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Gravitational Wilson Loop in Discrete Quantum Gravity

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ABSTRACT

Results for the gravitational Wilson loop, in particular the area law for large loops in the strong coupling region, and the argument for an effective positive cosmological constant, discussed in a previous paper, are extended to other proposed theories of discrete Euclidean quantum gravity in the strong coupling limit. We argue that the area law is a generic feature of almost all nonperturbative Euclidean lattice formulations, for sufficiently strong gravitational coupling. The effects on gravitational Wilson loops of the inclusion of various types of light matter coupled to lattice quantum gravity are discussed as well. One finds that significant modifications to the area law can only arise from extremely light matter particles. The paper ends with some general comments on possible, physically observable, consequences.

1 Introduction

The identification of possible observables is an important part of formulating a theory of quantum gravity. In general it is expected that these quantum observables will be represented by expectation values of operators which have physical interpretations in the context of a manifestly covariant formulation. In this paper, we focus on the gravitational analog of the Wilson loop [1,2], which provides physical information about the parallel transport of vectors, and therefore on the effective curvature, around large, near-planar loops. We will extend the analysis of earlier work [3, 4] to more general theories of discrete quantum gravity. A recent complementary discussion of the significance of physical observables in a quantum theory of gravity can be found, for example, in [5].

In classical gravity the parallel transport of a coordinate vector around a closed loop is described by a rotation, which is a given function of the affine connection along the space-time path. Then the total rotation matrix $\mathbf{U}(C)$ is given by the path-ordered ($\mathcal{P}$) exponential of the integral of the affine connection $\Gamma^{\lambda}_{\mu\nu}$ via

$$U^{\alpha}{}_{\beta}(C) \;=\; \Big[\,\mathcal{P}\,\exp\Big\{\oint_{\textbf{path C}} \Gamma^{\cdot}_{\lambda\,\cdot}\,dx^{\lambda}\Big\}\Big]^{\alpha}{}_{\beta}\quad . \tag{1}$$

The gravitational Wilson loop then represents naturally a quantum average of a suitable trace (or contraction) of the above nonlocal operator, as described in detail in [3]. Its large distance (i.e. for loops whose size is very large compared to the lattice cutoff) behavior can be estimated, provided one makes some suitable assumptions about the short distance fluctuations of the underlying geometry, with the key assumption being the use of a Haar integration measure for the local rotations at strong coupling. [1] A general result then emerges, at least for the Euclidean theory, which is that the Wilson loop generically exhibits an area law for sufficiently strong gravitational coupling (large G) and near-planar loops [3,4]. It should be noted here that in contrast to gauge theories, the Wilson loop in quantum gravity [6] does not provide useful information on the static potential, which is obtained instead from the correlation between particle world-lines [7,8]. Instrumental in deriving the results of [3] was the first-order Regge lattice [9] formulation of gravity, discussed

[1] In the following we will be dealing almost exclusively, unless stated otherwise, with the Euclidean theory. Thus, for example, we will be considering $O(4)$ rotations and not $O(3,1)$ rotations, for which convergence issues can arise when employing the Haar measure for the lattice theory at strong coupling. We note that in the context of the field-theoretic $2+\epsilon$ expansion for gravity in the continuum, as well as in the renormalizable higher derivative formulation in four dimensions, no differences appear in the relevant beta functions for gravity between the Lorentzian and Euclidean case, to *all orders* in the relevant expansion parameters. In the continuum a physical difference between the two cases would then have to originate from nonanalytic terms in the beta functions, possibly due to nontrivial saddle points in the Euclidean theory. Also, the nonperturbative treatment of the lattice Lorentzian case by numerical methods generally involves complex weights $\exp(iS)$, which are known to be very difficult to deal with reliably by statistical means.

originally in [10].

Furthermore, from a semiclassical point of view, a vector's rotation around a large macroscopic loop is expected to be directly related, by Stoke's theorem, to some sort of average curvature enclosed by the loop. In this semiclassical picture one would write for the rotation matrix $\mathbf{U}$

$$U^{\alpha}{}_{\beta}(C) \;\sim\; \Big[\exp \Big\{ \tfrac{1}{2} \int_{S(C)} R^{\,\cdot}{}_{\cdot\,\mu\nu} \, A_C^{\mu\nu} \Big\} \Big]^{\alpha}{}_{\beta} \;\; , \tag{2}$$

where $A_C^{\mu\nu}$ is the usual area bivector associated with the loop in question,

$$A_C^{\mu\nu} = \tfrac{1}{2} \oint dx^{\mu} \, x^{\nu} \;\; . \tag{3}$$

The use of semiclassical arguments in relating the above rotation matrix $\mathbf{U}(C)$ to the surface integral of the Riemann tensor assumes (as is usual in the classical context) that the curvature is slowly varying on the scale of the very large loop. Then, in such a semiclassical description of the parallel transport process, one can reexpress the connection in terms of a suitable coarse-grained, or semiclassical, Riemann tensor, and thus relate the quantum Wilson loop expectation value discussed previously to an observable large scale curvature. The latter is represented phenomenologically by the long distance, observed cosmological constant λ_{obs}.

It is important in this context to note, as an underlying theme, the close analogy between the Wilson loop in gravity and the one in gauge theories, both theories involving a connection as a fundamental entity. Furthermore, a lot is known about the behavior of the Wilson loop in non-Abelian gauge theories at strong coupling, some of it from analytical estimates and some from large-scale numerical simulations. Let us recall that in non-Abelian gauge theories, the Wilson loop expectation value for a closed planar loop C is defined by [1]

$$W(C) \; = < \mathrm{Tr} \, \mathcal{P} \exp \Big\{ ig \oint_C A_{\mu}(x) \, dx^{\mu} \Big\} > \;\; , \tag{4}$$

with $A_{\mu} \equiv t_a A_{\mu}^a$ and the t_a's the group generators of $SU(N)$ in the fundamental representation. In the pure gauge theory at strong coupling [1,2], it is easy to show that the leading contribution to the Wilson loop follows an area law for sufficiently large loops

$$< W(C) > \; \underset{A \to \infty}{\sim} \; \exp(-A_C / \xi^2) \tag{5}$$

where A_C is the minimal area spanned by the planar loop C and ξ the gauge field correlation length. Furthermore, it can be shown that the area law is fairly universal at strong coupling, in the sense that it is not too sensitive to specific short distance details of the $SU(N)$-invariant lattice action. Indeed one expects the result of Eq. (5) to have *universal* validity in the lattice continuum limit, the latter being taken in the vicinity of the ultraviolet fixed point at gauge coupling $g = 0$.

The fundamental renormalization group invariant quantity ξ appearing in the textbook result of Eq. (5) [2] represents the gauge field correlation length, defined, for example, from the exponential decay of connected Euclidean correlations of two infinitesimal loops separated by a distance $|x|$,

$$G_{\Box}(x) \;=\, < \mathrm{Tr}\,\mathcal{P}\exp\Big\{ig \oint_{C'_\epsilon} A_\mu(x')\,dx'^\mu\Big\}(x)\;\mathrm{Tr}\,\mathcal{P}\exp\Big\{ig \oint_{C''_\epsilon} A_\mu(x'')\,dx''^\mu\Big\}(0) >_c \quad . \tag{6}$$

Here the C_ϵ's are two infinitesimal loops centered around x and 0 respectively, suitably defined on the lattice as elementary square loops, and for which one has at sufficiently large separations

$$G_{\Box}(x) \;\underset{|x|\to\infty}{\sim}\; \exp(-|x|/\xi) \;\; . \tag{7}$$

Thus the inverse of the correlation length ξ is seen to correspond, via the Lehmann representation, to the lowest gauge invariant mass excitation in the gauge theory, the scalar glueball. [3]

Through the renormalization group ξ is related to the β- function of Yang-Mills theories, with ξ the renormalization group invariant obtained from integrating the Callan-Symanzik β-function,

$$\xi^{-1}(g) \;=\; \mathrm{const.}\;\Lambda\;\exp\left(-\int^g \frac{dg'}{\beta(g')}\right) \;\; , \tag{8}$$

with Λ the ultraviolet cutoff, so that ξ is then identified with the invariant gauge correlation length appearing in Eqs. (5) and (7).

In an earlier paper [3], we adapted the gauge definition of the Wilson loop to the gravitational case, specifically to the case of lattice gravity, and in the context of the discretization scheme due to Regge [9]. On the lattice, with each neighboring pair of simplices $s, s+1$ one can associate a Lorentz transformation $U^\mu{}_\nu(s,s+1)$, which describes how a given vector V^μ transforms between the local coordinate systems in these two simplices. This transformation is directly related to the continuum path-ordered (P) exponential of the integral of the local affine connection, with the connection here having support only on the common interface between two simplices. The lattice action itself only contains contributions from infinitesimal loops, but more generally one might want to consider near-planar, but noninfinitesimal, closed loops C (see Fig. 1). Along this closed loop the overall rotation matrix will be given by

$$U^\mu{}_\nu(C) \;=\; \Big[\prod_{s\subset C} U_{s,s+1}\Big]^\mu{}_\nu \;\; . \tag{9}$$

In analogy with the infinitesimal loop case, one would like to state that for the overall rotation matrix one has

$$U^\mu{}_\nu(C) \;\approx\; \Big[e^{\delta(C)B(C))}\Big]^\mu{}_\nu \;\; , \tag{10}$$

[2] See, for example, Peskin and Schroeder, *An Introduction to Quantum Field Theory*, p. 783, Eq. (22.3).

[3] We do not distinguish here, for the sake of simplicity, between the square root of the string tension and the mass gap. In $SU(N)$ Yang-Mill theories, and QCD in particular, these represent nearly the same mass scale, up to a constant of order one.

where $B_{\mu\nu}(C)$ is now an area bivector perpendicular to the loop and $\delta(C)$ the corresponding deficit angle, which will work only if the loop is close to planar so that $B_{\mu\nu}$ can be taken to be approximately constant along the path C. By a near-planar loop around the point P, we mean one that is constructed by drawing outgoing geodesics on a plane through P.

The matrix $U^{\mu}{}_{\nu}(C)$ in Eq. (9) then describes the parallel transport of a vector round the loop C. If that is true, then one can define an appropriate coordinate scalar by contracting the above rotation matrix $\mathbf{U}(C)$ with an appropriate bivector, namely

$$W(C) \;=\; \omega_{\alpha\beta}(C)\, U^{\alpha\beta}(C) \tag{11}$$

where the bivector, $\omega_{\alpha\beta}(C)$, is intended as being representative of the overall geometric features of the loop (for example, it can be taken as an average of the hinge bivector $\omega_{\alpha\beta}(h)$ along the loop). Finally, in the quantum theory one is interested in the quantum average or vacuum expectation value of the above loop operator $W(C)$, as in the gauge theory expression of Eq. (4).

The next step is to relate the so defined, and computed, quantum average to physical observable properties of the manifold. Indeed for any continuum manifold one can define locally the parallel transport of a vector around a near-planar loop C. Then parallel transporting a vector around a closed loop represents a suitable operational way of detecting curvature locally. Thus a direct calculation of the vacuum expectation of the quantum Wilson loop provides a way of determining an *effective* curvature at large distance scales, even in the case where short distance fluctuations in the metric may be significant.

For calculational convenience, the actual computation of the quantum gravitational Wilson loop in [3] was achieved by using a slight variant of Regge calculus, where the contribution to the action from the hinge h is given not by the original Regge expression

$$S_h \;=\; -\,k\, A_h\, \delta_h \quad , \tag{12}$$

with $k = 1/8\pi G$, but instead by the modified form

$$S_h \;=\; \frac{k}{4}\, A_h\ \mathrm{tr}[(B_h \,+\, \epsilon\, I_4)\,(\mathbf{U}_h \,-\, \mathbf{U}_h^{-1})] \quad , \tag{13}$$

where A_h is the area of the triangular hinge where the curvature is located, B_h (called U_h in [3,4]) is a bivector orthogonal to the hinge, ϵ is an arbitrary multiple of the unit matrix and $\mathbf{U}_h$ the product of rotation matrices relating the coordinate frames in the 4-simplices around the hinge. The motivation for this second choice was that analytical calculations could then be performed more easily in the strong coupling regime, using methods analogous to the ones used successfully for gauge theories [1,2]. Indeed it can be shown [3] that this second action contribution is equal to

$$S_h \;=\; -\,k\,A_h\,\sin(\delta_h) \;\;, \tag{14}$$

independently of the parameter ϵ, where δ_h is the deficit angle at the hinge. For small deficit angles one expects this to be a good approximation to the standard Regge action, and general universality arguments would suggest that the lattice continuum be the same in the two theories. The expectation values of gravitational Wilson loops were then defined by either

$$< W(C) > = < \;\mathrm{tr}(U_1\,U_2\,...\,U_n)\; > \;\;, \tag{15}$$

or

$$< W(C) > = < \;\mathrm{tr}[(B_C\;+\;\epsilon\,I_4)\,U_1\,U_2\,...\,...\,U_n]\; > \;\;, \tag{16}$$

where the U_is are the rotation matrices along the path, and, in the second expression, B_C is a suitable average direction bivector for the loop C, which is assumed to be near-planar. The values of $< W >$ in the strong coupling regime (i.e. for small k) can then be calculated for a number of loops, including some containing internal plaquettes. It was found that for large near-planar loops around n hinges, to lowest nontrivial order (i.e. corresponding to a tiling of the interior of the loop by a minimal surface),

$$< W > \approx \left(\frac{k\bar{A}}{16}\right)^n \epsilon^\alpha\;[\,p\,+\,q\,\epsilon^2\,]^\beta \;\;, \tag{17}$$

where $\alpha+\beta=n$, and $\bar{A}$ is the average area of the plaquettes. Then using $n=A_C/\bar{A}$, where A_C is the area of the loop, the area-dependent first factor can be written as

$$\exp[\,(A_C/\bar{A})\,\log(k\,\bar{A}/16)\,] \;=\; \exp\,(-\,A_C/\xi^2) \tag{18}$$

where we have set $\xi=[\bar{A}/|\log(k\,\bar{A}/16)|]^{1/2}$. Recall that for strong coupling, $k\to 0$, so ξ is real, and that the quantity ξ is in principle defined independently of the expectation value of the Wilson loop, through the correlation of suitable local invariant operators at a fixed geodesic distance.

In the following we shall assume, in analogy to what is known to happen in non-Abelian gauge theories, that even though the above form for the Wilson loop was derived in the extreme strong coupling limit, it will remain valid throughout the whole strong coupling phase and all the way up to the nontrivial ultraviolet fixed point, with the correlation length $\xi\to\infty$ the only relevant and universal length scale in the vicinity of the fixed point. The evidence for the existence of such a fixed point comes from three different sources, which have recently been reviewed, for example, in Ref.[4], and references therein. The first source is the $2+\epsilon$ expansion for gravity, which exhibits such a fixed point in G to one and two loops, shows that only one relevant direction exists to all orders in this expansion, and provides a quantitative estimate for the critical exponent ν at the nontrivial ultraviolet fixed point. The second source is the lattice gravity theory in $d = 4$ based on Regge's simplicial formulation, which also exhibits a phase transition, with a single calculable nontrivial relevant exponent ν. The third source is the Einstein-Hilbert truncation renormalization method in the continuum, which, although approximate in nature, provides a third rough independent estimate for the exponent ν at the nontrivial ultraviolet fixed point.

The next step was to interpret the result in semiclassical terms. By the use of Stokes's theorem, the parallel transport of a vector round a large loop depends on the exponential of a suitably-coarse-grained Riemann tensor over the loop. So by comparing linear terms in the expansion of this expression with the corresponding term in the expression of the area law, one can show [3] that the average curvature is of order $1/\xi^2$, at least in the strong coupling limit. Since the scaled cosmological constant is a measure of the intrinsic curvature of the vacuum, this also suggests that the cosmological constant is positive, and that the manifold is de Sitter at large distances.

The question now arises as to whether these results are peculiar to the particular formulation of discrete gravity used. This led to a study of other proposed formulations, most of which were written down more than twenty years ago. In this work we will show that where it seems possible to define and calculate gravitational Wilson loops, the same area law emerges, and automatically implies a positive cosmological constant.

Another key question we will address is whether these results are affected in any way by the presence of matter. After all the universe is not devoid of matter, and the pure gravity results should only be considered as a first-order approximation to the full quantum theory (in a spirit similar to the quenched approximation in non-Abelian gauge theories). This will be discussed here again in the context of the Regge formulation of discrete gravity used in [3], using the methods of coupling matter to gravity reviewed, for example, in Ref. [4].

An outline of the paper is as follows. In Sec. 2, we describe formulations of Einstein gravity as a gauge theory on a flat background lattice, and in Sec. 3, the MacDowell-Mansouri description of de Sitter gravity, as transcribed onto a flat background lattice by Smolin. More recent developments of discrete gravity, spin foam models, are discussed briefly in Sec. 4, and Sec. 5 contains mention of other relevant theories. We then turn to the effect of matter couplings on the gravitational Wilson loop results, and Secs. 6, 7, 8 and 9 contain systematic discussions of scalar matter, fermions, gauge fields and the lattice gravitino. Regarding these matter fields, the main conclusion is that the previous results are not affected, unless there are near massless spin 1/2 and spin 3/2 particles (i.e. whose mass is comparable to the exceedingly small gravitational scale ξ^{-1}). Sec. 10 consists of some conclusions.

2 Gauge-theoretical treatment of Einstein gravity on a flat background lattice

We will first look at formulations of Einstein gravity as a gauge theory on a flat hypercubical background lattice, and in particular expand on the work of Mannion and Taylor [11] and of Kondo [12]. In these cases, the standard machinery for calculating Wilson loops in lattice gauge theories [2] can be taken over without too many modifications. Although such formulations were not the first chronologically of those we consider in this paper, we treat them first because they are, in many respects, the simplest. The idea is to write Einstein gravity in four dimensions, without cosmological constant, on a flat background lattice, treating it as a gauge theory with gauge group $SL(2,C)$, and relating it to the Einstein-Cartan formalism. In fact, for simplicity, we shall consider an Euclidean version, replacing $SL(2,C)$ by $SO(4)$. The Minkowskian formulation presents new problems due to the noncompactness of the group, which will not be addressed here; basically the group-theoretic methods used below cannot be applied in the same fashion, and new convergence issues arise due to the different nature of the Haar measure.

In the following nearest neighbor sites are labeled by n and $n+\mu$, and their frames are related by

$$U_\mu(n) = \exp(iA_\mu(n)) = U_{-\mu}(n+\mu)^{-1}, \tag{19}$$

where

$$A_\mu(n) = \tfrac{1}{2}\, a\, A_\mu^{ab}(n)\, S_{ab} \quad , \tag{20}$$

with a the lattice spacing and S_{ab} the $O(4)$ generators, represented by the 4×4 matrices

$$S_{ab} \;=\; \frac{i}{4}\,[\gamma_a, \gamma_b] \quad , \tag{21}$$

with the Euclidean gamma matrices, γ_a satisfying

$$\{\gamma_a, \gamma_b\} \;=\; 2\,\delta_{ab}, \qquad \gamma_a^\dagger = \gamma_a, \qquad a = 1, ..., 4. \tag{22}$$

The curvature round an elementary plaquette spanned by the μ and ν directions is given as usual by

$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\mu)U_{-\mu}(n+\mu+\nu)U_{-\nu}(n+\nu) = U_\mu(n)U_\nu(n+\mu)U_\mu(n+\nu)^{-1}U_\nu(n)^{-1}, \tag{23}$$

and it can be shown that in the limit of small lattice spacing,

$$U_{\mu\nu}(n) \;\approx\; \exp(i\, a^2 R_{\mu\nu}), \tag{24}$$

where

$$R_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\,[A_\mu, A_\nu] \;\; . \tag{25}$$

One notices that the usual lattice gauge theory type action, consisting of sums of $U_{\mu\nu}$ terms, would give an $R_{\mu\nu}R^{\mu\nu}$ term in the limit of small a, so terms involving the vierbein $e^a_\mu(n)$ and the matrix $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ have to be introduced. One defines

$$S \;=\; \frac{1}{16\,\kappa^2} \sum_{n,\mu,\nu,\lambda,\rho} \epsilon^{\mu\nu\lambda\sigma}\;\mathrm{tr}[\gamma_5\, E_\lambda(n) U_{\mu\nu}(n) E_\sigma(n)] \tag{26}$$

where $E_\mu(n) = a\, e^a_\mu\, \gamma_a$ and κ is the Planck length in suitable units. It can then be shown that

$$S = \frac{a^4}{4\kappa^2} \sum_{n,\mu,\nu,\lambda,\rho} \epsilon^{\mu\nu\lambda\sigma}\, \epsilon_{abcd}\, R^{ab}_{\mu\nu}(n)\, e^c_\lambda(n)\, e^d_\sigma(n) + O(a^6), \tag{27}$$

which is the Einstein action in first-order form [13]. Furthermore by construction the action is invariant under local $O(4)$ rotations. For reasons which will become apparent, we shall consider a symmetrized form of the action: for each plaquette, rather than having the $E_\sigma\gamma_5 E_\lambda$ term inserted only at the base point, we shall consider the average of its insertion at all vertices of the plaquette.

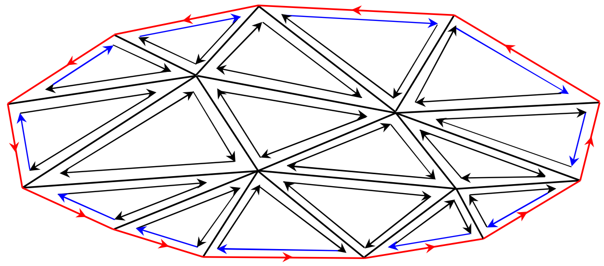

Figure 1. Illustration of the gravitational analog of the Wilson loop. A vector is parallel-transported along the larger outer loop. The enclosed minimal surface is tiled with parallel transport polygons, here chosen to be triangles for illustrative purposes. For each link of the dual lattice, the elementary parallel transport matrices $\mathbf{U}(s, s')$ are represented by arrows.

In the following the partition function is defined by the usual path integral expression

$$Z = \int [dA][dE] \; \exp(-S), \tag{28}$$

where $[dA] = \prod_{n,\mu} d_H U_\mu(n)$, $[dE] = \prod_{n,\lambda} dE_\lambda(n)$, and $d_H U$ is the Haar measure on $SO(4)$.

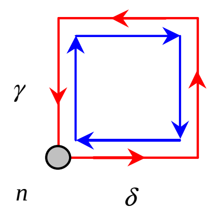

Figure 2. A parallel transport loop, spanned by the γ and δ directions, with four oriented links on the boundary. The parallel transport matrices U along the links, represented here by arrows, appear in pairs and are sequentially integrated over using the uniform measure.

Our interest here is in the definition and evaluation of Wilson loops in the strong coupling expansion. The authors of Ref. [11] define the loop around one plaquette, spanned by the γ and δ directions (see Fig. 2), by

$$W \; = \; \prod_{\kappa,\xi} \epsilon^{\delta\gamma\kappa\xi} \;\; \mathrm{Tr}[E_\kappa(n) U_{\delta\gamma}(n) E_\xi(n)] \;\; , \tag{29}$$

and so

$$< W > = \frac{1}{Z} \int [dA][dE] \prod_{\kappa,\xi} \epsilon^{\delta\gamma\kappa\xi} \; \mathrm{Tr}[E_\kappa(n) U_{\delta\gamma}(n) E_\xi(n)] \; \exp(-S) \; . \tag{30}$$

They go on to show that in the strong coupling expansion, the dominant term is proportional to

$$< W > = \int [dE] \; \epsilon^{\delta\gamma\lambda\sigma} \; \epsilon^{\delta\gamma\kappa\xi} \; \epsilon_{abst} \; e^s_\kappa \; e^t_\xi \; e^a_\lambda \; e^b_\sigma \; , \tag{31}$$

where there is no sum over γ and δ. Now suppose that $\gamma = 1, \delta = 2$. Then the sum over λ and σ leads to

$$\epsilon_{abst} \; e^s_3 \, e^t_4 \, e^a_3 \, e^b_4 \; , \tag{32}$$

which is zero on symmetry grounds. Therefore their definition needs some modification, or one has to go to higher orders in the strong coupling expansion. In the latter case, it is possible to get a nonzero contribution by going to order $1/k^6$, but here we concentrate on the first possibility. Omitting the Es from W also gives zero for $< W >$, so the modification we make is to insert a γ_5 into W. The lowest order contribution is then

$$\begin{aligned} &- \frac{1}{16\kappa^2} \int [dA][dE] \sum_{\kappa,\xi} \epsilon^{\delta\gamma\kappa\xi} \; \mathrm{Tr}[\gamma_5 \, E_\kappa(n) \, U_{\delta\gamma}(n) E_\xi(n)] \sum_{n',\mu,\nu,\lambda,\rho} \epsilon^{\mu\nu\lambda\sigma} \; \mathrm{Tr}[\gamma_5 \, E_\lambda(n') \, U_{\mu\nu}(n') \, E_\sigma(n')] \\ &= \frac{a^4}{16\kappa^2} \int [dA][dE] \sum_{\kappa,\xi} \epsilon^{\delta\gamma\kappa\xi} \; \mathrm{Tr}[\gamma_5 \, \gamma_s \, U_\delta(n) \, U_\gamma(n+\delta) \, U_\delta(n+\gamma)^{-1} \, U_\gamma(n)^{-1} \, \gamma_t] \\ &\times \sum_{n,\mu,\nu,\lambda,\rho} \epsilon^{\mu\nu\lambda\sigma} \; \mathrm{Tr}[\gamma_5 \, \gamma_a \, U_\mu(n') \, U_\nu(n'+\mu) \, U_\mu(n'+\nu)^{-1} \, U_\nu(n')^{-1} \, \gamma_b] \, e^s_\kappa(n) \, e^t_\xi(n) \, e^a_\lambda(n') \, e^b_\sigma(n') \; . \end{aligned} \tag{33}$$

The integration over the As is equivalent to the integration over U's in $SO(4)$ with the Haar measure:

$$\int d_H U \;\; U_{ij} \, U^{-1}_{kl} \; = \; \frac{1}{4} \, \delta_{il} \, \delta_{jk}, \tag{34}$$

and we obtain

$$\frac{a^4}{64\kappa^2} \int [dE] \sum_{\kappa\xi\lambda\sigma} \epsilon^{\delta\gamma\kappa\xi} \, \epsilon^{\delta\gamma\lambda\sigma} \; \mathrm{Tr}[\gamma_d \gamma_5 \gamma_c \gamma_b \gamma_5 \gamma_a] \, e^a_\lambda(n) \, e^b_\sigma(n) \, e^c_\kappa(n) \, e^d_\xi(n) \; . \tag{35}$$

Now we compute

$$\mathrm{Tr}[\gamma_d \gamma_5 \gamma_c \gamma_b \gamma_5 \gamma_a] = 4 \, (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \tag{36}$$

and, using

$$g_{\sigma\lambda} = \tfrac{1}{4} \, a^2 \; \mathrm{Tr}(\gamma_a \gamma_b) \, e^a_\lambda e^b_\sigma = a^2 \; \delta_{ab} \, e^a_\lambda \, e^b_\sigma \; , \tag{37}$$

we obtain

$$\frac{1}{16\kappa^2}\int[dE]\sum_{\kappa\xi\lambda\sigma}\epsilon^{\delta\gamma\kappa\xi}\,\epsilon^{\delta\gamma\lambda\sigma}\,(g_{\lambda\sigma}g_{\kappa\xi}-g_{\lambda\xi}g_{\kappa\sigma}-g_{\lambda\kappa}g_{\xi\sigma})\quad. \tag{38}$$

Suppose that $\gamma=1,\delta=2$, then the sum over the indices in the ϵ's and g's gives

$$4\,(\,g_{34}^2-g_{33}\,g_{44})\quad. \tag{39}$$

We expand the metrics in terms of the vierbeins and define the measure of integration to include a damping factor $(\lambda a^2/\pi)^8\exp[-\lambda a^2\Sigma_{b,\mu}(e^b_\mu)^2]$ at each point, with $Re\lambda>0$ [14], obtaining

$$-\frac{3}{4\kappa^2\lambda^2}\;. \tag{40}$$

(Note that we are ignoring a possible factor of the determinant of the vierbein in the measure.)

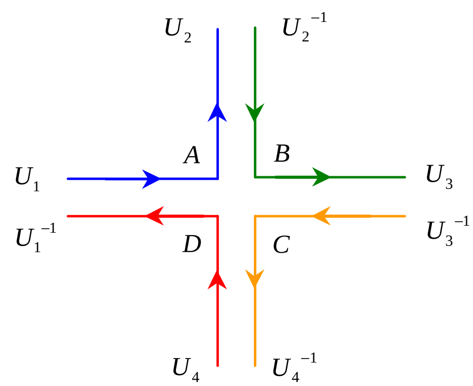

Figure 3. A vertex where various parallel transport matrices enter and leave, and where there are insertions on their paths.

Before considering larger loops, let us obtain an algorithm which simplifies the calculations considerably. Consider a vertex with the matrices A,B,C,D attached to it, and U-matrices attached to the lines entering and leaving the vertex, as shown in Fig. 3. Integration over the Us of the expression

$$(U_1)_{ab}\,A_{bc}\,(U_2)_{cd}\,(U_2^{-1})_{ef}\,B_{fg}\,(U_3)_{gh}\,(U_3^{-1})_{ij}\,C_{jk}\,(U_4)_{kl}\,(U_4^{-1})_{mn}\,D_{no}\,(U_1^{-1})_{op} \tag{41}$$

gives

$$\frac{1}{4^4}\,\mathrm{Tr}(ABCD)\,\delta_{de}\,\delta_{hi}\,\delta_{lm}\,\delta_{pa}\quad. \tag{42}$$

We see that the effect of the integration is to give a factor of $\frac{1}{4}$ for each U, and to give the trace of the product of factors at each vertex. For a vertex with no insertion, we obtain the trace of the identity matrix, 4, and for one insertion of $E\gamma_5 E$, the value is zero since it is traceless. For two insertions of $E\gamma_5 E$, we obtain $12/\lambda^2$, where the integration over the Es has been done. Recall that there is also a factor of $-1/16\kappa^2$ for each plaquette, corresponding to the relevant terms in the expansion of the exponential of minus the action. This means that within our loop, if it is to have a nonzero value, every vertex must have either no $E\gamma_5 E$ factors or two of them. This is why we took the average of insertions at all vertices of the plaquettes in the action; it would be impossible to get nonzero contribution from the internal plaquettes otherwise.

Before proceeding with the calculations, let us mention an alternative to the procedure of averaging the contribution of the action from a plaquette over its vertices, a possibility, similar to the procedure in [3]. If we replace γ_5 by $\gamma_5 + \epsilon I_4$ for some arbitrary parameter ϵ, the continuum limit of the action acquires a term proportional to $\epsilon^{\mu\nu\lambda\sigma} R_{\mu\nu\lambda\sigma}$, which is zero because of the symmetries of the Riemann tensor, so the action is unaltered. However, the value of the Wilson loop is still zero, not because of the traces but because of factors of the Kronecker delta, which give zero on symmetry grounds.

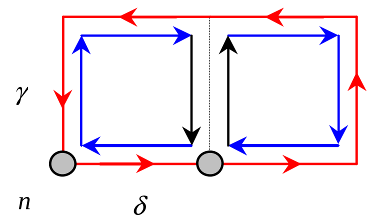

Figure 4. A parallel transport loop around two plaquettes, with insertions of $E\gamma_5 E$ on the loop shown by large dots.

For a loop around two plaquettes, we find that we obtain a nonzero value only if the $E\gamma_5 E$ is inserted at the place where the loop meets the second plaquette (see Fig. 4). The value of $< W >$ is then

$$\frac{1}{4}\left(\frac{3}{4\kappa^2\lambda^2}\right)^2 \quad . \tag{43}$$

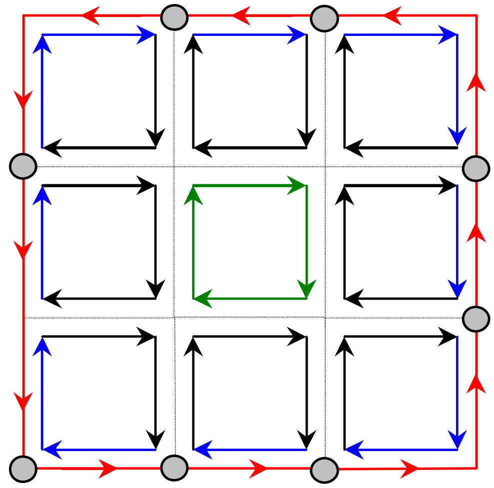

Figure 5. A larger parallel transport loop with 12 oriented links on the boundary. As before, the parallel transport matrices along the links appear in pairs and are sequentially integrated over using the uniform measure. The new ingredient in this configuration is an elementary loop at the center not touching the boundary. As in Fig. 4, the insertions of $E\gamma_5 E$ on the loop are shown by large dots.

For a loop around many plaquettes, we choose to insert the factors of $E\gamma_5 E$ in the loop wherever the loop meets a new plaquette. For a loop with internal plaquettes, there has to be an even number of internal plaquettes as the insertions need to be paired between them. (For example, see the loop around nine plaquettes in Fig. 5; there is no way the insertions on the one internal plaquette can give a nonzero value.) This means that we obtain nonzero values only for Wilson loops surrounding an even number of plaquettes; the simplest case, with 12 plaquettes, is shown in Fig. 6. There are two ways of getting nonzero values from the internal plaquettes, corresponding to pairings of the insertions at the two vertices they have in common, which gives a factor of 2 in the answer, which, when integrated over the vierbeins, is

$$\frac{\lambda^2}{4^{11}6}\left(\frac{3}{4\kappa^2\lambda^2}\right)^{12}. \tag{44}$$

Larger loops can then be treated in a similar way. From the results obtained so far, we deduce that, as the authors of [11] claimed, there is indeed an area law for large Wilson loops. The physical interpretation is of course very different, as discussed in the Introduction, and later in the Conclusion.

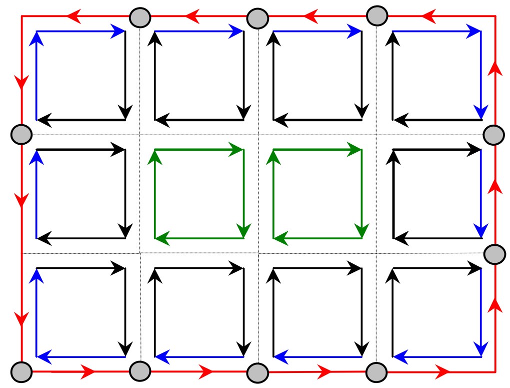

Figure 6. A Wilson loop around 12 plaquettes, of which two are internal, with the insertions of $E\gamma_5 E$ on the loop shown by large dots as before.

We now consider briefly the work of Kondo [12]. His basic formalism is very similar to that of [11], except that rather than introducing the vierbeins into the action directly, he introduces exponentials of them, with the action

$$S = -\frac{1}{4\kappa^2}\sum_{n,\mu,\nu,\lambda,\rho} \epsilon^{\mu\nu\lambda\sigma}\ \mathrm{Tr}[\gamma_5 U_{\mu\nu}(n) H_\lambda(n) H_\sigma(n)] \ , \tag{45}$$

where

$$H_\mu(n) = \exp[i\, a\, e^a_\mu(n)\gamma_a] \ . \tag{46}$$

This has the consequence that the action is bounded. (The minus sign, which appears different from the sign in the formalism of [11], is because of the different relative position of the γ_5 factor.) In practice, in calculations, it is impossible to work out traces without expanding the exponentials and retaining the lowest order terms in the lattice spacing, so the formalism reduces to that of [11] in this respect, and the same values are obtained for the Wilson loops. (We have checked that the lowest order contribution comes from the product of the linear terms in the expansions of the exponentials.) However, Kondo also aims to set up a formalism which has reflection positivity, so his action contains sums over reflections, and if this full action is used, it is very complicated to evaluate Wilson loops.

Note that this method of averaging the action contribution of each plaquette over the vertices of the plaquette needs to be used here, and could also be used in [3], eliminating the necessity for introducing the parameter ϵ.

3 Lattice formulation of MacDowell-Mansouri gravity

An earlier version of lattice gravity was given by Smolin [15], who transcribed the MacDowell and Mansouri [16] formulation of general relativity onto a flat background lattice. MacDowell and Mansouri built a gauge theory by defining ten (antisymmetric) gauge potentials by

$$A_\mu^{ab} = \omega_\mu^{ab}, \quad A_\mu^{5a} = \frac{1}{l}\, e_\mu^a \ , \tag{47}$$

where ω_μ^{ab} and e_μ^a are the usual gravitational connection and vierbein, and l is a lattice spacing. The curvature and torsion are defined in terms of the gauge potentials, and the action is of the form

$$S = \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \, \epsilon_{abcd} \, R_{\mu\nu}^{ab} \, R_{\rho\sigma}^{cd} \ , \tag{48}$$

where $R_{\mu\nu}^{ab}$ is the Riemann tensor for $O(3,2)$ or $O(4,1)$. This can be shown to be equivalent, after multiplication by $\mp 1/32l^2/\kappa^2$, with κ the bare Planck length, to

$$S = \int d^4x \left[\mp \frac{l^2}{32\kappa^2} \, \epsilon^{\mu\nu\rho\sigma} \, \epsilon_{abcd} \, R_{\mu\nu}^{0ab} \, R_{\rho\sigma}^{0cd} + \frac{1}{2\kappa^2} \, e \, R^0 \ \mp \frac{2}{\kappa^2 l^2} \, e \right] \ , \tag{49}$$

where $R_{\mu\nu}^{0ab}$ is the usual Riemann curvature tensor. The first term is a topological invariant, the Gauss-Bonnet term, while the second and third are obviously the Einstein term and the cosmological constant term respectively, with a scaled cosmological constant $\lambda = \pm 2/\kappa^2 l^2$. Note that in this formulation the relative coefficients of various action contributions are fixed in terms of the bare parameter κ and l.

Then the starting point in [15] is the continuum action

$$S = \mp \frac{1}{g^2} \int d^4x \, \epsilon^{\mu\nu\rho\sigma} \, R_{\mu\nu}^{AB} \, R_{\rho\sigma}^{CD} \, \epsilon_{ABCD5} \ , \tag{50}$$

where $R_{\mu\nu}^{AB}$ is the curvature associated with an $O(4,1)$ (minus sign) or $O(3,2)$ (plus sign) gauge connection, ϵ_{ABCD5} is the totally antisymmetric 5-tensor and $g = \sqrt{32}\,\kappa/l$ a dimensionless coupling constant. The parallel transport operators along the links of the lattice are defined by

$$U_\mu(n) = \mathcal{P} \exp \left[\tfrac{1}{2}\, g \int_n^{n+\mu} dx^\rho \, A_\rho^{AB}(x) \, T_{AB} \right] \ , \tag{51}$$

where the T^{AB} are matrix representations of the relevant Lie algebra. Then the curvature around a plaquette on a hypercubic lattice, $U_{\mu\nu}(n)$, is identical to the definition of Mannion and Taylor [11] [Eq. (23)], and this is related to the curvature by

$$\tfrac{1}{2} \, [U_{\mu\nu}(n)]_{ij} = a^2 g \, R_{\mu\nu}^{AB} \, (T_{AB})_{ij} + O(a^3). \tag{52}$$

The continuum action is then transcribed onto the lattice as

$$S = \mp \frac{1}{g^2} \sum_n \epsilon^{\mu\nu\rho\sigma}\, \epsilon^{ijkl5}\, [U_{\mu\nu}(n)]_{ij}\, [U_{\rho\sigma}(n)]_{kl}\, \epsilon_{ABCD5}\ . \tag{53}$$

It involves a sum over contributions from perpendicular plaquettes at each lattice vertex, in analogy to the construction of the $F\tilde{F}$ term in non-Abelian gauge theories. In order to maintain the discrete symmetries of the lattice (reflections and rotations through multiples of π), this is extended to a sum over all orientations of the dual plaquettes

$$S = \mp \frac{1}{16g^2} \sum_n \sum_{O,O'} \epsilon^{\mu\nu\rho\sigma}\, \epsilon^{ijkl5}\, [U^{O}_{\mu\nu}(n)]_{ij}\, [U^{O'}_{\rho\sigma}(n)]_{kl}\, \epsilon_{ABCD5}\ \ . \tag{54}$$

The partition function is then given by

$$Z = \int [dU]\ \exp(iS), \tag{55}$$

where we take $[dU]$ to be the normalized Haar measure. We restrict the integration to $O(5)$, rather than considering also $O(3,2)$ and $O(4,1)$ as in [15], since for the noncompact groups one has to define the measure by dividing through by the (infinite) volume of the gauge group. For the five-dimensional representations used, the relevant integrals are:

$$\int [d_H U] = 1\ \ , \tag{56}$$

$$\int [d_H U]\ [U_\mu(n)]_{ij} = 0\ \ , \tag{57}$$

$$\int [d_H U]\ [U_\mu(n)]_{ij}\ [U_\nu(n')]_{kl} = \frac{1}{5}\, \delta_{il}\, \delta_{jk}\, \delta_{nn'}\, \delta_{\mu\nu}\ \ . \tag{58}$$

The structure of the action, based on pairs of dual plaquettes, means that the calculations are somewhat different from the case in [11]. In particular, since we want eventually to evaluate Wilson loops for planar surfaces, we can take as our basic building block a combination of two pairs of dual plaquettes, put together so that one plaquette from each pair lies adjacent to the other in the plane or they meet at one point, and the other two are joined back-to-back (see Figs. 7 and 8). We then calculate the contribution from this configuration in both cases, when integration over the Us on the back-to-back faces is performed.

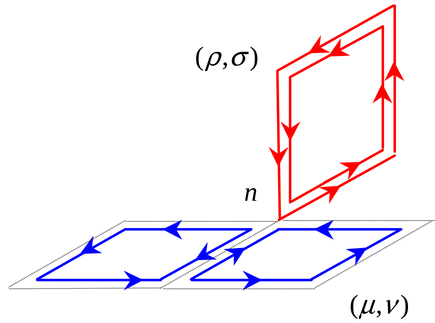

Figure 7. Two pairs of dual plaquettes joined together, with the ones in the (μ,ν)-plane lying side-by-side, and the ones in the (ρ,σ)-plane back to back.

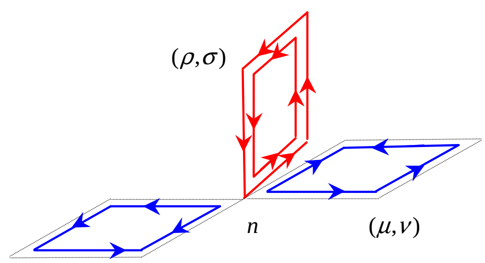

Figure 8. Two pairs of dual plaquettes joined together, with the ones in the (μ,ν)-plane sharing only the vertex n, and the ones in the (ρ,σ)-plane back-to back.

The quantity to evaluate in the first case is

$$\begin{aligned} S &= \frac{1}{16g^2}\int [d_H U]\,(\epsilon^{\mu\nu\rho\sigma})^2\,\epsilon^{ijkl5}\,\epsilon^{i'j'k'l'5}\,[U_\mu(n)U_\nu(n+\mu)U_\mu^{-1}(n+\nu)U_\nu^{-1}(n)]_{ij} \\ &\quad\times [U_\rho(n)U_\sigma(n+\rho)U_\rho^{-1}(n+\sigma)U_\sigma^{-1}(n)]_{kl}\,[U_\nu(n)U_\mu^{-1}(n+\nu-\mu)U_\nu^{-1}(n-\mu)U_\mu(n-\mu)]_{i'j'} \\ &\quad\times [U_\rho(n)U_\sigma(n+\rho)U_\rho^{-1}(n+\sigma)U_\sigma^{-1}(n)]_{k'l'}\quad , \end{aligned} \tag{59}$$

(with no summation over μ,ν). Integration over the U_ρs and U_σs gives

$$(\frac{1}{16g^2})^2(\epsilon^{\mu\nu\rho\sigma})^2\epsilon^{ijkl5}\epsilon^{jj'kl5}\,[U_\mu(n)U_\nu(n+\mu)U_\mu^{-1}(n+\nu)U_\mu^{-1}(n+\nu-\mu)U_\nu^{-1}(n-\mu)U_\mu(n-\mu)]_{ij'}. \tag{60}$$

Now

$$\epsilon^{ijkl5}\epsilon^{jj'kl5} = 2\,(\delta^{ij'}\;\delta^{jj} - \delta^{ij}\;\delta^{jj'}) \tag{61}$$

where

$$\not\delta_{ik} = \delta_{ik} - \delta_{i5}\delta_{k5} \ , \tag{62}$$

so the final contribution, including a factor of 4 from the summation over ρ and σ, is

$$(\frac{1}{16g^2})^2 \ \frac{24}{25}\epsilon^{ijkl5} \, \epsilon^{jj'kl5} \, [U_\mu(n)U_\nu(n+\mu) \, U_\mu^{-1}(n+\nu) \, U_\mu^{-1}(n+\nu-\mu) \, U_\nu^{-1}(n-\mu) \, U_\mu(n-\mu)]_{ij'} \ \ \not\delta_{ij'} \ . \tag{63}$$

In the second case, the calculation proceeds in a similar way, to give

$$\begin{aligned}&(\frac{1}{16g^2})^2 \ \frac{8}{5} \ (\not\delta_{ii'} \ \not\delta_{jj'} - \not\delta_{ij'} \ \not\delta_{ji'}) \ [U_\mu(n) \, U_\nu(n+\mu) \, U_\mu^{-1}(n+\nu) \, U_\nu^{-1}(n)]_{ij} \\ &\times [U_\mu^{-1}(n-\mu) \, U_\nu^{-1}(n-\mu-\nu) \, U_\mu(n-\mu-\nu) \, U_\nu(n-\nu)]_{i'j'} \ .\end{aligned} \tag{64}$$

We now define a Wilson loop as the product of the U factors around the given path, with no extra factors in this case, and we calculate its expectation value as usual:

$$< w > = \ \frac{1}{Z} \int \prod_i [d_H U_i] \ W \ \exp(iS) \ . \tag{65}$$

As explained in [15], calculations are done in this formalism on the assumption that one can ignore the zero-torsion constraint; the basis for this is that the torsion is suppressed by a factor of $\frac{1}{l}$, where l is large. As a result, one only needs to integrate over the U's, and there is no need to integrate over the vierbeins in this formalism. Note that because of the structure of the basic building blocks, we can define Wilson loops only around paths which contain an even number of plaquettes. The simplest of these is shown in Fig. 4, and the area can be tiled by only one of the two possible building blocks, giving the value $(1/(16g^2)^2)$ $(192/125)$.

The next most simple cases are shown in Fig. 9. The first of these can be tiled in four possible ways with the first of the building blocks, giving $(1/(16g^2)^4)$ $(2^{12}3^2/5^7)$, while in the second, which can be tiled in eight ways with the first building block and in one way with the second, the final contribution is $(1/(16g^2)^4)$ $(2^8 321/5^7)$. For the simplest configuration with internal plaquettes, a loop surrounding 12 plaquettes (see Fig. 6), there are many (1072) different ways of tiling it, so we need to add the contributions from all the different ways. The tiling shown in Fig. 6 gives $(1/(16g^2)^{12})$ $(2^{26}3^4/5^{21})$, and then combining this with the other contributions, we obtain $(1/(16g^2)^{12})$ $(2^{30}3^4 9481/5^{23})$. Notice the dependence on $1/g^2$ in the various cases evaluated. Again, larger loops can then be treated in a similar way although the calculations become increasingly tedious. This indicates the usual area law for the gravitational Wilson loop. We note here that the authors of Ref. [17] have performed numerical simulations using the action from [15], with an $SO(4)$ invariant action and a Haar measure over the group $SO(5)$, considering then both the weak and the strong coupling regimes.

We should state at this point that in this paper we have chosen to focus almost exclusively on the strong coupling limit of various models of lattice gravity, and in particular on the emergence of the area law for the Wilson loop. New problems can arise when approaching the lattice continuum limit in the vicinity of the critical point, if one exists. As an example, in some lattice models the transition appears to be first order [17], which would mean that either the lattice action has to be modified by adding second neighbor terms, or that the critical exponents have to be obtained by analytic continuation from the strong coupling phase, approaching in this way the fixed point located in the metastable phase. Within the limited framework of this work we shall not address these additional technical issues, and assume instead that a number of lattice theories examined here describe to some extent correctly at least the physics of the strong coupling phase of gravity.

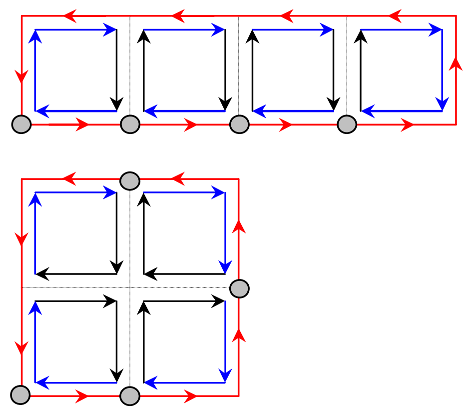

Figure 9. Two arrangements for Wilson loops around four plaquettes.

A formalism related to that of Smolin is described by Das, Kaku and Townsend [18]. They transcribe West's de Sitter invariant formulation of Einstein gravity onto the lattice, obtaining an action with plaquette contribution proportional to the square root of the trace of a square involving Smolin's action. They showed that their theory agrees with the one in [15] in the lattice continuum limit. The square root in the action makes it almost impossible to do any general analytical calculations.

To put the results of Wilson loop calculations in this and the previous section, together with the results of [3], into context, it is interesting to make comparisons by relating the coupling constants to that of the continuum action. The κ of Mannion and Taylor [11] and Kondo [12], and the g of Smolin [15] are related to the $k = 1/8\pi G$ of [4] by

$$\frac{1}{\kappa^2} = \frac{k}{2a^4}, \quad \frac{1}{g^2} = \frac{l^2 k}{32} \; . \tag{66}$$

Making a further normalization of the constants involved by equating the results for the smallest loop results, the answers of [11] and [12] agree with those of [3] until the loops contain internal plaquettes, and then, for example, the 12-plaquette results differ by a factor of $\lambda^2/6$. The results of [15] are of the same order of magnitude as those of [3].

4 Spin foam models

Spin foam models grew out of a combination of ideas from the Ponzano-Regge model of three-dimensional discrete Lorentzian quantum gravity, and from loop quantum gravity. In loop quantization, the fundamental excitations are loops created by Wilson loop operators analogous to the ones used in gauge theories [19], and one assumes that states can be written as power series in spatial Wilson loops of the connection [20]. What does this intimate connection between Wilson loops and spin foam models mean in the context of this paper?

In the three-dimensional formulation of Turaev and Viro [21], which is a regularized version of the Ponzano-Regge model, it has been shown [22] that the graph invariant defined by Turaev [23] coincides, in the semiclassical limit, with the expectation value of a Wilson loop. This is a consequence of the asymptotic behavior of $6j$-symbols, with certain arguments fixed, involving rotation matrices which combine to give parallel transport operators along the graph. The extension of this result to graph invariants in discrete four-manifolds has not been made (as far as we know) and it is not clear anyway whether an area law could be obtained for large loops since the concept of a planar loop is not well-defined.

One way of obtaining a spin foam model is from BF theory [24]. In four dimensions, representation labels are assigned to triangles and group elements to sections of the dual loop around each triangular hinge. The integral of the group elements around the dual loop gives the holonomy, which is a measure of the curvature, F. Thus an evaluation of Wilson loops is a basic ingredient in calculating the action, which is then conventionally expressed in terms of sums over amplitudes for the vertices, edges and faces of the spin foams. Alternatively, in group field theories, the action involves the integral over products of functions of the group variables, corresponding to a kinetic term and an interaction term. Here the evaluation of Wilson loops is somewhat similar to the way matter is inserted; certain edges are picked out (to form the loop) and are then treated differently in the summation process [25].

The authors of Ref. [26] have shown that there is an exact duality transformation mapping the strong coupling regime of a non-Abelian gauge theory to the weak coupling regime of a system of spin foams defined on the lattice. They obtain an expression for the expectation value of a non-Abelian Wilson loop (or spin network) in terms of integrals of expressions involving finite-dimensional unitary representations, intertwiners and characters of the gauge group, together with a gauge constraint factor for each lattice point. The integrals are done explicitly, leaving complicated products and sums over intertwiners, projectors and the character decomposition of the exponential of the action. Their calculation is very general, and to evaluate a Wilson loop in the usual sense, considerable simplification can be made. The links which form the Wilson loop can all be labeled with the same representation, and, as the loop has no multivalent vertices, the intertwiners all become trivial. Even so, the calculation is very complicated for a general gauge group,

To illustrate the ideas behind the work of these authors, we will describe the corresponding calculations in lower dimensions and with gauge group $SU(2)$ [27, 28]. We shall summarize the description in [28]. The partition function for gauge theory on a cubic lattice is written as usual as an integral over link variables U_l, with the action being a sum over plaquettes contributions

$$Z = \frac{1}{\beta} \int \prod_{\text{links}} dU_l \; \exp\left(\frac{\beta \sum_{\text{pl}} (\mathrm{Tr}\, U_{pl} + c.c.)}{2\, \mathrm{Tr}\, \mathbf{1}} \right), \tag{67}$$

with β being the dimensionless inverse coupling. The matrix U_{pl} is the standard product of four link matrices U_l around the plaquette. The idea of the duality transformation is to make a Fourier transform in the plaquette variables, by first inserting unity for each plaquette into the partition function, in the form

$$1 = \prod_{\text{pl}} \int dU_{pl} \; \delta(U_{pl}, U_1 U_2 U_3 U_4), \tag{68}$$

where $U_{1\ldots4}$ are the link variables around the plaquette. The δ-function can be realized by products of Wigner D-functions

$$\delta(U, V) = \sum_{J=0,\, \frac{1}{2},\, 1, \ldots} (2J+1)\; D^J_{m_1 m_2}(U^\dagger)\; D^J_{m_2 m_1}(V)\;. \tag{69}$$

The unity is then inserted into the partition function in the form

$$\begin{aligned} 1 \;\; = \;\; & \prod_{\text{pl}} \int dU_{pl} \; \sum_J (2J+1) D^J_{m_1 m_2}(U^\dagger) \\ & \times D^J_{m_2 m_3}(U_1)\, D^J_{m_3 m_4}(U_2)\, D^J_{m_4 m_5}(U_3)\, D^J_{m_5 m_1}(U_4)\;. \end{aligned} \tag{70}$$

The integration over the plaquette matrices is performed using

$$\int dU_{\rm pl}\ \exp\left(\frac{\beta\sum_{\rm pl}\left({\rm Tr}\,U_{pl}+c.c.\right)}{2\,{\rm Tr}\,\mathbf{1}}\right)D^{J}_{m_1 m_2}(U^{\dagger})\ =\ \frac{2}{\beta}\,\delta_{m_1,m_2}\,I_1(\beta)\,T_J(\beta), \tag{71}$$

where $T_J(\beta)\equiv I_{2J+1}(\beta)/I_1(\beta)$ [2,29] is the "Fourier transform" of the Wilson action and the I_n are modified Bessel functions. The partition function is then

$$\begin{aligned} Z\ &=\ \left[\frac{2}{\beta}\,I_1(\beta)\right]^{\text{no. of plaquettes}}\sum_{J_P}\prod_{\rm pl}(2J_P+1)\,T_{J_P}(\beta) \\ &\quad\times\prod_{\rm links\,l}\int dU_l\ D^{J_P}_{m_1m_2}(U_1)\,D^{J_P}_{m_2m_3}(U_2)\,D^{J_P}_{m_3m_4}(U_3)\,D^{J_P}_{m_4m_1}(U_4)\ . \end{aligned} \tag{72}$$

In two dimensions, each link is shared by two plaquettes and the integration over Ds gives Kronecker deltas, whereas in three dimensions, each link is shared by four plaquettes and the integration over Ds gives $6j$-symbols as in the Ponzano-Regge model. To compute the expectation value of a Wilson loop in representation j_s, a factor of $D^{j_s}(U)$ must be inserted for each link on the loop. In two dimensions, use of the asymptotics of $T_J(\beta)$ leads to the area law at large β (strong coupling) [28]. In three dimensions, the extra Ds along the link give rise to $9j$-symbols, and the asymptotic behavior of the Wilson loop has not been calculated explicitly.

The formulation of spin foam models which seems the most tractable for the calculation of Wilson loops is the one of Ref. [30]. (Their expressions are essentially identical to those written down earlier by Caselle, D'Adda and Magnea [10,3] (see also [31]). In the absence of a boundary, their action can be written as (see Eq. (13))

$$S=\sum_f\ {\rm Tr}[B_f(t)U_f(t)]\ \ , \tag{73}$$

where the sum is over triangular hinges, f, $U_f(t)$ is the product of rotation matrices linking the coordinate frames of the tetrahedra and four-simplices around the hinge and $B_f(t)$ is a bivector for the hinge, defined as the dual of $\Sigma_f(t)$. This in turn is the integral over the triangle f of the two-form $\Sigma(t)=e(t)\wedge e(t)$, formed from the vierbein in tetrahedron t. The action is independent of which tetrahedron is regarded as the initial one in the path around the hinge. There is a slight subtlety in the definition of $U_f(t)$, as the basic rotation variables are taken to be V_{tv}, which relates the frame in tetrahedron t to that in 4-simplex v, of which t is a face, which is crossed in the path around hinge$f$. Then

$$U_f(t)=V_{tv_1}V_{v_1t_1}...V_{v_nt}\ \ . \tag{74}$$

The action is sufficiently similar to that used by us in an earlier paper [3], that we may take over the formalism for calculating Wilson loops from there. The integration over the Vs, which are elements of $SO(4)$ in the Euclidean case, proceeds exactly as in [3], and the same problem arises with the unmodified action of [30], as the bivector B is traceless. Therefore the definition of the action and of the gravitational Wilson loop has to be modified by an addition of ϵI_4, as in [3], which again does not affect the value of the action. The results obtained are equivalent to those in our earlier paper, which indicates that the area law also holds for this formulation of spin foam models.

5 Other discrete models of quantum gravity

We now consider very briefly various other approaches to discrete quantum gravity and the possibility of evaluating the expectation values of gravitational Wilson loops in them. Kaku [32] has proposed a lattice version of conformal gravity, with action

$$S = \sum_n \epsilon^{\mu\nu\alpha\beta} \; \mathrm{Tr}[\gamma_5 \, P_{\mu\nu}(n) \, P_{\alpha\beta}(n)] \quad , \tag{75}$$

where $P_{\mu\nu}(n)$ gives the curvature round a plaquette and is related to the Us in our previous equations, with $U_\mu(n)$ given in terms of the $O(4,2)$ generators. The strong coupling expansion of the partition function is given by

$$Z = \int [dU][d\lambda] \sum_m \frac{1}{m!} \left[\frac{1}{\beta} \sum_n \epsilon^{\mu\nu\alpha\beta} \; \mathrm{Tr}[\gamma_5 \, P_{\mu\nu}(n) \, P_{\alpha\beta}(n)] \right]^m \exp\left\{ i\lambda^{a\mu\nu} \, \mathrm{Tr}[\gamma^a (1+\gamma_5) P_{\mu\nu}(n)] \right\} \quad , \tag{76}$$

where the last term is included to impose the zero-torsion constraint. The analytic calculation of Wilson loops is complicated considerably by the presence of this constraint. If it is ignored, and Wilson loops defined as a product of Us round the loop as usual, then comparison with other calculations suggests that an area law will be obtained. (The calculations are very similar to those of Ref. [15] if one assumes a form for the $O(4,2)$ integrals as in his paper. The γ_5s disappear in the process of evaluating the basic building blocks.) Again the caveats mentioned at the beginning of the paper in comparing the Lorentzian to the compact (Euclidean) case, and the ensuing differences in the group theoretic structures as they relate to the Haar measure, apply here as well.

Rather than considering conformal gravity, Tomboulis [33] has formulated a lattice version of the general higher derivative gravitational action in order to prove unitarity. He uses the gauge group $O(4)$ and considers vierbeins coupled as "additional matter fields", as in Mannion and Taylor [11] and Kondo [12], together with further auxiliary fields. After including reflections in order to preserve discrete rotation and reflection symmetry on the lattice, he squares and then takes a square root, to ensure scalar, rather than pseudoscalar, properties in the continuum limit, as in [18]. A torsion constraint is also necessary here. As in formulations discussed earlier, these features make calculations very complicated.

Finally the authors of Ref. [14] have presented a unified treatment of Poincaré, de Sitter and conformal gravity on the lattice. This shares many features with the formulations already described, so we will not discuss it further here. The main difference is that the lattice vierbein field is defined on the lattice links rather than at the vertices. The formulation is reflection positive, but the mode doubling problem seems to persist, as seen form the expansion about a flat background.

Causal dynamical triangulations [34] are based on the action of Regge Calculus, but the approach differs in that all simplices have identical spacelike edges and identical timelike edges, and the discrete path integral involves summing over triangulations. In this case it is not clear how to use the methods discussed here and in [3], which are based on the invariant Haar measure for continuous rotation matrices, since this formulation does not contain explicitly continuous degrees of freedom which could be used for such purpose.

The proposed formulation of Weingarten [35], based on squares, cubes and hypercubes, rather than simplices, involves six-index complex variables corresponding to cubes, so although it is possible to define a large planar loop, it is not clear how to evaluate a Wilson loop, except in the special case when the parameter ρ (the coefficient of the term in the action which gives the contribution from the boundaries of the 4-cells) is set equal to zero, which seems to correspond to the unphysical case of infinite cosmological constant.

A more radical approach to discrete quantum gravity, in which the ingredients are a set of points and the causal ordering between them, is known as causal sets. Recent progress includes a calculation of particle propagators from discrete path integrals [36]. In this formulation, it is not clear how to define a (closed) Wilson loop connecting points which are not causally related, and defining a near *planar* loop is also a problem here.

6 Effects of Scalar Matter Fields

In the next four sections, we consider whether the presence of matter affects the area law behavior of gravitational Wilson loops in the strong coupling limit. For each type of matter, we first describe briefly its transcription to the lattice [4].

A scalar field can be introduced as the simplest type of dynamical matter that can be coupled invariantly to gravity. In the continuum the scalar action for a single component field $\phi(x)$ is usually written as

$$I[g,\phi] = \tfrac{1}{2} \int dx \sqrt{g} \, [\, g^{\mu\nu} \, \partial_\mu \phi \, \partial_\nu \phi + (\, m^2 + \xi \, R \,) \, \phi^2 \,] + \dots \tag{77}$$

where the dots denote scalar self-interaction terms. Thus, for example, a scalar field potential $U(\phi)$ could be added containing quartic field terms, whose effects could then be of interest in the context of cosmological models where spontaneously broken symmetries play an important role. The dimensionless coupling ξ is arbitrary; two special cases are the minimal ($\xi = 0$) and the conformal ($\xi = \frac{1}{6}$) coupling case. In the following we shall mostly consider the case $\xi = 0$. It is straightforward to extend the treatment to the case of an N_s-component scalar field ϕ^a with $a = 1, ..., N_s$.

One way to proceed is to introduce a lattice scalar ϕ_i defined at the vertices of the simplices. The corresponding lattice action can then be obtained through a procedure by which the original continuum metric is replaced by the induced lattice metric. Within each n-simplex one defines a metric

$$g_{ij}(s) \; = \; e_i \cdot e_j \;\; , \tag{78}$$

with $1 \leq i, j \leq n$, and which in the Euclidean case is positive definite. In components one has $g_{ij} = \eta_{ab} \, e_i^a e_j^b$. In terms of the edge lengths $l_{ij} \; = \; |e_i - e_j|$, the metric is given by

$$g_{ij}(s) \; = \; \tfrac{1}{2} \left(l_{0i}^2 + l_{0j}^2 - l_{ij}^2 \right) \;\; . \tag{79}$$

The volume of a general n-simplex is then given by

$$V_n(s) \; = \; \frac{1}{n!} \sqrt{\det g_{ij}(s)} \;\; . \tag{80}$$

To construct the lattice action for the scalar field, one then performs the replacement

$$\begin{aligned} g_{\mu\nu}(x) \quad &\longrightarrow \quad g_{ij}(s) \\ \partial_\mu \phi \, \partial_\nu \phi \quad &\longrightarrow \quad \Delta_i \phi \, \Delta_j \phi \end{aligned} \tag{81}$$

with the scalar field derivatives replaced by finite differences

$$\partial_\mu \phi \longrightarrow (\Delta_\mu \phi)_i \; = \; \phi_{i+\mu} - \phi_i \;\; , \tag{82}$$

where the index μ labels the possible directions in which one can move away from a vertex within a given simplex. After some re-arrangements one finds a lattice expression for the action of a massless scalar field [37,38]

$$I(l^2, \phi) \; = \; \tfrac{1}{2} \sum_{<ij>} V_{ij}^{(d)} \left(\frac{\phi_i - \phi_j}{l_{ij}} \right)^2 \;\; . \tag{83}$$

Here $V_{ij}^{(d)}$ is the dual (Voronoi) volume [39] associated with the edge ij, and the sum is over all links on the lattice. Other choices for the lattice subdivision will lead to a similar formula for the lattice action, with the Voronoi dual volumes replaced by their appropriate counterparts for the new lattice. Mass and curvature terms such as the ones appearing in Eq. (77) can be added to the action, so that a more general lattice action is of the form

$$I = \tfrac{1}{2} \sum_{<ij>} V_{ij}^{(d)} \left(\frac{\phi_i - \phi_j}{l_{ij}} \right)^2 + \tfrac{1}{2} \sum_i V_i^{(d)} \, (m^2 + \xi R_i) \phi_i^2 \tag{84}$$

where the term containing the discrete analog of the scalar curvature involves

$$V_i^{(d)} R_i \equiv \sum_{h \supset i} \delta_h V_h^{(d-2)} \sim \sqrt{g}\, R \, . \tag{85}$$

In the expression for the scalar action, $V_i^{(d)}$ is the (dual) volume associated with the site i, and δ_h the deficit angle on the hinge h. The lattice scalar action contains a mass parameter m, which has to be tuned to zero in lattice units to achieve the lattice continuum limit for scalar correlations.

When considering whether the gravitational Wilson loop area law holds for large loops in the strong coupling limit, the matter considered must be almost massless, otherwise its effects will not propagate over large distances and so cannot change the large Wilson loop result found in the pure gravity case. In fact, since the lattice Lagrangian for the scalar matter involves only factors related to the lattice metric (functions of the edge lengths) and not the connection (provided the parameter $\xi = 0$), the integration over the connections, which is what gives the area law, is unaffected.

7 Effects of Lattice Fermions

On a simplicial manifold spinor fields ψ_s and $\bar{\psi}_s$ are most naturally placed at the center of each d-simplex s. In the following we will restrict our discussion for simplicity to the four-dimensional case, and largely follow the original discussion given in [40,41]. As in the continuum, the construction of a suitable lattice action requires the introduction of the Lorentz group and its associated tetrad fields $e_\mu^a(s)$ within each simplex labeled by s. Within each simplex one can choose a representation of the Dirac gamma matrices, denoted here by $\gamma^\mu(s)$, such that in the local coordinate basis

$$\{ \gamma^\mu(s), \gamma^\nu(s) \} \;=\; 2\, g^{\mu\nu}(s) \, . \tag{86}$$

These in turn are related to the ordinary Dirac gamma matrices γ^a, which obey

$$\left\{ \gamma^a, \gamma^b \right\} \;=\; 2\, \eta^{ab} \quad , \tag{87}$$

with η^{ab} the flat metric, by

$$\gamma^{\mu}(s) \;=\; e^{\mu}_{a}(s)\,\gamma^{a} \;, \tag{88}$$

so that within each simplex the tetrads $e^{a}_{\mu}(s)$ satisfy the usual relation

$$e^{\mu}_{a}(s)\; e^{\nu}_{b}(s)\;\eta^{ab} \;=\; g^{\mu\nu}(s) \;\;. \tag{89}$$

In general the tetrads are not fixed uniquely within a simplex, being invariant under local Lorentz transformations. In the following it will be preferable to discuss the Euclidean case, for which $\eta_{ab} = \delta_{ab}$.

In the continuum the action for a massless spinor field is given by

$$I \;=\; \int dx \sqrt{g}\; \bar{\psi}(x)\,\gamma^{\mu}\, D_{\mu}\,\psi(x) \tag{90}$$

where $D_{\mu} \;=\; \partial_{\mu} + \frac{1}{2}\,\omega_{\mu ab}\,\sigma^{ab}$ is the spinorial covariant derivative containing the spin connection $\omega_{\mu ab}$. In the absence of torsion, one can use a matrix $\mathbf{U}(s', s)$ to describe the parallel transport of any vector ϕ^{μ} from simplex s to a neighboring simplex s',

$$\phi^{\mu}(s') \;=\; U^{\mu}{}_{\nu}(s', s)\,\phi^{\nu}(s) \;. \tag{91}$$

$\mathbf{U}$ therefore describes a lattice version of the connection. Indeed in the continuum such a rotation would be described by the matrix

$$U^{\mu}{}_{\nu} \;=\; \left(e^{\Gamma \cdot dx}\right)^{\mu}{}_{\nu} \tag{92}$$

with $\Gamma^{\lambda}_{\mu\nu}$ the affine connection. The coordinate increment dx is interpreted as joining the center of s to the center of s', thereby intersecting the face $f(s, s')$. On the other hand, in terms of the Lorentz frames $\Sigma(s)$ and $\Sigma(s')$ defined within the two adjacent simplices, the rotation matrix is given instead by

$$U^{a}{}_{b}(s', s) \;=\; e^{a}{}_{\mu}(s')\, e^{\nu}{}_{b}(s)\; U^{\mu}{}_{\nu}(s', s) \tag{93}$$

(this last matrix reduces to the identity if the two orthonormal bases $\Sigma(s)$ and $\Sigma(s')$ are chosen to be the same, in which case the connection is simply given by $U(s', s)_{\mu}{}^{\nu} = e_{\mu}{}^{a}\, e^{\nu}{}_{a}$). Note that it is possible to choose coordinates so that $\mathbf{U}(s, s')$ is the unit matrix for one pair of simplices, but it will not then be unity for all other pairs in the presence of curvature.

One important new ingredient is the need to introduce lattice spin rotations. Given, in d dimensions, the above rotation matrix $\mathbf{U}(s', s)$, the spin connection $\mathbf{S}(s, s')$ between two neighboring simplices s and s' is defined as follows. Consider $\mathbf{S}$ to be an element of the 2^{ν}-dimensional representation of the covering group of $SO(d)$, $Spin(d)$, with $d = 2\nu$ or $d = 2\nu + 1$, and for which S is a matrix of dimension $2^\nu \times 2^\nu$. Then $\mathbf{U}$ can be written in general as

$$\mathbf{U} \;=\; \exp\left[\,\tfrac{1}{2}\,\sigma^{\alpha\beta}\theta_{\alpha\beta}\right] \tag{94}$$

where $\theta_{\alpha\beta}$ is an antisymmetric matrix. The σ's are $\frac{1}{2}d(d-1)$ $d \times d$ matrices, generators of the Lorentz group ($SO(d)$ in the Euclidean case, and $SO(d-1,1)$ in the Lorentzian case), whose explicit form is

$$[\sigma_{\alpha\beta}]^{\gamma}{}_{\delta} \;=\; \delta^{\gamma}{}_{\alpha}\,\eta_{\beta\delta} \;-\; \delta^{\gamma}{}_{\beta}\,\eta_{\alpha\delta} \tag{95}$$

so that, for example,

$$\sigma_{13} \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{96}$$

For fermions the corresponding spin rotation matrix is then obtained from

$$\mathbf{S} \;=\; \exp\left[\,\tfrac{i}{4}\,\gamma^{\alpha\beta}\theta_{\alpha\beta}\right] \tag{97}$$

with generators $\gamma^{\alpha\beta} = \frac{1}{2i}[\gamma^\alpha, \gamma^\beta]$. Taking appropriate traces, one can obtain a direct relationship between the original rotation matrix $\mathbf{U}(s,s')$ and the corresponding spin rotation matrix $\mathbf{S}(s,s')$

$$U_{\alpha\beta} \;=\; \mathrm{Tr}\left(\mathbf{S}^{\dagger}\,\gamma_\alpha\,\mathbf{S}\,\gamma_\beta\right) / \,\mathrm{Tr}\,\mathbf{1} \tag{98}$$

which determines the spin rotation matrix up to a sign. Now, if one assigns two spinors in two different contiguous simplices s_1 and s_2, one cannot in general assume that the tetrads $e^{\mu}_{a}(s_1)$ and $e^{\mu}_{a}(s_2)$ in the two simplices coincide. They will in fact be related by a matrix $\mathbf{U}(s_2, s_1)$ such that

$$e^{\mu}_{a}(s_2) \;=\; U^{\mu}{}_{\nu}(s_2, s_1)\; e^{\nu}_{a}(s_1) \tag{99}$$

and whose spinorial representation $\mathbf{S}$ is given in Eq. (98). Such a matrix $\mathbf{S}(s_2, s_1)$ is now needed to additionally parallel transport the spinor ψ from a simplex s_1 to the neighboring simplex s_2. The invariant lattice action for a massless spinor takes therefore the form

$$I \;=\; \tfrac{1}{2} \sum_{\text{faces f(ss}')} V(f(s,s'))\,\bar{\psi}_s\,\mathbf{S}(\mathbf{U}(s,s'))\,\gamma^{\mu}(s')\,n_{\mu}(s,s')\,\psi_{s'} \tag{100}$$

where the sum extends over all interfaces $f(s,s')$ connecting one simplex s to a neighboring simplex s', $n_\mu(s,s')$ is the unit normal to $f(s,s')$ and $V(f(s,s'))$ its volume. The above spinorial action can be considered closely analogous to the lattice Fermion action proposed originally by Wilson [1] for non-Abelian gauge theories. It is possible that it still suffers from the fermion doubling problem, although the situation is less clear for a dynamical lattice [42].

It is clear that the situation with gravitational Wilson loops is a bit more complicated than in the scalar field case, since the action now contains the spin connection matrix, which is a function of the matrices U which play the role of the connection. What is more, the generators of the spin rotation matrices are in a different representation from the generators of the rotation matrices, and it seems impossible to obtain, to lowest order, a spin zero object out of the combination of two objects of spin one-half (S) and spin one (U), unless one applies the fermion contribution twice to each link, in which case a nonzero contribution can arise. We note here that if the Wilson loop were to contain a perimeter contribution, it would be of the form

$$W(C) \;\sim\; const.\; (k_m)^{L(C)} \;\sim\; \exp\left[-m_p \, L(C)\right] \tag{101}$$

where $L(C)$ is the length of the perimeter of the near-planar loop C, m_p the particle's mass, equal here to $m_p = |\ln k_m|$ for small k_m, with k_m the weight of the single link contribution from the matter particle (sometimes referred to as the hopping parameter). Area and perimeter contributions to the near-planar Wilson loop would then become comparable only for exceedingly small particle masses, $m_P \sim L(C)/\xi^2$, i.e. for Compton wavelengths comparable to a macroscopic loop size (taking $A(C) \approx L(C)^2/4\pi$).

To demonstrate the perimeter behavior (see Fig. 10), one would need to show that the matrix S on the face between simplices s and s' would have a term proportional to the corresponding $U(s, s')$, with coefficient composed of γ-matrices, thereby possibly giving a nonzero contribution to the U-integration. (This does *not* seem to be true in the infinitesimal case to lowest order, where, for example, $S(\theta_{34}) = I_4 + \frac{1}{2}\gamma_4[U_{13}(\theta_{34}) - U_{24}(\theta_{34})]$.)

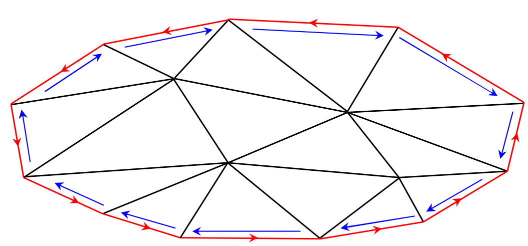

Figure 10. Illustration on how a perimeter contribution to the gravitational Wilson loop arises from matter field contributions. Note that now the arrows representing rotation matrices reside in principle in different representations.

8 Effects of Gauge Fields

In the continuum a locally gauge invariant action coupling an $SU(N)$ gauge field to gravity is

$$I_{\rm gauge} \;=\; -\frac{1}{4g^2}\int d^4x\,\sqrt{g}\,g^{\mu\lambda}\,g^{\nu\sigma}\,F^a_{\mu\nu}\,F^a_{\lambda\sigma} \tag{102}$$

with $F^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$ and $a, b, c = 1, \ldots, N^2-1$. On the lattice one can follow a procedure analogous to Wilson's construction on a hypercubic lattice, with the main difference that the lattice is now possibly simplicial. Given a link ij on the lattice one assigns group elements U_{ij}, with each U an $N\times N$ unitary matrix with determinant equal to one, and such that $U_{ji} = U_{ij}^{-1}$. Then with each triangle (plaquette) Δ, labeled by the three vertices ijk, one associates a product of three U matrices,

$$U_\Delta \;\equiv\; U_{ijk} \;= U_{ij}\,U_{jk}\,U_{ki}\;. \tag{103}$$

The discrete action is then given by [37]

$$I_{\rm gauge} \;=\; -\frac{1}{g^2}\sum_\Delta V_\Delta\,\frac{c}{A^2_\Delta}\,{\rm Re}\,\left[{\rm Tr}(1\;-\;U_\Delta)\right] \tag{104}$$

with 1 the unit matrix, V_Δ the 4-volume associated with the plaquette Δ, A_Δ the area of the triangle (plaquette) Δ, and c a numerical constant of order one. If one denotes by $\tau_\Delta = cV_\Delta/A_\Delta$ the $d-2$-volume of the dual to the plaquette Δ, then the quantity

$$\frac{\tau_\Delta}{A_\Delta} \;=\; c\,\frac{V_\Delta}{A^2_\Delta} \tag{105}$$

is simply the ratio of this dual volume to the plaquettes area. The edge lengths l_{ij} and therefore the metric enter the lattice gauge field action through these volumes and areas. One important property of the gauge lattice action of Eq. (104) is its local invariance under gauge rotations g_i defined at the lattice vertices., One can further show that the discrete action of Eq. (104) goes over in the lattice continuum limit to the correct Yang-Mills action for manifolds that are smooth and close to flat.

Regarding the effects of gauge fields on the gravitational Wilson loop one can make the following observation. Since the gauge action contains no factors related to the lattice connection, the Wilson loop area law for large gravitational loops will remain unaffected. In particular this will be true for the photon (which in principle could have led to important long-distance effects, since it is massless).

9 Effects from a Lattice Gravitino

Supergravity in four dimensions naturally contains a spin-3/2 gravitino, the supersymmetric partner of the graviton. In the case of $\mathcal{N} = 1$ supergravity these are the only 2 degrees of freedom present. The action contains, beside the Einstein-Hilbert action for the gravitational degrees of freedom, the Rarita-Schwinger action for the gravitino, as well as a number of additional terms (and fields) required to make the action manifestly supersymmetric off-shell [43].

A spin-3/2 Majorana fermion in four dimensions corresponds to self-conjugate Dirac spinors ψ_μ, where the Lorentz index $\mu = 1 \ldots 4$. In flat space the action for such a field is given by the Rarita-Schwinger term

$$\mathcal{L}_{\rm RS} \; = - \, \tfrac{1}{2} \, \epsilon^{\alpha\beta\gamma\delta} \, \psi_\alpha^T \, C \, \gamma_5 \, \gamma_\beta \, \partial_\gamma \, \psi_\delta \tag{106}$$

where C is the charge conjugation matrix. Locally the action is invariant under the gauge transformation

$$\psi_\mu(x) \; \rightarrow \; \psi_\mu(x) \, + \, \partial_\mu \, \epsilon(x) \tag{107}$$

where $\epsilon(x)$ is an arbitrary local Majorana spinor.

The construction of a suitable lattice action for the spin-3/2 particle proceeds in a way that is rather similar to what one does in the spin-1/2 case. On a simplicial manifold the Rarita-Schwinger spinor fields $\psi_\mu(s)$ and $\bar{\psi}_\mu(s)$ are most naturally placed at the center of each d-simplex s. Like the spin-1/2 case, the construction of a suitable lattice action requires the introduction of the Lorentz group and its associated vierbein fields $e^a_\mu(s)$ within each simplex labeled by s, together with representations of the Dirac gamma matrices (see the previous discussion of Dirac fields).

Now in the presence of gravity the continuum action for a massless spin-3/2 field is given by

$$I_{3/2} \; = \; -\tfrac{1}{2} \int dx \sqrt{g} \; \epsilon^{\mu\nu\lambda\sigma} \; \bar{\psi}_\mu(x) \, \gamma_5 \, \gamma_\nu \, D_\lambda \, \psi_\sigma(x) \tag{108}$$

with the Rarita-Schwinger field subject to the Majorana constraint $\psi_\mu = C \bar{\psi}_\mu(x)^T$. Here the covariant derivative is defined as

$$D_\nu \psi_\rho = \partial_\nu \psi_\rho - \Gamma^\sigma_{\nu\rho} \, \psi_\sigma + \tfrac{1}{2} \, \omega_{\nu ab} \, \sigma^{ab} \, \psi_\rho \tag{109}$$

and involves *both* the standard affine connection $\Gamma^\sigma_{\nu\rho}$, as well as the vierbein connection

$$\begin{aligned} \omega_{\nu \, ab} = \; & \tfrac{1}{2} \quad [e_a^{\ \mu} (\partial_\nu \, e_{b\mu} - \partial_\mu \, e_{b\nu}) + e_a^{\ \rho} e_b^{\ \sigma} (\partial_\sigma \, e_{c\rho}) \, e^c_{\ \nu}] \\ & - \quad (a \leftrightarrow b) \end{aligned} \tag{110}$$

with Dirac spin matrices $\sigma_{ab} = \frac{1}{2i}\,[\gamma_a, \gamma_b]$, and $\epsilon^{\mu\nu\rho\sigma}$ the usual Levi-Civita tensor, such that $\epsilon_{\mu\nu\rho\sigma} = -g\,\epsilon^{\mu\nu\rho\sigma}$.

It is easiest to just consider two neighboring simplices s_1 and s_2, covered by a common coordinate system x^μ. When the two vierbeins in s_1 and s_2 are made to coincide, one can then use a common set of gamma matrices γ^μ within both simplices. Then (in the absence of torsion) the covariant derivative D_μ in Eq. (108) reduces to just an ordinary derivative. The fermion field $\psi_\mu(x)$ within the two simplices can then be suitably interpolated, and one obtains a lattice action expression very similar to the spinor case. One can then relax the condition that the vierbeins $e_a^\mu(s_1)$ and $e_a^\mu(s_2)$ in the two simplices coincide. If they do not, then they will be related by a matrix $\mathbf{U}(s_2, s_1)$ such that

$$e_a^\mu(s_2) \;=\; U^\mu{}_\nu(s_2, s_1)\; e_a^\nu(s_1) \tag{111}$$

and whose spinorial representation $\mathbf{S}$ was given previously in Eq. (98). But the new ingredient in the spin-3/2 case is that, besides requiring a spin rotation matrix $\mathbf{S}(s_2, s_1)$, now one also needs the matrix $U_\mu^\nu(s, s')$ describing the corresponding parallel transport of the Lorentz *vector* $\psi_\mu(s)$ from a simplex s_1 to the neighboring simplex s_2. The invariant lattice action for a massless spin-3/2 particle takes therefore the form

$$I \;=\; -\tfrac{1}{2} \sum_{\text{faces f(ss}')} V(f(s,s'))\,\epsilon^{\mu\nu\lambda\sigma}\;\bar\psi_\mu(s)\;\mathbf{S}(\mathbf{U}(s,s'))\,\gamma_\nu(s')\,n_\lambda(s,s')\,U^\rho{}_\sigma(s,s')\,\psi_\rho(s') \tag{112}$$

with

$$\bar\psi_\mu(s)\;\mathbf{S}(\mathbf{U}(s,s'))\,\gamma_\nu(s')\,\psi_\rho(s') \;\equiv\; \bar\psi_{\mu\,\alpha}(s)\,S^\alpha{}_\beta(\mathbf{U}(s,s'))\,\gamma_\nu{}^\beta{}_\gamma(s')\,\psi_\rho^\gamma(s') \tag{113}$$

and the sum $\sum_{\text{faces f(ss}')}$ extends over all interfaces $f(s,s')$ connecting one simplex s to a neighboring simplex s'. When compared to the spin-1/2 case, the most important modification is the second rotation matrix $U^\nu{}_\mu(s,s')$, which describes the parallel transport of the fermionic vector ψ_μ from the site s to the site s', which is required in order to obtain locally a Lorentz scalar contribution to the action.

In this case again one expects the Wilson loop to follow a perimeter law, as in the spin one-half case of Eq. (101), because the matter action explicitly contains factors of U which will contribute when the Us and Ss around the loop are integrated over, which of course requires that one also take into account the spin connection matrices. These add complexity but are not expected, due to the nature of the interaction, to change the answer. The same general considerations then apply as in the spin-1/2 case: the perimeter contribution to the gravitational Wilson loop can significantly modify the area law result only if the corresponding particle mass is exceedingly small.

10 Possible Physical Consequences

In the previous sections we presented evidence for an area law behavior for a variety of different lattice discretizations of gravity, all studied in the strong coupling limit. We have not pursued yet the computation of higher order terms in the strong coupling expansion, which could be done. But we believe that the basic result, which we expect to be geometric in character, could be further tested by numerical means throughout the whole strong coupling phase. If the analogy with non-Abelian gauge theories and the concept of universal critical behavior continues to hold in Euclidean gravity, then one would expect that the area law result would hold not just at strong coupling but instead throughout the whole strong coupling region, up the nontrivial ultraviolet fixed point, if one can be found in the relevant lattice regularized theory, of which we have given here a few examples. Furthermore the $SO(4)$ lattice model of Sec. (2) is one example where the analogy with Wilson's non-Abelian gauge theory on the lattice is clearly seen as more than just superficial resemblance. The evidence for an ultraviolet fixed point for gravity has recently been reviewed in [4] and will not be repeated here. Our results and similar related lattice results could then be tested further in the case of gravity, for example, by numerical means, regarding their universal character and scaling behavior in the vicinity of the nontrivial fixed point.

In this section we wish to briefly discuss instead a possible physical interpretation of the Euclidean gravitational Wilson loop result, along the lines of the proposal in Refs. [3] and [4], and thus in terms of its relationship to a large-scale average curvature. Note that contrary to some earlier statements in the literature, the Wilson loop in gravity does not provide any useful information about the static gravitational potential [6-8]. The arguments presented below should therefore be taken with some clear caveats, namely, that (i) the results have been derived from the Euclidean theory, whose relationship to the Lorentzian case remains to be explored, that (ii) they assume concepts of universality of critical behavior which nevertheless are known to apply to just about any other quantum field theory except possibly gravity, and finally (iii) that it is assumed that the phase structure of Euclidean lattice gravity is such that a nontrivial fixed point can be found (which is not obvious at this point for some of the lattice models discussed previously in this paper).

Having then ascertained with some degree of confidence that in a number of different, and quite unrelated, Euclidean lattice discretizations of gravity the gravitational loop follows an area law at least for sufficiently strong coupling G, which we choose to write here as

$$< W(C) > \underset{A \to \infty}{\sim} \exp\left(-\, A_C / \xi^2\right) \qquad (114)$$

with ξ determined by scaling and dimensional arguments to be the unique nonperturbative gravitational correlation length, let us now turn to a possible physical interpretation of the result. Here the formula of Eq. (114), inspired by the analogy to gauge theories which gives Eq. (5) and by the well-established universality of critical behavior, is expected to summarize, at least for the purpose of our argument, the behavior of the gravitational Wilson loop throughout the whole strong coupling domain. In the same way that the analogous textbook result, Eq. (5), in a sense summarizes the long distance behavior of the Wilson loop for non-Abelian gauge theories in terms of the only admissible renormalization group invariant scale. Here we will therefore explore some possible ramifications of the above Ansatz in the context of the nontrivial fixed point in G, or asymptotic safety, scenario for quantum gravity, recently reviewed, for example, in Ref. [4]. This is perhaps not the only possible scenario, but it is the one we are most familiar with, and in our view also the most credible one at this point, supported by the $2+\epsilon$ expansion for gravity, by the nonperturbative Regge lattice calculations, and by the analogy with the much simpler but very well understood perturbatively nonrenormalizable nonlinear sigma model.

In particular, we intend to explore here briefly, following closely the arguments of Ref. [3], the connection of the lattice result of Eq. (114) to a semiclassical picture, describing the properties of curvature on very large, macroscopic distance scales. The procedure followed here and in [3] is simple and quite analogous to the original procedure proposed by Wilson for gauge theories [1]: the quantum Wilson loop average is computed in the full theory, and the answer is then compared to the result obtained when the path integral is dominated by a single classical configuration. In above quoted expression, ξ is therefore intended to be the renormalization group invariant quantity obtained by integrating the β-function for the Newtonian coupling G,

$$\xi^{-1}(G) \;=\; \text{const.}\; \Lambda \, \exp\left(- \int^{G} \frac{dG'}{\beta(G')} \right) \tag{115}$$

with Λ the ultraviolet cutoff (and thus analogous to Eq. (8) for gauge theories). In the vicinity of the ultraviolet fixed point at G_c

$$\beta(G) \,\equiv\, \mu \frac{\partial}{\partial \mu} G(\mu) \;\underset{G \to G_c}{\sim}\; \beta'(G_c)\,(G-G_c) \,+\, \dots \;\;, \tag{116}$$

which gives

$$\xi^{-1}(G) \propto \Lambda \,|\, (G-G_c)/G_c \,|^{\nu} \;, \tag{117}$$

with a correlation length exponent $\nu = -1/\beta'(G_c)$. In particular the correlation length $\xi(G)$ is related to the bare Newtonian coupling G, and diverges, in units of the cutoff Λ, as one approaches the fixed point at G_c. Thus for a bare G very close to G_c the two scales, Λ and ξ^{-1} can be vastly different. Furthermore the result of Eq. (114) was derived from the lattice theory of gravity in the strong coupling limit $G \rightarrow \infty$. But one would expect, based on general scaling arguments and the analogy with non-Abelian gauge theories, see Eq. (5), that such a behavior would persist throughout the whole strong coupling phase $G > G_c$, all the way up to the nontrivial ultraviolet fixed point at G_c. This is indeed what happens in non-Abelian gauge theories and spin systems such as the nonlinear sigma model: the only scale determining the nontrivial scaling properties in the vicinity of the fixed point is ξ; the corresponding behavior is known as universal renormalization group scaling.

As discussed at the beginning of this paper and in Refs. [3,4], the rotation matrix appearing in the gravitational Wilson loop can be related classically to a well-defined classical physical process, one in which a vector is parallel transported around a large loop, and at the end is compared to its original orientation. Then the vector's rotation is directly related to some sort of average curvature enclosed by the loop; the total rotation matrix $\mathbf{U}(C)$ is given by a path-ordered ($\mathcal{P}$) exponential of the integral of the affine connection $\Gamma^{\lambda}_{\mu\nu}$, as in Eq. (1). In a semiclassical description of the parallel transport process of a vector around a very large loop, one can reexpresses the connection in terms of a suitable coarse-grained, semiclassical slowly varying Riemann tensor, as in Eq. (2). Since the rotation is small for weak curvatures, one has for a macroscopic observer

$$U^{\alpha}{}_{\beta}(C) \;\sim\; \Big[1 \,+\, \tfrac{1}{2} \int_{S(C)} R^{\,\cdot}{}_{\cdot\,\mu\nu} \, A^{\mu\nu}_{C} \,+\, \dots \Big]^{\alpha}{}_{\beta} \;\; . \tag{118}$$

At this stage one can compare the above semiclassical expression to the quantum result of Eqs. (44), (66) and (114), and in particular one would like to relate the coefficients of the area terms. Since one expression [Eq. (118)] is a matrix and the other [Eq. (114)] is a scalar, one needs to take the trace after first contracting the rotation matrix with $(B_C \,+\, \epsilon\, I_4)$, as in our second definition of the Wilson loop of Eq. (16), giving

$$W(C) \,\sim\, \mathrm{Tr} \left((B_C \,+\, \epsilon\, I_4) \, \exp \left\{ \tfrac{1}{2} \int_{S(C)} R^{\,\cdot}{}_{\cdot\,\mu\nu} \, A^{\mu\nu}_{C} \right\} \right) \;\; . \tag{119}$$

Next, following Ref. [3], it is advantageous to consider the lattice analog of a background classical manifold with constant or near-constant large-scale curvature,

$$R_{\mu\nu\lambda\sigma} = \tfrac{1}{3} \, \lambda \, (g_{\mu\nu} \, g_{\lambda\sigma} \,-\, g_{\mu\lambda} \, g_{\nu\sigma}) \tag{120}$$

so that here one can set for the curvature tensor

$$R^{\,\alpha}{}_{\beta\,\mu\nu} \;=\; \bar{R} \, B^{\,\alpha}{}_{\beta} \, B_{\mu\nu}, \tag{121}$$

where $\bar{R}$ is some average curvature over the loop, and the area bivectors B here will be taken to coincide with B_C. The trace of the product of $(B_C + \epsilon I_4)$ with this expression then gives $Tr(\bar{R}\ B_C^2\ A_C) = -2\ \bar{R}\ A_C$. This is to be compared with the linear term from the other exponential expression, $-A_C/\xi^2$. Thus the average curvature is computed to be of the order

$$\bar{R} \sim +1/\xi^2 \tag{122}$$

at least in the small $k = 1/8\pi G$ limit. An equivalent way of phrasing the last result makes use of the classical field equations in the absence of matter $R = 4\lambda$. Then the rather surprising result emerges that $1/\xi^2$ should be identified, up to a constant of proportionality of order one, with the observed scaled cosmological constant λ_{obs},

$$\lambda_{obs} \simeq +\frac{1}{\xi^2}\ . \tag{123}$$

The latter can then be regarded either as a measure of the vacuum energy, or of the intrinsic curvature of the vacuum. It would seem therefore that a direct calculation of the gravitational Wilson loop, within the boundaries of our limited strong coupling results, could provide a direct insight into whether the manifold is de Sitter or anti-de Sitter *at large distances*. Moreover, in the case of lattice gravity at strong coupling, as has been shown in this work, it seems virtually impossible to obtain a negative sign in Eqs. (122) or (123), which would then suggest that Euclidean quantum gravity can only give a *positive* cosmological constant at large distances. (Again, the analogy with non-Abelian gauge theories comes to mind, where one has for the nonperturbative gluon condensate $< F_{\mu\nu}^2 > \sim 1/\xi^4$, where ξ is the nonperturbative QCD correlation length, $\xi_{QCD}^{-1} \sim \Lambda_{\overline{MS}}$; the analog of the vacuum condensate in non-Abelian field gauge theories is then naturally seen here as the vacuum expectation value of the curvature).

Let us explore this last point further. At first it would seem, from the nontrivial ultraviolet fixed point, or asymptotic safety, scenario point of view, [4] that in principle the scale ξ could take any value, including very small ones, based on the naive estimate $\xi \sim l_P$, where l_P is the Planck length whose magnitude is comparable to the (inverse of the) ultraviolet cutoff Λ. The last choice would of course preclude any observable quantum effects in the foreseeable future. But the relationship between ξ and large-scale curvature, or more precisely between ξ and λ_{obs}, arising out of the specific properties of the gravitational Wilson loop as proposed in Eqs. (122) and (123), opens up a new

[4] A nontrivial ultraviolet fixed point in fact implies the existence of such a new nonperturbative scale, which arises as an integration constant from the Callan-Symanzik renormalization group equations close to the UV fixed point [4], in the same way that a similar scale arises out of the renormalization group equations for asymptotically free Yang-Mills theories.

possibility. Namely a very large, cosmological value for $\xi \sim 10^{28} cm$, given the present observational bounds on λ_{obs}. Closely related possibilities exist, such as an identification of ξ with the Hubble constant as measured today, $\xi \simeq 1/H_0$; since this quantity is presumably time-dependent, a possible scenario is one in which $\xi^{-1} = H_\infty = \lim_{t\to\infty} H(t)$, with $H_\infty^2 = \frac{1}{3}\lambda_{obs}$. This in turn would suggest a number of other related observations, such as the fact that for distances $r \ll \xi$ one still resides in the short distance regime, where correlations are still expected to behave as power laws; significant deviations from classical gravity would then arise only for distance comparable or greater than ξ.

Finally we note that another physical consequence arises from the tentative identification of ξ with $1/\sqrt{\lambda_{obs}}$: as in gauge theories, one expects ξ to determine the scale dependence of the effective Newton's constant $G(\mu)$ appearing in the field equations, where the latter is obtained, for example, from solving the renormalization group equations for G, Eqs. (115) and (116). As discussed in [44], a running of the gravitational constant of the type discussed in [7] is best expressed in a fully covariant formulation, such as an effective classical, but nonlocal, set of field equations of the type

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = 8\pi G(\Box) T_{\mu\nu} \tag{124}$$

with $\lambda \simeq 1/\xi^2$, and $G(\Box)$ the running Newton's constant

$$G \to G(\Box) \tag{125}$$

with the running given by

$$G(\Box) = G_c \left[1 + a_0 \left(\frac{1}{\xi^2 \Box} \right)^{\frac{1}{2\nu}} + \dots \right] , \tag{126}$$

and $a_0 \simeq 42 > 0$ and $\nu \simeq 1/3$ [45]. G_c in the above expression should be identified to a first approximation with the laboratory scale value $\sqrt{G_c} \sim 1.6 \times 10^{-33} cm$ [44,4]. The running of G can then be worked out in detail for specific coordinate choices, and in the static isotropic case one finds a gradual slow increase in G with distance, in accordance with the formula

$$G \to G(r) = G \left(1 + \frac{a_0}{3\pi} m^3 r^3 \ln \frac{1}{m^2 r^2} + \dots \right) \tag{127}$$

in the regime $r \gg 2 M G$, where $2MG$ is the horizon radius, and $m \equiv 1/\xi$. The results of Eqs. (122) and (123) then open up a new possibility, and would suggest that the scale entering the quantum scale dependence of $G(r)$ is not of the order of the Planck length, but instead a very large-scale, comparable to the observed cosmological constant, $\xi = 1/\sqrt{\lambda_{obs}}$.

11 Conclusions

From our study of Wilson loops, where defined and calculable, in all theories of Euclidean discrete gravity that we have found, it seems that the area law holds for large loops in the strong coupling domain. This would suggest that one can infer, as in [3], that a universal prediction of strongly coupled Euclidean gravity without matter is that the scaled cosmological constant is positive. We have argued that the basic result, which appears to be geometric in character as in the better understood case of non-Abelian gauge theories, could be further tested by numerical means throughout the whole strong coupling phase. If the analogy with non-Abelian gauge theories and the concept of universal critical behavior continues to hold in Euclidean gravity, then one would expect that the area law result would hold not just at strong coupling but instead throughout the whole strong coupling region, up the nontrivial ultraviolet fixed point, if one can be found. But we wish to emphasize here again that the arguments connecting the area law result in the Euclidean theory to the physical scaled cosmological constant should be taken with some clear caveats, namely that they have been derived from the Euclidean theory, that they assume concepts of universality of critical behavior, and finally that they assume that the phase structure of various Euclidean lattice gravity models is such that a nontrivial fixed point can be found in all of them. Nevertheless we believe the value of our results might lie in the fact that they open the possiblity of (a) providing a set of explicit, unambiguous and presumably universal predictions which could be tested by numerical means, and (b) suggesting a new physical connection between two at first seemingly unrelated quantities, namely the scale for the running of the coupling G in the asymptotic safety scenario and the cosmological constant λ, leading possibly to a number of testable cosmological and astrophysical predictions.

We wish to make here a number of additional comments relating to the interpretation of the Euclidean lattice results. The effect on the Wilson loop of adding matter coupled to gravity is less clear-cut, although it is only massless or almost massless matter which propagates to sufficiently large distances to affect large gravitational Wilson loops. In that case, scalar matter and gauge fields (in particular the photon) do not affect the area law. For very low mass fermions (e.g. neutrinos), it is possible that the coupling gives rise to a perimeter contribution, which could replace the area law for suitable ratios of coupling constants, but this seems unlikely. Similarly, the lattice gravitino could produce a perimeter law. These possibilities will be investigated in future work. Numerical simulations of simplicial lattice gravity could provide vital clues here [45]. Numerical simulations in general require a general definition of the Wilson loop applicable to any geometry, a subject which has been discussed previously in a number of places, and which we will repeat here for completeness.

The argument relating the quantum vacuum expectation values of a gravitational Wilson loop to the corresponding classical quantity, namely the amount of rotation a vector experiences when parallel transported around a closed loop in a given classical background geometry, requires, as in ordinary gauge theories, that a connection be made between the full quantum domain dominated by large short distance field fluctuations on the one hand, and the semiclassical domain of smooth fields at large distances on the other. Originally it was thought that the gravitational Wilson loop, as computed in most of the original papers on hypercubic lattice gravity referred to in this work, would give information about the static potential, but this was shown later by Modanese to be incorrect[6]. Instead, the gravitational Wilson loop is now understood to provide physical information about the large-scale curvature of the fluctuating geometry in question [7,8].

Initially the discussion of the gravitational Wilson loop in the quoted papers focused on the weak field case, where the expectation of the loop is clearly well-defined. The calculation is then technically quite similar to the perturbative calculation of a square Wilson loop in non-Abelian gauge theories. A flat or near-flat background geometry is allowed to fluctuate locally, and a vector is parallel transported around a circular loop. An integration over the fluctuating part of the metric then yields an explicit and well-defined expression for the gravitational Wilson loop, suitably defined as a trace of the holonomy of the Levi-Civita connection. The limiting factor for such a calculation, already recognized at the time by the quoted author, is of course the fact that higher order radiative corrections are just as important as the leading contribution, due to the perturbative nonrenormalizability in four dimensions.

Nevertheless the gravitational Wilson loop for a regulated closed circular loop in flat space, or in a given near-flat background geometry (such as one that would arise from having to satisfy the classical field equations with a nonvanishing small classical cosmological constant term) is a completely well-defined object, to all orders in the weak field expansion, and in any dimension $d > 2$.

Similarly, one can argue based on semiclassical arguments, such as the ones advocated for example by Hartle [46,47] in conjunction with the emergence of a classical domain out of an underlying fluctuating geometry, that all which is required to define a gravitational Wilson loop is the existence of a smooth near-classical (and four-dimensional) geometry at very large distances, for which the parallel transport of a vector around a circular loop is well-defined according to classical general relativity (one could of course define loops of arbitrary sizes and shapes, but for the present argument a large circular loop of length L will suffice). Clearly such a definition breaks down if the notion of a circular loop cannot be stated, in which case though the geometry is not near flat at large distances, and no physically acceptable theory of gravity is recovered in this regime, making the whole exercise rather pointless. It would seem therefore that the computation of a Wilson loop in the lattice theory of gravity only makes sense if a semiclassical space-time is recovered at large distances, making a definition of a circular loop meaningful.

Nevertheless, irrespective of whether semiclassical spacetime is recovered at large distances, such a gravitational Wilson loop can still be defined in rather general terms. One way to proceed is to focus on a set point P located on a given fluctuating manifold, and consider a one-parameter family of geodesics emanating from that point, all lying in a given 2d plane sited at the point in question. Following the geodesics out to a distance R one obtains a suitable path over which to evaluate the trace of the holonomies; repeating the same procedure for many points and many field configurations one then would obtain a quantum average for the same quantity. The extent to which the corresponding loops are flat can then be determined by comparing the radius R with the length of the loop perimeter L; in a near-flat geometry at very large distances one would expect for large R and L the relationship $R \approx L/(2\pi)$.

A slightly more general way of defining a planar Wilson loop can be given as follows. Consider a point P on a d-dimensional manifold, and construct the $d-1$ dimensional surface around the point P defined as the locus of all points situated at a fixed geodesic R distance from P. Next consider the equator on this submanifold, defined as the set of all points equidistant from the point in question and its antipode (the point most distant, within the submanifold, from the chosen point). Its dimension will be $d-2$, making it suitable in three dimensions as a parallel transport path, with a given calculable length L, thus giving a useful and unambiguous definition of the gravitational Wilson loop in three dimensions. In dimensions higher than three the above geometric procedure needs to be iterated a sufficient number of times until the desired maximal near-planar one-dimensional path is obtained. Thus in four dimensions a point and its antipode need to be picked again within the compact submanifold of dimensions $d-2$, resulting in a one-dimensional Wilson loop path spanning the resulting equator (again defined as the locus of the points equidistant from the point picked and its antipode).

It is clear from the construction that many equivalent loops can be defined locally in this way. Of course in two dimensions only one such loop, centered at P and of size R, exists for a given fluctuating manifold. In three dimensions, given an origin P, there is on the other hand a two-parameter family of near-planar loops of size R associated with the center point in question, and in accordance with the loop's possible orientation. This would be the set of all great circles on a 2-sphere, parametrized by two angles. Then in four dimensions the corresponding statement is that given a point P, a three-parameter family of near-planar loops of size R centered at P can be constructed in the way described above.

Finally we should point out that if a timelike coordinate can somehow be defined, then the consideration of the gravitational Wilson could in principle be restricted, for example, to spacelike loops only, thus effectively reducing the dimension of the geometrical problem by one.

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