emphasis on using  $\phi$  as internal time and simply implement the group averaging procedure which uses the constraint operator as a whole, without having to single out a

<sup>1</sup> In LQG the basic geometric variable is an orthonormal triad and the physical metric  $q_{ab}$  is constructed from it. If the triad has the same orientation as the fiducial one, given by the coordinates  $x^i$ , the configuration variable  $\nu$  is positive and if the orientations are opposite,  $\nu$  is negative. Physics of the model is insensitive to the triad orientation and hence to the sign of  $\nu$ . In particular the kinematic and physical quantum states satisfy  $\Psi(\nu, \phi) = \Psi(-\nu, \phi)$ .

<sup>2</sup> One can also use a ‘polymer quantization’ of the scalar field at the kinematical level but the final physical theory turns out to be the same.

preferred time variable [20, 21]. This procedure plays an important role in sections III and IV. Therefore it is useful to summarize it in some detail. One begins by fixing a dense sub-space  $\mathcal{S}$  of  $\mathcal{H}_{\text{kin}}$ . In LQC, this is generally taken to be the Schwartz space of smooth functions  $f(\nu, \phi)$  which fall off to zero at infinity faster than any polynomial. The first step in the group averaging procedure is to extract a solution  $\Psi_f(\nu, \phi)$  to the quantum constraint operator (2.2) from each  $f \in \mathcal{S}$ . These solutions are not normalizable in  $\mathcal{H}_{\text{kin}}$  because the spectrum of the constraint  $C$  on  $\mathcal{H}_{\text{kin}}$  is continuous. The second step of the group averaging procedure provides an appropriate inner product between solutions  $\Psi_f(\nu, \phi)$ .

Denote by  $e_k(\nu)$ , with  $k \in (-\infty, \infty)$  a complete set of orthonormal eigenfunctions of  $\Theta$  on  $\mathcal{H}_{\text{kin}}^{\text{grav}}$ . We will denote the eigenvalues by  $\omega_k^2$  and, without loss of generality, assume that  $\omega_k \geq 0$  [35, 36]. (Eigenfunctions and operator functions of  $\Theta$  are discussed in Appendix C.) Any  $f(\nu, \phi) \in \mathcal{S}$  can be expanded as

$$f(\nu, \phi) = \int dk \frac{1}{2\pi} \int dp_\phi \tilde{f}(k, p_\phi) e^{ip_\phi \phi} e_k(\nu). \quad (2.4)$$

Here and in what follows the range of integrals will be from  $-\infty$  to  $\infty$  unless otherwise stated. Using this expansion, we can group-average any  $f(\nu, \phi)$  to obtain a distributional solution (in  $\mathcal{S}^*$ )  $\Psi_f(\nu, \phi)$  to the quantum constraint:

$$\Psi_f(\nu, \phi) := \int d\alpha [e^{i\alpha C} 2|p_\phi| f(\nu, \phi)] = \int dk \int dp_\phi \delta(p_\phi^2 - \omega_k^2) 2|p_\phi| \tilde{f}(k, p) e^{ip_\phi \phi} e_k(\nu), \quad (2.5)$$

where, the operator  $2|p_\phi|$  has been introduced just for later technical simplification. Had we dropped it, we would have associated with  $f$  the solution  $(2|p_\phi|)^{-1} \Psi_f$  and, in the end, obtained a unitarily equivalent representation of the algebra of Dirac observables.

By carrying out the integral over  $p_\phi$  the expression of  $\Psi_f$  can be brought to the desired form:

$$\begin{aligned} \Psi_f(\nu, \phi) &= \int dk [\tilde{f}(k, \omega_k) e^{i\omega_k \phi} e_k(\nu) + \tilde{f}(k, -\omega_k) e^{-i\omega_k \phi} e_k(\nu)] \\ &=: \Psi_f^+(\nu, \phi) + \Psi_f^-(\nu, \phi). \end{aligned} \quad (2.6)$$

By their very definition  $\Psi_f^\pm(\nu, \phi)$  satisfy

$$\Psi_f^\pm(\nu, \phi) = e^{\pm i\sqrt{\Theta}(\phi - \phi_o)} \Psi_f^\pm(\nu, \phi_o), \quad (2.7)$$

whence they can be interpreted as ‘positive and negative frequency solutions’ to (2.2) with respect to the relational time  $\phi$ . Thus the group average of  $f$  is a solution  $\Psi_f$  to the quantum constraint (2.2) which, furthermore, is naturally decomposed into positive and negative frequency parts.  $\Psi_f$  is to be regarded as a distribution in  $\mathcal{S}^*$  which acts on elements  $g \in \mathcal{S}$  via the kinematic inner product [20, 21]:

$$\begin{aligned} \langle \Psi_f | g \rangle &:= \langle \Psi_f | g \rangle \\ &= \int dk \int dp_\phi \delta(p_\phi^2 - \omega_k^2) 2\omega_k \tilde{f}(k, p_\phi) \tilde{g}(k, p_\phi) \\ &= \int dk [\tilde{f}(k, \omega_k) \tilde{g}(k, \omega_k) + \tilde{f}(k, -\omega_k) \tilde{g}(k, -\omega_k)]. \end{aligned} \quad (2.8)$$

Finally, the group averaged scalar product on solutions  $\Psi_f$  is given just by this action [20, 21]. Thus, given any elements  $f, g$  in  $\mathcal{S}$ , the scalar product between the corresponding

group averaged states  $\Psi_f, \Psi_g$  is given by

$$(\Psi_f, \Psi_g) := (\Psi_f|g\rangle = \overline{(\Psi_g|f\rangle}). \quad (2.9)$$

In section III we will obtain a vertex expansion for this scalar product.

A conceptually important observation is that, as in the Klein-Gordon case, there is a superselection. A complete set of Dirac observables is given by the scalar field momentum  $p_\phi = -i\partial_\phi$  and the volume  $V|_{\phi_o}$  (or, equivalently, the energy density operator  $\rho|_{\phi_o}$ ) at the value  $\phi = \phi_o$  of the internal time. (The factor of  $|p_\phi|$  introduced above simplifies the explicit expressions of  $V|_{\phi_o}$  and  $\rho|_{\phi_o}$  [35, 36, 39].) The action of these Dirac observables as well as time evolution leaves the space of positive and negative frequency solutions invariant. Therefore, as in the Klein-Gordon theory, we are led to work with either set. In LQC, one generally works with the positive frequency ones. Then the physical Hilbert space  $\mathcal{H}_{\text{phy}}$  of LQC consists of *positive frequency* solutions  $\Psi_+(\nu, \phi)$  to the quantum constraint (2.2), i.e. solutions satisfying

$$-i\partial_\phi \Psi_+(\nu, \phi) = \sqrt{\Theta} \Psi_+(\nu, \phi) \equiv H \Psi_+(\nu, \phi) \quad (2.10)$$

with inner-product (2.9). This inner product can be re-expressed simply as:

$$(\Psi_+, \Phi_+)_{\text{phy}} = \sum_{\nu=4n\ell_o} \bar{\Psi}_+(\nu, \phi_o) \Phi_+(\nu, \phi_o). \quad (2.11)$$

and is independent of the value  $\phi_o$  of  $\phi$  at which the right side is evaluated.

While this construction of  $\mathcal{H}_{\text{phy}}$  does not *require* us to think of  $\phi$  as internal time in quantum theory, this interpretation is natural in the light of final Eqs (2.10) and (2.11). For, these equations suggest that we can think of  $\nu$  as the sole configuration variable and introduce ‘Schrödinger states’  $\Psi(\nu)$  through the physical inner product (2.11). These ‘evolve’ via (2.10). This is the ‘deparameterized’ description to which we will return in section IV. In this picture, the restriction to positive frequency states has direct interpretation:  $p_\phi \equiv \sqrt{\Theta}$  is now a positive operator on  $\mathcal{H}_{\text{phy}}$  just as  $p_0$  is a positive operator on the traditional Klein-Gordon Hilbert space.

### III. THE TIMELESS FRAMEWORK

Recall that in the spin foam literature, one works with the timeless framework because a natural deparametrization is not available in general. To mimic the general spin foam constructions in LQC, in this section we will largely disregard the fact that the scalar field can be used as relational time and that the final constraint has the form of the Schrödinger equation. Instead, we will use the group averaging procedure for the full constraint

$$C = -\partial_\phi^2 - \Theta \equiv p_\phi^2 - \Theta \quad (3.1)$$

and incorporate the positive frequency condition in a second step. None of the steps in this analysis refer the evolution in relational time mentioned above. Thus, the primary object of interest will be the physical scalar product, rather than the transition amplitude for a Schrödinger state  $\Psi(\nu, \phi_i)$  at an initial ‘time instant’  $\phi_i$  to evolve to another state  $\Phi(\nu, \phi_f)$  at a final ‘time instant’  $\phi_f$ .

In section II we considered general kinematic states  $f(\nu, \phi)$ . In this section, by contrast,

we will focus on the basis vectors  $|\nu, \phi\rangle$  in  $\mathcal{H}_{\text{kin}}$  which are the LQC analogs of spin networks that are used to specify the boundary states in SFMs. Following the setup introduced in section I let us then fix two kinematic states,  $|\nu_i, \phi_i\rangle$  and  $|\nu_f, \phi_f\rangle$ . For notational simplicity, we will denote the group averaged solutions to (2.2) they define by  $||\nu_i, \phi_i\rangle\rangle$  and  $||\nu_f, \phi_f\rangle\rangle$ . The group averaged inner product between these states is given by

$$(|\nu_f, \phi_f\rangle, |\nu_i, \phi_i\rangle) = 2 \int d\alpha \langle \nu_f, \phi_f | e^{i\alpha C} | p_\phi | |\nu_i, \phi_i\rangle. \quad (3.2)$$

Our goal is to express this scalar product as a vertex expansion a la SFMs and study its properties. In section III A we will begin by rewriting it as a sum over histories a la Feynman [6] and then rearrange the sum as a vertex expansion. In section III B we will arrive at the same expansion using perturbation theory in a suitably defined interaction picture. This procedure is reminiscent of the perturbation expansion used in GFTs. As an important consistency check, in section III C we verify that this perturbative expansion does satisfy the constraint order by order. Finally, in section III D we observe that, in this simple example, the coupling constant  $\lambda$  used in the expansion is intimately related to the cosmological constant  $\Lambda$ . Although the precise relation we obtain is tied to LQC, the observation illustrates in a concrete fashion how one may be able to provide a gravitational interpretation to  $\lambda$  in GFTs and suggests an avenue for GFT to account for the smallness of  $\Lambda$ .

### A. Sum over Histories

Following Reisenberger and Rovelli [7], let us first focus on the amplitude

$$A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha) = 2 \langle \nu_f, \phi_f | e^{i\alpha C} | p_\phi | |\nu_i, \phi_i\rangle \quad (3.3)$$

which constitutes the integrand of (3.2). Mathematically one can choose to regard  $\alpha C$  as a Hamiltonian operator. Then  $A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha)$  can be interpreted as the probability amplitude for an initial kinematic state  $|\nu_i, \phi_i\rangle$  to evolve to a final kinematic state  $|\nu_f, \phi_f\rangle$  in a unit ‘time interval’ and we can follow Feynman’s procedure [6] to express it as a sum over histories. Technically, a key simplification comes from the fact that the constraint  $C$  is a sum of two commuting pieces that act separately on  $\mathcal{H}_{\text{kin}}^{\text{matt}}$  and  $\mathcal{H}_{\text{kin}}^{\text{grav}}$ . Consequently, the amplitude (3.3) factorizes as

$$A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha) = A_\phi(\phi_f, \phi_i; \alpha) A_G(\nu_f, \nu_i; \alpha) \quad (3.4)$$

with

$$A_\phi(\phi_f, \phi_i; \alpha) = 2 \langle \phi_f | e^{i\alpha p_\phi^2} | p_\phi | |\phi_i\rangle, \quad \text{and} \quad A_G(\nu_f, \nu_i; \alpha) = \langle \nu_f | e^{-i\alpha \Theta} | \nu_i \rangle. \quad (3.5)$$

It is easy to cast the first amplitude,  $A_\phi$ , in the desired form using either a standard Feynman expansion or simply evaluating it by inserting a complete eigen-basis of  $p_\phi$ . The result is:

$$A_\phi(\phi_f, \phi_i; \alpha) = 2 \int dp_\phi e^{i\alpha p_\phi^2} e^{ip_\phi(\phi_f - \phi_i)} |p_\phi| \quad (3.6)$$

The expansion of the gravitational amplitude  $A_G$  is not as simple. We will first express it as a sum over histories. In a second step, we will evaluate the total amplitude (3.3) by integrating over  $\alpha$  for each history separately. Although it is not a priori obvious, we will find



that the amplitude associated to each history is manifestly finite and the total amplitude can be written as a discrete sum that mimics the vertex expansion in SFMs.

### 1. The gravitational amplitude $A_G$

As mentioned above, to apply the standard Feynman procedure we will regard  $e^{-i\alpha\Theta}$  as an ‘evolution operator’ with ‘Hamiltonian’  $\alpha\Theta$  and a ‘time interval’  $\Delta\tau = 1$ . We emphasize that this ‘evolution’ is just a convenient mathematical construct and does not correspond to the physical evolution with respect to the relational time variables  $\phi$  normally used in LQC. Rather, since it is generated by the constraint  $C$ , physically it represents gauge transformations (or time reparameterizations).

Let us divide the interval  $\Delta\tau = 1$  into  $N$  parts each of length  $\epsilon = 1/N$  and write the gravitational amplitude  $A_G(\nu_f, \nu_i; \alpha)$  as

$$\langle \nu_f | e^{-i\alpha\Theta} | \nu_i \rangle = \sum_{\bar{\nu}_{N-1}, \dots, \bar{\nu}_1} \langle \nu_f | e^{-i\epsilon\alpha\Theta} | \bar{\nu}_{N-1} \rangle \langle \bar{\nu}_{N-1} | e^{-i\epsilon\alpha\Theta} | \bar{\nu}_{N-2} \rangle \dots \langle \bar{\nu}_1 | e^{-i\epsilon\alpha\Theta} | \nu_i \rangle \quad (3.7)$$

where we have first split the exponential into  $N$  identical terms and then introduced a decomposition of the identity operator at each intermediate ‘time’  $\tau = n\epsilon$ ,  $n = 1, 2, \dots, N-1$ . For notational simplicity, we will denote the matrix element  $\langle \bar{\nu}_n | e^{-i\epsilon\alpha\Theta} | \bar{\nu}_{n-1} \rangle$  by  $U_{\bar{\nu}_n \bar{\nu}_{n-1}}$  and set  $\nu_f = \bar{\nu}_N$  and  $\nu_i = \bar{\nu}_0$ . We then have

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{\bar{\nu}_{N-1}, \dots, \bar{\nu}_1} U_{\bar{\nu}_N \bar{\nu}_{N-1}} U_{\bar{\nu}_{N-1} \bar{\nu}_{N-2}} \dots U_{\bar{\nu}_1 \bar{\nu}_0}. \quad (3.8)$$

The division of  $\Delta\tau$  provides a skeletonization of this ‘time interval’. An assignment  $\sigma_N = (\bar{\nu}_N, \dots, \bar{\nu}_0)$  of volumes to the  $N+1$  time instants  $\tau = \epsilon n$  can be regarded as a discrete (gauge) history associated with this skeletonization since one can envision the universe going from  $\bar{\nu}_{n-1}$  to  $\bar{\nu}_n$  under a finite ‘evolution’. The matrix element is given by a sum of amplitudes over these discrete histories with fixed endpoints,

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{\sigma_N} A(\sigma_N) \equiv \sum_{\sigma_N} U_{\bar{\nu}_N \bar{\nu}_{N-1}} U_{\bar{\nu}_{N-1} \bar{\nu}_{N-2}} \dots U_{\bar{\nu}_2 \bar{\nu}_1} U_{\bar{\nu}_1 \bar{\nu}_0}. \quad (3.9)$$

The next step in a standard path integral construction is to take the ‘continuum’ limit,  $N \rightarrow \infty$ , of the skeletonization. In particle mechanics at this stage one uses a continuous basis (say the position basis  $|x\rangle$ ) to carry out this expansion. By contrast, our basis  $|\nu_n\rangle$  is discrete. As a result, one can make rigorous sense of the  $N \rightarrow \infty$  limit by reorganizing the well-defined sum (3.9) according to the number of volume transitions. The remainder of section III A 1 is devoted to carrying out this step.

This task involves two key ideas. Let us first note that along a path  $\sigma_N$ , the volume  $\bar{\nu}$  is allowed to remain constant along a number of time steps, then jump to another value, where it could again remain constant for a certain number of time steps, and so on. The first key idea is to group paths according to the number of *volume transitions* rather than time steps. Let us then consider a path  $\sigma_N^M$  which involves  $M$  volume transitions (clearly,

$M \leq N$ ):

$$\sigma_N^M = (\nu_M, \dots, \nu_M; \nu_{M-1}, \dots, \nu_{M-1}; \dots \dots; \overbrace{\nu_1, \dots, \nu_1}^{N_2}; \underbrace{\nu_0, \dots, \nu_0}_{N_1}). \quad (3.10)$$

Thus, the volume changes from  $\nu_{m-1}$  to  $\nu_m$  at ‘time’  $\tau = N_m \epsilon$  and remains  $\nu_m$  till time  $\tau = N_{m+1} \epsilon$ . Note that  $\nu_m$  is distinct from  $\bar{\nu}_m$  used in (3.9): While  $\nu_m$  is the volume after the  $m$ -th *volume transition* along the given discrete path,  $\bar{\nu}_m$  is the volume at the end of the  $m$ -th *time interval*, i.e., at  $\tau = m \epsilon$ .

These discrete histories can be labeled more transparently by two ordered sequences

$$\sigma_N^M = \{(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0); (N_M, N_{M-1}, \dots, N_2, N_1)\}, \quad \nu_m \neq \nu_{m-1}, \quad N_m > N_{m-1}. \quad (3.11)$$

where  $\nu_M, \dots, \nu_0$  denote the volumes that feature in the history  $\sigma_N^M$  and  $N_k$  denotes the number of time steps after which the volume changes from  $\nu_{k-1}$  to  $\nu_k$ . Note that while no two *consecutive* volume values can be equal, a given volume value can repeat in the sequence;  $\nu_m$  can equal some  $\nu_n$  if  $n \neq m \pm 1$ . The probability amplitude for such a history  $\sigma_N^M$  is given by:

$$A(\sigma_N^M) = [U_{\nu_M \nu_M}]^{N-N_{M-1}} U_{\nu_M \nu_{M-1}} \dots [U_{\nu_1 \nu_1}]^{N_2-N_1-1} U_{\nu_1 \nu_0} [U_{\nu_0 \nu_0}]^{N_1-1}. \quad (3.12)$$

The second key idea is to perform the sum over all these amplitudes in three steps. First we keep the ordered set of volumes  $(\nu_M, \dots, \nu_0)$  fixed, but allow the volume transitions to occur at *any* value  $\tau = n \epsilon$  in the interval  $\Delta \tau$ , subject only to the constraint that the  $m$ -th transition occurs before the  $(m+1)$ -th for all  $m$ . The sum of amplitudes over this group of histories is given by

$$A_N(\nu_M, \dots, \nu_0; \alpha) = \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \dots \sum_{N_1=1}^{N_2-1} A(\sigma_N^M). \quad (3.13)$$

Next we sum over all possible intermediate values of  $\nu_m$  such that  $\nu_m \neq \nu_{m-1}$ , keeping  $\nu_0 = \nu_i, \nu_M = \nu_f$  to obtain the amplitude  $A_N(M)$  associated with the set of all paths in which there are precisely  $M$  volume transitions:

$$A_N(M; \alpha) = \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \dots, \nu_0; \alpha) \quad (3.14)$$

Finally the total amplitude  $A_G(\nu_f; \nu_i, \alpha)$  is obtained by summing over all volume transitions that are permissible within our initially fixed skeletonization with  $N$  time steps:

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{M=0}^N A_N(M; \alpha) \quad (3.15)$$

This concludes the desired re-arrangement of the sum (3.9). The sum on the right side is manifestly finite. Furthermore, since  $A_G(\nu_f, \nu_i; \alpha) = \langle \nu_f | e^{-i\alpha \Theta} | \nu_i \rangle$ , the value of the amplitude (3.15) does not depend on  $N$  at all; the skeletonization was introduced just to express this well-defined amplitude as a sum over histories. Thus, while the range of  $M$  in the sum and

the amplitude  $A_N(M; \alpha)$  in (3.15) both depend on  $N$ , the sum does not.

Therefore we are well positioned to get rid of the skeletonization altogether by taking the limit  $N$  goes to infinity. Note first that with our fixed skeletonization, the gravitational amplitude is a finite sum of terms,

$$A_G(\nu_f, \nu_i; \alpha) = A_N(0; \alpha) + A_N(1; \alpha) + \dots + A_N(M; \alpha) + \dots + A_N(N; \alpha) \quad (3.16)$$

each providing the contribution of all discrete paths that contain a fixed number of volume transitions. Let us focus on the  $M$ th term in the sum:

$$A_N(M; \alpha) = \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \dots, \nu_0; \alpha) \quad (3.17)$$

Now, in Appendix A we show that the limit  $\lim_{N \rightarrow \infty} A_N(\nu_M, \dots, \nu_0; \alpha)$  exists and is given by

$$\begin{aligned} A(\nu_M, \dots, \nu_0; \alpha) &:= \lim_{N \rightarrow \infty} A_N(\nu_M, \dots, \nu_0; \alpha) \\ &= \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 A(\nu_M, \dots, \nu_0; \tau_M, \dots, \tau_1; \alpha) \end{aligned} \quad (3.18)$$

where

$$\begin{aligned} A(\nu_M, \dots, \nu_0; \tau_M, \dots, \tau_1; \alpha) &:= e^{-i(1-\tau_M)\alpha\Theta_{\nu_M\nu_M}} (-i\alpha\Theta_{\nu_M\nu_{M-1}}) \times \\ &\dots e^{-i(\tau_2-\tau_1)\alpha\Theta_{\nu_1\nu_1}} (-i\alpha\Theta_{\nu_1\nu_0}) e^{-i\tau_1\alpha\Theta_{\nu_0\nu_0}}. \end{aligned} \quad (3.19)$$

Note that the matrix elements  $\Theta_{\nu_m\nu_n} = \langle \nu_m | \Theta | \nu_n \rangle$  of  $\Theta$  in  $\mathcal{H}_{\text{kin}}^{\text{grav}}$  can be calculated easily from (2.3) and vanish if  $(\nu_m - \nu_n) \notin \{0, \pm 4\ell_0\}$ . Therefore, explicit evaluation of the limit is rather straightforward. We will assume that the limit  $N \rightarrow \infty$  can be interchanged with the sum over  $\nu_{M-1}, \dots, \nu_1$ . (This assumption is motivated by the fact that in the expression of  $A(\nu_M, \dots, \nu_0; \alpha)$  most matrix elements of  $\Theta$  vanish, and since the initial and final volumes are fixed, the sums over intermediate volumes  $\nu_{M-1}, \dots, \nu_1$  extend over only a finite number of non-zero terms.) Then it follows that

$$A_G(M; \alpha) := \lim_{N \rightarrow \infty} A_N(M; \alpha)$$

exists for each finite  $M$ . Note that the reference to the skeletonization disappears in this limit. Thus,  $A_G(M; \alpha)$  is the amplitude obtained by summing over all paths that contain precisely  $M$  volume transitions within the ‘time interval’  $\Delta\tau = 1$ , irrespective of precisely when and at what values of volume they occurred. Finally, (3.16) implies that the total gravitational amplitude can be written as an infinite sum:

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{M=0}^{\infty} A_G(M; \alpha) \quad (3.20)$$

While each partial amplitude  $A_G(M; \alpha)$  is well-defined and finite, it does not ensure that the infinite sum converges. A priori the infinite sum on the right hand side of (3.20) could be, for example, only an asymptotic series to the well-defined left side. Also, our derivation assumed that the limit  $N \rightarrow \infty$  commutes with the partial sums. Both these limitations will

be overcome in section III B: We will see that  $A_G(\nu_f, \nu_i; \alpha)$  is indeed given by a convergent sum (3.20).

The expression (3.18) still contains some integrals. These can be performed exactly. The case when all of  $(\nu_M, \dots, \nu_0)$  are distinct is straightforward and the result as given in [1]. The general case is a little more complicated and is analyzed in Appendix B. The final result is:

$$A(\nu_M, \dots, \nu_0; \alpha) = \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \times \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k - 1} \sum_{m=1}^p \frac{e^{-i\alpha \Theta_{w_m w_m} \Delta \tau}}{\prod_{j \neq m}^p (\Theta_{w_m w_m} - \Theta_{w_j w_j})} \quad (3.21)$$

where, since the volumes can repeat along the discrete path,  $w_m$  label the  $p$  distinct values taken by the volume and  $n_m$  the number of times that each value occurs in the sequence. The  $n_m$  satisfy  $n_1 + \dots + n_p = M + 1$ .

To summarize, we have written the gravitational part  $A_G(\nu_f, \nu_i; \alpha)$  of the amplitude as a ‘sum over histories’:

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{M=0}^{\infty} \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \dots, \nu_0; \alpha) \quad (3.22)$$

with  $A(\nu_M, \dots, \nu_0; \alpha)$  given by (3.21). This expression consists of a sum over  $M$ , the number of volume transitions, and a sum over the (finite number of) sequences of  $M - 1$  intermediate volumes that are consistent with the boundary conditions and the condition that  $\nu_m \neq \nu_{m+1}$ . In section III A 2 we will use this sum to generate the ‘vertex expansion’ of the physical inner product.

## 2. Vertex expansion of the physical inner product

Recall that the group-averaged scalar product can be expressed as

$$([\nu_f, \phi_f], [\nu_i, \phi_i]) = 2 \int d\alpha A_\phi(\nu_i, \phi_i; \alpha) A_G(\nu_f, \nu_i; \alpha). \quad (3.23)$$

The main assumption in our derivation —the only one that will be required also in section III B— is that one can interchange the integration over  $\alpha$  and the sum over  $M$  in the expression of  $A_G(\nu_f, \nu_i; \alpha)$ . Let us then use expressions (3.6) and (3.22) of  $A_\phi$  and  $A_G$ , make the interchange and carry out the integral over  $\alpha$ . The scalar product (3.23) is then re-expressed as a sum of amplitudes associated with discrete paths  $(\nu_M, \dots, \nu_0)$ :

$$([\nu_f, \phi_f], [\nu_i, \phi_i]) = \sum_{M=0}^{\infty} \left[ \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \dots, \nu_0; \phi_f, \phi_i) \right], \quad (3.24)$$

where,

$$A(\nu_M, \dots, \nu_0; \phi_f, \phi_i) = 2 \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \times \quad (3.25)$$

$$\prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k - 1} \sum_{m=1}^p \int dp_\phi e^{ip_\phi(\phi_f - \phi_i)} |p_\phi| \frac{\delta(p_\phi^2 - \Theta_{w_m w_m} \Delta\tau)}{\prod_{j \neq m}^p (\Theta_{w_m w_m} - \Theta_{w_j w_j})}.$$

The right side is a sum of distributions, integrated over  $p_\phi$ . It is straightforward to perform the integral and express  $A(\nu_M, \dots, \nu_0; \phi_f, \phi_i)$  in terms of the matrix elements of  $\Theta$ :

$$A(\nu_M, \dots, \nu_0; \phi_f, \phi_i) = \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \times \quad (3.26)$$

$$\prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k - 1} \sum_{m=1}^p \frac{e^{i\sqrt{\Theta_{w_m w_m}} \Delta\phi} + e^{-i\sqrt{\Theta_{w_m w_m}} \Delta\phi}}{\prod_{j \neq m}^p (\Theta_{w_m w_m} - \Theta_{w_j w_j})}$$

where  $\Delta\phi = \phi_f - \phi_i$ . Since by inspection each amplitude  $A(\nu_M, \dots, \nu_0, \phi_f, \phi_i)$  is real, the group averaged scalar product (3.24) is also real.

Finally, as explained in section II, the group averaging procedure yields a solution which has both positive and negative frequency components while the physical Hilbert space consists only of positive frequency solutions. Let us denote the positive frequency parts of the group averaged ket  $|\nu, \phi\rangle$  by  $|\nu, \phi\rangle_+$ . Then, the physical scalar product between these states in  $\mathcal{H}_{\text{phy}}$  is given by a sum over amplitudes  $A(M)$ , each associated with a fixed number of volume transitions:

$$([\nu_f, \phi_f]_+, [\nu_i, \phi_i]_+)_{\text{phy}} = \sum_{M=0}^{\infty} A(M) \quad (3.27)$$

$$= \sum_{M=0}^{\infty} \left[ \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \right.$$

$$\left. \times \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k - 1} \sum_{i=1}^p \frac{e^{i\sqrt{\Theta_{w_i w_i}} \Delta\phi}}{\prod_{j \neq i}^p (\Theta_{w_i w_i} - \Theta_{w_j w_j})} \right].$$

(Note that the right side is in general complex, a point to which we will return in section V.) This is the vertex expansion of the physical inner product we were seeking. It has two key features. First, the integral over the parameter  $\alpha$  was carried out and is not divergent. This is a non-trivial and important result if we are interested in computing the physical inner product perturbatively, i.e., order by order in the number of vertices. Second, the summand involves only the matrix elements of  $\Theta$  which are easy to compute. As remarked earlier, significant simplification arises because Eq (2.3) implies that  $\Theta_{\nu_m \nu_n}$  is zero if  $\nu_m - \nu_n \notin \{0, \pm 4\ell_0\}$ .

Let us summarize. We did not begin by postulating that the physical inner product is given by a formal path integral. Rather, we started with the kinematical Hilbert space and the group averaging procedure and *derived* a vertex expansion of the physical inner product. Because the Hilbert space framework is fully under control, we could pin-point the one assumption that is needed to arrive at (3.27): the sum over vertices and the integral over  $\alpha$  can be interchanged. In the full theory, one often performs formal manipulations which result in divergent individual terms in the series under consideration. (For instance sometimes one

starts by expanding the very first amplitude (3.3) in powers of  $\alpha$  even though the  $\alpha$  integral of each term is then divergent [7, 19]). *In our case, individual terms in the series are all finite*, and, as we will show in section III B, even the full series (3.22) representing the gravitational amplitude is convergent. Nonetheless, at present the interchange of the  $\alpha$ -integral and the infinite sum over  $M$  has not been justified. If this gap can be filled, we would have a fully rigorous argument that the well-defined physical inner product admits an exact, convergent vertex expansion (3.27). (This assumption is needed only in the timeless framework because the integration over  $\alpha$  never appears in the deparameterized framework of section IV.) In particular, there is no need to take a ‘continuum limit’.

## B. Perturbation Series

We will now show that the expression (3.27) of the transition amplitude can also be obtained using a specific perturbative expansion. Structurally, this second derivation of the vertex expansion is reminiscent of the perturbative strategy used in group field theory (see, e.g., [17, 18]).

Let us begin by considering the diagonal and off-diagonal parts  $D$  and  $K$  of the operator  $\Theta$  in the basis  $|\nu = 4n\ell_o\rangle$ . Thus, matrix elements of  $D$  and  $K$  are given by:

$$D_{\nu'\nu} = \Theta_{\nu\nu} \delta_{\nu'\nu}, \quad K_{\nu'\nu} = \begin{cases} \Theta_{\nu'\nu} & \nu' \neq \nu \\ 0 & \nu' = \nu \end{cases} \quad (3.28)$$

Clearly  $C = p_\phi^2 - D - K$ . The idea is to think of  $p_\phi^2 - D$  as the ‘main part’ of  $C$  and  $K$  as a ‘perturbation’. To implement it, introduce a 1-parameter family of operators

$$C_\lambda = p_\phi^2 - \Theta_\lambda := p_\phi^2 - D - \lambda K \quad (3.29)$$

as an intermediate mathematical step. The parameter  $\lambda$  will simply serve as a marker to keep track of powers of  $K$  in the perturbative expansion and we will have to set  $\lambda = 1$  at the end of the calculation.

Our starting point is again the decomposition (3.4) of the amplitude  $A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha)$  into a scalar field and a gravitational part. The  $\lambda$  dependence appears in the gravitational part:

$$A_G^{(\lambda)}(\nu_f, \nu_i, \alpha) := \langle \nu_f | e^{-i\alpha\Theta_\lambda} | \nu_i \rangle. \quad (3.30)$$

Let us construct a perturbative expansion of this amplitude. Again we think of  $e^{-i\alpha\Theta_\lambda}$  as a mathematical ‘evolution operator’ defined by the ‘Hamiltonian’  $\alpha\Theta_\lambda$  and a ‘time interval’  $\Delta\tau = 1$ . The ‘unperturbed Hamiltonian’ is  $\alpha D$  and the ‘perturbation’ is  $\lambda\alpha K$ . Following the textbook procedure, let us define the ‘interaction Hamiltonian’ as

$$H_I(\tau) = e^{i\alpha D\tau} \alpha K e^{-i\alpha D\tau}. \quad (3.31)$$

Then the evolution in the interaction picture is dictated by the 1-parameter family of unitary operators on  $\mathcal{H}_{\text{kin}}^{\text{grav}}$

$$\tilde{U}_\lambda(\tau) = e^{i\alpha D\tau} e^{-i\alpha\Theta_\lambda \tau}, \quad \text{satisfying} \quad \frac{d\tilde{U}_\lambda(\tau)}{d\tau} = -i\lambda H_I(\tau) \tilde{U}_\lambda(\tau). \quad (3.32)$$

The solution of this equation is given by a time-ordered exponential:

$$\begin{aligned}\tilde{U}_\lambda(\tau) &= \mathcal{T} e^{-i\int_0^\tau H_I(\tau) d\tau} \\ &= \sum_{M=0}^{\infty} \lambda^M \int_0^\tau d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 [-iH_I(\tau_M)] \dots [-iH_I(\tau_1)].\end{aligned}\quad (3.33)$$

Next we use the relation  $e^{-i\alpha\Theta_\lambda} = e^{-i\alpha D}\tilde{U}_\lambda(1)$ , with  $\tilde{U}_\lambda$  given by (3.33), take the matrix element of  $e^{i\alpha\Theta_\lambda}$  between initial and final states,  $|\nu_i \equiv \nu_0\rangle$  and  $|\nu_f \equiv \nu_M\rangle$ , and write out explicitly the product of the  $H_I$ 's. The result is

$$\begin{aligned}A_G^{(\lambda)}(\nu_f, \nu_i, \alpha) &= \sum_{M=0}^{\infty} \lambda^M \int_0^1 d\tau_M \dots \int_0^{\tau_2} d\tau_1 \sum_{\nu_{M-1}, \dots, \nu_1} [e^{-i(1-\tau_M)\alpha D_{\nu_M\nu_M}}] \times \\ &\quad (-i\alpha K_{\nu_M\nu_{M-1}}) \dots (-i\alpha K_{\nu_1\nu_0}) [e^{-i\tau_1\alpha D_{\nu_0\nu_0}}].\end{aligned}\quad (3.34)$$

We can now replace  $D$  and  $K$  by their definition (3.28). Because  $K$  has no diagonal matrix elements, only the terms with  $\nu_m \neq \nu_{m+1}$  contribute and the sum reduces precisely to

$$A_G^{(\lambda)}(\nu_f, \nu_i, \alpha) = \sum_{M=0}^{\infty} \lambda^M \left[ \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \dots, \nu_0; \alpha) \right], \quad (3.35)$$

where  $A(\nu_M, \dots, \nu_0; \alpha)$  is given by (3.21) as in the sum over histories expansion of section III A 1.

We can now construct the total amplitude by including the scalar field factor (3.6) and performing the  $\alpha$  integral as in section III A 2. Then the group averaged scalar product is given by

$$([\nu_f, \phi_f], [\nu_i, \phi_i])^{(\lambda)} = \sum_{M=0}^{\infty} \lambda^M \left[ \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \dots, \nu_0, \phi_f, \phi_i) \right] \quad (3.36)$$

where  $A(\nu_M, \dots, \nu_0, \phi_f, \phi_i)$  is given in (3.26). If we now set  $\lambda = 1$ , (3.36) reduces to (3.24) obtained independently in section III A 2.

Finally, let us restrict ourselves to the positive frequency parts  $[[\nu, \phi]_+]$  of  $[\nu, \phi]$  which provide elements of  $\mathcal{H}_{\text{phy}}$ . Reasoning of section III A 2 tells us that the physical scalar product  $([\nu_f, \phi_f]_+, [\nu_i, \phi_i]_+)_{\text{phy}}$  is given by (3.27).

Thus, by formally regarding the *volume changing*, off-diagonal piece of the constraint as a perturbation we have obtained an independent derivation of the vertex expansion for  $([\nu_f, \phi_f]_+, [\nu_i, \phi_i]_+)_{\text{phy}}$  as a power series expansion in  $\lambda$ , the power of  $\lambda$  serving as a bookmark that keeps track of the number of vertices in each term. In this sense this alternate derivation is analogous to the vertex expansion obtained using group field theory. This derivation has a technical advantage. Since  $H_I$  is self-adjoint on  $\mathcal{H}_{\text{kin}}^{\text{grav}}$ , it follows that the expansion (3.33) of  $\tilde{U}_\lambda(\tau)$  is convergent everywhere on  $\mathcal{H}_{\text{kin}}^{\text{grav}}$  [51]. This in turn implies that the right hand side of (3.35) converges to the well-defined gravitational amplitude  $A_G^{(\lambda)} = \langle \nu_f | e^{-i\alpha\Theta_\lambda} | \nu_i \rangle$ . However, to arrive at the final vertex expansion starting from (3.35) we followed the same procedure as in section III A 2. Therefore, this second derivation of the vertex amplitude also assumes that one can interchange the integral over  $\alpha$  with the (convergent but) infinite

sum over  $M$  in (3.35).

### C. Satisfaction of the constraint

The physical inner product between the basis states defines a 2-point function:

$$G(\nu_f, \phi_f; \nu_i \phi_i) := ([\nu_f, \phi_f]_+, [\nu_i, \phi_i]_+)_{\text{phy}} \quad (3.37)$$

and it follows from section II that it satisfies the constraint equation in each argument. Since  $G(\nu_f, \phi_f; \nu_i \phi_i) = \bar{G}(\nu_i, \phi_i; \nu_f \phi_f)$ , it suffices to focus just on one argument, say the final one. Then we have:

$$[\partial_{\phi_f}^2 - \Theta_f]G(\nu_f, \phi_f; \nu_i, \phi_i) = 0 \quad (3.38)$$

where  $\Theta_f$  acts as in (2.3) but on  $\nu_f$  in place of  $\nu$ . If one replaces  $\Theta$  by  $\Theta_\lambda$ , one obtains a 2-point function  $G_\lambda(\nu_f, \phi_f; \nu_i, \phi_i)$  which, as we saw in section III B admits a perturbative expansion:

$$G_\lambda(\nu_f, \phi_f; \nu_i, \phi_i) = \sum_{M=0}^{\infty} \lambda^M A_M(\nu_f, \phi_f; \nu_i, \phi_i), \quad (3.39)$$

where  $A_M$  is the amplitude defined in (3.27):

$$\begin{aligned} A_M(\nu_f, \phi_f; \nu_i, \phi_i) &= \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_+(\nu_M, \dots, \nu_0; \phi_f, \phi_i) \\ &\equiv \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \times \\ &\quad \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k - 1} \sum_{m=1}^p \frac{e^{i\sqrt{\Theta_{w_m w_m}} \Delta \phi}}{\prod_{j \neq m}^p (\Theta_{w_m w_m} - \Theta_{w_j w_j})} \end{aligned} \quad (3.40)$$

The suffix  $+$  in  $A_+(\nu_M, \dots, \nu_0; \phi_f, \phi_i)$  emphasizes that we have taken the positive frequency part.

As a non-trivial check on this expansion we will now show that  $G_\lambda$  satisfies (3.38) order by order. Since  $\Theta_\lambda = D + \lambda K$ , our task reduces to showing

$$(\partial_{\phi_f}^2 - D_f) A_M(\nu_f, \phi_f; \nu_i \phi_i) - K_f A_{M-1}(\nu_f, \phi_f; \nu_i \phi_i) = 0. \quad (3.41)$$

We'll show that the left hand side is zero path by path in the sense that for every path acted on by the off-diagonal part there are two paths acted on the diagonal part that cancel it.



Without loss of generality we assume that  $\nu_f = w_p$  in (3.40). Then we have

$$\begin{aligned}
(\partial_{\phi_f}^2 - D_f)A_+(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) &= \Theta_{\nu_f \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_i} \times \\
&\left[ \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k - 1} \sum_{m=1}^p \frac{\Theta_{w_m w_m} e^{i\sqrt{\Theta_{w_m w_m}} \Delta \phi}}{\prod_{j \neq m}^p (\Theta_{w_m w_m} - \Theta_{w_j w_j})} \right. \\
&\left. - \Theta_{w_p w_p} \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k - 1} \sum_{m=1}^p \frac{e^{i\sqrt{\Theta_{w_m w_m}} \Delta \phi}}{\prod_{j \neq m}^p (\Theta_{w_m w_m} - \Theta_{w_j w_j})} \right]. \tag{3.42}
\end{aligned}$$

If  $w_p$  occurs with multiplicity  $n_p = 1$ , if  $\nu_f$  is the only volume to take the value  $w_p$  then there are no derivatives in  $\Theta_{w_p w_p}$  in the above equation and it simplifies to

$$\begin{aligned}
(\partial_{\phi_f}^2 - D_f)A_+(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) &= \Theta_{\nu_f \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_i} \times \\
&\left[ \prod_{k=1}^{p-1} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k - 1} \sum_{m=1}^p \frac{(\Theta_{w_m w_m} - \Theta_{w_p w_p}) e^{i\sqrt{\Theta_{w_i w_i}} \Delta \phi}}{\prod_{j \neq i}^p (\Theta_{w_m w_m} - \Theta_{w_j w_j})} \right] \\
&= \Theta_{\nu_f \nu_{M-1}} A_+(\nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i). \tag{3.43}
\end{aligned}$$

Thus, on simple paths where the final volume occurs only once in the sequence, the action of  $[\partial_{\phi_f}^2 - D]$  is to give the amplitude of the path without  $\nu_f$ , times a matrix element of  $\Theta$  related to the transition from  $\nu_{M-1}$  to  $\nu_f$ . In general, the value of the final volume can be repeated in the discrete path;  $n_p \neq 1$ . In that case we need to push  $\Theta_{w_p w_p}$  under the derivatives but the final result is the same. Thus, in all cases we have

$$(\partial_{\phi_f}^2 - D_f) A_+(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) = \Theta_{\nu_f \nu_{M-1}} A_+(\nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i). \tag{3.44}$$

Finally, it is straightforward to evaluate the action of the off-diagonal part on  $A_{M-1}$  (see (3.41)):

$$K A_+(\nu_f, \nu_{M-2}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) = \sum_{\nu_{M-1}} \Theta_{\nu_f \nu_{M-1}} A_+(\nu_{M-1}, \nu_{M-2}, \dots, \nu_1, \nu_i; \phi_f, \phi_i). \tag{3.45}$$

Combining these results we see that Eq. (3.41) is satisfied. Thus the vertex expansion we obtained is a solution to the quantum constraint equation. Further it is a good perturbative solution in the sense that, if we only take paths in which the number of volume transitions is less than some  $M^*$ , then the constraint is satisfied to the  $M^*$  order in  $\lambda$

$$[\partial_{\phi_f}^2 - (D_f + \lambda K_f)] \sum_{M=0}^{M^*} \lambda^M A_M(\nu_f, \phi_f; \nu_i, \phi_i) = \mathcal{O}(\lambda^{M^*+1}) \tag{3.46}$$

Also in this calculation the cancelations occur in a simple manner; the off-diagonal part acting on paths with  $M - 1$  transitions gives a contribution for each path with  $M$  transitions that could be obtained by adding a single additional transition in the original path. These contributions cancel with the action of the diagonal part on the paths with  $M$  transitions.

This calculation provides an explicit check on our perturbative expansion of the physical

inner product. This is a concrete realization, in this simple example, of a central hope of SFMs: to show that the physical inner product between spin networks, expressed as a vertex expansion, does solve the Hamiltonian constraint of LQG order by order.

#### D. The ‘coupling constant’ $\lambda$ and the cosmological constant $\Lambda$

So far we have regarded the GFT inspired perturbation theory as a calculational tool and the coupling constant  $\lambda$  as a book-keeping device which merely keeps track of the number of vertices in the vertex expansion. From this standpoint values of  $\lambda$  other than  $\lambda = 1$  have no physical significance. However, if one regards GFT as fundamental and gravity as an emergent phenomenon, one is forced to change the viewpoint. From this new perspective, the coupling constant  $\lambda$  is physical and can, for example, run under a renormalization group flow. The question we raised in section I is: What would then be the physical meaning of  $\lambda$  from the *gravitational* perspective? Surprisingly, in the LQC model under consideration,  $\lambda$  can be regarded as (a function of) the cosmological constant  $\Lambda$ .

Let us begin by noting how the quantum constraint changes in presence of a cosmological constant  $\Lambda$ :

$$-C(\Lambda) = \partial_\phi^2 + \Theta(\Lambda) \equiv \partial_\phi^2 + \Theta - \pi G \gamma^2 \Lambda \nu^2. \quad (3.47)$$

Thus, only the diagonal part of  $\Theta$  is modified and it just acquires an additional term proportional to  $\Lambda$ . In the GFT-like perturbation expansion, then, we are led to decompose  $\Theta_\lambda(\Lambda)$  as

$$\Theta_\lambda(\Lambda) = D(\Lambda) + \lambda K \quad \text{where} \quad D(\Lambda) = \pi G \left( \frac{3}{2\ell_o^2} - \gamma^2 \Lambda \right) \nu^2. \quad (3.48)$$

It is now easy to check that  $\Psi(\nu, \phi)$  satisfies the constraint equation

$$[\partial_\phi^2 + D(\Lambda) + \lambda K] \Psi(\nu, \phi) = 0 \quad (3.49)$$

with cosmological constant  $\Lambda$  if and only if  $\tilde{\Psi}(\nu, \tilde{\phi})$  satisfies

$$[\partial_{\tilde{\phi}}^2 + D(\tilde{\Lambda}) + K] \tilde{\Psi}(\nu, \tilde{\phi}) = 0 \quad (3.50)$$

where

$$\tilde{\Lambda} = \frac{\Lambda}{\lambda} + \frac{3}{2\gamma^2 \ell_o^2 \lambda} (\lambda - 1), \quad \tilde{\phi} = \sqrt{\lambda} \phi, \quad \text{and} \quad \tilde{\Psi}(\nu, \tilde{\phi}) = \Psi(\nu, \phi). \quad (3.51)$$

Consequently the two theories are isomorphic.

Because of this isomorphism, the gravitational meaning of the coupling constant  $\lambda$  is surprisingly simple. Consider the FRW model with cosmological constant  $\Lambda$ . Evaluating the GFT-like perturbation theory at  $\lambda = \lambda_0$  would provide the spin foam-like vertex expansion for the theory with cosmological constant  $\tilde{\Lambda}_0 = \Lambda/\lambda_0 + (3/2\gamma^2 \ell_o^2 \lambda_0) (\lambda_0 - 1)$ . Thus, evaluation of the vertex expansion at  $\lambda \neq 1$  in GFT can be interpreted as a specific shift in the value of the cosmological constant in the Hamiltonian or the spin-foam formulations of LQC.

It is instructive to consider the reciprocal case and set  $\tilde{\Lambda} = 0$ . Then the tilde theory yields the spin-foam expansion with zero cosmological constant we obtained in section III A. The un-tilde theory provides its re-interpretation from the GFT perspective. Now the cosmological constant ‘runs with the coupling constant’ via  $\Lambda = 3(1 - \lambda)/2\gamma^2 \ell_o^2$ . In the weak coupling limit  $\lambda \approx 0$ , this theory has a positive but Planck scale cosmological constant  $\Lambda \approx 3/2\gamma^2 \ell_o^2$ .

This is what one would expect from the ‘vacuum energy’ considerations in quantum field theories in Minkowski space-time. If the coupling constant  $\lambda$  were to increase under some renormalization group flow and approach the SFM value  $\lambda = 1$ , then  $\Lambda$  would decrease. If we are just slightly away from the fixed point  $\lambda = 1$ , the cosmological constant  $\Lambda$  would be small and positive. These heuristics suggest an avenue by which a fully developed GFT could perhaps account for the smallness of the cosmological constant.

#### IV. DEPARAMETERIZED FRAMEWORK

In this section we will use the deparameterized framework which emphasizes the role of  $\phi$  as internal time. As explained in section II, now we can work in the Schrödinger picture, regarding  $\nu$  as the configuration variable and  $\phi$  as time. The physical states are now represented as functions  $\Psi(\nu)$  with a finite norm,

$$\|\Psi\|_{\text{phy}}^2 = \sum_{\nu=4n\ell_0} |\Psi(\nu)|^2, \quad (4.1)$$

and they evolve via Schrödinger equation:

$$-i\partial_\phi \Psi(\nu, \phi) = \sqrt{\Theta} \Psi(\nu, \phi) \equiv H \Psi(\nu, \phi). \quad (4.2)$$

In contrast to section III, in this section we will not be interested in the kinematical Hilbert space or the group averaging procedure. The primary object of interest will rather be the *transition amplitude*

$$A(\nu_f, \varphi; \nu_i, 0) = \langle \nu_f | e^{iH\varphi} | \nu_i \rangle \quad (4.3)$$

for the initial physical state  $|\nu_i\rangle$  at time  $\phi_i = 0$  to evolve to  $|\nu_f\rangle$  at time  $\phi_f = \varphi$ . From our discussion in section II, one would expect this amplitude to equal the physical scalar product  $([\nu_f, \varphi]_+, [\nu_i, 0]_+)_{\text{phy}} = G(\nu_f, \varphi; \nu_i, 0)$  considered in section III. This is indeed the case. For, the positive frequency solution  $\Psi_{\nu_i, \phi_i} \equiv [\nu_i, \phi_i]_+$  obtained by group averaging the kinematic basis vector  $|\nu_i, \phi_i\rangle$  is given by

$$\Psi_{\nu_i, \phi_i}(\nu, \phi) = \int dk (\bar{e}_k(\nu_i) e^{-i\omega_k \phi_i}) e^{i\omega_k(\phi)} e_k(\nu) \quad (4.4)$$

(see Eq.(2.6)) so that the physical scalar product between positive frequency solutions  $[\nu_i, \phi_i]_+$  and  $[\nu_f, \phi_f]_+$  is given by

$$([\nu_f, \phi_f]_+, [\nu_i, \phi_i]_+)_{\text{phy}} = \int dk e^{i\omega_k(\phi_f - \phi_i)} \bar{e}_k(\nu_i) e_k(\nu_f) \quad (4.5)$$

(see Eq (2.9)). The right hand side is precisely the expression of the transition amplitude  $\langle \nu_f | e^{iH\varphi} | \nu_i \rangle = \int dk \langle \nu_f | e^{iH\varphi} | k \rangle \langle k | \nu_i \rangle$ . Since  $e_k(\nu) = \langle \nu | k \rangle$ , we have the equality:  $G(\nu_f, \varphi; \nu_i, 0) = A(\nu_f, \varphi; \nu_i, 0)$ . However, the interpretation now emphasizes the *physical* time-evolution in  $\phi$  generated by  $H$  whence  $A(\nu_f, \varphi; \nu_i, 0)$  has the interpretation of a physical transition amplitude. Therefore, we can literally follow —not just mimic— the procedure Feynman used in non-relativistic quantum mechanics [6]. This will again lead to a vertex expansion but one which, if terminated at any finite order, is distinct from that obtained in section III.

In spite of important conceptual differences, the mathematical procedure used in this sec-

tion is completely analogous to that used in section III. Furthermore, this deparameterized framework was discussed in greater detail than the timeless framework in [1]. Therefore, in this section we will present only the main steps.

### A. Sum over histories

Following Feynman, let us divide the time interval  $(\varphi, 0)$  into  $N$  equal parts, each of length  $\epsilon = \varphi/N$ , and express the transition amplitude  $A(\nu_f, \varphi; \nu_i, 0)$  as a sum over discretized paths  $\sigma_N = (\nu_f = \nu_N, \bar{\nu}_{N-1}, \dots, \bar{\nu}_1, \bar{\nu}_0 = \nu_i)$ :

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{\sigma_N} A(\sigma_N) \quad \text{with} \quad A(\sigma_N) = U_{\bar{\nu}_N \bar{\nu}_{N-1}} U_{\bar{\nu}_{N-1} \bar{\nu}_{N-2}} \dots U_{\bar{\nu}_2 \bar{\nu}_1} U_{\bar{\nu}_1 \bar{\nu}_0} \quad (4.6)$$

where now  $U_{\bar{\nu}_{n+1} \bar{\nu}_n} \equiv \langle \bar{\nu}_{n+1} | e^{i\epsilon H} | \bar{\nu}_n \rangle$ . The structure of Eq (4.6) parallels that of Eq (3.9) in section III A. However, the mathematical ‘time interval’  $\Delta\tau = 1$  in section III A is now replaced by the *physical* time interval  $(\varphi, 0)$  and the mathematical ‘Hamiltonian’  $\alpha\Theta$  by the *physical* Hamiltonian  $H = \sqrt{\Theta}$ . Furthermore we no longer split the amplitude into a gravitational part and a scalar field part and the group averaging parameter  $\alpha$  will never appear in this section.

As in section III A, the next step is to make a convenient rearrangement of this sum, emphasizing volume-transitions, rather than what happens at each point  $\phi_n = n\epsilon$  of the skeletonized time interval. Thus, we first recognize that the volume could remain constant for a number of time steps and consider histories  $\sigma_N^M$  with precisely  $M$  volume transitions (where  $M < N$ ):

$$\sigma_N^M = \{ (\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0); (N_M, N_{M-1}, \dots, N_2, N_1) \}, \quad \nu_m \neq \nu_{m-1}, \quad N_m > N_{m-1}. \quad (4.7)$$

where  $\nu_M, \dots, \nu_0$  denote the volumes that feature in the history  $\sigma_N^M$  and  $N_k$  denotes the number of time steps after which the volume changes from  $\nu_{k-1}$  to  $\nu_k$ . The probability amplitude for such a history  $\sigma_N^M$  is given by:

$$A(\sigma_N^M) = [U_{\nu_M \nu_M}]^{N-N_M-1} U_{\nu_M \nu_{M-1}} \dots [U_{\nu_1 \nu_1}]^{N_2-N_1-1} U_{\nu_1 \nu_0} [U_{\nu_0 \nu_0}]^{N_1-1}. \quad (4.8)$$

As in section III A, we carry out the sum over all these amplitudes in three steps. First we keep the ordered set of volumes  $(\nu_M, \dots, \nu_0)$  fixed, but allow the volume transitions to occur at *any* value  $\phi = n\epsilon$  in the interval  $\mathcal{I}$ , subject only to the constraint that the  $m$ -th transition occurs before the  $(m+1)$ -th for all  $m$ . The sum of amplitudes over this group of histories is given by

$$A_N(\nu_M, \dots, \nu_0) = \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \dots \sum_{N_1=1}^{N_2-1} A(\sigma_N^M). \quad (4.9)$$

Next we sum over all possible intermediate values of  $\nu_m$  such that  $\nu_m \neq \nu_{m-1}$ , keeping  $\nu_0 = \nu_i, \nu_M = \nu_f$ , to obtain the amplitude  $A(M)$  associated with the set of all paths in

which there are precisely  $M$  volume transitions:

$$A_N(M) = \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \dots, \nu_0) \quad (4.10)$$

Finally the total amplitude  $A(\nu_f, \phi; \nu_i, 0)$  is obtained by summing over all volume transitions that are permissible within our initially fixed skeletonization with  $N$  time steps:

$$A(\nu_f, \phi; \nu_i, 0) = \sum_{M=0}^N A_N(M) \equiv \sum_{M=0}^N \left[ \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \dots, \nu_0) \right]. \quad (4.11)$$

As in section III A, since  $A(\nu_f, \phi; \nu_i, 0) = \langle \nu_f | e^{iH\phi} | \nu_i \rangle$ , the value of the amplitude (4.11) does not depend on  $N$  at all; the skeletonization was introduced just to express this well-defined amplitude as a sum over histories. Thus, while the range of  $M$  in the sum and the amplitude  $A_N(M)$  in (4.11) both depend on  $N$ , the sum does not. We can get rid of the skeletonization altogether by taking the limit as  $N$  goes to infinity, to express the total transition amplitude as a vertex expansion in the spirit of the timeless framework of spinfoams. Reasoning analogous to that in Appendix A shows that the limit does exist. In this limit the reference to the skeletonization of the time interval disappears and volume changes can now occur at *any* time in the continuous interval ( $\phi_i = 0$ ,  $\phi_f = \phi$ ). The contribution  $A_M$  from paths with precisely  $M$  volume changes has a well defined ‘continuous time’ limit and the total amplitude is given by a discrete sum over  $M$ :

$$A(\nu_f, \phi; \nu_i, 0) = \sum_{M=0}^{\infty} A_M(\nu_f, \phi; \nu_i, 0) \quad (4.12)$$

where the partial amplitudes  $A_M$  are given by

$$\begin{aligned} A_M(\nu_f, \phi; \nu_i, 0) &= \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i, \phi) \quad (4.13) \\ &= \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} H_{\nu_M \nu_{M-1}} H_{\nu_{M-1} \nu_{M-2}} \cdots H_{\nu_2 \nu_1} H_{\nu_1 \nu_0} \times \\ &\quad \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial H_{w_k w_k}} \right)^{n_k - 1} \sum_{m=1}^p \frac{e^{iH_{w_m w_m} \phi}}{\prod_{j \neq m}^p (H_{w_m w_m} - H_{w_j w_j})}. \end{aligned}$$

As one might expect, the final expression involves just the matrix elements of the Hamiltonian  $H = \sqrt{\Theta}$ . These are calculated in Appendix C.

Thus, the total transition amplitude has been expressed as a vertex expansion (4.12) a la SFMs. We provided several intermediate steps because, although the left hand sides are equal, the final vertex expansions is *different* from that obtained in section III A: While (4.12) features matrix elements of  $H = \sqrt{\Theta}$ , (3.27) features matrix elements of  $\Theta$  itself. The existence of distinct but equivalent vertex expansions is quite surprising. In each case we emphasized a distinct aspect of dynamics: the timeless framework and group averaging in (3.27), and relational time and deparametrization in (4.12).

## B. Perturbation expansion

This vertex expansion can also be obtained as a perturbation series that mimics GFTs. As in section III, the perturbative approach avoids skeletonization altogether and has the advantage that it guarantees a convergent series. Furthermore, since this deparametrization approach does not refer to an integral over  $\alpha$ , the assumption of interchange of the integral and the sum over  $M$  that was required in section III B is no longer necessary.

Let us now focus on the Hamiltonian operator  $H = \sqrt{\Theta}$  (rather than on  $\Theta$  used in section III B) and decompose it into a diagonal part  $D$  and the remainder, non-diagonal part  $K$  which is responsible for a volume change. Finally, let us set  $H_\lambda = D + \lambda K$  where  $\lambda$  will serve as a marker for powers of  $K$ , i.e., the number of volume changes in the expansion. Then, by working in the appropriate interaction picture, we obtain:

$$A_\lambda(\nu_f, \varphi; \nu_i, 0) = \sum_{M=0}^{\infty} \lambda^M A_M(\nu_f, \varphi; \nu_i, 0) \quad (4.14)$$

where  $A_M$  is again given by (4.13). This power series in  $\lambda$  is reminiscent of what one finds in GFTs. If we set  $\lambda = 1$  at the end of this derivation, we recover the vertex expansion (4.12) a la SFMs. For a discussion of the intermediate steps, see [1] and Appendix A.

## C. Satisfaction of the Schrödinger Equation

Recall that in the deparametrization scheme, the Schrödinger equation (4.2) incorporates both the quantum constraint and the positive frequency condition. By its very definition, the exact transition amplitude  $A(\nu_f, \varphi; \nu_i, 0)$  satisfies this Schrödinger equation. As a check on the perturbative expansion (4.14) we are led to ask whether the Schrödinger equation would be satisfied in a well-controlled approximate sense if we were to truncate the series on the right side of (4.14) at a finite value, say  $M^*$  of  $M$ . We will now show that this is indeed the case.

Since  $H_\lambda = D + \lambda K$ , the Schrödinger equation would be solved order by order in perturbation series if for each  $M$  we have:

$$(i\partial_\varphi + D_f) A_M(\nu_f, \varphi; \nu_i, 0) + K_f A_{M-1}(\nu_f, \varphi; \nu_i, 0) = 0. \quad (4.15)$$

Using the expression of the partial amplitudes  $A_M$  we are then led to ask if

$$\sum_{\substack{\nu_{M-2}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} \left[ \sum_{\substack{\nu_{M-1} \\ \nu_{M-1} \neq \nu_{M-2}}} (-i\partial_\varphi + D) A(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \varphi) + K A(\nu_f, \nu_{M-2}, \dots, \nu_1, \nu_i; \varphi) \right] \quad (4.16)$$

vanishes for each  $M$ . Using the expression (4.13) of  $A(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \varphi)$ , one can readily verify that this is indeed the case. As in section III C, the equation is satisfied ‘path by path’, i.e., already by the intermediate amplitudes  $A(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \varphi)$  and  $A(\nu_f, \nu_{M-2}, \dots, \nu_1, \nu_i; \varphi)$ .

Thus we have shown that the vertex expansion resulting from the perturbation series satisfies quantum dynamics in a well-controlled fashion: If we were to terminate the sum at

$M = M^*$ , we would have

$$(i\partial_\varphi + D_f + \lambda K) \left[ \sum_M^{M^*} \lambda^M A_M(\nu_f, \varphi; \nu_i, 0) \right] = \mathcal{O}(\lambda^{M^*+1}) \quad (4.17)$$

This brings out the precise sense in which a truncation to a finite order of the vertex expansion incorporates the quantum dynamics of the deparameterized theory approximately.

## V. DISCUSSION

Because LQC is well-developed in the Hamiltonian framework, it provides an interesting avenue to probe various aspects of the spin foam paradigm. For definiteness we focused on the Friedmann model with a massless scalar field as source. We used the group averaging procedure that is available for general constrained systems as well as the natural deparameterization, with  $\phi$  as the emergent time variable, that is often employed in LQC.

Group averaging provides a Green's function  $G(\nu_f, \phi_f; \nu_i, \phi_i)$  representing the inner product between physical states extracted from the kinematic kets  $|\nu_f, \phi_f\rangle$  and  $|\nu_i, \phi_i\rangle$ . The Schrödinger evolution of the deparameterized theory provides the transition amplitude  $A(\nu_f, \phi_f; \nu_i, \phi_i)$  for the physical state  $|\nu_i\rangle$  at the initial instant  $\phi_i$  to evolve to the state  $|\nu_f\rangle$  at the final instant of time  $\phi_f$ . We saw in section IV that the two quantities are equal. But they emphasize different physics. Following the general procedure invented by Feynman to pass from a Hamiltonian theory to a sum over histories, we were able to obtain a series expansion for each of these quantities —Eq (3.27) for  $G(\nu_f, \phi_f; \nu_i, \phi_i)$  and Eq (4.12) for  $A(\nu_f, \phi_f; \nu_i, \phi_i)$ — that mimic the vertex expansion of SFMs. In section III, we had to make one assumption in the derivation of the vertex expansion of  $G(\nu_f, \phi_f; \nu_i, \phi_i)$ : in the passage from (3.35) to (3.36) we assumed that the integration over  $\alpha$  of the group averaging procedure commutes with an infinite sum in (3.35). Since the integration over  $\alpha$  is by-passed in the deparameterized framework this assumption was not necessary in our derivation of the vertex expansion of  $A(\nu_f, \phi_f; \nu_i, \phi_i)$  in section IV.

Detailed parallels between our construction and SFMs are as follows. The analog of the manifold  $M$  with boundaries  $S_i, S_f$  in SFMs is the manifold  $\mathcal{V} \times \mathcal{I}$ , where  $\mathcal{V}$  is the elementary cell in LQC and  $\mathcal{I}$ , a closed interval in the real line (corresponding to  $\tau \in [0, 1]$  in the timeless framework and  $\phi \in [\phi_f, \phi_i]$  in the deparameterized). The analog of a triangulation in spinfoams is just a division of  $\mathcal{V} \times \mathcal{I}$  into  $M$  parts by introducing  $M - 1$  time slices. Just as the triangulation in SFMs is determined by the number of 4-simplices, what matters in LQC is the number  $M$ ; the precise location of slices is irrelevant. The analog of the dual-triangulation in SFMs is just a ‘vertical’ line in  $\mathcal{V} \times \mathcal{I}$  with  $M$  marked points or ‘vertices’ (not including the two end-points of  $\mathcal{I}$ ). Again, what matters is the number  $M$ ; the precise location of vertices is irrelevant. Coloring of the dual-triangulation in SFMs corresponds to an ordered assignment  $(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0)$  of volumes to edges bounded by these marked points (subject only to the constraints  $\nu_M = \nu_f$ ,  $\nu_0 = \nu_i$  and  $\nu_m \neq \nu_{m-1}$ ). Each vertex signals a change in the physical volume along the quantum history. The probability amplitude associated with the given coloring is given by  $A(\nu_f, \dots, \nu_0; \phi_f, \phi_i)$  in the group averaging procedure (see Eq (3.26)) and by  $A(\nu_f, \dots, \nu_0; \varphi)$  in the deparameterization procedure (see Eq (4.13)). A sum over colorings yields the partial amplitude associated with the triangulation with  $M$  ‘vertices’. The Green's function  $G(\nu_f, \phi_f; \nu_i, \phi_i)$  and the total transition amplitude

$A(\nu_f, \varphi; \nu_i, 0)$  are given by a sum over these  $M$ -vertex amplitudes.

Thus, the physical inner product of the timeless framework and the transition amplitude in the deparameterized framework can each be expressed as a *discrete sum* without the need of a ‘continuum limit’: A countable number of vertices suffices; the number of volume transitions does not have to become continuously infinite. This result supports the view that LQG and SFMs are not quite analogous to quantum field theories on classical space-times. Discrete quantum geometry at the Planck scale makes a key difference. In sections III B and IV B we were able to obtain the same vertex sum using a perturbative expansion, in a coupling constant  $\lambda$ , that is reminiscent of GFTs. In sections III C and IV C we showed that this is a useful expansion in the sense that the Green’s function and the transition amplitude satisfy the dynamical equations order by order in  $\lambda$ . Thus, if we were to truncate the expansion to order  $M$ , the truncated Green’s function and transition amplitude would satisfy the dynamical equations up to terms of the order  $O(\lambda^{M+1})$ . Finally in section III D we showed that the coupling constant  $\lambda$  inspired by GFTs is closely related to the cosmological constant. This interpretation opens a possibility that a detailed study of the renormalization group flow in GFT may be able to account for the very small, positive value of the cosmological constant.

Taken together, these results provide considerable concrete support for the general paradigms that underlie SFM and GFT.<sup>3</sup> However, we emphasize that this analysis has a key limitation: We did not begin with a SFM in full general relativity and then arrive at the LQC model through a systematic symmetry reduction of the full vertex expansion. Rather, we began with an already symmetry reduced model and recast the results in the spin foam language. Reciprocally, a key strength of these results is that we did not have to start by *postulating* that the physical inner product or the transition amplitude is given by a formal path integral. Rather, a rigorously developed Hamiltonian theory guaranteed that these quantities are well-defined. We simply recast their expressions as vertex expansions.

It is often the case that exactly soluble models not only provide support for or against general paradigms but they can also uncover new issues whose significance had not been realized before. The LQC analysis has brought to forefront three such issues.

First, it has revealed the advantage of adding matter fields. It is widely appreciated that on physical grounds it is important to extend SFMs beyond vacuum general relativity. However what was not realized before is that, rather than complicating the analysis, this generalization can in fact lead to interesting and significant technical simplifications. This point is brought out vividly by a recent analysis of Rovelli and Vidotto [53]. They considered a simple model on a finite dimensional Hilbert space where there is no analog of the scalar field or the possibility of deparameterization. There, individual terms in the vertex expansion turn out to be well defined only after a (natural) regularization. In our example, the presence of the scalar field simplified the analysis (in the transition from (3.24) to (3.26)) and individual terms in the vertex expansion are finite without the need of any regularization. Furthermore, this simplification is not an artefact of our restriction to the simplest cosmological model. For example, in the Bianchi I model the Hamiltonian theory is also well-developed in the *vacuum* case [52]. However, work in progress by Campiglia, Henderson, Nelson and Wilson-Ewing shows that technical problems illustrated by Rovelli

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<sup>3</sup> But it also brings out the fact that the term ‘third quantization’ that is sometimes used in GFTs is quite misleading especially in the cosmological context where it is often used to signify a Fock space of universes where the ‘single universe sector’ is described by the theory described here.



and Vidotto arise also in this case, making it necessary to introduce a regularization. These problems simply disappear if one also includes a scalar field.

Second, it came as a surprise that there are two *distinct* vertex expansions: Group averaging provides one that mainly uses the matrix elements of  $\Theta$  while the deparameterized framework provides one that uses only the matrix elements of  $\sqrt{\Theta}$ . This is not an artefact of using the simplest cosmological model. Work in progress indicates that the situation is similar in the anisotropic Bianchi models. Indeed, from a Hamiltonian perspective, it would appear that distinct vertex expansions can arise whenever a well-defined deparameterization is available. This raises an interesting and more general possibility. Can there exist distinct spin foam models —constructed by using, say, distinct vertex amplitudes— for which the complete vertex expansions yield the same answer? Finite truncations of these expansions could be inequivalent, but each could be tailored to provide an excellent approximation to the full answer for a specific physical question. One may then be able to choose which truncated expansion to use to probe a specific physical effect.

The third issue concerns three related questions in the spin foam literature: i) Should the physical inner products between states associated with spin networks be real rather than complex [31]? ii) In the classical limit, should one recover  $\cos S$  in place of the usual term  $e^{iS}$ , where  $S$  is the Einstein Hilbert action [32, 33]? iii) Should the choice of orientation play a role in the sum of histories [49]? In the LQC example we studied in this paper, these three questions are intimately related. The inner product between the physical states  $[\nu, \phi]_+$  determined by the kinematic basis vectors —which are the analogs of spin networks in this example— are in general complex (see Eq (3.27)). However, if we had dropped the positive frequency requirement, the group averaged inner products would have been real (see Eq (3.24)). The situation with action is analogous. And, as we show in the next paragraph, the positive frequency condition also selects a time-orientation.

Since this is an important issue, we will discuss it in some detail. Let us begin with the classical theory. The phase space is 4-dimensional and there is a single constraint:  $C(\nu, b; \phi, p_\phi) := G p_\phi^2 - 3\pi (\ell_{\text{Pl}}^2 \nu^2) b^2 = 0$ . Dynamics has two conceptually interesting features. First, given a solution  $(\nu(t), \phi(t))$  to the constraint and dynamical equations,  $(-\nu(t), \phi(t))$  is also a solution (where  $t$  denotes proper time). They define the same space-time metric and scalar field; only the parity of the spatial triad is reversed. Therefore  $(\nu(t), \phi(t)) \rightarrow (-\nu(t), \phi(t))$  is regarded as a gauge transformation. The second feature arises from the fact that the constraint surface has two ‘branches’,  $p_\phi > 0$  and  $p_\phi < 0$ , joined at points  $p_\phi = 0$  which represent Minkowski space-time. As is usual in quantum cosmology, let us ignore the trivial flat solution. Then each of the two portions  $\bar{\Gamma}^\pm$  of the constraint surface defined by the sign of  $p_\phi$  is left invariant by dynamics. Furthermore, there is a symmetry: Given a dynamical trajectory  $(\nu(t), \phi(t))$  in  $\bar{\Gamma}^+$ , there is a trajectory  $(\nu(t), -\phi(t))$  which lies in  $\bar{\Gamma}^-$ . This represents a *redundancy* in the description in the sense that we recover all physical space-time geometries  $g_{ab}(t)$  even if we restrict only to one of the two branches  $\bar{\Gamma}^\pm$ . In particular, the dynamical trajectories on  $\bar{\Gamma}^+$ , for example, include solutions which start with a big-bang and expand out to infinity *as well as* those which start out with infinite volume and end their lives in a big crunch. The difference is only in time orientation: If we regard  $\phi$  as an internal or relational time variable and *reconstruct* space-time geometries from phase space trajectories, space-times obtained from a trajectory on  $\bar{\Gamma}^+$  defines the same geometry as the one obtained from the corresponding trajectory on  $\bar{\Gamma}^-$  but with opposite time orientation. As in the Klein-Gordon theory of a free relativistic particle, this redundancy is removed by restricting oneself either to the  $p_\phi > 0$  sector or to the

$p_\phi < 0$  sector. In the quantum theory, then, the physical Hilbert space is given by solutions  $\Psi(\nu, \phi)$  to the quantum constraint (2.2) which in addition have only positive (or negative) frequency so that the operator  $p_\phi$  is positive (or negative) definite. (They are also invariant under parity,  $\Psi(\nu, \phi) = \Psi(-\nu, \phi)$ ). Thus, the LQC example suggests that in general SFMs one should fix the time-orientation, lending independent support to the new ideas proposed in [49]. Reality of the physical inner products between spin network states [31] and the emergence of  $\cos S$  in place of  $e^{iS}$  [32, 33] can be traced back to the fact that in most of the SFM literature one sums over both orientations. However, our analysis provides only a hint rather than an iron-clad argument because all our discussion is tied to LQC models where symmetry reduction occurs prior to quantization.

We conclude with an observation. We have recast LQC as a sum over histories. However, this is different from a Feynman path integral in which the integrand is expressed as  $e^{iS}$ , for a suitable action  $S$ . This step was not necessary for the goals of this paper. However, it is of considerable interest, especially in the cosmological context, for certain physical issues such as the emergence of the classical universe and semi-classical corrections to the classical theory. Such a path integral formulation of LQC does exist [54] and will be discussed elsewhere.<sup>4</sup>

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### Appendix A: Limit in Eq (3.18)

It is convenient to rewrite  $A_N(\nu_M, \dots, \nu_0; \alpha)$  defined in (3.13) in the following way:

$$A_N(\nu_M, \dots, \nu_0; \alpha) = U_{\nu_M \nu_{M-1}} \dots U_{\nu_1 \nu_0} [U_{\nu_M \nu_M}]^N [U_{\nu_M \nu_M} \dots U_{\nu_0 \nu_0}]^{-1} \times \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \dots \sum_{N_1=1}^{N_2-1} \left[ \frac{U_{\nu_{M-1} \nu_{M-1}}}{U_{\nu_M \nu_M}} \right]^{N_M} \dots \left[ \frac{U_{\nu_0 \nu_0}}{U_{\nu_1 \nu_1}} \right]^{N_1}. \quad (\text{A1})$$

Our aim is to calculate the limit  $N \rightarrow \infty$  of (A1) and show that is given by  $A(\nu_M, \dots, \nu_0; \alpha)$ , of Eq (3.18) which we rewrite as

$$A(\nu_M, \dots, \nu_0; \alpha) = (-i\alpha)^M \Theta_{\nu_M \nu_{M-1}} \dots \Theta_{\nu_1 \nu_0} e^{-i\alpha \Theta_{\nu_M \nu_M}} \times \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 e^{\tau_M b_M} \dots e^{\tau_1 b_1} \quad (\text{A2})$$

where

$$b_m := -i\alpha(\Theta_{\nu_{m-1} \nu_{m-1}} - \Theta_{\nu_m \nu_m}). \quad (\text{A3})$$

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<sup>4</sup> A path integral formulation of polymer quantum mechanics was carried out independently by Husain and Winkler [55].

We start by calculating the  $N \gg 1$  behavior of the terms appearing in (A1). These are:

$$U_{\nu_{m+1}\nu_m} = -\frac{i\alpha}{N}\Theta_{\nu_{m+1}\nu_m} + O(N^{-2}), \quad (\text{A4})$$

$$\begin{aligned} [U_{\nu_M\nu_M}]^N &= e^{N \log U_{\nu_M\nu_M}} \\ &= e^{N(-i\frac{\alpha}{N}\Theta_{\nu_M\nu_M} + O(N^{-2}))} \\ &= e^{-i\alpha\Theta_{\nu_M\nu_M}} + O(N^{-1}), \end{aligned} \quad (\text{A5})$$

$$[U_{\nu_M\nu_M} \cdots U_{\nu_0\nu_0}]^{-1} = 1 + O(N^{-1}), \quad (\text{A6})$$

$$\begin{aligned} \left[ \frac{U_{\nu_{m-1}\nu_{m-1}}}{U_{\nu_m\nu_m}} \right]^{N_m} &= e^{N_m(\log U_{\nu_{m-1}\nu_{m-1}} - \log U_{\nu_m\nu_m})} \\ &= e^{N_m(b_m/N + O(N^{-2}))} \\ &= e^{\frac{N_m}{N}b_m} + O(N_m N^{-2}), \end{aligned} \quad (\text{A7})$$

with  $b_m$  given in (A3). In (A5) and (A7) we have used the fact that the multivalued nature of the log function does not affect the final result:  $e^{N(\log x + 2\pi ik)} = e^{N \log x}$  where  $k \in \mathbb{Z}$  reflects the multiple values that log can take.

We now substitute expressions (A4) to (A7) in (A1) to obtain

$$\begin{aligned} A_N(\nu_M, \dots, \nu_0; \alpha) &= [(-i\alpha)^M \Theta_{\nu_M\nu_{M-1}} \cdots \Theta_{\nu_1\nu_0} e^{-i\alpha\Theta_{\nu_M\nu_M}} N^{-M} + O(N^{-M-1})] \times \\ &\quad \prod_{m=1}^M \left[ \sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N}b_m} + O(N_m N^{-2}) \right] \end{aligned} \quad (\text{A8})$$

where the product denotes the  $M$  nested sums in (A1). Each sum in (A8) has two terms. The first one gives a contribution of  $\sum_{N_m} e^{\frac{N_m}{N}b_m} \sim O(N)$  while the second one is  $\sum_{N_m} O(N_m N^{-2}) \sim O(1)$ . The  $M$  sums then give a contribution of order  $[O(N) + O(1)]^M \sim O(N^M) + O(N^{M-1})$ . By combining this with the first factor of (A8), we find that the non-vanishing contribution comes from the first terms of the sums:

$$\begin{aligned} A_N(\nu_M, \dots, \nu_0; \alpha) &= (-i\alpha)^M \Theta_{\nu_M\nu_{M-1}} \cdots \Theta_{\nu_1\nu_0} e^{-i\alpha\Theta_{\nu_M\nu_M}} \times \\ &\quad N^{-M} \prod_{m=1}^M \left[ \sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N}b_m} \right] + O(N^{-1}). \end{aligned} \quad (\text{A9})$$

Eq (A9) has all the pre-factors appearing in (A2). It then remains to show that  $N^{-M}$  times the sums in (A9) limits to the integrals in (A2). But this is rather obvious, as the

sums can be seen as Riemann sums for the integrals. Specifically,

$$\begin{aligned}
& \lim_{N \rightarrow \infty} N^{-M} \prod_{m=1}^M \left[ \sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N} b_m} \right] \\
&= \lim_{N \rightarrow \infty} N^{-M} \sum_{N_M=0}^N \sum_{N_{M-1}=0}^{N_M} \dots \sum_{N_1=0}^{N_2} e^{\frac{N_M}{N} b_M} \dots e^{\frac{N_1}{N} b_1} \\
&= \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 e^{\tau_M b_M} \dots e^{\tau_1 b_1} \tag{A10}
\end{aligned}$$

where, in the second line, we have slightly changed the limits on the sums, introducing an  $O(N^{-1})$ -term which vanishes in the limit. This concludes the proof of the limit (3.18).

### Appendix B: General Integrals in Eq (3.18)

The integrals over  $\tau$  appearing in the amplitude for a single discrete path (3.18) can be evaluated for a general sequence of volumes  $(\nu_M, \dots, \nu_0)$  with the result given by (3.21). In this appendix we will perform these integrals first for the case where all  $\nu_i$  are distinct and then for the general case. The amplitude for a single discrete path given by (3.18) and (3.19) is

$$\begin{aligned}
A(\nu_M, \dots, \nu_0, \alpha) &= \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 e^{-i(\Delta\tau - \tau_M)\alpha\Theta_{\nu_M\nu_M}} (-i\alpha\Theta_{\nu_M\nu_{M-1}}) \times \\
& \quad e^{-i(\tau_M - \tau_{M-1})\alpha\Theta_{\nu_{M-1}\nu_{M-1}}} \dots e^{-i(\tau_2 - \tau_1)\alpha\Theta_{\nu_1\nu_1}} (-i\alpha\Theta_{\nu_1\nu_0}) e^{i\tau_1\alpha\Theta_{\nu_0\nu_0}} \tag{B1}
\end{aligned}$$

This expression can be written in terms of the following integral.

$$\begin{aligned}
I(x_M, \dots, x_0, \Delta\tau) &= \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 (i)^M e^{i(\Delta\tau - \tau_M)x_M} e^{i(\tau_M - \tau_{M-1})x_{M-1}} \\
& \quad \dots e^{i(\tau_2 - \tau_1)x_1} e^{i\tau_1 x_0} \tag{B2}
\end{aligned}$$

We will first evaluate this integral for the case where all  $x_i$  are distinct. By induction on  $M$ —the number of vertices or the number of times that  $x$  changes value—we will show that when the  $x_i$  are all distinct the integral is given by

$$I(x_M, \dots, x_0, \Delta\tau) = \sum_{i=0}^M \frac{e^{ix_i\Delta\tau}}{\prod_{j \neq i}^M (x_i - x_j)} \tag{B3}$$

This is true by inspection for  $M = 0$ . If we assume that (B3) holds for  $M$  we can evaluate the integral with  $M + 1$  vertices.

$$\begin{aligned}
I(x_{M+1}, x_M, \dots, x_0, \Delta\tau) &= \int_0^{\Delta\tau} d\tau_{M+1} i e^{i(\Delta\tau - \tau_{M+1})x_{M+1}} I(x_M, \dots, x_0, \tau_{M+1}) \tag{B4} \\
&= \int_0^{\Delta\tau} d\tau_{M+1} i e^{i(\Delta\tau - \tau_{M+1})x_{M+1}} \sum_{i=0}^M \frac{e^{ix_i\tau_{M+1}}}{\prod_{j \neq i}^M (x_i - x_j)} \\
&= \sum_{i=0}^M \frac{e^{ix_i\Delta\tau}}{\prod_{j \neq i}^{M+1} (x_i - x_j)} - e^{i\Delta\tau x_{M+1}} \sum_{i=0}^M \frac{1}{\prod_{j \neq i}^{M+1} (x_i - x_j)}
\end{aligned}$$

In the first step we recognized that the  $M + 1$ -th integral contains the  $M$ -th and then, in

the second step, we inserted the assumed result for the  $M - th$  integral. In the second step the integral over  $\tau_{M+1}$  is carried out. Finally using the identity

$$\sum_{i=1}^{M+1} \frac{1}{\prod_{j \neq i}^{M+1} (x_i - x_j)} = 0 \quad (\text{B5})$$

The integral can be written as

$$I(x_{M+1}, x_M, \dots, x_0, \Delta\tau) = \sum_{i=0}^{M+1} \frac{e^{ix_i \Delta\tau}}{\prod_{j \neq i}^{M+1} (x_i - x_j)} \quad (\text{B6})$$

Therefore if (B3) holds for  $M$  it also holds for  $M + 1$ , thus by induction it holds for all  $M \geq 0$ .

If the  $x_i$  are not distinct, if there exist  $i, j$  such that  $x_i = x_j$ , then the proof follows in a similar way. The key element is that the integral  $I(x_M, \dots, x_0)$  is independent of the order of the  $x_i$ 's. This can be seen by rewriting the integral in terms of the time intervals  $\Delta\tau_i = \tau_{i+1} - \tau_i$  where  $\tau_0 = 0$  and  $\tau_{m+1} = \Delta\tau$ .

$$I(x_0, x_1, \dots, x_M, \Delta\tau) = \int_0^{\Delta\tau} d\Delta\tau_M d\Delta\tau_{M-1} \dots d\Delta\tau_0 \delta(\Delta\tau_M + \dots + \Delta\tau_0 - \Delta\tau) \quad (\text{B7})$$

$$(i)^M e^{i\Delta\tau_M x_M} e^{i\Delta\tau_{M-1} x_{M-1}} \dots e^{i\Delta\tau_1 x_1} e^{i\Delta\tau_0 x_0}$$

It is clear that this is symmetric under the interchange of  $x_i$  with  $x_j$  for all  $i, j$ , so the integral is independent of the order of the sequence  $x_i$ . Since the integral is independent of the order of the values  $x_i$  it should be characterized by the distinct values, labeled by  $y_i$  and their multiplicity  $n_i$ . Where  $n_1 + \dots + n_p = M + 1$ . Given a set of values  $x_i$  we will evaluate the integral for the case where they are organized such that any  $x_i$  sharing the same value are grouped together. Doing so the integral simplifies to

$$I(y_p, n_p, \dots, y_1, n_1, \Delta\tau) = \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \dots \int_0^{\tau_2} d\tau_1 (i)^M e^{i(\Delta\tau - \tau_{n_1} + \dots + \tau_{n_p-1}) y_p} \quad (\text{B8})$$

$$e^{i(\tau_{n_1} + \dots + \tau_{n_p-1} - \tau_{n_1} + \dots + \tau_{n_p-2}) y_{p-1}} \dots e^{i(\tau_{n_1} + \tau_2 - \tau_{n_1}) y_2} e^{i\tau_{n_1} y_1}$$

By induction on  $p$ , the number of distinct values, we show that this integral is given by

$$I(y_p, n_p, \dots, y_1, n_1, \Delta\tau) = \frac{1}{(n_p - 1)!} \left( \frac{\partial}{\partial y_p} \right)^{n_p - 1} \dots \frac{1}{(n_1 - 1)!} \left( \frac{\partial}{\partial y_1} \right)^{n_1 - 1} \sum_{i=1}^p \frac{e^{iy_i \Delta\tau}}{\prod_{j \neq i}^p (y_i - y_j)} \quad (\text{B9})$$

$$= \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial y_k} \right)^{n_k - 1} \sum_{i=1}^p \frac{e^{iy_i \Delta\tau}}{\prod_{j \neq i}^p (y_i - y_j)}$$

For  $p = 1$  (B8) can be easily evaluated giving

$$I(y_1, n_1) = \int_0^{\Delta\tau} d\tau_{n_1-1} \dots \int_0^{\tau_2} d\tau_1 (i)^{n_1-1} e^{iy_1 \Delta\tau} = \frac{(i\Delta\tau)^{n_1-1}}{(n_1-1)!} e^{iy_1 \Delta\tau} \quad (\text{B10})$$

$$= \left( \frac{\partial}{\partial y_1} \right)^{n_1-1} \frac{1}{(n_1-1)!} e^{iy_1 \Delta\tau}$$

If we assume that (B9) holds for  $p$  distinct values then we can evaluate it for  $p + 1$  distinct values as follows.

$$I(y_{p+1}, n_{p+1}, y_p, n_p, \dots, y_1, n_1, \Delta\tau) = \int_0^{\Delta\tau} d\tau_M \dots \int_0^{\tau_{M-n_{p+1}+2}} d\tau_{M-n_{p+1}+1} \quad (\text{B11})$$

$$(i)^{n_{p+1}-1} e^{i(\Delta\tau - \tau_{M-n_{p+1}+1})y_{p+1}} I(y_p, n_p, \dots, y_1, n_1, \tau_{M-n_{p+1}+1})$$

Plugging in the assumed result for  $p$  distinct values and performing the integrals over  $\tau$  we obtain

$$I(y_{p+1}, n_{p+1}, \dots, y_1, n_1, \Delta\tau) = \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial y_k} \right)^{n_k-1} \sum_{i=1}^p \frac{1}{\prod_{j \neq i}^p (y_i - y_j)} \quad (\text{B12})$$

$$\left[ \frac{e^{iy_i \Delta\tau}}{(y_i - y_{p+1})^{n_{p+1}}} - \sum_{m=0}^{n_{p+1}} \frac{e^{iy_{p+1} \Delta\tau}}{(y_i - y_{p+1})^m} \frac{(i \Delta\tau)^{n_{p+1}-m}}{(n_{p+1} - m)!} \right]$$

We recognize that the term in brackets can be written as derivatives with respect to  $y_{p+1}$  of a simple function.

$$I(y_{p+1}, n_{p+1}, y_p, n_p, \dots, y_1, n_1, \Delta\tau) = \prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial y_k} \right)^{n_k-1} \sum_{i=1}^p \frac{1}{\prod_{j \neq i}^p (y_i - y_j)} \quad (\text{B13})$$

$$\left[ \frac{1}{(n_{p+1} - 1)!} \left( \frac{\partial}{\partial y_{p+1}} \right)^{n_{p+1}-1} \left( \frac{e^{iy_i \Delta\tau}}{y_i - y_{p+1}} - \frac{e^{iy_{p+1} \Delta\tau}}{y_i - y_{p+1}} \right) \right]$$

Finally simplifying the expression and using eqn (B5) we obtain

$$I(y_{p+1}, n_{p+1}, \dots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p+1} \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial y_k} \right)^{n_k-1} \sum_{i=1}^{p+1} \frac{e^{iy_i \Delta\tau}}{\prod_{j \neq i}^p (y_i - y_j)} \quad (\text{B14})$$

Thus if (B9) holds for  $p$  then it also holds for  $p + 1$ , so it is true for all  $p \geq 0$ . Using this result we find that the contribution due to each discrete path is

$$A(\nu_M, \dots, \nu_0, \alpha) = (\Theta_{\nu_M \nu_{M-1}})(\Theta_{\nu_{M-1} \nu_{M-2}}) \dots (\Theta_{\nu_2 \nu_1})(\Theta_{\nu_1 \nu_0}) \quad (\text{B15})$$

$$\prod_{k=1}^p \frac{1}{(n_k - 1)!} \left( \frac{\partial}{\partial \Theta_{w_k w_k}} \right)^{n_k-1} \sum_{i=1}^p \frac{e^{-i\alpha \Theta_{w_i w_i} \Delta\tau}}{\prod_{j \neq i}^p (\Theta_{w_i w_i} - \Theta_{w_j w_j})}$$

where  $w_i$  label the distinct values taken by  $\nu$  along the path and  $n_i$  the multiplicity of each value.

### Appendix C: Eigenstates and Operator functions of $\Theta$

In the timeless framework of section III, the vertex expansion mostly featured matrix elements  $\Theta_{\nu_m \nu_n} = \langle \nu_m | \Theta | \nu_n \rangle$ . These are easy to evaluate directly from the definition (2.3) of  $\Theta$ . In the deparameterized framework of section IV, on the other hand, the vertex expansion involves matrix elements of  $\sqrt{\Theta}$ . To evaluate these one needs the spectral decomposition of  $\Theta$ . In the first part of this Appendix we construct eigenstates of  $\Theta$  and discuss their relevant

properties. In the second part we use these eigenstates to evaluate the matrix elements functions of  $\Theta$ , including  $\sqrt{\Theta}$ .

### 1. Eigenstates of $\Theta$

Recall that  $\Theta$  is a positive, self-adjoint operator on  $\mathcal{H}_{\text{kin}}^{\text{grav}}$ . By its definition (2.3), it follows that  $\Theta$  preserves each of the three sub-spaces in the decomposition  $\mathcal{H}_{\text{kin}}^{\text{grav}} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$ , spanned by functions with support on  $\nu < 0$ ,  $\nu = 0$  and  $\nu > 0$  respectively. In particular,  $|\nu = 0\rangle$  is the unique eigenvector of  $\Theta$ , with eigenvalue 0;  $\mathcal{H}_0$  is 1-dimensional. Our first task is to solve the eigenvalue equation for a general eigenvalue  $\omega_k^2$ :

$$\Theta e_k(\nu) = \omega_k^2 e_k(\nu). \quad (\text{C1})$$

This task becomes simpler in the representation in which states are functions  $\chi(b)$  of the variable  $b$  conjugate to  $\nu$ :<sup>5</sup>

$$\chi(b) := \sqrt{\frac{\ell_o}{\pi}} \sum_{\nu=4n\ell_o} e^{\frac{i}{2}\nu b} \frac{\Psi(\nu)}{\sqrt{|\nu|}}. \quad (\text{C2})$$

In this representation, the eigenvalue equation (C1) takes the form of a simple differential equation

$$(\Theta\chi_k)(b) = -12\pi G \left( \frac{\sin \ell_o b}{\ell_o} \partial_b \right)^2 \chi_k(b) = \omega_k^2 \chi_k(b), \quad (\text{C3})$$

whose solutions are

$$\chi_k(b) = A(k) e^{ik \log(\tan \frac{\ell_o b}{2})} \quad \text{with} \quad \omega_k^2 = 12\pi G k^2, \quad (\text{C4})$$

where  $A(k)$  is a normalization factor and  $k \in (-\infty, \infty)$ .  $k = 0$  yields a discrete eigenvalue  $\omega_k = 0$  and in the  $\nu$  representation the eigenvector can be expressed simply as  $e_0(\nu) = \delta_{0,\nu}$ . Eigenvectors with non-zero eigenvalues can also be expressed in the  $\nu$  representation by applying the inverse transformation of (C2) to (C4):

$$e_k(\nu) = A(k) \sqrt{\frac{\ell_o |\nu|}{\pi}} \int_0^{\pi/\ell_o} db e^{-\frac{i}{2}\nu b} e^{ik \log(\tan \frac{\ell_o b}{2})} \quad \text{where} \quad k \neq 0. \quad (\text{C5})$$

Let us note two properties of these eigenvectors. First,  $e_k$  and  $e_{-k}$  have the same eigenvalue and so the  $\omega_k^2$ -eigenspace is two-dimensional. Second, the vectors  $e_k(\nu)$  we have obtained have support on both  $\nu > 0$  and  $\nu < 0$ . However, since  $\Theta$  preserves the sub-spaces  $\mathcal{H}_{\pm}$ , it is natural to seek linear combinations  $e_k^{\pm}(\nu)$  of  $e_k(\nu)$  and  $e_{-k}(\nu)$  which lie in these sub-spaces. In particular, this will simplify the problem of normalization of eigenfunctions.

Let us begin by rewriting the integral in (C5) as a contour integral in the complex plane.

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<sup>5</sup> Our normalization is different from that in [39]. The wave function  $\tilde{\Psi}(\nu)$  in [39] is related to the one here by  $\Psi(\nu) = \sqrt{\frac{\ell_o}{\pi|\nu|}} \tilde{\Psi}(\nu)$ .

Recalling that  $\nu = 4\ell_o n$  and setting  $z = e^{ib\ell_o}$  we obtain

$$\frac{\ell_o}{\pi} \int_0^{\pi/\ell_o} db e^{-2ibn} e^{ik \log(\tan \frac{\ell_o b}{2})} = \frac{e^{-\pi k/2}}{\pi i} \int_{\mathcal{C}} z^{-2n-1} \left(\frac{1-z}{1+z}\right)^{ik} dz =: J(k, n), \quad (\text{C6})$$

where  $\mathcal{C}$  is the unit semicircle in counterclockwise direction in the upper half,  $\Im z > 0$ , of the complex plane. As remarked earlier,  $e_k(\nu) = A(k) \sqrt{\ell_o |\nu|/\pi} J(k, \nu/4\ell_o)$  has support on both positive and negative values of  $\nu = 4\ell_o n$ . Now, the second independent eigenfunction  $e_{-k}(\nu)$  with the same eigenvalue  $\omega_k^2$  can be represented in a similar fashion by setting  $z = -e^{ib\ell_o}$ . The result is a contour-integral along the unit semicircle in counterclockwise direction in the lower half,  $\Im z < 0$  of the complex plane. By combining the two integrals, we obtain a closed integral along the unit circle:

$$\frac{1}{2\pi i} \oint z^{-2n-1} \left(\frac{1-z}{1+z}\right)^{ik} dz = \frac{1}{2} (e^{\pi k/2} J(k, n) + e^{-\pi k/2} J(-k, n)) =: I(k, n). \quad (\text{C7})$$

Being a linear combination of  $e_k(\nu)$  and  $e_{-k}(\nu)$ , this  $I(k, n)$  gives also an eigenfunction of  $\Theta$  with eigenvalue  $\omega_k^2$ . Moreover, using elementary complex analysis, one finds that *it has support only on positive  $n$* :

$$I(k, n) = \begin{cases} \frac{1}{(2n)!} \left. \frac{d^{2n}}{ds^{2n}} \right|_{s=0} \left(\frac{1-s}{1+s}\right)^{ik} & n \geq 0 \\ 0 & n < 0. \end{cases} \quad (\text{C8})$$

Repeating the argument but taking  $z = e^{-ib\ell_o}$  and  $z = -e^{-ib\ell_o}$  one obtains

$$\frac{1}{2} (e^{-\pi k/2} J(k, n) + e^{\pi k/2} J(-k, n)) = \frac{1}{2\pi i} \oint z^{2n-1} \left(\frac{1-z}{1+z}\right)^{ik} dz = I(k, -n) \quad (\text{C9})$$

which has support only on negative  $n$ . Thus, the basis we are looking for is given by

$$e_k^\pm(\nu) := \frac{1}{2} (e^{\pm\pi k/2} e_k(\nu) + e^{\mp\pi k/2} e_{-k}(\nu)) = A(k) \sqrt{\frac{\pi |\nu|}{\ell_o}} I(k, \pm \frac{\nu}{4\ell_o}). \quad (\text{C10})$$

By construction,  $e_k^\pm \in \mathcal{H}_\pm$ .

Next, let us calculate the normalization of these vectors. It is convenient to introduce kets  $|k^\pm\rangle$  such that  $\langle \nu | k^\pm \rangle = e_k^\pm(\nu)$ . Then, it is clear that  $\langle k' \pm | k \mp \rangle = 0$ . To calculate the nontrivial inner product,  $\langle k' \pm | k \pm \rangle$ , let us return to the  $b$  representation. There, the functions describing the states  $|k^\pm\rangle$  are

$$\chi_k^\pm(b) = \frac{A(k)}{2} \left( e^{\pm\pi k/2} e^{ik \log(\tan \frac{\ell_o b}{2})} + e^{\mp\pi k/2} e^{-ik \log(\tan \frac{\ell_o b}{2})} \right) \quad (\text{C11})$$

and their inner product is given by [39]

$$\langle k' \pm | k \pm \rangle = \int_0^{\pi/\ell_o} db |A(k)|^2 \overline{\chi_{k'}^\pm(b)} |2i\partial_b| \chi_k^\pm(b) \quad (\text{C12})$$

where  $|2i\partial_b|$  is the absolute value of the volume operator  $\hat{\nu} = 2i\partial_b$ . Simplification occurs because  $e_k^\pm(\nu)$  have support only on positive/negative  $\nu$  values. Because of this property,



one can replace  $|\partial_b|$  in (C12) by  $\pm\partial_b$ . The calculation now reduces to a straightforward integration. The result is

$$\langle k' \pm |k\pm\rangle = |A(k)|^2 2\pi k \sinh(\pi k) \delta(k', k). \quad (\text{C13})$$

Using this with (C10) we can find the normalization of the eigenvectors  $e_k(\nu)$ , which turn out to be

$$\langle k'|k\rangle = |A(k)|^2 4\pi k \coth(\pi k) \delta(k', k). \quad (\text{C14})$$

Thus, by choosing  $A(k) = \sqrt{\tanh(\pi k)/(4\pi k)}$ , we get the standard continuum normalization for the vectors  $e_k(\nu)$ .

## 2. Matrix Elements for $f(\Theta)$

We will now use the eigenbasis  $|\pm k\rangle$  of  $\Theta$  to calculate the matrix elements  $\langle 4n\ell_o|f(\Theta)|4m\ell_o\rangle$ , of the operators of the form  $f(\Theta)$ , for a measurable function  $f$ . Throughout this section, the normalization factor  $A(k)$  is chosen to be unity. From the normalization condition (C13) with  $A(k) = 1$ , we have the following decomposition of the identity:

$$\mathbf{I} = \int_0^\infty \frac{dk}{2\pi k \sinh(\pi k)} (|k+\rangle\langle k+| + |k-\rangle\langle k-|). \quad (\text{C15})$$

which can be inserted in  $\langle 4n\ell_o|f(\widehat{\Theta})|4m\ell_o\rangle$ . If  $m$  and  $n$  have different signs, the result is zero. It suffices to consider the case where both are positive. By writing  $\langle 4n\ell_o|k+\rangle$  in terms of derivatives (see equations (C10) and (C8)), one obtains

$$\langle 4n\ell_o n|f(\widehat{\Theta})|4n\ell_o m\rangle = \frac{2\sqrt{mn}}{(2n)!(2m)!} \left. \frac{d^{2m}}{ds^{2m}} \frac{d^{2n}}{dt^{2n}} \right|_{s=t=0} F_{f(\Theta)} \left( \frac{1+s}{1-s} \frac{1-t}{1+t} \right) \quad (\text{C16})$$

with  $F_{f(\Theta)}$  the ‘generating function’ given by<sup>6</sup>

$$F_{f(\Theta)}(x) = \int_0^\infty dk \frac{f(12\pi Gk^2)x^{ik}}{k \sinh(\pi k)}. \quad (\text{C17})$$

We now give the generating function for  $\sqrt{\Theta}$ . It is also useful (at least to check normalization factors) to write down the generating functions for operators whose matrix elements are

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<sup>6</sup> For a general  $f$ , integral as defined may diverge. However the divergent terms (e.g., those which are  $x$ -independent) do not contribute to the expression of the matrix element and can therefore be discarded. This ‘finite part extraction’ is implicit in going from (C17) to (C18), (C19) and (C20).

known, namely  $\Theta$  and the identity  $I$ . These generating functions are given by,

$$F_I(x) = -2 \left( \log(1+x) + \log \Gamma(1/2 + i \frac{\log x}{2\pi}) \right) \quad (\text{C18})$$

$$F_{\sqrt{\Theta}}(x) = \sqrt{12\pi G} \left( \frac{2ix}{1+x} - \frac{1}{\pi} \psi(1/2 + i \frac{\log x}{2\pi}) \right) \quad (\text{C19})$$

$$F_{\Theta}(x) = 12\pi G \left( \frac{2x}{(1+x)^2} - \frac{1}{2\pi^2} \psi'(1/2 + i \frac{\log x}{2\pi}) \right) \quad (\text{C20})$$

where  $\Gamma(z)$  is the gamma function, and  $\psi(z) = \Gamma'(z)/\Gamma(z)$  the polygamma function.

In obtaining these functions, it is useful to observe the following relations among them:

$$F_{\sqrt{\Theta}}(x) = -i\sqrt{12\pi G} x \frac{d}{dx} F_I(x) \quad (\text{C21})$$

$$F_{\Theta}(x) = -i\sqrt{12\pi G} x \frac{d}{dx} F_{\sqrt{\Theta}}(x), \quad (\text{C22})$$

which can be derived from (C17).

We will conclude by noting that the matrix elements for the evolution operator  $U(\varphi) = e^{i\varphi\sqrt{\Theta}}$  are easy to find: From (C17) one sees that  $F_{U(\varphi)}(x) = F_I(e^{\sqrt{12\pi G}\varphi}x)$ .

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