

# Classical Group Field Theory

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## Abstract

The ordinary formalism for classical field theory is applied to dynamical group field theories. Focusing first on a local group field theory over one copy of  $SU(2)$  and, then, on more involved nonlocal theories (colored and non colored) defined over a tensor product of the same group, we address the issue of translation and dilatation symmetries and the corresponding Noether theorem. The energy momentum tensor and dilatation current are derived and their properties identified for each case.

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# 1 Introduction

Group field theories (GFTs) are usually defined as tensor field theories over a group manifold. Introduced in the beginning of the 90's [1], they rapidly become pertinent candidates for quantum gravity [2, 3, 4]. In a nutshell, GFTs provide a framework for addressing the problem of both the emergence spacetime topology and metric properties [4].

Since the GFT inception, focusing only on the quantum field theoretical point of view, several studies have been led using the path integral approach [5]. Interesting facts pertaining to the renormalization program have been highlighted such as power counting theorems [6]-[13], a new emergent locality principle [12, 14], Ward-Takahashi identities for unitary symmetries [15], to name but a few. Besides, a large  $1/N$  topological/combinatorial expansion [16]-[18] for colored GFT models [19, 20] portends new and fertile contacts with models in statistical mechanics.

Classical aspects of GFT have been also examined. For instance, equations of motion for topological models<sup>1</sup> have been solved providing an explicit class of instantons used to perform a perturbation theory and the subsequent effective prediction in [21]. Furthermore, using a group Fourier transform (for seminal works on this topic and more applications, [22] affords a recent and fair review), topological GFTs can be seen as noncommutative models with a set of diffeomorphism symmetries turning out to reduce to some deformed Poincaré group [23, 24]. Interestingly, it can be mentioned that in these works, the group manifold initially associated with the background onto which the fields live, becomes finally a curved momentum space via this group Fourier transform [25]. The direct space generated after inverse transform is flat whereas the algebra of fields is traded for a noncommutative algebra endowed with a  $\star$ -product. Nevertheless, other interesting properties even at the classical level have been yet investigated. A recent paper [14] emphasizes to what extent the dynamics matters in the renormalization program for GFT. Including a full dynamics in the game, the background group symmetries having so far trivial Noether currents should henceforth possess relevant properties. The present contribution addresses, in a more traditional field theory spirit by placing back the group as a base manifold, the issue of the classical formalism for GFTs.

We should emphasize that some anterior investigations have been carried out on field theories on a Lie group regarding both classical symmetries and also some of their quantum implications. For example, a  $\phi^4$  quantum field theory on the affine group has been studied in [26]. Furthermore, diffeomorphism and Weyl transformations in a curved  $\phi^4$  theory and their implications have a long history (see for instance [27][28]). We will not use, in the present contribution, the same route but will focus either on the gauge invariance of fields or on the nonlocal feature of GFT.

In this paper, we study the classical dynamical GFT over  $D = 1$  and then  $D = 3$  copies (that we shortly call dimensions or rank) of  $SU(2)$ . The case  $D = 2$  turns out to be equivalent to the situation  $D = 1$  due to the gauge invariance condition on fields. Our results can be extended without ambiguity in any dimension. As it will stand the action simply describes a gauge invariant tensor scalar field theory over  $D$  copies of the sphere  $S^3$  with a local (for  $D = 1$ ) or nonlocal (for  $D = 3$ ) interaction. The main issue in the present

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<sup>1</sup> In the sequel, topological models are referred to as GFT models with a trivial kinetic term in contradistinction with dynamical models having a nontrivial kinetic part. Typically, the nontrivial dynamics is governed by a Laplacian on the group manifold.

study is to show that the procedure for solving equations of motion or for studying the symmetries of the action finds an extension in this peculiar curved, tensored and nonlocal theory. The translation symmetry and the corresponding energy momentum tensor (EMT) have been worked out. We find that the EMT appears to be symmetric in a certain sense but not locally conserved for  $D > 1$  in ordinary GFT. However, surprisingly and positively, we find that for the colored GFT possesses a covariantly conserved quantity by integrating some EMT components. The ordinary improvement procedure in order to obtain a traceless EMT seems to be in conflict with the local conservation of the EMT. We then address another interesting question, in fact related to this latter issue, which is the implementation of dilatation symmetry or scale transformation at the GFT level. Requiring the invariance under dilatations yields a radically different action from the translation invariant action. We compute and characterize the current tensor associated with this transformation.

The content of this paper is the following: Section 2 is devoted to the model presentation and the first steps of the classical study by exhibiting and solving the equation of motion for free fields for the dynamical Boulatov model in  $D = 3$ . Then Section 3 thoroughly undertakes the Noether theorem for translation and dilatation symmetries for a  $1D$  GFT as a guiding path for more complicated higher rank GFTs. Section 4 discusses the same symmetries for general GFTs. Section 5 deals with the colored models and their particular characteristics. Section 6 summarizes our results and also provides outlooks of this work. Finally, a very detailed appendix collects the proofs of our claims and useful identities invoked along the text.

## 2 The dynamical Boulatov-Ooguri model

The prominent properties of the dynamical three dimensional GFT over  $G = SU(2)$  are quickly and pedagogically introduced (for more details on the general formulation see [2, 3, 4]). Then the equation of motion without coupling constant is solved. The Noether analysis of symmetries will be deferred to the next sections. This section admits a straightforward extension in any GFT dimension  $D > 1$ .

**The model** - The fields belong to the Hilbert space of square integrable and gauge invariant functions on  $G^3$  which satisfy

$$\phi(g_1h, g_2h, g_3h) = \phi(g_1, g_2, g_3) , \quad \forall h \in G . \quad (1)$$

The shorthand notation  $\phi(g_1, g_2, g_3) = \phi_{1,2,3}$  will be used henceforth.

The action  $S_{3D}$  is formed by a kinetic term and interacting part. The kinetic term has the form

$$S_{\text{kin}, 3D}[\phi] := \int \left[ \prod_{\ell=1}^3 dg_\ell \right] \left[ \frac{1}{2} \sum_{s=1}^3 \mathbf{g}_s^{ij} \nabla_{(s) i} \phi_{1,2,3} \nabla_{(s) j} \phi_{1,2,3} + \frac{1}{2} m^2 \phi_{1,2,3} \phi_{1,2,3} \right] , \quad (2)$$

where  $dg_i$  denotes the Haar measure on  $G = SU(2)$ , the operator  $\nabla_{(s)}^i$  represents the covariant derivative (acting here merely as a partial derivative on above fields) defined with the Levi-Civita connection on  $S^3 \simeq SU(2)$ . The index  $(s)$  will always refer to the tensor structure

and so to the particular group element  $g_s$  with respect to which one derivates.<sup>2</sup> The labels  $i, j$  refer to the local coordinates and, therefore, are lowered or raised by the  $S^3$  metric  $\mathbf{g}_{ij}$ . Note that the Haar measure of  $SU(2)$ ,  $dg$  can be written in a more standard fashion with respect to a theory on a curved background as  $dg = (2\pi^2)^{-1} \sqrt{|\det \mathbf{g}|} d\theta d\varphi_1 d\varphi_2$  with  $\mathbf{g} = d\theta^2 + \sin^2 \theta (d\varphi_1^2 + \sin^2 \varphi_1 d\varphi_2^2)$ .

The interaction in  $D$  dimensional GFTs is nonlocal and dually associated with a  $D$ -simplex. For  $D = 3$  dimensions, the interaction is

$$S_{\text{int}, 3D}[\phi] := \frac{\lambda}{4} \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \phi_{1,2,3} \phi_{3,4,5} \phi_{5,2,6} \phi_{6,4,1}, \quad (3)$$

with a particular pairing of the six variables according to the pattern of the edges of a tetrahedron.

An important remark is that the mass term enforces the gauge invariant constraint (1) on fields. Reducing the kinetic part to a pure massive term combined with the interaction term, one is led to a pure topological theory that is known as the Boulatov model [1].

Formally, a Lagrangian density can be defined as

$$\mathcal{L}_{3D} = \frac{1}{2} \sum_s \nabla_{(s)}^i \phi_{1,2,3} \nabla_{(s) i} \phi_{1,2,3} + \frac{1}{2} m^2 \phi_{1,2,3} \phi_{1,2,3} + \frac{\lambda}{4} \phi_{1,2,3} \phi_{3,4,5} \phi_{5,2,6} \phi_{6,4,1}. \quad (4)$$

The equation of motion for the field results from the action variation:

$$0 = \frac{\delta S_{\text{kin}, 3D}}{\delta \phi_{1,2,3}} + \frac{\delta S_{\text{int}, 3D}}{\delta \phi_{1,2,3}} = - \sum_s \Delta_{(s)} \phi_{1,2,3} + m^2 \phi_{1,2,3} + \lambda \int \left[ \prod_{\ell=4}^6 dg_i \right] \phi_{3,4,5} \phi_{5,2,6} \phi_{6,4,1}, \quad (5)$$

with  $\Delta_{(s)}$  being the Laplace operator on the group. Remark that to get (5), we implicitly used an integration by parts and the fact that the sphere does not have a boundary. Furthermore, one should also rename cyclically the group arguments in the interaction in order to vary properly this nonlocal term.

**Colored GFT** - Colored GFT models [19][20] have mainly the same definition as above with the crucial attribute that fields possess an extra ‘‘color’’ index  $\phi^a$ . We will choose them to be complex valued functions. The number of colors being the number of fields in the interaction. For the Boulatov colored model, we have  $a = 1, 2, 3, 4$ . More generally for a  $D$  dimensional GFT, the color indices are  $a = 1, 2, \dots, D + 1$ . All field properties remain the same as previously and a Lagrange density for the  $3D$  theory can be given as

$$\mathcal{L}^{\text{color}} = \sum_{a=1}^4 \left[ \sum_s \nabla_{(s)}^i \bar{\phi}_{1,2,3}^a \nabla_{(s) i} \phi_{1,2,3}^a + m^2 \bar{\phi}_{1,2,3}^a \phi_{1,2,3}^a \right] + \lambda \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \bar{\lambda} \bar{\phi}_{1,2,3}^1 \bar{\phi}_{3,4,5}^2 \bar{\phi}_{5,2,6}^3 \bar{\phi}_{6,4,1}^4. \quad (6)$$

Important quantum topological aspects lie in the ‘‘coloring’’ of GFT [20]. For the present work, we will indeed see that even at the level of classical analysis, implementing this extra

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<sup>2</sup>It will be also called strand index in the following,  $s = 1, 2, \dots, D$ .

color index to field might lead to an improvement of the features of the Noether currents for a given symmetry.

**Solving the tensor Klein-Gordon equation** - In [21] discussing the Boulatov model, a class of solutions for the equation of motion of this topological model have been found. In the present situation, another issue due to the dynamics arises. However the equivalent of Klein-Gordon equation can be again worked out. We have to solve

$$-\sum_s \Delta_{(s)} \phi_{1,2,3} + m^2 \phi_{1,2,3} = 0, \quad (7)$$

for gauge invariant fields. Using Peter-Weyl decomposition (see Appendix A for a summary of following notations), the above equation is equivalent to

$$\sum_{j_a, m_a, n_a} \phi_{m_a, n_a}^{j_a} \left( -\sum_{a=1}^3 C(j_a) + m^2 \right) \prod_{a=1}^3 \sqrt{d_{j_a}} D_{m_a, n_a}^{j_a}(g_a) = 0, \quad (8)$$

with  $C(j_a) = j_a(j_a + 1)$  denoting the Casimir or eigenvalue of  $\Delta_{(a)}$ . A solution of the Klein-Gordon equation for  $D = 3$  GFT is therefore

$$\phi_{1,2,3} = \sum_{j_a, m_a, n_a} \phi_{m_a, n_a}^{j_a} \delta_{\sum_{a=1}^3 C(j_a) - m^2, 0} \prod_{a=1}^3 \sqrt{d_{j_a}} D_{m_a, n_a}^{j_a}(g_a), \quad (9)$$

where  $\phi_{m_a, n_a}^{j_a}$  is assumed to satisfy also (A.3). For large spin  $j_a$ , the solutions (9) are such that only modes  $\phi^{j_a}$  with  $j_1^2 + j_2^2 + j_3^2 = m^2$  remain in the field expansion.

### 3 Translations and dilatations: 1D GFT

In this section, as a preliminary crucial study to the full picture for any GFT dimension, we first start by Noether theorem for translations and dilatations for GFT in one dimension. The latter theory is local and the subsequent analysis in this local framework will be compared to the analogous for a GFT in any dimension  $D \geq 3$  which is, in contrast, a nonlocal theory.

In 1D GFT, the gauge invariant condition (1) for fields should be abandoned for being equivalent to the requirement of constant fields. The bottom line is the data of an action over one copy of  $G$  of the form

$$S_{1D}[\phi] = \int dg \mathcal{L}_{1D}(\phi, \nabla \phi), \quad \mathcal{L}_{1D} = \frac{1}{2} \mathbf{g}^{ij} \nabla_i \phi(g) \nabla_j \phi(g) + \frac{1}{2} m^2 \phi^2(g) + \frac{\lambda}{4} \phi^4(g). \quad (10)$$

#### 3.1 Translations and EMT

A right translation<sup>3</sup> symmetry by an element  $h$  is simply the right group multiplication  $g \mapsto gh$ . Under this symmetry, a field transforms as

$$\phi(g) \mapsto \phi(gh). \quad (11)$$

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<sup>3</sup>Left translations can be carried out in a similar manner.

At the infinitesimal level, given a local coordinate system, the variation of any field is given by

$$\delta_X \phi = X \cdot \partial \phi = \sum_{i=1}^3 X^i \partial_i \phi . \quad (12)$$

The operator<sup>4</sup>

$$W(X)(\cdot) = \int d\theta d\varphi^1 d\varphi^2 \left( \delta_X \mathbf{g}^{ij} \frac{\delta}{\delta \mathbf{g}^{ij}}(\cdot) + \delta_X \phi \frac{\delta}{\delta \phi}(\cdot) \right) \quad (13)$$

acting on the action  $S_{1D}$  (10) allows one to define the Noether theorem for a given symmetry with parameter  $X$  for which an infinitesimal field variation  $\delta_X \phi$  is given. Operators of the kind (13) prove to be useful tool either in the situation of a gauge symmetry (and are indeed related to Ward identity operators when acting on a partition function), or when one deals with nonlocal interaction as appear in noncommutative geometry or matrix models [29]-[33]. In the following and according to the context, this operator will take different forms and will enable us to treat the nonlocal interaction properly.

Considering (12), one obtains after some algebra (Appendix B.1 provides details of the derivations)

$$\frac{\partial}{\partial X^i} W(X) S_{1D} = - \sum_k \int d\theta d\varphi^1 d\varphi^2 \partial_k (\sqrt{|\det \mathbf{g}|} \mathbf{g}^{kj} T_{ij}) , \quad (14)$$

where  $T_{ij}$  is the EMT given in a covariant form as

$$T_{ij} = \nabla_i \phi \nabla_j \phi - \mathbf{g}_{ij} \mathcal{L}_{1D} . \quad (15)$$

The properties of the EMT are quite straightforward:  $T_{ij}$  is symmetric and covariantly conserved. Using the equation of motion, it can be proved (see Appendix B.1) that

$$\nabla^i T_{ij} = 0 . \quad (16)$$

Nevertheless, the sense of conserved charges remains unclear at this level. Indeed, there is *a priori* no preferred coordinate embodying the ordinary role of time and no obvious partial integration on the remaining variables for which a correct conserved quantity could be generated from (16).

For a massless theory, the trace of the EMT (15) is not vanishing. Note that also the usual EMT in a massless  $\phi_4^4$  theory is not traceless. A traceless EMT can be only built by adding a correction to the original EMT. Here, the naive improvement procedure by adding an extra term to the EMT such that

$$\hat{T}_{ij} = T_{ij; m=0} + \frac{1}{\beta} (\mathbf{g}_{ij} \nabla^k \nabla_k - \nabla_i \nabla_j) \phi^2 \quad (17)$$

yields still a symmetric tensor but it is neither traceless nor covariantly conserved (the obstruction of that local conservation is well known to be expressed in terms of the Ricci

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<sup>4</sup>In the following, the normalization  $1/(2\pi^2)$  of the Haar measure will be dropped.

tensor associated with the connection). Another argument in order to understand why the EMT is not traceless is that the action (10) is not scale invariant in a sense that we will precise in the following section. Insisting on the traceless improvement procedure for the EMT (15), one can perform the following modification:

$$\hat{T}'_{ij} = T_{ij; m=0} + \frac{1}{\beta} \mathbf{g}_{ij} \phi \nabla^k \nabla_k \phi + \frac{1}{\beta'} \nabla_i \phi \nabla_j \phi, \quad (18)$$

such that  $\text{Tr } \hat{T}' = 0$  is recovered for  $\beta' = 2$  and  $\beta = 4$ . Note that,  $\hat{T}'$ , even though symmetric, is not covariantly conserved.

### 3.2 Dilatations and current vector

**Group dilatations** - Unlike in the flat and noncompact space case, the notion of dilatation symmetry on a compact manifold like the sphere is not an obvious concept. We use here an idea familiar to wavelet analysis on the two-sphere [34] for discussing the concept of dilatations on the sphere  $S^3$ . We will show that these dilatations can be implemented for particular GFT models.

Let  $a$  be a real strictly positive number. Given a group element  $g = g(\theta, \vec{n}) \in G \simeq S^3$ , characterized by the class angle  $\theta$  and the unit vector  $\vec{n} \in S^2$ , one defines the map  $d_a : G \rightarrow G$  such that  $g \mapsto g_a$  with

$$g_a = g_a(\theta_a, \vec{n}), \quad \tan \frac{\theta_a}{2} = a \tan \frac{\theta}{2}. \quad (19)$$

More intuitively, the group element  $g_a$  can be viewed as follows: given  $g \in S^3$ , project  $g$  on the tangent  $3D$  hyperplane at the North pole by a stereographic projection from the South pole; apply an usual Euclidean dilatation by  $a$  to the projected element in the flat space and then project back the result onto the sphere  $S^3$  by the inverse stereographic projection. Remark that the stereographic projection is not well defined at the South pole and this will also have consequences in the formulation with some undefined ratios.

Under this mapping, the  $\theta$  dependence of the Haar measure undergoes (Appendix C.1 provides justifications of the following results)

$$d\theta(\sin \theta)^2 \mapsto d\theta_a(\sin \theta_a)^2 = (\mu(a, \theta))^3 d\theta(\sin \theta)^2, \quad \mu(a, \theta) = \frac{2a}{(1 - a^2) \cos \theta + 1 + a^2}. \quad (20)$$

In fact, restricted to the two-sphere, dilatations of this kind together with translations belong to a subgroup of the Lorentz group  $SO_0(3, 1)$ , the component of  $SO(3, 1)$  connected to the identity, which acts conformally on  $S^2$  [34]. For our situation of the three-sphere, we foresee that dilatations and translations will reasonably belong to a subgroup of conformal group acting of  $S^3$ .

Discussing infinitesimal variations, the angle  $\theta$  transforms as

$$\delta_\epsilon \theta = 2 \arctan[(1 + \epsilon) \tan \frac{\theta}{2}] - \theta = 2\epsilon \cos \frac{\theta}{2} \sin \frac{\theta}{2} = \epsilon \sin \theta. \quad (21)$$

**Dilatations and current vector** - Scale invariance for fields for 1D GFT corresponds to the requirement

$$\phi(g) \mapsto \tilde{\phi}(g_a) = \mu(a, \theta)^{-1} \phi(g) . \quad (22)$$

Infinitesimally, the above transformation finds the variation (see (C.40) in Appendix C.2)

$$\delta_\epsilon \phi(g) = -\epsilon [\cos \theta + \sin \theta \partial_\theta] \phi(g) . \quad (23)$$

One notices that the group field dilatations (22) are different from the so-called canonical Weyl transformations considered in [28].

Considering the infinitesimal generator associated with this transformation

$$\mathcal{D} = \cos \theta + \sin \theta \partial_\theta \quad (24)$$

together with translation generators  $\partial_j$ , we have

$$[\partial_j, \mathcal{D}] = \delta_{j\theta} \mathcal{D}' , \quad [\mathcal{D}', \partial_j] = \delta_{j\theta} \mathcal{D} , \quad [\mathcal{D}, \mathcal{D}'] = -\partial_\theta , \quad \mathcal{D}' := (-\sin \theta + \cos \theta \partial_\theta) . \quad (25)$$

The generator  $\mathcal{D}'$  can be seen as a rotation of  $\mathcal{D}$  by an angle of  $\pi/2$  and so defines a generator of a distinct dilatation seen from another pole (West or East, up to a sign). Hence onto the algebra of fields, the translation generator  $\partial_\theta$ ,  $\mathcal{D}$  and  $\mathcal{D}'$  associated each with a different dilatation, form a closed  $\mathfrak{so}(2,1)$  Lie algebra of vector fields. This can be seen by first multiplying each generator by the complex  $i$  and then rename  $K_0 = i\partial_\theta$ ,  $K_1 = i\mathcal{D}$  and  $K_2 = i\mathcal{D}'$ . Note also that other translations generators  $\partial_{j'}$ ,  $j' \neq \theta$ , just span a central extension to be added to this Lie algebra.

Since the Haar measure transforms according to (20), a scale invariant action is of the form

$$\begin{aligned} S_{1D}^{\text{scale}}[\phi] &= \int dg \left[ (\sin \theta)^{-1} \frac{\mathbf{g}^{kl}}{2} (\partial_k (\sin \theta \phi)) (\partial_l (\sin \theta \phi)) + \frac{\lambda}{4} \sin \theta \phi^4 \right] \\ &= \int dg \left[ (\sin \theta)^{-1} \frac{\mathbf{g}^{kl}}{2} \{ \delta_{k,\theta} \delta_{l,\theta} [\cos \theta \phi]^2 + 2 \delta_{l,\theta} \cos \theta \sin \theta \phi \partial_k \phi + (\sin \theta)^2 \partial_k \phi \partial_l \phi \} \right. \\ &\quad \left. + \frac{\lambda}{4} \sin \theta \phi^4 \right] . \end{aligned} \quad (26)$$

It is worth emphasizing that, due to the explicit appearance of the coordinate  $\theta$  in the Lagrangian, we expect a breaking of the ordinary notion of local conservation of current in this theory. Note also that a mass term could be also included but, for simplicity, we will not consider it.

The field equation of motion reads

$$\begin{aligned} \frac{\delta S_{1D}^{\text{scale}}}{\delta \phi} = 0 &= (\bullet) \frac{(\cos \theta)^2}{\sin \theta} \phi + (\bullet) \cos \theta \partial_\theta \phi - \partial_\theta [(\bullet) \cos \theta \phi] - \tilde{\Delta} \phi + (\bullet) \lambda \sin \theta \phi^3 , \quad (27) \\ (\bullet) &:= \sqrt{|\det \mathbf{g}|} , \quad \tilde{\Delta} \phi := \partial_k \{ (\bullet) \sin \theta \mathbf{g}^{kl} \partial_l \phi \} , \end{aligned}$$

where  $\tilde{\Delta}$  is a modified Laplacian. The metric variation will be not considered this time and rather consider the functional operator for solely field dilatations given by

$$W(\epsilon)(\cdot) = \int d\theta d\varphi^1 d\varphi^2 \delta_\epsilon \phi \frac{\delta}{\delta \phi}(\cdot) . \quad (28)$$



We have to evaluate the variation of the action under (28)

$$\begin{aligned} \frac{\partial}{\partial \epsilon} W(\epsilon) S_{1D}^{\text{scale}} &= \frac{\partial}{\partial \epsilon_i} \int d\theta d\varphi^1 d\varphi^2 \left\{ (-\epsilon \mathcal{D}\phi) \times \right. \\ &\left. \left[ (\bullet) \frac{(\cos \theta)^2}{\sin \theta} \phi + (\bullet) \cos \theta \partial_\theta \phi - \partial_\theta [(\bullet) \cos \theta \phi] - \tilde{\Delta} \phi + (\bullet) \lambda \sin \theta \phi^3 \right] \right\} \end{aligned} \quad (29)$$

and will prove that this can be computed as a surface term. A direct, though lengthy, calculation (see Appendix C.3) yields the current

$$D_j = \sin \theta [\cos \theta + \sin \theta \nabla_\theta] \phi \nabla_j \phi + \mathbf{g}_{j\theta} \cos \theta \phi [\cos \theta + \sin \theta \nabla_\theta] \phi - \mathbf{g}_{j\theta} \sin \theta \mathcal{L}_{1D}^{\text{scale}}, \quad (30)$$

that we put in another form

$$D_j = \nabla_\theta (\sin \theta \phi) \nabla_j (\sin \theta \phi) - \mathbf{g}_{j\theta} \sin \theta \mathcal{L}_{1D}^{\text{scale}}. \quad (31)$$

Concerning the local conservation property, as expected, we find that the current is not covariantly conserved (a proof of this can be found in Appendix C.3). The breaking term for the covariant conservation to hold can be written

$$\begin{aligned} \nabla^j D_j &= \cos \theta \sin \theta \left[ -(\cot \theta)^2 \phi^2 + \nabla_\theta \phi \nabla_\theta \phi + \frac{\lambda}{2} \phi^4 \right] \\ &= 2 \cos \theta \left[ -\frac{1}{2} \frac{(\cos \theta)^2}{\sin \theta} \phi^2 + \frac{1}{2} \sin \theta \nabla_\theta \phi \nabla_\theta \phi + \frac{\lambda}{4} \sin \theta \phi^4 \right]. \end{aligned} \quad (32)$$

The breaking expression comes mainly from partial derivative in  $\theta$  of factors containing an explicit  $\theta$  dependence in the initial Lagrangian  $\mathcal{L}_{1D}^{\text{scale}}$ . A non-trivial task, going beyond the scope of this paper, is to understand the breaking (32) in terms of coordinate dependent regular Lagrangian systems [35].

As a remark, from the expression (31), one could be tempted to argue that the current  $D_j$  should be related to an EMT  $\tilde{T}_{ij}$  by just a field redefinition  $\phi \rightarrow \tilde{\phi} = \sin \theta \phi$ . Then one ought to check that  $\sin \theta \mathcal{L}_{1D}^{\text{scale}}$  is the correct Lagrangian of the form  $\widetilde{\mathcal{L}}_{1D} = (1/2) \mathbf{g}^{ij} \nabla_i \tilde{\phi} \nabla_j \tilde{\phi} + (\lambda/4) \tilde{\phi}^4$  so that the EMT in this theory can be related to the current by  $\tilde{T}_{\theta j} = D_j$  and thereby  $\nabla^j \tilde{T}_{\theta j} = \nabla^j D_j = 0$  should reasonably hold. We then compute

$$\sin \theta \mathcal{L}_{1D}^{\text{scale}} = \frac{\mathbf{g}^{kl}}{2} \partial_k \tilde{\phi} \partial_l \tilde{\phi} + \frac{\lambda}{4} (\sin \theta)^{-2} \tilde{\phi}^4 \quad (33)$$

and clearly find that  $\sin \theta \mathcal{L}_{1D}^{\text{scale}} \neq \widetilde{\mathcal{L}}_{1D}$ . The new Lagrangian  $\sin \theta \mathcal{L}_{1D}^{\text{scale}}$  as a function of  $\tilde{\phi}$  and  $\theta$  is not translation invariant even up to a surface term. This means that dilatation invariance cannot be reduced, at least in that naive way, to translation invariance.

Finally, in the ordinary  $\phi^4$  theory, the dilatation current  $d_j$  can be related to the improved local and traceless energy momentum tensor by  $d_\mu = x^\nu \hat{T}_{\nu\mu}$ , where  $\hat{T}_{\nu\mu} = T_{\nu\mu} + (1/6)(\eta_{\mu\nu} \partial^\kappa \partial_\kappa - \partial_\nu \partial_\mu) \phi^2$ ,  $T_{\nu\mu}$  being the ordinary EMT and  $\eta$  the Minkowski metric. Here trying to obtain an analogous relation, one may write at best

$$D_j = \sin \theta T_{\theta j}^{\text{scale}} + \sin \theta \cos \theta \phi \nabla_j \phi + \mathbf{g}_{j\theta} \cos \theta \phi (\cos \theta + \sin \theta \nabla_\theta) \phi, \quad (34)$$

and, remarkably,  $\sin \theta$  plays the old role of the coordinate position in flat spacetime.

## 4 Translations and dilatations: 3D case

### 4.1 Translations and EMT

Right translations conserve the sense of Section 3.1. In a higher rank tensor GFT, each group argument can be translated by a fixed quantity, but fields will transform according to a rule making that the tetrahedron representing the vertex interaction becomes shifted afterwards [23].

First, we need to introduce complex valued fields and consider the new Lagrangian density  $\mathcal{L}_{3D} = \mathcal{L}_{\text{kin}, 3D} + \mathcal{L}_{\text{int}, 3D}$  with

$$\mathcal{L}_{\text{kin}, 3D} = \sum_{s=1}^3 \mathbf{g}_s^{ij} \nabla_{(s) i} \bar{\phi}_{1,2,3} \nabla_{(s) j} \phi_{1,2,3} + m^2 \bar{\phi}_{1,2,3} \phi_{1,2,3} , \quad (35)$$

$$\mathcal{L}_{\text{int}, 3D} = \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} . \quad (36)$$

The action defined by

$$S_{\text{kin}, 3D}[\phi] := \int \left[ \prod_{\ell=1}^3 dg_{\ell} \right] \mathcal{L}_{\text{kin}, 3D} , \quad S_{\text{int}, 3D}[\phi] = \int \left[ \prod_{\ell=1}^6 dg_{\ell} \right] \mathcal{L}_{\text{int}, 3D} , \quad (37)$$

is real and may be written  $S_{3D} = S_{\text{kin}, 3D}[\phi] + S_{\text{int}, 3D}[\phi] := \int \left[ \prod_{\ell=1}^6 dg_{\ell} \right] \mathcal{L}_{3D}$ . The equation of motion for fields  $\phi_{1,2,3}$  can be inferred from (5) after renaming properly some variables in the interaction.

Under right translations by  $h_{\ell}$ ,  $\ell = 1, 2, 3$ , the fields transform according to

$$\begin{aligned} \phi(g_1, g_2, g_3) &\mapsto \phi(g_1 h_1, g_2 h_2, g_3 h_3) , \\ \bar{\phi}(g_5, g_4, g_3) &\mapsto \bar{\phi}(g_5 h_1, g_4 h_2, g_3 h_3) , \\ \phi(g_5, g_2, g_6) &\mapsto \phi(g_5 h_1, g_2 h_2, g_6 h_3) , \\ \bar{\phi}(g_1, g_4, g_6) &\mapsto \bar{\phi}(g_1 h_1, g_4 h_2, g_6 h_3) . \end{aligned} \quad (38)$$

Note that the group arguments with labels (1, 5), (2, 4) and (3, 6) are shifted by the same amount. This defines a correct field symmetry. Actually, there is a simpler field transformation which can be extracted from the above mappings by just shifting either the first, the second or the last field argument. Thus an ordinary  $D$  dimensional GFT will have  $D$  such basic translations. The results and conclusions obtained with the “3-translation” (38) are in some sense more general and will again hold for any of these simpler symmetries.

Infinitesimal variations of a tensor field can be inferred from the variations of a field defined over a single copy of the group. Three translations defined by the group elements  $h_{\ell}$ ,  $\ell = 1, 2, 3$ , yield the infinitesimal variations

$$\delta_{X_{(1)}, X_{(2)}, X_{(3)}} \phi_{1,2,3} = \sum_s X_{(s)} \cdot \partial_{(s)} \phi_{1,2,3} = \sum_{s,i} X_{(s)}^i \partial_{(s) i} \phi_{1,2,3} . \quad (39)$$

Henceforth,  $\delta_{X_{(1)}, X_{(2)}, X_{(3)}}$  will be simply denoted  $\delta_X$ . The following operator

$$W(X)(\cdot) = \int \left[ \prod_{\ell=1}^6 d\theta_{\ell} d\varphi_{\ell}^1 d\varphi_{\ell}^2 \right] \left[ \sum_{s=1}^6 \delta_X \mathbf{g}_s^{ij} \frac{\delta}{\delta \mathbf{g}_s^{ij}}(\cdot) + \delta_X \bar{\phi}_{1,2,3} \frac{\delta}{\delta \bar{\phi}_{1,2,3}}(\cdot) + \delta_X \phi_{1,2,3} \frac{\delta}{\delta \phi_{1,2,3}}(\cdot) \right] (40)$$

generalizing (13) will act again on the action  $S_{3D}$  in order to find the Noether current for translation symmetry with parameter  $X_{(s)}$ .

Considering (39), some calculations yield (see Appendix B.2)

$$\frac{\partial}{\partial X_{(s)}^i} W(X) S_{3D} = - \sum_{s',k} \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \partial_{(s')k} \left( \left[ \prod_{\ell=1}^6 \sqrt{|\det \mathbf{g}_\ell|} \right] \mathbf{g}_{s'}^{kj} T_{(s,s');(i,j)} \right), \quad (41)$$

where  $T_{(s,s');(i,j)}$  is the ‘‘stranded’’ EMT given by

$$T_{(s,s');(i,j)} = \partial_{(s)i} \phi_{1,2,3} \partial_{(s')j} \bar{\phi}_{1,2,3} + \partial_{(s)i} \bar{\phi}_{1,2,3} \partial_{(s')j} \phi_{1,2,3} - \delta_{s,s'} \mathbf{g}_{s'}^{ij} \mathcal{L}_{3D} - \delta_{s+[\alpha_s],s'} \mathbf{g}_{s'}^{ij} \mathcal{L}_{\text{int},3D}, \quad (42)$$

with indices such that  $s = 1, 2, 3$ ,  $s' = 1, 2, \dots, 6$  and  $([\alpha_1], [\alpha_2], [\alpha_3]) = (4, 2, 3)$ . In any dimension  $D$ , the EMT will hold this form, the strand indices will become  $s = 1, 2, \dots, D$  and  $s' = 1, 2, \dots, D(D+1)/2$  whereas the indices  $[\alpha_s]$  have to be combinatorially well chosen. For a basic field transformation acting, for example, only on the strands (1, 5), the corresponding EMT is nothing but the component  $T_{(1,s');(i,j)}$ .

First, the EMT (42) is symmetric under the permutation  $(s, i) \leftrightarrow (s', j)$ , if  $s' = 1, 2, 3$ . For  $s' = 4, 5, 6$ , the EMT breaks down to the interaction Lagrangian and therefore this component remains also symmetric. However, the EMT (42) turns out to be not covariantly conserved (see Appendix B.2). This is mainly due to the fact that the nonlocal interaction clashes with the specific way that the field symmetry is imposed in (38). Such an oddity prevents to form proper equations of motion for fields thanks to which, usually, the local conservation could be guaranteed. Although not clearly identified, an improvement procedure seems to be possible by imposing more constraints on the fields (see end of Appendix B.2).

Next, for a massless theory, let us evaluate the trace of the EMT (42) in the following sense:

$$\text{Tr } T_{m=0} = \sum_{s=1}^3 \left[ \mathbf{g}_s^{ij} T_{(s,s);(i,j); m=0} + \mathbf{g}_{s+[\alpha_s]}^{ij} T_{(s,s+[\alpha_s]);(i,j); m=0} \right], \quad (43)$$

where  $T_{(s,s');(i,j); m=0}$  is defined from  $\mathcal{L}_{3D,m=0}$ . A trace of this form is justified by the fact that a contribution for each strand represented in the Lagrangian is needed. The calculations of this trace yield (in covariant notations)

$$\text{Tr } T_{m=0} = \sum_{s=1}^3 [\nabla_{(s)}^i \bar{\phi}_{1,2,3} \nabla_{(s)i} \phi_{1,2,3}] - 9(\mathcal{L}_{3D,m=0} + \mathcal{L}_{\text{int},3D}) \quad (44)$$

which is not a vanishing quantity. A traceless EMT can be built by considering instead

$$\hat{T}_{(s,s');(i,j)} = T_{(s,s');(i,j); m=0} + \frac{1}{\beta} \delta_{s,s'} \mathbf{g}_{s'}^{ij} \sum_{s''=1}^3 \phi_{1,2,3} \Delta_{(s'')} \bar{\phi}_{1,2,3} + \frac{1}{\beta'} \nabla_{(s)i} \bar{\phi}_{1,2,3} \nabla_{(s')j} \phi_{1,2,3}. \quad (45)$$

The trace of this latter tensor is

$$\text{Tr } \hat{T} = \frac{-8\beta' + 1}{\beta'} \sum_{s=1}^3 \nabla_{(s)}^i \bar{\phi}_{1,2,3} \nabla_{(s)i} \phi_{1,2,3} - 9\lambda \phi_{123} \bar{\phi}_{543} \phi_{526} \bar{\phi}_{146} + \frac{9}{\beta} \sum_{s=1}^3 \phi_{1,2,3} \Delta_{(s)} \bar{\phi}_{1,2,3}. \quad (46)$$

The improved EMT  $\hat{T}$  is traceless if  $\beta' = 1/8$  and  $\beta = 1$  after integration of the variables coined by 4, 5, 6. We should emphasize that  $\hat{T}$  is not covariantly conserved.

## 4.2 Dilatations and current tensor

**Topological GFT** - We first study topological GFTs in order to make the following developments more comprehensible. Consider then the topological 3D GFT, i.e. a GFT model equipped with the quadratic part  $\mathcal{L}_{\text{kin}}^0[\phi] := \bar{\phi}_{1,2,3}\phi_{1,2,3}$  and an interaction part as given in (36) such that the total Lagrange density  $\mathcal{L}^0$  has a quadratic mass term with  $m = 1$  governing its dynamics.

Demanding scale invariance of the action implies that the fields transform as

$$\phi(g_1, g_2, g_3) \mapsto \tilde{\phi}(g_{a_1}, g_{a_2}, g_{a_3}) = \left[ \prod_{s=1}^3 \mu(a_s, \theta_s)^{c_s} \right] \phi(g_1, g_2, g_3). \quad (47)$$

Given (20), the scale invariance of the interaction quartic in fields and trivial kinetic term is achieved by  $3 + 2c_s = 0$ . Thus the scaling dimension for field is  $c = -3/2$ . Furthermore, as it was the case for translation symmetry, we use complex fields and require that group elements defined by the couples (1, 5), (2, 4) and (3, 6) are all submitted to the same dilatations.

Setting  $a_s = 1 + \epsilon_s$ , where  $\epsilon_s$  is an infinitesimal parameter, angle and field infinitesimal variations (Appendix C.2 gives useful details pertaining to the following identities) are of the form:

$$\begin{aligned} \delta_{\epsilon_s} \theta_s &= \epsilon_s \sin \theta_s, \\ \delta_{\epsilon} \phi_{1,2,3} &:= \delta_{\epsilon_1, \epsilon_2, \epsilon_3} \phi = \sum_{s=1}^3 \delta_{\epsilon_s} \phi_{1,2,3} = - \sum_{s=1}^3 \epsilon_s \left( -c \cos \theta_s + \sin \theta_s \partial_{(s) \theta} \right) \phi_{1,2,3}. \end{aligned} \quad (48)$$

We introduce the functional differential operator

$$W(\epsilon)(\cdot) = \int \left[ \prod_{\ell=1}^3 d\theta_{\ell} d\varphi_{\ell}^1 d\varphi_{\ell}^2 \right] \left[ \delta_{\epsilon} \phi_{1,2,3} \frac{\delta}{\delta \phi_{1,2,3}}(\cdot) + \delta_{\epsilon} \bar{\phi}_{1,2,3} \frac{\delta}{\delta \bar{\phi}_{1,2,3}}(\cdot) \right] \quad (49)$$

and evaluate for the action  $S^0 = \int [\prod dg_{\ell}] \mathcal{L}^0$

$$\begin{aligned} \frac{\partial}{\partial \epsilon_i} W(\epsilon) S^0 &= \frac{\partial}{\partial \epsilon_i} \int \left[ \prod_{\ell=1}^6 d\theta_{\ell} d\varphi_{\ell}^1 d\varphi_{\ell}^2 \right] \prod_{s=1}^6 \sqrt{|\det \mathbf{g}_s|} \left\{ \right. \\ &\left. \left[ - \sum_{s=1}^3 \epsilon_s \left[ -c \cos \theta_s + \sin \theta_s \partial_{(s) \theta} \right] \phi_{1,2,3} \right] \left[ \bar{\phi}_{1,2,3} + \lambda \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right] + (\phi \leftrightarrow \bar{\phi}) \right\}. \end{aligned} \quad (50)$$

In the last above line,  $(\phi \leftrightarrow \bar{\phi})$  is the symmetric the previous expression under complex conjugation. The expansion of the last expression yields

$$\begin{aligned} \frac{\partial}{\partial \epsilon_i} W(\epsilon) S^0 &= - \int \left[ \prod_{\ell=1}^6 d\theta_{\ell} d\varphi_{\ell}^1 d\varphi_{\ell}^2 \right] \\ &\left\{ \partial_{(i) \theta} \left[ \prod_{s=1}^6 \sqrt{|\det \mathbf{g}_s|} \right] \sin \theta_i \mathcal{L}^0 \right\} + \partial_{(i+[\alpha_i]) \theta} \left\{ \left[ \prod_{s=1}^6 \sqrt{|\det \mathbf{g}_s|} \right] \sin \theta_{i+[\alpha_i]} \mathcal{L}_{\text{int}, 3D} \right\} \end{aligned} \quad (51)$$

The dilatation current vector becomes a “reduced” and “stranded” quantity with two components

$$D_s = \sin \theta_s \mathcal{L}^0, \quad \tilde{D}_s = \sin \theta_{s+[\alpha_s]} \mathcal{L}_{\text{int}, 3D}. \quad (52)$$

This fact merely comes from the absence of a true dynamics in the model. In this “trivial” situation, the EMT reduces to the Lagrangian itself plus the interaction again, namely  $T^0 = (T_1^0, T_2^0) = (-\mathcal{L}^0, -\mathcal{L}_{\text{int}, 3D})$ . From this point, one infers a slightly generalized formula for dilatation current in a curved space and topological tensor theory in terms of

$$D_s = -\sin \theta_s T_1^0, \quad \tilde{D}_s = -\sin \theta_{s+[\alpha_s]} T_2^0 \quad (53)$$

where  $\sin \theta_s$  should be seen as the coordinate position. Properties of the EMT  $T^0$  and the current  $D_s$  are direct: they are not locally conserved.

**Dynamical GFT** - Incorporating nontrivial dynamical part, we have to define a correct scaling of derivative on fields. Consider a kinetic part of the form

$$\begin{aligned} S_{\text{kin}, 3D}^{\text{scale}}[\phi] &= \int \left[ \prod_{\ell=1}^3 dg_\ell \right] \mathcal{L}_{\text{kin}, 3D}^{\text{scale}} \\ \mathcal{L}_{\text{kin}, 3D}^{\text{scale}} &:= \sum_{s=1}^3 (\sin \theta_s)^{\gamma_s} \mathbf{g}_s^{ij} (\nabla_{(s) i} (\sin \theta_s)^{\beta_s} \bar{\phi}_{1,2,3}) (\nabla_{(s) j} (\sin \theta_s)^{\beta_s} \phi_{1,2,3}), \end{aligned} \quad (54)$$

where the degrees  $\gamma_s, \beta_s$  have to be chosen in order to satisfy the scale invariance. It can easily be inferred that  $\gamma_s = \gamma = -1$  and  $\beta_s = \beta = 3/2$ . The interaction part remains as  $S_{\text{int}, 3D}$  and  $\mathcal{L}_{\text{int}, 3D}$  so that  $S_{3D}^{\text{scale}} = S_{\text{kin}, 3D}^{\text{scale}} + S_{\text{int}, 3D}$  and  $\mathcal{L}_{3D}^{\text{scale}} = \mathcal{L}_{\text{kin}, 3D}^{\text{scale}} + \mathcal{L}_{\text{int}, 3D}$ .

A direct evaluation (see Appendix C.4) shows that the dilatation current is a tensor defined by

$$\begin{aligned} D_{(s,s');j} &= \\ \sin \theta_s &\left\{ \sin \theta_{s'}^{\frac{1}{2}} \left[ \partial_{(s)\theta} \phi_{1,2,3} \partial_{(s')j} (\sin \theta_{s'}^\beta \bar{\phi}_{1,2,3}) + \partial_{(s)\theta} \bar{\phi}_{1,2,3} \partial_{(s')j} (\sin \theta_{s'}^\beta \phi_{1,2,3}) \right] \right. \\ &\left. - \delta_{s,s'} \mathbf{g}_{s' j\theta} \mathcal{L}_{3D}^{\text{scale}} \right\} - \delta_{s+[\alpha_s], s'} \mathbf{g}_{s' j\theta} \sin \theta_{s+[\alpha_s]} \mathcal{L}_{\text{int}, 3D} + \beta \cos \theta_s \partial_{(s')j} \left( (\sin \theta_{s'})^2 \bar{\phi}_{1,2,3} \phi_{1,2,3} \right). \end{aligned} \quad (55)$$

This tensor is not covariantly conserved and its breaking involved both the nonlocal interaction and the fact that the Lagrangian contains explicit coordinate dependence.

Note that there exist certainly other types of GFT interactions which are scale invariant under (19). For instance, the following interaction

$$\tilde{S}_{\text{int}}[\phi] := \frac{\lambda}{4} \int \left[ \prod_{\ell=1}^4 dg_\ell \right] \phi_{1,2,3} \phi_{3,2,4} \phi_{3,4,1} \phi_{4,2,1}, \quad (56)$$

assigns each group variables  $g_i$  (appearing three time in the interaction) to a vertex in the tetrahedron. This vertex is indeed shared by three triangles, each triangle being represented by a field. Hence, this model should be equivalent to a colored GFT model. A straightforward inspection using (56) proves that the scaling dimension of the fields is  $c = -\beta_s = -1$  so that a kinetic term of the form (54) with  $\alpha_s = -1$  would be scale invariant. Remark that the problem of nonlocally conserved quantities will be not necessarily solved by considering these interactions.

## 5 Translations and dilatations: Colored GFT

### 5.1 Translations and EMT

We consider now a colored GFT with Lagrangian (6). Due the freedom of having colored fields, a right translation for only the fields  $\phi^1$  and  $\phi^4$  can be defined such that

$$\begin{aligned}\phi^1(g_1, g_2, g_3) &\mapsto \phi^1(g_1 h, g_2, g_3) , \\ \phi^4(g_6, g_4, g_1) &\mapsto \phi^4(g_6, g_4, g_1 h) ,\end{aligned}\quad (57)$$

whereas the fields of color 2, 3 remains unvarying. At the infinitesimal level, (57) gives

$$\delta_X \phi^{a=1,4} = (X^i \cdot \partial_{(1) i}) \phi^{a=1,4} , \quad (58)$$

where  $\partial_{(1)}$  refers to only to derivative with respect to the strand 1 involving the group element  $g_1$ .

Importantly, the coloring of GFT allows to define the “minimal” symmetry for the GFT action, in the sense that, within this formalism, we are able to transform only one group argument. This feature simplifies somehow the derivations. In order to recover the full symmetry of the action, one simply has to identify pairs of field arguments which can be transformed independently. Thus, besides of (57), these other possible field transformations are

$$\begin{aligned}\phi^1(g_1, g_2, g_3) &\mapsto \phi^1(g_1, g_2 h, g_3) , & \text{and} & & \phi^3(g_5, g_2, g_6) &\mapsto \phi^3(g_5, g_2 h, g_6) , \\ \phi^1(g_1, g_2, g_3) &\mapsto \phi^1(g_1, g_2, g_3 h) , & \text{and} & & \phi^2(g_3, g_4, g_5) &\mapsto \phi^2(g_3 h, g_4, g_5) , \\ \phi^2(g_3, g_4, g_5) &\mapsto \phi^2(g_3, g_4 h, g_5) , & \text{and} & & \phi^4(g_6, g_4, g_1) &\mapsto \phi^4(g_6, g_4 h, g_1) , \\ \phi^2(g_3, g_4, g_5) &\mapsto \phi^2(g_3, g_4, g_5 h) , & \text{and} & & \phi^3(g_5, g_2, g_6) &\mapsto \phi^3(g_5 h, g_2, g_6) , \\ \phi^3(g_5, g_2, g_6) &\mapsto \phi^3(g_5, g_2, g_6 h) , & \text{and} & & \phi^4(g_6, g_4, g_1) &\mapsto \phi^4(g_6 h, g_4, g_1) .\end{aligned}\quad (59)$$

For a  $D$  dimensional GFT, there will be  $D(D+1)/2$  of such basic transformations, one for each pair of group arguments in the interaction.

We write the equations of motion of the colors 1 and 4:

$$\begin{aligned}0 &= \frac{\delta S^{\text{color}}}{\delta \phi_{123}^1} = - \sum_{s=1}^3 \Delta_{(s)} \bar{\phi}_{1,2,3} + m^2 \bar{\phi}_{1,2,3}^1 + \lambda \int \left[ \prod_{\ell=4}^6 dg_\ell \right] \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 , \\ 0 &= \frac{\delta S^{\text{color}}}{\delta \phi_{6,4,1}^4} = - \sum_{s=1,4,6} \Delta_{(s)} \bar{\phi}_{6,4,1}^4 + m^2 \bar{\phi}_{6,4,1}^4 + \lambda \int \left[ \prod_{\ell \neq 1,4,6} dg_\ell \right] \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 .\end{aligned}\quad (60)$$

For the present purpose, the following functional operator will be used to compute the EMT:

$$\begin{aligned}W(X)(\cdot) &= \int \left[ \prod_{\ell=1}^3 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \delta \mathbf{g}_1^{ij} \frac{\delta(\cdot)}{\delta \mathbf{g}_1^{ij}} + \delta \phi_{1,2,3}^1 \frac{\delta(\cdot)}{\delta \phi_{1,2,3}^1} + \delta \bar{\phi}_{1,2,3}^1 \frac{\delta(\cdot)}{\delta \bar{\phi}_{1,2,3}^1} \right\} \\ &+ \int \left[ \prod_{\ell=1,4,6} d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \delta \phi_{6,4,1}^4 \frac{\delta(\cdot)}{\delta \phi_{6,4,1}^4} + \delta \bar{\phi}_{6,4,1}^4 \frac{\delta(\cdot)}{\delta \bar{\phi}_{6,4,1}^4} \right\} .\end{aligned}\quad (61)$$

Varying the action up to a surface term, the following two-component EMT has been identified (see Appendix B.3)

$$T_{(1,s);(i,j)}^{(1)} = \partial_{(1) i} \phi_{1,2,3}^1 \partial_{(s) j} \bar{\phi}_{1,2,3}^1 + \partial_{(1) i} \bar{\phi}_{1,2,3}^1 \partial_{(s) j} \phi_{1,2,3}^1 - \delta_{1,s} \mathbf{g}_s \mathbf{g}_s^{ij} \mathcal{L}^{\text{color}} ,$$

$$T_{(1,s);(i,j)}^{(4)} = \partial_{(1) i} \phi_{6,4,1}^4 \partial_{(s) j} \bar{\phi}_{6,4,1}^4 + \partial_{(1) i} \bar{\phi}_{6,4,1}^4 \partial_{(s) j} \phi_{6,4,1}^4 . \quad (62)$$

More generally, for any pair of colors  $(a, b)$  sharing a common group argument labelled by  $g_s$ , the EMT for a translation in  $g_s$  will be of the form:

$$\begin{aligned} T_{(s,s');(i,j)}^{(a)} &= \partial_{(s) i} \phi_{1,2,3}^a \partial_{(s') j} \bar{\phi}_{1,2,3}^a + \partial_{(s) i} \bar{\phi}_{1,2,3}^a \partial_{(s') j} \phi_{1,2,3}^a - \delta_{s,s'} \mathbf{g}_{s'}^{ij} \mathcal{L}^{\text{color}} , \\ T_{(s,s');(i,j)}^{(b)} &= \partial_{(s) i} \phi_{1',2',3'}^b \partial_{(s') j} \bar{\phi}_{1',2',3'}^b + \partial_{(s) i} \bar{\phi}_{1',2',3'}^b \partial_{(s') j} \phi_{1',2',3'}^b . \end{aligned} \quad (63)$$

Moreover, the components  $T^{(1)}$  and  $T^{(4)}$  satisfy the relation (see Appendix B.3)

$$\nabla_{(1)}^j \int \left[ \prod_{\ell=2}^6 dg_\ell \right] \left[ T_{(1,1);(i,j)}^{(1)} + T_{(1,1);(i,j)}^{(4)} \right] = 0 \quad (64)$$

and this means that

$$\int \left[ \prod_{\ell=2}^6 dg_\ell \right] \left[ T_{(1,1);(i,j)}^{(1)} + T_{(1,1);(i,j)}^{(4)} \right], \quad (65)$$

being still function of  $g_1$ , is a conserved current. It is quite natural to figure out why there exists such a conserved quantity in the colored model. This is mainly due to the way that translations were implemented therein: they involve an unique argument whereas all remaining field variables becomes integrated. Then, on the symmetry point of view, the colored theory with its minimal<sup>5</sup> symmetry acts a kind of local theory in  $\phi^1(g_1, -)$  and  $\phi^4(-, g_1)$  built on the only field argument relevant for the transformation.

## 5.2 Dilatations and current tensor

Let us assume once again that only  $\phi^1$  and  $\phi^4$  are subjected to the dilatation  $g_1 \rightarrow g_a 1$ :

$$\begin{aligned} \phi^1(g_1, g_2, g_3) &\mapsto \mu(a, \theta_1)^c \phi^1(g_1, g_2, g_3) , \\ \phi^4(g_6, g_4, g_1) &\mapsto \mu(a, \theta_1)^c \phi^4(g_6, g_4, g_1) , \end{aligned} \quad (66)$$

The action invariant under these dilatations is defined by the Lagrangian density

$$\begin{aligned} \mathcal{L}^{\text{color, scale}} = & \\ & (\sin \theta_1)^{-1} \mathbf{g}_1^{ij} \partial_{(1) i} [(\sin \theta_1)^{-c} \bar{\phi}_{1,2,3}^1] \partial_{(1) j} [(\sin \theta_1)^{-c} \phi_{1,2,3}^1] + \sum_{s=2}^3 \mathbf{g}_s^{ij} \partial_{(s) i} \bar{\phi}_{1,2,3}^1 \partial_{(s) j} \phi_{1,2,3}^1 \\ & + (\sin \theta_1)^{-1} \mathbf{g}_1^{ij} \partial_{(1) i} [(\sin \theta_1)^{-c} \bar{\phi}_{6,4,1}^4] \partial_{(1) j} [(\sin \theta_1)^{-c} \phi_{6,4,1}^4] + \sum_{s=4,6} \mathbf{g}_s^{ij} \partial_{(s) i} \bar{\phi}_{6,4,1}^4 \partial_{(s) j} \phi_{6,4,1}^4 \\ & + \sum_{s=3,4,5} \mathbf{g}_s^{ij} \partial_{(s) i} \bar{\phi}_{3,4,5}^2 \partial_{(s) j} \phi_{3,4,5}^2 + \sum_{s=5,2,6} \mathbf{g}_s^{ij} \partial_{(s) i} \bar{\phi}_{5,2,6}^3 \partial_{(s) j} \phi_{5,2,6}^3 \end{aligned}$$

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<sup>5</sup>Minimal in a sense that we previously gave, namely, a single translation symmetry on one group argument and keeping all the remaining variables fixed and afterwards integrated. But, at the end, collecting all these minimal symmetries, the colored theory will certainly fall into a category of theories with the maximal number of symmetries. For instance, the non colored case has  $D$  independent field translations/dilatations whereas the colored theory has  $D(D+1)/2$  of such transformations.

$$+ \lambda \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \bar{\lambda} \bar{\phi}_{1,2,3}^1 \bar{\phi}_{3,4,5}^2 \bar{\phi}_{5,2,6}^3 \bar{\phi}_{6,4,1}^4, \quad (67)$$

where we omit to write the mass terms even though they can also be included. Indeed, they possess the same scaling behaviour as the interaction itself.

Requiring an invariant action implies that  $c = -3/2$ . The associated current can be derived using

$$\begin{aligned} W(\epsilon)(\cdot) &= \int \left[ \prod_{\ell=1}^3 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left[ \delta_\epsilon \phi_{1,2,3}^1 \frac{\delta}{\delta \phi_{1,2,3}^1}(\cdot) + \delta_\epsilon \bar{\phi}_{1,2,3}^1 \frac{\delta}{\delta \bar{\phi}_{1,2,3}^1}(\cdot) \right] \\ &+ \int \left[ \prod_{\ell=1,4,6} d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left[ \delta_\epsilon \phi_{1,2,3}^4 \frac{\delta}{\delta \phi_{1,2,3}^4}(\cdot) + \delta_\epsilon \bar{\phi}_{1,2,3}^4 \frac{\delta}{\delta \bar{\phi}_{1,2,3}^4}(\cdot) \right]. \end{aligned} \quad (68)$$

The current tensor for this symmetry possesses distinct components (derivations are given in Appendix C.5)

$$\begin{aligned} D_{(1);j}^{(1)} &= \left[ (\sin \theta_1)^2 [\beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta}] \phi_{1,2,3}^1 \partial_{(1)j} \bar{\phi}_{1,2,3}^1 \right. \\ &\quad \left. + \beta \mathbf{g}_{1j\theta} \cos \theta_1 \sin \theta_1 [\beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta}] \phi_{1,2,3}^1 \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}) \right] \\ &\quad - \mathbf{g}_{1j\theta} \sin \theta_1 \mathcal{L}^{\text{color, scale } (1,4)}, \\ D_{(s);j}^{(1)} &= [\beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta}] \phi_{1,2,3}^1 \partial_{(s)j} \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}), \quad s = 2, 3, \end{aligned} \quad (69)$$

while the components  $D_{(s);j}^{(4)}$  can be obtained from  $D_{(1);j}^{(1)}$  and  $D_{(s=2,3);j}^{(1)}$  by taking the symmetry  $(\phi_{1,2,3}^1 \leftrightarrow \phi_{6,4,1}^4)$  without the Lagrangian term  $\mathcal{L}^{\text{color, scale } (1,4)}$  defined from  $\mathcal{L}^{\text{color, scale}}$  by only considering the fields of colors 1 and 4. The dilatation current component  $D_{(1);j}^{(1)}$  can be written also as

$$\begin{aligned} D_{(1);j}^{(1)} &= \partial_{(1)\theta} [(\sin \theta_1)^\beta \bar{\phi}_{1,2,3}^1] \partial_{(1)j} [(\sin \theta_1)^\beta \phi_{1,2,3}^1] + \partial_{(1)\theta} [(\sin \theta_1)^\beta \bar{\phi}_{1,2,3}^1] \partial_{(1)j} [(\sin \theta_1)^\beta \phi_{1,2,3}^1] \\ &\quad - \sin \theta_1 \mathcal{L}^{\text{color, scale } (1,4)}. \end{aligned} \quad (70)$$

Note finally that the dilatation current is not covariantly conserved due to the explicit coordinate dependence but not because of the nonlocal interaction:

$$\sum_{s=1,2,3} \nabla_{(s)}^j D_{(s)j}^{(1)} + \sum_{s=1,2,3} \nabla_{(s)}^j D_{(s)j}^{(4)} \neq 0. \quad (71)$$

The explicit expression of the breaking is given in Appendix C.5.

## 6 Conclusion

The classical formalism, i.e. the Klein-Gordon free field equation and group symmetry study, for dynamical GFT over tensor copies of  $SU(2)$  has been investigated in this paper. We find that the GFTs exhibit peculiarities that one could expect when dealing with nonlocal models. For translation symmetry, the EMT for the general GFT (without color) proves to be symmetric but not locally conserved for any dimension, save  $D = 1$ . As a matter of fact,



for  $D = 1$  locality is recovered and, from that, the EMT is covariantly conserved. In contrast and astonishingly, the genuine colored GFTs which are nonlocal also possess a covariantly conserved quantity obtained by integrating and summing some EMT components. This is another feature advocating in favor of these colored theories. In all situations (with and without color), the EMT possesses a nonvanishing trace, even though the latter property could be improved but the meaning of the resulting tensor remains unclear. It turns out that this nonvanishing trace property is more profound than one may think because of the specific way that group dilatations could be implemented at the GFT level. Indeed, dilatation symmetry can be consistently settled in GFT with cost to put an explicit dependence of the class angle  $\theta$  coordinate of the  $SU(2)$  group variable  $g(\theta, \vec{n})$  in the Lagrangian. In any dimension  $D \geq 1$  colored or not, this dissipative term explicitly breaks the conservation of the dilatation current.

It can be desirable to find some improvement procedures adapted for the non-colored GFTs in order to render the EMTs and other currents covariantly conserved for  $D \geq 1$ . In the case of noncommutative field theory defined with a Moyal  $\star$ -product with its induced nonlocality, improvement procedures have been highlighted to treat the breaking term of the EMT local conservation [31]-[33]. Nevertheless, these methods were successful due to the specificity of the Moyal field algebra. Here, clearly the issue is different and deserves a better understanding. Another important field symmetry, that we did not discuss and could be viewed as rotation in the context, would be the one associated with group adjoint transformation  $g \rightarrow hgh^{-1}$ . There are more infinitesimal vector field generators associated with such transformations and so more involved becomes the computation of the “angular momentum” tensor. The fact that the EMTs found here are symmetric is encouraging for the local conservation of the angular momentum tensor. Finally, one also should address the consequences of these symmetries at the quantum level. Typically, the formalism of Ward-Takahashi identities in GFT [15] could be investigated for the background symmetries unraveled in this work.

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## Appendix

### A Gauge invariant fields

Consider a gauge invariant field, namely a field satisfying

$$\phi(g_1, g_2, g_3) = \int dh \phi(g_1h, g_2h, g_3h) . \tag{A.1}$$

This condition translates in Fourier modes via Peter-Weyl theorem as

$$\sum_{j_a, m_a, n_a} \phi_{m_a, n_a}^{j_a} \prod_a \sqrt{d_{j_a}} D_{m_a, n_a}^{j_a}(g_a) = \sum_{j_a, m_a, k_a, n_a} \phi_{m_a, k_a}^{j_a} \left[ \prod_a \sqrt{d_{j_a}} D_{m_a, n_a}^{j_a}(g_a) \right] \int dh \prod_a D_{n_a, k_a}^{j_a}(h) \quad (\text{A.2})$$

where  $a = 1, 2, 3$ ,  $\phi_{m_a, n_a}^{j_a}$  is a notation for  $\phi_{m_1, n_1; m_2, n_2; m_3, n_3}^{j_1, j_2, j_3}$ ,  $j_a$  indexes half spin representation,  $j_a \in \frac{1}{2}\mathbb{N}$ ,  $n_a, m_a, k_a$  are ordinary associated magnetic momenta each constrained to be inside  $[-j_a, +j_a]$ ,  $d_{j_a} = 2j_a + 1$  is the dimension of the representation space.  $D_{mn}^j(g)$  denotes the Wigner matrix element of  $g$  in the representation  $j$ . The factor  $\sqrt{d_{j_a}}$  is chosen as a normalization convention.

Computing the last integral, one gets by simple identification:

$$\phi_{m_a, n_a}^{j_a} = \sum_{k_a} \phi_{m_a, k_a}^{j_a} \begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix}, \quad (\text{A.3})$$

with the arrays denoting the rotation invariant Wigner  $3j$ -symbols. A field with coefficients as (A.3) indeed exists and satisfies (A.1) due the orthogonality relation of  $3j$  symbols. A simple  $\phi_{m_a, n_a}^{j_a}$  is given for instance by the product of two  $3j$ -symbols with coefficients following the pattern (A.3). For the general  $D$  dimensional gauge invariant fields, the integral  $\int dh \prod_{a=1}^D D_{n_a, k_a}^{j_a}(h)$  gives an invariant intertwiner which can be recoupled in more involved Wigner  $3nj$ -symbols.

## B Group translations

In this appendix, we give the main identities leading to the formulas of EMTs in  $1D$ ,  $3D$  and colored GFTs and to the (non)local conservation property of these tensors. The first subsection explains also the method of computation of these quantities.

### B.1 Ward operator action for translations for 1D GFT

In this paragraph, an explicit calculation of the EMT is given for a  $1D$  GFT by applying the Ward operator method in a curved background like  $S^3$  parametrized by  $(\theta, \varphi_1, \varphi_2)$ . The ensuing tensor study could be deduced from this point.

For an infinitesimal translation of parameter  $X$ , we can set in a given local coordinate system  $\delta_X \phi = X^i \partial_i \phi$ . Then one introduces the operator

$$W(X)(\cdot) = \int d\theta d\varphi_1 d\varphi_2 \left( \delta_X \mathbf{g}^{ij} \frac{\delta(\cdot)}{\delta \mathbf{g}^{ij}} + \delta_X \phi \frac{\delta(\cdot)}{\delta \phi} \right) \quad (\text{B.4})$$

that acts on the action  $S_{1D}$  (10) such that

$$\begin{aligned} \frac{\partial}{\partial X^\rho} W(X) S_{1D} &= \frac{\partial}{\partial X^\rho} \int d\theta d\varphi_1 d\varphi_2 \left\{ \delta_X \mathbf{g}^{ij} \left[ \frac{\partial \sqrt{|\det \mathbf{g}|}}{\partial \mathbf{g}^{ij}} \mathcal{L}_{1D} + \sqrt{|\det \mathbf{g}|} \frac{\partial \mathcal{L}_{1D}}{\partial \mathbf{g}^{ij}} \right] \right. \\ &\quad \left. + \sqrt{|\det \mathbf{g}|} (X^\kappa \partial_\kappa \phi) (-\Delta \phi + m^2 \phi + \lambda \phi^3) \right\}. \end{aligned} \quad (\text{B.5})$$

Keeping in mind that the Laplacian contains an inverse factor of the metric determinant, one gets

$$\begin{aligned} \frac{\partial}{\partial X^\rho} W(X) S_{1D} &= \delta_\rho^\kappa \int d\theta d\varphi_1 d\varphi_2 \left\{ \partial_\kappa \mathbf{g}^{ij} \left[ -\frac{1}{2} \sqrt{|\det \mathbf{g}|} \mathbf{g}_{ij} \mathcal{L}_{1D} + \sqrt{|\det \mathbf{g}|} \frac{1}{2} \partial_i \phi \partial_j \phi \right] \right. \\ &\quad \left. - \partial_i (\partial_\kappa \phi \mathbf{g}^{ij} \sqrt{|\det \mathbf{g}|} \partial_j \phi) + (\partial_i \partial_\kappa \phi) \mathbf{g}^{ij} \sqrt{|\det \mathbf{g}|} \partial_j \phi + \sqrt{|\det \mathbf{g}|} \partial_\kappa (m^2 \frac{1}{2} \phi + \frac{\lambda}{4} \phi^4) \right\}. \end{aligned} \quad (\text{B.6})$$

It is customary in a field theory to identify the EMT from the variations of the metric  $\mathbf{g}^{ij}$ . The EMT already appears in the above expression up to some factor. However, in this paper, we will not use this route preferring instead to get a final surface term. To this end, further computations invoking both metric and field variations are in order:

$$\begin{aligned} \frac{\partial}{\partial X^\rho} W(X) S_{1D} &= \delta_\rho^\kappa \int d\theta d\varphi_1 d\varphi_2 \left\{ \partial_\kappa \mathbf{g}^{ij} \left[ -\frac{1}{2} \sqrt{|\det \mathbf{g}|} \mathbf{g}_{ij} \mathcal{L}_{1D} + \sqrt{|\det \mathbf{g}|} \frac{1}{2} \partial_i \phi \partial_j \phi \right] \right. \\ &\quad \left. - \partial_i (\partial_\kappa \phi \mathbf{g}^{ij} \sqrt{|\det \mathbf{g}|} \partial_j \phi) - \sqrt{|\det \mathbf{g}|} (\partial_\kappa \mathbf{g}^{ij}) (\frac{1}{2} \partial_i \phi \partial_j \phi) + \sqrt{|\det \mathbf{g}|} \partial_\kappa \mathcal{L}_{1D} \right\} \\ &= - \int d\theta d\varphi_1 d\varphi_2 \partial_i \{ \sqrt{|\det \mathbf{g}|} \mathbf{g}^{ij} (\partial_\rho \phi \partial_j \phi - \mathbf{g}_{\rho j} \mathcal{L}_{1D}) \}. \end{aligned} \quad (\text{B.7})$$

Finally, this expression is of the form of a surface term and we identify the EMT

$$T_{\rho j} = \partial_\rho \phi \partial_j \phi - \mathbf{g}_{\rho j} \mathcal{L}_{1D}. \quad (\text{B.8})$$

We verify, in a covariant script, that

$$\begin{aligned} \nabla^i T_{ij} &= \nabla^i (\nabla_i \phi \nabla_j \phi) - \nabla_j \left( \frac{1}{2} \mathbf{g}^{kl} \nabla_k \phi \nabla_l \phi + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right) \\ &= (\nabla^i \nabla_i \phi) \nabla_j \phi + \nabla_i \phi \nabla^i \nabla_j \phi - \frac{1}{2} \mathbf{g}^{kl} \nabla_j (\nabla_k \phi \nabla_l \phi) - \nabla_j \phi (m^2 \phi + \lambda \phi^3) = 0, \end{aligned} \quad (\text{B.9})$$

where, we use the property of the Levi-Civita connection, and, in last resort, the field equation of motion  $-\Delta \phi + m^2 \phi + \lambda \phi^3 = 0$ .

This method of deriving the EMT will be extended in the subsequent situations dealing with tensor models.

## B.2 EMT for GFT in 3D

**EMT calculation** - We provide here the main stages of calculations leading to (42).

First, we recall that the infinitesimal variation for a tensor field  $\phi$  under a “3-translation” is given by  $\delta_X \phi_{1,2,3} := \delta_{X(1), X(2), X(3)} \phi_{1,2,3} = \sum_{s=1}^3 X_{(s)}^i \partial_{(s) i} \phi_{1,2,3}$ . Then, we require an operator symmetrization for the interaction part:

$$\begin{aligned} &\lambda \int \left[ \prod_{\ell=1}^6 dg_\ell \right] (\delta_X \phi_{1,2,3}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} + (\delta_X \bar{\phi}_{1,2,3}) \phi_{5,4,3} \bar{\phi}_{5,2,6} \phi_{1,4,6} = \\ &\frac{\lambda}{2} \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \left\{ (\delta_X \phi_{1,2,3}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} + (\delta_X \phi_{5,2,6}) \phi_{1,2,3} \bar{\phi}_{5,4,3} \bar{\phi}_{1,4,6} \right\} \end{aligned}$$

$$\begin{aligned}
& + \left( \delta_X \bar{\phi}_{1,2,3} \right) \phi_{5,4,3} \bar{\phi}_{5,2,6} \phi_{1,4,6} + \left( \delta_X \bar{\phi}_{5,2,6} \right) \phi_{5,4,3} \phi_{1,4,6} \bar{\phi}_{1,2,3} \Big\} \\
& = \frac{\lambda}{2} \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \left\{ \sum_{s=1}^3 \left( X_{(s)}^i \partial_{(s)}^i \phi_{1,2,3} \right) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right. \\
& + \sum_{s=1}^3 \left( X_{(s)}^i \partial_{(s+\alpha_s)}^i \bar{\phi}_{5,4,3} \right) \phi_{1,2,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} + \sum_{s=1}^3 \left( X_{(s)}^i \partial_{(s+\alpha_s)}^i \phi_{5,2,6} \right) \bar{\phi}_{5,4,3} \phi_{1,2,3} \bar{\phi}_{1,4,6} \\
& + \left. \sum_{s=1}^3 \left( X_{(s)}^i \partial_{(s+\alpha_s)}^i \bar{\phi}_{1,4,6} \right) \bar{\phi}_{5,4,3} \phi_{5,2,6} \phi_{1,2,3} \right\} \\
& = \frac{\lambda}{2} \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \sum_{s=1}^3 X_{(s)}^i \left( \partial_{(s)}^i + \partial_{(s+\alpha_s)}^i \right) \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} . \tag{B.10}
\end{aligned}$$

where the index  $\alpha_s = 0, 2, 3, 4$  has to be chosen appropriately. In the last equality, the notation  $[\alpha_s]$  means that we fix  $([\alpha_1], [\alpha_2], [\alpha_3]) = (4, 2, 3)$ . It is remarkable that, under integral over all six variables, we can exchange  $\bar{\phi}_{1,2,3}$  for  $\bar{\phi}_{5,4,3}$  and  $\bar{\phi}_{5,2,6}$  for  $\bar{\phi}_{1,4,6}$  by just renaming the variables. One should keep in mind that there is set of discrete symmetries which will be also used in the sequel.

We introduce the notations  $(\bullet) = \prod_{s=1}^6 \sqrt{|\det \mathbf{g}_s|}$  and  $(\bullet)_{\bar{s}} = \prod_{k \neq s} \sqrt{|\det \mathbf{g}_k|}$ , so that

$$(\bullet) = (\bullet)_{\bar{s}} \sqrt{|\det \mathbf{g}_s|}, \quad \prod_{\ell=1}^6 [dg_\ell] = \frac{1}{2\pi^2} \prod_{\ell=1}^6 [d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2] (\bullet). \tag{B.11}$$

The factor  $1/(2\pi^2)$  will be omitted in the following. The EMT can be computed as follows

$$\begin{aligned}
W(X) S_{3D} & = \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left( \sum_{s=1}^6 \delta_X \mathbf{g}_s^{ij} \frac{\delta}{\delta \mathbf{g}_s^{ij}} S_{3D} \right. \\
& + \sum_{s'=1}^3 X_{(s')}^k \partial_{(s')}^k \phi_{1,2,3} \left[ - \sum_{s=1}^3 \partial_{(s)}^j \left( (\bullet)_{\bar{s}} \sqrt{|\det \mathbf{g}_s|} \mathbf{g}_s^{jl} \partial_{(s)}^l \bar{\phi}_{1,2,3} \right) \right] \\
& \left. + (\bullet) \sum_{s'=1}^3 X_{(s')}^k \partial_{(s')}^k \phi_{1,2,3} \left[ m^2 \bar{\phi}_{1,2,3} + \lambda \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right] + (\phi \leftrightarrow \bar{\phi}) \right), \tag{B.12}
\end{aligned}$$

where the part including the variation  $\delta_X \bar{\phi}$  is not explicitly displayed but appears symbolically as  $(\phi \leftrightarrow \bar{\phi})$ . The corresponding terms can be carried out in a symmetric manner. Adding all contributions, combining the mass terms and the interaction using (B.10), it can be seen that

$$\begin{aligned}
W(X) S_{3D} & = \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \sum_s \delta_X \mathbf{g}_s^{ij} \frac{\delta}{\delta \mathbf{g}_s^{ij}} S_{3D} \right. \\
& + \left. \left\{ - \sum_{s',s=1}^3 X_{(s')}^k \partial_{(s)}^j \left[ \partial_{(s')}^k \phi_{1,2,3} (\bullet) \mathbf{g}_s^{jl} \partial_{(s)}^l \bar{\phi}_{1,2,3} \right] \right\} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{s',s=1}^3 X_{(s')}^k \partial_{(s)j} [\partial_{(s')k} \phi_{1,2,3}] \left\{ (\bullet) \mathbf{g}_s^{jl} \partial_{(s)l} \bar{\phi}_{1,2,3} + (\phi \leftrightarrow \bar{\phi}) \right\} \\
& + (\bullet) \sum_{s'=1}^3 X_{(s')}^k \left[ m^2 \partial_{(s')k} (\bar{\phi}_{1,2,3} \phi_{1,2,3}) + \frac{\lambda}{2} (\partial_{(s')k} + \partial_{(s'+[\alpha_{s'})k}) \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right] \Big\}.
\end{aligned} \tag{B.13}$$

Focusing now on the dynamical part and metric variations, one has:

$$\begin{aligned}
K &= \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \int \left[ \prod_{\ell'=1}^6 d\theta_{\ell'} d\varphi_{\ell'}^1 d\varphi_{\ell'}^2 \right] \sum_{s=1}^6 \delta_X \mathbf{g}_s^{ij} \frac{\delta}{\delta \mathbf{g}_s^{ij}} ((\bullet)_s \sqrt{|\det \mathbf{g}_s|}) \mathcal{L}_{3D} \right. \\
& + \int \left[ \prod_{\ell'=1}^6 d\theta_{\ell'} d\varphi_{\ell'}^1 d\varphi_{\ell'}^2 \right] \sum_{s=1}^6 (\bullet) \delta_X \mathbf{g}_s^{ij} \frac{\delta \mathcal{L}_{\text{kin}, 3D}}{\delta \mathbf{g}_s^{ij}} \\
& + \left\{ - \sum_{s',s=1}^3 X_{(s')}^k \partial_{(s)j} [\partial_{(s')k} \phi_{1,2,3}] (\bullet) \mathbf{g}_s^{jl} \partial_{(s)l} \bar{\phi}_{1,2,3} + (\phi \leftrightarrow \bar{\phi}) \right\} \\
& + (\bullet) \sum_{s',s=1}^3 X_{(s')}^k \partial_{(s')k} [\mathbf{g}_s^{jl} \partial_{(s)j} \phi_{1,2,3} \partial_{(s)l} \bar{\phi}_{1,2,3}] \\
& \left. - (\bullet) \sum_{s',s=1}^3 X_{(s')}^k \partial_{(s')k} [\mathbf{g}_s^{jl}] \partial_{(s)j} \phi_{1,2,3} \partial_{(s)l} \bar{\phi}_{1,2,3} \right\}.
\end{aligned} \tag{B.14}$$

Canceling the variation  $\delta \mathcal{L}_{\text{kin}, 3D} / \delta \mathbf{g}_s^{ij}$  with its partner coming from the field variations and recomposing the Lagrangian, by injecting expression  $K$  (B.14) into (B.12), we get

$$\begin{aligned}
W(X) S_{3D} &= \\
& \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left[ \left\{ - \sum_{s',s=1}^3 X_{(s')}^k \partial_{(s)j} [(\bullet) \partial_{(s')k} \phi_{1,2,3} \mathbf{g}_s^{jl} \partial_{(s)l} \bar{\phi}_{1,2,3}] + (\phi \leftrightarrow \bar{\phi}) \right\} \right. \\
& \left. + \sum_{s'=1}^3 X_{(s')}^k \left\{ \partial_{(s')k} [(\bullet) \mathcal{L}_{3D}] + \partial_{(s'+[\alpha_{s'})k} [(\bullet) \mathcal{L}_{\text{int}, 3D}] \right\} \right].
\end{aligned} \tag{B.15}$$

We are now in position to provide the EMT for group translations of the GFT by just deriving the above expression by some infinitesimal parameter:  $X_{(s)}^k$ ,  $s, k = 1, 2, 3$ ,

$$\begin{aligned}
\frac{\partial}{\partial X_{(s)}^k} W(X) S_{3D} &= - \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \sum_{s'=1}^6 \partial_{(s')j} (\bullet) \mathbf{g}_{s'}^{jl} \left\{ \right. \\
& \left. \partial_{(s)k} \phi_{1,2,3} \partial_{(s')l} \bar{\phi}_{1,2,3} + \partial_{(s)k} \bar{\phi}_{1,2,3} \partial_{(s')l} \phi_{1,2,3} - \delta_{s,s'} \mathbf{g}_{s'}^{lk} \mathcal{L}_{3D} - \delta_{s+\alpha_s, s'} \mathbf{g}_{s'}^{lk} \mathcal{L}_{\text{int}, 3D} \right\}.
\end{aligned} \tag{B.16}$$

Hence the EMT is given by

$$T_{(s,s'); (i,j)} = \partial_{(s)i} \phi_{1,2,3} \partial_{(s')j} \bar{\phi}_{1,2,3} + \partial_{(s)i} \bar{\phi}_{1,2,3} \partial_{(s')j} \phi_{1,2,3} - \delta_{s,s'} \mathbf{g}_{s'}^{ij} \mathcal{L}_{3D} - \delta_{s+[\alpha_s], s'} \mathbf{g}_{s'}^{ij} \mathcal{L}_{\text{int}, 3D}, \tag{B.17}$$

for  $s = 1, 2, 3$ ,  $s' = 1, 2, \dots, 6$ , and  $i, j = 1, 2, 3$ .

**Covariant conservation** - The fact that the EMT is not covariantly conserved is proved here. We have in covariant notations:

$$\begin{aligned}
& \sum_{s'=1}^6 \nabla_{(s')}^j T_{(s,s');(ij)} = \\
& \left\{ \sum_{s'=1}^3 \left( \nabla_{(s')}^j \nabla_{(s)} i \phi_{1,2,3} \nabla_{(s')} j \bar{\phi}_{1,2,3} + \nabla_{(s)} i \phi_{1,2,3} \nabla_{(s')}^j \nabla_{(s')} j \bar{\phi}_{1,2,3} \right) + (\phi \leftrightarrow \bar{\phi}) \right\} \\
& - \nabla_{(s)} i \left( \sum_{s'=1}^3 \mathbf{g}_{s'}^{kl} \nabla_{(s')} k \phi_{1,2,3} \nabla_{(s')} l \bar{\phi}_{1,2,3} + m^2 \bar{\phi}_{1,2,3} \phi_{1,2,3} + \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right) \\
& - \nabla_{(s+[\alpha_s])} i \left( \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right) \\
& = \sum_{s'=1}^3 \left( \nabla_{(s)} i \phi_{1,2,3} \nabla_{(s')}^j \nabla_{(s')} j \bar{\phi}_{1,2,3} \right) - m^2 \bar{\phi}_{1,2,3} (\nabla_{(s)} i \phi_{1,2,3}) \\
& + \sum_{s'=1}^3 \left( \nabla_{(s)} i \bar{\phi}_{1,2,3} \nabla_{(s')}^j \nabla_{(s')} j \phi_{1,2,3} \right) - m^2 \phi_{1,2,3} (\nabla_{(s)} i \bar{\phi}_{1,2,3}) \\
& - \frac{\lambda}{2} (\nabla_{(s)} i \phi_{1,2,3}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} (\nabla_{(s)} i \bar{\phi}_{5,4,3}) \phi_{5,2,6} \bar{\phi}_{1,4,6} \\
& - \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} (\nabla_{(s)} i \phi_{5,2,6}) \bar{\phi}_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} (\nabla_{(s)} i \bar{\phi}_{1,4,6}) \\
& - \frac{\lambda}{2} (\nabla_{(s+[\alpha_s])} i \phi_{1,2,3}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} (\nabla_{(s+[\alpha_s])} i \bar{\phi}_{5,4,3}) \phi_{5,2,6} \bar{\phi}_{1,4,6} \\
& - \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} (\nabla_{(s+[\alpha_s])} i \phi_{5,2,6}) \bar{\phi}_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} (\nabla_{(s+[\alpha_s])} i \bar{\phi}_{1,4,6}) . \tag{B.18}
\end{aligned}$$

Using the equation of motion of  $\phi_{1,2,3}$  and  $\bar{\phi}_{1,2,3}$  after integrating by  $g_4, g_5$  and  $g_6$ , (B.18) becomes

$$\begin{aligned}
& \int \left[ \prod_{\ell=4}^6 dg_\ell \right] \left\{ + \frac{\lambda}{2} (\nabla_{(s)} i \phi_{1,2,3}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} (\nabla_{(s)} i \bar{\phi}_{5,4,3}) \phi_{5,2,6} \bar{\phi}_{1,4,6} \right. \\
& - \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} (\nabla_{(s)} i \phi_{5,2,6}) \bar{\phi}_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} (\nabla_{(s)} i \bar{\phi}_{1,4,6}) \\
& - \frac{\lambda}{2} (\nabla_{(s+[\alpha_s])} i \phi_{1,2,3}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} (\nabla_{(s+[\alpha_s])} i \bar{\phi}_{5,4,3}) \phi_{5,2,6} \bar{\phi}_{1,4,6} \\
& - \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} (\nabla_{(s+[\alpha_s])} i \phi_{5,2,6}) \bar{\phi}_{1,4,6} - \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} (\nabla_{(s+[\alpha_s])} i \bar{\phi}_{1,4,6}) \\
& \left. + \lambda (\nabla_{(s)} i \bar{\phi}_{1,2,3}) \phi_{5,4,3} \bar{\phi}_{5,2,6} \phi_{1,4,6} \right\} \tag{B.19}
\end{aligned}$$

which is not a vanishing quantity. Hence the EMT is not covariantly conserved and the breaking term is clearly a factor of the coupling constant  $\lambda$ . If we had considered the basic translation only on (1, 5), and compute for its symmetric sub-tensor  $T_{(1,s');(ij)}$ ,  $s' \leq 3$ ,

$$\int \left[ \prod_{\ell=4}^6 dg_\ell \right] \sum_{s'=1}^3 \nabla_{(s')}^j T_{(1,s');(ij)} = \frac{\lambda}{2} \int \left[ \prod_{\ell=4}^6 dg_\ell \right] \bar{\phi}_{5,4,3} \phi_{5,2,6} \left[ \nabla_{(1)} i \phi_{1,2,3} \bar{\phi}_{1,4,6} - \phi_{1,2,3} \nabla_{(1)} i \bar{\phi}_{1,4,6} \right].$$

This relation is vanishing, for instance, for  $\nabla_{(1)} i \phi_{1,2,3} \bar{\phi}_{1,4,6} = \phi_{1,2,3} \nabla_{(1)} i \bar{\phi}_{1,4,6}$  but, as a supplementary constraint on  $\phi$ , it may be not compatible with the equation of motion of  $\phi$ .

### B.3 EMT for the colored model

**EMT calculation** - Let us start by considering the functional operator (61) where the variations of the fields are  $\delta_X \phi_{1,2,3}^a = X^i \partial_i \phi_{1,2,3}^a$  and equations of motion (60). We introduce further notations  $(\bullet)_{a,b,c} = \prod_{s=a,b,c} \sqrt{|\det \mathbf{g}_s|}$ . Using the field variations and equations of motion, the interaction has to be reconstructed as follows

$$\begin{aligned} & \lambda \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \left[ \delta_X \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \delta_X \phi_{6,4,1}^4 \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 + (\phi \leftrightarrow \bar{\phi}) \right] \\ &= \lambda \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \left[ X^i \partial_{(1) i} (\phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4) + (\phi \leftrightarrow \bar{\phi}) \right]. \end{aligned} \quad (\text{B.20})$$

The EMT can be computed from a similar routine as previously performed. We sum up the main steps:

$$\begin{aligned} W(X)S^{\text{color}} &= \int \left[ \prod_{\ell=1}^3 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \delta_X \mathbf{g}_1^{ij} \frac{\delta}{\delta \mathbf{g}_1^{ij}} S^{\text{color}} \right. \\ &+ X^k \partial_{(1) k} \phi_{1,2,3}^1 \left[ - \sum_{s=1}^3 \partial_{(s) j} ((\bullet)_{\bar{s}} \sqrt{|\det \mathbf{g}_s|} \mathbf{g}_s^{jl} \partial_{(s) l} \bar{\phi}_{1,2,3}^1) \right] \\ &+ (\bullet)_{1,2,3} X^k \partial_{(1) k} \phi_{1,2,3}^1 \left[ m^2 \bar{\phi}_{1,2,3}^1 + \lambda \int \left[ \prod_{\ell=4}^6 dg_\ell \right] \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 \right] + (\phi \leftrightarrow \bar{\phi}) \left. \right\} \\ &+ \int \left[ \prod_{\ell=1,4,6} d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ X^k \partial_{(1) k} \phi_{6,4,1}^4 \left[ - \sum_{s=1}^3 \partial_{(s) j} ((\bullet)_{\bar{s}} \sqrt{|\det \mathbf{g}_s|} \mathbf{g}_s^{jl} \partial_{(s) l} \bar{\phi}_{6,4,1}^4) \right] \right. \\ &+ (\bullet)_{1,4,6} X^k \partial_{(1) k} \phi_{6,4,1}^4 \left[ m^2 \bar{\phi}_{6,4,1}^4 + \lambda \int \left[ \prod_{\ell \neq 1,4,6} dg_\ell \right] \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \right] + (\phi \leftrightarrow \bar{\phi}) \left. \right\} \end{aligned} \quad (\text{B.21})$$

Adding the contributions to the mass term and using (B.20), we get

$$\begin{aligned} W(X)S^{\text{color}} &= \int \left[ \prod_{\ell} d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \delta_X \mathbf{g}_1^{ij} \left[ (\bullet)_{\bar{1}} \frac{\delta \sqrt{|\det \mathbf{g}|_1}}{\delta \mathbf{g}_1^{ij}} \mathcal{L}^{\text{color}} + (\bullet) \frac{\delta \mathcal{L}_{\text{kin}}^{\text{color}}}{\delta \mathbf{g}_1^{ij}} \right] \right. \\ &+ \left\{ - X^k \sum_{s=1}^3 \partial_{(s) j} \left[ \partial_{(1) k} \phi_{1,2,3}^1 (\bullet) \mathbf{g}_s^{jl} \partial_{(s) l} \bar{\phi}_{1,2,3}^1 \right] \right. \\ &- X^k \sum_{s=1,4,6} \partial_{(s) j} \left[ \partial_{(1) k} \phi_{6,4,1}^4 (\bullet) \mathbf{g}_s^{jl} \partial_{(s) l} \bar{\phi}_{6,4,1}^4 \right] \\ &+ X^k \sum_{s=1}^3 (\bullet) \mathbf{g}_s^{jl} \partial_{(1) k} \left[ \partial_{(s) j} \phi_{1,2,3}^1 \partial_{(s) l} \bar{\phi}_{1,2,3}^1 \right] \end{aligned} \quad (\text{B.22})$$

$$\begin{aligned} &+ X^k \sum_{s=6,4,1} (\bullet) \mathbf{g}_s^{jl} \partial_{(1) k} \left[ \partial_{(s) j} \phi_{6,4,1}^4 \partial_{(s) l} \bar{\phi}_{6,4,1}^4 \right] + (\phi \leftrightarrow \bar{\phi}) \left. \right\} \quad (\text{B.23}) \\ &+ (\bullet) X^k \left[ m^2 \partial_{(1) k} (\bar{\phi}_{1,2,3}^1 \phi_{1,2,3}^1 + \bar{\phi}_{6,4,1}^4 \phi_{6,4,1}^4) \right. \\ &+ \left. \left( \lambda \partial_{(1) k} (\phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4) + (\phi \leftrightarrow \bar{\phi}) \right) \right] \left. \right\}, \end{aligned}$$

where we used the symmetric part in  $\bar{\phi}$  for completing the partial derivative. Expanding the metric variations, one has

$$\delta_X \mathbf{g}_1^{ij} \left[ (\bullet)_i \frac{\delta \sqrt{|\det \mathbf{g}|_1}}{\delta \mathbf{g}_1^{ij}} \mathcal{L}^{\text{color}} + (\bullet) \frac{\delta \mathcal{L}^{\text{color}}}{\delta \mathbf{g}_1^{ij}} \right] =$$

$$X^k \partial_{(1)k} \mathbf{g}_1^{ij} \left[ -\frac{1}{2} (\bullet)_i \sqrt{|\det \mathbf{g}|_1} \mathbf{g}_{1ij} \mathcal{L}^{\text{color}} \right] \quad (\text{B.24})$$

$$+ (\bullet) [\partial_{(1)i} \bar{\phi}_{1,2,3}^1 \partial_{(1)l} \phi_{1,2,3}^1 + \partial_{(1)i} \bar{\phi}_{6,4,1}^4 \partial_{(1)l} \phi_{6,4,1}^4]. \quad (\text{B.25})$$

The term (B.24) completes the derivative of  $(\bullet) \mathcal{L}^{\text{color}}$  while the second (B.25) cancels exactly the term with  $-\partial_{(1)k} \mathbf{g}_s^{ij}$  obtained after integrating by parts (B.22) and (B.23). We obtain

$$\frac{\partial}{\partial X^\rho} W(X) S = -\delta^{\rho k} \int \left[ \prod_\ell d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right.$$

$$\left[ \sum_{s=1}^3 \partial_{(s)j} [\partial_{(1)k} \phi_{1,2,3}^1 (\bullet) \mathbf{g}_s^{jl} \partial_{(s)l} \bar{\phi}_{1,2,3}^1] \right.$$

$$+ \sum_{s=1,4,6} \partial_{(s)j} [\partial_{(1)k} \phi_{6,4,1}^4 (\bullet) \mathbf{g}_s^{jl} \partial_{(s)l} \bar{\phi}_{6,4,1}^4] + (\phi \leftrightarrow \bar{\phi}) \left. \right]$$

$$- \partial_{(1)k} [(\bullet) \mathcal{L}^{(1,4)}] - [\partial_{(1)k} \mathbf{g}_1^{ij}] \left[ -\frac{1}{2} (\bullet)_i \sqrt{|\det \mathbf{g}|_1} \mathbf{g}_{1ij} \mathcal{L}^{(\bar{1},\bar{4})} \right] \left. \right\}, \quad (\text{B.26})$$

where, by definition,

$$\mathcal{L}^{(1,4)} := \sum_{s=1}^3 \mathbf{g}_s^{ij} \partial_{(s)i} \bar{\phi}_{1,2,3}^1 \partial_{(s)j} \phi_{1,2,3}^1 + \sum_{s=1,4,6} \mathbf{g}_s^{ij} \partial_{(s)i} \bar{\phi}_{6,4,1}^4 \partial_{(s)j} \phi_{6,4,1}^4$$

$$+ m^2 [\bar{\phi}_{1,2,3}^1 \phi_{1,2,3}^1 + \bar{\phi}_{6,4,1}^4 \phi_{6,4,1}^4] + \lambda \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \bar{\lambda} \bar{\phi}_{1,2,3}^1 \bar{\phi}_{3,4,5}^2 \bar{\phi}_{5,2,6}^3 \bar{\phi}_{6,4,1}^4,$$

$$\mathcal{L}^{(\bar{1},\bar{4})} := \sum_{s=3,4,5} \mathbf{g}_s^{ij} \partial_{(s)i} \bar{\phi}_{3,4,5}^2 \partial_{(s)j} \phi_{3,4,5}^2 + \sum_{s=5,2,6} \mathbf{g}_s^{ij} \partial_{(s)i} \bar{\phi}_{5,2,6}^3 \partial_{(s)j} \phi_{5,2,6}^3$$

$$+ m^2 [\bar{\phi}_{3,4,5}^2 \bar{\phi}_{3,4,5}^2 + \bar{\phi}_{5,2,6}^3 \bar{\phi}_{5,2,6}^3]. \quad (\text{B.27})$$

Since  $\mathcal{L}^{(\bar{1},\bar{4})}$  does not contain the variable  $g_1$ , the last term in (B.26) computes to a surface term  $\partial_{(1)k} [(\bullet) \mathcal{L}^{(\bar{1},\bar{4})}]$ . Thence, the variations (B.26) can be written

$$\frac{\partial}{\partial X^\rho} W(X) S =$$

$$- \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \sum_{s=1}^3 \partial_{(s)j} (\bullet) \mathbf{g}_s^{jl} \left[ \left( \partial_{(1)\rho} \phi_{1,2,3}^1 \partial_{(s)l} \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}) \right) - \delta_{s,1} \mathbf{g}_{s\rho} \mathcal{L}^{\text{color}} \right] \right.$$

$$+ \sum_{s=1,4,6} \partial_{(s)j} (\bullet) \mathbf{g}_s^{jl} \left[ \partial_{(1)\rho} \phi_{6,4,1}^4 \partial_{(s)l} \bar{\phi}_{6,4,1}^4 + (\phi \leftrightarrow \bar{\phi}) \right] \left. \right\}. \quad (\text{B.28})$$

From these last lines, the EMT can be readily identified as a two-component tensor

$$T_{(1,s);(i,j)}^{(1)} = \partial_{(1)i} \phi_{1,2,3}^1 \partial_{(s)j} \bar{\phi}_{1,2,3}^1 + \partial_{(1)i} \bar{\phi}_{1,2,3}^1 \partial_{(s)j} \phi_{1,2,3}^1 - \delta_{1,s} \mathbf{g}_{sij} \mathcal{L}^{\text{color}},$$



$$T_{(1,s);(i,j)}^{(4)} = \partial_{(1) i} \phi_{6,4,1}^4 \partial_{(s) j} \bar{\phi}_{6,4,1}^4 + \partial_{(1) i} \bar{\phi}_{6,4,1}^4 \partial_{(s) j} \phi_{6,4,1}^4 . \quad (\text{B.29})$$

Of course it is a matter of taste to put the Lagrangian term in one or the other component. **Covariant conservation** - The conservation property of the EMT ought to be checked. We first evaluate:

$$\begin{aligned} & \sum_{s=1}^3 \nabla_{(s)}^j T_{(1,s);(i,j)}^{(1)} + \sum_{s=1,4,6} \nabla_{(s)}^j T_{(1,s);(i,j)}^{(4)} = \\ & \sum_{s=1}^3 \left[ \nabla_{(1) i} \phi_{1,2,3}^1 \nabla_{(s) j} \nabla_{(s) j} \bar{\phi}_{1,2,3}^1 + \nabla_{(1) i} \bar{\phi}_{1,2,3}^1 \nabla_{(s) j} \nabla_{(s) j} \phi_{1,2,3}^1 \right] \\ & + \sum_{s=1,4,6} \left[ \nabla_{(1) i} \phi_{6,4,1}^4 \nabla_{(s) j} \nabla_{(s) j} \bar{\phi}_{6,4,1}^4 + \nabla_{(1) i} \bar{\phi}_{6,4,1}^4 \nabla_{(s) j} \nabla_{(s) j} \phi_{6,4,1}^4 \right] \\ & - \left[ m^2 \left[ \nabla_{(1) i} \bar{\phi}_{1,2,3}^1 \phi_{1,2,3}^1 + \nabla_{(1) i} \bar{\phi}_{6,4,1}^4 \phi_{6,4,1}^4 \right] \right. \\ & \left. + \lambda \left[ \nabla_{(1) i} \phi_{1,2,3}^1 \right] \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \lambda \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \left[ \nabla_{(1) i} \phi_{6,4,1}^4 \right] + (\phi \leftrightarrow \bar{\phi}) \right]. \quad (\text{B.30}) \end{aligned}$$

Integrating first by  $g_4, g_5$  and  $g_6$ , and using equations of motion of  $\phi^1$  and  $\bar{\phi}_1$ , we obtain

$$\begin{aligned} & \int \left[ \prod_{\ell=4}^6 dg_{\ell} \right] \left[ \sum_{s=1}^3 \nabla_{(s)}^j T_{(1,s);(i,j)}^{(1)} + \sum_{s=1,4,6} \nabla_{(s)}^j T_{(1,s);(i,j)}^{(4)} \right] = \quad (\text{B.31}) \\ & \int \left[ \prod_{\ell=4}^6 dg_{\ell} \right] \left\{ \nabla_{(1) i} \phi_{6,4,1}^4 \left[ \sum_{s=1,4,6} \nabla_{(s) j} \nabla_{(s) j} \bar{\phi}_{6,4,1}^4 - m^2 \bar{\phi}_{6,4,1}^4 - \lambda \int dg_5 \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \right] \right. \\ & \left. + (\phi \leftrightarrow \bar{\phi}) \right\} . \end{aligned}$$

Performing a second integration with respect to  $g_2$  and  $g_3$ , using this time equations of motion of  $\phi^4$  and  $\bar{\phi}^4$ , one gets

$$\int \left[ \prod_{\ell=2}^6 dg_{\ell} \right] \left[ \sum_{s=1}^3 \nabla_{(s)}^j T_{(1,s);(i,j)}^{(1)} + \sum_{s=1,4,6} \nabla_{(s)}^j T_{(1,s);(i,j)}^{(4)} \right] = 0 . \quad (\text{B.32})$$

Then, one infers that the following quantity

$$\int \left[ \prod_{\ell=2}^6 dg_{\ell} \right] \left[ T_{(1,1);(i,j)}^{(1)} + T_{(1,1);(i,j)}^{(4)} \right] \quad (\text{B.33})$$

is covariantly conserved. Indeed, starting from (B.32), a calculation yields

$$\begin{aligned} 0 &= \nabla_{(1)}^j \int \left[ \prod_{\ell=2}^6 dg_{\ell} \right] \left[ T_{(1,1);(i,j)}^{(1)} + T_{(1,1);(i,j)}^{(4)} \right] + \\ & \int \left[ \prod_{\ell=2}^6 dg_{\ell} \right] \left[ \sum_{s=2,3} \nabla_{(s)}^j T_{(1,s);(i,j)}^{(1)} + \sum_{s=4,6} \nabla_{(s)}^j T_{(1,s);(i,j)}^{(4)} \right] \end{aligned}$$

$$\begin{aligned}
0 &= \nabla_{(1)}^j \int \left[ \prod_{\ell=2}^6 dg_\ell \right] \left[ T_{(1,1);(i,j)}^{(1)} + T_{(1,1);(i,j)}^{(4)} \right] \\
&+ \sum_{s=2,3} \int \prod_{\ell=2,3} [d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2] \partial_{(s)k} [(\bullet) \mathbf{g}^{kj} T_{(1,s);(i,j)}^{(1)}] \\
&+ \sum_{s=4,6} \int \prod_{\ell=4,6} [d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2] \partial_{(s)k} [(\bullet) \mathbf{g}^{kj} T_{(1,s);(i,j)}^{(4)}], \tag{B.34}
\end{aligned}$$

where we used the fact that  $\nabla_{(s)j}$  and  $\nabla_{(s')i}$  commute for  $s \neq s'$ , and some integrations by parts for trading covariant derivatives for partial derivatives. Thus (B.33) is a conserved current.

## C Group dilatations

### C.1 Dilatations on the sphere $S^D$

Consider the sphere  $S^D$  with spherical local coordinates  $(\theta, \phi_1, \phi_2, \dots, \phi_{D-1})$ , and the transformation  $d_a : \theta \mapsto \theta_a$  such that

$$\tan \frac{\theta_a}{2} = a \tan \frac{\theta}{2}. \tag{C.35}$$

Note that this transformation is invertible  $(d_a)^{-1} = d_{\frac{1}{a}}$ . We define the mapping on  $S^D$

$$(\theta, \phi_1, \phi_2, \dots, \phi_{D-1}) \mapsto (y^0 = \theta_a, y^1 = \phi_1, y^2 = \phi_2, \dots, y^{D-1} = \phi_{D-1}) \tag{C.36}$$

$$\theta_a = 2 \arctan \left\{ a \tan \frac{\theta}{2} \right\} \tag{C.37}$$

The mapping defines a conformal transformation of the sphere with metric tensor  $\mathbf{g}$  if the metric induced by (C.36) satisfies,  $\forall p \in S^D$ ,

$$\mathbf{g}_{\mu\nu}|_{(\theta_a, \phi_i)} \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\nu}{\partial x^\beta} = \mu^2(\theta, \phi_i) \mathbf{g}_{\alpha\beta}|_{(\theta, \phi_i)}. \tag{C.38}$$

The first component of the induced metric can be computed as, for  $t = \theta/2$ ,

$$\mathbf{g}_{\theta\theta}|_{(\theta_a, \phi_i)} \frac{\partial \theta_a}{\partial \theta} \frac{\partial \theta_a}{\partial \theta} = 1 \cdot \left( \frac{\partial \theta_a}{\partial \theta} \right)^2 = \mu^2(a, \theta) \tag{C.39}$$

$$\frac{1}{2}(1 + \tan^2 t_a) d\theta_a = \frac{a}{2}(1 + \tan^2 t) d\theta, \quad \frac{d\theta_a}{d\theta} = \frac{2a}{(1 - a^2) \cos \theta + 1 + a^2} = \mu(a, \theta).$$

The other metric components are of the form  $\mathbf{g}_{\phi_i, \phi_i}|_{(\theta_a, \phi_i)} \cdot 1$ ,  $i = 1, \dots, D-1$ , such that one can prove that the metric tensor is conformally invariant. Indeed, the central points for that are: (1)  $\sin^2 \theta_a$  is a factor shared by all these components and (2)  $\sin \theta_a / \sin \theta$  scales as  $d\theta_a / d\theta$ . Hence the relation (C.38) is verified.

## C.2 Infinitesimal dilatations

Under an infinitesimal dilatation with parameter such that  $a = 1 + \epsilon$ ,  $\delta_\epsilon \theta = \epsilon \sin \theta$ , a field with scaling factor  $c$  defined on a single copy of  $G \simeq S^3$  transforms as

$$\begin{aligned} \delta_\epsilon \phi(g) &= \tilde{\phi}(d_a d_{\frac{1}{a}}(g)) - \phi(g) = \left[ \frac{2(1+\epsilon)}{(1-(1+2\epsilon)) \cos[\theta - \epsilon \sin \theta] + 1 + (1+2\epsilon)} \right]^c \phi(\theta - \epsilon \sin \theta) - \phi(\theta) \\ &= -\epsilon(-c \cos \theta + \sin \theta \partial_\theta) \phi(g). \end{aligned} \quad (\text{C.40})$$

Thus  $\mathcal{D}(\cdot) := [-c \cos \theta + \sin \theta \partial_\theta](\cdot)$  is the generator of this dilatation. For tensor fields, it can be shown similarly that the corresponding operator becomes “stranded”:

$$\delta_\epsilon \phi_{1,2,3} = \sum_{s=1}^3 \delta_{\epsilon_s} \phi_{1,2,3} = \sum_{s=1}^3 -\epsilon_s (-c \cos \theta_s + \sin \theta_s \partial_{(s)} \theta) \phi_{1,2,3}, \quad (\text{C.41})$$

$$\mathcal{D}_{(s)} := -c \cos \theta_s + \sin \theta_s \partial_{(s)} \theta. \quad (\text{C.42})$$

## C.3 Dilatation current for 1D GFT

**Current calculation** - Consider the operator (28), we evaluate the variation of the action under this operator using the equation of motion (27):

$$\begin{aligned} \frac{\partial}{\partial \epsilon} W(\epsilon) S_{1D}^{\text{scale}} &= \frac{\partial}{\partial \epsilon} \int d\theta d\varphi^1 d\varphi^2 \left\{ (-\epsilon \mathcal{D}\phi) \times \right. \\ &\left. \left[ (\bullet) \frac{(\cos \theta)^2}{\sin \theta} \phi + (\bullet) \cos \theta \partial_\theta \phi - \partial_\theta [(\bullet) \cos \theta \phi] - \tilde{\Delta} \phi + (\bullet) \lambda \sin \theta \phi^3 \right] \right\}. \end{aligned} \quad (\text{C.43})$$

First, we recombine the interaction in a surface term

$$A_0 = \int d\theta d\varphi^1 d\varphi^2 \left\{ (-\epsilon \mathcal{D}\phi) \left[ (\bullet) \lambda \sin \theta \phi^3 \right] \right\} = -\epsilon \int d\theta d\varphi^1 d\varphi^2 \partial_\theta \left[ (\bullet) \frac{\lambda}{4} (\sin \theta)^2 \phi^4 \right], \quad (\text{C.44})$$

and then reduce the following terms

$$\begin{aligned} B_0 &= \int d\theta d\varphi^1 d\varphi^2 \left\{ (-\epsilon \mathcal{D}\phi) \left[ (\bullet) \frac{(\cos \theta)^2}{\sin \theta} \phi + (\bullet) \cos \theta \partial_\theta \phi - \partial_\theta [(\bullet) \cos \theta \phi] \right] \right\} \\ &= -\epsilon \int d\theta d\varphi^1 d\varphi^2 \left\{ -\partial_\theta \left[ (\bullet) (\cos \theta + \sin \theta \partial_\theta) \phi \cos \theta \phi \right] \right. \\ &\quad + (\bullet) \frac{(\cos \theta)^3}{\sin \theta} \phi^2 + (\bullet) (\cos \theta)^2 \partial_\theta \frac{1}{2} \phi^2 \\ &\quad \left. + (\bullet) \left[ -\cos \theta \sin \theta \phi^2 + 3(\cos \theta)^2 \phi \partial_\theta \phi + \cos \theta \sin \theta \partial_\theta [\phi \partial_\theta \phi] \right] \right\} \\ &= \epsilon \int d\theta d\varphi^1 d\varphi^2 \left\{ \partial_\theta (\bullet) \left[ \cos \theta \{ (\cos \theta + \sin \theta \partial_\theta) \phi \} \phi - \cos \theta (\cos \theta + \sin \theta \partial_\theta) \frac{1}{2} \phi^2 \right] \right. \\ &\quad \left. - (\bullet) (\sin \theta)^2 \phi \partial_\theta \phi \right\}, \end{aligned} \quad (\text{C.45})$$

Evaluating the Laplacian term, one finds:

$$C_0 = \epsilon \int d\theta d\varphi^1 d\varphi^2 \mathcal{D}\phi \tilde{\Delta} \phi$$

$$\begin{aligned}
&= \epsilon \int d\theta d\varphi^1 d\varphi^2 \left\{ \partial_k \left\{ [\cos \theta + \sin \theta \partial_\theta] \phi (\bullet) \sin \theta \mathbf{g}^{kl} \partial_l \phi \right\} \right. \\
&\quad \left. - \partial_k \left\{ [\cos \theta + \sin \theta \partial_\theta] \phi \right\} (\bullet) \sin \theta \mathbf{g}^{kl} \partial_l \phi \right\} \\
&= \epsilon \int d\theta d\varphi^1 d\varphi^2 \left\{ \partial_k \left\{ [\cos \theta + \sin \theta \partial_\theta] \phi (\bullet) \sin \theta \mathbf{g}^{kl} \partial_l \phi \right\} \right. \\
&\quad \left. + (\bullet) \sin \theta \phi \partial_\theta \phi - \partial_\theta \left[ (\bullet) \frac{1}{2} (\sin \theta)^2 \mathbf{g}^{kl} \partial_k \phi \partial_l \phi \right] \right\}, \tag{C.46}
\end{aligned}$$

where we use some integrations by parts and the fact that  $\partial_\theta [\sin^2 \theta \mathbf{g}^{kl}] = 2\delta_{k,\theta} \delta_{l,\theta} \cos \theta \sin \theta$ .

One notices that in the last expression, the term which is not of the form of a surface term cancels exactly the similar expression in (C.45). Adding the three contributions  $A_0$  (C.44),  $B_0$  (C.45) and  $C_0$  (C.46), we get

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} W(\epsilon) S_{1D}^{\text{scale}} &= \int d\theta d\varphi^1 d\varphi^2 \left\{ \partial_k \left\{ (\bullet) \sin \theta \mathbf{g}^{kl} [\cos \theta + \sin \theta \partial_\theta] \phi \partial_l \phi \right\} \right. \\
&\quad \left. - \partial_\theta (\bullet) \left[ \sin \theta \left( \frac{1}{2} \sin \theta \mathbf{g}^{kl} \partial_k \phi \partial_l \phi + \frac{(\cos \theta)^2}{\sin \theta} \frac{1}{2} \phi^2 + \cos \theta \phi \partial_\theta \phi + \frac{\lambda}{4} \sin \theta \phi^4 \right) \right. \right. \\
&\quad \left. \left. - \cos \theta \{ (\cos \theta + \sin \theta \partial_\theta) \phi \} \phi \right] \right\} \\
&= \int d\theta d\varphi^1 d\varphi^2 \left\{ \partial_k \left\{ (\bullet) \sin \theta \mathbf{g}^{kl} [\cos \theta + \sin \theta \partial_\theta] \phi \partial_l \phi \right\} \right. \\
&\quad \left. - \partial_\theta (\bullet) \left[ \sin \theta \mathcal{L}_{1D}^{\text{scale}} - \cos \theta \{ (\cos \theta + \sin \theta \partial_\theta) \phi \} \phi \right] \right\} \\
&= \int d\theta d\varphi^1 d\varphi^2 \left\{ \partial_k \left\{ (\bullet) \mathbf{g}^{kl} \left[ \sin \theta [\cos \theta + \sin \theta \partial_\theta] \phi \partial_l \phi \right. \right. \right. \\
&\quad \left. \left. - \mathbf{g}_{l\theta} \sin \theta \mathcal{L}_{1D}^{\text{scale}} + \mathbf{g}_{l\theta} \cos \theta \phi (\cos \theta + \sin \theta \partial_\theta) \phi \right] \right\} \right\}. \tag{C.47}
\end{aligned}$$

The dilatation current can be written

$$\begin{aligned}
D_j &= \sin \theta [\cos \theta + \sin \theta \partial_\theta] \phi \partial_j \phi + \mathbf{g}_{j\theta} \cos \theta \phi (\cos \theta + \sin \theta \partial_\theta) \phi - \mathbf{g}_{j\theta} \sin \theta \mathcal{L}_{1D}^{\text{scale}}, \\
&= \partial_\theta (\sin \theta \phi) \partial_j (\sin \theta \phi) - \mathbf{g}_{j\theta} \sin \theta \mathcal{L}_{1D}^{\text{scale}}. \tag{C.48}
\end{aligned}$$

**Covariant conservation** - The equation of motion for this model can be calculated further as:

$$0 = -(\bullet) \frac{\cos^2 \theta - \sin^2 \theta}{\sin \theta} \phi - \tilde{\Delta} \phi + (\bullet) \lambda \sin \theta \phi^3. \tag{C.49}$$

We compute still using the Levi-Civita connection:

$$\begin{aligned}
\nabla^j D_j &= [\cos \theta + \sin \theta \nabla_\theta] \phi \frac{1}{(\bullet)} \tilde{\Delta} \phi \\
&+ [-\sin \theta + \cos \theta \nabla_\theta] \phi (\sin \theta \nabla_\theta \phi) + ([\cos \theta + \sin \theta \nabla_\theta] \nabla^j \phi) (\sin \theta \nabla_j \phi) \\
&- \sin \theta \phi [\cos \theta + \sin \theta \nabla_\theta] \phi + \cos \theta (\nabla_\theta \phi) [\cos \theta + \sin \theta \nabla_\theta] \phi \\
&+ \cos \theta \phi [-\sin \theta + \cos \theta \nabla_\theta] \phi + \cos \theta \phi [\cos \theta + \sin \theta \nabla_\theta] \nabla_\theta \phi \\
&- \cos \theta \left[ \frac{1}{2} \frac{(\cos \theta)^2}{\sin \theta} \phi^2 + \cos \theta \phi \partial_\theta \phi + \frac{1}{2} \sin \theta \mathbf{g}^{kl} \nabla_k \phi \nabla_l \phi + \frac{\lambda}{4} \sin \theta \phi^4 \right]
\end{aligned}$$

$$\begin{aligned}
& -\sin\theta \left[ \frac{1}{2} [(-2) - \cot^2\theta] \cos\theta\phi^2 + \frac{(\cos\theta)^2}{\sin\theta} \phi \nabla_\theta \phi \right. \\
& + (\nabla_\theta \phi) [\cos\theta \nabla_\theta \phi] + \phi [-\sin\theta \nabla_\theta \phi + \cos\theta \nabla_\theta \nabla_\theta \phi] \\
& \left. + \frac{1}{2} \nabla_\theta \{ \sin\theta \mathbf{g}^{kl} \nabla_k \phi \nabla_l \phi \} + \frac{\lambda}{4} [\cos\theta \phi^4 + 4(\sin\theta \phi^3) \nabla_\theta \phi] \right]. \tag{C.50}
\end{aligned}$$

Canceling the equation of motion, substituting the remaining term in  $\tilde{\Delta}$  making use of the equation of motion and performing some direct simplifications yields

$$\nabla^j D_j = \cos\theta \sin\theta \left[ -(\cot\theta)^2 \phi^2 + \nabla_\theta \phi \nabla_\theta \phi + \frac{\lambda}{2} \phi^4 \right], \tag{C.51}$$

which is not vanishing without further assumptions. Hence the dilatation current is not conserved as expected from a system with an explicit coordinate dependence in the Lagrangian unless the field satisfies both the equation of motion and (C.51) equated to zero. The latter statement is far to be an obvious issue or to even have nontrivial solutions for fields on  $SU(2)$ . In order to have a taste of that problem and for simplicity, by considering only class angle fields  $\phi = \phi(\theta)$ , the system to be solved is of the form

$$\begin{cases}
-(\cos\theta)^2 \phi^2 + (\sin\theta)^2 (\phi')^2 + \frac{\lambda}{2} (\sin\theta)^2 \phi^4 = 0 \\
(\cos^2\theta - \sin^2\theta) \phi + 3 \cos\theta \sin\theta \phi' + \sin^2\theta \phi'' - \lambda \sin^2\theta \phi^3 = 0
\end{cases} \tag{C.52}$$

## C.4 Dilatation current for GFT in 3D

In this section, we compute the dilatation current for a dynamical GFT in  $3D$ .

We need again to symmetrize the variation operator on the complex interaction, recalling that  $\delta_\epsilon \phi_{1,2,3}$  assumes the form (C.41):

$$\begin{aligned}
& \lambda \int \left[ \prod_{\ell=1}^6 dg_\ell \right] (\delta_\epsilon \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} + \delta_\epsilon \bar{\phi}_{1,2,3} \phi_{5,4,3} \bar{\phi}_{5,2,6} \phi_{1,4,6}) = \\
& \frac{\lambda}{2} \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \left\{ (\delta_\epsilon \phi_{1,2,3}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} + (\delta_\epsilon \phi_{5,2,6}) \phi_{1,2,3} \bar{\phi}_{5,4,3} \bar{\phi}_{1,4,6} \right. \\
& \left. + (\delta_\epsilon \bar{\phi}_{5,4,3}) \phi_{1,2,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} + (\delta_\epsilon \bar{\phi}_{1,4,6}) \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \right\} \\
& = -\frac{\lambda}{2} \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \left\{ \sum_{s=1}^3 (\epsilon_{(s)} \mathcal{D}_{(s)} \phi_{1,2,3}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right. \\
& + \sum_{s=1}^3 (\epsilon_{(s)} \mathcal{D}_{(s+\alpha_s)} \bar{\phi}_{5,4,3}) \phi_{1,2,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} + \sum_{s=1}^3 (\epsilon_{(s)} \mathcal{D}_{(s+\alpha_s)} \phi_{5,2,6}) \bar{\phi}_{5,4,3} \phi_{1,2,3} \bar{\phi}_{1,4,6} \\
& \left. + \sum_{s=1}^3 (\epsilon_{(s)} \mathcal{D}_{(s+\alpha_s)} \bar{\phi}_{1,4,6}) \bar{\phi}_{5,4,3} \phi_{5,2,6} \phi_{1,2,3} \right\} \\
& = -\frac{\lambda}{2} \int \left[ \prod_{\ell=1}^6 dg_\ell \right] \sum_{s=1}^3 \epsilon_{(s)} \left( \mathcal{D}_{(s)}^{(2)} + \mathcal{D}_{(s+[\alpha_s])}^{(2)} \right) \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6}, \tag{C.53}
\end{aligned}$$

$$\mathcal{D}_{(s)}^{(2)} := -2c \cos\theta_s + \sin\theta_s \partial_{(s)} \theta, \tag{C.54}$$

where the definition of indices  $\alpha_s$  and  $[\alpha_s]$  remains the same as in the section dealing with translations.

Let us consider the action  $S_{\text{kin},3D}^{\text{scale}}[\phi]$  (54) without mass,<sup>6</sup> with  $\gamma = -1$ ,  $\beta = 3/2$ . The equation of motion for the field  $\phi_{1,2,3}$  can be inferred as:

$$\begin{aligned} \frac{\delta S_{3D}^{\text{scale}}}{\delta \phi_{1,2,3}} &= \sum_{s=1}^3 \left\{ (\bullet)_{1,2,3} \beta^2 (\cos \theta_s)^2 \bar{\phi}_{1,2,3} + (\bullet)_{1,2,3} \beta \cos \theta_s \sin \theta_s \partial_{(s)} \theta \bar{\phi}_{1,2,3} \right. \\ &\quad \left. - \beta \partial_{(s)} \theta \left[ (\bullet)_{1,2,3} \cos \theta_s \sin \theta_s \bar{\phi}_{1,2,3} \right] - \tilde{\Delta}_{(s)} \bar{\phi}_{1,2,3} \right\} + (\bullet)_{1,2,3} \int \left[ \prod_{\ell=4}^6 dg_\ell \right] \lambda \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} , \\ (\bullet)_{1,2,3} &:= \prod_{s=1}^3 \sqrt{|\det \mathbf{g}_s|} , \quad \tilde{\Delta}_{(s)} \phi_{1,2,3} := \partial_{(s)k} \left\{ (\bullet)_{1,2,3} (\sin \theta_s)^2 \mathbf{g}_s^{kl} (\partial_{(s)l} \phi_{1,2,3}) \right\} , \end{aligned} \quad (\text{C.55})$$

where  $\tilde{\Delta}_{(s)}$  is again a modified Laplacian due the presence of the sine function. The functional operator for dilatations is given by (49) enables us to reduce the variations of the action up to a surface term:

$$\begin{aligned} \frac{\partial}{\partial \epsilon_i} W(\epsilon) S_{3D}^{\text{scale}} &= \frac{\partial}{\partial \epsilon_i} \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \left( - \sum_{s=1}^3 \epsilon_s \mathcal{D}_s \phi_{1,2,3} \right) \times \right. \\ &\quad \left[ \sum_{s=1}^3 \left\{ (\bullet) \beta^2 (\cos \theta_s)^2 \bar{\phi}_{1,2,3} + (\bullet) \beta \cos \theta_s \sin \theta_s \partial_{(s)} \theta \bar{\phi}_{1,2,3} \right. \right. \\ &\quad \left. \left. - \beta \partial_{(s)} \theta \left[ (\bullet) \cos \theta_s \sin \theta_s \bar{\phi}_{1,2,3} \right] - \tilde{\Delta}_{(s)} \bar{\phi}_{1,2,3} \right\} + (\bullet) \lambda \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right] + (\phi \leftrightarrow \bar{\phi}) \right\} . \end{aligned}$$

By first recombining the variations of the interaction, we get an expression like (C.53):

$$\begin{aligned} A &= \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ - \sum_{s=1}^3 \epsilon_s [\mathcal{D}_s \phi_{1,2,3}] \left[ (\bullet) \lambda \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right] + (\phi \leftrightarrow \bar{\phi}) \right\} \\ &= - \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \sum_{s=1}^3 \epsilon_s (\bullet) \left\{ \mathcal{D}_s^{(2)} + \mathcal{D}_{s+[\alpha_s]}^{(2)} \right\} \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \\ &= - \sum_{s=1}^3 \epsilon_s \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \partial_{(s)\theta} + \partial_{(s+[\alpha_s])\theta} \right\} \left[ (\bullet) \frac{\lambda}{2} \phi_{1,2,3} \bar{\phi}_{5,4,3} \phi_{5,2,6} \bar{\phi}_{1,4,6} \right] . \quad (\text{C.56}) \end{aligned}$$

Second, we treat the terms with no or an unique derivative in the kinetic part:

$$\begin{aligned} B &= - \sum_{s'=1}^3 \epsilon_{s'} \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \sum_{s=1}^3 \left\{ \right. \\ &\quad \left. \mathcal{D}_{s'} \phi_{1,2,3} \left[ (\bullet) \beta^2 (\cos \theta_s)^2 \bar{\phi}_{1,2,3} - \beta \partial_{(s)\theta} \left[ (\bullet) \cos \theta_s \sin \theta_s \right] \bar{\phi}_{1,2,3} \right] + (\phi \leftrightarrow \bar{\phi}) \right\} \\ &= - \sum_{s'=1}^3 \epsilon_{s'} \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \sum_{s=1}^3 \left\{ \right. \end{aligned}$$

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<sup>6</sup>In fact, a mass term can be included but for simplicity purpose, we do not consider a massive field.

$$\begin{aligned}
& + \left[ -\beta \partial_{(s)} \theta [(\bullet) (\beta \cos \theta_{s'} + \sin \theta_{s'} \partial_{(s')} \theta) \phi_{1,2,3} \cos \theta_s \sin \theta_s \bar{\phi}_{1,2,3}] + (\phi \leftrightarrow \bar{\phi}) \right] \\
& + (\bullet) 3\beta^2 (\cos \theta_s)^2 \cos \theta_{s'} \bar{\phi}_{1,2,3} \phi_{1,2,3} + (\bullet) \beta^2 (\cos \theta_s)^2 \sin \theta_{s'} \partial_{(s')} \theta [\bar{\phi}_{1,2,3} \phi_{1,2,3}] \\
& + (\bullet) \delta_{s,s'} \beta \cos \theta_s \sin \theta_s (-2\beta \sin \theta_{s'} \bar{\phi}_{1,2,3} \phi_{1,2,3}) \tag{C.57}
\end{aligned}$$

$$\begin{aligned}
& + (\bullet) \delta_{s,s'} \beta \cos \theta_s \sin \theta_s \cos \theta_{s'} (\bar{\phi}_{1,2,3} \partial_{(s')} \theta \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s')} \theta \bar{\phi}_{1,2,3}) \\
& + (\bullet) 2\beta^2 \cos \theta_s \sin \theta_s \cos \theta_{s'} (\bar{\phi}_{1,2,3} \partial_{(s)} \theta \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s)} \theta \bar{\phi}_{1,2,3}) \\
& + (\bullet) \beta \sin \theta_{s'} \cos \theta_s \sin \theta_s [\partial_{(s')} \theta (\bar{\phi}_{1,2,3} \partial_{(s)} \theta \phi_{1,2,3}) + \partial_{(s')} \theta (\phi_{1,2,3} \partial_{(s)} \theta \bar{\phi}_{1,2,3})] \}. \tag{C.58}
\end{aligned}$$

Since  $\partial_{(s')} \theta [(\bullet) \sin \theta_{s'} \cos^2 \theta_s] = (\bullet) (3 \cos \theta_{s'} \cos^2 \theta_s - 2\delta_{s,s'} \sin^2 \theta_{s'} \cos \theta_{s'})$ , the intermediate line (C.57) reduces to a surface term

$$\begin{aligned}
& (\bullet) 3 \cos \theta_{s'} \sum_{s=1}^3 \beta^2 (\cos \theta_s)^2 \bar{\phi}_{1,2,3} \phi_{1,2,3} + (\bullet) \sin \theta_{s'} \sum_{s=1}^3 \beta^2 (\cos \theta_s)^2 \partial_{(s')} \theta [\bar{\phi}_{1,2,3} \phi_{1,2,3}] \tag{C.59} \\
& - 2(\bullet) \sin \theta_{s'} \sum_{s=1}^3 \delta_{s,s'} \beta^2 \cos \theta_s \sin \theta_s \bar{\phi}_{1,2,3} \phi_{1,2,3} = \partial_{(s')} \theta \left[ \sin \theta_{s'} (\bullet) \sum_{s=1}^3 \beta^2 (\cos \theta_s)^2 \bar{\phi}_{1,2,3} \phi_{1,2,3} \right]
\end{aligned}$$

Furthermore, using

$$\partial_{(s')} \theta [(\bullet) \sin \theta_{s'} \cos \theta_s \sin \theta_s] = (\bullet) [3 \cos \theta_{s'} \cos \theta_s \sin \theta_s + \delta_{s,s'} \sin \theta_{s'} (-\sin^2 \theta_{s'} + \cos^2 \theta_{s'})], \tag{C.60}$$

we have

$$\begin{aligned}
& 3 \cos \theta_{s'} (\bullet) \sum_{s=1}^3 \beta \cos \theta_s \sin \theta_s (\bar{\phi}_{1,2,3} \partial_{(s)} \theta \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s)} \theta \bar{\phi}_{1,2,3}) \\
& + \sum_{s=1}^3 (\bullet) \delta_{s,s'} \beta \cos \theta_s \sin \theta_s \cos \theta_{s'} (\bar{\phi}_{1,2,3} \partial_{(s)} \theta \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s)} \theta \bar{\phi}_{1,2,3}) \\
& + \sin \theta_{s'} (\bullet) \sum_{s=1}^3 \beta \cos \theta_s \sin \theta_s \partial_{(s')} \theta (\bar{\phi}_{1,2,3} \partial_{(s)} \theta \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s)} \theta \bar{\phi}_{1,2,3}) \\
& = \partial_{(s')} \theta \left[ \sin \theta_{s'} \sum_{s=1}^3 (\bullet) \beta \cos \theta_s \sin \theta_s (\bar{\phi}_{1,2,3} \partial_{(s)} \theta \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s)} \theta \bar{\phi}_{1,2,3}) \right] \\
& + \beta \sin^3 \theta_{s'} (\bar{\phi}_{1,2,3} \partial_{(s')} \theta \phi_{1,2,3} + \phi_{1,2,3} \partial_{(s')} \theta \bar{\phi}_{1,2,3}) \}. \tag{C.61}
\end{aligned}$$

Hence, the quantity  $B$  can be rewritten as

$$\begin{aligned}
B & = - \sum_{s'=1}^3 \epsilon_{s'} \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \{ \tag{C.62} \\
& \left[ - \sum_{s=1}^3 \beta \partial_{(s)} \theta [(\bullet) (\beta \cos \theta_{s'} + \sin \theta_{s'} \partial_{(s')} \theta) \phi_{1,2,3} \cos \theta_s \sin \theta_s \bar{\phi}_{1,2,3}] + (\phi \leftrightarrow \bar{\phi}) \right] \\
& + \partial_{(s')} \theta \left[ \sin \theta_{s'} (\bullet) \sum_{s=1}^3 \beta^2 (\cos \theta_s)^2 \bar{\phi}_{1,2,3} \phi_{1,2,3} \right]
\end{aligned}$$

$$+\partial_{(s')} \theta \left[ \sin \theta_{s'} (\bullet) \sum_{s=1}^3 \beta \cos \theta_s \sin \theta_s \partial_{(s)} \theta (\bar{\phi}_{1,2,3} \phi_{1,2,3}) \right] + \beta \sin^3 \theta_{s'} \partial_{(s')} \theta (\bar{\phi}_{1,2,3} \phi_{1,2,3}) \Big\}.$$

Last, the Laplacian terms have to be calculated as follows:

$$\begin{aligned} C &= \sum_{s'=1}^3 \epsilon_{s'} \int \left[ \prod_{\ell=1}^3 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \sum_{s=1}^3 \left[ \mathcal{D}_{s'} \phi_{1,2,3} \tilde{\Delta}_{(s)} \bar{\phi}_{1,2,3} + (\phi \leftrightarrow \bar{\phi}) \right] \\ &= \sum_{s'=1}^3 \epsilon_{s'} \int \left[ \prod_{\ell=1}^3 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \sum_{s=1}^3 \left\{ \right. \\ &\quad \left[ \partial_{(s)} k \left\{ \left[ \beta \cos \theta_{s'} + \sin \theta_{s'} \partial_{(s')} \theta \right] \phi_{1,2,3} (\bullet) (\sin \theta_s)^2 \mathbf{g}_s^{kl} \partial_{(s)} l \bar{\phi}_{1,2,3} \right\} \right. \\ &\quad + (\bullet) \delta_{s,s'} \delta_{k,\theta} \beta (\sin \theta_{s'})^3 \phi_{1,2,3} \mathbf{g}_s^{kl} \partial_{(s)} l \bar{\phi}_{1,2,3} \\ &\quad - (\bullet) \beta \cos \theta_{s'} (\sin \theta_s)^2 \partial_{(s)} k \phi_{1,2,3} \mathbf{g}_s^{kl} \partial_{(s)} l \bar{\phi}_{1,2,3} \\ &\quad - (\bullet) \delta_{s,s'} \delta_{k,\theta} \cos \theta_{s'} (\sin \theta_s)^2 \partial_{(s)} \theta \phi_{1,2,3} \mathbf{g}_s^{kl} \partial_{(s)} l \bar{\phi}_{1,2,3} + (\phi \leftrightarrow \bar{\phi}) \left. \right] \\ &\quad - (\bullet) \sin \theta_{s'} \partial_{(s')} \theta \left[ \sin^2 \theta_s \mathbf{g}_s^{kl} \partial_{(s)} k \phi_{1,2,3} \partial_{(s)} l \bar{\phi}_{1,2,3} \right] \\ &\quad \left. + (\bullet) \sin \theta_{s'} \partial_{(s')} \theta \left[ \sin^2 \theta_s \mathbf{g}_s^{kl} \right] \partial_{(s)} k \phi_{1,2,3} \partial_{(s)} l \bar{\phi}_{1,2,3} \right\}. \end{aligned} \quad (\text{C.63})$$

The identity  $\partial_{(s')} \theta (\sin^2 \theta_s \mathbf{g}_s^{kl}) = \delta_{s,s'} \delta_{k,\theta} \delta_{l,\theta} 2 \cos \theta_{s'} \sin \theta_{s'}$ , allows one to rewrite (C.63) as

$$\begin{aligned} C &= \sum_{s'=1}^3 \epsilon_{s'} \int \left[ \prod_{\ell=1}^3 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right. \\ &\quad \left[ \sum_{s=1}^3 \partial_{(s)} k \left\{ \left[ \beta \cos \theta_{s'} + \sin \theta_{s'} \partial_{(s')} \theta \right] \phi_{1,2,3} (\bullet) (\sin \theta_s)^2 \mathbf{g}_s^{kl} \partial_{(s)} l \bar{\phi}_{1,2,3} \right\} + (\phi \leftrightarrow \bar{\phi}) \right] \\ &\quad \left. - \partial_{(s')} \theta \left( (\bullet) \sin \theta_{s'} \sum_{s=1}^3 \sin^2 \theta_s \mathbf{g}_s^{kl} \partial_{(s)} k \phi_{1,2,3} \partial_{(s)} l \bar{\phi}_{1,2,3} \right) + (\bullet) \beta (\sin \theta_{s'})^3 \partial_{(s')} \theta (\bar{\phi}_{1,2,3} \phi_{1,2,3}) \right\}. \end{aligned} \quad (\text{C.64})$$

The non-like surface term appearing in (C.64) cancels the extra term appearing in (C.61). Summing all contributions,  $A$  (C.56),  $B$  (C.62) and  $C$  (C.64), one ends up with

$$\begin{aligned} \frac{\partial}{\partial \epsilon_q} W(\epsilon) S_{3D}^{\text{scale}} &= \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right. \\ &\quad \left[ \sum_{s=1}^3 \partial_{(s)} k \left\{ (\bullet) (\sin \theta_s)^2 \left[ \beta \cos \theta_q + \sin \theta_q \partial_{(q)} \theta \right] \phi_{1,2,3} \mathbf{g}_s^{kl} \partial_{(s)} l \bar{\phi}_{1,2,3} \right\} \right. \\ &\quad \left. + \sum_{s=1}^3 \beta \partial_{(s)} \theta \left[ (\bullet) \cos \theta_s \sin \theta_s \left( \beta \cos \theta_q + \sin \theta_q \partial_{(q)} \theta \right) \phi_{1,2,3} \bar{\phi}_{1,2,3} \right] + (\phi \leftrightarrow \bar{\phi}) \right] \\ &\quad - \partial_q \theta \left[ (\bullet) \sin \theta_q \mathcal{L}_{3D}^{\text{scale}} \right] - \partial_{q+[\alpha_q]} \theta \left[ (\bullet) \sin \theta_{q+[\alpha_q]} \mathcal{L}_{\text{int}, 3D} \right] \left. \right\} \\ &= \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right. \\ &\quad \left[ \sum_{s=1}^3 \partial_{(s)} k (\bullet) \mathbf{g}_s^{kl} \left\{ (\sin \theta_s)^2 \left[ \beta \cos \theta_q + \sin \theta_q \partial_{(q)} \theta \right] \phi_{1,2,3} \partial_{(s)} l \bar{\phi}_{1,2,3} \right. \right. \end{aligned}$$



$$\begin{aligned}
& +\beta \mathbf{g}_s{}_{l\theta} \cos \theta_s \sin \theta_s [\beta \cos \theta_q + \sin \theta_q \partial_{(q)} \theta] \phi_{1,2,3} \bar{\phi}_{1,2,3} + (\phi \leftrightarrow \bar{\phi}) \\
& -\delta_{q,s} \mathbf{g}_s{}_{l\theta} \sin \theta_q \mathcal{L}_{3D}^{\text{scale}} - \delta_{q+[\alpha_q],s} \mathbf{g}_s{}_{l\theta} \sin \theta_{q+[\alpha_q]} \mathcal{L}_{\text{int},3D} \} . \tag{C.65}
\end{aligned}$$

The current for this symmetry becomes a stranded tensor expressed by

$$\begin{aligned}
& D_{(s,s');j} = \\
& \sin \theta_s \left\{ \sin \theta_{s'}^{\frac{1}{2}} \left[ \partial_{(s)} \theta \phi_{1,2,3} \partial_{(s')} j (\sin \theta_{s'}^\beta \bar{\phi}_{1,2,3}) + \partial_{(s)} \theta \bar{\phi}_{1,2,3} \partial_{(s')} j (\sin \theta_{s'}^\beta \phi_{1,2,3}) \right] \right. \\
& \left. - \delta_{s,s'} \mathbf{g}_{s'}{}_{j\theta} \mathcal{L}_{3D}^{\text{scale}} \right\} - \delta_{s+[\alpha_s],s'} \mathbf{g}_{s'}{}_{j\theta} \sin \theta_{s+[\alpha_s]} \mathcal{L}_{\text{int},3D} + \beta \cos \theta_s \partial_{(s')} j \left( (\sin \theta_{s'})^2 \bar{\phi}_{1,2,3} \phi_{1,2,3} \right) . \tag{C.66}
\end{aligned}$$

Again due to both the presence of the nonlocal interaction and the explicit coordinate appearance in the Lagrangian, the dilatation current is not covariantly conserved.

## C.5 Dilatation current for the colored model

**Current calculation** - We start by giving the equations of motion for the fields  $\phi^1$  and  $\phi^4$ , using  $\mathcal{L}^{\text{color,scale}} = \mathcal{L}^{\text{color,scale}(1,4)} + \mathcal{L}^{\text{color,scale}(\bar{1},\bar{4})}$  in the form (67), with  $-c = \beta = 3/2$ , with

$$\begin{aligned}
& \mathcal{L}^{\text{color,scale}(1,4)} = \\
& (\sin \theta_1)^{-1} \mathbf{g}_1^{ij} \partial_{(1) i} [(\sin \theta_1)^\beta \bar{\phi}_{1,2,3}^1] \partial_{(1) j} [(\sin \theta_1)^\beta \phi_{1,2,3}^1] + \sum_{s=2}^3 \mathbf{g}_s^{ij} \partial_{(s) i} \bar{\phi}_{1,2,3}^1 \partial_{(s) j} \phi_{1,2,3}^1 \\
& + (\sin \theta_1)^{-1} \mathbf{g}_1^{ij} \partial_{(1) i} [(\sin \theta_1)^\beta \bar{\phi}_{6,4,1}^4] \partial_{(1) j} [(\sin \theta_1)^\beta \phi_{6,4,1}^4] + \sum_{s=4,6} \mathbf{g}_s^{ij} \partial_{(s) i} \bar{\phi}_{6,4,1}^4 \partial_{(s) j} \phi_{6,4,1}^4 \\
& + \lambda \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \bar{\lambda} \bar{\phi}_{1,2,3}^1 \bar{\phi}_{3,4,5}^2 \bar{\phi}_{5,2,6}^3 \bar{\phi}_{6,4,1}^4 , \\
& \mathcal{L}^{\text{color,scale}(\bar{1},\bar{4})} = \sum_{s=3,4,5} \mathbf{g}_s^{ij} \partial_{(s) i} \bar{\phi}_{3,4,5}^2 \partial_{(s) j} \phi_{3,4,5}^2 + \sum_{s=5,2,6} \mathbf{g}_s^{ij} \partial_{(s) i} \bar{\phi}_{5,2,6}^3 \partial_{(s) j} \phi_{5,2,6}^3 . \tag{C.67}
\end{aligned}$$

The equation of motion obtained for  $\phi_{1,2,3}^1$  is

$$\begin{aligned}
& \frac{\delta S^{\text{color,scale}}}{\delta \phi_{1,2,3}^1} = (\bullet)_{1,2,3} \beta^2 (\cos \theta_1)^2 \bar{\phi}_{1,2,3}^1 + (\bullet)_{1,2,3} \beta \cos \theta_1 \sin \theta_1 \partial_{(1) \theta} \bar{\phi}_{1,2,3}^1 \\
& - \beta \partial_{(1) \theta} \left[ (\bullet)_{1,2,3} \cos \theta_1 \sin \theta_1 \bar{\phi}_{1,2,3}^1 \right] - \tilde{\Delta}_{(1)} \bar{\phi}_{1,2,3}^1 - (\bullet)_{1,2,3} \sum_{s=2,3} \Delta_{(s)} \bar{\phi}_{1,2,3}^1 \\
& + \lambda (\bullet)_{1,2,3} \int \left[ \prod_{\ell=4}^6 dg_\ell \right] \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 , \tag{C.68} \\
& (\bullet)_{a,b,c} := \prod_{s=a,b,c} \sqrt{|\det \mathbf{g}_s|} , \quad \tilde{\Delta}_{(1)} \phi_{1,2,3} := \partial_{(1) k} \left\{ (\bullet)_{1,2,3} (\sin \theta_1)^2 \mathbf{g}_1^{kl} (\partial_{(1) l} \phi_{1,2,3}) \right\} .
\end{aligned}$$

The equation of motion of  $\phi^4$  and complex conjugate fields are therefore obvious from (C.68). The functional operator for dilatations is given by (68) where the infinitesimal field variations possess an unique parameter  $\epsilon$ . Let us evaluate the variations of the action up to the point we obtain a surface term:

$$\frac{\partial}{\partial \epsilon} W(\epsilon) S^{\text{color,scale}} = \frac{\partial}{\partial \epsilon} (-\epsilon) \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \{$$

$$\begin{aligned}
& \mathcal{D}_{(1)}\phi_{1,2,3}^1 \left[ (\bullet)\beta^2(\cos\theta_1)^2\bar{\phi}_{1,2,3}^1 + (\bullet)\beta\cos\theta_1\sin\theta_1\partial_{(1)\theta}\bar{\phi}_{1,2,3}^1 \right. \\
& \left. -\beta\partial_{(1)\theta}[(\bullet)\cos\theta_1\sin\theta_1\bar{\phi}_{1,2,3}^1] - \tilde{\Delta}_{(1)}\bar{\phi}_{1,2,3}^1 - (\bullet)\sum_{s=2,3}\Delta_{(s)}\bar{\phi}_{1,2,3}^1 + (\bullet)\lambda\phi_{3,4,5}^2\phi_{5,2,6}^3\phi_{6,4,1}^4 \right] \\
& +\mathcal{D}_{(1)}\phi_{6,4,1}^4 \left[ (\bullet)\beta^2(\cos\theta_1)^2\bar{\phi}_{6,4,1}^4 + (\bullet)\beta\cos\theta_1\sin\theta_1\partial_{(1)\theta}\bar{\phi}_{6,4,1}^4 \right. \\
& \left. -\beta\partial_{(1)\theta}[(\bullet)\cos\theta_1\sin\theta_1\bar{\phi}_{6,4,1}^4] - \tilde{\Delta}_{(1)}\bar{\phi}_{6,4,1}^4 - (\bullet)\sum_{s=4,6}\Delta_{(s)}\bar{\phi}_{6,4,1}^4 + (\bullet)\lambda\phi_{1,2,3}^1\phi_{3,4,5}^2\phi_{5,2,6}^3 \right] \\
& +(\phi\leftrightarrow\bar{\phi}) \Big\}. \tag{C.69}
\end{aligned}$$

Following the same steps as in Appendix C.4, the variations of the interaction can be recombined as

$$\begin{aligned}
A &= -\epsilon \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] (\bullet) \left[ \mathcal{D}_{(1)}^{(2)}[\lambda\phi_{1,2,3}^1\phi_{3,4,5}^2\phi_{5,2,6}^3\phi_{6,4,1}^4] + (\phi\leftrightarrow\bar{\phi}) \right] \tag{C.70} \\
&= -\epsilon \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \partial_{(1)\theta} \left\{ (\bullet)\sin\theta_1 [\lambda\phi_{1,2,3}^1\phi_{3,4,5}^2\phi_{5,2,6}^3\phi_{6,4,1}^4] \right\} + (\phi\leftrightarrow\bar{\phi}) \right\}.
\end{aligned}$$

Second, we treat the terms with no or a single derivative in the kinetic part:

$$\begin{aligned}
B &= -\epsilon \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right. \\
& \left[ \left( \mathcal{D}_{(1)}\phi_{1,2,3}^1 \left[ (\bullet)\beta^2(\cos\theta_1)^2\bar{\phi}_{1,2,3}^1 - \beta\partial_{(1)\theta}[(\bullet)\cos\theta_1\sin\theta_1]\bar{\phi}_{1,2,3}^1 \right] \right. \right. \\
& \left. \left. + (\phi_{1,2,3}^1 \leftrightarrow \phi_{6,4,1}^4) \right) + (\phi\leftrightarrow\bar{\phi}) \right\} \\
&= -\epsilon \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right. \\
& \left[ -\beta\partial_{(1)\theta}[(\bullet)(\beta\cos\theta_1 + \sin\theta_1\partial_{(1)\theta})\phi_{1,2,3}^1\cos\theta_1\sin\theta_1\bar{\phi}_{1,2,3}^1] + (\phi\leftrightarrow\bar{\phi}) \right] \\
& +\partial_{(1)\theta} \left[ \beta^2(\bullet)\sin\theta_1(\cos\theta_1)^2\bar{\phi}_{1,2,3}^1\phi_{1,2,3}^1 \right] \\
& +\partial_{(1)\theta} \left[ \beta(\bullet)\cos\theta_1(\sin\theta_1)^2\partial_{(1)\theta}(\bar{\phi}_{1,2,3}^1\phi_{1,2,3}^1) \right] + \beta(\bullet)\sin^3\theta_1\partial_{(1)\theta}(\bar{\phi}_{1,2,3}^1\phi_{1,2,3}^1) \\
& \left. + (\phi_{1,2,3}^1 \leftrightarrow \phi_{6,4,1}^4) \right\}. \tag{C.71}
\end{aligned}$$

Last, the Laplacian terms have to be calculated following the same steps as done for the case without color. We find:

$$\begin{aligned}
C &= \epsilon \int \left[ \prod_{\ell=1}^3 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right. \\
& \left[ \mathcal{D}_1\phi_{1,2,3}^1 \tilde{\Delta}_{(1)}\bar{\phi}_{1,2,3}^1 + (\bullet)_{1,2,3} \sum_{s=2,3} \mathcal{D}_1\phi_{1,2,3}^1 \Delta_{(s)}\bar{\phi}_{1,2,3}^1 + (\phi\leftrightarrow\bar{\phi}) \right] + (\phi_{1,2,3}^1 \leftrightarrow \phi_{6,4,1}^4) \Big\} \\
&= \epsilon \int \left[ \prod_{\ell=1}^3 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right.
\end{aligned}$$

$$\begin{aligned}
& \left[ \partial_{(1)k} \left\{ \left[ \beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta} \right] \phi_{1,2,3}^1 (\bullet) (\sin \theta_1)^2 \mathbf{g}_1^{kl} \partial_{(1)l} \bar{\phi}_{1,2,3}^1 \right\} \right. \\
& + \sum_{s=2,3} \partial_{(s)k} \left[ \left( \beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta} \right) \phi_{1,2,3}^1 (\bullet) \mathbf{g}_s^{kl} \partial_{(s)l} \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}) \right] \\
& + (\bullet) \beta (\sin \theta_1)^3 \partial_{(1)\theta} (\phi_{1,2,3}^1 \bar{\phi}_{1,2,3}^1) \\
& - \partial_{(1)\theta} \left[ \sum_{s=2,3} (\bullet) \sin \theta_1 \mathbf{g}_s^{kl} \partial_{(s)k} \phi_{1,2,3}^1 \partial_{(s)l} \bar{\phi}_{1,2,3}^1 \right] \\
& \left. - \partial_{(1)\theta} \left[ (\bullet) \sin \theta_1 \left[ (\sin \theta_1)^2 \mathbf{g}_1^{kl} \partial_{(1)k} \phi_{1,2,3}^1 \partial_{(1)l} \bar{\phi}_{1,2,3}^1 \right] + (\phi_{1,2,3}^1 \leftrightarrow \phi_{6,4,1}^4) \right] \right\}. \quad (\text{C.72})
\end{aligned}$$

and again the non-like surface term in (C.72) cancels the extra term in (C.71). By adding all contributions,  $A$  (C.70),  $B$  (C.71) and  $C$  (C.72), one writes

$$\begin{aligned}
\frac{\partial}{\partial \epsilon} W(\epsilon) S^{\text{color, scale}} &= \int \left[ \prod_{\ell=1}^6 d\theta_\ell d\varphi_\ell^1 d\varphi_\ell^2 \right] \left\{ \right. \\
& \partial_{(1)k} (\bullet) \mathbf{g}_1^{kl} \left\{ \left[ \left( (\sin \theta_1)^2 \left[ \beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta} \right] \phi_{1,2,3}^1 \partial_{(1)l} \bar{\phi}_{1,2,3}^1 \right. \right. \right. \\
& + \beta \mathbf{g}_1^{l\theta} \cos \theta_1 \sin \theta_1 \left[ \beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta} \right] \phi_{1,2,3}^1 \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}) \right. \\
& \left. \left. \left. + (\phi_{1,2,3}^1 \leftrightarrow \phi_{6,4,1}^4) \right] - \mathbf{g}_1^{l\theta} \sin \theta_1 \mathcal{L}^{\text{color, scale}(1,4)} \right\} \\
& + \sum_{s=2,3} \partial_{(s)k} (\bullet) \mathbf{g}_s^{kl} \left[ \left( \left[ \beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta} \right] \phi_{1,2,3}^1 \partial_{(s)l} \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}) \right) \right. \\
& \left. \left. + (\phi_{1,2,3}^1 \leftrightarrow \phi_{6,4,1}^4) \right] \right\}. \quad (\text{C.73})
\end{aligned}$$

The current tensor for this symmetry possesses the distinct components:

$$\begin{aligned}
D_{(1);j}^{(1)} &= \left[ (\sin \theta_1)^2 \left[ \beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta} \right] \phi_{1,2,3}^1 \partial_{(1)j} \bar{\phi}_{1,2,3}^1 \right. \\
& \quad \left. + \beta \mathbf{g}_1^{j\theta} \cos \theta_1 \sin \theta_1 \left[ \beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta} \right] \phi_{1,2,3}^1 \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}) \right] \\
& \quad - \mathbf{g}_1^{j\theta} \sin \theta_1 \mathcal{L}^{\text{color, scale}(1,4)}, \\
D_{(s);j}^{(1)} &= \left[ \beta \cos \theta_1 + \sin \theta_1 \partial_{(1)\theta} \right] \phi_{1,2,3}^1 \partial_{(s)j} \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}), \quad s = 2, 3, \quad (\text{C.74})
\end{aligned}$$

and the other components  $D_{(s);j}^{(4)}$  can be obtained from  $D_{(1);j}^{(1)}$  and  $D_{(s=2,3);j}^{(1)}$  by taking the symmetry  $(\phi_{1,2,3}^1 \leftrightarrow \phi_{6,4,1}^4)$  and omitting the Lagrangian part. We can rewrite the dilatation tensor component  $D_{(1);j}^{(1)}$  in the more compact form:

$$\begin{aligned}
D_{(1);j}^{(1)} &= \partial_{(1)\theta} \left[ (\sin \theta_1)^\beta \bar{\phi}_{1,2,3}^1 \right] \partial_{(1)j} \left[ (\sin \theta_1)^\beta \phi_{1,2,3}^1 \right] + \partial_{(1)\theta} \left[ (\sin \theta_1)^\beta \phi_{1,2,3}^1 \right] \partial_{(1)j} \left[ (\sin \theta_1)^\beta \bar{\phi}_{1,2,3}^1 \right] \\
& - \mathbf{g}_1^{j\theta} \sin \theta_1 \mathcal{L}^{\text{color, scale}(1,4)}. \quad (\text{C.75})
\end{aligned}$$

**Covariant conservation** - We write the equation of motion for the color 1 field as

$$\begin{aligned}
0 &= \beta \left[ -\beta \cos^2 \theta_1 + \sin^2 \theta \right] \bar{\phi}_{1,2,3}^1 - \frac{1}{(\bullet)_{1,2,3}} \tilde{\Delta}_{(1)} \bar{\phi}_{1,2,3}^1 - \sum_{s=2,3} \Delta_{(s)} \bar{\phi}_{1,2,3}^1 \\
& + \lambda \int \left[ \prod_{\ell=4}^6 dg_\ell \right] \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4.
\end{aligned}$$

In a covariant form, we evaluate

$$\begin{aligned}
& \sum_{s=1,2,3} \nabla_{(s)}^j D_{(s)j}^{(1)} + \sum_{s=1,4,6} \nabla_{(s)}^j D_{(s)j}^{(4)} = \\
& \nabla_{(1)}^j \left[ [\beta \cos \theta_1 + \sin \theta_1 \nabla_{(1)} \theta] \bar{\phi}_{1,2,3}^1 [(\sin \theta_1)^2 \nabla_{(1)j} \phi_{1,2,3}^1 \right. \\
& \left. + \beta \delta_{\theta j} \sin \theta_1 \cos \theta_1 \phi_{1,2,3}^1 [\beta \cos \theta_1 + \sin \theta_1 \nabla_{(1)} \theta] \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}) \right] \\
& + \sum_{s=2,3} \left[ [\beta \cos \theta_1 + \sin \theta_1 \nabla_{(1)} \theta] \nabla_{(s)}^j \phi_{1,2,3}^1 \nabla_{(s)j} \bar{\phi}_{1,2,3}^1 \right. \\
& \left. + [\beta \cos \theta_1 + \sin \theta_1 \nabla_{(1)} \theta] \phi_{1,2,3}^1 \nabla_{(s)}^j \nabla_{(s)j} \bar{\phi}_{1,2,3}^1 + (\phi \leftrightarrow \bar{\phi}) \right] \\
& + \nabla_{(1)}^j \left[ [\beta \cos \theta_1 + \sin \theta_1 \nabla_{(1)} \theta] \bar{\phi}_{6,4,1}^4 [(\sin \theta_1)^2 \nabla_{(1)j} \phi_{6,4,1}^4 \right. \\
& \left. + \beta \delta_{\theta j} \sin \theta_1 \cos \theta_1 \phi_{6,4,1}^4 [\beta \cos \theta_1 + \sin \theta_1 \nabla_{(1)} \theta] \bar{\phi}_{6,4,1}^4 + (\phi \leftrightarrow \bar{\phi}) \right] \\
& + \sum_{s=4,6} \left[ [\beta \cos \theta_1 + \sin \theta_1 \nabla_{(1)} \theta] \nabla_{(s)}^j \phi_{6,4,1}^4 \nabla_{(s)j} \bar{\phi}_{6,4,1}^4 \right. \\
& \left. [\beta \cos \theta_1 + \sin \theta_1 \nabla_{(1)} \theta] \phi_{6,4,1}^4 \nabla_{(s)j} \nabla_{(s)}^j \bar{\phi}_{6,4,1}^4 + (\phi \leftrightarrow \bar{\phi}) \right] \\
& - \cos \theta_1 \mathcal{L}^{(1,4)} \\
& - \sin \theta_1 \left[ \right. \\
& \left( 2 \cos \theta_1 \sin \theta_1 \nabla_{(1)}^j \bar{\phi}_{1,2,3}^1 \nabla_{(1)j} \phi_{1,2,3}^1 \right. \\
& \left. + (\sin \theta_1)^2 \mathbf{g}_1^{jk} [\nabla_{(1)\theta} \nabla_{(1)j} \bar{\phi}_{1,2,3}^1] \nabla_{(1)k} \phi_{1,2,3}^1 + (\sin \theta_1)^2 \mathbf{g}_1^{jk} \nabla_{(1)j} \bar{\phi}_{1,2,3}^1 [\nabla_{(1)\theta} \nabla_{(1)k} \phi_{1,2,3}^1] \right. \\
& \left. + \beta [-(\sin \theta_1)^2 + (\cos \theta_1)^2] \bar{\phi}_{1,2,3}^1 \nabla_{(1)\theta} \phi_{1,2,3}^1 \right. \\
& \left. + \beta \cos \theta_1 \sin \theta_1 [\nabla_{(1)\theta} \bar{\phi}_{1,2,3}^1 \nabla_{(1)\theta} \phi_{1,2,3}^1 + \bar{\phi}_{1,2,3}^1 \nabla_{(1)\theta} \nabla_{(1)\theta} \phi_{1,2,3}^1] \right. \\
& \left. + \beta [-(\sin \theta_1)^2 + (\cos \theta_1)^2] \phi_{1,2,3}^1 \nabla_{(1)\theta} \bar{\phi}_{1,2,3}^1 \right. \\
& \left. + \beta \cos \theta_1 \sin \theta_1 [\nabla_{(1)\theta} \phi_{1,2,3}^1 \nabla_{(1)\theta} \bar{\phi}_{1,2,3}^1 + \phi_{1,2,3}^1 \nabla_{(1)\theta} \nabla_{(1)\theta} \bar{\phi}_{1,2,3}^1] \right. \\
& \left. + \beta^2 [-2 \sin \theta_1 \cos \theta_1] \bar{\phi}_{1,2,3}^1 \phi_{1,2,3}^1 + \beta^2 (\cos \theta)^2 [\nabla_{(1)\theta} \bar{\phi}_{1,2,3}^1 \phi_{1,2,3}^1 + \bar{\phi}_{1,2,3}^1 \nabla_{(1)\theta} \phi_{1,2,3}^1] \right. \\
& \left. + \sum_{s=2,3} [\nabla_{(1)\theta} \nabla_{(s)}^j \bar{\phi}_{1,2,3}^1 \nabla_{(s)j} \phi_{1,2,3}^1 + \nabla_{(s)}^j \bar{\phi}_{1,2,3}^1 \nabla_{(1)\theta} \nabla_{(s)j} \phi_{1,2,3}^1] \right. \\
& \left. + (\phi^1 \leftrightarrow \phi^4) \right) \\
& \lambda [\nabla_{(1)\theta} \phi_{1,2,3}^1] \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 [\nabla_{(1)\theta} \phi_{6,4,1}^4] \\
& \left. + \bar{\lambda} [\nabla_{(1)\theta} \bar{\phi}_{1,2,3}^1] \bar{\phi}_{3,4,5}^2 \bar{\phi}_{5,2,6}^3 \bar{\phi}_{6,4,1}^4 + \bar{\phi}_{1,2,3}^1 \bar{\phi}_{3,4,5}^2 \bar{\phi}_{5,2,6}^3 [\nabla_{(1)\theta} \bar{\phi}_{6,4,1}^4] \right] \tag{C.76}
\end{aligned}$$

which yields after canceling equations of motion of  $\phi^1$ ,  $\bar{\phi}^1$ ,  $\phi^4$  and  $\bar{\phi}^4$  by integrating all variables save  $g_1$  and trading the remaining modified Laplacian using once again the equations of motion:

$$\int \left[ \prod_{\ell=2}^6 dg_\ell \right] \left[ \sum_{s=1,2,3} \nabla_{(s)}^j D_{(s)j}^{(1)} + \sum_{s=1,4,6} \nabla_{(s)}^j D_{(s)j}^{(4)} \right] =$$

$$\begin{aligned}
&= \int \left[ \prod_{\ell=2}^6 dg_{\ell} \right]_2^6 \left\{ \left[ \left( \beta \cos \theta_1 \bar{\phi}_{1,2,3}^1 \left[ \frac{1}{(\bullet)_{123}} \tilde{\Delta}_{(1)} \phi_{1,2,3}^1 + \sum_{s=2,3} \nabla_{(s)}^j \nabla_{(s)j} \phi_{1,2,3}^1 \right] + (\phi \leftrightarrow \bar{\phi}) \right) \right. \right. \\
&\quad + 2 \cos \theta_1 (\sin \theta_1)^2 \nabla_{(1)\theta} \bar{\phi}_{1,2,3}^1 \nabla_{(1)\theta} \phi_{1,2,3}^1 + \beta^2 \cos \theta [(\cos \theta_1)^2 - 2(\sin \theta_1)^2] \phi_{1,2,3}^1 \bar{\phi}_{1,2,3}^1 \\
&\quad \left. \left. + 2 \cos \theta_1 \sum_{s=2,3} \nabla_{(s)}^j \bar{\phi}_{1,2,3}^1 \nabla_{(s)j} \phi_{1,2,3}^1 + (\phi^1 \leftrightarrow \phi^4) \right] \right\} \\
&\quad - \left[ \lambda \cos \theta_1 \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \bar{\lambda} \cos \theta_1 \bar{\phi}_{1,2,3}^1 \bar{\phi}_{3,4,5}^2 \bar{\phi}_{5,2,6}^3 \bar{\phi}_{6,4,1}^4 \right] \Big\} \\
&= \int \left[ \prod_{\ell=2}^6 dg_{\ell} \right]_2^6 \left\{ \left[ 2 \cos \theta_1 (\sin \theta_1)^2 \nabla_{(1)\theta} \bar{\phi}_{1,2,3}^1 \nabla_{(1)\theta} \phi_{1,2,3}^1 \right. \right. \\
&\quad + 2 \cos \theta_1 \sum_{s=2,3} \nabla_{(s)}^j \bar{\phi}_{1,2,3}^1 \nabla_{(s)j} \phi_{1,2,3}^1 - \frac{9}{2} (\cos \theta_1)^3 \bar{\phi}_{1,2,3}^1 \phi_{1,2,3}^1 + (\phi^1 \leftrightarrow \phi^4) \Big] \\
&\quad \left. + \frac{1}{2} \left[ \lambda \cos \theta_1 \phi_{1,2,3}^1 \phi_{3,4,5}^2 \phi_{5,2,6}^3 \phi_{6,4,1}^4 + \bar{\lambda} \cos \theta_1 \bar{\phi}_{1,2,3}^1 \bar{\phi}_{3,4,5}^2 \bar{\phi}_{5,2,6}^3 \bar{\phi}_{6,4,1}^4 \right] \right\}. \tag{C.77}
\end{aligned}$$

One can compare the latter expression with the breaking (C.51) for the 1D case and discover than they have in fact the same structure.

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