On the "Spin-Connection Foam" Picture of Quantum Gravity from Precanonical Quantization

Igor V. Kanatchikov*

School of Physics and Astronomy, University of St Andrews,
St Andrews KY16 9SS, Scotland
**E-mail: ik25@st-andrews.ac.uk*

Precanonical quantization uses a different generalization of Hamiltonian formalism to field theory, the so-called De Donder–Weyl (DW) theory, which does not require a space-time decomposition and treats the space-time variables on the equal footing. Quantum dynamics is described by precanonical wave functions on the finite dimensional space of field coordinates and space-time coordinates, which satisfy a partial derivative precanonical Schrödinger equation. Based on the analysis of constraints within the De Donder–Weyl Hamiltonian formulation of Einstein-Palatini vielbein formulation of GR and quantization of generalized Dirac brackets defined on differential forms, we derived the covariant precanonical Schrödinger equation for quantum gravity. The resulting dynamics of quantum gravity is encoded in the wave function or transition amplitudes on the bundle of spin-connections over the space-time. Thus the precanonical quantization leads to the "spin-connection foam" picture of quantum geometry represented by a non-Gaussian random field of spin connection coefficients. We also argue that the normalizability of precanonical wave functions with respect to the scalar product, which involves an operator-valued invariant measure on the space of spin connection coefficients, naturally leads to the quantum gravitational avoidance of curvature singularities.

Keywords: Quantum gravity; precanonical quantization; De Donder – Weyl theory; vielbein gravity; Dirac brackets.

1. Precanonical Quantization of Fields

Contemporary quantum field theory originates from canonical quantization which is based on the canonical Hamiltonian formalism. The latter dictates a picture of fields as infinite dimensional Hamiltonian systems. It also restricts the consideration to the globally hyperbolic space-times, as it implies a different role of the time dimension, along which the evolution proceeds, and the space dimensions, which label the continuum of degrees of freedom of fields. Many problems we encounter in quantum gravity theories can be traced back to this very origin of QFT.

However, the canonical Hamiltonian formalism is not the only possibility to extend the Hamiltonian formalism from mechanics to field theory. The alternative "Hamiltonizations" (i.e. writing the field equations in the first order form using some generalization of the Legendre transform) known in the calculus of variations [1] are more geometrical than the canonical formalism and they treat the space and time variables (i.e. the independent variables of the multiple integral variational problem) on the equal footing, i.e. essentially as multidimensional generalizations of the one-dimensional time parameter in mechanics.

In a sense, those Hamiltonizations of field theories are intermediate between the Lagrangean description and the canonical Hamiltonian formalism. Besides, in the case of mechanics, i.e. one-parameter variational problems, all those formulations reduce to the Hamiltonian formalism in mechanics. For this reason we name those formulations "precanonical" and the resulting procedure of quantization of fields "precanonical quantization".

The simplest precanonical formulation is the so-called De Donder–Weyl (DW) theory[1]: for a Lagrangian density $L = L(\phi^a, \phi^a_\mu, x^\nu)$, which is a function of the fields variables ϕ^a, their first space-time derivatives ϕ^a_μ, and the space-time variables x^μ, one defines the *polymomenta* $p^\mu_a := \frac{\partial L}{\partial \phi^a_\mu}$ and the *DW Hamiltonian function* $H(\phi^a, p^\mu_a, x^\mu) := \phi^a_\mu(\phi, p)p^\mu_a - L$. Then, if the Legendre transform $\phi^a_\nu \to p^\nu_a$ is regular, the field equations can be cast into the *DW Hamiltonian form*:

$$\partial_\mu \phi^a(x) = \partial H/\partial p^\mu_a, \quad \partial_\mu p^\mu_a(x) = -\partial H/\partial \phi^a. \tag{1}$$

This formulation requires neither a splitting into the space and time nor infinite-dimensional spaces of field configurations. Here the analogue of the extended configuration space is a finite dimensional space of field variables ϕ^a and space-time variables x^μ, and the analogue of the extended phase space is a finite dimensional space of p^μ_a, ϕ^a and x^μ. Those spaces are bundles over the space-time (see e.g. Ref. 4) whose sections are classical field configurations.

The Poisson brackets in DW Hamiltonian formulation of field theory in n dimensions[2] are based on the construction using the *polysymplectic* $(n+1)$*–form* on the extended polymomentum phase space: $\Omega := dp^\mu_a \wedge d\phi^a \wedge \varpi_\mu$, where $\varpi_\mu := \partial_\mu \lrcorner\, (dx^1 \wedge ... \wedge dx^n)$, as the fundamental underlying structure generalizing the symplectic 2–form of the canonical Hamiltonian formalism. The Poisson brackets are defined on differential forms representing dynamical variables and they lead to a Gerstenhaber algebra structure, which generalizes the Poisson algebra to the DW Hamiltonian formulation of field theory.[2] Precanonical quantization of fields[3] is based on quantization of Poisson-Gerstenhaber brackets of forms[2] according to the Dirac quantization rule.

A formulation of quantum scalar field theory based on quantization of Poisson-Gerstenhaber brackets of forms naturally leads to a description of quantum fields in terms of Clifford (Dirac) algebra valued wave function on the space of field variables and space-time variables, $\Psi(\phi^a, x^\mu)$, which fulfills a Dirac-like *precanonical Schrödinger equation* with the mass term replaced by the DW Hamiltonian operator $\widehat{H}$:

$$i\hbar\varkappa\gamma^\mu\partial_\mu\Psi = \widehat{H}\Psi, \tag{2}$$

where $\varkappa$ is an ultraviolet constant of the dimension of the inverse spatial volume and $\widehat{H}$ is a partial derivative operator with respect to the field variables. The natural appearance of Clifford algebra valued functions and operators can be argued already on the level of geometric prequantization generalized to the DW Hamiltonian formalism.[4] Note that the DW Hamiltonian equations (1) can be derived from (2) as the equations on the expectation values of the corresponding precanonical quantum operators.[5] The explicit form of a generalization of equation (2) in quantum gravity will be presented below.

A relation of this description to the standard QFT in the functional Schrödinger representation has been established in the limit of infinite $\varkappa$ or, more precisely, when $\frac{1}{\varkappa}\gamma^0 \rightarrow \varpi_0$.[6] In this limiting case one can construct the Schrödinger wave functional as the multidimensional Volterra product integral[7] of precanonical wave functions and derive the canonical functional derivative Schrödinger equation from the precanonical Schrödinger equation (2).[6] Thus the standard QFT turns out to be a singular limiting case of quantum theory of fields obtained via precanonical quantization. One can view the latter as an "already regularized" quantum field theory in which the regularizing UV scale $\varkappa$ is itself a part of the field quantization procedure.

2. Precanonical quantization of vielbein gravity and the spin-connection foam

A central role of the Dirac operator in precanonical quantization, which generalizes $i\partial_t$ in quantum mechanics (c.f. eq. (2)), implies that gravity has to be quantized in vielbein formulation (c.f. Ref. 8 for an earlier work on a metric formulation). Here we follow our earlier work.[10]

The Einstein-Palatini Lagrangian density with the cosmological term

$$\mathfrak{L} = \tfrac{1}{\kappa}\mathfrak{e} e^{[\alpha}_I e^{\beta]}_J \left(\partial_\alpha \omega_\beta{}^{IJ} + \omega_\alpha{}^{IK}\omega_{\beta K}{}^J\right) + \tfrac{1}{\kappa}\Lambda\mathfrak{e}, \tag{3}$$

where the vielbein components e^μ_I and the spin-connection coefficients ω^{IJ}_α are independent field variables, and $\mathfrak{e} := (\det ||e^\mu_I||)^{-1}$, is easily seen to lead to a singular DW Hamiltonian theory with the primary constraints

$$\mathfrak{p}^\alpha_{e^\beta_I} := \frac{\partial \mathfrak{L}}{\partial\, \partial_\alpha e^\beta_I} \approx 0, \quad \mathfrak{p}^\alpha_{\omega^{IJ}_\beta} := \frac{\partial \mathfrak{L}}{\partial\, \partial_\alpha \omega^{IJ}_\beta} \approx \tfrac{1}{\kappa}\mathfrak{e} e^{[\alpha}_I e^{\beta]}_J. \tag{4}$$

Those are second class, as it follows from the Poisson-Gerstenhaber brackets of $(n-1)$–forms of constraints $\mathfrak{C}_{e^\beta_I} := \mathfrak{p}^\alpha_{e^\beta_I}\varpi_\alpha$ and $\mathfrak{C}_{\omega^{IJ}_\beta} := \left(\mathfrak{p}^\alpha_{\omega^{IJ}_\beta} - \tfrac{1}{\kappa}\mathfrak{e} e^{[\alpha}_I e^{\beta]}_J\right)\varpi_\alpha$:

$$\{\![\mathfrak{C}_e, \mathfrak{C}_{e'}]\!\} = 0 = \{\![\mathfrak{C}_\omega, \mathfrak{C}_{\omega'}]\!\}, \ \{\![\mathfrak{C}_{e^\gamma_K}, \mathfrak{C}_{\omega^{IJ}_\beta}]\!\} = -\tfrac{1}{\kappa}\partial_{e^\gamma_K}\left(\mathfrak{e} e^{[\alpha}_I e^{\beta]}_J\right)\varpi_\alpha. \tag{5}$$

The DW Hamiltonian density obtained from (3) reads

$$\mathfrak{H} := \frac{\partial \mathfrak{L}}{\partial\, \partial_\alpha \omega}\mathfrak{p}^\alpha_\omega + \frac{\partial \mathfrak{L}}{\partial\, \partial_\alpha e}\mathfrak{p}^\alpha_e - \mathfrak{L} = -\tfrac{1}{\kappa}\mathfrak{e} e^{[\alpha}_I e^{\beta]}_J \omega^{IK}_\alpha \omega^J_{\beta K} - \tfrac{1}{\kappa}\Lambda\mathfrak{e}. \tag{6}$$

By using a generalization of the Dirac bracket to the DW theory[9] we obtain an amazingly simple algebra of fundamental brackets on the subalgebra of $(n-1)$– and

0–forms:[10]

$$\{\![\mathfrak{p}^\alpha_\omega\varpi_\alpha, \omega'\varpi_{\alpha'}]\!\}^{\mathrm{D}} = \{\![\mathfrak{p}^\alpha_\omega\varpi_\alpha, \omega'\varpi_{\alpha'}]\!\} \; = \; \delta^{\omega'}_\omega\varpi_{\alpha'}, \tag{7}$$

$$\{\![\mathfrak{p}^\alpha_e\varpi_\alpha, e'\varpi_{\alpha'}]\!\}^{\mathrm{D}} = 0 = \{\![\mathfrak{p}^\alpha_e\varpi_\alpha, \mathfrak{p}_\omega]\!\}^{\mathrm{D}} = \{\![\mathfrak{p}^\alpha_e\varpi_\alpha, \omega'\varpi_{\alpha'}]\!\}^{\mathrm{D}} = \{\![\mathfrak{p}^\alpha_\omega\varpi_\alpha, e'\varpi_{\alpha'}]\!\}^{\mathrm{D}} \tag{8}$$

$$\{\![\mathfrak{p}^\alpha_\omega\varpi_\alpha, \omega']\!\}^{\mathrm{D}} = \{\![\mathfrak{p}^\alpha_\omega\varpi_\alpha, \omega']\!\} \; = \; \delta^{\omega'}_\omega, \tag{9}$$

$$\{\![\mathfrak{p}^\alpha_e\varpi_\alpha, e']\!\}^{\mathrm{D}} = 0 = \{\![\mathfrak{p}^\alpha_e\varpi_\alpha, \mathfrak{p}_\omega]\!\}^{\mathrm{D}} = \{\![\mathfrak{p}^\alpha_e\varpi_\alpha, \omega]\!\}^{\mathrm{D}} = \{\![\mathfrak{p}^\alpha_\omega\varpi_\alpha, e]\!\}^{\mathrm{D}}, \tag{10}$$

$$\{\![p^\alpha_\omega, \omega'\varpi_\beta]\!\}^{\mathrm{D}} = \{\![p^\alpha_\omega, \omega'\varpi_\beta]\!\} = \delta^\alpha_\beta\delta^\omega_{\omega'}, \tag{11}$$

$$\{\![\mathfrak{p}^\alpha_e, e'\varpi_{\alpha'}]\!\}^{\mathrm{D}} = 0 = \{\![\mathfrak{p}^\alpha_e, \mathfrak{p}_\omega\varpi_{\alpha'}]\!\}^{\mathrm{D}} = \{\![\mathfrak{p}^\alpha_e, \omega\varpi_{\alpha'}]\!\}^{\mathrm{D}} = \{\![\mathfrak{p}^\alpha_\omega, e'\varpi_{\alpha'}]\!\}^{\mathrm{D}} = 0. \tag{12}$$

The fundamental brackets are quantized according to the generalized Dirac's quantization rule: $[\hat{A}, \hat{B}] = -i\hbar\widehat{\mathfrak{e}\{\![A,B]\!\}^{\mathrm{D}}}$, in which the presence of the operator of $\mathfrak{e}$ guarantees that tensor densities are quantized as density-valued operators. The quantization of fundamental Dirac brackets (7)-(12) and the equations of constraints (4) leads to a representation of the operators of vielbeins: $\hat{e}^\beta_I = -i\hbar\varkappa\kappa\bar{\gamma}^J\frac{\partial}{\partial\omega^{IJ}_\beta}$, and the polymomenta of spin-connection: $\hat{\mathfrak{p}}^\alpha_{\omega^{IJ}_\beta} = -\hbar^2\varkappa^2\kappa\,\hat{\mathfrak{e}}\,\bar{\gamma}^{KL}\frac{\partial}{\partial\omega^{KL}_{[\alpha}}\frac{\partial}{\partial\omega^{IJ}_{\beta]}}$, where $\bar{\gamma}^J$ are the fiducial Minkowskian Dirac matrices. Those Clifford-valued operators act on Clifford-valued quantum gravitational precanonical wave functions which do not depend on vielbein variables: $\Psi = \Psi(\omega^{IJ}_\alpha, x^\mu)$, i.e. they are wave functions on the configuration bundle of spin-connections over the space-time.

We can also construct the operator of DW Hamiltonian density $\mathfrak{H} =: \mathfrak{e}H$ restricted to the surface of constraints C. From (6) and (4) we obtain $(\mathfrak{e}H)|_C = -\mathfrak{p}^\alpha_{\omega^{IJ}_\beta}\omega^{IK}_\alpha\omega_{\beta K}{}^J - \frac{1}{\kappa}\Lambda\mathfrak{e}$ and, using the above representations,

$$\widehat{H} = \hbar^2\varkappa^2\kappa\bar{\gamma}^{IJ}\omega_{[\alpha}{}^{KM}\omega_{\beta]M}{}^L\frac{\partial}{\partial\omega^{IJ}_\alpha}\frac{\partial}{\partial\omega^{KL}_\beta} - \frac{1}{\kappa}\Lambda. \tag{13}$$

Now the covariant precanonical Schrödinger equation for quantum gravity (c.f. (2))

$$i\hbar\varkappa\widehat{\not\nabla}\Psi = \widehat{H}\Psi, \tag{14}$$

where $\widehat{\not\nabla} := \widehat{\gamma}^\mu(\partial_\mu + \frac{1}{4}\omega_{\mu IJ}\bar{\gamma}^{IJ})$ and $\widehat{\gamma}^\mu := \bar{\gamma}^I\hat{e}^\mu_I = -i\hbar\varkappa\kappa\bar{\gamma}^{IJ}\frac{\partial}{\partial\omega^{IJ}_\mu}$, can be written in an explicit form:

$$\bar{\gamma}^{IJ}\left(\partial_\mu + \frac{1}{4}\omega_{\mu KL}\bar{\gamma}^{KL} - \omega_{\mu M}{}^K\omega_\beta{}^{ML}\frac{\partial}{\partial\omega_\beta{}^{KL}}\right)\frac{\partial}{\partial\omega_\mu{}^{IJ}}\Psi = -\lambda\Psi, \tag{15}$$

where $\lambda := \frac{\Lambda}{(\hbar\varkappa\kappa)^2}$ is a dimensionless constant which involves three different scales: cosmological, Planck, and the UV scale $\varkappa$ introduced by precanonical quantization.

The fact that all physical constants have been absorbed in a single dimensionless constant λ, which is present as an eigenvalue of the operator in the l.h.s. of (2), seems to suggest that the latter, and the theory of quantum gravity derived from precanonical quantization, represent a certain purely mathematical statement concerning the sections of the Clifford bundle over the bundle of spin-connections over the space-time, which is actually the essence of "quantum geometry" in the present formulation.

We assume that the Hilbert space of precanonical wave functions is defined by the scalar product

$$\langle \Phi | \Psi \rangle := \mathrm{Tr} \int \overline{\Phi}\, \widehat{[d\omega]}\, \Psi, \quad \widehat{[d\omega]} = \hat{\mathfrak{e}}^{-n(n-1)} \prod_{\mu, I<J} d\omega_\mu^{IJ}, \tag{16}$$

where $\overline{\Psi} := \bar{\beta}\Psi^\dagger\bar{\beta}$ and a Misner-like diffeomorphism invariant measure on the space of spin-connection coefficients (i.e. a fiber of the bundle of spin-connections over the space-time) is operator valued, because $\hat{\mathfrak{e}}^{-1} = \frac{1}{n!}\epsilon^{I_1 \ldots I_n}\epsilon_{\mu_1 \ldots \mu_n}\hat{e}^{\mu_1}_{I_1}\ldots\hat{e}^{\mu_n}_{I_n}$ is a differential operator itself.

As a consequence of such definition of the scalar product (and quite independently from its details) the normalizability of precanonical wave functions, which requires $\hat{\mathfrak{e}}^{-\frac{1}{2}n(n-1)}\Psi$ to vanish at large ω-s, i.e. at large space-time curvatures, actually implies the quantum singularity avoidance in the sense that the probability density of observing the regions of space-time with extremely high curvatures is vanishing. However, in spite of its plausibility, this resolution of the singularity problem depends on the actual existence of the properly normalized solutions of Eq. (15), which is not yet proven. Moreover, the argument based on the normalizability of precanonical wave functions in its present form ignores the intricalities related to the indefiniteness of $\mathrm{Tr}[\overline{\Psi}\Psi]$ and the gauge fixing, i.e. the choice of the coordinate systems and local orientations of vielbeins on the average when extracting a physical information from the solutions of (15).

Eq. (15) can be seen as a generalized hypergeometric equation of several matrix variables $Z_\mu := \omega_\mu^{IJ}\bar{\gamma}_{IJ}$. The theory of such equations is not yet developed in mathematics, so that even the existence of properly normalized solutions (see below), which is essential for the present formulation of quantum theory of gravity to be physically viable and mathematically well-defined is still to be proven.

The latter problem can be addressed in the simplified case of space-times with less than three independent nonvanishing spin-connection components. In this case the $\omega\omega\partial_\omega\partial_\omega$ term in (13) is vanishing and the remaining part can be written in the form of (15)

$$\left(ik_\mu \partial_{Z_\mu} + \frac{1}{4} Z_\mu \partial_{Z_\mu} + \lambda \right) \Psi = 0, \tag{17}$$

assuming $\Psi = e^{ik_\mu x^\mu}\Phi(Z_\mu)$. By separating variables Z_μ, so that $\Psi(Z) = \Pi_\mu \Psi(Z_\mu)$ we obtain the following equation for each μ

$$(ik\partial_Z + \frac{1}{4}Z\partial_Z)\Psi + c\Psi = 0. \tag{18}$$

Its formal solution (written in terms of the function of the fractional degree of the matrix under the bracket) is

$$\Psi \sim (4k - iZ)^{-4c} \tag{19}$$

The Green functions of (15) are the transition amplitudes between different values of spin-connection at different points, $\langle \omega, x | \omega', x' \rangle$. They describe a quantum space-time geometry which generalizes classical geometry described in terms of smooth spin-connection fields $\omega(x)$. By noticing the analogy with the statistical hydrodynamics approach to turbulence, which replaces the description in terms of the smooth velocity fields $\mathbf{v}(\mathbf{x})$ by the statistical description in terms of the velocity correlators at different points, we can speak of the picture of quantum gravity derived from precanonical quantization as a *space-time turbulence* or *spin-connection foam*. In fact, this picture is even closer to the original Wheeler's intuition about the "space-time foam" than his quantum geometrodynamics based on the Wheeler–De Witt equation and the notion of infinite-dimensional superspace of all metrics. Besides, the metric tensor in the present formulation is the operator given by $\widehat{g}^{\mu\nu} = -\hbar^2 \varkappa^2 \kappa^2 \eta^{IJ} \eta^{KL} \frac{\partial^2}{\partial \omega_\mu^{IK} \partial \omega_\nu^{JL}}$, hence the distances between points given by $\widehat{ds}^2 = \widehat{g}_{\mu\nu} dx^\mu dx^\nu$ are operator-valued. This makes the notion of the distances between points or, consequently, the operational notion of the points themselves, fuzzy. Thus, precanonical quantization of general relativity leads to a description of quantum geometry of space-time which complements the contemporary ideas about quantum space-time originating from LQG, string theory and non-commutative geometry.

3. Some connections with LQG

Historically, the paper by Esposito e.a.[11] was the first one to discuss the constraints in Ashtekar's formulation using a version of DW (multisymplectic) Hamiltonian theory applied to the vielbein Einstein-Palatini Lagranigian (3).

In spite of the obvious differences of our precanonical approach from the LQG programme, which uses 3+1 decomposition vs. our explicitly space-time symmetric approach, functionals and functional derivative operators vs. our use of functions and partial derivative operators etc., there are some striking similarities as well, one of them being the emergence, after Hamiltonization, of a formulation based on connections, with the vielbeins, or densitized inverse triads/dreibeins in Ashtekar's formulation, represented as differential operators with respect to the connections.

Whereas our approach to quantization is based on the fundamental brackets in the Weyl subalgebra in the subspace of Hamiltonian 0– and $(n-1)$–forms, our construction of Poisson-Gerstenhaber brackets in field theory[2] offers another, yet unexplored opportunity based on the Dirac bracket between the spin-connection 1–form $\omega^{IJ}_\alpha dx^\alpha$ and its conjugate polymomenta $(n-2)$–form $\mathfrak{p}^\mu_{\omega_\nu^{KL}} \varpi_{\mu\nu}$, where $\varpi_{\mu\nu} := \partial_\mu \lrcorner\, \partial_\nu \lrcorner\, \varpi$:

$$\{\![\omega^{IJ}_\alpha dx^\alpha, \mathfrak{p}^\mu_{\omega_\nu^{KL}} \varpi_{\mu\nu}]\!\}^{\mathrm{D}} = \{\![\omega^{IJ}_\alpha dx^\alpha, \mathfrak{p}^\mu_{\omega_\nu^{KL}} \varpi_{\mu\nu}]\!\} \sim n\, \delta^{[I}_K \delta^{J]}_L . \qquad (20)$$

This bracket generalizes the bracket between the potential 1–form A and the $(n-2)$–form of the dual field strength $*\mathsf{F} = *\frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ in Electrodynamics found in Ref. 2 (see also Refs. 14, 15). After a (3+1)-decomposition, restriction to the initial data surfaces, and integration over them (c.f. Refs. 2, 14), the bracket can be related to the fundamental Poisson bracket underlying the Ashtekar formulation in (3+1)-dimensions, and the forms involved in (20) can be used to construct the holonomy-flux variables underlying the LQG approach (see e.g. Ref. 12 for a review). However, when using the bracket (20) one should keep in mind that the set of $(n-2)-$ and $1-$forms is not closed with respect to the Poisson-Gerstenhaber bracket operation, unlike the subalgebra of $0-$ and $(n-1)-$forms in (7)–(12) our consideration is based on.

A further understanding of possible connections of precanonical quantization with LQG in (3+1) dimensions would require an inclusion of the Holst term[16] with the Barbero-Immirzi parameter in the Einstein-Palatini Lagrangian (3), which does not, however, appear to be necessary within our approach. It is interesting to note in this connection that a multidimensional generalization of the connection formulation of pure gravity discussed by Bodendorfer e.a.[17] also does not require the Barbero-Immirzi parameter in general.

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