Neutron star interiors and topology change

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Quark matter is believed to exist in the center of neutron stars. A combined model consisting of quark matter and ordinary matter is used to show that the extreme conditions existing in the center could result in a topology change, that is, in the formation of wormholes.

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I. INTRODUCTION

Wormholes are handles or tunnels in the spacetime topology connecting different parts of our Universe or of different universes. The meticulous analysis in Ref. [1] has shown that such wormholes may be actual physical objects permitting two-way passage.

Also introduced in Ref. [1] is the metric for a static spherically symmetric wormhole, namely

$$ds^2 = -e^{\phi(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \quad (1)$$

where $e^{\lambda(r)} = 1/(1 - b(r)/r)$. Here $b = b(r)$ is the *shape function* and $\phi = \phi(r)$ is the *redshift function*, which must be everywhere finite to prevent an event horizon. For the shape function we must have $b(r_0) = r_0$, where $r = r_0$ is the radius of the *throat* of the wormhole; also, $b'(r_0) < 1$ and $b(r) < r$ to satisfy the flare-out condition [1]. We are using geometrized units where $G = c = 1$.

While Morris and Thorne concentrated on traversable wormholes suitable for humanoid travelers (and possible construction by an advanced civilization), a more general question is the possible existence of naturally occurring wormholes [2]. It has also been suggested that the extreme conditions inside a massive compact star may result in a topology change [3]. It is proposed in this paper that a neutron star is a candidate for such a topology change as a result of *quark matter* [4–6], which is believed to exist at the center of neutron stars [7]. To that end, we will use a two-fluid model consisting of ordinary matter and quark matter based on the MIT bag model [8]. Two cases will be considered: the non-interacting and interacting two-fluid models. (We will concentrate mainly on the former case.) The reason for having two cases is that quark matter is a viable candidate for dark matter and so may be weakly interacting [9].

A class of exact solutions of the Einstein-Maxwell field equations describing a wormhole with an anisotropic matter distribution is discussed in Ref. [10].

A combined model consisting of neutron-star matter and of a phantom/ghost scalar field yielding a nontrivial topology is discussed in Ref. [11]. The model is referred to as a neutron-star-plus-wormhole system.

II. BASIC EQUATIONS

For our basic equations we follow Ref. [12]. The energy momentum tensor of the two-fluid model is given by

$$T_0^0 \equiv \rho_{\text{effective}} \;=\; \rho + \rho_q, \quad (2)$$

$$T_1^1 = T_2^2 \equiv -p_{\text{effective}} \;=\; -(p + p_q). \quad (3)$$

In Eqs. (2) and (3), ρ and p correspond to the respective energy density and pressure of the baryonic matter, while ρ_q and p_q correspond to the energy density and pressure of the quark matter. The left-hand sides are the effective energy density and pressure, respectively, of the composition.

The Einstein's field equations are listed next [12]:

$$e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi\left(\rho + \rho_q\right), \quad (4)$$

$$e^{-\lambda}\left(\frac{\phi'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi\left(p + p_q\right), \quad (5)$$

$$\frac{e^{-\lambda}}{2}\left[\frac{(\phi')^2 - \lambda'\phi'}{2} + \frac{\phi' - \lambda'}{r} + \phi''\right] = 8\pi\left(p + p_q\right), \quad (6)$$

recalling that $T_1^1 = T_2^2$.

In the MIT bag model, the matter equation of state is given by

$$p_q = \frac{1}{3}(\rho_q - 4B), \quad (7)$$

where B is the bag constant, which we take to be $145\,\text{MeV}/(\text{fm})^3$ [8, 13]. Since quarks are part of the standard model, we assume that $p_q > 0$; hence $\rho_q > 4B$. For normal matter we use the equation of state [10]

$$p = m\rho, \quad 0 < m < 1. \quad (8)$$

Since we are assuming the pressure to be isotropic, the conservation equation is [12]

$$\frac{d(p_{\text{effective}})}{dr} + \frac{1}{2}\phi'\left(\rho_{\text{effective}} + p_{\text{effective}}\right) = 0. \quad (9)$$

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III. SOLUTIONS OF BASIC EQUATIONS

As noted in the Introduction, we will concentrate mainly on the non-interacting fluid model. This means that the two fluids, normal matter and quark matter, do not interact. The resulting conservation equations are therefore independent of each other. Using Eqs. (7) and (8), we have

$$\frac{d\rho}{dr} + \phi' \left(\frac{1+m}{2m}\right)\rho = 0 \tag{10}$$

and

$$\frac{d\rho_q}{dr} + 2\phi'(\rho_q - B) = 0. \tag{11}$$

The solutions are

$$\rho = \rho_0 e^{-\phi(1+m)/2m} \tag{12}$$

and

$$\rho_q = B + \rho_{(q,0)} e^{-2\phi}, \tag{13}$$

where ρ_0 and $\rho_{(q,0)}$ are integration constants. Equations (4), (5), and (6) now give

$$e^{-\lambda}\left[-\frac{\lambda'\phi'}{2} + \frac{(\phi')^2}{2} + \phi'' + \frac{2\phi'}{r}\right] = 8\pi\left[\rho(1+3m) + (2\rho_q - 4B)\right] \tag{14}$$

It becomes apparent that this equation is linear in $e^{-\lambda}$ once it is rewritten in the following form:

$$(e^{-\lambda})' + e^{-\lambda}\left(\phi' + \frac{2\phi''}{\phi'} + \frac{4}{r}\right) = \frac{2}{\phi'}(8\pi)[\rho(1+3m) + (2\rho_q - 4B)].$$

The integrating factor is

$$\text{I.F.} = e^{\phi + 2\ln\phi' + 4\ln r} = e^{\phi}(\phi')^2 r^4,$$

leading at once to

$$\frac{d}{dr}[e^{-\lambda}e^{\phi}(\phi')^2 r^4] = e^{\phi}(\phi')^2 r^4\left(\frac{2}{\phi'}\right)(8\pi)[\rho(1+3m) + (2\rho_q - 4B)].$$

To write the solution for $e^{-\lambda}$, we first define $F(r) = \rho(1+3m) + (2\rho_q - 4B)$, which becomes, after substituting the solutions ρ and ρ_q,

$$F(r) = \rho_0 e^{-\phi(1+m)/2m}(1+3m) + 2\rho_{(q,0)}e^{-2\phi} - 2B.$$

We now have

$$e^{-\lambda} = \frac{16\pi}{e^{\phi}(\phi')^2 r^4}\int_{r_0}^{r} e^{\phi}\phi' r_1^4 F(r_1)dr_1. \tag{15}$$

The lower limit $r = r_0$ is the radius of the throat of the wormhole, as we will see.

As noted in Sec. I, the shape function $b = b(r)$ is obtained from $e^{-\lambda(r)}$, so that

$$b(r) = r(1 - e^{-\lambda(r)}). \tag{16}$$

We obtain

$$b(r) = r\left[1 - \frac{16\pi}{e^{\phi}(\phi')^2 r^4}\int_{r_0}^{r} e^{\phi}\phi' r_1^4 F(r_1)dr_1\right]. \tag{17}$$

Note especially that $b(r_0) = r_0$, which characterizes the throat of the wormhole.

To study the other requirement, $b'(r_0) < 1$, we start with

$$b'(r) = 1 - e^{-\lambda(r)} + r\left[-\frac{d}{dr}e^{-\lambda(r)}\right],$$

which leads to

$$b'(r_0) = 1 + r_0\frac{16\pi}{\phi'(r_0)}\left[-\rho_0 e^{-\phi(r_0)(1+m)/2m}(1+3m) - 2\rho_{(q,0)}e^{-2\phi(r_0)} + 2B\right]. \tag{18}$$

Given that ϕ', the gradient of the redshift function, is related to the tidal force, one would expect its magnitude to be extremely large. So $b'(r_0)$ will indeed be less than unity if the expression inside the brackets in Eq. (18) is negative and sufficiently small in absolute value. If this is the case, then the flare-out condition is met, that is,

$$\frac{r_0 b'(r_0) - b(r_0)}{2[b(r_0)]^2} < 0. \tag{19}$$

Since it indicates a violation of the weak energy condition, the flare-out condition is the primary prerequisite for the existence of wormholes [1]. (The condition $b(r)/r < 1$ will be checked later.)

Unfortunately, the constants B and $\rho_{(q,0)}$ are also large, so that Eq, (18) must be examined more closely.

IV. THE GRADIENT OF THE REDSHIFT FUNCTION

To analyze Eq. (18), we start with the gradient of ϕ [14]:

$$\frac{d\phi}{dr} = \left[\frac{2Gm(r)}{c^2r^2} + 8\pi r\frac{G}{c^4}p\right]\frac{1}{1 - 2Gm(r)/c^2r},$$

where $m(r)$ is the total mass-energy inside the radial distance r. (This formula is also used in the derivation of the Tolman-Oppenheimer-Volkoff equation.) In geometrized units we get for $r = r_0$,

$$\phi'(r_0) = \frac{2m(r_0)/r_0^2 + 8\pi r_0 p(r_0)}{1 - 2m(r_0)/r_0}. \tag{20}$$

Since we are now concerned with what is going to become the throat at $r = r_0$, we need to evaluate ρ_0 and $\rho_{(q,0)}$ in Eqs (12) and (13) to obtain

$$\rho(r) = \rho(r_0) e^{\phi(r_0)(1+m)/2m} e^{-\phi(r)(1+m)/2m} \tag{21}$$

and

$$\rho_q(r) = e^{2\phi(r_0)}[\rho_q(r_0) - B]e^{-2\phi(r)} + B. \tag{22}$$

So Eq. (18) becomes

$$b'(r_0) = 1 + \frac{16\pi r_0[-\rho(r_0)(1+3m) - 2\rho_q(r_0) + 4B]}{[2m(r_0)/r_0^2 + 8\pi r_0 p(r_0)]/[1 - 2m(r_0)/r_0]} \tag{23}$$

We already know that $p = m\rho$ and that $\rho_q > 4B$, so that the numerator is indeed negative. For the purpose of comparison, we will simply assume for now that $\rho_q(r_0) = 4B$. Then

$$b'(r_0) = 1 + \frac{16\pi[-\rho(r_0)(1+3m) - 4B]}{[2m(r_0)/r_0^3 + 8\pi p(r_0)]/[1 - 2m(r_0)/r_0]}. \tag{24}$$

Disregarding r_0 for the time being, suppose we determine $m(r_0)$ so that $2m(r_0)$ exceeds $16\pi(4B)$. We obtain

$$16\pi(4B) = 16\pi(4)(145)\frac{\text{MeV}}{(\text{fm})^3}\frac{G}{c^4}\ \text{m}^{-2},$$

which leads to $m(r_0) \approx 10^{-8}$ m. The other term in the denominator, $8\pi p(r_0)$, could be close to zero (since $0 < m < 1$). However, by simply doubling the value of $m(r_0)$, we can compensate for $-\rho(r_0)(1+3m)$ in the numerator, since this term is smaller than the other term. Moreover, taking ρ_q to be 2×10^{18} kg/m^3, we see that in geometrized units, $\rho_q(r_0)$ is in fact larger than and reasonably close to $4B$, thereby justifying the assumption $\rho_q(r_0) \approx 4B$.

Next, given the enormous density near the center of a neutron star, $m(r_0) \approx 10^{-8}$ m would correspond to a fairly small value for r_0: using $\rho_q(r_0) \approx 4B$ again,

$$\frac{4}{3}\pi r_0^3 \rho_q(r_0) = 10^{-8}$$

yields $r_0 \approx 1$ m, although larger values would be allowed. So the above estimate holds.

Finally, observe that the other factor in the denominator, $1 - 2m(r_0)/r_0$, is now seen to be very close to unity.

We conclude that $b'(r_0) < 1$. Moreover, using Eq. (22) in Eq. (17), we obtain $b(r) - r < 0$ for $r > r_0$, so that $b(r)/r < 1$.

V. COMPLETING THE WORMHOLE STRUCTURE

As we have seen, $m(r_0)$ is the mass inside the radial distance $r = r_0$. Since $r = r_0$ is now the throat of the wormhole, the interior $r < r_0$ is outside the manifold. Although not part of the wormhole spacetime, it still contributes to the gravitational field. This can be compared to a thin-shell wormhole from a Schwarzschild black hole [15]: while not part of the manifold, the black hole generates the gravitational field. This comparison explains why our wormhole requires a minimum throat radius to help produce a sufficiently strong gravitational field.

A. The need for quark matter

If quark matter is eliminated from the model, then the right-hand side of Eq. (14) becomes $8\pi\rho(1+3m)$. As a consequence, the $4B$-term in Eq. (24) is also eliminated. Since

$$|16\pi[-\rho(r_0)(1+3m)]| > 2 \times \frac{4}{3}\pi\rho(r_0) = \frac{2m(r_0)}{r_0^3}\rho(r_0),$$

the flare-out condition $b'(r_0) < 1$ is no longer satisfied. It now becomes apparent that our conclusion depends critically on the fact that B is a constant: increase the throat size until $2m(r_0)$ surpasses $16\pi(4B)$. So based on our model, a topology change is possible only if there is a sufficient amount of quark matter in the center. More precisely, the assumption $\rho(r_0) \approx 4B$ implies that the quark matter has to extend at least to $r = r_0$.

B. The interacting case

We conclude with some brief comments on the interacting case. If the two fluids are assumed to interact, then the conservation equations take on the following forms [12, 16]:

$$\frac{d\rho}{dr} + \phi'\left(\frac{1+m}{2m}\right)\rho = Q \tag{25}$$

and

$$\frac{d\rho_q}{dr} + 2\phi'(\rho_q - B) = -3Q. \tag{26}$$

The quantity Q expresses the interaction between the two types of matter and falls off rapidly as $r \to \infty$. Furthermore, since the interaction is assumed to be relatively weak, Q is a very small quantity compared to B. So in comparing $m(r_0)$ and $\rho_q(r_0)$ in the expression for $b'(r_0)$, the presence of the quantity Q is going to have little effect.

VI. CONCLUSION

Quark matter is believed to exist in the center of neutron stars. The analysis in this paper is therefore based on a two-fluid model comprising ordinary and quark matter with an isotropic matter distribution. It is shown that the extreme conditions may result in a topology change, that is, a neutron star could give rise to a wormhole.

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