Chapter 1

Mathematical Concepts and Physical Objects

Introduction

With this text, we will first of all propose and discuss a distinction, internal to mathematics, between "construction principles" and "proof principles." In short, it will be a question of grasping the difference between the construction of mathematical concepts and structures and the role of proof, more or less formalized. The objective is also to analyze the methods of physics from a similar viewpoint and, from the analogies and differences that we shall bring to attention, to establish a parallel between the foundations of mathematics and the foundations of physics.

When proposing a mathematical structure, for example the integer numbers or the real numbers, the Cartesian space or a Hilbert space, we use a plurality of concepts often stemming from different conceptual experiences: the construction of the integers evokes the generalized successor operation, but at the same time we make sure they are "well-ordered," in space or time. That is, that they form a strictly increasing sequence, with no (backward) descending chains. This apparently obvious property (doesn't it appear so?) yields this well-ordered "line of integer numbers," a non-obvious logical property, yet one we easily "see" within our mental space (can't you see it?). And we construct the rationals, as ratios of integers modulo ratio equivalence, and then the real numbers, as convergent sequences (modulo equiconvergence), for example. The mathematician "sees" this Cantor–Dedekind-styled construction of the continuum, the modern real line and continuum, a remarkable and very difficult mathematical reconstruction of the phenomenal continuum. It is nevertheless not unique: different continua may be more effective for certain applications, albeit that their structures are locally and globally very different, non-isomorphic to

MATHEMATICS AND THE NATURAL SCIENCES - The Physical Singularity of Life

this very familiar standard continuum (see Bell, 1998). And this construction is so important that the "objectivity" of real numbers is "all there," it depends solely upon this very construction, based on the well-order of integers, the passage to the quotients (the rationals) and then the audacious limit operation by Cantor (add all limits of converging sequences). One could say as much about the most important set-theoretic constructions, the cumulative hierarchies of sets, the sets constructed from the empty set (a key concept in mathematics) by the iterated exponent operations, and so on. These conceptual constructions therefore obey well-explicated "principles" (of construction, as a matter of fact): add one (the successor), order and take limits in space (thus, iteration, the well-order of the integers, limits of converging sequences).

But how may one grasp the properties of these mathematical structures? How may one "prove them"? The great hypothesis of logicism (Frege) as well as of formalism (Hilbert's program) has been that the logico-formal proof principles could have completely described the properties of the most important mathematical structures. Induction, particularly, as a logical principle (Frege) or as a potentially mechanizable formal rule (Hilbert), should have permitted us to demonstrate all the properties of integers (for Frege, the logic of induction coincided, simply, with the structure of the integers – it should have been "categorical," in modern terms). Now it happens that logico-formal deduction is not even "complete," as we will recall (let's put aside Frege's implicit hypothesis of categoricity); particularly, many of the integers' "concrete" properties elude it. We will evoke the "concrete" results of incompleteness from the last decades: the existence of quite interesting properties, demonstrably realized by the well-ordering of integers, and which formal proof principles are unable to grasp. But that also concerns the fundamental properties of sets, the continuum hypothesis, and of the axiom of choice, for example, demonstrably true within the framework of certain constructions, as shown by Gödel in 1938, or demonstrably false in others (constructed by Cohen in 1964), thus unattainable by the sole means of formal axiomatics and deductions.

To summarize this, the distinction between "construction principles" and "proof principles" shows that theorems of incompleteness prohibit the reduction (theoretical and epistemic) of the former to the latter (or also of semantics – proliferating and generative – to strictly formalizing syntax).

Can we find, this time, and in what concerns the foundations of physics, some relevance to such a distinction? In what would it consist and would it play an epistemologically similar role? Indeed, if the contents and the methods of these two disciplines are eminently different, the fact that mathematics plays a constitutive role for physics should nevertheless allow us to establish some conceptual and epistemological correspondences regarding their respective foundations. This is the question we shall attempt to examine here. To do so, we will try to describe the same level of "construction principles" for mathematics and physics, that of mathematical structures. This level is common to both disciplines, because the mathematical organization of the real world is a constitutive element of all modern physical knowledge (in short, but we will return to this, the constitution of the "physical object" *is* mathematical).

However, the difference becomes very clear at the level of the *proof* principles. The latter are of a logico-formal nature in mathematics, whereas in physics they refer to observation or to experience; shortly, they refer to measurement. This separation is of an epistemic nature and refers, from a historical viewpoint, to the role of logicism (and of formalism) in mathematics and of positivism in physics. We will therefore base ourselves upon the following table:

Disciplines	**Mathematics**	**Physics**
1. Construction principles	Mathematical structures and their relationships	
2. Proof principles	Formal/Logical proofs	Experience/observation
Reduction of 1 to 2	Logicism/Formalism	Positivism/empiricism

Let's comment this schema with more detail. The top level corresponds to the construction principles, which have their effectiveness and their translation in the elaboration of mathematical structures as well as in the various relationships they maintain. These structures may be relative to mathematics as such or to the mathematical models which retranscribe, organize, and give rise to physical principles – and by that, partly at least, the phenomena that these principles "legalize" by provoking and often guiding experiments and observation. This community of level between the two disciplines, in what concerns the construction of concepts, does not only come from the constitutive character of mathematics for physics, which we just evoked and which would almost suffice to justify it, but it also allows us to understand the intensity of the theoretical exchanges (and not only the instrumental ones) between these disciplines. Physics certainly obtains elements of generalization, modelization, and generativity from mathematical structures and their relationships, but physics' own developments also suggest and propose to mathematics the construction of novel concepts, of which physics, in some cases, already makes use, without waiting to be rigorously founded. Historical examples abound: be it the case of Leibnizian infinitesimals, which appeared to be so paradoxical at the moment they were introduced – and for a long while after that – and which were never theoretically validated elsewise than by modern non-standard analysis, be it Dirac's "function" which was rigorously dealt with only in the subsequent theory of distributions, be it the case of Feynman's path integrals – which have not yet found a sufficiently general rigorous mathematical treatment, while revealing themselves to be completely operable – or be it the birth of non-commutative geometry inspired by the properties of quantum physics.

The second level, corresponding to that of the proof principles, divides itself into two distinct parts according to whether it concerns mathematics or physics (in that their referents are obviously different). For mathematics, what works as such are the corresponding syntaxes and logico-formal languages which, since Frege, Russell, Hilbert, have been presented as the foundations of mathematics. In fact, the logicism and formalism which have thus developed themselves at the expense of any other approach never stopped to identify the construction principles level with the proof principles level by reducing the first to the second. The incompleteness theorems having shown that this program could not be fulfilled for reasons internal to formalism (they prove that the formal proof principles produce valid but unprovable statements), the paradoxical effect was to completely disjoin one level from the other in the foundations of mathematics, by leading syntax to oppose semantics or, by contrast, by refusing to satisfy oneself with proofs not totally formalized (in the sense of this formalism) as can exist in geometry for example. In fact, it appears, conversely, that, as all of the practice of mathematics demonstrates, it is the coupling and circulation between these two levels that make this articulation between innovative imagination and rigor which characterizes the generativity of mathematics and the stability of its concepts.

Let's now consider physics, where the emergence of invariants (and symmetries) also constitutes a methodological turning point, as well as the constitution of objects and of concepts (see Chapters 4 and 5). But this time, at the level of the proof principles, we no longer find a formal language, but the empiricism of phenomena: experiences, observations, even simulations, validate the theoretical predictions or insights provided by the mathematical models and prove their relevance. As constructed as they may have been by anterior theories and interpretations, it is the physical facts which constitute the referents and the instruments of proof. And there again, a particular philosophical option, related to the stage of development of the discipline and to the requirement of rigor in relation to physical factuality, has played, for the latter, a similar role to that of logicism and especially to that of formalism for mathematics. It consists in the positivism and the radical empiricism which, believing to be able to limit themselves only to "facts," attempted to reduce the level of construction, characterized, namely, by interpretative debates, to that of proof, identified to pure empiricity. The developments of contemporary physics, that of quantum physics particularly, of course, but also that of the theory of dynamical systems, have shown that this position was no longer tenable and that the same paradoxical effect has led, doubtlessly by reaction, to the epistemological disjunction between the levels of conceptual and mathematical construction and of empirical proof (a transposed trace is its opposition between "nominalists" and "realists" in the epistemology of physics). While, there again, all the practice of physicists shows that it is in the coupling and the circulation between these levels that lies the fecundity of the discipline, where empirical practices are rich of theoretical commitments and, conversely, theories are heavily affected by the methods of empirical proofs. And, since for us the analysis of the genesis of concepts is part of foundational analysis, it is this productivity itself that feeds off interactions and which takes root within cognitive processes, which must be analyzed.

It is thus in this sense, summarized by the above schema, despite their very different contents and practices, that the foundations of mathematics and the foundations of physics can be considered as presenting some common structural traits. That is, this distinction between two conceptual instances are qualifiable in both cases as construction principles and as proof principles, and the necessity of their coupling – against their disjunction or conversely, their confusion – is important to also be able to account for the effective practice of researchers in each of these disciplines. Moreover, that they share the same level as for the constitution of mathematical structures characterizing the dynamics of construction principles and feeding off the development of each of them.

If we now briefly address the case of this other discipline of natural sciences which is biology, it appears, in what concerns the structure of its own foundations, to distinguish itself from this schema, though we may consider that it shares with physics the same level of proof principles, that is, the constraint of reference to the empiricity of observation and of experience. However, we are led, at the level of this proof principle, to operate a crucial distinction between what is a matter of *in vivo* (biological as such in that it is integrated and regulated by biological functions), and what is a matter of *in vitro* (and which practically confounds itself with the physico-chemical). But what manifestly changes the most depends, it seems, on two essential factors. On the one hand, the level of what we may call (conceptual) "construction principles" in biology still does not seem well characterized and stabilized (despite models of evolution, autonomy or autopoiesis[1]). On the other hand, it seems that another conceptual level adds itself, one specific to the epistemology of the living, and to which is confronted any reflection in biology and which we may qualify, to use Monod's terminology, as the level of the teleonomic principle. This principle in some way makes the understanding of the living depend not only upon that of its past and current relationships to its relevant environment, but also upon that of the anticipations relative to the future of what this environment will become under the effect of its own activity of living. And this temporality lays itself beside the temporality treated by physical theories. This regulates the physico-chemical action-reaction relation, but, on purely theoretical grounds, we must consider also a biological temporality specific to the organism which manifests itself as the existence and the activity of "biological clocks" which time its functions (see Chapter 3). This conceptual situation then leads us to consider, for biology, the characterization of an extra, specific concept, in interaction with the first two, which we like to call "contingent finality"; meaning by that the regulations induced by the implications of these anticipations, and which themselves open the way to the accounting for "significations" as we will argue below.

As mentioned in the book's introduction, this chapter (and only this chapter) will be based on an explicit distinction of the author's contribution, following the dialog which started this work. The preliminary questions concerning foundational issues in mathematics and physics will then be raised.

[1] This is defined as a "process that produces the components that produce the process, ..." typically, in a cell, the metabolic activities are a process of this kind, see Varela (1989), (Bourgine and Stewart, 2004).

1.1 On the Foundations of Mathematics. A First Inquiry

(by Francis Bailly)

1.1.1 *Terminological issues?*

1.1.1.1 *Regarding the term "structure"*

There is often confusion, in physics, on the use of terms such as "mathematical structure" and "mathematical formalism": a physicist claims to be "formalizing" or "mathematizing" in a rather polysemic way (which is of course sufficient for his/her everyday work, but not for our foundational analysis of common construction principles). It very well appears that the term "structure" (and its derivatives) in mathematics may be interpreted in two distinct ways, which may be clarified in the light of our previous distinction (proof vs construction principles). The first sense refers to the general usage of the term: a formal mathematical structure characterized by axiomatic determinations and associated rules of deduction, for instance, the formalized structure of the field of real numbers, the structure of numbers defined by the axioms of Peano's arithmetic, the axiomatic structure of transformation groups, etc. The second sense rather refers to a structure as characterized by properties of content more than by formal axiomatic determinations and therefore presents more of a semantic aspect, that is proposed by construction principles (we will detail at length in this book symmetries, ordering principles, etc). As, for example, when we refer to the structure of continuous mathematics or to the connectedness of space. It is in this second sense that Giuseppe Longo and many others seem to employ the term in their criticism of the formalist and set-theoretic approach and it is according the latter framework that we are then led to question ourselves regarding a possible dialectic between the rigidity of an excess of structure and the dispersal entailed by a complete lack of structure.

Using the example of space, will one refer to a sort of space which is completely determined in its topology, its differential properties, in its metric – a space that is very "structured"? Or, conversely, would one refer to a very "unstructured" space, composed of simple sets of points, which enable, in the manner of Cantor, even if that means a loss of all continuity, of all notion of neighborhood, to establish a bijection between the plane and the straight line that is far removed from the first phenomenological intuition of the space in which our body is located and evolves?

We must note, at this stage, that the reference systems used in contemporary physics have recourse to "spaces" presenting somewhat intermediary properties: they do in fact renounce the absoluteness of a very strong if not complete determination (the space of Newtonian and of classical mechanics), but they do not go as far as the complete parcellation as presented by Cantorian sets of independent points. Hence, they conserve an important structure in terms of continuity or connexity all the while losing, via the properties of homogeneity or of isotropy, a number of possible structural determinations. Moreover, invariance by symmetry is sufficiently constrictive or structuring to preclude the definition of an absolute origin, of a privileged direction, or of many other "rigid" properties. Noether's theorem, which we will address in length particularly in Chapters 3 and 5, establishes an essential correlation between these properties of time/space symmetry and the conservation of certain physical magnitudes (energy, kinetic moment, electric charge, ...) that characterize the system, in its profound identity and in its evolution. It appears that it is these properties of symmetry (of invariance) which lead us to characterize the reference space's relevant structure, which is neither too strong nor too weak.

So there would be a sort of theoretical equivalence, in physics, of this type of mathematics where a too strong "structure" would only enable us to construct "isolated" and specific objects, whereas a little bit of structural relaxing would enable us to characterize similarities and to elaborate categories and relations between them.

1.1.1.2 *Concerning the term "foundation"*

The term "foundation" raises similar questions. It appears indeed that the term may be articulated into two quite different concepts.

It could stand for the evocation of an *a priori* origin that is associated with a first intuition, and from which would historically be deployed the theoretical edifice (the phenomenological intuition of continuity or of number, for example) and which would remain as an hermeneutic insistence of the question originally posed. Foundation would then have a genetic status, and it would provide a proof of the hermeneutic relevance of an issue by the ensuing theoretical fecundity. In this sense, the foundation would appear as the basis for any ulterior development or construction. We may ask ourselves if it is not in this epistemological or even genetic sense that we should use the term here.

In contrast, however, and historically, the term “foundation” may also designate an *a posteriori* structure that is quite formal or even completely logicized, and which presents itself as the result of a theoretical evolution and as the very elaborate product of a rational reconstruction which enables us to reinterpret and to restitute all of the concept’s anterior work (as in the case of a Hilbertian axiomatic reconstruction, for example). It is partly in these terms that the problematic of foundations was articulated after the crisis at the beginning of the XXth century.

Beyond their contrasting positions regarding the “temporality” of the theoretical/conceptual work (an origin difficult to assign in one case, posteriority never achieved in the other), do these two meanings not underlie different representations of essentiality? As indeterminate as it may still appear although being rich in terms of the ulterior developments it is likely to generate, do we not find different philosophies of knowledge in the very nature of the question which is posed, it being in the first case genetic and in the second case formal? And do we not find differing philosophies of knowledge in the nature of the answer, capable of reinterpreting and of conferring meaning to the effort of which it embodies the outcome and which it summarizes?

Besides these issues, there also arises the question relative to *invariance*: structural invariance and conceptual stability, which Longo makes into one of the most important characteristics of mathematics as a discipline, and which needs to be addressed in length. Now, in the problem at hand, invariance itself seems to take two aspects: on the one hand in the insistence of the original and still relevant question (the question of the continuum, for example, which spurs increasingly profound research), and on the other hand, in the formal structure revealed by research and which, once constructed, presents itself in a quasi atemporal manner (the proof-theoretic invariance related to the validity of Pythagoras’ theorem, for example).

Would it not be this which would enable us to explain the double characteristic associated with the concept of foundation but also the double aspect of structure, according to whether it has recourse to a richness founded upon a semantic intuition or to a rigor compelled by formalism?

1.1.2 *The genesis of mathematical structures and of their relationships – a few conceptual analogies*

Still concerning mathematical structures, it is a question here of raising and briefly discussing the issues relating to their genesis and not only to their history. So operating an important distinction between the genesis of mathematical structures themselves and the reconstructed genesis of their relationships, the one and the other appear to stem from different approaches, conceptualizations, and relational processes and to involve distinct cognitive resources.

As for the genesis of mathematical structures, one may distinguish an historical genesis as such which can be retraced and located within a timeline as well as a conceptual genesis of which the temporality is clearly more complex. Naturally, the first is an object of the history of mathematics, in terms of the discoveries and inventions which do not need to be addressed here. The second is of a quite different nature: a given concept or approach having had its time of preponderance is re-proposed by others and then reappears, is developed and is forgotten again until it resurges later (as was the case in physics with the atomic hypothesis, for example, and as is the case with mathematical infinity, which also has had a complex historical specification). Another concept appearing to be autonomous and original, even unique like Euclidean geometry, may finally prove to constitute a particular thematization of a more general trend to which it will be associated from then on and which endows it with a different coloration (this was also often the case in number theory, firstly with the appearance of negative numbers, and then of complex numbers, or was also the case with prime numbers and ideal numbers). The work underlying this evolution is that of the concept, of its delimitation and of its generalization. This work is not linear and unidirectional: it returns to previous definitions and developments, enriches them, modifies them, uses them to generate different ramifications, reunifies them and retranslates them, one into the other. In this way, it internalizes the historical temporality which generated these concepts and makes it into an interpreting and interpretive temporality. As recalled earlier, this is what justifies, with the philosophical approach the author associates with it, the qualification of *formal hermeneutic*, which Salanskis (1991) conferred to it, using as characteristic examples the theories of continua, of infinity, and of space. The approach proposed by Longo, and to which we will return in this book, also consecrates the existence of a hermeneutic dimension to the genesis of mathematical structures, as would suggest the importance he gives to *meaning*, beyond that which is purely syntactic.

The issue of the genesis of *relationships* between structures stems from a quite different problem. This genesis also presents two aspects according to the approach one would favor: the aspect usually characterized as *foundational* and the aspect which we qualify as *relational*.

The foundational aspect corresponds in sum to the formal/set-theoretical approach. It is characterized by the search for the most simple, intuitive or elementary foundations possible, from which mathematics as a whole can be re-elaborated in the manner of an edifice, progressively and deductively, going from the simplest and most elementary to the most complicated or sophisticated. From this point of view, this genesis may be considered as marked by the irreversibility of the process and it is quite naturally associated to a typification of stages. This is supposed – in the first formalist programs – to put into correspondence what we qualify as proof principles on the one hand, and construction principles on the other. Revealing that the former did not coincide with the latter was one of the effects of incompleteness theorems.

In contrast, the relational aspect is more of an intuitional/categorical nature: the structures mutually refer the former to the latter in a network more than they follow one another while overlapping. Interreducibility manifests itself diagrammatically and that which is fundamental lies in the isomorphy of correspondences much more than in a presumed elementarity. There, the mismatch between proof principles and construction principles revealed by the incompleteness theorems, to which Longo will need to return in his response, is no longer really a problem because the issue is no longer to make progression from the foundational coincide with the elaboration of structures. Nor does the recourse to impredicative definitions pose a problem, as we will emphasize, since the network in question is not meant to be conceptually hierarchized in the sense of set theories.

So, just as the foundational genesis – of the set-theoretical type – of the relations between structures evokes the correspondence with a sort of external (logical), one-dimensional and irreversible temporality, relational genesis – of the categorical type – refers to a specific, internal temporality, which is characteristic of the network it contributes to weaving. How could we characterize this characteristic temporality in a way that would not be purely intuitive and which would engage a process of objectification?

Seen from this angle, the genesis of structures and the genesis of relationships between structures, despite their profound differences, appear to form a pair, articulating two distinct temporalities, the one defined somewhat externally and the other derived from an internality (or regulating it). Thus, these temporalities specifically relating to mathematics offer troubling analogies with certain aspects of theoretical biology which itself copes with two types of temporality: the physical temporality of the external relationships of the organism to its environment (which presents all the characteristics of physical time, modulo the solely biological relationships as such between stimulus and response) and the intrinsic temporality of its own iterative rhythms defined not only by dimensional physical magnitudes (seconds, hours, ...) but by pure numbers (number of heart beats over the life of a mammal, number of corresponding breaths, etc). And in the case of the relationships between structures, the parallel may appear to be particularly significant: the irreversible "time" of the foundational derivations associated with set theories echoes the external physical-biological time of the succession of forms of life, in a sort of common logic of *thus, ... hence ...*; while the temporality specific to the categorical relationships of networking rather resonates with the biological time specific to the "biological clocks" (which we will address in Chapter 3) – morphogenesis, genetic activations, physiological functionings – according, this time, to an apparently more restrictive logic in terms of ontological engagement but which is more supple in the opening of possibilities, of the *if ..., then ...* .

Furthermore, it must be noted, curiously (but no doubt fortuitously, given the dynamic and temporalized representations which are often at the origin of the intuitionist and constructivist approaches), the relational construction of the relationships between structures tends to mobilize a semanticity quite akin to the auto-organizational version of the biological theories (Varela, 1989). Indeed, the tolerance relative to impredicativity and self-reference is in tune with the self-organizing (and thus self-referential) approach to the organism (reevaluated relationships of the self and non-self, couplings between life and knowledge, recourse to "looped" recursion, etc). The categorical closure involved with this same constructive approach evokes the organizational closure associated with the "self" paradigm in order to delimitate the identities and qualify the exchanges. Here lies one of the possible sources of the conceptual connections we propose to operate between mathematical foundations and possible theorizations in biology.

In conclusion, if one accepts this analysis, then the term of *construction*, of which the scope proved to be so important in both the epistemology of mathematics and in philosophy (Kant notes that if philosophy proceeds by means of concepts, mathematics proceeds on its part by means of the construction of concepts), is both destabilized and enriched. In particular, it is brought to bear two distinct meanings which do not mutually reduce themselves to one another.

On the one hand, indeed, we would have a construction that is irreversible in a way (at least *a posteriori*) which leads from infrastructures to superstructures: the construction of ordinals from the empty set, for example, or of rational and real numbers from natural numbers, etc. as presented by the formalist/set-theoretical approach. And on the other hand, we would have a construction that is much closer to that which extends to characterize intuitionism and constructivism as such, which principally concerns the relationships between mathematical structures as presented by category theory: not only from structural genesis to structures but also restitution of the effective processes of constitution. Integer numbers, we will claim, are constructed and grounded on the manifold experiences of ordering and sequencing, both in space and time: the intuition of the discrete sequence of a time moment, for example, is at the core of the intuitionistic foundation of mathematics. Yet, the constituted invariant, the concept, we will argue, requires many active experiences to attain the objective status of maximally stable intersubjective knowledge.

The role of time in the construction of knowledge will lead us to raise the issue of the resulting concepts of temporality, by attempting to put them into relationship with the concepts of temporality that can be found in physics, but also in other disciplinary fields, namely in biology. More generally, these considerations will lead to questions relative to putting into perspective the term of "conceptual construction" and to the delimitations of the meanings the latter may bear in distinct scientific situations.

1.1.3 *Formalization, calculation, meaning, subjectivity*

1.1.3.1 *About "formalization"*

The first question still seems to concern terminological aspects, but its clarification may have a greater epistemological dimension. In many disciplines, notably in physics, the term "formalization" (and its derivatives) is virtually equivalent to that of "mathematization" (or, more restrictively, of "modelization"). This is visibly not the case in mathematics and logic, where this term takes a meaning which is much stronger and much more defined. Indeed, in the tradition of prevailing foundational programs, this term is used in a strictly formalist sense which is, moreover, resolutely finitary. It is not surprising that the term "formalizable" is almost synonymous with "mechanizable" or "algorithmizable." How then may we qualify other "formalizations" which, all the while remaining within the framework of logic, do not really conform to these very constraining norms, while nevertheless presenting the same rigor in terms of reasoning and proof? If, as observed by Longo (2002) the notion of proof in mathematics, as opposed to Hilbertian certitude, is not necessarily decidable (a consequence of Gödelian incompleteness, to which we will return), then, what will explicitly be, if this is possible to explain, the objective criteria (or at least those which are shared consensually, intersubjectively) that will enable the validation of a deduction?

Moreover, in a somewhat similar order of ideas in what distinguishes the formal from the calculable, how is the situation, regarding the issue of infinity, which appears in non-standard analysis where it is possible to have formally finite sets (in that they are not equipotent to any of their own parts) that are, however, calculably infinite, in that they comprise, for instance, infinitely large integers? Would it be an abusive use of the concept of "formal" (or of "calculable")? Is there an abusive mixture between distinct logical types, or is it of no consequence whatsoever? Would it suffice to redefine the terms? Besides, concerning the reference to (and use of) "actual infinity" in mathematics, questions arise regarding Longo's precise stance, a position which we would like to develop in this text. His intuitionist and constructivist references seem to lead to a necessity to eliminate this concept, whereas he appears to validate it in its existence and its usage within mathematical structures as in proofs.

1.1.3.2 *Regarding the status of "calculation"*

In what concerns the issue of "calculation", let's note that in physics, each calculation step associated with the underlying mathematical model does not necessarily have an "external" correlate, in physical objectivity (moreover, it is possible to go from the premises to the same results by means of very different calculations). Conversely, it would appear that from a mathematically specific point of view, each stage of a calculation must have an "external" correlate (external to the calculation as such), that is the rules of logic and reasoning which authorize it.

Similarly, mathematical "inputs" (axiomatic, for example) and the corresponding "outputs" (theorems) seem relatively arbitrary (they only satisfy the implication of the "*if ... then ...*"), whereas physical "inputs" (the principles), like the outputs (observational or experimental predictions) are narrowly constrained by the physical objectivity and phenomenality which constrain the mathematical model.

Hence the following questions: how and where does the construction of meaning in mathematics occur, as we present it with regards to calculation? What about the "significations" associated with the very rules which regulate it?

1.1.3.3 *On the independence of certain results and the role of significations*

Apart from many other reasons, it is necessary to relate here (see also Chapter 2) results of incompleteness and undecidability in order to produce a critique of the formalist program and of the approach it induces with regard to structure (strictly of a syntactic nature) and to the absence of signification of proof. Likewise for the independence of the Continuum Hypothesis (the impossibility of constructing cardinalities between the countable infinity and the continuum of the real numbers), to which Longo confers somewhat the same critical role, but rather in relationship to formal set theory. However, one can only observe that the history of geometry is marked by similar problems, for instance, with the fact that after centuries of unfruitful research of the independence of Euclid's fifth axiom of which the negation opens the way to non-Euclidean geometries. Hence the question: would we make the same sort of critique regarding the geometric axiomatic, or would we consider that it is in fact a different approach inasmuch as it directly involves "significations" (and in what way)?

But we do know, on the other hand, that such significations mainly based on perception or on language habits may be misleading (illusions, language effects, ...). So how may we preserve what we gain from taking distances from a sort of spontaneous semantic, all the while conserving the dimensions of "meaning"? Would we not be brought to relativize this "meaning" itself, to make it a tributary of the genesis of mathematical structures and proof, to historicize it, in the way that mathematical (or physical) intuition historicizes itself in close relationship with the conceptual evolution of which the forms, content, and avenues change in function of the knowledge cumulated?

1.1.3.4 *On the relationships between rationality, affect, and objectivity*

If all do agree to acknowledge the importance and operativity of affects and emotions for scientific creativity and imaginativeness, is there not, however, a double stake of separation between the two, at the two opposite poles of the scientific approach? At the origin of this approach, including from an historical standpoint, the separation would be between the rational approach and magic (or myth), in the view of achieving objectivity and rationality. Following the process of creation (or of comprehension, that is, in a certain sense, of re-creation), the separation would be between the singular specificity of the subject involved by emotions and the constraint of the communication necessary to the establishment of an intersubjectivity, which would enable the construction of objectivity.

The excess of formalism, from this standpoint, would have been to confuse the condition which enables this latter distinction with the elimination of the significations themselves by seeking to reduce the objective construction to a play of "pure" syntax. In this regard, the formalist approach is not exempt of an interesting historical paradox in that it was initially conceived in relationship to an indubitable ethical dimension, very rich in significations. Leibniz, in his search of a universal characteristic explicitly had as one of his objectives the elimination of violence from human inter-relationships, the *calculemus* being meant to replace power struggles, by the very fact that subjective interests would be eliminated. And this concern – if not its effective actualization in the form of logicism – should be authenticated. In this sense, the only alternative would indeed appear to reside in the construction and cognitive determination of these *cognitive invariants* such as evoked by Longo.

But do we currently have the capacity to identify the invariants (of a cognitive nature and no longer only disciplinary) which would enable for their part to ensure this construction of objectivity in a way that would preserve these significations (or at least, in a way which would preserve its generativity) without, however, including singular idiosyncrasies? The effective practice of many disciplines – including mathematics where the referent is purely conceptual – does seem to indicate that this should be possible. Yet, a more specific and general statement of the ins and outs of these cognitive invariants remains much more problematic.

1.1.4 *Between cognition and history: Towards new structures of intelligibility*

The attempt to analyze processes of construction of knowledge and of constitution of scientific objects, which we will discuss here, does not exactly fall under the framework of the cognitive sciences, nor *a fortiori* within those of the neurosciences. It is rather to be considered as an approach which we could call "complementary," all the while referring often to human cognition. We will do this by analyzing the foundations of mathematics and the steps governing the construction of scientific objectivity and knowledge in their contemporary dynamism inherited from a history clarifying its sources.

First, the development of mathematics, then of physics, and of biology today, presents characteristics, which are sometimes similar, but often different, and which illustrate our cognitive capacities and their operativity in fields which stand among the most advanced; thus, the analysis of these developments provides an area of study and reflection which seems particularly adapted to our analyses and to the conceptual frameworks which we propose.

The history of sciences such as those being conducted today demonstrate quite well, during the stages of their deployment, the extent to which, by means of various evocative metaphors as well as by thorough inquiries, these disciplines have served as a source of inspiration and as examples for the investigation of nature and of the modes of functioning of human cognition. Not that everything of the human would resemble or conform to this: far from it, actually, because there is for instance a lack of the other dimensions of a specifically ethical or artistic nature (although a certain form of estheticism of theories will be present and have its effect on the scientist), but the scientific domain presents itself as the locus of a major advance in terms of human rationality and we believe that this is called to be reflected in most human activities within society.

So this was for the global perspective. Regarding the more specific contents, if we want to be both rigorous and operative in our approach, we cannot make as if they escaped from their constitutive history or to the interpretive controversies which they have caused and continue to cause. We may recall foundational crises in mathematics and formalist drifts, foundational crises in physics and the emergence of very novel and counter-intuitive theories that are subject to conflicts of interpretation, the eruption of biology as a new and expanding field, which is, however, threatened by the reductionism of the "all genetic" of which the success are nevertheless obvious, etc. All these aspects will therefore be the more present in our discussions inasmuch as they enable us to better understand the stages to which these disciplines have developed and that their trace remains within the very content of the theories. As there is no *plausible epistemology* without a quasi-technical mastery and quasi-internal analysis of the scientific contents, we believe that there is no plausible approach to the conditions of development and the properties of operativity of human cognition without a thorough investigation of the fields where it has been deployed in a privileged fashion.

1.1.4.1 *Memory and forgetfulness in mathematics*

An example of this paradigmatic play between human cognition and history is the reference we will make to "memory." We will return to this in Chapter 2 and address the issue of mathematical temporality via that of memory (of concepts, of the methods of its elaboration, its transformations, etc). For now, let's emphasize that we are proposing here a venue aiming to identify the characteristics of this specific temporality which is related to the construction of mathematical objectivity and which manifests in the genesis of its structures. What this approach suggests, with regard to significations, appears to be quite interesting and convincing. We will also address, but much more briefly, the issue of forgetfulness in the constitution of conceptual and mathematical invariants, namely by invoking the relationships between consciousness and unconsciousness. It is, however, in regard to this aspect of forgetfulness (and of a certain form of atemporality which appears to be related to it) that an apparently important question may be posed. Now, as constituents of the invariance and conceptual stability which characterize the mathematics often highlighted by Longo, memory and forgetfulness are to be understood as cognitive phenomena, specific to the human individual all the way from his animality to the communicating community that characterizes his humanity, but they must also and therefore be understood as historical phenomena. Forgetfulness of "that which is not important" (relative to a point of view, an objective, an intention) and selective memory of "that which is relevant" contributes in constituting the objectivity and very object of knowledge itself, as an invariant of manifold active experiences.

The culminant point of forgetfulness and of atemporality, in what concerns the foundations of mathematics, seems to have been reached with the formalist/logicist program and the elimination of the meanings which it explicitly advocated. This approach did obliterate memory, but intended on the other hand to offer the advantage of complete communication and of an alleged universality (independently of distinct cultures, of particular traditions, of subjective singularities and thus, in its spirit, independently of sterile confrontations as well as of conceptual ambiguities). Without returning to previous questions, how may a revelation of these cognitive and historical invariants that are still theoretically problematic, but apparently empirically proven, enable us to articulate the plays and interactions of this necessary constitutive memory and of this forgetfulness which is also constructive? How would it enable us to articulate this temporality specific to mathematical genesis (or even to scientific concepts) and this coefficient of atemporality enabling both intercultural validation and accumulation?

1.2 Mathematical Concepts: A Constructive Approach

(by Giuseppe Longo)[2]

1.2.1 *Genealogies of concepts*

Let's more closely tackle now the idea of a parallel between the constitution of mathematical concepts and of physical objects. We will only be able to respond partially to this inquiry and shall rather reflect upon the meaning of the relativizing constructions specific to mathematics and to physics, within an explicative and foundational framework inspired by the arisen questions. We already hinted at the identification of mathematical and physical "construction principles." But our project is wider, because it is a question of grounding the two "constitutive histories" within our worldly living being, to grasp this biological and historical "cognitive subject," which we share and which guarantees us the objectivity of our forms of knowledge. It is not a question of unifying by force the epistemologies of differing disciplines, but to make them "exchange between themselves," to reveal the reciprocal dependencies, the several common roots. The analysis we propose here will thus base itself upon the following principles:

- The problem of the foundations of mathematics is (also) an epistemological problem.

[2] This section by Longo, jointly with the Introduction above, appeared in **Rediscovering Phenomenology** (L. Boi, P. Kerszberg, F. Patras ed.), Kluwer, 2005.

- Any epistemology (of mathematics) must refer to a conceptual genesis, as a "process of construction of knowledge."
- The epistemology of mathematics is an integral part of the epistemology of the sciences (the exact sciences, at least).
- A constitutive element of our scientific knowledge is the relationship, established in the different sciences, to space and to time.

In short, a sensible epistemology of mathematics must try to explicate a "philosophy of nature," a term which is dear to the great minds of the XIXth century. As it is, mathematics is one of the pillars of our forms of knowledge, it helps to constitute the objects and the objectivity as such of knowledge (exact knowledge), because it is the locus where "thought stabilizes itself"; by this device, its foundation "blends" itself to other forms knowledge and to their foundations. Moreover, the conceptual stability of mathematics, its relative simplicity (it can be profound all the while basing itself upon stable and elementary, sometimes quite simple, principles) can provide the connection which we are looking for with the elementary cognitive processes, those which reflect some of the world's regularities in our active presence within that same world, as living beings (and living in intersubjectivity and in history). For these same reasons, the theories of knowledge, from Plato to Descartes, to Kant, Husserl or Wittgenstein, have all addressed the question of the foundations of mathematics, this "purified knowledge," both mysterious and simple, where notions of "truth" and of "proof" (reasoning) are posed with extreme clarity. The problem of the cognitive foundations of mathematics must therefore be analyzed as an essential component of the analysis of human cognition. Within that framework, we will attempt to analyze in what sense "foundations" and "genesis" (cognitive and historic) are strictly related. The very notion of "cognitive foundations" explicitly juxtaposes foundations and genesis.

In this study, the notions of time and space, which we use, do not refer to "natural entities," but rather to the play between sensible experience and conceptual frameworks which allow the natural sciences to manifest themselves. That was in fact the inquiry of the great geometers (Riemann, Helmholtz, Poincaré, Enriques, Weyl, ...) who tried to pose the problem of the foundations of mathematics within the framework of a philosophy of nature. But the analysis, which came to dominate afterwards, stemmed from a very clear division between logical (or formal) foundations and epistemological problems, particularly those presented under the form of this relationship to time and space which ground mathematics in this world.

Frege explicitly denounces the "delirious" situation in which the problem of space finds itself, because of the emergence of non-Euclidean geometries (Frege, 1884), and proposes a "royal way out," by laying the bases of a new discipline, mathematical logic. Mathematics itself is the development of "absolute laws of thought," logical rules outside of this world and independent of any cognitive subject. For that, Frege introduces a very clear distinction between "foundations" and "genesis," he breaks any epistemological ambition, all the while attacking "psychologism" (as of Herbart/Riemann) and "empiricism" (as of John Stuart Mill). The former try to understand which "hypotheses" (which "*a priori*") allow us to make physical space (and time) intelligible to the knowing subject, while the latter relates mathematics to a theory, alas too naive, of perception. Faced with all these first attempts at a "cognitive analysis" of mathematics, Frege proposes a philosophy centered upon a very inflexible dogma, the logicist dogma, according to which mathematics has no psychologico-historical or empirical genesis. It is, according to him, a constituted knowledge, concepts without conceptors. This philosophy, this dogma, is at the origin of the fundamental split, which will accompany all of the XXth century, between foundational analysis and epistemological problems, between mathematics and this very world it organizes and makes intelligible.[3]

Moreover, for Frege, geometry itself, as given by numerical ratios (Frege, 1884), bases itself on arithmetics; and the latter is but the expression of logical laws, because the concept of number is a logical concept and induction, a key rule of arithmetics, is a logical rule. Finally, the continuum, this difficult stake of phenomenal time and space, is also very well mathematized, in Cantor-Dedekind style, from arithmetics.

So there are the problems of time and space and of their mathematization, neglected to the benefit of their indirect foundation, via arithmetic, upon logic; pure concepts, with no relationship whatsoever to sensible experience nor to physical construction. Conversely, this relationship was at the center of the inquiry of the inventors of non-Euclidean geometries: Gauss, Lobatchevsky or Riemann did not play the logical negation of Euclid's fifth axiom and of its formal developments, but they proposed a "new physics,"

[3] For us, however, the "almighty dogma of the severance of principle between epistemological elucidation and historical explicitation as well as psychological explicitation within the sciences of the mind, of the rift between epistemological origin and genetic origin; this dogma, inasmuch as we do not inadmissibly limit, as it is often the case, the concepts of "history", of "historical explicitation" and of "genesis", this dogma is turned heads over heels" (Husserl, 1933: p.201).

a different organization of the world (see Lobachevskij, 1856; Riemann, 1854). It also happens that the numerical relationships may possibly found Euclidean geometry, but surely not other geometries, because Euclidean geometry is the only one which preserves these relationships (it is the only one whose group of transformations – of automorphisms – which defines it, contains the homotheties[4]).

Now it is doubtless that mathematics has a logical as well as a formal foundation (a distinction will need to be made here), but it is in fact a "three-dimensional" construction. It constitute itself within the interactions of the logical and totally essential "if ... then" (first dimension), of perfectly formal, even mechanic calculus (second dimension), but also in a third conceptual dimension, these constructions of (and in) time and space, which mingle it, even more so than the two others, with the different forms of knowledge. And the epistemological problem then poses itself as an analysis of the constitution of the invariants of language and of proof, these invariants which we call "logic" and "formal systems," as well as the invariants of time, and of space, upon which we construct our geometries, these "human constructs ... in our spaces of humanity" as Husserl says in the "Origin of Geometry" (see below). The problem is thus posed from the analysis of this very peculiar form of knowledge which is mathematics, from its cognitive roots, be they pre-human, to its communicable display, with its thousands of mediating levels.

Axiomatic conventions and logico-formal proof are actually but the ultimate results of a constitution of meaning, common notations of concepts rooted in "our living practices," to put it as Wittgenstein would do, in our "acts of experience" (Weyl): logico-formal analysis is a necessary accompaniment to this latter part of the epistemological process, the analysis of proof, of certain proofs, but it is insufficient (it is essentially "incomplete," some theorems tell us). The foundational analyses of mathematics must thus be extended from the study of deduction and of axiomatics to that

[4]Hilbert, as a great mathematician, will manage quite well otherwise. Thanks to the Beltrami-Klein interpretation of non-Euclidean geometries within Euclidean geometry, he will give a correct immersion (interpretation) of his axioms for geometry within arithmetic via analysis (Hilbert, 1899). But, for the latter, he will not look for a "logical meaning," unlike Frege. Indeed, once geometry (ies) is (are) interpreted within arithmetic, a finitary proof of its (thus, their) coherence (of non-contradiction) would suffice for its foundational analysis, entirely and exclusively centered around its problem of coherence; its a pity it doesn't work, because arithmetic does not have, itself, any arithmetic (finitary) proof of coherence. To the contrary, we manage with an infinite piling of infinities, or by proof founded upon geometrical judgments, see Chapter 2.

of the constitution of concepts and of structures; but this is impossible without a parallel analysis of the constitution of the physical object and of perception.

1.2.2 *The "transcendent" in physics and in mathematics*

There is no doubt that there exists a reality beyond ourselves, which enters into "friction" with our actions upon it and which, moreover, "canalizes" them. Husserl uses a word from the idealist tradition to designate this reality: he considers the notion of transcendence. In a very common interpretation of this word, and quite independently from Husserl, the following deduction is usually made, first in physics, then in mathematics: the "properties" of the world (physical, numerical, mathematical, ...) are transcendent and, moreover, are not all known. They are therefore "already there," they pre-exist. The objects of the world around us have well-established properties that are quite stable and invariant in relation to our senses: I look at this pencil, I touch it, even its odour confirms its "objectivity," independently of the specific sense I use to explore it ..., it is thus already there, it pre-exists my explorations, with all its properties. In a completely analogous manner, the properties of numbers, of mathematical structures do not depend on notation (for numbers: decimal, binary ...) nor on other details of representation, of the mathematician exploring them ... therefore they pre-exist.

Now it is the word "property" – in physics, in mathematics – that must first be agreed upon: a property is "talked about," it is first of all an expression in these languages through which we try to speak of the world, to organize it and to give it meaning, a meaning shared with others. But the world canalizes our efforts to obtain knowledge and displays some resistance (causes friction) to our propositions to organize it. "Properties," as we render them through intersubjectivity by words, are not in themselves isomorphic to absolute facts that are "already there," possibly well established or that would manifest themselves under well established forms of linguistic structures; by our active gaze, in our exchange with others, we propose a structure with hints of a reality which is there, as unorganized frictional matter. Thus, through language, pictures, gesture, we unify certain phenomena, we draw contours upon a phenomenal veil, which is an interface between the world and us. The transcendent is constituted, it is the result of a constitutive activity, of a process which precedes the individual or that the individual performs mostly with others. This process is best synthesized as the result of a *transcendental* (and not transcendent) activity, and such is the lesson we draw from Husserl.

It is no coincidence if the many examples of objects proposed by ontologizing philosophies, in mathematics, in physics, refer to medium size manufactured objects, all the while attempting to escape the problem of cognitive relativism. These thinkers of ontology, of essences, rarely refer to the "objects" of quantum physics, for example, in order to propose an ontology that is much more difficult to take on, of the electron, of the photon ... But even these medium size manufactured objects, of an apparently such simple ontology, if it is true that they are really there, are just as much constituted as the concept they are associated with. The pencil is constructed, in history, at the same time as the concept of the pencil. Both are related to drawing, to writing, as human activities. They are pre-existent, the object and the concept, for the individual subject, they are not so for humanity, in its history. There was no pencil, nor table, nor a pot such as the one laid on Kurt Gödel's table, before the beginning of our human acting and thinking. On the other hand, there was surely already a physical "reality" (for Galileo, less so for Tales), but its organization and its interpretation as photon, electron ... robust, stable, in fact mathematical, was not yet there, nor was it's organiation into pots, pencils, and tables before the blossoming of our humanity. And this approach, we think, does not face the dangers of relativism, because the objectivity of the constructed, of the concept, of the object, lies in the constitutive process, which is itself objective.

Cassirer, quoted by Parrini in a work whose goal is to overcome the fracture between absolutism and relativism, partially addresses this theme (Parrini 1995, p.118): "if we determine the object not as absolute substance beyond all knowledge, but as object which takes form within the progression of knowledge itself," then, "this object, from the viewpoint of the psychological individual, can be said to be transcendent," despite that "from the viewpoint of logic and of its supreme principles," it must "be considered as immanent." Ideality, the concept as "conceived," "a cut-out" ("decoupage") performed upon the world in order to give it contours, to structure it, will thus detach itself from subjective representation, despite that it may have its origins within the community of subjects, in what they share: similar bodies and brains from the start, in the same world, and all that which they build in common, in their common history. It is thus not a question of writing a history of individuals, but of tracing back the origin of an idea; not historicizing relativism, but a reference to history as an explication of our "being together in the world," the locus of the active constitution of all our forms of knowledge.

In the case of the objects of physics, of microphysics in particular, this activity of the construction of objects by "conceptual carving" is rather clear: electrons, muons, fermions, quantum fields ... are not already there. They are concepts that are proposed in order to unify, to organize, to understand the signals the world sends us. These signals are not arbitrary and they are also the result of an active exploration. In order to obtain them it was necessary to develop rather complex measurement instruments, which are themselves the result of a theory. All the instruments for physical measurement, and more so those of microphysics, are constructed after an enormous theoretical commitment: I want to measure this but not that, by using these materials but not other ones, I "look" here and not there. The "facts" which result from this, as Goodman would say, are thus "small-scale theories" themselves.

Let's consider for example the wave–particle duality in quantum physics. The photon, the electron, present themselves as "waves" or "particles" depending upon the experimental context: specific instruments are put into place, in fact the experiment is prepared from the viewpoint of a certain theory .... The object that will result from this will depend as much upon the theoretico-experimental framework as it will upon friction – "the canalization of thought" that nature imposes upon and through these tools. A certain viewpoint will show us the particle, another will show us the wave. More precisely, we will obtain macroscopical properties on a screen, on a detecting device, and by a process just as important, we will interpret them as symptoms of the "existence" of a particle or of a wave. There is no duality as such for the physical object, but a context of reconstruction of the world where we are as present as the object under observation.

Properties, then, are the "explicated" result of an organizing of clues, of a group of facts, which are themselves "little theories." But reality is there, doubtlessly, because it canalizes our efforts to obtain knowledge in non-arbitrary directions, it causes friction, by opposing itself to our theoretical propositions, great and small, these "properties" spoken of in our languages. The transcendency of these properties, as if they were already constituted, as "ontologies," is a "*flatus vocis*" to which we contrapose the constitutive process of the transcendental which is at the center of Husserl's philosophy. It is our task, when referring to different forms of scientific knowledge, to enrich and to specify this so very fuzzy word, the notion of "property" for the physical world, as well as that of mathematical property.

1.2.2.1 *Transcendence vs transcendental constitution: Gödel vs Husserl*

So let's move on to mathematics. In this discussion we refer to one of the most interesting among thinkers having an "ontologizing" tendency (and one of the greatest mathematicians of the XXth century), K. Gödel. Actually, Gödel also proposes a strict parallel between physical objects and mathematical concepts, although from a perspective different from ours (the similarity of "ontologies" or of "independent existence"): "It seems to me that the assumption of [mathematical] objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence" (Gödel, 1944) ..."the properties of these concepts are something quite as objective and independent of our choice as physical properties of matter ... since we can create [them] as little as the constituent properties of matter" (Gödel, 1947). So, physical bodies, and constituent properties of matter, as well as mathematical concepts are all preconstituted entities, possibly the ultimate building blocks, independent of or transcending the cognitive subject (not "created"). Again, even the word property, as referring to outside objective states of affairs, is used in a naive, ordinary way, even for constituent elements, it seems, whose analysis belongs to the entangled constructions of microphysics, where the constitutive polarity "subject/object" is at the core of the modern perspective in quantum mechanics (indeed, since the 1930s).

In his masterpiece about the foundations of mathematics, "The Origin of Geometry," Husserl frequently emphasizes the role of the transcendental constitution of mathematical objects. The epistemological problem they pose is, for him, a "problem of genesis," a "historical problem" (see the footnote above). Geometry, as an attempt (and mankind makes many) to make space intelligible is the result of an activity by "our communicating community"; it is "the constituted," the result of a non-arbitrary process, which grounds our constitutive hypotheses within certain regularities of the world, regularities, "donations" which impose themselves upon us; these regularities are themselves "already there" (the connectivity of space, isotropy, symmetries – inspiring ourselves by Riemann and Weyl). But it is us who choose to see them.

I have a Jovian friend who has five legs, three and a half eyes and no, absolutely no, symmetry to his body. He sees not or does not give any importance to the symmetries of light reflected by a surface, or to crystals, for example, these symmetries which are before our eyes, before his eyes;

and his mathematical structures are not imbued with symmetries like ours (from Greek geometry of the dualities and adjunctions so well described in the Theory of Categories). They are rather constructed around "zurabs," an essential regularity from his perspective, but which we do not see or which we neglect. It goes likewise for colors; he sees a bandwidth beyond violet, where one can find, as a matter of fact, splendid colors. He, therefore, cannot appreciate this marvelous human construction, rich in history, that we call "painting": Titian's colors are invisible for him. Just like we do not see his masterpieces, of such beautiful ultraviolet colors.

The two constructions are not arbitrary, light's waves (or the reality we categorize as such) "are there," just as are the symmetries of crystals or of light bounces, but our active presence interacts with these elements of reality in order to choose, emphasize, correlate some of them, but not others, to gives names, not arbitrary names because they are rich in history and in meaning, to certain color bandwidths and not to others. Moreover, our action interpolates the missing elements, proposes links by analogies derived from other experiences; it integrates a variety of acts of experience in order to create a new structure, an inexisting network between "the things" of the world. To figure out, among the regularities of the world and among the foundational acts of any form of knowledge, which ones are at the origin of mathematics, is one of the tasks of the analysis of the cognitive foundations of mathematics. Husserlian phenomenal analysis may be one tool, if we do not limit ourselves to a fuzzy notion of "transcendence," but if we recover the richness of "transcendental constitution," as we did. Unfortunately, most anti-formalist mathematicians, and even the greatest of mathematical logic, such as Frege and Gödel, insist upon the "transcendent" ("the properties and the objects of mathematics pre-exist, just as do the properties and the objects of physics"). In fact, Gödel, while knowing Husserl, does not refer to the "genesis," to the "history" (in the sense of Husserl (1933) of this constitution at the center of our conceptual constructions.[5] Gödel thus

[5]See the discussions reproduced in Wang (1987). Follesdal (1999) makes the generous effort of reading some of Husserl in Gödel. However, for Gödel, the existence of mathematical objects is as external to us as that of physical objects, in that both types pre-exist: "they are independent of our definitions and our constructions"; the intuition of mathematical objects (sets, actually) is a form of "physical perception" (Gödel, 1944, supplement in 1964), in the most naive sense of the term "perception," a sensorial "input" that reaches us as -is. We will hint to the profoundness of Poincaré's sketch of a theory of perception, for example, where we find a true attempt at epistemology in mathematics, rooted in a "philosophy of nature." In the many papers that had a major influence on the contemporary philosophy of mathematics, there is nothing but transcendence in Gödel, even in the quotations chosen by Follesdal, without all the remainder of the phenome-

remains, in mathematics and in physics, at a stage of a realism, which neither specifies the notion of property nor that of object: it has only the objects and the properties derived "from sensations," properties of a physics of "medium size objects" (this table, a pencil ...), a physics which no longer exists, decades after work and debate in relativity, in the physics of critical systems and in quantum physics. The failure of this "realist" epistemology of mathematics is parallel to the absence of an epistemology of physics.

It should be clear though that we have been mainly discussing of Gödel's "realist" position, not only as a tribute to the mathematician (of whom the work on types, in 1958, as well as that on recursion and incompleteness, in 1931, made its mark on XXth century mathematical logic, as well as on the work of this author), but also because his philosophy is by far the most profound among philosophies of mathematical "realism/Platonism." Alain Badiou (Badiou, 1990) emphasizes the richness of this Platonism, alone, in mathematics, resembling that of Plato: *thought envelops* the object, while the idea *is* "already there," but as the name of that which is thought and which would remain unthinkable if not activated within thought. Moreover, for Gödel, as we are reminded, "the objective existence of the objects of mathematical intuition ... is an exact replica of the question of the objective existence of the outside world" (Gödel, 1947). This approach, all the while bringing the question of a mathematical ontology closer to that of an ontology of physics, is far more promising than the realism common in mathematics, a funny mix of vulgar empiricism and of idealism, with the worst shortcomings of each of these two philosophies. However, the difference, relative to the approach sketched here, is given by the understanding of the object as constituted; it is not the existence of physical objects or of mathematical concepts that is at stake, but their constitution, as *their objectivity is entirely in their constitutive path*. It is thus necessary to take Gödel's philosophy, for what it puts into mathematical and physical relation, and to turn it head over heels, to bring it back to earth: one must not start "from above," from objects, as being already constituted (existing), but from the constitutive process of these objects and

nal analysis characterizing Husserl; transcendence without transcendental constitution, as a constitutive process of knowledge, without this "Ego" that is co-constituted with the world, which is at the center of Husserlian philosophy, particularly of its maturity. Gödel's late and unpublished writings seem to broaden these views and account for an approach carrying more attention to constitutive paths; yet, a philosophy matters also for the role it has had in history.

concepts. This requires a non-naive analysis of the object and of physical objectivity, as well as a non-passive theory of perception.

1.2.2.2 *Conceptual constructions: history vs games*

To summarize, the objects of mathematics are "outside of ourselves" (transcendent) only as much as they belong to a constituted, which precedes our subject: they are a co-constituted, at the same time as the very intelligibility of the world, by our "living and communicating community." They are not arbitrary because they are rooted in the regularities of reality, to which are confronted our living beings in the world. They are (relative) invariants, first, of time and space, that we then develop by constructing a whole universe derived from conceptual structures, with the most stable tools of our understanding, these invariants of language and of intersubjectivity that we call "logic" and "formalisms": these as well are *the result of a praxis*, the practice of human reasoning, beginning with the Greek agora, in human interaction. In this sense of a previous phylogenetic and historic constitution of their construction principles, and not any another, the objects of mathematics may have properties of which "we do not know," as not yet engendered properties within a more or less precisely given conceptual universe. Take the integers, for example. Once presented, by 0 and the successor operation, as the mental construct of an infinite sequence, discrete and well-ordered (you can picture it, aligned from left to right in a mental space, right?), we can surely give ourselves a language (that of Peano-Dedekind, for example) and enounce an infinity of properties for the elements of this sequence which "we do not know." We will then need to exercise some "friction" between these properties, in that language, and the given construction; and to prove by the most varied methods or tools (arithmetic induction, but also complex variable functions, for example) if they are "realized" upon this well-ordered, infinite structure. In other words, we need to compare construction principles and proof principles. It is thus like this that we may understand the essential incompleteness of the formal theory of numbers: the (formal) proof principles are weaker than the (conceptual/structural)construction principles, which give us the well-ordering of integer numbers, in this case (see Chapter 2). It should then be clear that this absolutely does not imply that this infinite sequence "pre-exists" as a conceptorless concept: if five stones were surely already there, at the foot of this mountain, one billion years ago, what was not there was the concept of the number 5, something completely different, nor were the infinitary properties of that number, ordered within the infinite sequence with the others, as for example the solvability of fifth degree equations or the results of many other linguistic/algebraic constructions we know how to make; constructions that are far from being arbitrary, because rooted in a creative mix of significant conceptual methods (logico-formal, results of spatial invariants, regularities, etc.), but not pre-existing human activities. Yet, they are objective, as rooted in invariants and regularities (order, symmetries, ...) that we conceptualize after and by a friction over this world.

Also consider a variant of chess I am inventing right now: a 100 times 100 square, with 400 pieces that have quite varied but not arbitrary rules: very symmetrical finite movements and simulations of natural movements. I then scatter the pieces randomly; what must be demonstrated is that the configuration thus obtained is compatible with (attainable by) the given rules. Can we say that that configuration (a property of the game) was already there, a billion years ago? What is the meaning of that sentence? Worse, I propose a game with an infinity of squares and pieces, ordered with great originality in the three dimensions, but by effective rules (spirals, fractals ...). I call them "spiralu numbers" or "zamburus," and give you infinitary relationships upon these conceptual objects (I describe, using words, infinite subsets, relationships upon this structure or I scatter the pieces randomly). What sense does it make to say that these properties/relationships were already there? That the compatibility of the distributions of the pieces thus obtained were already decided or were valid since ever? Surely, proof will be necessary in order to "verify" it (I prefer: to check if these distributions are "realized" upon the structure, that is to establish friction, by means of proof, between given properties in the language or the geometry of the squares and the game's construction principles). But as long as the infinitary structure, my construction, built in history, a non-arbitrary extension of a practice of squares and of order, is not posed with the rigor of its construction principles, as the locus where to realize, by the friction of proof, this other construction given in the language of the properties to verify, what sense does it make to say that the conceptual structure and the properties of its infinite subsets "pre-existed"? Conversely for the games which I just proposed, which are my own individual construction, the grounding in the world, within a very ancient intersubjectivity, of the concept of the number, of zero, of the successor, of the infinite well-order gives them a "transcendent" status with respect to my individual existence. Yet, this must not lead us to forget that also these mathematical "objects" are concepts, the results of a very structured, phylogenetic and historical conceptual construction, determined by its constitutive hypotheses; they are not a "pre-existing ontology," they do not transcend our human, actually animal existence (as counting is a pre-human activity). The fact that we ignore the totality (what does "totality" mean?) of their "properties" (careful with this word) in no way demonstrates this ontology we so easily confer upon them: we ignore them, just as we ignore the totality of the scatterings of our whimsical chess games on the infinite chessboard above. There is no transcendence in mathematics, or, rather, there is no transcendence which is not the result of non arbitrary constitutive processes (for example, the construction of algebrico-formal enunciations or of the well-ordering of integers), constructions needing to be compared (relatively realized) with one another, by means of this "friction" between and upon conceptual structures, which is called mathematical proof. More specifically, between principles of proof (that we give ourselves, by non-arbitrary choices) and construction principles (that participate in our own cognitive determination, in the relationship with the world).

Continue, for example, and start with the construction of the integers and pass on to the rationals, as ratios of integers, modulo an equivalence of ratios; then consider the convergent sequences (of Cauchy) of these new numbers, modulo equiconvergence. There are the real numbers, constituted using a mathematical method which reconstructs and links together, in its own way, different histories, by distilling the key concepts. The real numbers do not exist, in any sense of a plausible ontology, but their constitution is as objective as are many other conceptual organizations of the world which render it intelligible to us. And they propose to us a very efficient conceptual structure for the phenomenal continuum of time and space. In short, the very existence and the objectivity of Cantor's real numbers is entirely in their construction.

1.2.3 *Laws, structures, and foundations*

In the first part of this chapter, Francis Bailly, from the perspective of physics, poses other important questions, among which I now retain those concerning the terms of "structure" and of "foundation." What I deny is that one can identify the notion of mathematical structure with its axiomatic presentation and, then, that the analysis of proof, within these axiomatic frameworks, can be a sufficient foundational analysis. To discuss this last point, we will also speak of "laws."

Physicists sometimes confuse "formalism" with "mathematization"; it is customary of their language. The mathematical structuration of the world, of a physical experiment, that they propose is often called "formalization." That is quite understandable, because in what concerns the "very concrete" about which they are thinking (physical "reality"), the mathematical structure is surely abstract and symbolic. But with a bit of experience with the debate about the foundations of mathematics, where these terms are employed with rigor (and philosophical relentlessness, I would say), one understands that rigorous, abstract, and symbolic does not mean formal. In fact, a formal system must work without reference to meaning; it is constructed and manipulated thanks only to mechanical rules. These rules are also and surely used during a physico-mathematical calculus, but the formula about which the physicist thinks has nothing to do with that of logical "formalism": the formula is significative from the onset, because the physicist constructed it with permanent reference to its meaning, to his or her physical experience, he or she inserts it into a mathematical context rich with explicative connections. The physicist proposes mathematical structures to make his or her experience intelligible, the physicist does not invent a set of formal rules disconnected from the world, as would do the formalist, whose foundational analysis lies only in consistency. He or she thus proposes mathematical structures, and not formal systems. Between the two there are at least the great theorems of incompleteness, which separate structural construction principles from formal deductions.

Let's try to exemplify this distinction within mathematics themselves. Consider, as "construction principles," translations, and rotations of figures constructed by rule and compass; if one fixes the unit of length, one will easily construct a segment of length the square root of two. And there, a very first challenge for mathematical understanding: the theory of linear equations with integer coefficients, and with its formal rules of calculus, is demonstrably incomplete with regards to this construction (the segment is not a ratio of integers). With the same principles of construction, including the absence of gaps and jumps within the Euclidean continuum, construct the limit of the polygons inscribed in and circumscribed around a circle. It will then be the formal theory of rational coefficient algebraic equations, which is incomplete with regards to this construction of π.

If we move on to the XXth century, Gödel demonstrated that the formal theory of numbers, with its proof principles, is incomplete with regards to the well-order of integers as a construction principle. By analogy to the role of symmetries in physics, one could say in that regard that Hilbert's conjecture of the completeness of formal arithmetic was a mirror-symmetry hypothesis between formal language and ontologizing semantics (the first accurately reflects the second). Gödel's theorem of incompleteness breaks this alleged symmetry and initiates modern logic. In more constructive and recent terms, the breaking of the symmetry between proof principles and construction principles, of an essentially geometric nature, leads us to understand the insufficiency of a sole logico-formal language as the foundation of mathematics and brings back to the center of our forms of knowledge a constitutive mathematics of time and space, thanks to its construction principles. There is concrete incompleteness, a modern version of Gödelian incompleteness, a discrepancy or breaking in provable symmetry between construction principles and proof principles.

Mathematical structures are, in fact, the result of a reconstruction which organizes reality, all the while stemming from concepts, such as the pre-mathematical concept of the infinite (the theological concept, for example), or, even, from pre-conceptual practices (the invariants of memory), the experience and *practice* of order, of comparison, the structurations of the visual and perceptual in general gestalts. These lead to a structuration, explicated in language, of these (pre-)concepts and of their relationships: the well-order of the integers, the Cantorian infinite, the continuum of the real numbers, the notion of a Riemannian manifold. The concept of infinity gets involved, because it is the result of a profound and ancient conceptual practice, as solid as many other mathematical constructions; these practices are not arbitrary and each may be understood and justified by the process of the construction of scientific objectivity to which it is related.

After the construction of these abstract structures that are symbolic yet rich in meaning, because they refer to the underlying practical and conceptual acts of experience, we may continue and establish axiomatic frameworks that we attempt to grasp at a formal level, whose manipulation may disregard meaning. This process is important, because it adds a possible level of generality and especially highlights certain, possible, "proof principles" which enable us to work, upon these structures, by using purely logico-formal deductions, within well specified languages. But these principles are essentially incomplete, that is what the great results of incompleteness of the last 70 years, in particular the recent "concrete" ones, tell us, as we will explain in Chapter 2. Moreover, as we said in the introduction, the analysis of proof, particularly if this analysis is only formal, is but the last part of an epistemology of mathematics: it is also necessary to account for the constitution of the concepts and of the structures which are manipulated during these proofs. But there is more to this usual and fallacious identification of "axioms" with "structures," of "foundations" with "logico-formal rules." In order to understand this, let's return to physics. Husserl, in an extraordinary epistolary exchange with Weyl (see Tonietti, 1988), grasps a central point of relativistic physics, highlighted, particularly, by the mathematical work of Weyl (but also by the reflections of Becker, a philosopher of physics and student of Husserl, see Mancosu and Ryckman, 2002). The passing from classical physics to the new relativistic framework first bases itself upon the following change in perspective: we go from *causal lawfulness* to the structural organization of time and space (*structural lawfulness*), nay, from causal lawfulness to intelligibility by mathematical (geometric) structures. In fact, Riemann is at the base of this revolutionary transformation, all the while developing the ideas of Gauss. In his habilitation memoir, (Riemann, 1854), a pillar of modern mathematics and of their applications to physics, he aims to unify the different physical fields (gravitation and electromagnetism) through the geometrical structure of space. He throws out the hypothesis that the local structure of space (its metric, its curvature) may be "linked to the cohesive forces between bodies." "Divination" Weyl will call it in 1921, for it is effectively the viewpoint peculiar to this geometrization of physics which at least begins with Riemann, finds its physical meaning with Einstein, and, with Weyl, its modern mathematical analysis.

It thus seems to me that the attempt to mathematize the foundational analysis of mathematics by only referring to the "laws of thought" is comparable to a reconstruction of the unique, absolute classical universe in physics, with its Newtonian laws. It is not *a priori* laws that regulate mathematics, but they do constitute themselves as structures, conceptual plays, that are not arbitrary. The "cohesive forces," in mathematics, would correspond to an "interactive dynamic of meaning," a structuration of concepts and of deduction itself.

In category theory, for example, we propose a new conceptual structure, by novel objects (invariants) and morphisms (transformations); we link it to other structures by using functors, that we analyze in terms of transformations ("natural," their technical name), all the while following/reconstructing the open dynamic of mathematics, of which the unity manifests itself through these reciprocal translations of theories (interpretation functors). And the relative (functorial) interpretations relate the ongoing conceptual constructions (categories): unity is an ongoing conquest and not given by a pre-existing set-theoretic background universe.

Moreover, certain of these categories have strong properties of closure, a bit like rational numbers that are closed for multiplication and division, as real numbers are for particular limits. One of the logically interesting properties, among many others, is "small completeness," that is, the closure with regards to products which interpret the universal quantification, among which, in particular, is second-order quantification (quantification upon collection of collections). Through this device, some categories confer mathematical meaning to the challenges of impredicativity (Asperti and Longo, 1991), the great bogeyman of "stratified" worldviews and of logic: the formal certitudes constructed upon elementary and simple building blocks, one level independent of the other. The world, however, seems to build itself upon essential circularities, from the merest dynamical system (three bodies interacting in a gravitational field) or the local/global interaction (non-locality) in quantum physics, up to the "impredicative" unity of any living organism, of which the parts have no meaning and are out of place outside of the organism as a whole. Maybe the emergence of that which is new, in physics, in biology, only takes place under the presence of strong circularities, sorts of internal interactions within complex systems.

Mathematics is thus not a logico-formal deduction, nicely stratified from these axioms of set theory that are as absolute as Newton's universe, but is structurations of the world, abstract and symbolic, doubtless, yet not formal, because significant; its meaning is constructed in a permanent resonance to the very world it helps us understand. They then propose collections of "objects" as conceptual invariants, of which the important thing is the individuation of the transformations which preserve them, exactly like (iso-)morphisms and functors preserve categorical structures (properties of objects of a category).

There are no absolutes given by logical rules, beyond the world and the cognitive subject, by definite rules (but then why not those of scholastics or of Euclid's key rule: "a part always has less elements than does the whole," which is false in the case of our infinite sets?), but there is a dynamic of structures (of categories), emergent from a mathematical practice, then linked by those interpretation functors which unify them, which explain the ones by the others, which confer upon them meaning within a "reflexive equilibrium" of theories (and of categories, particularly those which correspond to deductive systems (Lambek and Scott, 1986; Asperti and Longo, 1991)).

Surely there is a temporality in the construction of the meaning we confer to the world through mathematics; and it is a "rich" temporality, because it is not that of sequential deduction, of Turing machines: it is closer to the evolution of space-distributed dynamical-type systems, as we shall see. We must let go of this myth of pre-existing "laws of thought" and immerse mathematics into the world while appreciating its constitutive dynamics of which the analysis is an integral part of the foundational project. The laws or "rules" of mathematical deduction, which are surely at the center of proof, are themselves also the constituted of a praxis, of language, as invariants of the reasoning and of the practice of proof itself.

The foundation, so, as the constitutive process of a piece of knowledge, is constructed responsively to the world, the physical world and that of our sensations. But ... where does this process begin? It is surely not a case of reascending "to the mere stuff of perception, as many positivists assert," since physical objects are "intentional objects of acts of consciousness" (Weyl, 1918a). There is a very Husserlian remark, a constitution of objects which we have called a conceptual "decoupage" (cutting-off). And this deecoupage is performed (and produced) by the mathematical concept, a conscious (intentional) act towards the world. Then, reasoning, sometimes rooted in a whole different practice, in the language of social interaction, that of the rules of logical coherence or of the aesthetic of symmetries, for example, generates new mathematical concepts, which may themselves, but not necessarily, propose new physical objects (positrons, for example, derived from electrons by a pure symmetry in equations in microphysics).

The autonomy of mathematics, thanks to the generativity of reasoning, even of the formal type (calculus for example), is indubitable, there lies its predictive force in physics. The integration of these different conceptual dimensions, of these different praxes (geometrical structuration of the world, logical and formal deduction, even far removed from any physical meaning), also confers upon mathematics its explicative and normative character with regards to reality: one goes, in space, let's say, from a physical invariant to another by purely logico-formal means (an algebraic transformation applied to this invariant and which preserves it, a symmetry ...) and a new physical object is thus proposed. The physical proof will be a new experience to invent, with instruments to be invented.

Obviously, in this grounding of our sciences in the world, perception also plays an essential role, but we must then develop a solid theory of perception, rooted in a cognitive science that allows us to go far beyond the positivist's "passive perception," of which Weyl speaks about. We shall return to this point.

The approach we propose, of course, causes the loss of the absolute certitude of logico-formal, decidable proof. But we know, since Gödel, that any formal theory, be it slightly ambitious and of which the notion of proof is decidable, is essentially incomplete. So logicism's and formalism's "unshakeable certainties" (the absolute certification of proof) are lost ... since a long time. There remains the risk of the construction of scientific objectivity, thoroughly human, even in mathematics, the adventure of thought which constitutes its own structures of the intelligibility of the world, by the interaction with the former and with the thoughts of others. The risk we will take in Chapter 2 of acknowledging the foundational role of the well-ordering of integers, by a geometric judgment constituted in history, action, language, and intersubjectivity, will be to certify the coherence of arithmetic.

1.2.4 *Subject and objectivity*

In various works, Weyl develops a very interesting philosophical analysis concerning the passage in physics from the subjective to the objective, on the basis of references to his own mathematical works in relativity theory. This analysis is emphasized by Mancosu and Ryckman (2002), who refer mostly to Weyl (1918a-b, 1927). The importance of Weyl's remarks obviously extends way beyond the philosophical stakes in physics and in mathematics, because it touches upon a central aspect of any philosophy of knowledge, the tension between the "cult of the absolute" and "relativism." Husserl seeks to move beyond this split in all of his work and in his reading of the history of philosophy (see, for example, Husserl, 1956). XXth century physics can provide tools for contributing to that debate, and those are Weyl's motivations.

For Weyl, immediate experience is "subjective and absolute," or, better, it claims to be absolute; the objective world, conversely, that the natural sciences "crystallise out of our practical lives ... this objective world is necessary relative." So, it is the immediate subjective experience which proposes absolutes, while the scientific effort towards objectivity is relativizing, because "it is only presentable in a determined manner (through numbers or other symbols) after a coordinate system is arbitrarily introduced in the world. This oppositional pair: subjective-absolute and objective-relative seems to me to contain one of the most fundamental epistemological insights that can be extracted from natural sciences".

Following his works in relativity, Weyl thus gives a central role to reference frames. The subject lays, chooses, a reference frame and in this manner organizes time and space. That choice is the very first step performed by the knowing subject. But the operation of measurement, by means of its own definition, also implies the subject: any physical size is relative to (and set by) a "cognizing ego." The passage to objectivity is given, in quantum physics, by the analysis of "gauge invariants," for example, one of Weyl's great mathematical contributions to this field: they are given as invariants in relation to the passing from one reference and measurement system to another. More generally, the passage from subjectivity to scientific objectivity implies the explicit and explicated choice of a reference frame, including for mathematical measurements. The analysis of the invariants with respect to different reference frames gives then the constituted and objective knowledge.

Weyl thus emphasizes, in Husserlian fashion, that any object in the physical world is the result of an intentional act, of the awareness "of a pure, sense giving ego." For both thinkers, it is a matter of the Cartesian, "Ego" to which Husserl so often returns to, which "is, since it thinks"; and it is, because, as a consciousness, it has "objects of consciousness" (consciousness is "intentional," it has an "aim"). It is the subject, this conscious Cartesian "Ego," that chooses the reference frame and who, afterwards, is set aside. It poses the origin, the 0 and the measurement, and it mathematically structures time and space (as a Cantor-Dedekind continuum, for example, or as a Riemannian manifold with its curvature tensors); by that act (the construction of a space as a mathematical manifold), it poses a framework of objectivity, independently of the subject, objectivity nevertheless consciously relativized to that choice. Because the choice of viewpoint, of the frame, is relativizing and breaks the absolute characteristic of the subject before the passage to scientific objectivity; this passing of subjectivity, which claims to be absolute, to relativizing objectivity, is the meaning of the scientific approach central to relativity. Just as it is very well put in (Mancosu and Ryckman, 2002): "The significance of [Weyl's] 'problem of relativity' is that objectivity in physics, that is, the purely symbolic world of the tensor field of relativistic physics, is constituted or constructed via subjectivity, neither postulated nor inferred as mind-independent or transcendent to consciousness." But this symbolic world of mathematics is in turn itself the result of an interaction of the knowing subject(s), within intersubjectivity, with the regularities of the world, these regularities, which we see and which are the object of intentional acts, of a view directed with "fullness and willing," as Husserl and Weyl say.[6]

The subject is thus at the origin of scientific knowledge, and it is with the subject that any mathematical construction begins. However, it will be necessary to push the analysis of the subject's role further: today we can pose the problem of objectivity at the very center of the knowing subject, because this subject is not the psychological subject, which is also disputed by the seekers of the absolute, of transcendental truths, of configurations or properties which are already there, true prior to any construction/specification, even in my infinite chessboard or in the sequence of integers. In fact, it is a question of the "cognitive subject," of this "Ego" that we share as living, biological creatures, living in a common history that is co-constituted with the world, at the same time as its activity in the world. There is the next issue we will have to deal with, in the dialog with cognitive sciences, basing ourselves on non-naive (and non passive) theories of perception, on theories of the objective co-constitution of the subject. The scientific analysis of the subject must, by these means, underline what is common to subjective, psychological variability: more than a simple "intersection of subjectivities" it is a question of grasping in that way what

[6]The profoundness of Weyl's philosophy of sciences is extraordinary and his philosophy of mathematics is but a part of it (a small one). I found quite misleading, with regards to this profoundness and its originality, the many attempts of many, including some leading "predicativists" to make him into a predecessor of their formalist philosophy of mathematics. Briefly, in Weyl (1918a), a remarkable Husserlian analysis of the phenomenal continuum of time and space, Weyl also feels concerned with the problem of "good definitions," a problem that preoccupied all mathematicians of the time (including Poincaré and Hilbert, of course): the XIXth century was a great period for mathematics, but, so very often, ... what a confusion, what a lack of rigor! Particularly, it was necessary to watch out for definitions that may have implied circularities, such as impredicative definitions. Weyl noticed that Russell's attempt to give mathematics a framework of "stratified" certitudes does not work ("he performs a hara-kiri with the axiom of reductiveness," (Weyl, 1918a)). In the manner of the great mathematician he was, Weyl proposed, in a few pages, a formally "predicativist" approach that works a thousand times better than that of Russell with his theory of types. An approach and an interesting exercise in clarification that Weyl will never follow in his mathematical practice; to the contrary (Weyl, 1918a), he criticizes, quite a few times, Hilbertian formalism of which the myth of complete formalization "trivializes mathematics" and comes to conjecture the incompleteness of formal arithmetic (!). Feferman (1987) took up these ideas only slightly brushed upon by Weyl (and not the predicatively incoherent heaviness of Russell's theory of types), to make it into an elegant and coherent predicative formal theory for analysis. Remarkable technical work, but accompanied by an abusive and quite incomplete reading of Weyl's philosophy, which is a much broader philosophy of natural sciences, never reduced to stratified predicativisms nor their corresponding formal or logicist perspectives.

lies behind individual variabilities, what directs them and allows them to communicate and to understand/construct the world together.

Foundational analysis, in mathematics and in physics, must therefore propose a scientific analysis of the cognitive subject and, then, highlight the objectivity of the construction of knowledge within its referential systems or reference frames.

In what concerns the foundations of mathematics, a process analogous to this "choice of reference frame" is well explicated, in category theory, by choosing the right "topos" (as referential category for a logic or with an "internal logic" (Johnstone, 1977)), to relate, through interpretation functors, other categorical constructions, in a dynamic of these structures by which we give mathematical meaning to the world (algebraic, geometrical, manifolds' categories, ...). This has nothing to do, as we have already emphasized, with the absoluteness of the axioms of set theory, a Newtonian universe that has dominated mathematical logic and that has contributed for a century to the separation of mathematical foundations from epistemology and from the philosophy of natural sciences. That was a matter, indeed, of an absolute, that of sets, intuition of which is compared, as we said, by the "realists" in mathematical philosophy, to the perception of physical objects (quite naively described in its passivity), sets and objects also being transcendent, with their properties all listed there, "pre-existing." A typical example of that which Husserl, de Ideen, and Weyl (taken up by Becker, see Mancosu and Ryckman, 2002) call the "dogmatism" of those who speak of absolute reality, an infinite list of already constituted properties, constituted before any pre-conscious and conscious access, before projecting our regularities, interpreting, acting on the world, before the shared practices in our communicating community.

1.2.5 *From intuitionism to a renewed constructivism*

Quite fortunately, within the same mathematical logic, we begin to hear different voices: "Realism: No doubt that there is reality, whatever this means. But realism is more than the recognition of reality, it is a simple-minded explanation of the world, seen as made out of solid bricks. Realists believe in determinism, absoluteness of time, refuse quantum mechanics: a realist cannot imagine 'the secret darkness of milk.' In logic, realists think that syntax refers to some pre-existing semantics. Indeed, there is only one thing which definitely cannot be real: reality itself" (Girard, 2001). The influence of Brouwer, the leader of intuitionism, and of Kreisel, as well as the mathematical experience with intuitionist systems, is surely present in the mathematical work and in the rare philosophical reflections of Girard, but without the slip, characteristic of Brouwer, into a senseless solipsism, nor with the *a priori* limitations of our proof tools. Moreover, time and space are included in Girard's proof analysis: the connectivity, the symmetries of proof as network, time as irreversible change in polarity in Girard (2001), have nothing to do with "time as secreted by clocks" (his expression), the time of sequential proof, of Turing machines, which is beyond the world (see Chapter 5).

Brouwer's intuitionism, among the different trends in the philosophy of mathematics (formalist, Platonic realist, intuitionist), is possibly the only foundational analysis that has attempted to propose an epistemology of mathematics (and a role for the knowing subject). The discrete sequence of numbers, as a trace of the passing of time in memory ((Brouwer, 1948), see also (Longo, 2002a, 2005)), is posed as constitutive element of mathematics. It is exactly this vision of mathematics as conceptual construction that has made Weyl appreciate Brouwer's approach for a long time. In fact, the analyses of the mathematical continuum for Brouwer and Weyl (as well as for Husserl, see (Weyl, 1918a; Tonietti, 1988; Longo, 1999)) are quite similar in many respects. However, Weyl had to distance himself from Brouwer, during the 1920s, when he realized that the latter excessively limits the tools of proof in mathematics and does not know how to go beyond the "psychological subject," to the point of renouncing the constitutive role of language and of intersubjectivity and to propose a "languageless mathematics" (a central theme of Brouwer's solipsism, see (Brouwer, 1948; van Dalen, 1991).

Conversely, and as we have tried to see, the relativity problem for Weyl, as a passage from "causal lawfulness" to "structural lawfulness" in physics, as well as a play between subjectivity-absolute and objectivity-relative, is at the center of an approach that poses the problem of knowledge in its unity, particularly as it is the relationship between physical objectivity and the mathematical structures that make time and space intelligible, thanks, among other things, to language. All the while following Weyl, we have made a first step towards an extension of foundational analysis in mathematics by a cognitive analysis of what should precede purely logical analyses: only the last segment is without doubt constituted by the logico-formal analysis of proof. But upstream there remains the problem of the constitution of structures and of concepts, a problem which is strictly related to the structuration of the physical world and to its objectivity. The project of a cognitive analysis of the foundations of mathematics thus requires an explication of the cognitive subject. As a living brain/body unit, dwelling in intersubjectivity and in history, this subject outlines the objects and the structures, the spaces and the concepts common to mathematics and to physics on the phenomenal veil, while constituting itself. In short, parallel constitutive history, in physics, begins with perception as action: we construct an object by an active viewing, by the presence of all of our body and of our brain, as integrator of the plurality of sensations and actions, as there is no perception without action: starting with Merleau-Ponty's "vision as palpation by sight," perception is the result of a comparison between sensorial input and a hypothesis performed by the brain (Berthoz, 1997). In fact, any invariant is an invariant in relation to one or more transformations, so in relation to action. And we isolate, we "single out," invariants from the praxis that language, the exchange with others, forces us to transform into concepts, independently, as communicables, from the constitutive subject, from invariants constituted with others, with those who differ from us but who share the same world with us, and the same type of body. From the act of counting, the appreciation of the dimensionless trajectory – dimensionless since it is a pure direction – we arrive at the mathematical concepts of number, of aunidimensional line, with no thickness, and, then, of a point. Invariants quite analogous to the physical concepts of energy, force, gravitation, electron. The latter are the result of a similar process, they are conceptual invariants which result from a very rich and "objective" praxis, that of physics, inconceivable without a close interaction with mathematics. They organize the cues that we select through perception and through action upon the world, through our measurement instruments; the geometrical structuration of those invariants is the key organizing tool, because it explicates in time and space our action and our comprehension. These are the cognitive origins of the common construction principles in mathematics and physics.

Individual and collective memory is an essential component to this process constitutive of the conceptual invariants (spatial, logical, temporal, ...). The capacity to forget in particular, which is central to human (and animal) memory, helps us erase the "useless" details; useless with regards to intentionality, to a conscious or unconscious aim. The capacity to forget thus contributes in that way to the constitution of that which is stable, of that which matters to our goals, which we share: in short, to the determination of these invariant structures and concepts, which are invariant because filtered of all which may be outside our intentional acts of knowledge. Their intercultural universality is the result of a shared or "sharable" praxis, in the sense that these invariants, these concepts, may very well be proposed in one specific culture (think about Greek geometry or Arabic algebra), but their rooting in fundamental human cognitive processes (our relationship to measurement and to the space of the senses, basic counting and ordering, ...) make them accessible to other cultures. This widening of a historic basis of usage is not neutral, it may require the blotting out of other experiences specific to the culture which assimilates them, but confers on them this universality that accompanies and which results from the maximal stability and conceptual invariance specific to mathematics. But this universal is posed with relation to human forms of life and does not mean absolute; it is itself a cultural invariant, between cultures that take shape through interaction. Because universality is the result a common evolutive history as well as of these communicating communities. Historical demise is a factor of it: oblivion or expulsion from mathematics of magical numbers, of "zombalo" (whatever) structures ... of that which does not have the generality of method and results we call, *a posteriori*, mathematical.

As for the mathematical organization of space, both physical and sensible, it begins very early, probably as soon as space is described by gesticulation and words, or with the spatial perspective and width of the pictorial images of Lascaux, 20,000 years ago, or from the onset of the play of Euclid's rigid bodies, which structures geometrical space. Euclid's axiomatics indeed summarize the minimal actions, indispensable to geometry, with their rule and compass, as construction and measurement instruments: "trace a straight line from one point to another," "extend a finite line to a continuous line," "construct a circle from a point and a distance" ... (note that all these constructions are based and/or preserve symmetries). His first theorem is the "vision of a construction" (in Greek, theorem means "sight," it has the same root as "theater"): he instructs how to "construct an equilateral triangle from a segment," by symmetric tracing with a compass.[7]

[7] That's what the first theorem of the first book is and the point constructed during the proof is the result of the intersection of two lines traced using a compass. Its existence has not been forgotten, as claimed by the formalist reading of this theorem, it is constructed: Euclid's geometry presupposes and embraces a theory of the continuum, a cohesive entity without gaps or jumps (see Parmenides and Aristotle). Note that the concept of the dimensionless point is a consequence of the extraordinary Greek invention of the line with no thickness: points (semeia, to be precise – signs) are at the extreme of a segment or are obtained by the intersection of two lines, a remark by Wittgenstein.

This history leads to Weyl's symmetries, regularities of the world which impose themselves (donations that, in this sense, pre-exist or that reality imposes on us), but that we see or decide to see. We then transform them into concepts and choose to pose them as organizing criteria of reality, even in microphysics, far removed from sensorial space.

1.3 Regarding Mathematical Concepts and Physical Objects

(by Francis Bailly)

Giuseppe Longo suggests that we establish a parallel between *mathematical concept* and *physical object*. The massive mathematization of physics, the source of new mathematical structures it is thus likely to generate, the aptitude of mathematics to base its constructions on the physicality of the world, possibly in the view of later surpassing or even of discarding this physicality in its movement of abstraction and its generativity proper, all these entail questions regarding the relationships that the mathematical concepts thus constructed may entertain with the highly formalized physical objects of classical or contemporary physics. So what could be the static or dynamic traits specific to each of these determinations and of these methods which would permit such a parallel?

In the introduction, we have already stated the analogies and differences between the foundations of physics and the foundations of mathematics in the respective corresponding of their construction principles and of their proof principles. In short, they seem to share similar construction principles and to have recourse to different proof principles – formal for mathematics *via* logic, and experimental or observational for physics *via* measurement. Is it possible to go further without limiting ourselves to the simple observation of their reciprocal transferals?[8]

[8] There is no need to return to the obvious and constitutive transfer from mathematical structures to physics. The transfer in the other direction, from physics towards mathematics, is on the other hand more sensitive. Recall the introduction in physics of the Dirac "function," which led to distribution theory; or the introduction of Feynman path integrals and the corresponding mathematical research aiming to provide them with a rigorous foundation; more recently, the Heisenberg non-commutative algebra of quantum measurement and Connes' invention of non-commutative geometry. Without speaking of the convergences between the physical theory of quasi-crystals and combinatory theory in mathematics or between physical turbulence theory and the mathematical theory of non-linear dynamical systems, etc.

1.3.1 *"Friction" and the determination of physical objects*

To address a different level, let's note that in the epistemological discussion regarding the relationships between the foundations of physics and the foundations of mathematics, Giuseppe Longo proposes that we consider that which causes "*friction*" in the set of determinations of their respective objects. For physical friction, that which validates it (or serves as proof), can be found first of all in the relationship to physical phenomenality and its measurement: experience or observation are determinant in the last instance, even if at the same time more abstract frictions continue to operate, frictions which are more "cognitive" with regard to mathematical theorization. In contrast, for mathematics, it very well appears that the dominant friction may be found in the relationship to our cognitive capacities as such (in terms of coherence of proof, of exactitude of calculation), even if mathematical intuition sometimes feeds on the friction with physical phenomenality and may also be canalized by it.

Let's explain. "Canalization" and "friction," in the constitution of the mathematical concepts and structures of which Longo speaks, seem mainly related to "*a reality*" resulting from the play between the knowing subject and the world, a play which imposes certain unorganized regularities. Mathematical construction then and again enters into friction with the world, by its organization of reality. In physics, where prevails the "*blinding proximity of reality*" (Bitbol, 2000a), friction and canalization seem to operate within distinct fields: if friction, as we have just highlighted, remains related to the conditions of experience, of observation, of measurement – in short, of physical phenomenality – canalization, on its part, now results much more from the nature and the generativity of the mathematical structures which organize this phenomenality, modelize it and finally enable us to constitute it into an objectivity. In a provocative, but sound, mood many physicists consider the electron to be just the solution of Dirac's equation.

If we refer to the aphorism according to which "*reality is that which resists*," it appears that, by interposed friction, physical reality, all the while constituting itself now *via* mathematization, finds its last instance in the activity of the measurement, while mathematical phenomenality may be found essentially in the activity associated with our own cognitive processes and to our abstract imagination. What relationship is there between each of these types of friction, between both of these realities? At a first glance, it does appear that there is none: the reality of the physical world seems totally removed from that of the cognitive world and we will not have recourse to the easy solution which consists in arguing that in one case as in the other, we have to face material supports, relative to a unique substantiality. Indeed, even if it is the same matter, it manifests at very different levels of organization which are far removed the one from each other according to whether we are considering physical phenomenality or our cognitive structures. On the other hand, if we are nevertheless guided by a somewhat monist vision of our investigational capacities, we can legitimately wonder about the coupling between these levels enabling knowledge to constitute itself: coupling dominated by one of the poles involved (phenomenal or cognitive) according to whether we use a physical or a mathematical approach. In fact, it is this coupling itself which appears to constitute knowledge, as well as life itself, if we agree that cognition begins with life (Varela, 1989). The relationship that could then be established between the physical object and the mathematical concept would therefore stem from the fact that it would be a question of accounting for the same coupling (between cognitive structures and physical phenomenality) but taken from different angles depending on whether it is a question of a theory of physical matter or a theory of abstract structures. The fact that it is the same coupling would then manifest within the community of construction principles which we have already described, whereas the difference in disciplinary viewpoints regarding this coupling (from the phenomenal to the cognitive-structural or conversely) would manifest in the difference in nature of the proof principles, demonstration, or measurement.

Another approach for conducting a comparison between physical object and mathematical concept, which seems to ignore the phenomenal friction characteristic of physics, consists in considering that the scientific concept, be it attached to a physical object or to a mathematical ideality, has lost a number of its determinations of "concept" to become an abstract formal structure. This would only translate the increasing apparentness between contemporary physical objects and mathematical structures modelizing them. It could possibly be a way to thematize the constitutive role of mathematics for physics, which we have already presented. Such an appreciation is probably well founded, but it does not completely do justice to this particular determination of physical objects to be found in the second part of the expression "the mathematical structures *which modelize them*." Indeed, the necessity of adding this precision refers to this "something" which needs to be modelized, to a referent which is not the mathematical structure itself. And it is this change of level of determination, this heteronomy contrasting with the autonomy of the mathematical structure, which probably specifies the objectivity of the physical object and suggests that it is likely to respond to other types of determination than those attached to the sole mathematical structure, to that which we can henceforth call a "new" friction.

1.3.2 *The absolute and the relative in mathematics and in physics*

The physicist can only fully agree with what Weyl had indicated when producing a critique, from the mathematical standpoint, of the so-called absoluteness of the subjective in order to confront it with what he would call the relativity of objectivity (see the text by Longo above). All of his work consists indeed in breaking away from the illusion of this subjective "absoluteness" in the apprehension of phenomena in order to achieve the construction of objective invariants likely to be communicated. By doing this, he obviously succeeds in qualifying the subjective as relative and in qualifying the objective if not as absolute, at least as stable invariant. Moreover, the results of this construction of physical objectivity prove to be sometimes incredibly counter-intuitive. Consider for example quantum non-separability which prohibits speaking of two distinct quanta once they have interacted; but already, in the age of Copernicus and Galileo, the roundness of the world and its movement relative to the sun was a challenge to intuitive spontaneous perception and common sense: language perpetuates traces of this, it continues to see the sun rise.

What is at stake in the analysis one can make of this situation, can doubtlessly be found in the relationships between the use of natural language on the one hand and the mathematization presiding over the elaboration of mathematical models on the other hand. To put it briefly, the relativity of the subjective is relative to language and may thus appear to be absolute since language then plays a referential role, whereas the relativity of the objective is relative to the model itself that is presented as a source of the stable invariants which, by this fact, are likely to play a role of "absolutes" in that they are (in principle) completely communicable. Hence the possibility for a sort of chiasma in the qualifications according to the first referral, meaning also according to the involvement of the person speaking: in his or her intuitive and singular grasp only the absoluteness of the intuition is perceived and not the relativity of the "ego." But in the rational reconstruction aiming for objectivity, relativity is not included within the mathematical model and the invariant objectives constructed by the intersubjective community are taken for quasi "ontological" absolutes. It is these referrals (to language and to the mathematical model) which we will therefore attempt to discuss more precisely.

1.3.3 *On the two functions of language within the process of objectification and the construction of mathematical models in physics*

Despite the observation of the increasingly mathematical and abstract character of the scientific object of physics, it would nevertheless be wrong to conclude that the corresponding scientification movement only comprises a process of removal with regard to natural language and a discredit of its usage. This would be to completely ignore that scientific intuitions, however formal they may be, continue to be rooted in this linguistic usage and that the interpretations which contribute to making them intelligible, including for the constitutive intersubjectivity of the scientific community, cannot do without these intuitions. The references to ordinary language and common intuitions are needed in order to communicate not only with the non-specialists of the field, but also in the heuristic of the disciplinary research itself, notably in its imaginative and creative moment, as singular as it may be.

Thus, under deeper analysis, objectifying mathematization appears in fact not to be exclusive to ordinary language and its usage, even if in the moment of the technical deployment it may be as such. It rather appears to be an insert between two distinct functions of language, which it contributes to distinguish all the while articulating them. By doing this, the hermeneutical dimension is reactivated and the components of history and genesis are reintroduced also to where mainly dominates the mathematical structures and their conceptual and theoretical organizations. It is this point we would like to argue and develop here by asserting that in the same way that we are led to consider, for reason, a double status – constituting and constituted reason – we are led to distinguish two functions for language, relatively to mathematical formalism: a *referring* function and a *referred* function. Let's try to clarify this.

In its referring function, language provides the means of formulating and establishing, for physics (but this also applies to other disciplines), the major theoretical principles around which it organizes itself. Relatively to the norm-setting subject, in a certain sense it governs objectifying activity. In contrast, in its referred function relatively to these modelizations, language tends more to use *terms* (mathematical terms, typically) than words.[9] Similarly, it uses conceptual or even formal relationships rather than references to signification. It is then submitted to the determinations specific to these abstract mathematical structures that it had contributed in implementing and of which it triggered the specific generativity. This is done until the movement of scientific theorization uses this referred state of language to confer it with a new referring function in view of the elaboration of new models, of new principles, that are more general or more abstract, the "final state" of a stage becoming in a way the "initial state" of the following stage. In this always active dialectic process, mathematics maintains the gap and the distinction – which are essential for the construction of objectivity – between these two functions of language, all the while ensuring the necessary mediation between them. Its role is reinforced and modified by its informal use in a referring function, while it continuously transforms the referred function by means of the internal dynamic specific to it, due to the generativity of the mathematical model. In doing so, mathematics contributes in generating the language of knowledge by means of the functions it confers to language, between which it ensures the regulated circulation (though in the realm of objectifying rigor, a bit like what poetry achieves in the realm of the involvement of subjectivity).

A physical example of this process, in direct continuity with the innovations by Kepler, Copernicus, and Galileo, may be found in the status of Newton's universal gravitational theory. The referring state of language had recourse in the past to an "Aristotelian" representation of the world according to which the "supralunar" constituted an *absolute* of perfection and of permanence (invariability of the course of planets describing perfect circles, and the corresponding mathematical model of Ptolemaic epicycles). It was, however, from within its referring function (of which the still quasi-mythical state can also be found in Newton's alchemical or Biblical works (Verlet, 1993)) that the mathematical model of universal gravitation was constructed, as Galileo had wanted. This model governs all bodies, be they infra- or supralunar. Thanks to mathematization, this radical relativization as for interaction forces, is indeed accompanied by the maintaining (or even, by the introduction) of another *absolute*, that of time and space. However, it also redefines the language of the course of planets in a state that is

[9]In the sense that it often uses, as terms, more or less precise, more or less polysemic, words from ordinary language to designate much more rigorous (of no or restricted polysemy) and abstract notions which have meaning only within the "technical" context within which they are used.

henceforth referred to in this model where elliptic orbits and empirical observations are "explained" by the law of universal gravitation. Even more: the relevant physical invariants that will serve as support structure for any ulterior consideration and which will model the language of this new cosmology are identified. It is this referred state of the language of Newtonian cosmology (the mathematical model thus constructed) that will afterwards serve as the new foundation for following research, having from that moment a referring role. The geometrization of the Newtonian language of forces will lead to the relativization of the absolutes of space and of time themselves. This will be obtained by its referring use, as the background language for conceiving the Einsteinian theory of general relativity.

Let's return to this distinction from a complementary point of view, which is closer to the procedures, closer also to more specifically logical formalisms. As referred, language must in one way or another, in order to make sense and to avert paradoxes, conform to a sort of theory of types capable of discrimination between the different levels of its statements: it must clearly distinguish between terms and words as well as between different *types* of terms. But the construction of such a theory of types has recourse to the referring function of language. This function guides the conceptual elaboration and the formulation of formal statements. Hence, the referring function invents and is normative, beyond the normative creativity of the referred mathematical model. It governs creative and organizing *activity*. Since the referred function is the object of study and of analysis, it requires the mediation of a logical-mathematical language which objectifies and enables us to process it with rigor also from the point of view of its own referring function.

Let's note at this stage that the requirement of an "effective logic," sometimes formulated by constructivist logicians, concerns essentially the referred role of language. With intuitionism, the activity of thought, beyond language, is at work in the process of mathematical elaboration and construction. While it innovates and creates, that is, as it brings forth new referrals, the activity of thinking does not respond to criteria or norms of constructability: it creates them.

We agree on this creative role, but, by allowing a constitutive and double function to language, our analysis of knowledge construction in science describes the modelization of (physical) theory itself as derived but nevertheless as determinant. While the reflection, as abstract and rigorous as it may already be and of which the anteriority may confer it with a status of apparent absoluteness, enunciates the principles – what is to be modelized

– the mathematical model governs any ulterior theoretical advance. Moreover, these advances may go as far as contradicting principles considered beforehand as evident.

It is this conceptual configuration which enables us to understand how such counter-intuitive situations addressed by contemporary physical theories (quantum mechanics, typically) may nevertheless be "spoken" in a natural language which continues to spontaneously claim the opposite of the results obtained. This so-called natural language is no longer as natural as that: terms have been substituted for its words. And in any case it is no longer based on its specific linguistic structures and grammar (and the corresponding mentalities) but rather on the mathematics of the model which it interprets and comments. However, this use of natural language does not restitute their profoundness and, most important, their generativity which is only proper to the mathematical model, but nevertheless it continues to ensure cultural communication.

1.3.4 *From the relativity to reference universes to that of these universes themselves as generators of physical invariances*

What Weyl highlights is a process of conceptual emancipation: moving away from the "absolutist" illusion of the subjective and of the language-related, scientific concepts are objectified *via* their relativization to the reference universes and their mathematization. But contemporary physics doubtlessly enables a supplementary passing, a transition in this process of emancipation relative to "sensible" constraints, this time through gauge theories of which the very same Weyl was one of the proponents. Indeed, these theories amount to relativizing reference universes by stripping them of most of their properties which were previously conceived as "absolute": space is such that there is no assignable origin for translations or for rotations, resulting in the invariances of the kinetic and rotational moments; time is such that there is no privileged point enabling an absolute measurement, resulting in the invariance of energy in conservational systems. This is for external reference universes. But for internal quantum universes, it goes as follows: the global absence of an assignable origin for the phases of an electronic wave function will entail the conservation of the electric charge; its local absence is the source of the electromagnetic field. Hence, the same fields of interaction are associated with gauge changes authorized by these relativizations of the reference universes. And these intrinsic relativizations are none other than the symmetries presented by these universes (connexity, isotropy, homogeneity, ...). To summarize, *the less, by these symmetries, we are able to specify these reference universes in an absolute way, the more the physical invariances (and abstract determinations) we are able to bring forth.* Add to this the complementary aspect in which the particular specifications of the objects in question are increasingly referred to spontaneous breaking of symmetry. It is as if, beyond the construction of objectivity itself, it was the *very identity* of the object so constructed – in its stability and in its specific "properties" – which was being determined.

1.3.5 *Physical causality and mathematical symmetry*

Longo also notes that with relativistic physics, according to Weyl, among others, there occurs a "*change of perspective: we pass from 'causal laws' to the structural organization of space and time, or even from causal laws to the 'legality/normativity' of geometric structures.*"

One can only observe that this movement has since only reasserted and amplified itself, while producing new and difficult epistemological questions. Indeed, we have witnessed, with quantum physics, an apparently paradoxical development: on the one hand, geometric concepts have become omnipresent (be it an issue of topology, of algebraic geometry or, most of all, of symmetries) at the same time that, on the other hand and as we have abundantly emphasized in the "physical" section of this text concerning space and time, quantum events find their most adequate description in unexpected spaces, increasingly removed from our intuitive space-time, or even from relativistic space-time (functional Hilbert spaces, Fock spaces, etc.). Moreover, as we know, the very notion of trajectory is problematic in quantum physics and that causality *stricto sensu* (that which may be associated with relativistic theories) finds itself to be profoundly thrown into question, particularly in the process of measurement; hence, the difficulties in terms of unification between general relativity and quantum theory.

It is therefore necessary to be clear on this: the massive geometrization of quantum physics does in fact amount to having recourse to and working with concepts of a geometrical origin, but the geometry in question is increasingly removed from that of our habitual space-times, be they four-dimensional. In fact, this geometrization is much more associated with the use of symmetries and symmetry breaking, which enable us to both identify invariants and conserved quantities and to mathematically and conceptually construct gauge theories, as we have seen, disjoining yet attempting to articulate internal space-times and external space-times. From this point, we indeed witness the aforementioned observation of "causal laws" by these "mathematical structural organizations," but which apply this time to spaces and times presenting new characteristics.

As noted by Chevalley in his *Presentation* in the work by van Fraassen (1989), this observation is massive, to an extent, in the analysis of contemporary physics and in its very functioning, of " *substituting to the concept of law that of symmetry*," thus extending the appreciation of the author who, while adopting what he calls the "semantic approach" in the analysis of physics, does not hesitate to assert regarding symmetry that: "*I consider this concept as being the principal means of access to the world we create in theories.*"

Such radicalism could surprise at first sight but may, however, be nicely explained if one realizes that it is an issue of considering essential elements of the very process of the construction of physical objectivity and of the determinations of the corresponding scientific objects. Indeed, as conservations of physical quantities are associated with the principles of relativity and of symmetry and that all of physics bases itself on the measurement of quantities related to properties which must remain stable in order to be observed, one may go as far as asserting that these relativities and symmetries, while appearing to reduce the possible information relative to the systems under study, are constitutive of the very *identity* of these systems. As if it were a question, to use the old vocabulary of medieval scholastics, of identifying the primary qualities (that is, their essential identity structures) while leaving with the "laws" the care of regulating their secondary qualities (to which would correspond here their actual behavior). To continue a moment on this path, we could even go so far in the analysis of the relationships between symmetry and identity by considering that all information is a (relative) breaking of symmetry and that, reciprocally, any relative breaking of symmetry constitutes an objective element of information. According to such a schema, it would then be relevant to consider that to the metaphysical substance/form pair, to which was partially substituted during the scientific era the energy/information (or entropy) pair, finally corresponds the symmetry/breaking of symmetry pair as constituent of the identity of the scientific object, which we have just mentioned.

Let's specify our argument concerning the relationships between causal laws and mathematical structures of geometry. Relativistic theories – general relativity in particular – constitute the privileged domain where was first identified, in modern terms, the relevance of causal laws by their identification to structural organizations. This stems essentially from the intrinsic duality existing between the characterization of the geometry of the universe and that of energy-momentum within that universe. By this duality and the putting into effect of the principle of invariance under the differentiable transformations of space-time, the "forces" are relativized to the nature of this geometry: they will even appear or disappear according to the geometric nature of the universe chosen *a priori* to describe physical behaviors. Now, it is similar for quantum physics, in gauge theories. Here, gauge groups operate upon internal variables, such as in the case of relativity, where the choice of local gauges and their changes enable us to define, or conversely, to make disappear, the interactions characterizing the reciprocal effects of fields upon one another. For example, it is the choice of the Lorentz gauge which enables us to produce the potential for electromagnetic interactions as correlates to gauge invariances.

Consequently, if one considers that one of the modalities of expression and observation of the causal processes is to be found in the precise characterization of the forces and fields "causing" the phenomena observed, then it is apparent that this modality is profoundly thrown into question by the effects of these transformations. Not that the causal structure itself will as a result be intrinsically subverted, but the description of its effects is profoundly relativized. This type of observation therefore leads to having a more elaborate representation of causality than that resulting from the first intuition stemming from classical behaviors. Particularly, the causality of contemporary physics seems much more associated with the manifestation of a formal solidarity of the phenomena between themselves, as well as between the phenomena and the referential frameworks chosen to describe them, than to an object's "action" oriented towards another in inert space-time, as classical mechanics could have accredited the idea. After Kant, it already had ceased claiming to restitute the functioning of reality "as such," whereas with contemporary results it goes so far as to presenting itself as technically dependent upon the models which account for the phenomena under study. Causes, in this sense, become interactions and these interactions themselves constitute the fabric of the universe of their manifestations, its geometry: modifying this fabric appears to cause the interactions to change; changing the interactions modifies the fabric.

These considerations may extend to theories of the critical type in that, among other things, the spontaneous breaking of symmetry partially subvert Curie's principle which stated that the symmetry of causes were to be found in those of the effects. And if we want to generalize the Curie principle to the case of the effects which manifest symmetry breaking relative to causes, then we are led to consider not only a singular experience which manifests this breaking, but the entire class of equivalent experiences and their results. As breakings of symmetry are singularized at random by fluctuations which orient them (this is the fundamental hypothesis which enables us to do without the existence of other "causes" which would not have been considered in the problem), the accounting for all possible experiences contributes in eliminating this random character by averaging it and contributes to restore the symmetry of potentialities in actuality and in average. It nevertheless remains that the precise predictability of the result of a given experiment remains limited: if we know that it necessarily belongs to the class of symmetries authorized by the Curie principle, we do not know which possibility it actualizes, that precisely in which the symmetry finds itself to be spontaneously broken and the usual causality jeopardized.[10]

To conclude, it therefore appears that, at the same time as causal laws are replaced in their theoretical fecundity and their explanatory scope by analyses of "geometric" structures and transformations, it is the very concept of causality which regains its status of regulatory concept and which moves away from the constitutive role we have more or less consciously conferred to it. To return to the remarks we have already formulated according to a different perspective, if the phenomenal explanation and the description of *sensible manifestations* continue in physics to have recourse to the concept of cause, on the other hand, the *constitution* of identity and of physical objectivity is increasingly related to the mathematical structures which theorize them and now confer upon them predictive power. This provides renewed topicality to the considerations voiced by Weyl and his friends (such as my interlocutor).

1.3.6 *Towards the "cognitive subject"*

Let's return to another aspect of Longo's "Mathematical concepts and physical objects." It contains remarks which may constitute the basis of a novel inquiry into human cognition. Let's report two of these indications which deserve further reflections. He writes: *"Foundational analysis, in mathematics and in physics, must propose a scientific analysis of the cognitive*

[10]Note that the spontaneous symmetry breaking, in classical frames, has a random origin, which, in contrast to what happens with intrinsic quantum fluctuations, does not throw causality as such into question, but concerns only observability (it is epistemic).

subject and then highlight the objectivity of the construction of knowledge within its reference frameworks or systems". And further on, from a more specific angle: *"The project of a cognitive analysis of the foundations of mathematics requires an explanation of the cognitive subject, as living unity of body and mind, living within intersubjectivity and within history. This subject who traces on the phenomenal veil objects and structures, spaces and concepts which are common to mathematics and to physics, lays outlines which are not 'already there,' but which result from the interaction between ourselves and the world."*

This approach, in a way, takes the direction opposite to Boole, Frege, and their logicist and formalist upholders who wanted to find in logic and formalism "the laws of thought," well, at least those governing mathematical thinking. Without denying the advances enabled by the developments within these fields, Longo proposes on the other hand to see and find essential elements enabling the characterization and objective analysis of the cognitive subject in the development of the practice and in mathematical structures themselves. He does not base this proposition solely upon mathematical rigor and demonstrative capacities (prevalence of proof principles), but also on the conceptual stability they authorize as well as on their aptitude to objectively thematize and categorize the quasi-kinesthetic relationships to experiences as primitive as those related to movement, to space, to order (role of construction principles). Beyond their spontaneous language-related apprehensions (which we know to vary according to culture and civilization), inquiries on the basis of the analysis of "behaviors" and mathematical approaches would enable us in a way to stabilize these experiences and intuitions by abstracting them, by objectivizing and universalizing them, and also by revealing the deepest of cognitive workings which animate them.