

Introduction to Quantum Gravity

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In this talk^a, we give a glimpse of the problems with quantum gravity and some possible solutions.

I. INTRODUCTION

Over the centuries, physicists try to unify the fundamental interactions. An essential ingredient for these unification theories is the gauge symmetry, in particular, the (special) unitary group.

Unification	Group
Electromagnetic	$U(1)$
Electroweak	$SU(2) \times U(1)$
Standard Model	$SU(3) \times SU(2) \times U(1)$
Grand Unified Theory ?	$\times SU(3) \times SU(2) \times U(1)$

However, among the 4 fundamental interactions in nature, only the gravitational interaction is related to other group than the unitary. Furthermore, it is described by other lagrangian than the Yang-Mills (despite the “accident” in $(2+1)$ -dimensions [1]).

Interaction	Group	Lagrangian
Strong	$SU(3)$	} Yang-Mills
Weak	$SU(2)$	
Electromagnetic	$U(1)$	
Gravitational	General Covariance	Einstein-Hilbert

Therefore, from the unification viewpoint, it would be reasonable to argue that general covariance is equivalent to some unitary group and/or is the Einstein-Hilbert lagrangian tantamount to Yang-Mills lagrangian? As we will see, the answer to both questions is no.

II. COMPARISON BETWEEN YANG-MILLS AND EINSTEIN-HILBERT LAGRANGIANS

The Yang-Mills lagrangian

$$\mathcal{L}_{YM} = \frac{1}{\kappa^2} \text{tr}(\star F \wedge F),$$

where the curvature $F = dA + A \wedge A$, \star is the Hodge dual operator, tr is the trace over $\mathfrak{su}(N)$ algebra and κ is the coupling constant, can be rewritten, after rescaling the gauge field $A \rightarrow \kappa A$, symbolically as

$$\mathcal{L}_{YM} = \underbrace{(dA)^2}_{\mathcal{L}_{\text{free}}} + \underbrace{\kappa(dA)A^2 + \kappa^2 A^4}_{\mathcal{L}_{\text{interaction}}}, \quad (1)$$

and thus is easy to quantize — a free propagator and a finite number of (two) vertices — at least in the high-energy limit, where κ is small and perturbation theory makes sense.

On the other hand, the Einstein-Hilbert lagrangian

$$\mathcal{L}_{EH} = \frac{1}{\kappa^2} \sqrt{g} R,$$

where $\kappa^2 = 16\pi G$, G is Newton’s constant, the scalar curvature $R = g^{\mu\nu}(\Gamma_{\mu\alpha,\nu}^\alpha - \Gamma_{\mu\nu,\alpha}^\alpha + \Gamma_{\mu\alpha}^\beta \Gamma_{\beta\nu}^\alpha - \Gamma_{\mu\nu}^\beta \Gamma_{\beta\alpha}^\alpha)$, the Christoffel symbol $\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\beta}(g_{\nu\beta,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta})$ and $g = \det(g_{\mu\nu})$ is the determinant of the metric $g_{\mu\nu}$, is distinct from the Yang-Mills lagrangian. Even the equivalent $\Gamma\Gamma$ lagrangian [2]

$$\mathcal{L}_{\Gamma\Gamma} = \frac{1}{\kappa^2} \sqrt{g} g^{\mu\nu} (\Gamma_{\mu\nu}^\beta \Gamma_{\beta\alpha}^\alpha - \Gamma_{\mu\alpha}^\beta \Gamma_{\beta\nu}^\alpha)$$

is completely different from Yang-Mills, as can be seen explicitly in some D -dimensional examples

$$D = 1, \quad \mathcal{L}_{\Gamma\Gamma} = 0$$

$$D = 2, \quad \mathcal{L}_{\Gamma\Gamma} = \frac{1}{2\kappa^2} \frac{\begin{vmatrix} g_{11} & g_{12} & g_{22} \\ g_{11,1} & g_{12,1} & g_{22,1} \\ g_{11,2} & g_{12,2} & g_{22,2} \end{vmatrix}}{\begin{vmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{vmatrix}^{3/2}} \quad (2)$$

$$D = 3, \quad \mathcal{L}_{\Gamma\Gamma} = \dots$$

(after some algebra, the 2-dimensional case simplifies to this fraction of determinants, this simplification doesn’t seem to occur for higher dimensions). Proceeding as before, by rescaling the metric $g_{\mu\nu} \rightarrow \kappa g_{\mu\nu}$, is useless, specially in the 2-dimensional case which is scale invariant. So, how to quantize?

III. THE QUANTIZATION PROBLEM

A. Quantization methods

There exist several methods of quantization: canonical, constrained, path-integral, stochastic, etc. There are pros and cons to each method, but all are solvable, roughly, only for quadratic terms in the lagrangian. For example, the path-integral of the free Yang-Mills la-

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grangian, Eq. (1) with $\kappa = 0$, plus a source term $J(x)$,

$$\begin{aligned} Z_{\kappa=0}[J] &= \int DA e^{-\int d^4x (\mathcal{L}_{free} + J \cdot A)} \\ &= \int DA e^{-\int d^4x [(dA)^2 + J \cdot A]}, \end{aligned}$$

can be done exactly, since these are gaussian integrals. From the functional generator $Z[J]$, through differentiation in J , the propagator, vertices, etc. can be calculated. Some details still need attention, such as ghosts, owing to gauge invariance, although the remaining is just a power series expansion in κ . How about the free Einstein-Hilbert lagrangian? Eqs. (2) with $\kappa = 0$?

B. Early attempts

Historically, the first attempt to quantize gravity was done by Rosenfeld [3] through a “scale-shift” transformation, best known as a fixed background field, i.e.,

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \kappa h_{\mu\nu}(x),$$

(in this case the expansion is around a flat spacetime, where $g_{\mu\nu}^{\kappa=0}(x) = \delta_{\mu\nu}$ is the euclidean metric) such that, after expansion in κ , the Einstein-Hilbert or the $\Gamma\Gamma$ lagrangian become

$$\mathcal{L}_{EH} = \underbrace{(dh)^2}_{\mathcal{L}_{free}} + \underbrace{\kappa(dh)^2 h + \kappa^2(dh)^2 h^2 + \kappa^3(dh)^2 h^3 + \dots}_{\mathcal{L}_{interaction}}$$

which resembles the Yang-Mills lagrangian, Eq. (1), but has an infinite number of vertices.

Other attempts employing different types of transformations, for example, another choices of scale-shift such as $\sqrt{g}g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$, $g^{\mu\nu} = \delta^{\mu\nu} - \kappa h^{\mu\nu}$, ... Or distinct background fields $g_{\mu\nu} = \varphi_{\mu\nu} + \kappa h_{\mu\nu}$, where $\varphi_{\mu\nu}$ is, e.g., the (anti) de Sitter metric. Still another type using the first order formalism as $e^i_\mu = \delta^i_\mu + \kappa h^i_\mu$, where e^i_μ is the tetrad field, etc. All have an identical problem — the renormalization problem — after all, they share the same properties, i.e., a fixed background field and a perturbative expansion in a dimensional coupling constant $\kappa \propto \sqrt{G}$ with length dimension [4].

C. The renormalization problem

Despite the initial miracle [5] in the pure gravitational sector, in the presence of matter fields the one-loop calculation has a divergence of the form $c_1 R^2 + c_2 R^{\mu\nu} R_{\mu\nu}$, where c_1 and c_2 are constants. One possible way out was to modify the Einstein-Hilbert lagrangian to a higher-derivative lagrangian

$$\mathcal{L} = \frac{1}{\kappa^2} \sqrt{g} (R + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu}),$$

which makes the theory renormalizable [6] but, unfortunately, suffers from other problems [7]. Another way out is to consider a more general function in the lagrangian $f(R) = R + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu} + \dots$, although it becomes more difficult the interpretation of the constants α , β , ... because of the lack of experimental data, since the energy scale is the Planck energy $1/\kappa \propto 1/\sqrt{G} \sim 10^{28}$ eV.

IV. SOME RECENT THEORIES

A. Perturbative theories

The renormalization problem found using perturbation theory has alternative solutions, to wit: the semiclassical [8] and effective [9] methods. In the former, gravity is a classical field and everything else is quantized; now in the latter, everything is quantized, including gravity, but the Feynman amplitude is expanded in terms of the momentum exchanged. Despite being quite different approaches, they are equivalent [10] in the sense that both give results strictly valid in the low-energy limit.

B. Non-perturbative theories

We do not intend to discuss these theories, although some comments are in order. As far as we know, even in the 2-dimensional toy model, there is no choice of variables that makes the fraction of determinants, Eq. (2), into a fourth order polynomial, Eq. (1), at least without ruining the measure in the path-integral as in the unimodular gravity [11]. Therefore, other choices of variables seem unhelpful, specially in the 4-dimensional case. Besides that, theories of everything have struggled to overcome worse problems than the renormalization problem. In view of Occam’s razor, it is reasonable to search for a simpler solution.

C. Modified BF theory

A possible solution to the presented problems is a expansion in the dimensionless and extremely small coupling constant $\kappa^2 = G\Lambda \sim 10^{-120}$, where Λ is the cosmological constant. This is achieved through a modification of the BF theory lagrangian $\mathcal{L} = \text{tr}(-iB \wedge F)$, namely,

$$\mathcal{L} = \text{tr}(-iB \wedge F - \kappa^2 B \wedge \Gamma B), \quad (3)$$

which is polynomial, and for $\Gamma = \gamma_5$ reproduces the Einstein-Hilbert lagrangian [12], and for $\Gamma = \star$ also reproduces the Yang-Mills theory [13].

In this approach, the metric is a derived object, i.e., the $SO(5)$ gauge field A_μ^{IJ} breaks into a $SO(4)$ spin connection $A_\mu^{ij} = \omega_\mu^{ij}$ and a tetrad field $A_\mu^{i5} = \sqrt{\Lambda} e^i_\mu$, and thus

$$g_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij} = \frac{1}{\Lambda} A_\mu^{i5} A_\nu^{j5} \delta_{ij}. \quad (4)$$

Even in the lorentzian case the gauge group is $SO(4, 1)$ or $SO(3, 2)$, and hence the general covariance group seems related to the (broken symmetry) special orthogonal group. Moreover, as an expansion around a topological theory, it is background independent.

Despite some discussions [14], common to new ideas, it is worth noting that Eq. (4) is a sign of the conjecture that gravity is the square of a gauge field [15].

V. CONCLUSIONS

In this talk, we give a glimpse of the quantization problem of gravity. As argued, this problem is due to the non-

polynomial form of the Einstein-Hilbert lagrangian. Attempts to recast this expression into a polynomial form, as the fixed background field, seem unhelpful and lead to another problem — the renormalization problem — that is also related to the dimensional coupling constant $\kappa^2 \propto G$.

Therefore, a simple theory that can be written in polynomial form, with background independence and a dimensionless coupling constant, is a serious candidate as a possible solution to these problems. As we have seen, a theory that fulfills all these prerequisites is the Modified BF. Furthermore, from the unification point of view, this theory is worthwhile, inasmuch as it reproduces the Einstein-Hilbert and the Yang-Mills lagrangians.

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