# On precanonical quantization of gravity

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**Abstract**

Precanonical quantization is based on the mathematical structures of the De Donder-Weyl Hamiltonization of field theories. The resulting formulation of quantum gravity describes the quantum geometry of space-time in terms of operator-valued distances and the transition amplitudes between the values of spin connection at different points of space-time, which obey the covariant precanonical analogue of the Schrödinger equation. In the context of quantum cosmology the theory predicts a probability distribution of a cosmological spin-connection field, which may have an observable impact on the large scale structures in the universe.

**Introduction.** The attempts to construct quantum theory of gravity using the methods of QFT originating from canonical quantization in Minskowski space-time are known to lead to certain technical and conceptual difficulties. One of them is the so-called "problem of time" which can be traced back to the distinguished role of time in the canonical Hamiltonian formalism. The approach of precanonical quantization is based on a different Hamiltonization in field theory, which does not distinguish between the space and time variables. The space-time variables are treated on the equal footing as a multidimensional analogue of the time parameter in mechanics. This Hamiltonization is known in the calculus of variations as the De Donder-Weyl (DW) theory (see e.g. [1]).

**DW Hamiltonization.** For a Lagrangian density $L = L(y^a, y^a_\mu, x^\nu)$, which is a function of the fields variables $y^a$, their first space-time derivatives $y^a_\mu$, and the space-time variables $x^\mu$, one defines the *polymomenta*: $p^\mu_a := \frac{\partial L}{\partial y^a_\mu}$, and the *DW Hamiltonian function*: $H(y^a, p^\mu_a, x^\mu) := y^a_\mu(y,p)p^\mu_a - L$. Then, in the regular case $\det(\partial^2 L/\partial y^a_\mu \partial y^b_\nu) \neq 0$, the Euler-Lagrange field equation can be written in the *DW Hamiltonian form*:

$$\partial_\mu y^a(x) = \partial H/\partial p^\mu_a, \quad \partial_\mu p^\mu_a(x) = -\partial H/\partial y^a, \tag{1}$$

which requires neither a splitting into the space and time nor infinite-dimensional spaces of field configurations. Here the analogue of the extended configuration space is the space of field variables $y^a$ and space-time variables $x^\mu$, and the analogue of the extended phase space is a finite dimensional space of $p^\mu_a, y^a$ and $x^\mu$. Classical fields are sections in the corresponding bundles over the space-time.

**DW theory and precanonical quantization.** Field quantization based on the above Hamiltonization uses the mathematical structures of DW Hamiltonian formalism which were found in our earlier papers [2]. The *polysymplectic form* on the polymomentum phase space: $\Omega := dp^\mu_a \wedge dy^a \wedge \varpi_\mu$, where $\varpi_\mu := \partial_\mu \rfloor \varpi$ and $\varpi{:=}dx^1 \wedge ... \wedge dx^n$ is the volume form on $n$-dimensional space-time, leads to the definition of Poisson brackets on forms of different degrees $p$ and $q$ which represent dynamical variables: $\{\!\![ \overset{p}{F}_1, \overset{q}{F}_2 ]\!\!\} = (-)^{(n-p)} \overset{n-p}{X}_1 \rfloor d\overset{q}{F}_2$, where $\overset{n-p}{X}$ is a Hamiltonian multivector field related to the $p$-form $\overset{p}{F}$ via the map: $\overset{n-p}{X} \rfloor \Omega = d\overset{p}{F}$, $p = 0, 1, ..., (n-1)$. The space of forms for which this map exists is closed with respect to the $\bullet$-product: $\overset{p}{F} \bullet \overset{q}{F} := *^{-1}(*\overset{p}{F} \wedge *\overset{q}{F})$, and the bracket operation equips it with the structure of the *Gerstenhaber algebra*, which appears here as a generalization of the Poisson algebra structure to the DW Hamiltonian formulation. Precanonical quantization relies on the fundamental brackets [2,3]:

$$\{\!\![ p^\mu_a \varpi_\mu, y^b ]\!\!\} = \delta^b_a, \ \ \{\!\![ p^\mu_a \varpi_\mu, y^b \varpi_\nu ]\!\!\} = \delta^b_a \varpi_\nu, \ \ \{\!\![ p^\mu_a, y^b \varpi_\nu ]\!\!\} = \delta^b_a \delta^\mu_\nu. \qquad (2a, b, c)$$

Their quantization leads to the representation of polymomenta and $(n-1)$-forms $\varpi_\mu$ as Clifford-valued operators [3]:

$$\hat{p}^\nu_a = -i\hbar\varkappa\gamma^\nu \frac{\partial}{\partial y^a}, \quad \widehat{\varpi}_\nu = \frac{1}{\varkappa}\gamma_\nu, \qquad (3a, b)$$

where the parameter $\frac{1}{\varkappa}$ appears on the dimensional grounds as a very small quantity of the dimension of $(n-1)$-volume; one could dub it a quantum of space.

The precanonical analogue of the Schrödinger equation [3,4]:

$$i\hbar\varkappa\gamma^\mu \partial_\mu \Psi = \widehat{H}\Psi, \qquad (4)$$

where $\widehat{H}$ is the operator of DW Hamiltonian and $\Psi(y^a, x^\mu)$ is a Clifford-valued wave function, is suggested by the fact that the DW Hamiltonian equations can be writen in terms of the bracket of the fundamental variables in (2) with $H$ [2], which will generate their total co-exterior differential [3,4] (similarly to the generation of the total time derivative by the Poisson bracket with the Hamilton's function in mechanics). We can also argue [3] that (4) allows us to obtain the classical field equations in DW form as the equations for the expectation values of the corresponding precanonical operators, and to reproduce the Hamilton-Jacobi equation of DW theory [1] in the classical limit. The scalar product is related to the conservation law of (4): $\partial_\mu \int dy\, \mathrm{Tr}\left[\overline{\Psi}\gamma^\mu\Psi\right]=0$, where $\overline{\Psi}:=\gamma^0\Psi^\dagger\gamma^0$.

When applied to the scalar field theory [3] with $L=\frac{1}{2}\partial_\mu y \partial^\mu y - V(y)$, we obtain $\hat{H}=-\frac{1}{2}\hbar^2\varkappa^2\frac{\partial^2}{\partial y^2}+V(y)$. For the free field theory with $V(y)=\frac{1}{2}\frac{m^2}{\hbar^2}y^2$ the spectrum of normal ordered $\frac{1}{\varkappa}\hat{H}$ reproduces the mass spectrum of free particles: $mN$, where $N$ is the quantum number of $\hat{H}$. By writing (4) in the form $i\hbar\partial_\mu\Psi=\hat{P}_\mu\Psi$ and defining $\hat{y}(x):=e^{i\hat{P}_\nu x^\nu}y\,e^{-i\hat{P}_\nu x^\nu}$, we can derive the standard correlators of $\hat{y}(x)$ from the precanonical theory [5].

**Standard QFT as a limiting case.** The comparison of probabilistic interpretations of precanonical $\Psi(y,x)$ and the canonical Schrödinger wave functional $\mathbf{\Psi}([y(\mathbf{x})],t)$, and the corresponding equations, allows us to establish a relation between them [6] in terms of the Volterra's multidimensional product integral [7]:

$$\mathbf{\Psi}=\mathsf{Tr}\left\{\prod_{\mathbf{x}} e^{-iy(\mathbf{x})\alpha^i\partial_i y(\mathbf{x})d\mathbf{x}}\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)\big|_{\frac{1}{\varkappa}\beta\mapsto d\mathbf{x}}\right\}, \tag{5}$$

where $\Psi_\Sigma(y(\mathbf{x}),\mathbf{x},t)$ is the restriction of $\Psi(y,x)$ to the subspace $\Sigma$: $(y=y(\mathbf{x}), x^0=t)$, and the notation $\Psi_\Sigma\big|_{\frac{1}{\varkappa}\beta\mapsto d\mathbf{x}}$ means that every $\beta/\varkappa$ in the expression of $\Psi$ is replaced by $d\mathbf{x}$ before the product integral is evaluated. In [6b] it is explicitly demonstrated how this product integral formula leads to the vacuum state wave functional of free scalar field from the ground state solution of precanonical Schrödinger equation. Formula (5) also tells us that the standard QFT obtained from canonical quantization is a limiting case of vanishing $\frac{1}{\varkappa}$ of the theory obtained from precanonical quantization. To be more precise, the limiting transition involves the inverse of the quantization map in (3b) at $\nu=0$: $\frac{\beta}{\varkappa}\mapsto d\mathbf{x}$, that implies an infinitesimal quantum of space $\frac{1}{\varkappa}$.

**Precanonical quantization of gravity.** While precanonical quantization of metric gravity was discussed by us earlier [8], the appearance of the Dirac operator in (4) makes the vielbein formulation of general relativity a preferable starting point for precanonical quantization. Here the Lagrangian density

$$\mathfrak{L}=\ \tfrac{1}{\kappa_E}\mathfrak{e}e^{[\alpha}_I e^{\beta]}_J\left(\partial_\alpha\omega_\beta{}^{IJ}+\omega_\alpha{}^{IK}\omega_{\beta K}{}^J\right)+\tfrac{1}{\kappa_E}\Lambda\mathfrak{e} \tag{6}$$

with the vielbein components $e^\mu_I$, the torsion-free spin-connection coefficients $\omega^{IJ}_\alpha$, the Einstein's gravitational constant $\kappa_E:=8\pi G$, and $\mathfrak{e}:=\det||e^I_\mu||$ leads to the singular DW Hamiltonization with the primary constraints

$$\mathfrak{p}^\alpha_{e^I_\beta}:=\frac{\partial\mathfrak{L}}{\partial_\alpha e^I_\beta}\approx 0,\quad \mathfrak{p}^\alpha_{\omega^{IJ}_\beta}:=\frac{\partial\mathfrak{L}}{\partial_\alpha\omega^{IJ}_\beta}\approx\tfrac{1}{\kappa_E}\mathfrak{e}e^{[\alpha}_I e^{\beta]}_J. \tag{7}$$

We use our generalization of Dirac's approach to constrained systems and the

Dirac bracket to singular DW theories [9]. The Poisson brackets of $(n-1)$-forms constructed from the constraints: $\mathfrak{C}_{e^I_\beta} := \mathfrak{p}^\alpha_{e^I_\beta}\varpi_\alpha$, $\mathfrak{C}_{\omega^{IJ}_\beta} := \mathfrak{p}^\alpha_{\omega^{IJ}_\beta}\varpi_\alpha - \frac{1}{\kappa_E}\mathfrak{e}e_I^{[\alpha}e_J^{\beta]}\varpi_\alpha$:

$$\{\![\mathfrak{C}_e, \mathfrak{C}_{e'}]\!\} = 0, \; \{\![\mathfrak{C}_\omega, \mathfrak{C}_{\omega'}]\!\} = 0, \; \{\![\mathfrak{C}_{e^K_\gamma}, \mathfrak{C}_{\omega^{IJ}_\beta}]\!\} = -\frac{1}{\kappa_E}\frac{\partial}{\partial e^K_\gamma}\left(\mathfrak{e}e_I^{[\alpha}e_J^{\beta]}\right)\varpi_\alpha, \tag{8}$$

indicate that the primary constraints of DW formulation are **second class**. Using our generalization of the Dirac bracket to DW theory we were able to show [10] that the Dirac brackets between the vielbeins and their polymomenta vanish, e.g. $\{\![\mathfrak{p}^\alpha_e\varpi_\alpha, e']\!\}^D = 0$, and the Dirac brackets between the spin connection coefficients and their polymomenta are the same as if there were no constraints, e.g. $\{\![\mathfrak{p}^\alpha_\omega\varpi_\alpha, \omega']\!\}^D = \delta^{\omega'}_\omega$. This fact simplifies quantization performed in [10] using the generalized Dirac's quantization rule: $[\hat{A}, \hat{B}] = -i\hbar\widehat{\mathfrak{e}\{\![A,B]\!\}^D}$, where the operator of $\mathfrak{e}$ ensures that tensor densities are quantized as density-valued operators.

From quantization of fundamental Dirac brackets and using the equations of constraints (7) we conclude that the precanonical wave function does not depend on vielbein variables, i.e. $\Psi = \Psi(\omega^{IJ}_\alpha, x^\mu)$, and obtain a represenation of the operators of vielbeins: $\hat{e}^\beta_I = -i\hbar\varkappa\kappa_E\bar{\gamma}^J\frac{\partial}{\partial\omega^{IJ}_\beta}$, and the polymomenta of spin-connection: $\hat{\mathfrak{p}}^\alpha_{\omega^{IJ}_\beta} = -\hbar^2\varkappa^2\kappa_E\,\hat{\mathfrak{e}}\,\bar{\gamma}^{KL}\frac{\partial}{\partial\omega^{KL}_{[\alpha}}\frac{\partial}{\partial\omega^{IJ}_{\beta]}}$, where $\bar{\gamma}^J$ are the fiducial Minkowskian Dirac matrices and $\hat{\mathfrak{e}} = \left(\frac{1}{n!}\epsilon^{I_1\ldots I_n}\epsilon_{\mu_1\ldots\mu_n}\hat{e}^{\mu_1}_{I_1}\ldots\hat{e}^{\mu_n}_{I_n}\right)^{-1}$. This allows us to construct the operator of DW Hamiltonian density $\mathfrak{e}H$ restricted to the constraints surface $C$: $(\mathfrak{e}H)|_C = -\mathfrak{p}^\alpha_{\omega^{IJ}_\beta}\omega^{IK}_\alpha\omega_{\beta K}{}^J - \frac{1}{\kappa_E}\Lambda\mathfrak{e}$, which is derived from (6), so that

$$\widehat{H} = \hbar^2\varkappa^2\kappa_E\bar{\gamma}^{IJ}\omega_{[\alpha}{}^{KM}\omega_{\beta]M}{}^L\frac{\partial}{\partial\omega^{IJ}_\alpha}\frac{\partial}{\partial\omega^{KL}_\beta} - \frac{1}{\kappa_E}\Lambda, \tag{9}$$

and to obtain the covariant precanonical analogue of the Schrödinger equation for quantum gravity:

$$i\hbar\varkappa\widehat{\not{\nabla}}\Psi = \widehat{H}\Psi, \tag{10}$$

where $\widehat{\not{\nabla}} := \widehat{\gamma}^\mu(\partial_\mu + \frac{1}{4}\omega_{\mu IJ}\bar{\gamma}^{IJ})$, in the explicit form:

$$\bar{\gamma}^{IJ}\left(\partial_\mu + \frac{1}{4}\omega_{\mu KL}\bar{\gamma}^{KL} - \omega^K_{\mu M}\omega^{ML}_\beta\frac{\partial}{\partial\omega^{KL}_\beta}\right)\frac{\partial}{\partial\omega^{IJ}_\mu}\Psi + \lambda\Psi = 0, \tag{11}$$

where $\lambda := \Lambda/(\hbar^2\varkappa^2\kappa_E^2)$ is a dimensionless constant.

The Hilbert space of the theory is defined by the scalar product with the operator-valued invariant measure on the space of spin-connection coefficients:

$$\langle\Phi|\Psi\rangle := \mathrm{Tr}\int\overline{\Phi}\,\widehat{[d\omega]}\,\Psi, \quad \widehat{[d\omega]} = \hat{\mathfrak{e}}^{-n(n-1)}\prod_{\mu, I<J}d\omega^{IJ}_\mu, \tag{12}$$

which is obtained using the arguments similar to those in [11]. It is interesting to note that the normalizability of precanonical wave functions actually implies the quantum singularity avoidance, because $\Psi$ should vanish at large $\omega$-s, i.e. at large space-time curvatures.

Note that the potential issues related to the indefiniteness of $\mathrm{Tr}[\overline{\Psi}\Psi]$ and the gauge fixing, i.e. the choice of the coordinate systems and local orientations of vielbeins on the average, when extracting a physical information from the solutions of (11), are not yet sufficiently clarified.

The Green functions of (11): $\langle \omega, x|\omega', x'\rangle$, which are the transition amplitudes from the values of the spin-connection components $\omega'$ at the point $x'$ to the values $\omega$ at the point $x$, provide an inherently quantum description of space-time geometry, which generalizes the classical description of geometry in terms of smooth spin-connection fields $\omega(x)$. Besides, the distances between points are given by quantum operators, because the metric tensor in the present formulation is operator-valued: $\widehat{g^{\mu\nu}} = -\hbar^2\varkappa^2\kappa_E^2\eta^{IJ}\eta^{KL}\frac{\partial^2}{\partial\omega_\mu^{IK}\partial\omega_\nu^{JL}}$. This type of description of quantum geometry of space-time in terms of the transition amplitudes on the connection bundle and the operator-valued metric structure on the space-time complements the current intuitive ideas about the quantum space-time suggested by quantum geometrodynamics, loop quantum gravity, string theory and non-commutative geometry.

The fact that all dimensionful constants in (11) are absorbed in one dimensionless constant $\lambda$, which depends on the ordering of operators $\omega$ and $\partial_\omega$, seems to be important. Knowing $\lambda$ we would be able to determine the value of our constant $\varkappa$. A naive estimation yields $\lambda \sim n^6$ and then $\varkappa$ at $n = 4$ is at the nuclear scale, which is unexpected. If, however, we assume that $\varkappa$ is Planckian, then the estimated value of $\Lambda$ is $\sim 10^{120}$ times higher than the observable one, which is a usual problem in naive QFT-based estimations of $\Lambda$. This coincidence confirms that $\varkappa$ of precanonical quantization is related to the ultra-violet cutoff scale in standard QFT and indicates that the cosmological constant is not likely to be related to the ground state of pure quantum gravity alone.

**Precanonical quantum cosmology.** For $n = 4$ flat FLRW metric with a harmonic time $\tau$:

$$ds^2 = a(\tau)^6 d\tau^2 - a(\tau)^2 d\mathbf{x}^2, \tag{13}$$

let us choose $e^0_\nu = a^3\delta^0_\nu$, $e^I_\nu = a\delta^I_\nu$, so that $\omega^{I0}_\nu = \dot{a}/2a^3\delta^I_\nu =: \omega\delta^I_\nu$ $(I = 1, 2, 3)$. Then the precanonical Schrödinger equation, eq. (11), takes the form

$$\left(2\sum_{i=I=1}^{3}\gamma^{0I}\partial_\omega\partial_i + 3\omega\partial_\omega + c\right)\Psi = 0, \tag{14}$$

where, if $\omega\partial_\omega$ is Weyl-ordered, $c = \frac{3}{2} + \frac{\Lambda}{(\hbar\varkappa\kappa_E)^2}$ is the effective cosmological constant. By separation of variables $\Psi := u(\mathbf{x})f(\omega)$ we obtain: $2\sum_{i=I=1}^{3}\gamma^{0I}\partial_i u = iqu$, and $(iq\partial_\omega + 3\omega\partial_\omega + c)f = 0$. The solution of the latter: $f \sim (3\omega + iq)^{-c/3}$, yields the probability density

$$\rho(\omega) := \bar{f}f \sim \frac{1}{(9\omega^2 + \bar{q}q)^{c/3}}. \tag{15}$$

One can either interpret it as a distribution of quantum universes according to the value of $\omega = \dot{a}/2a^3$, i.e. essentially the expansion rate $\dot{a}$, or as a spatially homogeneous distribution function of quantum fluctuations of the random cosmological spin-connection field $\omega$. The possibility of the latter point of view within the precanonical approach makes the usual interpretational issues of quantum cosmology much less troublesome.

Note that our discussion is based on a toy quantum cosmology model, where no influence of matter fields is taken into account so far. It would be interesting to investigate if the probability distribution of spin connection (15) predicted by precanonical quantum gravity theory manifests itself in the large scale structures in the universe and can be tested by cosmological observations.

**Acknowledgement.** I thank Prof. Yu. Kurochkin and the organizing committee of Zeldovich-100 in Minsk for the opportunity to present this talk and enjoy the meeting.

## References

[1] H. Kastrup, *Phys. Rep.* **101** (1983) 1.

[2] I.V. Kanatchikov, *Rep. Math. Phys.* **41** (1998) 49, hep-th/9709229;
I.V. Kanatchikov, *Rep. Math. Phys.* **40** (1997) 225, hep-th/9710069.

[3] I.V. Kanatchikov, *AIP Conf. Proc.* **453** (1998) 356, hep-th/9811016;
I.V. Kanatchikov, *Rep. Math. Phys.* **43** (1999) 157, hep-th/9810165.

[4] I.V. Kanatchikov, in: *Differential Geometry and its Applications,* ed. O. Kowalski e.a. (Opava, 2001) p. 309, hep-th/0112263.

[5] I.V. Kanatchikov, Precanonical quantization: from foundations to quantum gravity, *in preparation.*

[6] I.V. Kanatchikov, *Adv. Theor. Math. Phys.* **18** (2014), arXiv:1112.5801;
I.V. Kanatchikov, arXiv:1312.4518.

[7] V. Volterra, B. Hostinský, *Opérations Infinitésimales Linéaires,* Gauthier-Villars, Paris (1938).

[8] I.V. Kanatchikov, *Int. J. Theor. Phys.* **40** (2001) 1121, gr-qc/0012074.
See also: I.V. Kanatchikov, gr-qc/9810076, gr-qc/9912094, gr-qc/0004066.

[9] I.V. Kanatchikov, in: *Differential Geometry and its Applications,* ed. O. Kowalski e.a. (World Sci., Singapore, 2008) p. 615, arXiv:0807.3127.

[10] I.V. Kanatchikov, *AIP Conf. Proc.* **1514** (2012) 73, arXiv:1212.6963;
I.V. Kanatchikov, *J. Phys. Conf. Ser.* **442** (2013) 012041, arXiv:1302.2610.

[11] C. Misner, *Rev. Mod. Phys.* **29** (1957) 497.