Three Principles for Quantum Gravity

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Abstract

We postulate that the fundamental principles of Quantum Gravity are diffeomorphism symmetry, unitarity, and locality. Local observables are compatible with diffeomorphism symmetry in the presence of diff anomalies, which modify the symmetry algebra upon quantization. We describe the generalization of the Virasoro extension to the diffeomorphism algebra in several dimensions, and its off-shell representations. These anomalies can not arise in QFT, because the Virasoro-like cocycles are functionals of the observer's spacetime trajectory, which is not present in QFT. Possible implications for physics are discussed.

1 The postulate

All known physical phenomena are described by two theories: General Relativity (GR), which describes gravity, and Quantum Field Theory (QFT), which describes everything else. For the past 85 years, physicists have seeked to unify these two theories into a single theory of Quantum Gravity (QG). Alas, GR and QFT are mutually incompatible, and despite an immense amount of work by many leading physicists, there has been no clear progress. In particular, the origin of mass quantization (why is $m_p \approx 1836 \cdot m_e$?) remains a complete mystery.

In view of this failure, I propose to take a step back and reexamine the fundamental principles that QG should rest upon. A radical possibility is that QG simply combines the fundamental properties of GR and QFT:

Postulate 1 (Main postulate, physical version) *Quantum Gravity has the following properties:*

1. *Spacetime diffeomorphism symmetry (the gravity property).*
2. *Unitarity and energy bounded from below (the quantum property).*
3. *Locality (the field property).*

None of the currently popular QG candidates satisfy all three properties. There is of course an excellent reason for this: according to standard wisdom, the three properties in the main postulate are mutually incompatible.

Theorem 1 (No-go theorem, physical version) *There are no local observables in QG. In QFT, local observables are gauge-invariant unitary operators. Since diffeomorphisms are part of the gauge group of GR, any observable must be invariant under arbitrary diffeomorphisms, and hence it can not be local. The three properties of Postulate 1 are mutually exclusive.*

To gain some further insight, let us rephrase the postulate in terms of the representation theory of the diffeomorphism group.

Postulate 2 (Main postulate, representation theory version) *Quantum Gravity has the following properties:*

1. *All objects in the theory carry representations of the spacetime diffeomorphism group (the gravity property).*
2. *The representations are unitary and of lowest-energy type (the quantum property).*
3. *At least some representations are non-trivial (the field property).*

The no-go theorem can now be formulated as follows:

Theorem 2 (No-go theorem, representation theory version) *The spacetime diffeomorphism group has no non-trivial, proper, unitary representations of lowest-energy type.*

This theorem is correct as stated, but no theorem is stronger than its axioms. The keyword is "proper"; if we relax that condition, the theorem no longer holds, as the following example illustrates.

Consider the group of diffeomorphisms on the circle, and its Lie algebra of vector fields $\mathfrak{vect}(S^1) \equiv \mathfrak{vect}(1)$; for brevity, the notation only indicates the number of dimensions. The infinitesimal generators $L_m = -i\exp(imx)\partial/\partial x$, $m \in \mathbb{Z}$, satisfy

$$[L_m, L_n] = (n-m)L_{m+n}. \tag{1}$$

The only unitary lowest-energy representation of $\mathfrak{vect}(1)$ is the trivial one, in accordance with Theorem 2. However, it is well known from conformal field theory (CFT) how to solve this problem. $\mathfrak{vect}(1)$ admits a non-trivial central extension, the Virasoro algebra:

$$[L_m, L_n] = (n-m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n}, \tag{2}$$

where δ_m denotes the Kronecker delta and c is the central charge. A lowest-energy representation has a unique vacuum vector $|h\rangle$, which satisfies

$$\begin{aligned} L_0|h\rangle &= h|h\rangle, \\ L_{-m}|h\rangle &= 0, \qquad \text{for all } -m < 0. \end{aligned} \tag{3}$$

The Virasoro algebra has non-trivial unitary representations of lowest-energy type, e.g. the entire Verma modules for $c > 1, h > 0$ or the discrete unitary series [3]:

$$\begin{aligned} c &= 1 - \frac{6}{m(m+1)}, \\ h &= \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}, \qquad 1 \le r < m, 1 \le s \le r. \end{aligned} \tag{4}$$

For these values of c and h, CFT satisfies all conditions in Postulate 2:

1. The theory has a symmetry under the diffeomorphism group on the circle.

2. The theory is unitary and the energy is bounded from below - the L_0 eigenvalue is at least h for every state in the Hilbert space.

3. The theory is local in the sense that correlation functions depend on separation. E.g., the correlator between two primary fields behaves like

$$\langle \phi(z)\phi(w)\rangle \approx (z-w)^{-2h} \tag{5}$$

when $z \to w$.

It is now clear how the no-go theorem can be avoided: allow projective representations of the spacetime diffeomorphism group.

Theorem 3 *To satisfy all desiderata in the main postulate it is necessary that symmetry of QG is some group extension of the spacetime diffeomorphism group. This converts the classical diffeomorphism gauge symmetry into a quantum global symmetry, which does not need to commute with observables.*

On the Lie algebra level, this amounts to replacing $\mathfrak{vect}(d)$, the Lie algebra of vector fields in d dimensional spacetime, with a Lie algebra extension thereof. Since this extension generalizes the Virasoro algebra to multi-dimensional manifolds, we call it the multi-dimensional Virasoro algebra and denote it by $Vir(d)$; $Vir(1)$ is the ordinary Virasoro algebra.

2 The objections

Replacing $\mathfrak{vect}(d)$ with $Vir(d)$ is a drastic step, which may potentially lead to several objections.

1. $\mathfrak{vect}(d)$ does not possess any central extension at all when $d > 1$.
2. An extension of the diffeomorphism algebra is a diff anomaly. In QFT, there are no diff anomalies in four dimensions [1].
3. Diffeomorphisms are part of the gauge symmetries of gravity. In QFT observables are gauge-invariant operators, and hence all observables must commute with diffeomorphisms.
4. A diff anomaly is a kind of gauge anomaly, which automatically renders the theory inconsistent.

The first three objections are correct as formulated, but the statements contain assumptions that are overly strong. The last objection is manifestly false.

1. The diffeomorphism algebra in $d > 1$ dimensions does not possess any **central** extension, but it does possess non-central extensions that reduce to the Virasoro algebra in the case $d = 1$. $Vir(d)$ is an extension of $\mathfrak{vect}(d)$ by its module of one-forms modulo exact forms. When $d = 1$, all one-forms on the circle are exact except the constant one,

and the extension is central. When $d > 1$, the extension does not commute with diffeomorphisms, but there are still non-trivial Lie algebra extensions.

The multi-dimensional Virasoro algebra is described explicitly in section 3. For a classification of abelian extensions of $\mathfrak{vect}(d)$ by modules of tensor fields, see [4].

2. There are no diff anomalies in four dimensions **within the framework of QFT**. However, the multi-dimensional Virasoro extensions described in section 3 certainly exist. Hence there are diff anomalies in arbitrary dimensions, in the same sense as the Virasoro central charge is a conformal anomaly in two dimensions, but these anomalies can not arise in QFT.

   The off-shell representations of $\mathfrak{vect}(d)$ act on tensor fields and tensor densities. However, tensor densities are not a good starting point for quantization when $d > 1$; in higher dimensions, normal ordering gives rise to infinities coming from unrestricted sums over spatial degrees of freedom. Instead we must start from histories in the space of tensor-valued p-jets, p finite; locally, a p-jet is the same as a Taylor series truncated at order p. Since a p-jet history consists of finitely many functions of a single variable, normal ordering can be done without introducing any infinitities.

   A p-jet can be thought of as a regularization of the field, but not only so. A Taylor series does not only depend on the function being expanded, but also on the choice of expansion point, a.k.a. the observer's position. This is essential, because in all known representations of $Vir(d)$, the extension is a functional of the observer's trajectory. The Virasoro-like diff anomalies can not arise in QFT, because they depend on degrees of freedom not available. To construct these diff anomalies, we must replace QFT with a theory that depends on the observer's trajectory in addition to the fields. This theory is tentatively labelled Quantum Jet Theory (QJT).

   The off-shell representations of $Vir(d)$ are explicitly described in section 4, and some conjectured physical consequences of QJT in section 6.

3. Diffeomorphisms generate a gauge symmetry **in the absense of diff anomalies**. A gauge anomaly converts a classical gauge symmetry into a quantum global symmetry, which acts on the Hilbert space

rather than reducing it. Hence there may be local observables in QG in the presence of diff anomalies.

4. It is simply not true that every theory with gauge anomalies is inconsistent. Counterexample: according to the no-ghost theorem, the free subcritical string can be quantized with a ghost-free spectrum despite its conformal gauge anomaly ([5], section 2.4). A gauge anomaly simply means that the classical and quantum theories have different symmetry groups.

   This does of course not mean that every theory with a gauge anomaly can be rendered consistent, but the crucial consistency criterion is unitarity, not triviality. E.g., the gauge anomalies that appear in the standard model are related to the Mickelsson-Faddeev (MF) algebra[1] [11], which is known to lack good quantum representations; more precisely, the MF algebra has no non-trivial, unitary representations acting on a separable Hilbert space [12]. Gauge anomalies of this type must therefore cancel, which is also the case in the standard model. In contrast, $Vir(d)$ may well have non-trivial unitary representations (this is at least the case when $d = 1$), and such diff anomalies are not necessarily a sign of inconsistency.

   Treating an anomalous gauge symmetry as a redundancy is of course inconsistent, since it becomes a global symmetry after quantization.

3 Multi-dimensional Virasoro algebra

Denote by $Vir(d)$ the Virasoro algebra in d dimensions. In a Fourier basis on the d-torus, the generators are $L_\mu(m)$ and $S^\mu(m)$, $m = (m_0, m_1, ...m_{d-1}) \in \mathbb{Z}^d$, which satisfy

$$\begin{aligned}
[L_\mu(m), L_\nu(n)] &= n_\mu L_\nu(m+n) - m_\nu L_\mu(m+n) \\
&- (c_1 m_\nu n_\mu + c_2 m_\mu n_\nu) m_\rho S^\rho(m+n), \\
[L_\mu(m), S^\nu(n)] &= n_\mu S^\nu(m+n) + \delta^\nu_\mu m_\rho S^\rho(m+n), \\
[S^\mu(m), S^\nu(n)] &= 0, \\
m_\mu S^\mu(m) &= 0.
\end{aligned} \tag{6}$$

To see that this algebra indeed reduces to the usual Virasoro algebra when $d = 1$, we notice that the condition $m_0 S^0(m_0) = 0$ implies that $S^0(m_0)$ is

[1] Note that the MF algebra is substantially different from the multi-dimensional affine algebra $Aff(d, \mathfrak{g})$ described in Section 5 below.

proportional to the Kronecker delta, which indeed commutes with diffeomorphisms. So the Virasoro extension is central when $d = 1$ but not otherwise. Nevertheless, (6) defines a well-defined and non-trivial Lie algebra extension of $\mathfrak{vect}(d)$ for every d.

The cocycle proportional to c_1 was discovered by Rao and Moody [14], and the one proportional to c_2 by myself [6]. We refer to c_1 and c_2 as *abelian charges*, in analogy with the central charge of $Vir(1)$.

In the sequel we will use a different formulation not specific to tori. Let $\xi = \xi^\mu(x)\partial_\mu$ be a vector field, with commutator $[\xi, \eta] \equiv \xi^\mu\partial_\mu\eta^\nu\partial_\nu - \eta^\nu\partial_\nu\xi^\mu\partial_\mu$. The Lie derivatives $\mathcal{L}_\xi$ are the generators of $\mathfrak{vect}(d)$. $Vir(d)$ is defined by the following brackets

$$
\begin{aligned}
[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi,\eta]} + \frac{1}{2\pi i}\int dt\ \dot{q}^\rho(t)\Big\{c_1\ \partial_\rho\partial_\nu\xi^\mu(q(t))\partial_\mu\eta^\nu(q(t))\ + \\
&\qquad + c_2\ \partial_\rho\partial_\mu\xi^\mu(q(t))\partial_\nu\eta^\nu(q(t))\Big\}, \\
[\mathcal{L}_\xi, q^\mu(t)] &= \xi^\mu(q(t)), \\
[q^\mu(t), q^\nu(t')] &= 0.
\end{aligned} \tag{7}
$$

The connection between (6) and (7) is given by

$$
\begin{aligned}
L_\mu(m) &= \mathcal{L}_{-i\exp(im\cdot x)\partial_\mu}, \\
S^\mu(m) &= \frac{1}{2\pi}\int dt\ \ \exp(im\cdot q(t))\ \dot{q}^\mu(t).
\end{aligned} \tag{8}
$$

In particular, the last condition in (6) becomes $\int dt\ \frac{d}{dt}(\exp(im\cdot q(t))) \equiv 0$.

4 Off-shell representations

To construct Fock representations of $Vir(1)$ is straightforward:

- Start from classical fields, i.e. primary fields = scalar densities.
- Introduce canonical momenta.
- Normal order.

The first two steps of this procedure generalize nicely to higher dimensions, but the third leads to infinitites due to unrestricted sums over spatial directions. This is the reason why the representations of $Vir(d)$, $d \geq 2$, do not act on quantum fields.

Instead, we notice that $\mathfrak{vect}(d)$ can be embedded into a Heisenberg algebra with $2d$ generators q^μ and p_ν, and brackets

$$[q^\mu, p_\nu] = i\delta^\mu_\nu, \qquad [q^\mu, q^\nu] = [p_\mu, p_\nu] = 0. \tag{9}$$

The embedding is given by

$$\mathcal{L}_\xi = i\xi^\mu(q)p_\mu. \tag{10}$$

Hence $\mathfrak{vect}(d)$ acts on the corresponding Fock module, which can be identified with the space of spacetime fields:

$$\mathcal{L}_\xi \Phi(q) = \xi^\mu(q)\partial_\mu \Phi(q). \tag{11}$$

Since the Heisenberg algebra (9) is finite-dimensional, the Fock representation of $\mathfrak{vect}(d)$ is proper. To obtain the extensions in (6), we need to find an embedding into an infinite-dimensional Heisenberg algebra. To this end, introduce infinitely many oscillators $q^\mu(t)$ and $p_\nu(t)$, $t \in S^1$, with non-zero brackets

$$[q^\mu(t), p_\nu(t')] = i\delta^\mu_\nu \delta(t - t'). \tag{12}$$

The embedding is given by

$$\mathcal{L}_\xi = i\int dt\ \xi^\mu(q(t))p_\mu(t), \tag{13}$$

where the integral runs over $0 \leq t < 2\pi$.

Unlike the finite-dimensional case, the infinite-dimensional Heisenberg algebra (12) has several inequivalent Fock representations. To satisfy the quantum property, we must choose the one with energy bounded from below, where energy is identified with the frequency dual to the circle variable t. The Fock module consists of all functions of the positive-frequency Fourier components, plus half of the zero-frequency components.

However, the operators (13) do not act in a well-defined manner on this Fock space, because the action on the Fock vacuum is infinite. To remove this infinity, we must normal order. Because the oscillators $q^\mu(t)$ commute among themselves, this amounts to moving the positive-frequency components of $p_\mu(t)$ in (13) to the left. The normal ordered-operators satsify the multi-dimensional Virasoro algebra (7) with $c_1 = 2d$, $c_2 = 0$. $q^\mu(t)$ is the same in both (7) and (13).

More general Fock representations act on histories the the space of p-jets [7], which locally can be identified with the space of Taylor series truncated at order p. Consider a spacetime field $\phi(x)$, expand it in a Taylor series around q^μ, and truncate at order p.

$$\phi(x) = \sum_{|\mathbf{m}|\leq p} \frac{1}{\mathbf{m}!}\phi_{\mathbf{m}}(x-q)^{\mathbf{m}}, \tag{14}$$

where $\mathbf{m} = (m_0, m_1, ..., m_{d-1})$, all $m_\mu \geq 0$, is a multi-index of length $|\mathbf{m}| = \sum_{\mu=0}^{d-1} m_\mu$, $\mathbf{m}! = m_0!m_1!...m_{d-1}!$, and

$$(x-q)^{\mathbf{m}} = (x^0-q^0)^{m_0}(x^1-q^1)^{m_1}...(x^{d-1}-q^{d-1})^{m_{d-1}}. \tag{15}$$

The space of p-jets is spanned by the Taylor coefficients $\phi_{\mathbf{m}}$, $|\mathbf{m}| \leq p$ and the expansion point q^μ.

Now consider p-jet histories by letting everything depend on an extra circle parameter $t \in S^1$. The Heisenberg algebra is spanned by the oscillators $q^\mu(t)$, $p_\nu(t)$, $\phi_{\mathbf{m}}(t)$, and $\pi^{\mathbf{n}}(t)$, obeying (12) and

$$[\phi_{\mathbf{m}}(t), \pi^{\mathbf{n}}(t')] = i\delta^{\mathbf{n}}_{\mathbf{m}}\delta(t-t'). \tag{16}$$

After normal ordering, denoted by double dots : :, we obtain a projective Fock representation of the diffeomorphism algebra

$$\mathcal{L}_\xi = i\int dt\, \Big\{ :\xi^\mu(q(t))p_\mu(t): - \sum_{\mathbf{m},\mathbf{n}} :\pi^{\mathbf{n}}(t)T^{\mathbf{m}}_{\mathbf{n}}(\xi(q(t)))\phi_{\mathbf{m}}(t): \Big\}, \tag{17}$$

where the sum runs over all $\mathbf{m}$ and $\mathbf{n}$ such that $|\mathbf{m}| \leq |\mathbf{n}| \leq p$. $T^{\mathbf{m}}_{\mathbf{n}}(\xi)$ are some functions of ξ^μ and its derivatives up to order $p+1$, explicitly written down in [7].

The construction is readily generalized to fermionic fields, but the expansion point $q^\mu(t)$ is of course always bosonic.

A major shortcoming of this construction is that only linear representations have been considered. In physics, we are ultimately interested in unitary representations, but this leads to complications. One problem is is that there is that the Fock space is too large – momenta and velocities are unrelated.

Consider the observer's position and momentum. We could try to impose the constraint

$$p_\mu(t) \approx i\dot{q}^\mu(t), \tag{18}$$

but this does not work because the two sides do not transform in the same way (covariant vs. contravariant vectors). Nor can we lower indices with the Minkowski metric, which is meaningless in the context of general diffeomorphisms (a two-tensor does not transform trivially under arbitrary diffeomorphisms). To construct a meaningful phase space, we hence need to consider on-shell representations. Let $g_{\mu\nu}(x)$ be a metric field and $g_{\mu\nu,\mathbf{m}}(t)$ the corresponding Taylor coefficients. Then we may impose the constraint

$$p_\mu(t) \approx i g_{\mu\nu,\mathbf{0}}(t)\dot{q}^\nu(t), \tag{19}$$

which involves the zeroth order Taylor coefficient with multi-index $\mathbf{m} = \mathbf{0}$. Unlike (18), the two sides of this equation transform identically under $\mathfrak{vect}(d)$, so the condition is meaningful.

The same type of condition must be imposed on the Taylor coefficients as well. Morally we want to identify $\pi^{\mathbf{m}}(t)$ and $\dot{\phi}_{\mathbf{m}}(t)$, but this is complicated and no explicit results have been found.

5 Multi-dimensional affine algebra

There is an analogous multi-dimensional affine algebra ([13] section 4). Let $\mathfrak{map}(d, \mathfrak{g})$ be the algebra of maps from d-dimensional space to a Lie algebra $\mathfrak{g}$ with basis J^a, structure constants f^{abc}, and Killing metric δ^{ab}. $Aff(d, \mathfrak{g})$ is defined by the brackets

$$[\mathcal{J}_X, \mathcal{J}_Y] \;=\; \mathcal{J}_{[X,Y]} - \frac{k}{2\pi i}\,\delta^{ab}\int dt\ \dot{q}^\rho(t)\partial_\rho X_a(q(t))Y_b(q(t)), \tag{20}$$

where $X = X_a(x)J^a$ is a $\mathfrak{g}$-valued function. In the Fourier basis, this becomes

$$[J^a(m), J^b(n)] = if^{abc}J^c(m+n) - k\delta^{ab}m_\mu S^\mu(m+n). \tag{21}$$

The $Aff(d, \mathfrak{g})$ generators commute with $q^\mu(t)$ and $S^\mu(m)$, and admit an intertwining action of $Vir(d)$.

Note that this cocycle is proportional to the second Casimir operator. $Aff(d, \mathfrak{g})$ is thus unrelated to the gauge anomalies appearing in the standard model, which are proportional to the third Casimir.

Off-shell representations of $Aff(d, \mathfrak{g})$ are constructed in analogy with $Vir(d)$. Let M^a denote matrices in some finite-dimensional representation of $\mathfrak{g}$. The following expression defines an embedding of $Aff(d, \mathfrak{g})$ into the Heisenberg algebra:

$$\mathcal{J}_X = -i\int dt\ \sum_{\mathbf{m},\mathbf{n}} :\pi^{\mathbf{n}}(t)J^{\mathbf{m}}_{\mathbf{n}}(X(q(t)))\phi_{\mathbf{m}}(t):, \tag{22}$$

where

$$J_{\mathbf{n}}^{\mathbf{m}}(X) \quad = \quad \binom{\mathbf{n}}{\mathbf{m}} \partial_{\mathbf{n}-\mathbf{m}} X_a M^a. \qquad (23)$$

Hence $Aff(d, \mathfrak{g})$ acts on the Fock space.

Whereas the off-shell representations of $Vir(d)$ are only understood at the linear level, an exhaustive classification of unitary irreps of $Aff(d, \mathfrak{g})$ was made long ago ([13], section 4). Consider the representation induced from a unitary irrep of $Aff(1, \mathfrak{g}) = \widehat{\mathfrak{g}}$, living on some circle embedded in spacetime. It turns out that each such representation is unitary and irreducible, and that this exhausts the possibilities.

In the present formalism, this corresponds to specializing (22) to zero-jets:

$$\mathcal{J}_X \quad = \quad \int dt\ X(q(t)) J^a(t), \qquad (24)$$

where

$$J^a(t) = -i : \pi^{\mathbf{0}}(t) M^a \phi_{\mathbf{0}}(t) : . \qquad (25)$$

To verify that (24) satisfies $Aff(d, \mathfrak{g})$, we only need to use that the $J^a(t)$ satisfies $\widehat{\mathfrak{g}}$, which conversely means that we obtain an $Aff(d, \mathfrak{g})$ representation for every $\widehat{\mathfrak{g}}$ representation.

Pressley and Segal [13] found this result rather disappointing, but the reason why the irreps only require zero-jets is that the current algebra does not explore neighboring spacetime points. The circle $q^\mu(t)$ commutes with everything in sight and can therefore be replaced with a c-number; the extension becomes central. In physics, there is always an intertwining action of diffeomorphisms or some subgroup thereof, such as the Poincaré group. Once such spacetime groups are taken into consideration, $q^\mu(t)$ becomes an operator, the extension is no longer central, and interesting representations depend on more than $\widehat{\mathfrak{g}}$.

6 Quantum Jet Theory

In order to fulfill all three desiderata in the main postulate, we must construct projective representations of the diffeomorphism algebra. A necessary condition is that we quantize histories in the space of p-jets rather than the fields themselves. This theory will be named Quantum Jet Theory (QJT).

Since diff anomalies of the type in (6) can not arise in QFT, QJT is substantially different from QFT.

The Pressley-Segal classification of $Aff(d, \mathfrak{g})$ irreps mentioned in Section 5 may be regarded as a result in QJT.

In this section we explore some physical consequences of this insight, ranging from the obvious to the speculative.

6.1 QJT as a regularization

p-jet histories arise naturally in the time evolution of physical systems. Recall that a spacetime tensor field is the history of a tensor field in space, or a collection of such fields. Similarly, a p-jet history is the time evolution of a p-jet in space. From this point of view, QJT is merely a regularization. A spacetime field $\phi(x)$, which depends on d variables x^μ, is replaced with finitely many Taylor coefficients $\phi_{\mathbf{m}}(t)$, which depend on a single variable t. A problem in QFT is thus replaced by a problem in ordinary quantum mechanics, and hence infinities are regularized.

What make the QJT regularization unique is that it is compatible with spacetime diffeomorphism symmetry. The operators (17) act on the truncated jet space, and satisfy the full diffeomorphism algebra without any modification, apart from the abelian extensions. No other regularization has this property.

An analogous statement holds for $Aff(d, \mathfrak{g})$. It is often claimed that lattice gauge theory preserves the gauge symmetries of Yang-Mills theory, but this is not quite true. When replacing spacetime $\mathbb{R}^d$ with a lattice Λ, the gauge group is changed from $Map(\mathbb{R}^d, G)$ to $Map(\Lambda, G)$. Since the former is infinite-dimensional and the latter finite-dimensional, the two groups are not the same. In contrast, the full infinite-dimensional Lie algebra of gauge transformations acts projectively in the QJT regularization of Yang-Mills theory.

6.2 Field theory limit and anomaly almost-cancellation

At the end of the day, the regulator must be removed. In QJT this amounts to take the truncation order p to infinity, assuming that an infinite jet can be identified with the field itself. However, this limit is problematic because the abelian charges c_1 and c_2 in (7) diverge in this limit; they typically behave like p^d for large p. E.g., for a scalar field, a p-jet has $\binom{d+p}{d}$ Taylor coefficients, each of which contributes a finite amount to the abelian charges. The sum thus diverges when $p \to \infty$.

This is not surprising, since normal ordering the fields directly formally yields an infinite extension, i.e. nonsense. The p-jet regularization made it possible to formulate the anomalies, but when the regulator is removed the field theory problems resurface in the form of divergent anomalies.

A way to avoid this problem was discovered in [8]. If we start with several fields, the leading divergencies can be cancelled, because the abelian charges are polynomials in p. There must be both fermionic and boson fields to obtain terms with different signs, and different Taylor series must be truncated at different orders; we need jets of order p, $p-1$, ..., $p-d+1$ to cancel all terms in the abelian charges except for the p-independent one. Explicit expressions can be found in [8].

The diff anomalies in QJT correspond to new degrees of freedom (dofs). These dofs can not be local, because a field has infinitely many dofs and accordingly yields infinite anomalies. If the local parts of the anomalies cancel, the become invisible in field theory. The finite remainder must hence be associated with finitely many non-local, distributed dofs.

The key theme of this paper is that diff anomalies are necessary to combine diffeomorphism symmetry and locality. This statement must now be refined. The diff anomalies in (7) do not cancel, but they almost cancel in the sense that they remain finite in the field theory limit.

6.3 The origin of the anomalies

Gauge anomalies arise when it is impossible to make a regularization that preserves gauge symmetries. However, we noted that passage to jet space is the unique regularization that preserves diffeomorphism symmetry, so how can anomalies arise in QJT? The answer is that although QJT preserves the symmetries, it does not preserve the equations of motion.

To make the point, it suffices to consider a free scalar field $\phi(x)$ in Minkowski space. The equations of motion read

$$\mathcal{E}(x) = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi(x) - \omega^2\phi(x) = 0. \tag{26}$$

Pass to the corresponding space of p-jets (14). The Taylor coefficients obey the equations of motion

$$\mathcal{E}_{\mathbf{m}}(t) = \sum_{\mu\nu} \eta^{\mu\nu}\phi_{\mathbf{m}+\mu+\nu}(t) - \omega^2\phi_{\mathbf{m}}(t) = 0, \tag{27}$$

where we identified $t = x^0$. $\mathbf{m}+\mu+\nu$ is the multi-index obtained from $\mathbf{m}$ by adding unity to both m_μ and m_ν.

The key observation is now that $\mathcal{E}_{\mathbf{m}}(t)$ does not belong to p-jet space unless $|\mathbf{m}| \leq p-2$. If $|\mathbf{m}| = p-1$ or p, $|\mathbf{m}+\mu+\nu| > p$ and $\phi_{\mathbf{m}+\mu+\nu}(t)$ is not a Taylor coefficient of order at most p. Hence the equations of motion are undefined for the two highest orders of Taylor coefficients. The correct configuration space is hence spanned by

- $\phi_{\mathbf{m}}(0)$ for $|\mathbf{m}| \leq p-2$. This part is finite-dimensional.
- $\phi_{\mathbf{m}}(t)$ for $|\mathbf{m}| = p-1, p$. This part is infinite-dimensional.

The elements in the second group must be normal ordered, which gives rise to anomalies.

The analysis was carried out for the free scalar field, but the conclusion is general. In a theory where the equations of motions have order n, only Taylor coefficients up to order $p-n$ have equations of motion. The top n orders have no well-defined equations of motion within p-jet space, and their histories are not fixed by their values at $t=0$.

6.4 Four dimensions

On-shell representations on the p-jet phase space are natural candidates to cancel the divergent part of the abelian charges. Starting with fermionic and bosonic p-jets, the equations of motion are $(p-1)$-jets and $(p-2)$-jets, respectively, and the continuity equation associated with a gauge symmetry is a $(p-3)$-jet. We can therefore hope to cancel the leading terms of the anomalies as in Section 6.2, leaving only a finite contribution when $p \to \infty$.

Assuming that the theory has fermions, bosons and irreducible gauge symmetries, but no reducible gauge symmetries, the field content can be chosen to make the divergent parts of all anomalies cancel in exactly four dimensions. Details can be found in [9, 10]. That QJT seems to prefer four dimensions is a quite robust prediction independent of the details of the model.

Alas, applying the same reasoning to gauge symmetries in Yang-Mills theory leads to serious problems, which casts some doubt on this argument. Nevertheless, it is encouraging that the multi-dimensional Virasoro algebra not only seems to predict the number of dimensions, but actually the correct number.

6.5 Ultralocality and the observer

QJT is a regularization of QFT, but not only so. A p-jet depends not only on the field being expanded, but also on the choice of expansion point.

Classically, the numerical value of a Taylor series in the limit $p \to \infty$ is independent of the expansion point, but for finite p this is not the case. Moreover, we have seen that diff anomalies are functionals of the expansion point, so the quantum theory must depend on it even in the field theory limit.

The expansion point has a natural physical interpretation: it is the observer's position in spacetime.

QJT enjoys a stronger notion of locality than QFT: ultralocality. The observables in QJT are built from Taylor coefficients, which are located on the observer's trajectory. Points away from the observer's trajectory can only be accessed in the field theory limit $p \to \infty$. Not only must interaction terms be local, but observables are local to the observer.

Ultralocality is quite natural from a physical point of view, because every experiment is located inside a detector. E.g., a terrestial observer can not observe the sun directly. Instead, a detector interacts with photons. The observer may then deduce that the photons emanated from the sun eight minutes ago, but a physical theory only needs to be concerned with the primary observation of photons inside the detector.

6.6 The physical observer and QG

The previous section suggests that the physical problem with QG is that the observer's dynamics is ignored in present theories. The following simple argument makes this explicit.

Every real experiment is an interaction between a system and an observer, and the outcome of the experiment depends on the physical properties of both. In particular, it depends on the observer's mass. Neither QFT nor GR depend on the observer's mass, so some tacit assumptions have been made:

- In GR, the observer's heavy mass is assumed to vanish, so the observer does not disturb the fields.
- In QFT, the observer's inert mass is assumed to be infinite, so the fields do not disturb the observer. More precisely, the observer's position and velocity at equal times commute, so he can know his position at all times with arbitary precision.

This suggests that the quantization of gravity does not necessarily involve any new physics. Instead the crucial new ingredient is to include the observer, and in particular the observer's trajectory, into the dynamics. This is automatically taken care of in QJT.

6.7 Cosmological constant

How can the diff anomalies in QJT manifest themselves in physical experiments? The abelian charges in (7) are generalizations of the Virasoro central charge, which is known to couple to length scales. E.g. in Einsteins gravity in three-dimensional AdS space, the central charge $c = -3\ell/2G$, where ℓ is the AdS radius and G is Newtons constant [2]. By analogy, we expect that the abelian charges are manifested in the large-scale structure of the universe, e.g. as a cosmological constant.

At the very least the anomalies can not correspond to new local fields, since the number of new dofs is finite if the abelian charges remain finite in the field theory limit.

6.8 Towards the end of physics

The situation in experimental physics today is somewhat schizophrenic. On the one hand, the standard model (SM) describes all laboratory experiments perfectly, making all BSM (beyond the SM) models look increasingly contrieved. On the other hand, cosmological data suggest that visible matter only accounts for 5% of the universe's mass, so most of the universe should be described by BSM physics.

I suggest that one should take the laboratory data seriously, and that (almost) all experimental physics has already been found, but that the results must be interpreted within the context of QJT. Let us address the most serious objections to this conjecture:

- We know that 95% of the universe's mass consists of dark matter and dark energy. However, medieval astronomers knew that the universe was a mechanical clockwork with at least thirteen epicycles. The result of an observation is always interpreted within some context, be it epicycles or dark matter/energy. The diff anomalies in QJT must manifest themselves in some way, quite likely in the large-scale structure of the universe. If the observation is parametrized in terms of dark matter and energy, the diff anomalies could be mistaken for those.

- Gravity is not part of the SM, so there is at least some BSM physics. However, gravity is very different from flat-space BSM physics. As we argued in Section 6.6, the conceptual problem with gravity has to do with the observer. Once the physical observer has been properly included into the theory, the problems with quantizing gravity may well disappear.

This hypothesis would be invalid if the Higgs particle had turned out to be a few GeV lighter. As far as I understand, the observed Higgs mass of 125-126 GeV is exactly at the lower boundary of the region where the vacuum is stable. That the SM seems to balance at the verge of inconsistency is an interesting observation, and may be an important hint, as the following historical parallel suggests. In the 1940s Lars Onsager wrote down some inequalities that critical exponents must satify for consistency. Twenty years later people realized that these inequalities were in fact equalities, i.e. critical exponents balance on the verge of inconsistency. The underlying reason for this is scale symmetry. By analogy, the almost inconsistent Higgs mass may be due to some symmetry principle. The multi-dimensional Virasoro and affine algebras are natural candidates.

7 Conclusion

Local observables are compatible with spacetime diffeomorphism symmetry, provided that the latter is represented projectively, i.e. with diff anomalies. Generalizations of the Virasoro extensions to the diffeomorphism algebra in arbitrary dimension have been presented, and the off-shell representations have been constructed. There are no definite results on unitarity, because that requires of on-shell representations, which remain an elusive goal. Finally, we speculated on the physical implications of QJT.

It should be emphasized that QJT is substantially different from QFT, as well as from all other approaches to QG. This is proved by the existence of the non-trivial diffeomorphism cocycles in $Vir(d)$. Only in the presence of such diff anomalies can diffeomorphism symmetry be compatible with locality.

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