

Quanta of Geometry

Ali H. Chamseddine,^{1,2} Alain Connes,^{3,2,4} and Viatcheslav Mukhanov^{5,6}

¹Physics Department, American University of Beirut, Lebanon

²I.H.E.S. F-91440 Bures-sur-Yvette, France

³College de France, 3 rue Ulm, F75005, Paris, France

⁴Department of Mathematics, The Ohio State University, Columbus OH 43210 USA

⁵Theoretical Physics, Ludwig Maxmillians University, Theresienstr. 37, 80333 Munich, Germany

⁶MPI for Physics, Foehringer Ring, 6, 80850, Munich, Germany*

In the construction of spectral manifolds in noncommutative geometry, a higher degree Heisenberg commutation relation involving the Dirac operator and the Feynman slash of real scalar fields naturally appears and implies, by equality with the index formula, the quantization of the volume. We first show that this condition implies that the manifold decomposes into disconnected spheres which will represent quanta of geometry. We then refine the condition by involving the real structure and two types of geometric quanta, and show that connected manifolds with large quantized volume are then obtained as solutions. When this condition is adopted in the gravitational action it leads to the quantization of the four volume with the cosmological constant obtained as an integration constant. Restricting the condition to a three dimensional hypersurface implies quantization of the three volume and the possible appearance of mimetic dark matter. When restricting to a two dimensional hypersurface, under appropriate boundary conditions, this results in the quantization of area and has many interesting applications to black hole physics.

PACS numbers: 02.40.Gh, 02.40.Ky, 04.60.-m, 04.70.-s

Introduction.—In noncommutative geometry a geometric space is encoded by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where the algebra \mathcal{A} is the algebra of functions which interacts with the inverse line element D by acting in the same Hilbert space \mathcal{H} , where D is an unbounded self-adjoint operator. There is, in the even case, an additional decoration given by the chirality operator $\gamma = \gamma^*$, $\gamma^2 = 1$, $D\gamma = -\gamma D$ [1]. For a compact spin Riemannian manifold M the algebra \mathcal{A} is the algebra of functions on M , the Hilbert space \mathcal{H} is the Hilbert space of L^2 -spinors and the operator D is the Dirac operator. This operator theoretic data encodes not only the geometry (the Riemannian metric) but also the K -homology fundamental class of M which is represented by the K -homology class of the spectral triple. Among the operator theoretic properties fulfilled by the special spectral triples coming from Riemannian geometries, one of them called the *orientability condition* asserts that, in the even dimensional case, one can recover the chirality operator γ as an expression of the form

$$\gamma = \sum a_0 [D, a_1] \cdots [D, a_n], \quad (1)$$

where the $a_j \in \mathcal{A}$ and the formal expression is a totally antisymmetric Hochschild cycle that represents the (oriented) volume form dv of the manifold [2]. Our goal in this letter is to study a quantized form of the orientability condition. It was observed in [2] that in the particular case of even spheres the trace of the Chern character of an idempotent e , i.e. $e^2 = e$, leads to a remarkably simple operator theoretic equation which takes the form (up to a normalization factor $\frac{1}{2^{n/2}n!}$)

$$\langle Y [D, Y] \cdots [D, Y] \rangle = \sqrt{\kappa} \gamma \quad (n \text{ terms } [D, Y]) \quad (2)$$

Here $\kappa = \pm 1$ and $C_\kappa \subset M_s(\mathbb{C})$, $s = 2^{n/2}$, is the Clifford algebra on $n + 1$ gamma matrices Γ_a , $0 \leq a \leq n$ [12]

$$\Gamma_a \in C_\kappa, \quad \{\Gamma^a, \Gamma^b\} = 2\kappa \delta^{ab}, \quad (\Gamma^a)^* = \kappa \Gamma^a$$

We let $Y \in \mathcal{A} \otimes C_\kappa$ be of the Feynman slashed form $Y = Y^a \Gamma_a$, and fulfill the equations

$$Y^2 = \kappa, \quad Y^* = \kappa Y \quad (3)$$

When we write $[D, Y]$ in (2), we mean $[D \otimes 1, Y]$. Finally $\langle \rangle$ applied to a matrix M_s of operators is its trace.

Note that here the components $Y^a \in \mathcal{A}$ but it is true in general that (3) implies that the components Y^a are self-adjoint commuting operators.

As shown in [2] for even n , equation (2), together with the hypothesis that the eigenvalues of D grow as in dimension n , imply that the volume, expressed as the leading term in the Weyl asymptotic formula for counting eigenvalues of the operator D , is “quantized” by being equal to the index pairing of the operator D with the K -theory class of \mathcal{A} defined by the projection $e = (1 + \sqrt{\kappa} Y)/2$.

In this letter we shall take equation (2), and its two sided refinement (4) below using the real structure, as a geometric analogue of the Heisenberg commutation relations $[p, q] = i\hbar$ where D plays the role of p (momentum) and Y the role of q (coordinate) and use it as a starting point of quantization of geometry with quanta corresponding to irreducible representations of the operator relations. The above integrality result on the volume is a hint of quantization of geometry. We first use the one-sided (2) as the equations of motion of some field theory on M and describe the solutions as follows. (For details and proofs see [3]).

Theorem 1 *Let M be a spin Riemannian manifold of even dimension n and $(\mathcal{A}, \mathcal{H}, D)$ the associated spectral triple. Then a solution of the one-sided equation exists if and only if M breaks as the disjoint sum of spheres of unit volume. On each of these irreducible components the unit volume condition is the only constraint on the Riemannian metric which is otherwise arbitrary for each component.*

Each geometric quantum is a sphere of arbitrary shape and unit volume (in Planck units).

It would seem at this point that only disconnected geometries fit in this framework but this is ignoring an essential piece of structure of the NCG framework, which allows one to refine (2). It is the real structure J , an antilinear isometry in the Hilbert space H which is the algebraic counterpart of charge conjugation. This leads to refine the quantization condition by taking J into account as the two-sided equation [13]

$$\langle Z[D, Z] \cdots [D, Z] \rangle = \gamma \quad Z = 2EJEJ^{-1} - 1, \quad (4)$$

where E is the spectral projection for $\{1, i\} \subset \mathbb{C}$ of the double slash $Y = Y_+ \oplus Y_- \in C^\infty(M, C_+ \oplus C_-)$. It is the classification of finite geometries of [4] which suggested to use the direct sum $C_+ \oplus C_-$ of two Clifford algebras and the algebra $C^\infty(M, C_+ \oplus C_-)$. It turns out moreover that in dimension 4 one has $C_+ = M_2(\mathbb{H})$ and $C_- = M_4(\mathbb{C})$ which is in perfect agreement with the algebraic constituents of the Standard Model. One now gets two maps $Y_\pm : M \rightarrow S^n$ while, for $n = 2, 4$, (4) becomes,

$$\det(e_\mu^a) = \Omega_+ + \Omega_-. \quad (5)$$

with Ω_\pm the Jacobian of Y_\pm (the pullback of the volume form of the sphere).

In the next theorem the algebraic relations between $Y_\pm, D, J, C_\pm, \gamma$ are assumed to hold.

Theorem 2 *Let $n = 2$ or $n = 4$.*

(i) *In any operator representation of the two sided equation (4) in which the spectrum of D grows as in dimension n the volume (the leading term of the Weyl asymptotic formula) is quantized.*

(ii) *Let M be a compact oriented spin Riemannian manifold of dimension n . Then a solution of (5) exists if and only if the volume of M is quantized to belong to the invariant $q_M \subset \mathbb{Z}$ defined as the subset of \mathbb{Z}*

$$q_M = \{\deg(\phi_+) + \deg(\phi_-) \mid \phi_\pm : M \rightarrow S^n, \\ |\phi_+|(x) + |\phi_-|(x) \neq 0, \forall x \in M, \quad (6)$$

where \deg is the topological degree of the smooth maps and $|\phi|(x)$ is the Jacobian of ϕ at $x \in M$.

The invariant q_M makes sense in any dimension. For $n = 2, 3$, and any M , it contains all sufficiently large integers. The case $n = 4$ is more difficult but for our

purposes it will suffice to know that q_M contains arbitrarily large numbers in the two relevant cases $M = S^4$ and $M = S^3 \times S^1$.

Quantization of four-volume.—We now specialize to a four dimensional Euclidian manifold and for simplicity consider only one set of maps, as this does not effect the analysis, and write

$$Y = Y^A \Gamma_A, \quad A = 1, 2, \dots, 5, \quad (7)$$

where Y^A are real and Γ_A are the Hermitian gamma matrices satisfying $\{\Gamma^A, \Gamma^B\} = 2\delta^{AB}$. The condition $Y^2 = 1$ implies

$$Y^A Y^A = 1, \quad (8)$$

thus defining the coordinates on the sphere S^4 . Notice that Y^A are functions on the Euclidian manifold M_4 which depend on the coordinates x^μ . The Dirac operator on M_4 is

$$D = \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + \omega_\mu \right), \quad (9)$$

where $\gamma^\mu = e_a^\mu \gamma^a$ and $\gamma^1 \gamma^2 \gamma^3 \gamma^4 = \gamma$, and ω_μ is the connection, so that $[D, Y] = \gamma^\mu \frac{\partial Y^A}{\partial x^\mu} \Gamma_A$. Using the properties of gamma matrices one can check that the condition (2) reduces to

$$\det(e_\mu^a) = \frac{1}{4!} \epsilon^{\mu\nu\kappa\lambda} \epsilon_{ABCDE} Y^A \partial_\mu Y^B \partial_\nu Y^C \partial_\kappa Y^D \partial_\lambda Y^E. \quad (10)$$

Integrating over the volume of the manifold we find that

$$V = \int \frac{1}{4!} \epsilon^{\mu\nu\kappa\lambda} \epsilon_{ABCDE} Y^A \partial_\mu Y^B \partial_\nu Y^C \partial_\kappa Y^D \partial_\lambda Y^E d^4x \\ = \int \det(e_\mu^a) d^4x = \frac{8\pi^2}{3} n. \quad (11)$$

By Theorem 1 this number n is the number of components when using the one-sided equation (2) but using (4) one gets the sum of the degrees of the maps $Y_\pm : M \rightarrow S^n$ [5]. Thus, we conclude that in noncommutative geometry the volume of the compact manifold is quantized in terms of Planck units. This solves a basic difficulty of the spectral action [1] whose huge cosmological term is now quantized and no longer contributes to the field equations.

Gravitational action and cosmological constant.—Let us study consequences of the four volume quantization for Einstein gravity. For simplicity we shall utilize one set of maps $Y^A(x)$ since most of the details of what follows do not change when two sets $Y_\pm^A(x)$ are used instead. First we consider Euclidian compact spacetime and implement the kinematic constraints (8) and (10) in the action for gravity through Lagrange multipliers. This action then

becomes

$$I = -\frac{1}{2} \int d^4x \sqrt{g} R + \frac{1}{2} \int d^4x \sqrt{g} \lambda' (Y^A Y^A - 1) + \int d^4x \frac{\lambda}{2} \left(\sqrt{g} - \frac{\epsilon^{\mu\nu\kappa\lambda}}{4!} \epsilon_{ABCDE} Y^A \partial_\mu Y^B \partial_\nu Y^C \partial_\kappa Y^D \partial_\lambda Y^E \right), \quad (12)$$

where $8\pi G = 1$. Notice that the last term is a four-form and represents the volume element of a unit four-sphere and can be written in differential forms as $-\frac{1}{4!} \frac{\lambda}{8} \text{Tr}(Y dY \wedge dY \wedge dY \wedge dY)$ and is independent of variation of the metric. Taking into account the constraints (8) and (10), variation of the action with respect to the metric gives

$$G_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \lambda = 0, \quad (13)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor. Tracing this equation gives

$$\lambda = -\frac{1}{2} G, \quad (14)$$

and as a result equations for the gravitational field become traceless

$$G_{\mu\nu} - \frac{1}{4} g_{\mu\nu} G = 0. \quad (15)$$

Using the Bianchi identity these equations imply that $\partial_\mu G = 0$, and hence

$$G = 4\Lambda, \quad (16)$$

where Λ is the cosmological constant arising as a *constant of integration* (compare to [6]). Variation of the action with respect to Y^A does not lead to any new equations because the last term in equation (12) is a topological invariant if $Y^A Y^A = 1$.

One immediate application is that, in the path integration formulation of gravity, and in light of having only the traceless Einstein equation (15), integration over the scale factor is now replaced by a sum of the winding numbers with an appropriate weight factor. We note that for the present universe, the winding number equal to the number of Planck quanta is of the order of 10^{61} [7]

Three-volume quantization and mimetic matter.—In reality spacetime is Lorentzian and generically has one noncompact dimension corresponding to time. Therefore, the condition for the volume quantization is literally non-applicable there. However being implemented in the Euclidian action it leads nevertheless to the appearance of the cosmological constant as an integration constant even in the Lorentzian spacetime. To show this let us first make a Wick rotation and then decompactify M_4 to $\mathbb{R} \times S^3$. With this purpose we set $Y^5 = \eta X$ and one of the coordinates $x^4 = \eta t$ and take the limit $\eta \rightarrow 0$. In this

limit equation (8) becomes $Y^a Y^a = 1$, $a = 1, \dots, 4$, and the constraint (10) turns to

$$\begin{aligned} \sqrt{g} &= \lim_{\eta \rightarrow 0} \left(\frac{1}{4!} \kappa^4 \epsilon^{\mu\nu\kappa\lambda} \epsilon_{ABCDE} Y^A \partial_\mu Y^B \partial_\nu Y^C \partial_\kappa Y^D \partial_\lambda Y^E \right) \\ &= \frac{1}{3!} \epsilon^{\mu\nu\kappa\lambda} \epsilon_{abcd} (\partial_\mu X) Y^a \partial_\nu Y^b \partial_\kappa Y^c \partial_\lambda Y^d. \end{aligned} \quad (17)$$

The Lorentzian action for the gravity is

$$I = -\frac{1}{2} \int d^4x \sqrt{-g} R + \frac{1}{2} \int d^4x \sqrt{-g} \lambda' (Y^a Y^a - 1) + \int d^4x \frac{\lambda}{2} \left(\sqrt{-g} - \frac{1}{3!} \epsilon^{\mu\nu\kappa\lambda} \epsilon_{abcd} (\partial_\mu X) Y^a \partial_\nu Y^b \partial_\kappa Y^c \partial_\lambda Y^d \right). \quad (18)$$

The equations of motion in this case are the same as before and the cosmological constant arises as a constant of integration. The variable X in (18) is a priori unrestricted. We will show now that the requirement of the volume quantization of S^3 in the mapping $M_4 \rightarrow \mathbb{R} \times S^3$ leads to the following normalization condition for this variable,

$$g^{\mu\nu} \partial_\mu X \partial_\nu X = 1. \quad (19)$$

Let us consider the 3 + 1 splitting of space-time, so that

$$ds^2 = h_{ij} (dx^i + N^i dt) (dx^j + N^j dt) - N^2 dt^2 \quad (20)$$

where $N(x^i, t)$ and $N^i(x^i, t)$ are the lapse and shift functions respectively and

$$\sqrt{-g} = N \sqrt{h}. \quad (21)$$

Considering a compact hypersurfaces Σ defined by the condition $t = \text{const}$ and taking

$$\partial_i X = 0, \quad \partial_t X = N, \quad (22)$$

which satisfy (19) we have (for details see [3])

$$\left(N \sqrt{h} \right)_\Sigma = \frac{1}{3!} N (\epsilon^{ijk} Y^a \partial_i Y^b \partial_j Y^c \partial_k Y^d \epsilon_{abcd}), \quad (23)$$

and therefore

$$\int_\Sigma \sqrt{h} d^3x = n \left(\frac{4}{3} \pi^2 \right), \quad (24)$$

where n is the winding number for the mapping $\Sigma \rightarrow S^3$. Thus we have shown that (19) implies quantization of the volume of compact 3d hypersurfaces in 4d spacetime. This condition can be understood as a restriction of the maps $Y^A(x)$ along directions orthogonal to the hypersurface Σ , to be length preserving. To incorporate this condition in the action (18) we add to it the term

$$+ \int d^4x \sqrt{-g} \lambda'' (g^{\mu\nu} \partial_\mu X \partial_\nu X - 1) \quad (25)$$

which corresponds to mimetic dark matter [10],[11]. Thus the resulting action describes both dark matter and dark energy. Both substances arise automatically when the kinematic 4d and 3d compact volume quantization in noncommutative geometry is incorporated in the gravity action. We note that the same field X could be used when we consider two different set of maps $Y_+^A(x)$ and $Y_-^A(x)$ from Σ to S^3 .

Area quantization and black holes.—We can determine the conditions under which the area of any compact 2d submanifold of the 4d manifold must also be quantized. One can show (see ([3] for details) that by writing $Y^4 = \eta X^1$, $Y^5 = \eta X^2$ and rescaling the coordinates transverse to the 2d hypersurface Σ as $x^\alpha \rightarrow \eta x^\alpha$, we get

$$\sqrt{{}^{(2)}g} = \det(e_i^a) = \frac{1}{2!} \epsilon^{ij} \epsilon_{ABC} Y^A \partial_i Y^B \partial_j Y^C, \quad (26)$$

provided that the area preserving condition on the hypersurface

$$\det(g^{\mu\nu} \partial_\mu X^a \partial_\nu X^b)|_\Sigma = 1, \quad (27)$$

is satisfied, where $A, \dots = 1, 2, 3$ and x^i are coordinates on the hypersurface. Hence, the area of a two dimensional manifold is quantized

$$\begin{aligned} S &= \int \sqrt{{}^{(2)}g} d^2x \\ &= \int \frac{1}{2!} \epsilon^{\mu\nu} \epsilon_{ABC} Y^A \partial_\mu Y^B \partial_\nu Y^C d^2x = 4\pi n, \end{aligned} \quad (28)$$

where n is the winding number for the mapping of two-dimensional manifold to the sphere S^2 , defined by $Y^A Y^A = 1$.

This can have far-reaching consequences for black holes and de Sitter space. In particular, the area of the black hole horizon must be quantized in integers of the Planck area (see also [8]). Because the area of a black hole of mass M is equal to

$$A = 16\pi M^2, \quad (29)$$

this implies mass quantization

$$M_n = \frac{\sqrt{n}}{2} \quad (30)$$

As it was shown in [9] Hawking radiation in this case can be considered as a result of quantum transitions from the level n to the nearby levels $n-1, n-2, \dots$. As a result even for large black holes Hawking radiation is emitted in discrete lines and the spectrum with the thermal envelope is not continuous. The distance between the nearby lines for large black holes is of order

$$\omega = M_n - M_{n-1} \simeq \frac{1}{4\sqrt{n}} = \frac{1}{8M}, \quad (31)$$

and proportional to Hawking temperature, while the width of the line is expected to be at least ten times less

than the distance between the lines [9]. Note that taking the minimal area to be α larger than the Planck area changes the distance between the lines by a factor α^2 . Thus area quantization can be experimentally verified if evaporating black hole will be discovered.

Applying the same reasoning to the event horizon in de Sitter universe we find that the cosmological constant in this case must be quantized as

$$\Lambda_n = \frac{3}{n}. \quad (32)$$

It is likely that this quantization can have drastic consequences for the inflationary universe, in particular, in the regime of self-reproduction.

AHC is supported in part by the National Science Foundation under Grant No. Phys-1202671. The work of VM is supported by TRR 33 “The Dark Universe” and the Cluster of Excellence EXC 153 “Origin and Structure of the Universe”

* Electronic address: chams@aub.edu.lb, alain@connes.org, viatcheslav.n

- [1] A. H. Chamseddine and A. Connes, *Noncommutative Geometry as a framework for unification of all fundamental interactions including gravity*, Fortsch. Phys. **58** (2010) 553.
- [2] A. Connes, *A short Survey of Noncommutative Geometry*, arXiv: hep-th/0003006.
- [3] A. H. Chamseddine, A. Connes and V. Mukhanov, In preparation.
- [4] A. H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. **58**, (2008) 38.
- [5] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature and Cohomology*, volume 2, pages 347-351, Academic Press, New York and London, 1973.
- [6] M. Henneaux and C. Teitelboim, *The Cosmological Constant and General Covariance*, Phys.Lett. **B222** (1989) 195.
- [7] A. H. Chamseddine and A. Connes, *The uncanny precision of the spectral action*, Comm. Math. Phys. **293**, (2010) 867.
- [8] J. D. Bekenstein, *The quantum mass spectrum of the Kerr black hole*, Lett. Nuovo Cimento **11**, 467 (1974); V. Mukhanov, *Are black holes quantized?* JETP Letters **44**, (1986) 63.
- [9] J. D. Bekenstein and V. Mukhanov, *Spectroscopy of the quantum black hole*, Phys.Lett. **B360** (1995) 7.
- [10] A. H. Chamseddine and V. Mukhanov, *Mimetic dark matter*, JHEP **1311** (2013) 135.
- [11] A. H. Chamseddine, V. Mukhanov and A. Vikman, *Cosmology with Mimetic Matter*, JCAP **1406** (2014) 017.
- [12] It is $n+1$ and not n the first Γ_0 is up to normalization the product of the n others.
- [13] The γ involved here commutes with the Clifford algebras and does not take into account an eventual $\mathbb{Z}/2$ -grading γ_F of these algebras, yielding the full grading $\gamma \otimes \gamma_F$.