# Twisted spectral triple for the Standard Model and and spontaneous breaking of the Grand Symmetry

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**Abstract**

Grand symmetry models in noncommutative geometry have been introduced to explain how to generate minimally (i.e. without adding new fermions) an extra scalar field beyond the standard model, which both stabilizes the electroweak vacuum and makes the computation of the mass of the Higgs compatible with its experimental value. In this paper, we use Connes-Moscovici twisted spectral triples to cure a technical problem of the grand symmetry, that is the appearance together with the extra scalar field of unbounded vectorial terms. The twist makes these terms bounded, and also permits to understand the breaking to the standard model as a dynamical process induced by the spectral action, as conjectured in [20]. This is a spontaneous breaking from a pre-geometric Pati-Salam model to the almost-commutative geometry of the standard model, with two Higgs-like fields: scalar and vector.

## 1 Introduction

Noncommutative geometry [NCG] provides a description of the standard model of elementary particles [SM] in which the mass of the Higgs − at unification scale $\Lambda$ − is a function of the other parameters of the theory, especially the Yukawa coupling of fermions [7]. Assuming there is no new physics between the electroweak and the unification scales (the "big desert hypothesis"), the flow of this mass under the renormalization group yields a prediction for the Higgs observable mass $m_H$. It is well known that in the absence of new physics the three constants of interaction fail to meet at a single unification scale, but form a triangle which lays between $10^{13}$ and $10^{17}$ GeV. The situation can be improved by taking into account higher order term in the NCG action [19], or gravitational effects [18]. Nevertheless, the prediction of $m_H$ is not much sensible on the choice of the unification scale. Since the beginning of the model in the early 90' [12, 13], for $\Lambda$ between $10^{13}$ and $10^{17}$GeV this prediction had been around 170 GeV, a value ruled out by Tevratron in 2008. Consequently, either the model should be abandoned, or the big desert hypothesis questioned.

The recent discovery of the Higgs boson with a mass $m_H \simeq 126$ Gev suggests the big desert hypothesis should be questioned. There is indeed an instability in the electroweak vacuum which is meta-stable rather than stable (see [3] for the most recent update). There does not seem to be a consensus in the community whether this is an important problem or not: on the one hand the mean time of this meta-stable state is longer than the age of the universe, on the other hand in some cosmological scenario the meta-stabililty may be problematic [23, 24]. Still, the fact that $m_H$ is almost at the boundary value between the stable and meta-stable phases

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of the electroweak vacuum suggests that "something may be going on". In particular, particle physicists have shown how a new scalar field suitably coupled to the Higgs - usually denoted $\sigma$ - can cure the instability (e.g. [11, 22]).

Taking into account this extra field in the NCG description of the SM induces a modification of the flow of the Higgs mass, governed by the parameter $r = \frac{k_\nu}{k_t}$, which is the ratio of the Dirac mass of the neutrino and of the Yukawa coupling of the quark top. Remarkably, for any value of $\Lambda$ between $10^{12}$ and $10^{17}$ Gev, there exists a realistic value $r \simeq 1$ which brings back the computed value of $m_H$ to 126 Gev [6].

The question is then to generate the extra field $\sigma$ in agreement with the tools of noncommutative geometry. Early attempts in this direction have been done in [29], but they require the adjunction of new fermions (see [30] for a recent state of the art). In [6], a scalar $\sigma$ correctly coupled to the Higgs is obtained without touching the fermionic content of the model, simply by turning the Majorana mass $k_R$ of the neutrino into a field

$$k_R \to k_R \, \sigma. \tag{1.1}$$

Usually the bosonic fields in NCG are generated by inner fluctuations of the geometry. However this does not work for the field $\sigma$ because of the first-order condition

$$[[D, a], JbJ^{-1}] = 0 \quad \forall a, b \in \mathcal{A} \tag{1.2}$$

where $\mathcal{A}$ and $D$ are the algebra and the Dirac operator of the spectral triple of the standard model, and $J$ the real structure.

In [9, 10] it was shown how to obtain $\sigma$ by a inner fluctuation that does not satisfy the first-order condition, but in such a way that the latter is retrieved dynamically, as a minimum of the spectral action. The field $\sigma$ is then interpreted as an excitation around this minimum. Previously in [20] another way had been investigated to generate $\sigma$ in agreement with the first-order condition, taking advantage of the fermion doubling in the Hilbert space $\mathcal{H}$ of the spectral triple of the SM [26–28].

More specifically, under natural assumptions on the representation of the algebra and an ad-hoc symplectic hypothesis, it is shown in [5] that the algebra in the spectral triple of the SM should be a sub-algebra of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_F$, where $\mathcal{M}$ is a Riemannian compact spin manifold (usually of dimension 4) while

$$\mathcal{A}_F = M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C}) \quad a \in \mathbb{N}. \tag{1.3}$$

The algebra of the standard model

$$\mathcal{A}_{sm} := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \tag{1.4}$$

is obtained from $\mathcal{A}_F$ for $a = 2$, by the grading and the first-order conditions. Starting instead with the "grand algebra" ($a = 4$)

$$\mathcal{A}_G := M_4(\mathbb{H}) \oplus M_8(\mathbb{C}), \tag{1.5}$$

one generates the field $\sigma$ by a inner fluctuation which respects the first-order condition imposed by the part $D_\nu$ of the Dirac operator that contains the Majorana mass $k_R$ [20]. The breaking to $\mathcal{A}_{sm}$ is then obtained by the first-order condition imposed by the free Dirac operator $\not{D} := \not{\partial} \otimes \mathbb{I}$.

Unfortunately, before this breaking not only is the first-order condition not satisfied, but the commutator

$$[\not{D}, A] \quad A \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_G \tag{1.6}$$

is never bounded. This is problematic both for physics, because the connection 1-form containing the gauge bosons is unbounded; and from a mathematical point of view, because the construction of a Fredholm module over $\mathcal{A}$ and Hochschild character cocycle depends on the boundedness of the commutator (1.6).

In this paper, we solve this problem by using instead a *twisted spectral triple* $(\mathcal{A}, \mathcal{H}, D, \rho)$ [14]. Rather than requiring the boundedness of the commutator, one asks that there exists a automorphism $\rho$ of $\mathcal{A}$ such that the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D \tag{1.7}$$

is bounded for any $a \in \mathcal{A}$. Accordingly, we introduce in Def. 3.1 a *twisted first-order condition*

$$[[D, a]_\rho, JbJ^{-1}]_\rho := [D, a]_\rho JbJ^{-1} - J\rho(b)J^{-1}[D, a]_\rho = 0 \quad \forall a, b \in \mathcal{A}. \tag{1.8}$$

We then show that a for a suitable choice of a subalgebra $\mathcal{B}$ of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$, a *twisted fluctuation* of $\not{D} + D_\nu$ that satisfies (1.8) generates a field $\boldsymbol{\sigma}$ - slightly different from the one of [6] - together with an additional vector field $X_\mu$.

Furthermore, the breaking to the standard model is now spontaneous, as conjectured by Lizzi in [20]. Namely the reduction of the grand algebra $\mathcal{A}_G$ to $\mathcal{A}_{sm}$ is obtained dynamically, as a minimum of the spectral action. The scalar and the vector fields then play a role similar as the one of the Higgs in the electroweak symmetry breaking.

Mathematically, twists make sense as explained in [14], for the Chern character of finitely summable spectral triples extends to the twisted case, and lands in ordinary (untwisted) cyclic cohomology. Twisted spectral triples have been introduced to deal with type III examples, such as those arising from transverse geometry of codimension one foliation. It is quite surprising that the same tool allows a rigorous implementation in NCG of the idea of a "bigger symmetry beyond the SM".

The main results of the paper are summarized in the following theorem.

**Theorem 1.1.** *Let $\mathcal{H}$ be the Hilbert space of the standard model described in §2.1. There exists a sub-algebra $\mathcal{B}$ of the grand algebra $\mathcal{A}_G$ containing $\mathcal{A}_{sm}$ together with an automorphism $\rho$ of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ such that*

*i) $(C^\infty(\mathcal{M}) \otimes \mathcal{B}, \mathcal{H}, \not{D} + D_\nu; \rho)$ is a twisted spectral triple satisfying the twisted $1^{st}$-order condition* (1.8)*;*

*ii) a twisted fluctuation of $\not{D} + D_\nu$ by $\mathcal{B}$ generates an extra scalar field $\boldsymbol{\sigma}$, together with an additional vector field $X_\mu$;*

*iii) the spectral triple of the standard model is obtained as the minimum of the spectral action induced by a twisted fluctuation of $\not{D}$. The same result is obtained from a twisted fluctuation of $\not{D} + D_\nu$, neglecting the interaction term between $\boldsymbol{\sigma}$ and $X_\mu$.*

Explicitly, $\mathcal{B}$ is a sub-algebra $\mathbb{H}^2 \oplus \mathbb{C}^2 \oplus M_3(\mathbb{C})$ of $\mathcal{A}_G$. Labelling the two copies of the quaternion and complex algebras by the left/right spinorial indices $l, r$ and the left/right internal indices L/R, that is

$$\mathcal{B} = \mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{C}^l_R \oplus \mathbb{C}^r_R \oplus M_3(\mathbb{C}), \tag{1.9}$$

the automorphism $\rho$ is the exchange of the left/right spinorial indices:

$$\rho\,(\,q^l_L,\, q^r_L,\, c^l_R,\, c^r_R,\, m) \to (q^r_L,\, q^l_L,\, c^r_R,\, c^l_R,\, m) \tag{1.10}$$

where $m \in M_3(\mathbb{C})$ while the $q$'s and $c$'s are quaternions and complex numbers belonging to their respective copy of $\mathbb{H}$ and $\mathbb{C}$.

The paper is organized as follows. In section 2 we recall briefly the spectral triple of the standard model (§2.1), the tensorial notation used all along the paper (§2.2), and the results of [20] on the grand algebra (§2.3). We discuss the unboundedness of the commutator (1.6) in §2.4. Section 3 deals with the twist. It begins with the definition of the twisted first-order condition in Def. 3.1. In §3.1 we fix the representation of the grand algebra, which differs from the one used in [20]. It is used in §3.2 to build a twisted spectral triple with the free Dirac operator (Prop. 3.4). In §3.3 the twisted first-order condition for $D_\nu$ yields the reduction to the algebra $\mathcal{B}$ (Prop. 3.5). In section 4 we compute the twisted fluctuations $D_X$ of the free Dirac operator $\not{D}$ (§4.1), and $D_\sigma$ of the Majorana-Dirac operator $D_\nu$ (§4.2). The additional vector field is obtained in Prop. 4.1, the extra scalar field $\boldsymbol{\sigma}$ in Prop. 4.4. In section 5, after some generalities on the spectral action in §5.1, we show that the reduction of $\mathcal{B}$ to the standard model is dynamical, first by showing in §5.2 how to get $\mathcal{A}_{sm}$ as a minimum of the spectral action for $D_X$, then proving in §5.3 that the standard model also corresponds to the minimum of the potential of the field $\boldsymbol{\sigma}$. These results are discussed in section 6. In §6.1 we stress how twisting the almost commutative geometry of the SM may open the way to models where the algebra is not the tensor product of a manifold by a finite dimensional geometry. This justifies the choice of the representation of $\mathcal{A}_G$ made in the present paper, but we show in §6.2 that the results are the same with the representation used in [20].

# 2 Standard model and the grand algebra

## 2.1 The spectral triple of the standard model

The main tools of NCG [15] are encoded within a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra acting on a Hilbert space $\mathcal{H}$, and $D$ is a selfadjoint operator on $\mathcal{H}$. These three elements come with two more operators, a real structure $J$ [16] and a graduation $\Gamma$ that are generalizations to the noncommutative setting of the charge conjugation and the chirality operators of quantum field theory. These five objects satisfy a set of properties guaranteeing that given any spectral triple with $\mathcal{A}$ unital and commutative, then there exists a closed Riemannian spin manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$ [17]. These conditions still make sense in the noncommutative case [12], hence the definition of a *noncommutative geometry* as a spectral triple where the algebra $\mathcal{A}$ is non necessarily commutative.

Among these conditions, the ones that play an important role in this work are the first-order condition (1.2), the boundedness and the grading conditions

$$[D, a] \in \mathcal{B}(\mathcal{H}), \quad [\Gamma, a] = 0 \quad \forall a \in \mathcal{A}, \tag{2.1}$$

as well as the order-zero condition

$$[a, Jb^* J^{-1}] = 0 \quad \forall a, b \in \mathcal{A}. \tag{2.2}$$

A gauge theory is described by an *almost commutative geometry*

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F,\ \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F,\ D = \not{\partial} \otimes \mathbb{I}_F + \gamma^5 \otimes D_F, \tag{2.3}$$

which is the product of the canonical spectral triple $\left(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{\partial}\right)$ associated to a oriented closed spin manifold $\mathcal{M}$ of (even) dimension $m$, by a finite dimensional spectral triple

$$(\mathcal{A}_F, \mathcal{H}_F, D_F). \tag{2.4}$$

Here $L^2(\mathcal{M}, S)$ is the space of square integrable spinors on $\mathcal{M}$, and $\partial\!\!\!/ = -i\sum_{\mu=1}^m \gamma^\mu \partial_\mu$ is the Dirac operator with $\gamma^\mu = \gamma^{\mu\dagger}$ the selfadjoint Dirac matrices. The chirality operator $\gamma^5$ is a graduation of $L^2(\mathcal{M}, S)$ which commutes with $C^\infty(\mathcal{M})$ and anticommutes with $\partial\!\!\!/$. The notation is justified assuming $\mathcal{M}$ has dimension 4 (what we do from now on): $\gamma^5$ is then the product of the four Dirac matrices.

The choice of the finite dimensional spectral triple (2.4) is dictated by the physical contains of the theory. For the SM, the algebra is $\mathcal{A}_{sm}$ given in (1.4), whose group of unitary elements yields the gauge group of the standard model. The finite dimensional Hilbert space $\mathcal{H}_F$ is spanned by the particle content of the theory. The standard model has 96 such degrees of freedom: 8 fermions (electron, neutrino, up and down quarks with three colors each) for N=3 generations and two chiralities $L$, $R$, plus antiparticles. Therefore one takes

$$\mathcal{H}_F = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^c = \mathbb{C}^{96}. \tag{2.5}$$

The finite dimensional Dirac operator $D_F = D_0 + D_R$ is a $96 \times 96$ matrix where

$$D_0 := \begin{pmatrix} 0_{8N} & \mathcal{M}_0 & 0_{8N} & 0_{8N} \\ \mathcal{M}_0^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & \mathcal{M}_0^* \\ 0_{8N} & 0_{8N} & \mathcal{M}_0^T & 0_{8N} \end{pmatrix} \quad \text{and} \quad D_R := \begin{pmatrix} 0_{8N} & 0_{8N} & \mathcal{M}_R & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & 0_{8N} \\ \mathcal{M}_R^\dagger & 0_{8N} & 0_{8N} & 0_{8N} \\ 0_{8N} & 0_{8N} & 0_{8N} & 0_{8N} \end{pmatrix}. \tag{2.6}$$

The matrix $\mathcal{M}_0$ contains the Yukawa couplings of fermions, the Dirac mass of neutrinos, the Cabibbo matrix and the mixing matrix for neutrinos. The matrix $\mathcal{M}_R$ contains the Majorana mass of neutrinos. Explicitly

$$\mathcal{M}_0 = \begin{pmatrix} M_u & 0_4 \\ 0_4 & M_d \end{pmatrix} \otimes \mathbb{I}_N \qquad \mathcal{M}_R = \begin{pmatrix} M_R & 0_4 \\ 0_4 & 0_4 \end{pmatrix} \otimes \mathbb{I}_N \tag{2.7}$$

where, for the first generation, $M_u$ is a diagonal matrix containing the Yukawa coupling of the up quark and the Dirac mass of $\nu_e$, $M_d$ is a diagonal matrix containing the down quark and the electron masses, $M_R$ contains the Majorana mass of $\nu_e$. The structure is repeated for the other two generations.

The real structure

$$J = \mathcal{J} \otimes J_F \tag{2.8}$$

acts as the charge conjugation operator $\mathcal{J} = i\gamma^0\gamma^2 cc$ on $L^2(\mathcal{M}, S)$, and as

$$J_F := \begin{pmatrix} 0 & \mathbb{I}_{16N} \\ \mathbb{I}_{16N} & 0 \end{pmatrix} cc \tag{2.9}$$

on $\mathcal{H}_F$, where it exchanges the blocks $\mathcal{H}_R \oplus \mathcal{H}_L$ of particles with the block $\mathcal{H}_R^c \oplus \mathcal{H}_L^c$ of antiparticles. The graduation is

$$\Gamma = \gamma^5 \otimes \gamma_F \quad \text{where} \quad \gamma_F := \begin{pmatrix} \mathbb{I}_{8N} & & & \\ & -\mathbb{I}_{8N} & & \\ & & -\mathbb{I}_{8N} & \\ & & & \mathbb{I}_{8N} \end{pmatrix}. \tag{2.10}$$

The operators $\gamma_F, J_F$ and $D_F$ are such that $J_F^2 = \mathbb{I}$, $J\gamma_F = -\gamma_F J_F$, $J_F D_F = D_F J_F$, meaning that the finite part of the spectral triple of the standard model has $KO$-dimension 6 [2, 7]. Meanwhile the continuous part of the spectral triple has $KO$-dimension 4, that is $\mathcal{J}^2 = -\mathbb{I}$, $\mathcal{J}\gamma = \gamma\mathcal{J}$ and $\mathcal{J}\partial\!\!\!/ = \partial\!\!\!/\mathcal{J}$.

Gauge fields are obtained by fluctuating the operator $D$ by $\mathcal{A}$, that is substituting it with the *covariant Dirac operator*

$$D_{\mathbb{A}} := D + \mathbb{A} + J\mathbb{A}J^{-1} \tag{2.11}$$

where

$$\mathbb{A} = \sum_i a_i[D, b_i] \quad a_i, b_i \in \mathcal{A} \tag{2.12}$$

is a selfadjoint 1-form of the almost commutative manifold.

As stressed in the introduction, the field $\sigma$ cannot be generated by a fluctuation of the Majorana part

$$D_\nu := \gamma^5 \otimes D_R \tag{2.13}$$

of the Dirac operator, because of the first-order condition: one easily checks [20] that for any $a, b \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}$

$$[[D_\nu, A],\ JbJ^{-1}] = 0 \text{ if and only if } [D_\nu, A] = 0. \tag{2.14}$$

Hence the necessity to make the first-order condition more flexible [10], or to enlarge the algebra one is starting with, in order to have enough space to generate the field $\sigma$ without violating the first-order condition. This enlargement is made possible by mixing the internal degrees of freedom of $\mathcal{H}_F$ with the spinorial degrees of freedom of $L^2(\mathcal{M}, S)$. This has been done in [20] and is recalled in the next two paragraphs.

## 2.2 Mixing of spinorial and internal degrees of freedom

The total Hilbert space $\mathcal{H}$ of the almost commutative geometry (2.3) is the tensor product of four dimensional spinors by the 96-dimensional elements of $\mathcal{H}_F$. Any of its element is a $\mathbb{C}^{384}$-vector valued function on $\mathcal{M}$. From now on we work with $N = 1$ generation only, and consider instead $384/3 = 128$ components vector. The total Hilbert space can thus be written - at least in a local trivialization - in two ways:

$$\mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F = L^2(\mathcal{M}) \otimes \mathsf{H}_F \tag{2.15}$$

where $\mathsf{H_F} \simeq \mathbb{C}^{128}$ takes into account both external (i.e. spin) and internal (i.e. particle) degrees of freedom. We label the basis of $\mathsf{H_F}$ with a multi-index $s\dot{s}\mathsf{C}\mathrm{I}\alpha$ where:

- $s, \dot{s}$ are the four spinor indices: $s = r, l$ runs over the right, left parts and $\dot{s} = \dot{0}, \dot{1}$ over the particle, antiparticle parts of the spinors.
- $\mathsf{C}$ indicates wether we are considering “particles” ($\mathsf{C} = 0$) or “antiparticles” ($\mathsf{C} = 1$).
- $\mathrm{I}$ is a “lepto-colour” index: $\mathrm{I} = 0$ identifies leptons while $\mathrm{I} = 1, 2, 3$ are the three colors of QCD.
- $\alpha$ is the flavor index. It runs over the set $u_R, d_R, u_L, d_L$ when $\mathrm{I} = 1, 2, 3$, and $\nu_R, e_R, \nu_L, e_L$ when $\mathrm{I} = 0$.

On this basis, an element $\Psi$ of $\mathcal{H}$ has components $\Psi^{\mathsf{C}\mathrm{I}}_{s\dot{s}\alpha} \in L^2(\mathcal{M})$. Notice that the position of the indices is arbitrary: $\Psi$ evaluated at $x \in \mathcal{M}$ is a column vector, so all the indices are raw indices.

This choice of indices yields the chiral basis for the Euclidean Dirac matrices:*

$$\gamma^\mu = \begin{pmatrix} 0_2 & \sigma^{\mu \dot{t}}_{\ \dot{s}} \\ \tilde{\sigma}^{\mu \dot{t}}_{\ \dot{s}} & 0_2 \end{pmatrix}_{st}, \quad \gamma^5 = \begin{pmatrix} \mathbb{I}_2 & 0_2 \\ 0_2 & -\mathbb{I}_2 \end{pmatrix}_{st}, \tag{2.16}$$

where for $\mu = 0, 1, 2, 3$ one defines

$$\sigma^\mu = \{\mathbb{I}_2, -i\sigma_i, \}, \quad \tilde{\sigma}^\mu = \{\mathbb{I}_2, i\sigma_i\} \tag{2.17}$$

with $\sigma_i$, $i = 1, 2, 3$ the Pauli matrices. Explicitly,

$$\sigma^0 = \mathbb{I}_2,\ \sigma^1 = -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}_{\dot{s}\dot{t}},\ \sigma^2 = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\dot{s}\dot{t}},\ \sigma^3 = -i\sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}_{\dot{s}\dot{t}}.$$

The free Dirac operator $\partial\!\!\!/$ extended to $\mathcal{H}$ as $D\!\!\!\!/ := \partial \otimes \mathbb{I}_F$ acts as †

$$D\!\!\!\!/ := \delta^{\mathsf{CI}\beta}_{\mathsf{DJ}\alpha} \partial\!\!\!/ = -i \begin{pmatrix} \delta^{\mathrm{I}\beta}_{\mathrm{J}\alpha}\, \gamma^\mu \partial_\mu & 0_{64} \\ 0_{64} & \delta^{\mathrm{I}\beta}_{\mathrm{J}\alpha}\, \gamma^\mu \partial_\mu \end{pmatrix}_{\mathsf{CD}}. \tag{2.18}$$

In tensorial notation, the charge conjugation operator is

$$\mathcal{J} = i\gamma^0\gamma^2 cc = i \begin{pmatrix} \overline{\sigma}^{2\dot{t}}_{\ \dot{s}} & 0_2 \\ 0_2 & \sigma^{2\dot{t}}_{\ \dot{s}} \end{pmatrix}_{st} cc = -i\eta^t_s \tau^{\dot{t}}_{\dot{s}}\, cc, \tag{2.19}$$

while

$$J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}_{\mathsf{CD}} cc, \tag{2.20}$$

hence

$$(J\Psi)^{\mathsf{CI}}_{s\dot{s}\alpha} = -i\eta^t_s\, \tau^{\dot{t}}_{\dot{s}}\, \xi^{\mathsf{C}}_{\mathsf{D}}\, \delta^{\mathrm{I}\beta}_{\mathrm{J}\alpha}\, \bar{\Psi}^{\mathsf{DJ}}_{t\dot{t}\beta} \tag{2.21}$$

where for any pair of indices $x, y \in [1, ..., n]$ one defines

$$\xi^x_y = \begin{pmatrix} 0_n & \mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix}, \quad \eta^x_y = \begin{pmatrix} \mathbb{I}_n & 0_n \\ 0_n & -\mathbb{I}_n \end{pmatrix}, \quad \tau^x_y = \begin{pmatrix} 0_n & -\mathbb{I}_n \\ \mathbb{I}_n & 0_n \end{pmatrix}. \tag{2.22}$$

The chirality acts as $\gamma^5 = \eta^t_s \delta^{\dot{t}}_{\dot{s}}$ on the spin indices, and as $\gamma_F = \eta^{\mathsf{C}}_{\mathsf{D}}\, \delta^{\mathrm{I}}_{\mathrm{J}}\, \eta^\beta_\alpha$ on the internal indices:

$$(\Gamma\Psi)^{\mathsf{CI}}_{s\dot{s}\alpha} = \eta^t_s \delta^{\dot{t}}_{\dot{s}}\ \eta^{\mathsf{C}}_{\mathsf{D}}\, \delta^{\mathrm{I}}_{\mathrm{J}}\, \eta^\beta_\alpha\ \Psi^{\mathsf{DJ}}_{t\dot{t}\beta}. \tag{2.23}$$

## 2.3 The grand algebra

Under natural assumptions (irreducibility of the representation, existence of a separating vector), a "symplectic hypothesis" and the requirement that the $KO$-dimension is 6, the most general finite algebra that satisfies the conditions for the real structure is [5]

$$\mathcal{A}_F = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \quad a \in \mathbb{N}, \tag{2.24}$$

*The multi-index $st$ after the closing parenthesis is to recall that the block-entries of the $\gamma$'s matrices are labelled by indices $s, t$ taking values in the set $\{l, r\}$. For instance the $l$-raw, $l$-column block of $\gamma^5$ is $\mathbb{I}_2$. Similarly the entries of the $\sigma$'s matrices are labelled by $\dot{s}, \dot{t}$ indices taking value in the set $\{\dot{0}, \dot{1}\}$: for instance $\sigma^{2\dot{0}}_{\ \dot{0}} = \sigma^{2\dot{1}}_{\ \dot{1}} = 0$.

†We use Einstein summation on alternated up/down indices. For any $n$ pairs of indices $(x_1, y_1)$, $(x_2, y_2)$, ... $(x_n, y_n)$, we write $\delta^{y_1 y_2 \dots y_n}_{x_1 x_2 \dots x_n}$ instead of $\delta^{y_1}_{x_1}\delta^{y_2}_{x_2}...\delta^{y_n}_{x_n}$. The indices $t, \dot{t}$, J and $\beta$ are column indices with the same range as $s, \dot{s}, \mathrm{I}, \alpha$.

acting on a Hilbert space of dimension $2(2a)^2$. To have a non-trivial grading on $\mathbb{M}_a(\mathbb{H})$ the integer $a$ must be at least 2, meaning the simplest possibility is $\mathbb{M}_2(\mathbb{H}) \oplus \mathbb{M}_4(\mathbb{C})$. The dimension of the Hilbert space is thus $2(2 \cdot 2)^2 = 32$, which is precisely the dimension of $\mathcal{H}_\mathcal{F}$ for one generation. The grading condition $[a, \Gamma] = 0$ imposes the reduction to the left-right algebra,

$$\mathcal{A}_{LR} := \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C}), \tag{2.25}$$

and the order one condition $[[D_F, a],\, JbJ^{-1}] = 0$ reduces further the algebra to $\mathcal{A}_{sm}$ in (1.4).

The case $a = 3$ requires an Hilbert space of dimension $2(2 \cdot 3)^2 = 72$, which has no obvious physical interpretation so far.

For $a = 4$, the dimension is $2(2 \cdot 4)^2 = 128$, which turns out to be precisely the dimension of the "fermion doubled" space $\mathsf{H}_F$. In other terms, the mixing of the internal and the spin degrees of freedom provides exactly the space required to represent the "grand algebra"

$$\mathcal{A}_G = \mathbb{M}_4(\mathbb{H}) \oplus \mathbb{M}_8(\mathbb{C}). \tag{2.26}$$

Any elements of $\mathcal{A}_G$ is seen as a pair of $8 \times 8$ complex matrices $Q \in \mathbb{M}_4(\mathbb{H}), M \in \mathbb{M}_8(\mathbb{C})$, each having a block structure of four $4 \times 4$ matrices

$$Q = \begin{pmatrix} Q_1^1 & Q_1^2 \\ Q_1^2 & Q_2^2 \end{pmatrix}, \quad M = \begin{pmatrix} M_1^1 & M_1^2 \\ M_1^2 & M_2^2 \end{pmatrix} \tag{2.27}$$

where $Q_i^j \in M_2(\mathbb{H})$ and $M_i^j \in M_4(\mathbb{C})$ for any $i, j = 1, 2$. By further imposing all the conditions defining a spectral triple, one intends to find back the algebra $\mathcal{A}_{sm}$ of the standard model acting suitably on $\mathcal{H}_F$. This imposes that $Q$ acts on the particle subspace $\mathsf{C} = 0$, trivially on the lepto-colour index I, meaning the complex components of each of the four $4 \times 4$ matrices $Q_i^j$ are labelled by the flavor index $\alpha$. Similarly, one asks that $M$ acts on antiparticles $\mathsf{C} = 1$, trivially on the flavor index, meaning the components of each of the four $M_i^j$ are labelled by the lepto-color index I. Identifying a matrix with its components, namely

$$Q = Q_{i\alpha}^{j\beta}, \quad M = M_{i\mathrm{I}}^{j\mathrm{J}}, \tag{2.28}$$

this means that any element $(Q, M) \in \mathcal{A}_G$ acts on $\mathsf{H}_F$ as

$$\delta_{\mathsf{C}\mathrm{I}}^{0\mathrm{J}}\, Q_{i\alpha}^{j\beta} + \delta_{\mathsf{C}\alpha}^{1\beta}\, M_{i\mathrm{I}}^{j\mathrm{J}}. \tag{2.29}$$

The representation of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ is obtained viewing $Q_{i\alpha}^{j\beta}$, $M_{i\mathrm{I}}^{j\mathrm{J}}$ no longer as constants but as $L^2$ functions on $\mathcal{M}$.

There is still some freedom on how to label the blocks of the matrices $Q$ and $M$. One simply needs indices $i, j$ that lives on the $s\dot{s}$ spinorial space, take two values each and are compatible with the order-zero condition (2.2). The natural choice is to label the blocks of either $Q$ or $M$ by the chiral index $s = r, l$ and the other blocks by (anti)-particle index $\dot{s} = \dot{0}, \dot{1}$ (although in principle one could also consider combinations of them). In [20] we chose to label the quaternions by the anti-(particle) index and the complex matrices by the chiral index,

$$Q = Q_{\dot{s}\alpha}^{\dot{t}\beta}, \quad M = M_{s\mathrm{I}}^{t\mathrm{J}}. \tag{2.30}$$

The reduction of $\mathcal{A}_G$ to the algebra of the standard model is then obtained as follows

$$\begin{aligned}
\mathcal{A}_G &= M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \\
&\Downarrow \text{ grading condition} \\
\mathcal{A}'_G &= \mathbb{M}_2(\mathbb{H})_L \oplus \mathbb{M}_2(\mathbb{H})_R \oplus M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C}) \\
&\Downarrow \text{ 1}^{\text{st}}\text{-order for the Majorana-Dirac operator } D_\nu \\
\mathcal{A}''_G &= (\mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}_R \oplus \mathbb{C}'_R) \oplus (\mathbb{C}^l \oplus M_3^l(\mathbb{C}) \oplus \mathbb{C}^r \oplus M_3^r(\mathbb{C})) \text{with } \mathbb{C}_R = \mathbb{C}^r = \mathbb{C}^l \\
&\Downarrow \text{ 1}^{\text{st}}\text{-order for the free Dirac operator } \partial\!\!\!/ \\
\mathcal{A}_{sm} &= \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})
\end{aligned} \tag{2.31}$$

The interest of the grand algebra is the possibility to generate the field $\sigma$ thanks to a fluctuation of the Majorana mass term $D_\nu$ (2.13) which respects the first-order condition imposed by this same Majorana mass term. Namely [20], and this has to be put in contrast with (2.14):

$$\text{for } A \in \mathcal{A}''_G,\ [D_\nu, A] \text{ is not necessarily zero.} \tag{2.32}$$

## 2.4 Unboundedness of the commutator

As explained in [21], there is no spectral triple for the grand algebra because the commutator $[\partial\!\!\!/, A]$ of any of its element with the free Dirac operator is never bounded. This can be seen from eq. (5.3) in [20] and has been pointed out to us by W. v. Suijlekom. In order to have bounded commutators, the action of $\mathcal{A}_G$ has to be diagonal on spinors.

**Proposition 2.1.** *Let $\mathsf{A}_F$ be a finite dimensional algebra acting on the Hilbert space $\mathsf{H}_F$ in (2.15). The commutator $[\partial\!\!\!/, A]$ of any $A \in C^\infty(\mathcal{M}) \otimes \mathsf{A}_F$ with the free Dirac operator $\partial\!\!\!/$ is bounded if and only if $\mathsf{A}_F$ acts trivially on the spinors indices $s\dot{s}$. In particular, the biggest sub-algebra of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ acting as in (2.29) and whose commutator with $\partial\!\!\!/$ is bounded is $C^\infty(\mathcal{M}) \otimes (M_2(\mathbb{H}) \oplus M_4(\mathbb{C}))$.*

*Proof.* In tensorial notation, a generic element of $\mathsf{A}_F$ is $A = A_{\mathsf{D}s\mathsf{J}\dot{s}\alpha}^{\mathsf{C}t\mathsf{I}\dot{t}\beta}$. For any such $A$, by (2.18) and omitting the indices $st\dot{s}\dot{t}$ for the Dirac matrices, one gets

$$[\partial\!\!\!/, A] = [\delta_{\mathsf{DJ}\alpha}^{\mathsf{CI}\beta}\partial\!\!\!/, A_{\mathsf{D}s\mathsf{J}\dot{s}\alpha}^{\mathsf{C}t\mathsf{I}\dot{t}\beta}] = -i[\delta_{\mathsf{DJ}\alpha}^{\mathsf{CI}\beta}\gamma^\mu, A_{\mathsf{D}s\mathsf{J}\dot{s}\alpha}^{\mathsf{C}t\mathsf{I}\dot{t}\beta}]\partial_\mu - i\gamma^\mu(\partial_\mu A_{\mathsf{D}s\mathsf{J}\dot{s}\alpha}^{\mathsf{C}t\mathsf{I}\dot{t}\beta}). \tag{2.33}$$

This is bounded if and only if the first term in the r.h.s. of the equation above is zero. The only matrices that commute with all the Dirac matrices are the multiple of the identity, hence $[\partial\!\!\!/, A]$ is bounded if and only if $A = \lambda\delta_{s\dot{s}}^{t\dot{t}} A_{\mathsf{DJ}\alpha}^{\mathsf{CI}\beta}$ for some scalar $\lambda$. This means that in (2.28) one has $Q = \lambda\delta_{s\dot{s}}^{t\dot{t}} Q_\beta^\alpha \in M_2(\mathbb{H})$ and $M = \lambda\delta_{s\dot{s}}^{t\dot{t}} M_{\mathrm{I}}^{\mathrm{J}} \in M_4(\mathbb{C})$. ■

In other term, to build a spectral triple with the grand algebra ($a = 4$ in (2.24)), one has to consider its subalgebra given by $a = 2$, that acts without mixing spinorial and internal indices. This is of course not interesting from our perspective, since the aim of the grand algebra is precisely to mix spinorial with internal degrees of freedom. A solution is to consider instead *twisted spectral triples*. They have been introduced in [14] precisely to solve the problem of the unboundedness of the commutator, which may occur in very elementary situations such as the lift to spinors of a conformal transformation. Using twists to make $[\partial\!\!\!/, A]$ bounded has been suggested independently to the second author by J.-C. Wallet, and to the first author by W. v. Suijlekom, who also brought our attention on ref. [14].

# 3 Twisting the standard model

A *twisted spectral triple*[‡] is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra acting on a Hilbert space $\mathcal{H}$ and $D$ a selfadjoint operator on $\mathcal{H}$ with compact resolvent, together with an automorphism $\rho$ of $\mathcal{A}$ such that

$$[D, a]_\rho = Da - \rho(a)D \tag{3.1}$$

is bounded for any $a \in \mathcal{A}$. It is graded if, in addition, there is a selfadjoint operator $\Gamma$ of square $\mathbb{I}$ which commutes the algebra and anticommutes with $D$.

As far as we know, the other conditions satisfied by a spectral triple have not been adapted to the twisted case yet. As long as the commutator between the algebra and the Dirac operator is not involved, one can keep the definitions of an ordinary spectral triple, for instance the order-zero condition. In the $1^{\text{st}}$-order condition (1.2) it is natural to substitute $[D, a]$ with the twisted commutator $[D, a]_\rho$. The question is whether to twist the commutator with $JbJ^{-1}$. We adopt here the first solution (this choice is discussed below proposition 3.4), assuming moreover the real structure $J$ commutes with $\rho$.

**Definition 3.1.** *A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D, \rho)$ with real structure $J$ satisfies the twisted $1^{st}$-order condition if and only if*

$$[[D, a]_\rho,\, JbJ^{-1}]_\rho = [D, a]_\rho\, JbJ^{-1} - J\rho(b)J^{-1}[D, a]_\rho = 0 \qquad \forall a, b \in \mathcal{A}. \tag{3.2}$$

## 3.1 Representation

For reasons explained in § 6.1, it is convenient to work with the other natural representation of the grand algebra than the one used in [20]. Namely instead of (2.30) one asks that quaternions carry the chiral index $s$ of spinors while the complex matrices carry the (anti)-particle index:

$$Q = Q_{s\alpha}^{t\beta}, \quad M = M_{\dot{s}\mathrm{I}}^{\dot{t}\mathrm{J}}. \tag{3.3}$$

Explicitly, the representation of the grand algebra $\mathcal{A}_G$ is

$$Q = \begin{pmatrix} Q_r^r & Q_r^l \\ Q_l^r & Q_l^l \end{pmatrix}_{st} \in M_4(\mathbb{H}), \quad M = \begin{pmatrix} M_{\dot{0}}^{\dot{0}} & M_{\dot{0}}^{\dot{1}} \\ M_{\dot{1}}^{\dot{0}} & M_{\dot{1}}^{\dot{1}} \end{pmatrix}_{\dot{s}\dot{t}} \in M_8(\mathbb{C}), \tag{3.4}$$

where for any $s, t \in \{l, r\}$ and $\dot{s}, \dot{t} \in \{\dot{0}, \dot{1}\}$ one defines

$$Q_s^t = \begin{pmatrix} Q_{sa}^{ta} & Q_{sa}^{tb} & Q_{sa}^{tc} & Q_{sa}^{td} \\ Q_{sb}^{ta} & Q_{sb}^{tb} & Q_{sb}^{tc} & Q_{sb}^{td} \\ Q_{sc}^{ta} & Q_{sc}^{tb} & Q_{sc}^{tc} & Q_{sc}^{td} \\ Q_{sd}^{ta} & Q_{sd}^{tb} & Q_{sd}^{tc} & Q_{sd}^{td} \end{pmatrix}_{\alpha\beta} \in M_2(\mathbb{H}), \quad M_{\dot{s}}^{\dot{t}} = \begin{pmatrix} M_{\dot{s}0}^{\dot{t}0} & M_{\dot{s}0}^{\dot{t}1} & M_{\dot{s}0}^{\dot{t}2} & M_{\dot{s}0}^{\dot{t}3} \\ M_{\dot{s}1}^{\dot{t}0} & M_{\dot{s}1}^{\dot{t}1} & M_{\dot{s}1}^{\dot{t}2} & M_{\dot{s}1}^{\dot{t}3} \\ M_{\dot{s}2}^{\dot{t}0} & M_{\dot{s}2}^{\dot{t}1} & M_{\dot{s}2}^{\dot{t}2} & M_{\dot{s}2}^{\dot{t}3} \\ M_{\dot{s}3}^{\dot{t}0} & M_{\dot{s}3}^{\dot{t}1} & M_{\dot{s}3}^{\dot{t}2} & M_{\dot{s}3}^{\dot{t}3} \end{pmatrix}_{\mathrm{IJ}} \in M_4(\mathbb{C}).$$

[‡]Also called $\sigma$-triple, but to avoid confusion with the field $\sigma$, we denote by $\rho$ the automorphism called $\sigma$ in [14]

Here we use $a, b, c, d$ to denote the value of the flavor index $\alpha$. On the remaining indices, $Q$ and $M$ act trivially, that is as the identity operator. The representation of $A = (Q, M) \in \mathcal{A}_G$ on $\mathsf{H}_F$ is thus

$$A^{\mathsf{C}t\mathrm{I}\dot{t}\beta}_{\mathsf{D}s\mathrm{J}\dot{s}\alpha} = \left(\delta^{\mathsf{C}\dot{t}\mathrm{I}}_{0\dot{s}\mathrm{J}}\, Q^{t\beta}_{s\alpha} + \delta^{\mathsf{C}}_{1} M^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} \delta^{t\beta}_{s\alpha}\right) = \begin{pmatrix} \delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\, Q^{t\beta}_{s\alpha} & 0_{64} \\ 0_{64} & M^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\, \delta^{t\beta}_{s\alpha} \end{pmatrix}_{\mathsf{CD}}. \tag{3.5}$$

One easily checks the order-zero condition (2.2): with $A = (R, N) \in \mathcal{A}_G$, a generic element of the opposite algebra is

$$JAJ^{-1} = -JAJ = \begin{pmatrix} -\delta^{t\beta}_{s\alpha}\, (\tau\bar{N}\tau)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} & 0_{64} \\ 0_{64} & \delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\, (\eta\bar{R}\eta)^{t\beta}_{s\alpha} \end{pmatrix}_{\mathsf{CD}} \tag{3.6}$$

where the bar denotes the complex conjugate and we used

$$\mathcal{J}R\mathcal{J} := (\tau^2)^{\dot{t}}_{\dot{s}}\, (\eta\bar{R}\eta)^{t\beta}_{s\alpha} = -\delta^{\dot{t}}_{\dot{s}}\, (\eta\bar{R}\eta)^{t\beta}_{s\alpha}, \quad \mathcal{J}N\mathcal{J} := (\eta^2)^{t}_{s}\, (\tau\bar{N}\tau)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} = \delta^{t}_{s}(\tau\bar{N}\tau)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}. \tag{3.7}$$

Obviously (3.5) commutes with (3.6).

**Lemma 3.2.** *The biggest subalgebra of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ that satisfies the grading condition* (2.1) *and has bounded commutator with $\not{D}$ is the left-right algebra $\mathcal{A}_{LR}$ given in* (2.25).

*Proof.* By (2.23), for the quaternion sector $[\Gamma, A] = 0$ amounts to asking $[\eta^t_s \eta^\beta_\alpha, Q^{t\beta}_{s\alpha}] = 0$. This imposes

$$Q = \begin{pmatrix} Q^r_r & 0_4 \\ 0_4 & Q^l_l \end{pmatrix}_{st} \tag{3.8}$$

where

$$Q^r_r = \begin{pmatrix} q^r_R & 0_2 \\ 0_2 & q^r_L \end{pmatrix}_{\alpha\beta}, \; Q^l_l = \begin{pmatrix} q^l_R & 0_2 \\ 0_2 & q^l_L \end{pmatrix}_{\alpha\beta} \quad \text{with } q^r_R, q^r_L, q^l_R, q^l_L \in \mathbb{H}. \tag{3.9}$$

For matrices, one asks $[\delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}, M^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}] = 0$ which is trivially satisfied. So the grading condition $[\Gamma, A] = 0$ imposes the reduction of $\mathcal{A}_G$ to

$$\mathcal{B}_{LR} := (\mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{H}^l_R \oplus \mathbb{H}^r_R) \oplus M_8(\mathbb{C}). \tag{3.10}$$

For $A = (Q, M) \in C^\infty(\mathcal{M}) \otimes \mathcal{B}_{LR}$, the boundedness of the commutator [§]

$$[\not{D}, A] = \begin{pmatrix} \delta^I_J\, [\not{\partial}, Q] & 0_{64} \\ 0_{64} & \delta^\beta_\alpha\, [\not{\partial}, M] \end{pmatrix}_{\mathsf{CD}} \tag{3.11}$$

means that

$$[\not{\partial}, Q] = -i\gamma^\mu(\partial_\mu Q) - i[\gamma^\mu, Q]\partial_\mu \quad \text{and} \quad [\not{\partial}, M] = -i\gamma^\mu(\partial_\mu M) - i[\gamma^\mu, M]\partial_\mu \tag{3.12}$$

are bounded. This is obtained when $Q$ and $M$ commute with all the Dirac matrices, that is are proportional to $\delta^{t\dot{t}}_{s\dot{s}}$. For $Q$ this means $Q^r_r = Q^l_l$ in (3.8), hence the reductions

$$\mathbb{H}^r_R \oplus \mathbb{H}^l_R \to \mathbb{H}_R, \quad \mathbb{H}^r_L \oplus \mathbb{H}^l_L \to \mathbb{H}_L. \tag{3.13}$$

For $M$, this means that all the components $M^{\dot{t}}_{\dot{s}}$ in (3.4) are equal, that is the reduction

$$M_8(\mathbb{C}) \to M_4(\mathbb{C}). \tag{3.14}$$

Therefore $\mathcal{B}_{LR}$ is reduced to $\mathcal{A}_{LR}$, acting diagonally on spinors. ■

[§] To lighten notation, we omit the trivial indices in the product (hence in the commutators) of operators. From (3.3) one knows that $Q$ carries the indices $s\alpha$ while $\gamma^\mu$ carries $s\dot{s}$, hence $[\not{\partial}, Q]$ carries indices $s\dot{s}\alpha$ and should be written $[\delta^\beta_\alpha \not{\partial}, \delta^{\dot{t}}_{\dot{s}} Q]$. As well, $[\not{\partial}, M]$ carries indices $s\dot{s}I$ and holds for $[\delta^J_I \not{\partial}, \delta^t_s M]$.

This lemma is nothing but a restatement of Prop. 2.1 in the peculiar representation (3.5) and taking into account the grading condition. Nevertheless, it is useful to have it explicitly, in order to understand how to get rid of the unboundedness of the commutator. It is also worth stressing the difference with the representation (2.30), for which the grading breaks both matrices and quaternions and reduces $\mathcal{A}_G$ to $\mathcal{A}'_G$. Here only quaternions are broken by the grading.

To cure the unboundedness of the commutator, the idea we propose is the following: impose the reduction (3.14) by hand, and deal with the unboundedness of $[\partial\!\!\!/, Q]$ thanks to a twist. This is a "middle term solution": imposing by hand both reductions (3.14) and (3.13) is not interesting from the grand algebra point of view, since it brings us back to an almost commutative geometry where spinorial and internal indices are not mixed; solving both the unboundedness of $[\partial\!\!\!/ Q]$ and $[\partial\!\!\!/ M]$ by a twist yields some complications discussed in §6.1. The remarkable point is that this middle term solution is sufficient to obtain the $\sigma$-field by a fluctuation that respects the twisted first-order condition of definition 3.1.

## 3.2 Twisted first-order condition for the free Dirac operator

Imposing (3.14) on the grand algebra $\mathcal{A}_G$ reduced by the grading to $\mathcal{B}_{LR}$ yields

$$\mathcal{B}' := (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r) \oplus M_4(\mathbb{C}). \tag{3.15}$$

An element $A = (Q, M)$ of $\mathcal{B}'$ is given by (3.5) where $Q$ is as in (3.8) while $M$ in (3.3) is proportional to $\delta^{\dot t}_{\dot s}$:

$$M = \delta^{\dot t}_{\dot s} M^{\mathrm{I}}_{\mathrm{J}} \in M_4(\mathbb{C}). \tag{3.16}$$

The algebra $\mathcal{B}'$ contains the algebra of the standard model $\mathcal{A}_{sm}$, and still has a part (the quaternion) that acts in a non-trivial way on the spin degrees of freedom. In this sense $\mathcal{B}'$ is still from the grand algebra side, even if it is "not so grand".

Let $\rho$ be the automorphism of $(\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r)$ that exchanges $Q_r^r$ and $Q_l^l$ in (3.8), that is the exchange

$$\mathbb{H}_R^r \leftrightarrow \mathbb{H}_R^l, \quad \mathbb{H}_L^r \leftrightarrow \mathbb{H}_L^l. \tag{3.17}$$

**Lemma 3.3.** *Denote by the same letter the extension of $\rho$ to $C^\infty(\mathcal{M}) \otimes (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r)$. Then*

$$[\partial\!\!\!/, Q]_\rho = -i\gamma^\mu(\partial_\mu Q). \tag{3.18}$$

*Proof.* One has

$$\rho\left(\begin{pmatrix} Q_r^r & 0_4 \\ 0_4 & Q_l^l \end{pmatrix}_{st}\right) = \begin{pmatrix} Q_l^l & 0_4 \\ 0_4 & Q_r^r \end{pmatrix}_{st}. \tag{3.19}$$

From (3.5), the representation of $Q$ commutes with $\sigma^\mu$ hence

$$\begin{aligned}[\gamma^\mu\partial_\mu, Q]_\rho &= \begin{pmatrix} 0_8 & \sigma^\mu\partial_\mu Q_l^l - Q_l^l\sigma^\mu\partial_\mu \\ \bar\sigma^\mu\partial_\mu Q_r^r - Q_r^r\bar\sigma^\mu\partial_\mu & 0_8 \end{pmatrix}_{st} \\ &= \begin{pmatrix} 0_8 & \sigma^\mu(\partial_\mu Q_l^l) \\ \bar\sigma^\mu\partial_\mu(Q_r^r) & 0_8 \end{pmatrix}_{st} = \gamma^\mu(\partial_\mu Q).\end{aligned} \tag{3.20}$$

∎

We still denote by the same letter the extension of $\rho$ to $C^\infty(\mathcal{M}) \otimes \mathcal{B}'$:

$$\rho((Q, M)) := ((\rho(Q), M). \tag{3.21}$$

**Proposition 3.4.** *$(C^\infty(\mathcal{M}) \otimes \mathcal{B}', \mathcal{H}, \not{D}, \rho)$ together with the graduation $\Gamma$ in (2.10) and the real structure $J$ in (2.8) is a graded twisted spectral triple which satisfies the twisted first-order condition of definition 3.1.*

*Proof.* Let $A = (Q, M) \in C^\infty(\mathcal{M}) \otimes \mathcal{B}'$. The twisted version of (3.11) is

$$[\not{D}, A]_\rho = \begin{pmatrix} \delta^I_J \, [\not{\partial}, Q]_\rho & 0_{64} \\ 0_{64} & \delta^\beta_\alpha \, [\not{\partial}, M] \end{pmatrix}_{\mathsf{CD}}. \tag{3.22}$$

From (3.16) and (3.5) $M$ commutes with $\gamma^\mu$, so that the second equation in (3.12) reduces to

$$[\not{\partial}, M] = -i\gamma^\mu(\partial_\mu M), \tag{3.23}$$

which is a bounded operator. By lemma 3.3, $[\not{\partial}, Q]_\rho = -i\gamma^\mu(\partial_\mu Q)$ is bounded as well. Hence $(C^\infty(\mathcal{M}) \otimes \mathcal{B})', \mathcal{H}, \not{D}, \rho)$ together with $\Gamma$ form a graded twisted spectral triple.

We now examine the twisted first-order condition (3.1). Let $B = (R, N) \in C^\infty(\mathcal{M}) \otimes \mathcal{B}'$. A generic element of the algebra opposite to $C^\infty(\mathcal{M}) \otimes \mathcal{B}'$ is

$$JBJ^{-1} = -JBJ = \begin{pmatrix} \delta^{t\beta}_{s\alpha} \bar{N} & 0_{64} \\ 0_{64} & \delta^{\dot{t}\mathrm{J}}_{\dot{s}\mathrm{I}} \bar{R} \end{pmatrix}_{\mathsf{CD}} \tag{3.24}$$

where we used (3.6) and noticed that for $R$ as in (3.8) and $N$ as in (3.16) one has

$$(\eta \bar{R} \eta)^{t\beta}_{s\alpha} = \bar{R}^{t\beta}_{s\alpha}, \qquad (\tau \bar{N} \tau)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} = -\bar{N}^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}. \tag{3.25}$$

As well, one has

$$J\rho(B)J^{-1} = -J\rho(B)J = \begin{pmatrix} \delta^{t\beta}_{s\alpha} \bar{N} & 0_{64} \\ 0_{64} & \delta^{\dot{t}\mathrm{J}}_{\dot{s}\mathrm{I}} \rho(\bar{R}) \end{pmatrix}_{\mathsf{CD}}. \tag{3.26}$$

Thus $[\not{D}, A]_\rho JBJ^{-1} - J\rho(B)J^{-1}[\not{D}, A]_\rho$ is a diagonal matrix with components

$$[\delta^I_J \, [\not{\partial}, Q]_\rho, \, \delta^{t\beta}_{s\alpha} \bar{N}], \qquad \delta^\beta_\alpha \, [\not{\partial}, M] \, \delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} \bar{R} - \delta^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} \rho(\bar{R}) \, \delta^\beta_\alpha \, [\not{\partial}, M]. \tag{3.27}$$

The first term vanishes because the only non-trivial index carries by $\bar{N}$ is IJ. The second term is (omitting the deltas and a global $-i$ factor)

$$\begin{aligned} &\begin{pmatrix} 0_8 & \sigma^\mu(\partial_\mu M) \\ \bar{\sigma}^\mu(\partial_\mu M) & 0_8 \end{pmatrix}_{st} \begin{pmatrix} \bar{R}^r_r & 0_8 \\ 0_8 & \bar{R}^l_l \end{pmatrix}_{st} - \begin{pmatrix} \bar{R}^l_l & 0_8 \\ 0_8 & \bar{R}^r_r \end{pmatrix}_{st} \begin{pmatrix} 0_8 & \sigma^\mu(\partial_\mu M) \\ \bar{\sigma}^\mu(\partial_\mu M) & 0_8 \end{pmatrix}_{st} \\ &= \begin{pmatrix} 0_8 & [\sigma^\mu(\partial_\mu M), \bar{R}^l_l] \\ [\bar{\sigma}^\mu(\partial_\mu M), \bar{R}^r_r] & 0_8 \end{pmatrix}_{st} \end{aligned} \tag{3.28}$$

which vanishes because $R$ only non-trivial index is $\alpha\beta$ while $\left[\bar{\sigma}^\mu(\partial_\mu M), \bar{R}^r_r\right]$ is proportional to $\delta^\beta_\alpha$. ∎

### 3.3 Twisted first-order condition for the Majorana-Dirac operator

We individuate a subalgebra $\mathcal{B}$ of $\mathcal{B}'$ such that a twisted fluctuation of the *Majorana-Dirac operator* (2.13)

$$D_\nu = \gamma^5 \otimes D_R = \eta^t_s \, \delta^{\dot{t}}_{\dot{s}} \Xi^{\mathrm{I}\beta}_{\mathrm{J}\alpha} \begin{pmatrix} 0 & k_R \\ \bar{k}_R & 0 \end{pmatrix}_{\mathsf{CD}} \tag{3.29}$$

by $\mathcal{B}$ satisfies the twisted first-order condition.

**Proposition 3.5.** *A subalgebra of $\mathcal{B}'$ wich satisfies the twisted first-order condition induced by the Majorana-Dirac operator*

$$[[D_\nu, A]_\rho, JBJ^{-1}]_\rho = 0 \tag{3.30}$$

*is*

$$\mathcal{B} := \mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{C}^l_R \oplus \mathbb{C}^r_R \oplus M_3(\mathbb{C}). \tag{3.31}$$

*Proof.* Consider first the subalgebra $\tilde{\mathcal{B}} := (\mathbb{H}^l_L \oplus \mathbb{H}^r_L \oplus \mathbb{C}^l_R \oplus \mathbb{C}^r_R) \oplus (M_3(\mathbb{C}) \oplus \mathbb{C})$ of $\mathcal{B}'$ obtained by asking that $q^l_R, q^r_R$ in (3.9) are diagonal quaternions, namely

$$q^l_R = \begin{pmatrix} c^l_R & 0 \\ 0 & \bar{c}^l_R \end{pmatrix}, \; q^r_R = \begin{pmatrix} c^r_R & 0 \\ 0 & \bar{c}^r_R \end{pmatrix} \text{ with } c^l_R, c^r_R \in \mathbb{C}; \tag{3.32}$$

while $M$ in (3.16) is of the form

$$M = \delta^{\dot{t}}_{\dot{s}} \begin{pmatrix} m & 0 \\ 0 & \mathbf{M} \end{pmatrix}_{\mathrm{IJ}} \text{ with } m \in \mathbb{C}, \mathbf{M} \in M_3(\mathbb{C}). \tag{3.33}$$

This means that $Q$ carries non-trivial indices $\dot{s}, \alpha$, while $M$ is non-trivial only in the I index. We define similarly $B = (R, N) \in \tilde{\mathcal{B}}$ with components $d^l_R, d^r_R \in \mathbb{C}$, $n \in \mathbb{C}$, $\mathbf{N} \in M_3(\mathbb{C})$. For any $A, B \in \tilde{\mathcal{B}}$, one has

$$[D_\nu, A]_\rho = \begin{pmatrix} 0_{64} & k_R(\mathsf{D}_\nu M - \rho(Q)\mathsf{D}_\nu) \\ \bar{k}_R(\mathsf{D}_\nu Q - M\mathsf{D}_\nu) & 0_{64} \end{pmatrix}_{\mathsf{CD}} \tag{3.34}$$

where we write $\mathsf{D}_\nu := \eta^t_s \, \delta^{\dot{t}}_{\dot{s}} \, \Xi^{\mathrm{I}\beta}_{\mathrm{J}\alpha}$. By (3.24), (3.26) and omitting the deltas,

$$[[D_\nu, A]_\rho, JBJ^{-1}]_\rho = \begin{pmatrix} 0_{64} & k_R\left((\mathsf{D}_\nu M - \rho(Q)\mathsf{D}_\nu)\bar{R} - \bar{N}(\mathsf{D}_\nu M - \rho(Q)\mathsf{D}_\nu)\right) \\ \bar{k}_R\left((\mathsf{D}_\nu Q - M\mathsf{D}_\nu)\bar{N} - \rho(\bar{R})(\mathsf{D}_\nu Q - M\mathsf{D}_\nu)\right) & 0_{64} \end{pmatrix}_{\mathsf{CD}}$$

The various terms entering the upper-right components of this matrix are (omitting a global $k_R$ factor)

$$\bar{N}\mathsf{D}_\nu M = (\bar{N}\Xi M)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} (\eta\Xi)^{t\beta}_{s\alpha} = \begin{pmatrix} \bar{\mathsf{n}}\mathsf{m} & 0_4 \\ 0_4 & \bar{\mathsf{n}}\mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st}, \tag{3.35}$$

$$\bar{N}\rho(Q)\mathsf{D}_\nu = (\bar{N}\Xi)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} \, (\rho(Q)\eta\Xi)^{t\beta}_{s\alpha} = \begin{pmatrix} \bar{\mathsf{n}} & 0_4 \\ 0_4 & \bar{\mathsf{n}} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}^l_R & 0_4 \\ 0_4 & -\mathsf{c}^r_R \end{pmatrix}_{st}, \tag{3.36}$$

$$\mathsf{D}_\nu M\bar{R} = (\Xi M)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} \, (\eta\Xi\bar{R})^{t\beta}_{s\alpha} = \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \bar{\mathsf{d}}^r_R & 0_4 \\ 0_4 & -\bar{\mathsf{d}}^l_R \end{pmatrix}_{st}, \tag{3.37}$$

$$\rho(Q)\mathsf{D}_\nu\bar{R} = (\Xi\delta)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}} \, (\rho(Q)\eta\Xi R)^{t\beta}_{s\alpha} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}^l_R\bar{\mathsf{d}}^r_R & 0_4 \\ 0_4 & -\mathsf{c}^r_R\bar{\mathsf{d}}^l_R \end{pmatrix}_{st}, \tag{3.38}$$

where we defined

$$\mathsf{m} := \begin{pmatrix} m & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{c}^r_R = \begin{pmatrix} c^r_R & 0 \\ 0 & 0_3 \end{pmatrix}_{\mathrm{IJ}}, \quad \mathsf{c}^l_R = \begin{pmatrix} c^l_R & 0 \\ 0 & 0_3 \end{pmatrix}_{\mathrm{IJ}} \tag{3.39}$$

and similarly for $\mathsf{d}^r_R$, $\mathsf{d}^l_R$ and $\mathbf{n}$. Collecting the various terms, one finds that the upper-right component of $[[D_\nu, A]_\rho, JBJ^{-1}]_\rho$ vanishes if and only if

$$(c^l_R - m)(\bar{d}^r_R - \bar{n}) = 0, \quad (c^r_R - m)(\bar{d}^l_R - \bar{n}) = 0. \tag{3.40}$$

Similarly, for the lower-left component of $[[D_\nu, A]_\rho, JBJ^{-1}]_\rho$ one has

$$\rho(\bar{R})M\mathsf{D}_\nu = (\Xi M)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}(\rho(\bar{R})\eta\Xi)^{t\beta}_{s\alpha} = \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \bar{\mathsf{d}}^l_R & 0_4 \\ 0_4 & -\bar{\mathsf{d}}^r_R \end{pmatrix}_{st}, \tag{3.41}$$

$$\rho(\bar{R})\mathsf{D}_\nu Q = (\Xi\delta)^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\,(\rho(\bar{R})\,\eta\Xi\,Q)^{t\beta}_{s\alpha} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}^r_R\bar{\mathsf{d}}^l_R & 0_4 \\ 0_4 & -\mathsf{c}^l_R\bar{\mathsf{d}}^r_R \end{pmatrix}_{st}, \tag{3.42}$$

$$M\mathsf{D}_\nu\bar{N} = (M\Xi\bar{N})^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\,(\eta\Xi)^{t\beta}_{s\alpha} = \begin{pmatrix} \bar{\mathsf{n}}\mathsf{m} & 0_4 \\ 0_4 & \bar{\mathsf{n}}\mathsf{m} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st}, \tag{3.43}$$

$$\mathsf{D}_\nu Q\bar{N} = (\Xi\bar{N})^{\dot{t}\mathrm{I}}_{\dot{s}\mathrm{J}}\,(\eta\Xi Q)^{t\beta}_{s\alpha} = \begin{pmatrix} \bar{\mathsf{n}} & 0_4 \\ 0_4 & \bar{\mathsf{n}} \end{pmatrix}_{\dot{s}\dot{t}} \otimes \begin{pmatrix} \mathsf{c}^r_R & 0_4 \\ 0_4 & -\mathsf{c}^l_R \end{pmatrix}_{st}, \tag{3.44}$$

yielding the same condition (6.23). Hence the twisted first-order condition is satisfied as soon as

$$c^r_R = m,\ d^r_R = n, \tag{3.45}$$

which amounts to identify $\mathbb{C}^r_R$ with $\mathbb{C}$. Hence the reduction of $\mathcal{B}'$ to $\mathcal{B}$ as defined in (6.13). ■

One could identify $\mathbb{C}^l_R$ with $\mathbb{C}$ instead of $\mathbb{C}^r_R$. This does not change the result.

As discussed before definition 3.1, one may also consider a first-order condition where only the commutator with $D$ is twisted, that is

$$[[D_\nu, A]_\rho, JBJ^{-1}] = 0. \tag{3.46}$$

This is not pertinent in our case however, for this amounts to permuting $\bar{R}^l_l$ with $\bar{R}^r_r$ in - and only in - the second term in (3.28), which then no longer vanishes as soon as $R^r_r \neq R^l_l$.

Proposition 3.5 deals only with the finite dimensional part of the spectral triple, but (3.30) is still satisfied with $A, B \in C^\infty(\mathcal{M})$ (despite the slight abuse of language in calling (3.30) a "twisted first-order condition for $D_\nu$", since on $L^2(\mathcal{M}) \otimes \mathbb{C}^{128}$ the operator $D_\nu$ does not have a compact resolvent). Proposition 3.4 is true for the subalgebra $C^\infty(\mathcal{M}) \otimes \mathcal{B}$. The twisted first-order condition (3.2) is thus true for $\not{\partial} + D_\nu$ since it is true for $\not{\partial}$ and $D_\nu$ independently. This proves the first statement of theorem 1.1.

# 4 Twisted covariant Dirac operators

In analogy with gauge fluctuation of almost commutative geometries, we call *twisted fluctuation of $D$ by $C^\infty(\mathcal{M}) \otimes \mathcal{B}$* the substitution of $D = \not{\partial} + D_\nu$ with

$$D_{\mathbb{A}} = D + \mathbb{A} + J\,\mathbb{A}\,J^{-1} \tag{4.47}$$

where $\mathbb{A}$ is twisted 1-form

$$\mathbb{A} = B^i[D, A_i]_\rho \quad A_i, B^i \in C^\infty(\mathcal{M}) \otimes \mathcal{B}. \tag{4.48}$$

We do not require $\mathbb{A}$ to be selfadjoint, we only ask that $D_{\mathbb{A}}$ is selfadjoint and called it *twisted-covariant Dirac operator*. It is the sum

$$D_{\mathbb{A}} = D_X + D_\sigma \tag{4.49}$$

of the twisted-covariant free Dirac operator

$$D_X := \not{D} + \not{\mathbb{A}} + J\not{\mathbb{A}}J^{-1} \qquad \not{\mathbb{A}} := B^i[\not{D}, A_i] \tag{4.50}$$

with the twisted-covariant Majorana-Dirac operator

$$D_\sigma := D_\nu + \mathbb{A}_\nu + J\mathbb{A}_\nu J^{-1} \qquad \mathbb{A}_\nu := B^i[D_\nu, A_i]. \tag{4.51}$$

In this section, we compute explicitly $D_X$ and $D_\sigma$, and show that they are parametrized by a vector and a scalar field.

In the following, $A_i = (Q_i, M_i)$ and $B^i = (R^i, N^i)$ are arbitrary elements of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$, where $i$ a summation index and

$$Q_i = \begin{pmatrix} Q^r_{ri} & 0_4 \\ 0_4 & Q^l_{li} \end{pmatrix}_{st}, \quad M_i = \delta^{\dot{t}}_{\dot{s}} \begin{pmatrix} c^r_i & 0 \\ 0 & \mathsf{M}_i \end{pmatrix}_{\mathsf{IJ}} \tag{4.52}$$

with¶ $\mathsf{M}_i \in M_3(\mathbb{C})$ and

$$Q^r_{ri} = \begin{pmatrix} q^r_{Ri} & 0_2 \\ 0_2 & q^r_{Li} \end{pmatrix}_{\alpha\beta}, \; Q^l_{li} = \begin{pmatrix} q^l_{Ri} & 0_2 \\ 0_2 & q^l_{Li} \end{pmatrix}_{\alpha\beta} \tag{4.53}$$

with $q^l_{Li} \in \mathbb{H}^l_L$, $q^r_{Li} \in \mathbb{H}^r_L$ and

$$q^r_{Ri} = \mathrm{diag}\,(c^r_i, \bar{c}^r_i), \quad q^l_{Ri} = \mathrm{diag}\,(c^l_i, \bar{c}^l_i) \quad \text{with} \quad c^r_i \in \mathbb{C}^r_R,\; c^l_i \in \mathbb{C}^l_R. \tag{4.54}$$

The components $R^i, N^i$ of $B^i$ are defined similarly, with

$$d^{ri} \in \mathbb{C}^r_R,\; d^{li} \in \mathbb{C}^l_R, \quad r^{ri}_L \in \mathbb{H}^r_L,\; r^{ri}_L \in \mathbb{H}^l_L \quad \text{and} \quad \mathsf{N}_i \in M_3(\mathbb{C}). \tag{4.55}$$

### 4.1 Twisted-covariant free Dirac operator $D_X$

**Proposition 4.1.** *The twisted fluctuation* (4.50) *of the free Dirac operator* $\not{D} = \not{\partial} \otimes \mathbb{I}$ *by* $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ *is*

$$D_X = -i \begin{pmatrix} \gamma^\mu \left(\delta^{I\beta}_{J\alpha}\partial_\mu + X_\mu\right) & 0_{64} \\ 0_{64} & \gamma^\mu \left(\delta^{I\beta}_{J\alpha}\partial_\mu - \bar{X}_\mu\right) \end{pmatrix}_{\mathsf{CD}} \tag{4.56}$$

*where we define the bounded-operator valued vector field* ‖

$$X_\mu := \delta^I_J \,\rho(R^i)\, \partial_\mu Q_i - \delta^\beta_\alpha\, \bar{N}^i \partial_\mu \bar{M}_i. \tag{4.57}$$

*Proof.* Given $A_i = (Q_i, M_i)$ and $B^i = (R^i, N^i)$ in $\mathcal{B}$, one gets from (3.22), (3.23) and (3.18)

$$\not{\mathbb{A}} = -iB^i[\not{D}, A_i]_\rho = -i \begin{pmatrix} \delta^I_J\, \gamma^\mu \rho(R^i)\partial_\mu Q_i & 0_{64} \\ 0_{64} & \delta^\beta_\alpha\, \gamma^\mu N^i \partial_\mu M_i \end{pmatrix}_{\mathsf{CD}} \tag{4.58}$$

where we used that $N^i$ commutes with $\gamma^\mu$ while, by explicit computation and remembering that $R^i$ commutes with the $\sigma$'s matrices, one has

$$R^i \gamma^\mu = \gamma^\mu \rho(R^i). \tag{4.59}$$

¶In all this section, the components of the matrices are functions on $\mathcal{M}$. To lighten notation we write $M_3(\mathbb{C})$ instead of $C^\infty(\mathcal{M}) \otimes M_3(\mathbb{C})$. The same is true for the various copies of $\mathbb{H}$ and $\mathbb{C}$.

‖To lighten notations we omit the parenthesis around $\partial_\mu Q_i$ and $\partial_\mu \bar{M}_i$: the latter are bounded operators and act as matrices, not as differential operators.

By (2.21) one gets

$$J\mathbb{A}J^{-1} = -J\mathbb{A}J = i\begin{pmatrix} \delta_\alpha^\beta\,\gamma^\mu \bar{N}^i\partial_\mu\bar{M}_i & 0_{64} \\ 0_{64} & \delta_J^I\gamma^\mu\rho(\bar{R}^i)\partial_\mu\bar{Q}_i \end{pmatrix}_{\mathsf{CD}} \tag{4.60}$$

where we used that $\mathcal{J}$ anti-commutes with the $\gamma$'s matrices,** so that

$$\mathcal{J}\gamma^\mu N^i\partial_\mu M_i\mathcal{J} = -\gamma^\mu\mathcal{J}N^i\partial_\mu M_i\mathcal{J} = \gamma^\mu\bar{N}^i\partial_\mu\bar{M}_i, \tag{4.61}$$
$$\mathcal{J}\gamma^\mu\rho(R^i)\partial_\mu Q_i = -\gamma^\mu\mathcal{J}\rho(R^i)\partial_\mu Q_i\mathcal{J} = \gamma^\mu\rho(\bar{R}^i)\partial_\mu\bar{Q}_i. \tag{4.62}$$

Collecting all the terms, one obtains the lemma. ■

**Lemma 4.2.** *$D_X$ is selfadjoint, and called twisted-covariant free Dirac operator, if and only if for any $\mu = 0, 1, 2, 3$ one has*

$$\rho(X_\mu) = -X_\mu^\dagger. \tag{4.63}$$

*Proof.* In the $st$ indices, $X_\mu$ is a block diagonal matrix which is proportional to $\delta_{\dot{s}}^{\dot{t}}$,

$$X_\mu = \delta_{\mathrm{J}\dot{s}}^{\mathrm{I}\dot{t}}\begin{pmatrix} R^{il}_{\;\;l}\partial_\mu Q^{\;\;r}_{ir} & 0_4 \\ 0_4 & R^{ir}_{\;\;r}\partial_\mu Q^{\;\;l}_{il} \end{pmatrix}_{st} - \delta^{\beta t\dot{t}}_{\alpha s\dot{s}}\bar{N}^i\partial_\mu\bar{M}_i =: \delta_{\dot{s}}^{\dot{s}}\begin{pmatrix} X_\mu^r & 0_{32} \\ 0_{32} & X_\mu^l \end{pmatrix}_{st}, \tag{4.64}$$

so that

$$\gamma^\mu X_\mu = \begin{pmatrix} 0_{32} & \sigma^\mu X_\mu^l \\ \tilde{\sigma}^\mu X_\mu^r & 0_{32} \end{pmatrix}_{st} \tag{4.65}$$

Since $X_\mu$ commutes with the $\sigma$'s matrices and $(\sigma^\mu)^\dagger = \tilde{\sigma}^\mu$, one has

$$(\gamma^\mu X_\mu)^\dagger = \begin{pmatrix} 0_{32} & \sigma^\mu(X_\mu^r)^\dagger \\ \tilde{\sigma}^\mu(X_\mu^l)^\dagger & 0_{32} \end{pmatrix}_{st} = \gamma^\mu\rho(X_\mu^\dagger), \tag{4.66}$$

so that $\gamma^\mu X_\mu$ is selfadjoint iff

$$\sigma^\mu(X_\mu^r)^\dagger = \sigma^\mu X_\mu^l. \tag{4.67}$$

Since $\mathrm{Tr}\,\bar{\sigma}^\nu\sigma^\mu = 2\delta_\nu^\mu$ and both $X_\mu^r$ and $X_\mu^l$ are proportional to $\delta_{\dot{s}}^{\dot{t}}$, the partial trace on the $\dot{s}\dot{t}$ indices of the above equation, where both side have been multiplied by $\bar{\sigma}^\lambda$, yields $(X_\mu^r)^\dagger = X_\mu^l$ for any $\mu$, that is

$$X_\mu^\dagger = \rho(X_\mu). \tag{4.68}$$

The lemma is obtained noticing that $D_X$ is selfadjoint if and only if $iX^\mu$ is selfadjoint, that is $X^\mu$ is anti-selfadjoint. ■

### 4.2 Twisted-covariant Majorana-Dirac operator $D_\sigma$

**Lemma 4.3.** *For $A = (Q, M) \in \mathcal{B}$ with components $c^r, c^l \in \mathbb{C}$ as in (4.54), one has*

$$[D_\nu, A]_\rho = \begin{pmatrix} 0_2 & k_R(c^r - c^l)\mathcal{S} \\ \bar{k}_R(c^r - c^l)\mathcal{S}' & 0_2 \end{pmatrix}_{\mathsf{CD}}\delta_{\dot{s}}^{\dot{t}}\,\Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \tag{4.69}$$

*where*

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{st}, \quad \mathcal{S}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{st}. \tag{4.70}$$

** $\{\mathcal{J}, \gamma^\mu\} = i(\gamma^0\gamma^2\bar{\gamma}^\mu + \gamma^\mu\gamma^0\gamma^2) = 0$ because $\bar{\gamma}^\mu = -\gamma^\mu$ for $\mu = 1, 3$, $\bar{\gamma}^\mu = \gamma^\mu$ for $\mu = 0, 2$.

*Proof.* Computing explicitly (3.34) with notations (3.39) yields

$$\mathsf{D}_\nu M - \rho(Q)\mathsf{D}_\nu = (\Xi M)^{t\mathrm{I}}_{\dot{s}\mathrm{J}} (\kappa \Xi)^{t\beta}_{s\alpha} - (\Xi\delta)^{t\mathrm{I}}_{\dot{s}\mathrm{J}} \, (\rho(Q)\kappa\Xi)^{t\beta}_{s\alpha} \tag{4.71}$$

$$= \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}t} \otimes \begin{pmatrix} k_R \Xi & 0_4 \\ 0_4 & -k_R\Xi \end{pmatrix}_{st} - \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}t} \otimes \begin{pmatrix} k_R \mathsf{c}^l_R & 0_4 \\ 0_4 & -k_R \mathsf{c}^r_R \end{pmatrix}_{st}$$

$$= \begin{pmatrix} \begin{pmatrix} k_R(m - c^l_R)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0 \\ 0 & k_R(m - c^l_R)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\dot{s}t} & 0_{32} \\ 0_{32} & \begin{pmatrix} -k_R(m - c^r_R)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0 \\ 0 & -k_R(m - c^r_R)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\dot{s}t} \end{pmatrix}_{st}$$

$$\mathsf{D}_\nu{}^\dagger Q - M \mathsf{D}^\dagger_\nu = (\Xi\delta)^{t\mathrm{I}}_{\dot{s}\mathrm{J}} \, (\kappa \Xi Q)^{t\beta}_{s\alpha} - (\Xi M)^{t\mathrm{I}}_{\dot{s}\mathrm{J}} (\kappa\Xi)^{t\beta}_{s\alpha} \tag{4.72}$$

$$= \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}t} \otimes \begin{pmatrix} \bar{k}_R \mathsf{c}^r_R & 0_4 \\ 0_4 & -\bar{k}_R \mathsf{c}^l_R \end{pmatrix}_{st} - \begin{pmatrix} \mathsf{m} & 0_4 \\ 0_4 & \mathsf{m} \end{pmatrix}_{\dot{s}t} \otimes \begin{pmatrix} \bar{k}_R \Xi & 0_4 \\ 0_4 & -\bar{k}_R\Xi \end{pmatrix}_{st}$$

$$= \begin{pmatrix} \begin{pmatrix} \bar{k}_R(c^r_R - m)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0 \\ 0 & \bar{k}_R(c^r_R - m)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\dot{s}t} & 0_{32} \\ 0_{32} & \begin{pmatrix} -\bar{k}_R(c^l_R - m)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} & 0 \\ 0 & -\bar{k}_R(c^l_R - m)\, \Xi^{\beta\mathrm{J}}_{\alpha\mathrm{I}} \end{pmatrix}_{\dot{s}t} \end{pmatrix}_{st}.$$

Identifying $c^r_R$ with $m$ following (3.45) yields the result, where we drop the index $R$ to match notation (4.54). ■

**Proposition 4.4.** *The selfadjoint twisted fluctuation* (4.51) *of the Majorana-Dirac operator* $D_\nu = \gamma^5 \otimes\ D_R$ *by* $\mathcal{B}$*, called twisted-covariant Majorana-Dirac operator, is*

$$D_\sigma = \boldsymbol{\sigma}\gamma^5 \otimes D_R \tag{4.73}$$

*where*

$$\boldsymbol{\sigma} = (\mathbb{I} + \gamma^5 \phi) \tag{4.74}$$

*with* $\phi$ *a real scalar field.*

*Proof.* Let $B^i = (R^i, N^i)$ as in (4.55). From lemma 4.3 one gets

$$\mathbb{A}_\nu = B^i [D_\nu, A_i]_\rho = \ \phi \begin{pmatrix} 0_2 & k_R\, \mathcal{S} \\ \bar{k}_R\, \mathcal{S}' & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\ \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha} \tag{4.75}$$

where

$$\phi := d^{ir}(c^r_i - c^l_i). \tag{4.76}$$

One has $\mathcal{J}(\mathcal{S}\delta^{\dot{t}}_{\dot{s}})\mathcal{J} = -\mathcal{S}\delta^{\dot{t}}_{\dot{s}}$ and $\mathcal{J}(\mathcal{S}'\delta^{\dot{t}}_{\dot{s}})\mathcal{J} = -\mathcal{S}'\delta^{\dot{t}}_{\dot{s}}$. Hence

$$J\mathbb{A}_\nu J^{-1} = -J\mathbb{A}_\nu J = \bar{\phi} \begin{pmatrix} 0_2 & k_R \mathcal{S}' \\ \bar{k}_R \mathcal{S} & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\ \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha} \tag{4.77}$$

so that

$$D_\nu + \mathbb{A}_\nu + J\mathbb{A}_\nu J^{-1} = \begin{pmatrix} 0_2 & k_R\,(\eta^t_s + \phi\,\mathcal{S} + \bar{\phi}\,\mathcal{S}') \\ \bar{k}_R\,(\eta^t_s + \phi\,\mathcal{S}' + \bar{\phi}\mathcal{S}) & 0_2 \end{pmatrix}_{\mathsf{CD}} \delta^{\dot{t}}_{\dot{s}}\ \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha}. \tag{4.78}$$

It is selfadjoint if and only if $\phi = \bar{\phi}$. Then

$$D_\sigma := D_\nu + \mathbb{A}_\nu + J\mathbb{A}_\nu J^{-1} = \begin{pmatrix} 0_4 & k_R(\gamma^5 + \phi\mathbb{I}_4) \\ \bar{k}_R(\gamma^5 + \phi\mathbb{I}_4) & 0_4 \end{pmatrix}_{\mathsf{CD}} \Xi^{\mathrm{J}\beta}_{\mathrm{I}\alpha}, \tag{4.79}$$

$$= (\gamma^5 + \phi\mathbb{I}) \otimes D_R. \tag{4.80}$$

Factorizing by $\gamma^5$, one gets the result. ■

Propositions 4.1 and 4.4 prove the second statement of theorem 1.1. The field $\boldsymbol{\sigma}$ in (4.74) is slightly different from the one obtained in [20] from a non-twisted fluctuation of $D_\nu$ by $\mathcal{A}_{sm} \otimes C^\infty(\mathcal{M})$, namely

$$\sigma = (1+\phi)\mathbb{I}. \tag{4.81}$$

We comment on that in the conclusion.

# 5 Breaking of the grand symmetry to the standard model

We now prove the last point of theorem 1.1, namely that the breaking of the grand algebra to the standard model is dynamical.

## 5.1 Spectral action

A striking application of noncommutative geometry to physics is to give a gravitational interpretation of the standard model [12]. By this, one intends that the bosonic part of the SM Lagrangian is deduced from an action which is purely geometric, that is which depends only the spectrum of the covariant Dirac operator $D_A$. The most obvious way to define such an action consists in counting the eigenvalues lower than a given energy scale $\Lambda$. This is the spectral action [4]

$$S = \operatorname{Tr} f\left(\frac{D_A^2}{\Lambda^2}\right) \tag{5.1}$$

where $f$ is a cutoff function, usually the (smoothened) characteristic function on the interval $[0,1]$, and $\Lambda$ is an energy scale. It has an asymptotic expansion in power series of $\Lambda^{-1}$,

$$S = \sum_{n\geq 0} f_{4-n}\, a_n(D_A^2/\Lambda^2) \tag{5.2}$$

where the $f_n$ are the momenta of $f$ and the $a_n$ the Seeley-de Witt coefficients which are nonzero only for $n$ even. Writing $D_A^2$ as an elliptic operator of Laplacian type,

$$D_A^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu + \alpha^\mu\partial_\mu + \beta) \tag{5.3}$$

these coefficients are functions of

$$\begin{aligned} \omega_\mu &= \frac{1}{2} g_{\mu\nu}\left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\right), \quad \Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \\ E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right). \end{aligned} \tag{5.4}$$

The first coefficients are [25,32]

$$a_0 = \frac{\Lambda^4}{16\pi^2}\int dx^4\sqrt{g}\,\operatorname{Tr}(Id), \tag{5.5}$$

$$a_2 = \frac{\Lambda^2}{16\pi^2}\int dx^4\sqrt{g}\,\operatorname{Tr}\left(-\frac{R}{6} + E\right)$$

$$a_4 = \frac{1}{16\pi^2}\frac{1}{360}\int dx^4\sqrt{g}\,\operatorname{Tr}(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \tag{5.6}$$

$$+2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu}) \tag{5.7}$$

where $R_{\mu\nu}$ is the Ricci tensor and $-R$ the scalar curvature. Applied to the spectral triple (2.3) of the standard model, fluctuated according to (2.11), the expansion (5.1) yields the bosonic part of Lagrangian of the standard model - including the Higgs - minimally coupled with gravity [7, Sect. 4.1]. For the fermionic action and how it can be related to the spectral action see [1], and for a complete and pedagogical treatment of the subject, see the recent book [31].

We prove the third point of theorem 1.1, namely that the breaking of the grand algebra to the standard model is dynamical, by computing the spectral action (5.1) for the twisted-covariant free Dirac operator $D_X$. More precisely we show that the potential of the vector field $X_\mu$, that is the part of

$$V := \Lambda^2 f_2 \mathrm{Tr}\, E + \frac{1}{2} f_0 \mathrm{Tr}\; E^2 \tag{5.8}$$

that does not depend on the derivative of $X_\mu$, is minimum when $\not{D}$ is fluctuated by a subalgebra of $\mathcal{B} \otimes C^\infty(\mathcal{M})$ which is invariant under the automorphism $\rho$. The biggest such subalgebra is $\mathcal{A}_{sm} \otimes C^\infty(\mathcal{M})$, since by (3.21) an element $(Q, M)$ of $\mathcal{B}$ is invariant by the automorphism $\rho$ if and only if

$$\rho(Q) = Q, \tag{5.9}$$

which means $\mathbb{H}_R^r = \mathbb{H}_R^l$ and $\mathbb{C}_L^r = \mathbb{C}_L^l$, that is $\mathcal{B} \to \mathcal{A}_{sm}$.

### 5.2 Breaking by $D_X$

For simplicity we restrict to the flat case, so that (5.4) reduces to

$$\nabla_\mu = \partial_\mu + \omega_\mu\, ,\ \omega_\mu = \frac{1}{2} g_{\mu\nu}\alpha^\nu\, ,\ E = \beta - g^{\mu\nu}\left(\partial_\mu \omega_\nu + \omega_\mu\omega_\nu\right). \tag{5.10}$$

We take as a dynamical parameter the vector field $X_\mu$, and work out the part of the potential $E$ that does not depend on its derivative.

**Lemma 5.1.** *One has* $D_X^2 = -\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \alpha_X^\mu\partial_\mu + \beta_X\right)$ *where*

$$\beta_X \;:=\; \begin{pmatrix} \gamma^\mu\gamma^\nu(\partial_\mu X_\nu) + \gamma^\mu\gamma^\nu\rho(X_\mu)X_\nu & 0_{64} \\ 0_{64} & X_\mu \leftrightarrow -\bar{X}_\mu \end{pmatrix}_{\mathsf{CD}} \tag{5.11}$$

$$\alpha_X^\mu \;:=\; \begin{pmatrix} \gamma^\mu\gamma^\nu\Delta_\nu + 2g^{\mu\nu}\rho(X_\nu) & 0_{64} \\ 0_{64} & X_\mu \leftrightarrow -\bar{X}_\mu \end{pmatrix}_{\mathsf{CD}} \tag{5.12}$$

*with*

$$\Delta_\mu := X_\mu - \rho(X_\mu). \tag{5.13}$$

*Proof.* Using

$$X_\mu\gamma^\nu = \gamma^\nu\rho(X_\mu) \tag{5.14}$$

which follows from (4.59) and the definition of $X^\mu$, the square of (4.56) writes

$$D_X^2 = \begin{pmatrix} -\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - \gamma^\mu\gamma^\nu\rho(X_\mu)\partial_\nu - \gamma^\mu\gamma^\nu(\partial_\mu X_\nu) - \gamma^\mu\gamma^\nu X_\nu\partial_\mu - \gamma^\mu\gamma^\nu\rho(X_\mu)X_\nu & 0_{64} \\ 0_{64} & X_\mu \leftrightarrow -\bar{X}_\mu. \end{pmatrix}_{\mathsf{CD}}$$

This is of the form claimed in the corollary, with $\beta_X$ as in (5.11) and

$$\alpha_X^\mu = \begin{pmatrix} \gamma^\nu\gamma^\mu\rho(X_\nu) + \gamma^\mu\gamma^\nu X_\nu & 0_{64} \\ 0_{64} & X_\mu \leftrightarrow -\bar{X}_\mu \end{pmatrix}_{\mathsf{CD}}. \tag{5.15}$$

The form (5.12) of $\alpha_X^\mu$ is obtained thanks to the anticommutation rules of Dirac matrices:

$$\gamma^\nu\gamma^\mu\rho(X_\nu) + \gamma^\mu\gamma^\nu X_\nu = -\gamma^\mu\gamma^\nu\rho(X_\nu) + 2g^{\mu\nu}\rho(X_\nu) + \gamma^\mu\gamma^\nu X_\nu = \gamma^\mu\gamma^\nu\Delta_\nu + 2g^{\mu\nu}\rho(X_\nu). \qquad \blacksquare$$

**Lemma 5.2.** *Define* $\omega_\mu^X := g_{\mu\nu}\alpha_X^\nu$. *One has*

$$g^{\mu\nu}\omega_\mu^X\omega_\nu^X = \begin{pmatrix} W(X) & 0_{64} \\ 0_{64} & W(-\bar{X}) \end{pmatrix}_{\mathsf{CD}} \tag{5.16}$$

*with*

$$W(X) := \frac{1}{2}\not{\Delta}\not{\Delta} + \frac{1}{2}\not{\Delta}\not{\rho}(X) + \Delta\cdot\rho(X) + \frac{1}{2}\not{X}\not{\Delta} + \rho(X)\cdot\rho(X) \tag{5.17}$$

*where*

$$\not{X} = \gamma^\mu X_\mu, \quad \not{\rho}(X) = \gamma^\mu\rho(X_\mu)), \quad \not{\Delta} = \gamma^\mu\Delta_\mu \tag{5.18}$$

*and* $\cdot$ *is the inner product defined by* $g^{\mu\nu}$.

*Proof.* We make the computation for the first component of the matrix $\alpha_X^\mu$, that we still denote $\alpha_X^\mu$. The computation for the other component is similar. Using $g_{\mu\nu}g^{\nu\tau} = \delta_\mu^\tau$, one gets

$$\omega_\mu^X = \frac{1}{2}g_{\mu\nu}\alpha_X^\nu = \frac{1}{2}g_{\mu\nu}\left(\gamma^\nu\gamma^\tau\Delta_\tau + 2g^{\nu\tau}\rho(X_\tau)\right) = \frac{1}{2}g_{\mu\nu}\gamma^\nu\gamma^\tau\Delta_\tau + \rho(X_\mu).$$

Thus

$$\begin{aligned} \omega_\mu^X\omega_\nu^X &= \left[\frac{1}{2}g_{\mu\rho}\gamma^\rho\gamma^\tau\Delta_\tau + \rho(X_\mu)\right]\left[\frac{1}{2}g_{\nu\sigma}\gamma^\sigma\gamma^\delta\Delta_\delta + \rho(X_\nu)\right] \\ &= \frac{1}{4}g_{\mu\rho}\gamma^\rho\gamma^\tau\Delta_\tau g_{\nu\sigma}\gamma^\sigma\gamma^\delta\Delta_\delta + \frac{1}{2}g_{\mu\rho}\gamma^\rho\gamma^\tau\Delta_\tau\rho(X_\nu) + \\ &\quad + \frac{1}{2}\rho(X_\mu)g_{\nu\sigma}\gamma^\sigma\gamma^\delta\Delta_\delta + \rho(X_\mu)\rho(X_\nu) \end{aligned} \tag{5.19}$$

so that

$$g^{\mu\nu}\omega_\mu^X\omega_\nu^X = \frac{1}{4}\gamma^\nu\gamma^\tau\Delta_\tau\gamma_\nu\gamma^\delta\Delta_\delta + \frac{1}{2}\gamma^\nu\gamma^\tau\Delta_\tau\rho(X_\nu) + \frac{1}{2}\rho(X_\mu)\gamma^\mu\gamma^\delta\Delta_\delta + \rho(X)\cdot\rho(X) \tag{5.20}$$

where we used that $g^{\mu\nu}$ acts as $g^{\mu\nu}\mathbb{I}$, hence commutes with all other operators.

Using

$$\Delta_\tau\gamma_\nu = \gamma_\nu\rho(\Delta_\tau) = -\gamma_\nu\Delta_\tau \tag{5.21}$$

together with $\gamma^\nu\gamma^\tau\gamma_\nu = -2\gamma^\tau$, the first term in the equation above is

$$\frac{1}{4}\gamma^\nu\gamma^\tau\Delta_\tau\gamma_\nu\gamma^\delta\Delta_\delta = \frac{1}{2}\gamma^\tau\Delta_\tau\gamma^\delta\Delta_\delta = \frac{1}{2}\not{\Delta}\not{\Delta}. \tag{5.22}$$

The second term is

$$\begin{aligned} \frac{1}{2}\gamma^\nu\gamma^\tau\Delta_\tau\rho(X_\nu) &= \frac{1}{2}\left(-\gamma^\tau\gamma^\nu + 2g^{\tau\nu}\right)\Delta_\tau\rho(X_\nu) \\ &= -\frac{1}{2}\gamma^\tau\rho(\Delta_\tau)\gamma^\nu\rho(X_\nu) + g^{\tau\nu}\Delta_\tau\rho(X_\nu) = \frac{1}{2}\not{\Delta}\not{\rho}(X) + \Delta\cdot\rho(X). \end{aligned}$$

The third term is $\frac{1}{2}\not{X}\not{\Delta}$ since $\rho(X_\mu)\gamma^\mu = \gamma^\mu X_\mu$. Hence the result. ■

**Corollary 5.3.** *The part of the potential* $E$ *that does not depend on the derivative of the* $X_\mu$ *is*

$$E_X = \begin{pmatrix} \frac{1}{2}\not{X}\not{\rho}(X) + \frac{1}{2}\not{\rho}(X)\not{X} - X\cdot\rho(X) & 0_{64} \\ 0_{64} & X_\mu \leftrightarrow -\bar{X}_\mu \end{pmatrix}_{\mathsf{CD}}. \tag{5.23}$$

*Proof.* As in lemma 5.2, we write the proof for the first component of the matrices in the CD indices. The part of $\beta$ that does not depend on the derivative of $X_\mu$ is

$$\beta_X^0 := \gamma^\mu\gamma^\nu\rho(X_\mu)X_\nu = \gamma^\mu X_\mu \gamma^\nu X_\nu = \not{X}\not{X}. \tag{5.24}$$

By lemma 5.2 one has

$$E_X = \beta_X^0 - W(X) = \not{X}\not{X} - \frac{1}{2}\not{\Delta}\not{\Delta} - \frac{1}{2}\not{\Delta}\rho(\not{X}) - \Delta\cdot\rho(X) - \frac{1}{2}\not{X}\not{\Delta} - \rho(X)\cdot\rho(X). \tag{5.25}$$

The result is obtained substituting $\not{\Delta}$ with $\not{X} - \not{\rho}(X)$, in agreement with definition (5.13). ∎

**Proposition 5.4.** *The trace of $E_X$ and $E_X^2$ are positive, and vanish for $\rho(X_\mu) = X_\mu$.*

*Proof.* As above we work only on the first component of the CD matrix. Obvious manipulations on (5.23) using (5.14) yields

$$\mathrm{Tr}(E_X) = \frac{1}{2}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\left(\rho(X_\mu)\rho(X_\nu) + X_\mu X_\nu\right) - 2g^{\mu\nu}X_\mu\rho(X_\nu)\right] \tag{5.26}$$

In the $st$ indices, $X_\mu$ is the block diagonal matrix which is proportional to $\delta_{\dot{s}}^{\dot{t}}$:

$$X_\mu = \delta_{\mathrm{J}\dot{s}}^{\mathrm{I}\dot{t}}\begin{pmatrix} R_l^{il}\partial_\mu Q_{ir}^{\;r} & 0_4 \\ 0_4 & R_l^{il}\partial_\mu Q_{ir}^{\;r}\end{pmatrix}_{st} - \delta_{\alpha s\dot{s}}^{\beta t\dot{t}}\bar{N}^i\partial_\mu\bar{M}_i =: \delta_{\dot{s}}^{\dot{s}}\begin{pmatrix} X_\mu^r & 0_{32} \\ 0_{32} & X_\mu^l\end{pmatrix}_{st}. \tag{5.27}$$

Let tr denote the partial trace on the spinorial indices $s\dot{s}$. One obtains

$$\begin{aligned}
\mathrm{tr}\left(\gamma^\mu\gamma^\nu X_\mu X_\nu\right) &= \mathrm{tr}\begin{pmatrix}\sigma^\mu\bar{\sigma}^\nu X_\mu^r X_\nu^r & 0_{64} \\ 0_{64} & \bar{\sigma}^\mu\sigma^\nu X_\mu^l X_\nu^l\end{pmatrix}_{st} && (5.28)\\
&= \begin{pmatrix}\dot{\mathrm{tr}}(\sigma^\mu\bar{\sigma}^\nu)\, X_\mu^r X_\nu^r & 0_{64} \\ 0_{64} & \dot{\mathrm{tr}}(\bar{\sigma}^\mu\sigma^\nu)\, X_\mu^l X_\nu^l\end{pmatrix}_{st} && (5.29)\\
&= \begin{pmatrix}2\delta^{\mu\nu}\, X_\mu^r X_\nu^r & 0_{64} \\ 0_{64} & 2\delta^{\mu\nu}\, X_\mu^l X_\nu^l\end{pmatrix}_{st} = 2\,\mathsf{X}^\mu\mathsf{X}_\mu && (5.30)
\end{aligned}$$

where $\dot{\mathrm{tr}}$ denote the trace on the $\dot{s}$ index only and $\mathsf{X}_\mu = \dot{\mathrm{tr}}X_\mu$. Similarly

$$\mathrm{tr}\left[\gamma^\mu\gamma^\nu\rho(X_\mu)\rho(X_\nu)\right] = 2\rho(\mathsf{X}^\mu)\rho(\mathsf{X}_\mu) \tag{5.31}$$

and

$$\mathrm{tr}\left[g^{\mu\nu}X_\mu\rho(X_\nu)\right] = 2\mathsf{X}^\mu\rho(\mathsf{X}_\mu). \tag{5.32}$$

Therefore

$$\mathrm{tr}\,E_X = \mathsf{X}^\mu\mathsf{X}_\mu + \rho(\mathsf{X}^\mu)\rho(\mathsf{X}_\mu) - 2\mathsf{X}^\mu\rho(\mathsf{X}_\mu). \tag{5.33}$$

Taking the trace on the remaining indices, one gets

$$\begin{aligned}
\mathrm{Tr}\,E_X &= \mathrm{Tr}\left(\mathsf{X}^\mu\mathsf{X}_\mu + \rho(\mathsf{X}^\mu)\rho(\mathsf{X}_\mu) - 2\mathsf{X}^\mu\rho(\mathsf{X}_\mu)\right) && (5.34)\\
&= \mathrm{Tr}\left(\mathsf{X}^\mu - \rho(\mathsf{X}^\mu)\right)\left(\mathsf{X}_\mu - \rho(\mathsf{X}_\mu)\right) && (5.35)\\
&= \sum_\mu \mathrm{Tr}\;\Delta_\mu^\dagger\Delta_\mu. && (5.36)
\end{aligned}$$

where we observed that lemma 4.2 yields

$$\Delta_\mu^\dagger = (X_\mu - \rho(X_\mu))^\dagger = -\rho(X_\mu) + X_\mu = \Delta_\mu. \tag{5.37}$$

Being the sum of traces of positive operators, (5.36) is positive. It is zero if and only if $X_\mu = \rho(X_\mu)$ for any $\mu = 0, 1, 2, 3$.

By lemma 4.2, $E_X$ is a selfadjoint matrix, hence $E_X^2$ is a positive operator and its trace is never negative. It vanishes when $E_X = 0$, that is $\rho(X_\mu) = X_\mu$. ■

The last point is to check that the invariance of $X_\mu$ under the twist implies the invariance of its components $R^i, Q_i$.

**Proposition 5.5.** *The biggest unital subalgebra of $\mathcal{B} \otimes \mathcal{C}^\infty(\mathcal{M})$ for wich any combination*

$$X_\mu = \delta_J^I\, \rho(R^i)\, \partial_\mu Q_i - \delta_\alpha^\beta\, \bar{N}^i \partial_\mu \bar{M}_i \tag{5.38}$$

*is invariant under the twist is $\mathcal{A}_{SM} \otimes C^\infty(\mathcal{M})$.*

*Proof.* Let $\mathcal{G}$ be any subalgebra of $\mathcal{B} \otimes \mathcal{C}^\infty(\mathcal{M})$ such that any linear combinations $X_\mu$ with $(R^i, N^i)$ and $(Q_i, M_i)$ in $\mathcal{G}$ is invariant under the automorphism $\rho$. This means in particular that for $X = R\partial_\mu Q - Q\partial_\mu R$ with $R, Q$ arbitrary elements in $\mathcal{G}$, one has

$$\rho(X_\mu) - X_\mu = \rho(R)\partial_\mu Q - R\partial_\mu \rho(Q) = 0. \tag{5.39}$$

Taking $R = \mathbb{I}$, this implies

$$\partial_\mu(Q - \rho(Q)) = 0. \tag{5.40}$$

So any element of $\mathcal{G}$ is $(Q, M)$ where

$$Q = \begin{pmatrix} Q_r^r & 0 \\ 0 & Q_r^r + c \end{pmatrix}_{st} \tag{5.41}$$

with $c$ a constant. For $\mathcal{G}$ to be an algebra, (5.41) must be true also for $Q^2$, that is there must exists a constant $c'$ such that

$$Q^2 = \begin{pmatrix} (Q_r^r)^2 & 0 \\ 0 & (Q_r^r)^2 + c^2 + 2cQ_r^r \end{pmatrix}_{st} = \begin{pmatrix} (Q_r^r)^2 & 0 \\ 0 & (Q_r^r)^2 + c'^2 \end{pmatrix}_{st}. \tag{5.42}$$

This is possible if and only if $c = c' = 0$. Thus $\rho(Q) = Q$ for any $(Q, M) \in \mathcal{G}$. The proposition follows from the identification of $\mathcal{A}_{sm}$ as the biggest $\rho$-invariant sub-algebra of $\mathcal{B}$, as explained below. ■

We thus obtain that the breaking of the grand algebra to the standard model is dynamical. This proves the first statement of point iii. of theorem 1.1.

### 5.3 Potential of the scalar field $\boldsymbol{\sigma}$

We now consider the spectral action for the full twisted-covariant Dirac operator $D_{\mathbb{A}} = D_X + D_\sigma$. In analogy with $\Delta_\mu$ which measures how much the vector field $X_\mu$ varies under the twist, we define

$$\Delta_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \rho(\boldsymbol{\sigma}) \tag{5.43}$$

where

$$\rho(\boldsymbol{\sigma}) = \mathbb{I} - \gamma^5 \phi \tag{5.44}$$

is obtained by extending the automorphism $\rho$ to $\mathcal{B}(\mathcal{H})$, as the conjugate action of the unitary operator that exchanges the indices $l$ and $r$ in the basis of $\mathsf{H}$ (in particular one has $\rho(\gamma^5) = -\gamma^5$).

**Lemma 5.6.**

$$D_{\mathbb{A}}^2 = -\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + (\alpha_X^\mu + \alpha_{X\boldsymbol{\sigma}}^\mu)\partial_\mu + \beta_X + \beta_{X\boldsymbol{\sigma}} + \beta_{\boldsymbol{\sigma}}\right) \tag{5.45}$$

*where $\alpha_X^\mu$ and $\beta_X$ are given in lemma 5.1, while*

$$\alpha_{X\boldsymbol{\sigma}}^\mu := i\gamma^\mu\gamma^5\Delta_{\boldsymbol{\sigma}} \otimes D_R, \quad \beta_{\boldsymbol{\sigma}} := -\boldsymbol{\sigma}^2 \otimes D_R^2, \quad \beta_{X\boldsymbol{\sigma}} := i\partial_\mu\boldsymbol{\sigma} \otimes D_R + \beta_{X\boldsymbol{\sigma}}^0 \tag{5.46}$$

*with*

$$\beta_{X\boldsymbol{\sigma}}^0 := i\Xi_{J\alpha}^{I\beta}\gamma^\mu\gamma^5 \begin{pmatrix} 0_4 & k_R\left(X_\mu\boldsymbol{\sigma} + \rho(\boldsymbol{\sigma})\bar{X}_\mu\right) \\ \bar{k}_R\left(-\bar{X}_\mu\boldsymbol{\sigma} - \rho(\boldsymbol{\sigma})X_\mu\right) & 0_4 \end{pmatrix}_{\mathsf{CD}}. \tag{5.47}$$

*Proof.* One has

$$D_{\mathbb{A}}^2 = D_X^2 + D_\sigma^2 + D_X D_\sigma + D_\sigma D_X. \tag{5.48}$$

The first term $D_X^2 = -(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \alpha_X^\mu\partial_\mu + \beta_X)$ has been computed in lemma 5.1. From (4.73), the second term is $-\beta_{\boldsymbol{\sigma}}$. From propositions 4.1 and 4.4, noticing that $\Xi_{J\alpha}^{I\beta}$ commute with $X_\mu$ and $\boldsymbol{\sigma}$, the interaction term writes

$$D_X D_\sigma + D_\sigma D_X = -i\Xi_{J\alpha}^{I\beta} \begin{pmatrix} 0_4 & k_R\, F(X_\mu, \boldsymbol{\sigma}) \\ \bar{k}_R\, F(-\bar{X}_\mu, \boldsymbol{\sigma}) & 0_4 \end{pmatrix}_{\mathsf{CD}} \tag{5.49}$$

with

$$\begin{aligned} F(X_\mu, \boldsymbol{\sigma}) &:= \gamma^\mu\left(\partial_\mu + X_\mu\right)\gamma^5\boldsymbol{\sigma} + \gamma^5\boldsymbol{\sigma}\gamma^\mu\left(\partial_\mu - \bar{X}_\mu\right) & (5.50)\\ &= \gamma^\mu\gamma^5\left(\partial_\mu\boldsymbol{\sigma} + \boldsymbol{\sigma}\partial_\mu\right) + \gamma^\mu\gamma^5 X_\mu\boldsymbol{\sigma} - \gamma^\mu\gamma^5\rho(\boldsymbol{\sigma})\left(\partial_\mu - \bar{X}_\mu\right) & (5.51)\\ &= \gamma^\mu\gamma^5\left(\partial_\mu\boldsymbol{\sigma} + X_\mu\boldsymbol{\sigma} + \rho(\boldsymbol{\sigma})\bar{X}_\mu\right) + \gamma^\mu\gamma^5\Delta_{\boldsymbol{\sigma}}\partial_\mu, & (5.52) \end{aligned}$$

where we use that $\gamma^5$ anti-commutes with $\gamma^\mu$ so that $\gamma^\mu\boldsymbol{\sigma} = \rho(\boldsymbol{\sigma})\gamma^\mu$, and commutes with $X_\mu$. Writing explicitly the r.h.s. of (5.49) yields the proposition. ∎

**Lemma 5.7.** *Let $\omega_\mu = \omega_\mu^X + \omega_\mu^{X\boldsymbol{\sigma}}$ where $\omega_\mu^{X\boldsymbol{\sigma}} = \frac{1}{2}g_{\mu\nu}\alpha_{X\boldsymbol{\sigma}}^\nu$ and $\omega_\mu^X = \frac{1}{2}g_{\mu\nu}\alpha_X^\nu$ as defined in lemma 5.2. One has*

$$g^{\mu\nu}\omega_\mu\omega_\nu = g^{\mu\nu}\omega_\mu^X\omega_\nu^X - \delta_{\boldsymbol{\sigma}}^2 \otimes D_R^2 + \omega(X, \boldsymbol{\sigma}) \tag{5.53}$$

*where $\omega(X, \boldsymbol{\sigma})$ is an interaction term.*

*Proof.* One has

$$g^{\mu\nu}\omega_\mu\omega_\nu = g^{\mu\nu}\omega_\mu^X\omega_\nu^X + g^{\mu\nu}\omega_\mu^{X\boldsymbol{\sigma}}\omega_\nu^{X\boldsymbol{\sigma}} + g^{\mu\nu}(\omega_\mu^X\omega_\nu^{X\boldsymbol{\sigma}} + \omega_\mu^{X\boldsymbol{\sigma}}\omega_\nu^X). \tag{5.54}$$

The second term is

$$\begin{aligned} g^{\mu\nu}\omega_\mu^{X\boldsymbol{\sigma}}\omega_\nu^{X\boldsymbol{\sigma}} = \frac{1}{4}g_{\mu\nu}\alpha_{X\boldsymbol{\sigma}}^\mu\alpha_{X\boldsymbol{\sigma}}^\nu &= -\frac{1}{4}g_{\mu\nu}\gamma^\mu\gamma^5\Delta_{\boldsymbol{\sigma}}\gamma^\nu\gamma^5\Delta_{\boldsymbol{\sigma}} \otimes D_R^2 & (5.55)\\ &= -\frac{1}{4}g_{\mu\nu}\gamma^\mu\gamma^\nu\Delta_{\boldsymbol{\sigma}}^2 \otimes D_R^2 = -\Delta_{\boldsymbol{\sigma}}^2\kappa^2 & (5.56) \end{aligned}$$

where we use that $\Delta_{\boldsymbol{\sigma}}$ anti-commutes with $\gamma^\nu$ and commutes with $\gamma^5$. ∎

**Proposition 5.8.** *The potential of the field $\boldsymbol{\sigma}$ is*

$$V_{\boldsymbol{\sigma}} = 4f_0\Phi^4 + 8(3\Lambda^2 f_2 - f_0|k|^2)\,\Phi^2 + \mathit{constant} \tag{5.57}$$

*where $\Phi := \frac{|k_R|}{4}\sqrt{Tr\ \Delta_{\boldsymbol{\sigma}}^2}$.*

*Proof.* The potential $V_{\boldsymbol{\sigma}}$ is given by (5.8), taking for $E$ the part $E_{\boldsymbol{\sigma}}$ of $\beta_X + \beta_{X\boldsymbol{\sigma}} + \beta_{\boldsymbol{\sigma}} - g^{\mu\nu}\omega_\mu\omega_\nu$ that depends solely on $\boldsymbol{\sigma}$ but not on its derivative. With the two lemma above, this part reduces to

$$E_{\boldsymbol{\sigma}} = \beta_{\boldsymbol{\sigma}} + \Delta^2_{\boldsymbol{\sigma}} \otimes D_R^2 = \left(\Delta^2_{\boldsymbol{\sigma}} - \boldsymbol{\sigma}^2\right) \otimes D_R^2. \tag{5.58}$$

Noticing that

$$\Delta_{\boldsymbol{\sigma}} = 2\gamma^5 \phi\, \mathbb{I}_4 \tag{5.59}$$

and that all term with an odd power of $\gamma^5$ have zero trace, one gets

$$\begin{aligned} \mathrm{Tr}\, E_{\boldsymbol{\sigma}} &= (4\phi^2 - 1 - \phi^2)\ \mathrm{Tr}\left(\,\mathbb{I}_4 \otimes D_R^2\right) && (5.60)\\ &= 8|k_R|^2\,(3\phi^2 - 1). && (5.61)\end{aligned}$$

Similarly,

$$E^2_{\boldsymbol{\sigma}} = \left(\Delta^4_{\boldsymbol{\sigma}} + \boldsymbol{\sigma}^4 - 2\Delta^2_{\boldsymbol{\sigma}}\boldsymbol{\sigma}^2\right) \otimes D_R^4 \tag{5.62}$$

whose trace is

$$\begin{aligned} \mathrm{Tr}\, E^2_{\boldsymbol{\sigma}} &= \Big((2\phi)^4 + (1 + 6\phi^2 + \phi^4) - 2(2\phi)^2(1+\phi)^2\Big)\ \mathrm{Tr}\left(\mathbb{I} \otimes D_R^4\right) && (5.63)\\ &= 8|k_R|^4(\phi^2 - 1)^2. && (5.64)\end{aligned}$$

The result easily follows. ■

For large $\Lambda$, one has $3\Lambda^2 f_2 \geq f_0|k_R|^2$ so that $V_\sigma$ is minimum when $\Phi = 0$, that is $\phi = 0$ by (5.59). From the definition of $\phi$ (4.76) and the same argument as in Prop. 5.5, the biggest subalgebra of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ for which any fluctuation of $D_\nu$ gives a vanishing $\phi$ is $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{sm}$. This ends the proof of theorem 1.1.

# 6 Twist and representations

We discuss the choices made in the construction of the twisted spectral triple for the standard model, that is the middle-term solution consisting in imposing by hand the reduction $M_8(\mathbb{C}) \to M_4(\mathbb{C})$, and the representation of $\mathcal{A}_G$.

## 6.1 Global twist

Instead of reducing by hand $\mathcal{B}_{LR}$ to $\mathcal{B}'$ by imposing the reduction $M_8(\mathbb{C}) \to M_4(\mathbb{C})$, one could twist $\mathcal{B}_{LR}$ as well. This means finding an automorphism $\rho$ of $M_8(\mathbb{C})$ such that

$$\sigma^\mu M\, \partial_\mu - \sigma(M)\sigma^\mu \partial_\mu = 0, \qquad \bar{\sigma}^\mu M\, \partial_\mu - \bar{\sigma}(M)\bar{\sigma}^\mu \partial_\mu = 0. \tag{6.1}$$

Using $\sigma^\mu \bar{\sigma}^\nu \partial_\mu \partial_\nu = \nabla^2$, the first expression yields

$$\sigma(M) = \sigma^\mu M \bar{\sigma}^\nu \frac{1}{\nabla^2}\partial_\mu \partial_\nu. \tag{6.2}$$

This does not define an automorphism of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$. Indeed, writing $T_{\mu\nu} \equiv \frac{1}{\nabla^2}\partial_\mu\partial_\nu$ and $M_1^{\mu\nu} \equiv \sigma^\mu M_1 \bar{\sigma}^\nu$, one gets

$$\begin{aligned} \sigma(M_1)\sigma(M_2) &= (M_1^{\mu\nu} T_{\mu\nu})\left(M_2^{\alpha\beta} T_{\alpha\beta}\right) && (6.3)\\ &= M_1^{\mu\nu}\left[T_{\mu\nu}, M_2^{\alpha\beta}\right] T_{\alpha\beta} + M_1^{\mu\nu} M_2^{\alpha\beta} T_{\mu\nu} T_{\alpha\beta}, && (6.4)\\ &= \sigma(M_1 M_2) + M_1^{\mu\nu}\left[T_{\mu\nu}, M_2^{\alpha\beta}\right] T_{\alpha\beta} && (6.5)\end{aligned}$$

where we compute

$$\begin{aligned}
M_1^{\mu\nu} M_2^{\alpha\beta} T_{\mu\nu} T_{\alpha\beta} &= \sigma^\mu M_1 \bar{\sigma}^\nu \sigma^\alpha M_2 \bar{\sigma}^\beta \frac{1}{\nabla^2}\frac{1}{\nabla^2}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta \\
&= \sigma^\mu M_1 M_2 \bar{\sigma}^\beta \frac{1}{\nabla^2}\partial_\mu\partial_\beta \\
&= \sigma(M_1 M_2). \qquad (6.6)
\end{aligned}$$

A possible solution is to look for a $\star$ product such that

$$\sigma(M_1) \star \sigma(M_2) = \sigma(M_1 \star M_2), \qquad (6.7)$$

that would encode the intrinsic mixing between the manifold (space-time) and the matrix part (gauge sector) that is the core of the Grand Symmetry. This would also force us to consider an algebra $\mathcal{A}_0$ of pseudo-differential operators bigger than $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$. This point is particularly interesting if one believes that almost commutative geometries are an effective low energy description of a more fundamental theory, based on a "truly" non-commutative algebra (that is with a finite dimensional center). This idea has been often advertised by D. Kastler, and it could be that $\mathcal{A}_0$ is not so far from the "noncommutative salmon" he aims at fishing. All this will be investigated in future works.

The reason why we choose the representation (3.3) instead of (2.30) as in [20] is that while it is right that (6.2) is still in $\mathbb{M}_4(\mathbb{C})$, it would not be true for an element $Q = Q_{\dot{s}\alpha}^{\dot{t}\beta} \in M_2(\mathbb{H})$ that $\sigma^\mu Q \bar{\sigma}^\nu$ is still in $M_2(\mathbb{H})$. However, all the results presented in this paper would also be true with the representation (2.30), as explained in the next paragraph.

## 6.2 Invariance of the constraints

The grand algebra in the representation (3.3) is broken by the grading to [20, eq. (3.17)]

$$\mathcal{A}'_G = M_2(\mathbb{H})_L \oplus M_2(\mathbb{H})_R \oplus M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C}). \qquad (6.8)$$

To have bounded commutators with $\not{D}$, we impose by hand that quaternions act trivially on the $\dot{s}$ index, yielding the reduction to

$$\mathcal{A}' := \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C}) \qquad (6.9)$$

whose elements are $(Q, M)$ where

$$Q = \delta_{s\dot{s}}^{t\dot{t}} \begin{pmatrix} q_R & 0_2 \\ 0_2 & q_L \end{pmatrix}_{\alpha\beta}, \quad M = \begin{pmatrix} M_l^l & 0_4 \\ 0_4 & M_r^r \end{pmatrix}_{st} \quad \text{with } q_r \in \mathbb{H}, M_l^l, M_r^r \in M_4(\mathbb{C}). \qquad (6.10)$$

The twist $\rho$ is still defined as the exchange of the left and right part of spinors, but it now acts on the matrix part

$$\rho(M) = \begin{pmatrix} M_r^r & 0_4 \\ 0_4 & M_l^l \end{pmatrix}_{st}. \qquad (6.11)$$

This guarantees that

$$[\not{D}, M]_\rho = (\not{\partial} M) + [\gamma^\mu, M]_\rho = (\not{\partial} M) \qquad (6.12)$$

is bounded, so that $(C^\infty(\mathcal{M}) \otimes \mathcal{A}', \mathcal{H}, \not{D} + D_\nu; \rho)$ is a twisted spectral triple. The twisted first-order condition for $\not{D}$ is checked as in proposition 3.4.

For the twisted first-order condition imposed by $D_\nu$, one first consider the subalgebra of $\mathcal{A}'$

$$\tilde{\mathcal{A}} := \mathbb{H}_L \oplus \mathbb{C}_R \oplus M_3^l(\mathbb{C}) \oplus \mathbb{C}^l \oplus M_3^r(\mathbb{C}) \oplus \mathbb{C}^r \tag{6.13}$$

obtained by asking

$$q_R = \begin{pmatrix} c_R & 0 \\ 0 & \bar{c}_R \end{pmatrix} \quad \text{with } c_R \in \mathbb{C} \tag{6.14}$$

in (6.10) and

$$M_r^r = \begin{pmatrix} m^r & 0_2 \\ 0_2 & \mathbf{M}^r \end{pmatrix}_{\mathrm{IJ}}, \quad M_l^l = \begin{pmatrix} m^l & 0_2 \\ 0_2 & \mathbf{M}^l \end{pmatrix}_{\mathrm{IJ}} \text{ with } \mathbf{M}^r, \mathbf{M}^l \in M_3(\mathbb{C}),\ m^r, m^l \in \mathbb{C}. \tag{6.15}$$

Let $B = (R, N) \in \tilde{\mathcal{B}}$ be another element of $\tilde{\mathcal{A}}$, with components $d_r, n^r, n^l \in \mathbb{C}$ and $\mathbf{N}^r, \mathbf{N}^l \in M_3(\mathbb{C})$. The double twisted commutator $[[D_\nu, A]_\rho,\ JBJ^{-1}]_\rho$ is an off-diagonal matrix with components

$$(\mathsf{D}_\nu M - Q\mathsf{D}_\nu)\bar{R} - \rho(\bar{N})(\mathsf{D}_\nu M - Q\mathsf{D}_\nu), \tag{6.16}$$

$$(\mathsf{D}_\nu Q - \rho(M)\mathsf{D}_\nu)\bar{N} - \bar{R}(\mathsf{D}_\nu Q - \rho(M)\mathsf{D}_\nu). \tag{6.17}$$

One has

$$\rho(\bar{N})\mathsf{D}_\nu M = (\rho(\bar{N})\eta\Xi M)^{t\mathrm{I}}_{s\mathrm{J}}(\Xi\delta)^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \bar{\mathsf{n}}^l \mathsf{m}^{\mathsf{r}} & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^r \mathsf{m}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.18}$$

$$\rho(\bar{N})Q\mathsf{D}_\nu = (\rho(\bar{N})\eta\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(Q\Xi)^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \bar{\mathsf{n}}^l & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^r \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R & 0_4 \\ 0_4 & \mathsf{c}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.19}$$

$$\mathsf{D}_\nu M\bar{R} = (\eta\Xi M)^{t\mathrm{I}}_{s\mathrm{J}}\,(\Xi\bar{R})^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \mathsf{m}^r & 0_4 \\ 0_4 & -\mathsf{m}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.20}$$

$$Q\mathsf{D}_\nu\bar{R} = (\eta\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(Q\Xi\bar{R})^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R\bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \mathsf{c}_R\bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.21}$$

where we defined

$$\mathsf{m}^{\mathsf{r}} := \begin{pmatrix} m^r & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{m}^{\mathsf{l}} := \begin{pmatrix} m^l & 0 \\ 0 & 0_3 \end{pmatrix}_{\alpha\beta}, \quad \mathsf{c}_R = \begin{pmatrix} c_R & 0 \\ 0 & 0_3 \end{pmatrix}_{\mathrm{IJ}} \tag{6.22}$$

and similarly for $\mathsf{n}^r$, $\mathsf{n}^l$ and $\mathsf{d}_R$. Collecting the various terms, one finds that (6.16) is zero if and only if

$$(c_R - m^r)(\bar{d}_R - \bar{n}^l) = 0, \quad (c_R - m^l)(\bar{d}_R - \bar{n}^r) = 0 \tag{6.23}$$

which are the same constraints (6.23) coming from the other representation. The same is true for (6.17), using

$$\bar{R}\rho(M)\mathsf{D}_\nu = (\rho(M)\eta\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(\Xi\bar{R})^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \mathsf{m}^l & 0_4 \\ 0_4 & -\mathsf{m}^r \end{pmatrix}_{st} \otimes \begin{pmatrix} \bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.24}$$

$$\bar{R}\mathsf{D}_\nu Q = (\eta\Xi)^{t\mathrm{I}}_{s\mathrm{J}}\,(\bar{R}\Xi Q)^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \Xi & 0_4 \\ 0_4 & -\Xi \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R\bar{\mathsf{d}}_R & 0_4 \\ 0_4 & \mathsf{c}_R\bar{\mathsf{d}}_R \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.25}$$

$$\rho(M)\mathsf{D}_\nu\bar{N} = (\rho(M)\eta\Xi\bar{N})^{t\mathrm{I}}_{s\mathrm{J}}(\Xi)^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \mathsf{m}^l\,\bar{\mathsf{n}}^r & 0_4 \\ 0_4 & -\mathsf{m}^r\,\bar{\mathsf{n}}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \Xi & 0_4 \\ 0_4 & \Xi \end{pmatrix}_{\dot{s}\dot{t}}, \tag{6.26}$$

$$\mathsf{D}_\nu Q\bar{N} = (\eta\Xi\bar{N})^{t\mathrm{I}}_{s\mathrm{J}}\,(\Xi Q)^{\dot{t}\beta}_{\dot{s}\alpha} = \begin{pmatrix} \bar{\mathsf{n}}^r & 0_4 \\ 0_4 & -\bar{\mathsf{n}}^l \end{pmatrix}_{st} \otimes \begin{pmatrix} \mathsf{c}_R & 0_4 \\ 0_4 & \mathsf{c}_R \end{pmatrix}_{\dot{s}\dot{t}}. \tag{6.27}$$

Solving (6.23) by asking $m^r = c_R$, that is identifying $\mathbb{C}^r$ and $\mathbb{C}_R$ with a single copy $\mathbb{C}_R^r$ of the complex numbers, one reduces $\tilde{A}$ to

$$\mathcal{A} := \mathbb{H}_L \oplus \mathbb{C}_R^r \oplus \mathbb{C}^l \oplus M_3^l(\mathbb{C}) \oplus M_3^r(\mathbb{C}). \tag{6.28}$$

This algebra plays for the representation (2.30) the same role as the algebra $\mathcal{B}$ for the representation (3.3). Repeating the computation of §4.2, one finds a scalar field similar to $\boldsymbol{\sigma}$. Thus, except for the hope of a global twist described in §6.1, there is at the moment no motivation to prefer one or the other of the two natural representations of the grand algebra.

# 7 Conclusion

Let us summarize our results by the following chain of breaking, to be compared with (2.31):

$$\begin{array}{rcl}
\mathcal{A}_G & = & M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \\
& \Downarrow & \text{grading condition} \\
\mathcal{B}_{LR} & = & (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r) \oplus M_8(\mathbb{C}) \\
& \Downarrow & \text{ bounded commutator for } M_8(\mathbb{C}) \\
\mathcal{B}' & = & (\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r) \oplus M_4(\mathbb{C}) \\
& \Downarrow & 1^{\text{st}}\text{-order for the Majorana-Dirac operator} D_\nu \\
\mathcal{B} & = & (\mathbb{H}_L^l \oplus \mathbb{C}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{C}_R^r) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \text{ with } \mathbb{C} = \mathbb{C}_R^r \\
& \Downarrow & \text{minimum of the spectral action} \\
\mathcal{A}_{sm} & = & \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})
\end{array}$$

Starting with the "not so grand algebra" $\mathcal{B}$, one builds a twisted spectral triple whose fluctuations generate both an extra scalar field $\boldsymbol{\sigma}$ and an additional vector field $X_\mu$. This is a Pati-Salam like model - the unitary of $\mathcal{B}$ yields both an $SU(2)_R$ and an $SU(2)_L$ - but in a pre-geometric phase since the Lorentz symmetry (in our case: the Euclidean $SO(n)$ symmetry) is not explicit. The spectral action spontaneously breaks this model to the standard model, with both a scalar and a vector field playing a role similar as the one of Higgs field. We thus have a dynamical model of emergent geometry.

The idea that the scalar field $\sigma$ is associated to the spontaneous breaking of a bigger symmetry to the standard model has been formulated in [20], but, it was not fully implemented, because the fluctuation of the free Dirac operator by the grand algebra $\mathcal{A}_G$ yields an operator whose square is a non-minimal Laplacian. The heat kernel expansion of such operators is notably difficult to compute. Almost simultaneously, a similar idea has been implemented in [10], where the bigger symmetry does not come from a bigger algebra, but follows from relaxing the first-order condition. It would be interesting to understand if the twisted fluctuations are a particular case of those inner fluctuation without first oder condition.

The twist $\rho$ is remarkably simple, and its mathematical significance should be studied more in details, in particular how it should be incorporated in the axioms of noncommutative geometry, like the orientability condition. Also, the physical meaning of the twist is intriguing: the untwisting of $\mathcal{B}$ forces the action of the algebra to be the same on the left and right components of spinors. In this sense the breaking of the grand algebra to the standard model looks like a "primordial" chiral symmetry breaking.

Full phenomenology and comparison with [9] require to take into account all fermions, not only the right neutrino. This means to compute the spectral action of $\not{D} + D_\nu + \gamma^5 \otimes D_0$. This would also allow to check that our $\boldsymbol{\sigma}$ couples to the Higgs as $\sigma$ does in [6]. This will be the object of a future work.

Finally, let us mention a very recent work of Chamseddine, Connes and Mukhanov [8] where the algebra $\mathcal{A}_F$ for $a = 2$ is obtained without the ad-hoc symplectic hypothesis, but from an higher degree Heisenberg relation for the space-time coordinates. It would be interesting to understand whether the case $a = 4$ enters this framework.

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