

# ‘What is a Thing?’: Topos Theory in the Foundations of Physics<sup>1</sup>

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## Abstract

*“From the range of the basic questions of metaphysics we shall here ask this one question: “What is a thing?” The question is quite old. What remains ever new about it is merely that it must be asked again and again [36].”*

**Martin Heidegger**

The goal of this paper is to summarise the first steps in developing a fundamentally new way of constructing theories of physics. The motivation comes from a desire to address certain deep issues that arise when contemplating quantum theories of space and time. In doing so we provide a new answer to Heidegger’s timeless question “What is a thing?”.

Our basic contention is that constructing a theory of physics is equivalent to finding a representation in a topos of a certain formal language that is attached to the system. Classical physics uses the topos of sets. Other theories involve a different topos. For the types of theory discussed in this paper, a key goal is to represent any physical quantity  $A$  with an arrow  $\hat{A}_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$  where  $\Sigma_\phi$  and  $\mathcal{R}_\phi$  are two special objects (the ‘state-object’ and ‘quantity-value object’) in the appropriate topos,  $\tau_\phi$ .

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We discuss two different types of language that can be attached to a system,  $S$ . The first,  $\mathcal{P}\mathcal{L}(S)$ , is a propositional language; the second,  $\mathcal{L}(S)$ , is a higher-order, typed language. Both languages provide deductive systems with an intuitionistic logic. With the aid of  $\mathcal{P}\mathcal{L}(S)$  we expand and develop some of the earlier work<sup>4</sup> on topos theory and quantum physics. A key step is a process we term ‘daseinisation’ by which a projection operator is mapped to a sub-object of the spectral presheaf  $\underline{\Sigma}$ —the topos quantum analogue of a classical state space. The topos concerned is  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ : the category of contravariant set-valued functors on the category (partially ordered set)  $\mathcal{V}(\mathcal{H})$  of commutative sub-algebras of the algebra of bounded operators on the quantum Hilbert space  $\mathcal{H}$ .

There are two types of daseinisation, called ‘outer’ and ‘inner’: they involve approximating a projection operator by projectors that are, respectively, larger and smaller in the lattice of projectors on  $\mathcal{H}$ .

We then introduce the more sophisticated language  $\mathcal{L}(S)$  and use it to study ‘truth objects’ and ‘pseudo-states’ in the topos. These objects topos play the role of states: a necessary development as the spectral presheaf has no global elements, and hence there are no microstates in the sense of classical physics.

One of the main mathematical achievements is finding a topos representation for self-adjoint operators. This involves showing that, for any bounded, self-adjoint operator  $\hat{A}$ , there is a corresponding arrow  $\delta^o(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\succeq}$  where  $\underline{\mathbb{R}}^{\succeq}$  is the quantity-value object for this theory. The construction of  $\delta^o(\hat{A})$  is an extension of the daseinisation of projection operators.

The object  $\underline{\mathbb{R}}^{\succeq}$  is a monoid-object only in the topos,  $\tau_\phi = \mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , of the theory, and to enhance the applicability of the formalism we discuss another candidate,  $\underline{\mathbb{R}}^{\leftrightarrow}$ , for the quantity-value object. In this presheaf, both inner- and outer-daseinisation are used in a symmetric way. Another option is to apply to  $\underline{\mathbb{R}}^{\succeq}$  a topos analogue of the Grothendieck extension of a monoid to a group. The resulting object,  $k(\underline{\mathbb{R}}^{\succeq})$ , is an abelian group-object in  $\tau_\phi$ .

Finally we turn to considering a *collection* of systems: in particular, we are interested in the relation between the topos representation of a composite system, and the representations of its constituents. Our approach to these matters is to construct a *category* of systems and to find coherent topos representations of the entire category.

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<sup>4</sup>By CJI and collaborators.

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# 1 Introduction

Many people who work in quantum gravity would agree that a deep change in our understanding of foundational issues will occur at some point along the path. However, opinions differ greatly on whether a radical revision is necessary at the very *beginning* of the process, or if it will emerge ‘along the way’ from an existing, or future, research programme that is formulated using the current paradigms. For example, many (albeit not all) of the current generation of string theorists seem inclined to this view, as do a, perhaps smaller, fraction of those who work in loop quantum gravity.

In this article we take the iconoclastic view that a radical step is needed at the very outset. However, for anyone in this camp the problem is always knowing where to start. It is easy to talk about a ‘radical revision of current paradigms’—the phrase slips lightly off the tongue—but converting this pious hope into a concrete theoretical structure is a problem of the highest order.

For us, the starting point is quantum theory itself. More precisely, we believe that this theory needs to be radically revised, or even completely replaced, before a satisfactory theory of quantum gravity can be obtained.

In this context, a striking feature of the various current programmes for quantising gravity—including superstring theory and loop quantum gravity—is that, notwithstanding their disparate views on the nature of space and time, they almost all use more-or-less standard quantum theory. Although understandable from a pragmatic viewpoint (since all we have *is* more-or-less standard quantum theory) this situation is nevertheless questionable when viewed from a wider perspective.

For us, one of the most important issues is the use in the standard quantum formalism of critical mathematical ingredients that are taken for granted and yet which, we claim, implicitly assume certain properties of space and/or time. Such an *a priori* imposition of spatio-temporal concepts would be a major category<sup>5</sup> error if they turn out to be fundamentally incompatible with what is needed for a theory of quantum gravity.

A prime example is the use of the *continuum*<sup>6</sup> by which, in this context, is meant the real and/or complex numbers. These are a central ingredient in all the various mathematical frameworks in which quantum theory is commonly discussed. For example, this is clearly so with the use of (i) Hilbert

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<sup>5</sup>The philosophy of Kant runs strongly in our veins.

<sup>6</sup>When used in this rather colloquial way, the word ‘continuum’ suggests primarily the cardinality of the sets concerned, and, secondly, the topology that is conventionally placed on these sets.



spaces or  $C^*$ -algebras; (ii) geometric quantisation; (iii) probability functions on a non-distributive quantum logic; (iv) deformation quantisation; and (v) formal (i.e., mathematically ill-defined) path integrals and the like. The *a priori* imposition of such continuum concepts could be radically incompatible with a quantum-gravity formalism in which, say, space-time is fundamentally discrete: as, for example, in the causal-set programme.

As we shall argue later, this issue is closely connected with the question of what is meant by the ‘value’ of a physical quantity. In so far as the concept is meaningful at all at the Planck scale, why should the value be a real number defined mathematically in the usual way?

Another significant reason for aspiring to change the quantum formalism is the peristalithic problem of deciding how a ‘quantum theory of cosmology’ could be interpreted if one was lucky enough to find one. Most people who worry about foundational issues in quantum gravity would probably place the quantum-cosmology/closed-system problem at, or near, the top of their list of reasons for re-envisioning quantum theory. However, although we are deeply interested in such conceptual issues, the primary motivation for our research programme is not to find a new interpretation of quantum theory. Rather, our main goal is to find a novel structural framework within which new *types* of theories of physics can be constructed.

However, having said that, in the context of quantum cosmology it is certainly true that the lack of any external ‘observer’ of the universe ‘as a whole’ renders inappropriate the standard Copenhagen interpretation with its instrumentalist use of counterfactual statements about what *would* happen *if* a certain measurement is performed. Indeed, the Copenhagen interpretation is inapplicable for *any*<sup>7</sup> system that is truly ‘closed’ (or ‘self-contained’) and for which, therefore, there is no ‘external’ domain in which an observer can lurk. This problem has motivated much research over the years and continues to be of wide interest.

The philosophical questions that arise are profound, and look back to the birth of Western philosophy in ancient Greece, almost three thousand years ago. Of course, arguably, the longevity of these issues suggests that these questions are ill-posed in the first place, in which case the whole enterprise is a complete waste of time! This is probably the view of most, if not all, of our colleagues at Imperial College; but we beg to differ<sup>8</sup>.

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<sup>7</sup>The existence of the long-range, and all penetrating, gravitational force means that, at a fundamental level, there is only *one* truly closed system, and that is the universe itself.

<sup>8</sup>Of course, it is also possible that our colleagues are right.

When considering a closed system, the inadequacy of the conventional instrumentalist interpretation of quantum theory encourages the search for an interpretation that is more ‘realist’ in some way. For over eighty years, this has been a recurring challenge for those concerned with the conceptual foundations of modern physics. In rising to this challenge we join our Greek ancestors in confronting once more the fundamental question:<sup>9</sup>

“What is a thing?”

Of course, as written, the question is itself questionable. For many philosophers, including Kant, would assert that the correct question is not “What is a thing?” but rather “What is a thing as it appears to *us*?” However, notwithstanding Kant’s strictures, we seek the thing-in-itself, and, therefore, we persevere with Heidegger’s form of the question.

Nevertheless, having said that, we can hardly ignore the last three thousand years of philosophy. In particular, we must defend ourselves against the charge of being ‘naive realists’.<sup>10</sup> At this point it become clear that theoretical physicists have a big advantage over professional philosophers. For we are permitted/required to study such issues in the context of specific mathematical frameworks for addressing the physical world; and one of the great fascinations of this process is the way in which various philosophical positions are implicit in the ensuing structures. For example, the exact meaning of ‘realist’ is infinitely debatable but, when used by a classical physicist, it invariably means the following:

1. The idea of ‘a property of the system’ (for example, ‘the value of a physical quantity at a certain time’) is meaningful, and mathematically representable in the theory.
2. Propositions about the system (typically asserting that the system has this or that property) are handled using Boolean logic. This requirement is compelling in so far as we humans are inclined to think in a Boolean way.

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<sup>9</sup>“What is a thing?” is the title of one of the more comprehensible of Heidegger’s works [36]. By this, we mean comprehensible to the authors of the present article. We cannot speak for our colleagues across the channel: from some of them we may need to distance ourselves.

<sup>10</sup>If we were professional philosophers this would be a terrible insult. :-)

3. There is a space of ‘microstates’ such that specifying a microstate<sup>11</sup> leads to unequivocal truth values for all propositions about the system: i.e., a state<sup>12</sup> encodes “the way things are”. This is a natural way of ensuring that the first two conditions above are satisfied.

The standard interpretation of classical physics satisfies these requirements and provides the paradigmatic example of a realist philosophy in science. Heidegger’s answer to his own question adopts a similar position [36]:

*“A thing is always something that has such and such properties, always something that is constituted in such and such a way. This something is the bearer of the properties; the something, as it were, that underlies the qualities.”*

In quantum theory, the situation is very different. There, the existence of any such realist interpretation is foiled by the famous Kochen-Specker theorem [50]. This asserts that it is impossible to assign values to all physical quantities at once if this assignment is to satisfy the consistency condition that the value of a function of a physical quantity is that function of the value. For example, the value of ‘energy squared’ is the square of the value of energy.

Thus, from a conceptual perspective, the challenge is to find a quantum formalism that is ‘realist enough’ to provide an acceptable alternative to the Copenhagen interpretation, with its instrumentally-construed intrinsic probabilities, whilst taking on board the implications of the Kochen-Specker theorem.

So, *in toto* what we seek is a formalism that is (i) free of *prima facie* prejudices about the nature of the values of physical quantities—in particular, there should be no fundamental use of the real or complex numbers; and (ii) ‘realist’, in at least the minimal sense that propositions are meaningful, and are assigned ‘truth values’, not just instrumentalist probabilities of what would happen if appropriate measurements are made.

However, finding such a formalism is not easy: it is notoriously difficult to modify the mathematical framework of quantum theory without destroying

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<sup>11</sup>In simple non-relativistic systems, the state is specified at any given moment of time. Relativistic systems (particularly quantum gravity!) require a more sophisticated understanding of ‘state’, but the general idea is the same.

<sup>12</sup>We are a little slack in our use of language here and in what follows by frequently referring to a microstate as just a ‘state’. The distinction only becomes important if one wants to introduce things like mixed states (in quantum theory), or macrostates (in classical physics) all of which are often just known as ‘states’. Then one must talk about microstates (pure states) to distinguish them from the other type of state.

the entire edifice. In particular, the Hilbert space structure is very rigid and cannot easily be changed; and the formal path-integral techniques do not fare much better.

To seek inspiration let us return briefly to the situation in classical physics. There, the concept of realism (as asserted in the three statements above) is encoded mathematically in the idea of a space of states,  $\mathcal{S}$ , where specifying a particular state (or ‘micro-state’),  $s \in \mathcal{S}$ , determines entirely ‘the way things are’ for the system. In particular, this suggests that each physical quantity  $A$  should be associated with a real-valued function  $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$  such that when the state of the system is  $s$ , the value of  $A$  is  $\check{A}(s)$ . Of course, this is indeed precisely how the formalism of classical physics works.

In the spirit of general abstraction, one might wonder if this formalism can be generalised to a structure in which  $A$  is represented by an arrow  $\check{A} : \Sigma \rightarrow \mathcal{R}$  where  $\Sigma$  and  $\mathcal{R}$  are objects in some category,  $\tau$ , other than the category of sets, **Sets**? In such a theory, one would seek to represent propositions about the ‘values’ (whatever that might mean) of physical quantities with sub-objects of  $\Sigma$ , just as in classical physics propositions are represented by subsets of the state space  $\mathcal{S}$  (see Section 2.2 for more detail of this).

Our central conceptual idea is that such a categorial structure constitutes a generalisation of the concept of ‘realism’ in which the ‘values’ of a physical quantity are coded in the arrow  $\check{A} : \Sigma \rightarrow \mathcal{R}$ .

Clearly the propositions will play a key role in any such theory, and, presumably, the minimum required is that the associated sub-objects of  $\Sigma$  form some sort of ‘logic’, just as the subsets of  $\mathcal{S}$  form a Boolean algebra.

This rules out most categories since, generically, the sub-objects of an object do not have any logical structure. However, if the category  $\tau$  is a ‘topos’ then the sub-objects of any object *do* have this property, and hence the current research programme.

Our suggestion, therefore, is to try to construct physical theories that are formulated in a topos other than **Sets**. This topos will depend on both the theory-type and the system. More precisely, if a theory-type (such as classical physics, or quantum physics) is applicable to a certain class of systems, then, for each system in this class, there is a topos in which the theory is to be formulated. For some theory-types the topos is system-independent: for example, classical physics always uses the topos of sets. For other theory-

types, the topos varies from system to system: as we shall see, this is the case in quantum theory.

In somewhat more detail, any particular example of our suggested scheme will have the following ingredients:

1. There are two special objects in the topos  $\tau_\phi$ : the ‘state-object’<sup>13</sup>,  $\Sigma_\phi$  and the ‘quantity-value object’,  $\mathcal{R}_\phi$ . Any physical quantity,  $A$ , is represented by an arrow  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$  in the topos. Whatever meaning can be ascribed to the concept of the ‘value’ of a physical quantity is encoded in (or derived from) this representation.
2. Propositions about a system are represented by sub-objects of the state-object  $\Sigma_\phi$ . These sub-objects form a Heyting algebra (as indeed do the sub-objects of any object in a topos): a distributive lattice that differs from a Boolean algebra only in that the *law of excluded middle* need not hold, i.e.,  $\alpha \vee \neg\alpha \preceq 1$ . A Boolean algebra is a Heyting algebra with strict equality:  $\alpha \vee \neg\alpha = 1$ .
3. Generally speaking (and unlike in set theory), an object in a topos may not be determined by its ‘points’. In particular, this may be so for the state-object, in which case the concept of a microstate is not so useful.<sup>14</sup> Nevertheless, truth values can be assigned to propositions with the aid of a ‘truth object’ (or ‘pseudo-state’). These truth values lie in another Heyting algebra.

Of course, it is not instantly obvious that quantum theory can be written in this way. However, as we shall see, there *is* a topos reformulation of quantum theory, and this has two immediate implications. The first is that we acquire a new type of ‘realist’ interpretation of standard quantum theory. The second is that this new approach suggests ways of generalising quantum theory that make no fundamental reference to Hilbert spaces, path integrals, etc. In particular, there is no *prima facie* reason for introducing standard continuum quantities. As emphasised above, this is one of our main motivations for developing the topos approach. We shall say more about this later.

From a conceptual perspective, a central feature of our scheme is the ‘neo-realist’ structure reflected mathematically in the three statements above.

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<sup>13</sup>The meaning of the subscript ‘ $\phi$ ’ is explained in the main text. It refers to a particular topos-representation of a formal language attached to the system.

<sup>14</sup>In quantum theory, the state-object has no points/microstates at all. As we shall see, this statement is equivalent to the Kochen-Specker theorem.

This neo-realism is the conceptual fruit of the fact that, from a categorial perspective, a physical theory expressed in a topos ‘looks’ like *classical* physics expressed in the topos of sets.

The fact that (i) physical quantities are represented by arrows whose domain is the state-object,  $\Sigma_\phi$ ; and (ii) propositions are represented by sub-objects of  $\Sigma_\phi$ , suggests strongly that  $\Sigma_\phi$  can be regarded as the topos-analogue of a classical state space. Indeed, for any classical system the topos is just the category of sets, **Sets**, and the ideas above reduce to the familiar picture in which (i) there is a state space (set)  $\mathcal{S}$ ; (ii) any physical quantity,  $A$ , is represented by a real-valued functions  $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$ ; and (iii) propositions are represented by subsets of  $\mathcal{S}$  with a logical structure given by the associated Boolean algebra.

Evidently the suggested mathematical structures could be used in two different ways. The first is that of the ‘conventional’ theoretical physicist with little interest in conceptual matters. For him/her, what we and our colleagues are developing is a new tool-kit with which to construct novel types of theoretical model. Whether or not Nature has chosen such models remains to be seen, but, at the very least, the use of topoi certainly suggests new techniques.

For those physicists who are interested in conceptual issues, the topos framework gives a radically new way of thinking about the world. The neo-realism inherent in the formalism is described mathematically using the *internal* language that is associated with any topos. This describes how things look from ‘within’ the topos: something that should be particularly useful in the context of quantum cosmology<sup>15</sup>.

On the other hand, the pragmatic theoretician with no interest in conceptual matters can use the ‘external’ description of the topos in which the category of sets provides a metalanguage with which to formulate the theory. From a mathematical perspective, the interplay between the internal and external languages of a topos is one of the fascinations of the subject. However, much remains to be said about the significance of this interaction for real theories of physics.

This present article is partly an amalgam of a series of four papers that we placed on the ArXiv server<sup>16</sup> in March, 2007 [21, 22, 23, 24]. However, we have added a fair amount of new material, and also made a few minor

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<sup>15</sup>In this context see the work of Markopoulou who considers a topos description of the universe as seen by different observers who live inside it [58].

<sup>16</sup>These are due to published in *Journal of Mathematical Physics* in the Spring of 2008.

corrections (mainly typos).<sup>17</sup> We have also added some remarks about developments made by researchers other than ourselves since the ArXiv preprints were written. Of particular importance to our general programme is the work of Heunen and Spitters [38] which adds some powerful ingredients to the topoi-in-physics toolkit. Finally, we have included some background material from the earlier papers that formed the starting point for the current research programme [44, 45, 35, 13].

We must emphasise that this is *not* a review article about the general application of topos theory to physics; this would have made the article far too long. For example, there has been a fair amount of study of the use of synthetic differential geometry in physics. The reader can find references to much of this on the, so-called, ‘Siberian toposes’ web site<sup>18</sup>. There is also the work by Mallios and collaborators on ‘Abstract Differential Geometry’ [59, 60, 66, 61]. Of course, as always these days, Google will speedily reveal all that we have omitted.

But even less is this paper a review of the use of category theory in general in physics. For there are many important topics that we do not mention at all. For example, Baez’s advocacy of  $n$ -categories [5, 6]; ‘categorical quantum theory’ [2, 73]; Takeuti’s theory<sup>19</sup> of ‘quantum sets’ [71]; and Crane’s work on categorical models of space-time [17].

Finally, a word about the style in which this article is written. We spent much time pondering on this, as we did before writing the four ArXiv preprints. The intended audience is our colleagues who work in theoretical physics, especially those whose interests included foundational issues in quantum gravity and quantum theory. However, topos theory is not an easy branch of mathematics, and this poses the dilemma of how much background mathematics should be assumed of the reader, and how much should be explained as we go along.<sup>20</sup> We have approached this problem by including a short mathematical appendix on topos theory. However, reasons of space precluded a thorough treatment, and we hope that, fairly soon, someone will write an introductory review of topos theory in a style that is accessible to a typical theoretical-physicist reader.

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<sup>17</sup>Some of the more technical theorems have been placed in the Appendix with the hope that this makes the article a little easier to read.

<sup>18</sup>This is <http://users.univer.omsk.su/~topoi/>. See also Cecilia Flori’s website that deals more generally with topos theory and physics: <http://topos-physics.org/>

<sup>19</sup>Takeuti’s work is not exactly about category theory applied to quantum theory: it is more about the use of formal logic, but the spirit is similar. For a recent paper in this genre see [64].

<sup>20</sup>The references that we have found most helpful in our research are [55, 29, 52, 8, 56, 48].



This article is structured in the following way. We begin with a discussion of some of the conceptual background, in particular the role of the real numbers in conventional theoretical physics. Then in Section 3 we introduce the idea of attaching a propositional language,  $\mathcal{PL}(S)$ , to each physical system  $S$ . The intent is that each theory of  $S$  corresponds to a particular representation of  $\mathcal{PL}(S)$ . In particular, we show how classical physics satisfies this requirement in a very natural way.

Propositional languages have limited scope (they lack the quantifiers ‘ $\forall$ ’ and ‘ $\exists$ ’), and in Section 4 we propose the use of a higher-order language  $\mathcal{L}(S)$ . Languages of this type are a central feature of topos theory and it is natural to consider the idea of representing  $\mathcal{L}(S)$  in different topoi. Classical physics always takes place in the topos, **Sets**, of sets but our expectation is that other areas of physics will use a different topos.

This expectation is confirmed in Section 5 where we discuss in detail the representation of  $\mathcal{PL}(S)$  for a quantum system (the representation of  $\mathcal{L}(S)$  is discussed in Section 8). The central idea is to represent propositions as sub-objects of the ‘spectral presheaf’  $\underline{\Sigma}$  which belongs to the topos,  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , of presheaves (set-valued, contravariant functors) on the category,  $\mathcal{V}(\mathcal{H})$ , of abelian sub-algebras of the algebra  $B(\mathcal{H})$  of all bounded operators on  $\mathcal{H}$ . This representation employs the idea of ‘daseinisation’ in which any given projection operator  $\hat{P}$  is represented at each context/stage-of-truth  $V$  in  $\mathcal{V}(\mathcal{H})$  by the ‘closest’ projector to it in  $V$ . There are two variants of this: (i) ‘outer’ daseinisation, in which  $\hat{P}$  is approached from above (in the lattice of projectors in  $V$ ); and (ii) ‘lower’ daseinisation, in which  $\hat{P}$  is approached from below.

The next key move is to discuss the ‘truth values’ of propositions in a quantum theory. This requires the introduction of some analogue of the microstates of classical physics. We say ‘analogue’ because the spectral presheaf  $\underline{\Sigma}$ —which is the quantum topos equivalent of a classical state space—has no global elements, and hence there are no microstates at all: this is equivalent to the Kochen-Specker theorem. The critical idea is that of a ‘truth object’, or ‘pseudo-state’ which, as we show in Section 6, is the closest one can get in quantum theory to a microstate.

In Section 7 we introduce the ‘de Groote’ presheaves and the associated ideas that lead to the concept of daseinising an arbitrary bounded self-adjoint operator, not just a projector. Then, in Section 8, the spectral theorem is used to construct several possible models for the quantity-value presheaf in quantum physics. The simplest choice is  $\underline{\mathbb{R}}^{\succeq}$ , but this uses only outer daseinisation, and a more balanced choice is  $\underline{\mathbb{R}}^{\leftrightarrow}$  which uses both inner and



outer daseinisation. Another possibility is  $k(\underline{\mathbb{R}}^{\supset})$ : the Grothendieck topos extension of the monoid object  $\underline{\mathbb{R}}^{\supset}$ . A key result is the ‘non-commutative spectral theorem’ which involves showing how each bounded, self-adjoint operator  $\hat{A}$  can be represented by an arrow  $\check{A} : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\leftarrow}$ .

In Section 10 we discuss the way in which unitary operators act on the quantum topos objects. Then, in Sections 11, 12 and 13 we discuss the problem of handling ‘all’ possible systems in a single coherent scheme. This involves introducing a category of systems which, it transpires, has a natural monoidal structure. We show in detail how this scheme works in the case of classical and quantum theory.

Finally, in Section 14 we discuss/speculate on some properties of the state object, quantity-value object, and truth objects that might be present in any topos representation of a physical system.

To facilitate reading this long article, some of the more technical material has been put in Appendix 1. In Appendix 2 there is a short introduction to some of the relevant parts of topos theory.

## 2 The Conceptual Background of our Scheme

### 2.1 The Problem of Using Real Numbers a Priori

As mentioned in the Introduction, one of the main goals of our work is to find new tools with which to develop theories that are significant extensions of, or developments from, quantum theory but without being tied *a priori* to the use of the standard real or complex numbers.

In this context we note that real numbers arise in theories of physics in three different (but related) ways: (i) as the values of physical quantities; (ii) as the values of probabilities; and (iii) as a fundamental ingredient in models of space and time (especially in those based on differential geometry). All three are of direct concern vis-a-vis our worries about making unjustified, *a priori* assumptions in quantum theory. We shall now examine them in detail.

#### 2.1.1 Why Are Physical Quantities Assumed to be Real-Valued?

One reason for assuming physical quantities are real-valued is undoubtedly grounded in the remark that, traditionally (i.e., in the pre-digital age), they are measured with rulers and pointers, or they are defined operationally in terms of such measurements. However, rulers and pointers are taken to be

classical objects that exist in the physical space of classical physics, and this space is modelled using the reals. In this sense there is a direct link between the space in which physical quantities take their values (what we call the ‘quantity-value space’) and the nature of physical space or space-time [41].

If conceded, this claim means the assumption that physical quantities are real-valued is problematic in any theory in which space, or space-time, is not modelled by a smooth manifold. Admittedly, if the theory employs a *background* space, or space-time—and if this background is a manifold—then the use of real-valued physical quantities *is* justified in so far as their value-space can be related to this background. Such a stance is particularly appropriate in situations where the background plays a central role in giving meaning to concepts like ‘observers’ and ‘measuring devices’, and thereby provides a basis for an instrumentalist interpretation of the theory.

But even here caution is needed since many theoretical physicists have claimed that the notion of a ‘space-time point in a manifold’ is intrinsically flawed. One argument (due to Penrose) is based on the observation that any attempt to localise a ‘thing’ is bound to fail beyond a certain point because of the quantum production of pairs of particles from the energy/momentum uncertainty caused by the spatial localisation. Another argument concerns the artificiality<sup>21</sup> of the use of real numbers as coordinates with which to identify a space-time point. There is also Einstein’s famous ‘hole argument’ in general relativity which asserts that the notion of a space-time point (in a manifold) has no physical meaning in a theory that is invariant under the group of space-time diffeomorphisms.

Another cautionary caveat concerning the invocation of a background is that this background structure may arise only in some ‘sector’ of the theory; or it may exist only in some limiting, or approximate, sense. The associated instrumentalist interpretation would then be similarly limited in scope. For this reason, if no other, a ‘realist’ interpretation is more attractive than an instrumentalist one.

In fact, in such circumstances, the phrase ‘realist interpretation’ does not really do justice to the situation since it tends to imply that there are other interpretations of the theory, particularly instrumentalism, with which the realist one can contend on a more-or-less equal footing. But, as we just argued, the instrumentalist interpretation may be severely limited as compared to the realist one. To flag this point, we will sometimes refer to a

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<sup>21</sup>The integers, and associated rationals, have a ‘natural’ interpretation from a physical perspective since we can all count. On the other hand, the Cauchy-sequence and/or the Dedekind-cut definitions of the reals are distinctly un-intuitive from a physical perspective.

‘realist formalism’, rather than a ‘realist interpretation’.<sup>22</sup>

### 2.1.2 Why Are Probabilities Required to Lie in the Interval $[0, 1]$ ?

The motivation for using the subset  $[0, 1]$  of the real numbers as the value space for probabilities comes from the relative-frequency interpretation of probability. Thus, in principle, an experiment is to be repeated a large number,  $N$ , times, and the probability associated with a particular result is defined to be the ratio  $N_i/N$ , where  $N_i$  is the number of experiments in which that result was obtained. The rational numbers  $N_i/N$  necessarily lie between 0 and 1, and if the limit  $N \rightarrow \infty$  is taken—as is appropriate for a hypothetical ‘infinite ensemble’—real numbers in the closed interval  $[0, 1]$  are obtained.

The relative-frequency interpretation of probability is natural in instrumentalist theories of physics, but it is not meaningful if there is no classical spatio-temporal background in which the necessary measurements could be made; or, if there is a background, it is one to which the relative-frequency interpretation cannot be adapted.

In the absence of a relative-frequency interpretation, the concept of ‘probability’ must be understood in a different way. In the physical sciences, one of the most discussed approaches involves the concept of ‘potentiality’, or ‘latency’, as favoured by Heisenberg [37], Margenau [57], and Popper [65] (and, for good measure, Aristotle). In this case there is no compelling reason why the probability-value space should necessarily be a subset of the real numbers. The minimal requirement is that this value-space is an ordered set, so that one proposition can be said to be more or less probable than another. However, there is no *prima facie* reason why this set should be *totally* ordered: i.e., there may be pairs of propositions whose potentialities cannot be compared—something that seems eminently plausible in the context of non-commensurable quantities in quantum theory.

By invoking the idea of ‘potentiality’, it becomes feasible to imagine a quantum-gravity theory with no spatio-temporal background but where probability is still a fundamental concept. However, it could also be that the concept of probability plays no fundamental role in such circumstances, and can be given a meaning *only* in the context of a sector, or limit, of the

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<sup>22</sup>Of course, such discussions are unnecessary in classical physics since, there, if knowledge of the value of a physical quantity is gained by making a (ideal) measurement, the reason why we obtain the result that we do, is because the quantity *possessed* that value immediately before the measurement was made. In other words, “epistemology models ontology”.

theory where a background does exist. This background could then support a limited instrumentalist interpretation which would include a (limited) relative-frequency understanding of probability.

In fact, most modern approaches to quantum gravity aspire to a formalism that is background independent [4, 15, 67, 68]. So, if a background space does arise, it will be in one of the restricted senses mentioned above. Indeed, it is often asserted that a proper theory of quantum gravity will not involve *any* direct spatio-temporal concepts, and that what we commonly call ‘space’ and ‘time’ will ‘emerge’ from the formalism only in some appropriate limit [12]. In this case, any instrumentalist interpretation could only ‘emerge’ in the same limit, as would the associated relative-frequency interpretation of probability.

In a theory of this type, there will be no *prima facie* link between the values of physical quantities and the nature of space or space-time, although, of course, this cannot be totally ruled out. In any event, part of the fundamental specification of the theory will involve deciding what the ‘quantity-value space’ should be.

These considerations suggest that quantum theory must be radically changed if one wishes to accommodate situations where there is no background space/space-time, manifold within which an instrumentalist interpretation can be formulated. In such a situation, some sort of ‘realist’ formalism is essential.

These reflections also suggest that the quantity-value space employed in an instrumentalist realisation of a theory—or a ‘sector’, or ‘limit’, of the theory—need not be the same as the quantity-value space in a neo-realist formulation. At first sight this may seem strange but, as is shown in Section 8, this is precisely what happens in the topos reformulation of standard quantum theory.

## 2.2 The Genesis of Topos Ideas in Physics

### 2.2.1 Why are Space and Time Modelled with Real Numbers?

Even setting aside the more exotic considerations of quantum gravity, one can still query the use of real numbers to model space and/or time. One might argue that (i) the use of (triples of) real numbers to model space is based on empirically-based reflections about the nature of ‘distances’ between objects; and (ii) the use of real numbers to model time reflects our experience that ‘instants of time’ appear to be totally ordered, and that intervals of time are

always divisible<sup>23</sup>.

However, what does it really mean to say that two particles are separated by a distance of, for example,  $\sqrt{2}$ cms? From an empirical perspective, it would be impossible to make a measurement that could unequivocally reveal precisely that value from among the continuum of real numbers that lie around it. There will always be experimental errors of some sort: if nothing else, there are thermodynamical fluctuations in the measuring device; and, ultimately, uncertainties arising from quantum ‘fluctuations’. Similar remarks apply to attempts to measure time.

Thus, from an operational perspective, the use of real numbers to label ‘points’ in space and/or time is a theoretical abstraction that can never be realised in practice. But if the notion of a space/time/space-time ‘point’ in a continuum, is an abstraction, why do we use it? Of course it works well in theories used in normal physics, but at a fundamental level it must be seen as questionable.

These operational remarks say nothing about the structure of space (or time) ‘in itself’, but, even assuming that this concept makes sense, which is debatable, the use of real numbers is still a metaphysical assumption with no fundamental justification.

Traditionally, we teach our students that measurements of physical quantities that are represented theoretically by real numbers, give results that fall into ‘bins’, construed as being subsets of the real line. This suggests that, from an operational perspective, it would be more appropriate to base mathematical models of space or time on a theory of ‘regions’, rather than the real numbers themselves.

But then one asks “What is a region?”, and if we answer “A subset of triples of real numbers for space, and a subset of real numbers for time”, we are thrown back to the real numbers. One way of avoiding this circularity is to focus on relations between these ‘subsets’ and see if they can be axiomatised in some way. The natural operations to perform on regions are (i) take intersections, or unions, of pairs of regions; and (ii) take the complement of a region. If the regions are modelled on Borel subsets of  $\mathbb{R}$ , then the intersections and unions could be extended to countable collections. If they are modelled on open sets, it would be arbitrary unions and finite intersections.

From a physical perspective, the use of open subsets as models of regions is attractive as it leaves a certain, arguably desirable, ‘fuzziness’ at the edges, which is absent for closed sets. Thus, following this path, we would axioma-

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<sup>23</sup>These remarks are expressed in the context of the Newtonian view of space and time, but it is easy enough to generalise them to special relativity.

tise that a mathematical model of space or time (or space-time) involves an algebra of entities called ‘regions’, and with operations that are the analogue of unions and intersections for subsets of a set. This algebra would allow arbitrary ‘unions’ and finite ‘intersections’, and would distribute<sup>24</sup> over these operations. In effect, we are axiomatising that an appropriate mathematical model of space-time is an object in the category of locales.

However, a locale is the same thing as a complete Heyting algebra (for the definition see below), and, as we shall, Heyting algebras are inexorably linked with topos theory.

### 2.2.2 Another Possible Role for Heyting Algebras

The use of a Heyting algebra to model space/time/space-time is an attractive possibility, and was the origin of the interest in topos theory of one of us (CJI) some years ago. However, there is another motivation which is based more on logic, and the desire to construct a ‘neo-realist’ interpretation of quantum theory.

To motivate topos theory as the source of neo-realism let us first consider classical physics, where everything is defined in the category, **Sets**, of sets and functions between sets. Then (i) any physical quantity,  $A$ , is represented by a real-valued function  $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$ , where  $\mathcal{S}$  is the space of microstates; and (ii) a proposition of the form “ $A \in \Delta$ ” (which asserts that the value of the physical quantity  $A$  lies in the subset  $\Delta$  of the real line  $\mathbb{R}$ )<sup>25</sup> is represented by the subset<sup>26</sup>  $\check{A}^{-1}(\Delta) \subseteq \mathcal{S}$ . In fact any proposition  $P$  about the system is represented by an associated subset,  $\mathcal{S}_P$ , of  $\mathcal{S}$ : namely, the set of states for which  $P$  is true. Conversely, every (Borel) subset of  $\mathcal{S}$  represents a proposition.<sup>27</sup>

It is easy to see how the logical calculus of propositions arises in this picture. For let  $P$  and  $Q$  be propositions, represented by the subsets  $\mathcal{S}_P$  and

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<sup>24</sup>If the distributive law is dropped we could move towards the quantum-set ideas of [71]; or, perhaps, the ideas of non-commutative geometry instigated by Alain Connes [14].

<sup>25</sup>In the rigorous theory of classical physics, the set  $\mathcal{S}$  is a symplectic manifold, and  $\Delta$  is a *Borel* subset of  $\mathbb{R}$ . Also, the function  $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$  may be required to be measurable, or continuous, or smooth, depending on the quantity,  $A$ , under consideration.

<sup>26</sup>Throughout this article we will adopt the notation in which  $A \subseteq B$  means that  $A$  is a subset of  $B$  that could equal  $B$ ; while  $A \subset B$  means that  $A$  is a *proper* subset of  $B$ ; i.e.,  $A$  does not equal  $B$ . Similar remarks apply to other pairs of ordering symbols like  $\prec, \preceq$ ; or  $\succ, \succeq$ , etc.

<sup>27</sup>More precisely, every Borel subset of  $\mathcal{S}$  represents *many* propositions about the values of physical quantities. Two propositions are said to be ‘physically equivalent’ if they are represented by the same subset of  $\mathcal{S}$ .

$\mathcal{S}_Q$  respectively, and consider the proposition “ $P$  and  $Q$ ”. This is true if, and only if, both  $P$  and  $Q$  are true, and hence the subset of states that represents this logical conjunction consists of those states that lie in both  $\mathcal{S}_P$  and  $\mathcal{S}_Q$ —i.e., the set-theoretic intersection  $\mathcal{S}_P \cap \mathcal{S}_Q$ . Thus “ $P$  and  $Q$ ” is represented by  $\mathcal{S}_P \cap \mathcal{S}_Q$ . Similarly, the proposition “ $P$  or  $Q$ ” is true if either  $P$  or  $Q$  (or both) are true, and hence this logical disjunction is represented by those states that lie in  $\mathcal{S}_P$  plus those states that lie in  $\mathcal{S}_Q$ —i.e., the set-theoretic union  $\mathcal{S}_P \cup \mathcal{S}_Q$ . Finally, the logical negation “not  $P$ ” is represented by all those points in  $\mathcal{S}$  that do not lie in  $\mathcal{S}_P$ —i.e., the set-theoretic complement  $\mathcal{S}/\mathcal{S}_P$ .

In this way, a fundamental relation is established between the logical calculus of propositions about a physical system, and the Boolean algebra of subsets of the state space. Thus the mathematical structure of classical physics is such that, *of necessity*, it reflects a ‘realist’ philosophy, in the sense in which we are using the word.

One way to escape from the tyranny of Boolean algebras and classical realism is via topos theory. Broadly speaking, a topos is a category that behaves very much like the category of sets; in particular, the collection of sub-objects of an object forms a *Heyting algebra*, just as the collection of subsets of a set form a Boolean algebra. Our intention, therefore, is to explore the possibility of associating physical propositions with sub-objects of some object  $\Sigma$  (the analogue of a classical state space) in some topos.

A Heyting algebra,  $\mathfrak{h}$ , is a distributive lattice with a zero element,  $0$ , and a unit element,  $1$ , and with the property that to each pair  $\alpha, \beta \in \mathfrak{h}$  there is an implication  $\alpha \Rightarrow \beta$ , characterized by

$$\gamma \preceq (\alpha \Rightarrow \beta) \text{ if and only if } \gamma \wedge \alpha \preceq \beta. \quad (2.1)$$

The negation is defined as  $\neg\alpha := (\alpha \Rightarrow 0)$  and has the property that the *law of excluded middle* need not hold, i.e., there may exist  $\alpha \in \mathfrak{h}$ , such that  $\alpha \vee \neg\alpha \prec 1$  or, equivalently, there may exist  $\alpha \in \mathfrak{h}$  such that  $\neg\neg\alpha \succ \alpha$ . This is the characteristic property of an intuitionistic logic.<sup>28</sup> A Boolean algebra is the special case of a Heyting algebra in which there is the strict equality: i.e.,  $\alpha \vee \neg\alpha = 1$  for all  $\alpha$ . It is known from Stone’s theorem [70] that each Boolean algebra is isomorphic to an algebra of (clopen, i.e., closed and open) subsets of a suitable (topological) space.

<sup>28</sup>Here,  $\alpha \Rightarrow \beta$  is nothing but the category-theoretical exponential  $\beta^\alpha$  and  $\gamma \wedge \alpha$  is the product  $\gamma \times \alpha$ . The definition uses the adjunction between the exponential and the product,  $\text{Hom}(\gamma, \beta^\alpha) = \text{Hom}(\gamma \times \alpha, \beta)$ . A slightly easier, albeit ‘less categorical’ definition is: a Heyting algebra,  $\mathfrak{h}$ , is a distributive lattice such that for any two elements  $\alpha, \beta \in \mathfrak{h}$ , the set  $\{\gamma \in \mathfrak{h} \mid \gamma \wedge \alpha \leq \beta\}$  has a maximal element, denoted by  $(\alpha \Rightarrow \beta)$ .



The elements of a Heyting algebra can be manipulated in a very similar way to those in a Boolean algebra. One of our claims is that, as far as theories of physics are concerned, Heyting logic is a viable<sup>29</sup> alternative to Boolean logic.

To give some idea of the difference between a Boolean algebra and a Heyting algebra, we note that the paradigmatic example of the former is the collection of all measurable subsets of a measure space  $X$ . Here, if  $\alpha \subseteq X$  represents a proposition, the logical negation,  $\neg\alpha$ , is just the set-theoretic complement  $X \setminus \alpha$ .

On the other hand, the paradigmatic example of a Heyting algebra is the collection of all open sets in a topological space  $X$ . Here, if  $\alpha \subseteq X$  is open, the logical negation  $\neg\alpha$  is defined to be the *interior* of the set-theoretical complement  $X \setminus \alpha$ . Therefore, the difference between  $\neg\alpha$  in the topological space  $X$ , and  $\neg\alpha$  in the measurable space generated by the topology of  $X$ , is just the ‘thin’ boundary of the closed set  $X \setminus \alpha$ .

### 2.2.3 Our Main Contention about Topos Theory and Physics

We contend that, for a given theory-type (for example, classical physics, or quantum physics), each system  $S$  to which the theory is applicable is associated with a particular topos  $\tau_\phi(S)$  within whose framework the theory, as applied to  $S$ , is to be formulated and interpreted. In this context, the ‘ $\phi$ ’-subscript is a label that changes as the theory-type changes. It signifies the representation of a system-language in the topos  $\tau_\phi(S)$ : we will come to this later.

The conceptual interpretation of this formalism is ‘neo-realist’ in the following sense:

1. A physical quantity,  $A$ , is to be represented by an arrow  $A_{\phi,S} : \Sigma_{\phi,S} \rightarrow \mathcal{R}_{\phi,S}$  where  $\Sigma_{\phi,S}$  and  $\mathcal{R}_{\phi,S}$  are two special objects in the topos  $\tau_\phi(S)$ . These are the analogues of, respectively, (i) the classical state space,  $\mathcal{S}$ ; and (ii) the real numbers,  $\mathbb{R}$ , in which classical physical quantities take their values.

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<sup>29</sup>The main difference between theorems proved using Heyting logic and those using Boolean logic is that proofs by contradiction cannot be used in the former. In particular, this means that one cannot prove that something exists by arguing that the assumption that it does not leads to contradiction; instead it is necessary to provide a *constructive* proof of the existence of the entity concerned. Arguably, this does not place any major restriction on building theories of physics. Indeed, over the years, various physicists (for example, Bryce DeWitt) have argued that constructive proofs should always be used in physics.



In what follows,  $\Sigma_{\phi,S}$  and  $\mathcal{R}_{\phi,S}$  are called the ‘state object’, and the ‘quantity-value object’, respectively.

2. Propositions about the system  $S$  are represented by sub-objects of  $\Sigma_{\phi,S}$ . These sub-objects form a Heyting algebra.
3. Once the topos analogue of a state (a ‘truth object’) has been specified, these propositions are assigned truth values in the Heyting logic associated with the global elements of the sub-object classifier,  $\Omega_{\tau_\phi(S)}$ , in the topos  $\tau_\phi(S)$ .

Thus a theory expressed in this way *looks* very much like classical physics except that whereas classical physics always employs the topos of sets, other theories—including quantum theory and, we conjecture, quantum gravity—use a different topos.

One deep result in topos theory is that there is an *internal language* associated with each topos. In fact, not only does each topos generate an internal language, but, conversely, a language satisfying appropriate conditions generates a topos. Topoi constructed in this way are called ‘linguistic topoi’, and every topos can be regarded as a linguistic topos. In many respects, this is one of the profoundest ways of understanding what a topos really ‘is’.<sup>30</sup>

These results are exploited in Section 4 where we introduce the idea that, for any applicable theory-type, each physical system  $S$  is associated with a ‘local’ language,  $\mathcal{L}(S)$ . The application of the theory-type to  $S$  is then involves finding a representation of  $\mathcal{L}(S)$  in an appropriate topos; this is equivalent to finding a ‘translation’ of  $\mathcal{L}(S)$  into the internal language of that topos.

Closely related to the existence of this linguistic structure is the striking fact that a topos can be used as a *foundation* for mathematics itself, just as set theory is used in the foundations of ‘normal’ (or ‘classical’) mathematics. In this context, the key remark is that the internal language of a topos has a form that is similar in many ways to the formal language on which normal set theory is based. It is this internal, topos language that is used to *interpret* the theory in a ‘neo-realist’ way.

The main difference with classical logic is that the logic of the topos language does not satisfy the principle of excluded middle, and hence proofs by contradiction are not permitted. This has many intriguing consequences. For example, there are topoi in which there exist genuine *infinitesimals* that

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<sup>30</sup>This aspect of topos theory is discussed at length in the books by Bell [8], and Lambek and Scott [52]

can be used to construct a rival to normal calculus. The possibility of such quantities stems from the fact that the normal proof that they do *not* exist is a proof by contradiction.

Thus each topos carries its own world of mathematics: a world which, generally speaking, is *not* the same as that of classical mathematics.

Consequently, by postulating that, for a given theory-type, each physical system carries its own topos, we are also saying that to each physical system plus theory-type there is associated a framework for mathematics itself! Thus classical physics uses classical mathematics; and quantum theory uses ‘quantum mathematics’—the mathematics formulated in the topos of quantum theory. To this we might add the conjecture: “Quantum gravity uses ‘quantum gravity’ mathematics”!

## 3 Propositional Languages and Theories of Physics

### 3.1 Two Opposing Interpretations of Propositions

Attempts to construct a naïve realist interpretation of quantum theory founder on the Kochen-Specker theorem. However, if, despite this theorem, some degree of realism is still sought, there are not that many options.

One approach is to focus on a particular, maximal commuting subset of physical quantities and declare by fiat that these are the ones that ‘have’ values; essentially, this is what is done in ‘modal’ interpretations of quantum theory. However, this leaves open the question of why Nature should select this particular set, and the reasons proposed vary greatly from one scheme to another.

In our work, we take a completely different approach and try to formulate a scheme which takes into account *all* these different choices for commuting sets of physical quantities; in particular, equal ontological status is ascribed to all of them. This scheme is grounded in the topos-theoretic approach that was first proposed in [44, 45, 35, 13]. This uses a technique whose first step is to construct a category,  $\mathcal{C}$ , the objects of which can be viewed as *contexts* in which the quantum theory can be displayed: in fact, they are just the commuting sub-algebras of operators in the theory. All this will be explained in more detail in Section 5.

In this earlier work, it was postulated that the logic for handling quan-

tum propositions from this perspective is that associated with the topos of presheaves<sup>31</sup> (contravariant functors from  $\mathcal{C}$  to **Sets**),  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ . The idea is that a single presheaf will encode quantum propositions from the perspective of *all* contexts at once. However, in the original papers, the crucial ‘daseinisation’ operation (see Section 5) was not known and, consequently, the discussion became rather convoluted in places. In addition, the generality and power of the underlying procedure was not fully appreciated by the authors.

For this reason, in the present article we return to the basic questions and reconsider them in the light of the overall topos structure that has now become clear.

We start by considering the way in which propositions arise, and are manipulated, in physics. For simplicity, we will concentrate on systems that are associated with ‘standard’ physics. Then, to each such system  $S$  there is associated a set of physical quantities—such as energy, momentum, position, angular momentum etc.<sup>32</sup>—all of which are real-valued. The associated propositions are of the form “ $A \varepsilon \Delta$ ”, where  $A$  is a physical quantity, and  $\Delta$  is a subset<sup>33</sup> of  $\mathbb{R}$ .

From a conceptual perspective, the proposition “ $A \varepsilon \Delta$ ” can be read in two, very different, ways:

- (i) **The (naïve) realist interpretation:** “The physical quantity  $A$  has a value, and that value lies in  $\Delta$ .”
- (ii) **The instrumentalist interpretation:** “If a measurement is made of  $A$ , the result will be found to lie in  $\Delta$ .”

The former is the familiar, ‘commonsense’ understanding of propositions in both classical physics and daily life. The latter underpins the Copenhagen interpretation of quantum theory. Of course, the instrumentalist interpretation can also be applied to classical physics, but it does not lead to anything new. For, in classical physics, what is measured is what *is* the case: “Epistemology models ontology”.

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<sup>31</sup>In quantum theory, the category  $\mathcal{C}$  is just a partially-ordered set, which simplifies many manipulations.

<sup>32</sup>This set does not have to contain ‘*all*’ possible physical quantities: it suffices to concentrate on a subset that are deemed to be of particular interest. However, at some point, questions may arise about the ‘completeness’ of the set.

<sup>33</sup>As was remarked earlier, for various reasons, the subset  $\Delta \subseteq \mathbb{R}$  is usually required to be a *Borel* subset, and for the most part we will assume this without further comment.

We will now study the role of propositions in physics more carefully, particularly in the context of ‘realist’ interpretations.

## 3.2 The Propositional Language $\mathcal{P}\mathcal{L}(S)$

### 3.2.1 Intuitionistic Logic and the Definition of $\mathcal{P}\mathcal{L}(S)$

We are going to construct a formal language,  $\mathcal{P}\mathcal{L}(S)$ , with which to express propositions about a physical system,  $S$ , and to make deductions concerning them. Our intention is to interpret these propositions in a ‘realist’ way: an endeavour whose mathematical underpinning lies in constructing a representation of  $\mathcal{P}\mathcal{L}(S)$  in a Heyting algebra,  $\mathfrak{H}$ , that is part of the mathematical framework involved in the application of a particular theory-type to  $S$ .

In constructing  $\mathcal{P}\mathcal{L}(S)$  we suppose that we have first identified some set,  $\mathcal{Q}(S)$ , of *physical quantities*: this plays a fundamental role in our language. In addition, for *any* system  $S$ , we have the set,  $P_B\mathbb{R}$  of (Borel) subsets of  $\mathbb{R}$ . We use the sets  $\mathcal{Q}(S)$  and  $P_B\mathbb{R}$  to construct the ‘primitive propositions’ about the system  $S$ . These are of the form “ $A \varepsilon \Delta$ ” where  $A \in \mathcal{Q}(S)$  and  $\Delta \in P_B\mathbb{R}$ .

We denote the set of all such strings by  $\mathcal{P}\mathcal{L}(S)_0$ . Note that what has been here called a ‘physical quantity’ could better (but more clumsily) be termed the ‘name’ of the physical quantity. For example, when we talk about the ‘energy’ of a system, the word ‘energy’ is the same, and functions in the same way in the formal language, irrespective of the details of the actual Hamiltonian of the system.

The primitive propositions “ $A \varepsilon \Delta$ ” are used to define ‘sentences’. More precisely, a new set of symbols  $\{\neg, \wedge, \vee, \Rightarrow\}$  is added to the language, and then a *sentence* is defined inductively by the following rules (see Ch. 6 in [29]):

1. Each primitive proposition “ $A \varepsilon \Delta$ ” in  $\mathcal{P}\mathcal{L}(S)_0$  is a sentence.
2. If  $\alpha$  is a sentence, then so is  $\neg\alpha$ .
3. If  $\alpha$  and  $\beta$  are sentences, then so are  $\alpha \wedge \beta$ ,  $\alpha \vee \beta$ , and  $\alpha \Rightarrow \beta$ .

The collection of all sentences,  $\mathcal{P}\mathcal{L}(S)$ , is an elementary formal language that can be used to express and manipulate propositions about the system  $S$ . Note that, at this stage, the symbols  $\neg$ ,  $\wedge$ ,  $\vee$ , and  $\Rightarrow$  have no explicit meaning, although of course the implicit intention is that they should stand for ‘not’,

‘and’, ‘or’ and ‘implies’, respectively. This implicit meaning becomes explicit when a representation of  $\mathcal{P}\mathcal{L}(S)$  is constructed as part of the application of a theory-type to  $S$  (see below). Note also that  $\mathcal{P}\mathcal{L}(S)$  is a *propositional* language only: it does not contain the quantifiers ‘ $\forall$ ’ or ‘ $\exists$ ’. To include them requires a higher-order language. We shall return to this in our discussion of the language  $\mathcal{L}(S)$ .

The next step arises because  $\mathcal{P}\mathcal{L}(S)$  is not only a vehicle for expressing propositions about the system  $S$ : we also want to *reason* with it about the system. To achieve this, a series of axioms for a deductive logic must be added to  $\mathcal{P}\mathcal{L}(S)$ . This could be either classical logic or intuitionistic logic, but we select the latter since it allows a larger class of representations/models, including representations in topoi in which the law of excluded middle fails.

The axioms for intuitionistic logic consist of a finite collection of sentences in  $\mathcal{P}\mathcal{L}(S)$  (for example,  $\alpha \wedge \beta \Rightarrow \beta \wedge \alpha$ ), plus a single rule of inference, *modus ponens* (the ‘rule of detachment’) which says that from  $\alpha$  and  $\alpha \Rightarrow \beta$  the sentence  $\beta$  may be derived.

Others axioms might be added to  $\mathcal{P}\mathcal{L}(S)$  to reflect the implicit meaning of the primitive proposition “ $A \varepsilon \Delta$ ”: i.e., (in a realist reading) “ $A$  has a value, and that value lies in  $\Delta \subseteq \mathbb{R}$ ”. For example, the sentence “ $A \varepsilon \Delta_1 \wedge A \varepsilon \Delta_2$ ” (‘ $A$  belongs to  $\Delta_1$ ’ and ‘ $A$  belongs to  $\Delta_2$ ’) might seem to be equivalent to “ $A$  belongs to  $\Delta_1 \cap \Delta_2$ ” i.e., “ $A \varepsilon \Delta_1 \cap \Delta_2$ ”. A similar remark applies to “ $A \varepsilon \Delta_1 \vee A \varepsilon \Delta_2$ ”.

Thus, along with the axioms of intuitionistic logic and detachment, we might be tempted to add the following axioms:

$$A \varepsilon \Delta_1 \wedge A \varepsilon \Delta_2 \Leftrightarrow A \varepsilon \Delta_1 \cap \Delta_2 \quad (3.1)$$

$$A \varepsilon \Delta_1 \vee A \varepsilon \Delta_2 \Leftrightarrow A \varepsilon \Delta_1 \cup \Delta_2 \quad (3.2)$$

These axioms are consistent with the intuitionistic logical structure of  $\mathcal{P}\mathcal{L}(S)$ .

We shall see later the extent to which the axioms (3.1–3.2) are compatible with the topos representations of classical and quantum physics. However, the other obvious proposition to consider in this way—“It is *not* the case that  $A$  belongs to  $\Delta$ ”—is clearly problematical.

In classical logic, this proposition<sup>34</sup>, “ $\neg(A \varepsilon \Delta)$ ”, is equivalent to “ $A$  belongs to  $\mathbb{R} \setminus \Delta$ ”, where  $\mathbb{R} \setminus \Delta$  denotes the set-theoretic complement of  $\Delta$  in  $\mathbb{R}$ . This might suggest augmenting (3.1–3.2) with a third axiom

$$\neg(A \varepsilon \Delta) \Leftrightarrow A \varepsilon \mathbb{R} \setminus \Delta \quad (3.3)$$

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<sup>34</sup>The parentheses ( ) are not symbols in the language; they are just a way of grouping letters and sentences.

However, applying ‘ $\neg$ ’ to both sides of (3.3) gives

$$\neg\neg(A \varepsilon \Delta) \Leftrightarrow A \varepsilon \Delta \quad (3.4)$$

because of the set-theoretic result  $\mathbb{R} \setminus (\mathbb{R} \setminus \Delta) = \Delta$ . But in an intuitionistic logic we do not have  $\alpha \Leftrightarrow \neg\neg\alpha$  but only  $\alpha \Rightarrow \neg\neg\alpha$ , and so (3.3) could be false in a Heyting-algebra representation of  $\mathcal{PL}(S)$  that is not Boolean. Therefore, adding (3.3) as an axiom in  $\mathcal{PL}(S)$  is not indicated if representations are to be sought in non-Boolean topoi.

### 3.2.2 Representations of $\mathcal{PL}(S)$ .

To use a language  $\mathcal{PL}(S)$  ‘for real’ for some specific physical system  $S$  one must first decide on the set  $\mathcal{Q}(S)$  of physical quantities that are to be used in describing  $S$ . This language must then be represented in the concrete mathematical structure that arises when a theory-type (for example: classical physics, quantum physics, DI-physics, ...) is applied to  $S$ . Such a representation,  $\pi$ , maps each primitive proposition,  $\alpha$ , in  $\mathcal{PL}(S)_0$  to an element,  $\pi(\alpha)$ , of some Heyting algebra (which could be Boolean),  $\mathfrak{H}$ , whose specification is part of the theory of  $S$ . For example, in classical mechanics, the propositions are represented in the Boolean algebra of all (Borel) subsets of the classical state space.

The representation of the primitive propositions can be extended recursively to all of  $\mathcal{PL}(S)$  with the aid of the following rules [29]:

$$(a) \quad \pi(\alpha \vee \beta) := \pi(\alpha) \vee \pi(\beta) \quad (3.5)$$

$$(b) \quad \pi(\alpha \wedge \beta) := \pi(\alpha) \wedge \pi(\beta) \quad (3.6)$$

$$(c) \quad \pi(\neg\alpha) := \neg\pi(\alpha) \quad (3.7)$$

$$(d) \quad \pi(\alpha \Rightarrow \beta) := \pi(\alpha) \Rightarrow \pi(\beta) \quad (3.8)$$

Note that, on the left hand side of (3.5–3.8), the symbols  $\{\neg, \wedge, \vee, \Rightarrow\}$  are elements of the language  $\mathcal{PL}(S)$ , whereas on the right hand side they denote the logical connectives in the Heyting algebra,  $\mathfrak{H}$ , in which the representation takes place.

This extension of  $\pi$  from  $\mathcal{PL}(S)_0$  to  $\mathcal{PL}(S)$  is consistent with the axioms for the intuitionistic, propositional logic of the language  $\mathcal{PL}(S)$ . More precisely, these axioms become tautologies: i.e., they are all represented by the maximum element, 1, in the Heyting algebra. By construction, the map  $\pi : \mathcal{PL}(S) \rightarrow \mathfrak{H}$  is then a representation of  $\mathcal{PL}(S)$  in the Heyting algebra  $\mathfrak{H}$ . A logician would say that  $\pi : \mathcal{PL}(S) \rightarrow \mathfrak{H}$  is an  $\mathfrak{H}$ -valuation, or  $\mathfrak{H}$ -model, of the language  $\mathcal{PL}(S)$ .

Note that different systems,  $S$ , can have the same language. For example, consider a point-particle moving in one dimension, with a Hamiltonian function  $H(x, p) = \frac{p^2}{2m} + V(x)$  and state space  $T^*\mathbb{R}$ . Different potentials  $V$  correspond to different systems (in the sense in which we are using the word ‘system’), but the physical quantities for these systems—or, more precisely, the ‘names’ of these quantities, for example, ‘energy’, ‘position’, ‘momentum’—are the same for them all. Consequently, the language  $\mathcal{P}\mathcal{L}(S)$  is independent of  $V$ . However, the *representation* of, say, the proposition “ $E \in \Delta$ ” (where ‘ $E$ ’ is the energy), with a specific subset of the state space *will* depend on the details of the Hamiltonian.

Clearly, a major consideration in using the language  $\mathcal{P}\mathcal{L}(S)$  is choosing the Heyting algebra in which the representation is to take place. A fundamental result in topos theory is that the set of all sub-objects of any object in a topos is a Heyting algebra, and these are the Heyting algebras with which we will be concerned.

Of course, beyond the language,  $\mathcal{S}$ , and its representation  $\pi$ , lies the question of whether or not a proposition is ‘true’. This requires the concept of a ‘state’ which, when specified, yields ‘truth values’ for the primitive propositions in  $\mathcal{P}\mathcal{L}(S)$ . These can then be extended recursively to the rest of  $\mathcal{P}\mathcal{L}(S)$ . In classical physics, the possible truth values are just ‘true’ or ‘false’. However, as we shall see, the situation in topos theory is more complex.

### 3.2.3 Using Geometric Logic

The inductive definition of  $\mathcal{P}\mathcal{L}(S)$  given above means that sentences can involve only a *finite* number of primitive propositions, and therefore only a finite number of disjunctions (‘ $\vee$ ’) or conjunctions (‘ $\wedge$ ’). An interesting variant of this structure is the, so-called, ‘propositional geometric logic’. This is characterised by modifying the language and logical axioms so that:

1. There are arbitrary disjunctions, including the empty disjunction (‘0’).
2. There are finite conjunctions, including the empty conjunction (‘1’)
3. Conjunction distributes over arbitrary disjunctions; disjunction distributes over finite conjunctions.

This structure does *not* include negation, implication, or infinite conjunctions.

From a conceptual viewpoint, this set of rules is obtained by considering what it means to actually ‘affirm’ the propositions in  $\mathcal{P}\mathcal{L}(S)$ . A careful

analysis of this concept is given by Vickers [72]; the idea itself goes back to work by Abramsky [1]. The conclusion is that the set of ‘affirmable’ propositions should satisfy the rules above.

Clearly such a logic is tailor-made for seeking representations in the open sets of a topological space—the paradigmatic example of a Heyting algebra. The phrase ‘geometric logic’ is normally applied to a *first-order* logic with the properties above, and we will return to this in our discussion of the typed language  $\mathcal{L}(S)$ . What we have here is just the propositional part of this logic.

The restriction to geometric logic would be easy to incorporate into our languages  $\mathcal{P}\mathcal{L}(S)$ : for example, the axiom (3.2) (if added) could be extended to read<sup>35</sup>

$$\bigvee_{i \in I} (A \varepsilon \Delta_i) = A \varepsilon \bigcup_{i \in I} \Delta_i \quad (3.9)$$

for all index sets  $I$ .

The move to geometric logic is motivated by a conception of truth that is grounded in the actions of making real measurements. This resonates strongly with the logical positivism that seems still to lurk in the collective unconscious of the physics profession, and which, of course, was strongly affirmed by Bohr in his analysis of quantum theory. However, our drive towards ‘neo-realism’ involves replacing the idea of observation/measurement with that of ‘the way things are’, albeit in a more sophisticated interpretation than that of the ubiquitous cobbler-in-the-market. Consequently, the conceptual reasons for using ‘affirmative’ logic are less compelling. This issue deserves further thought: at the moment we are open-minded about it.

The use of geometric logic becomes more interesting in the context of the typed language  $\mathcal{L}(S)$ , and we shall return to this in Section 4.2.2

### 3.2.4 Introducing Time Dependence

In addition to describing ‘the way things are’ there is also the question of how the-way-things-are changes in time. In the form presented above, the language  $\mathcal{P}\mathcal{L}(S)$  may seem geared towards a ‘canonical’ perspective in so far as the propositions concerned are implicitly taken to be asserted at a particular moment of time. As such,  $\mathcal{P}\mathcal{L}(S)$  deals with the values of physical quantities at that time. In other words, the underlying spatio-temporal

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<sup>35</sup>Note that the bi-implication  $\Leftrightarrow$  used in, for example, (3.1–3.2), is not available if there is no implication symbol. Thus we have assumed that we are now working with a logical structure in which ‘equality’ is a meaningful concept; hence the introduction of ‘=’ in (3.9).



perspective seems thoroughly ‘Newtonian’.

However, this is only partly true since the phrase ‘physical quantity’ can have meanings other than the canonical one. For example, one could talk about the ‘time average of momentum’, and call that a physical quantity. In this case, the propositions would be about *histories* of the system, not just ‘the way things are’ at a particular moment in time.

In practice, the question of time dependence can be addressed in various ways. One is to attach a (external) time label,  $t$ , to the physical quantities, so that the primitive propositions become of the form “ $A_t \varepsilon \Delta$ ”. This can be interpreted in two ways. The first is to think of  $\mathcal{Q}(S)$  as including the symbols  $A_t$  for all physical quantities  $A$  and all values of time  $t \in \mathbb{R}$ . The second is to keep  $\mathcal{Q}(S)$  fixed, but instead let the language itself become time-dependent, so that we should write  $\mathcal{P}\mathcal{L}(S)_t$ ,  $t \in \mathbb{R}$ .

In the former case,  $\mathcal{P}\mathcal{L}(S)$  would naturally include *history* propositions of the form

$$(A_{1t_1} \varepsilon \Delta_1) \wedge (A_{2t_2} \varepsilon \Delta_2) \wedge \cdots \wedge (A_{nt_n} \varepsilon \Delta_n) \quad (3.10)$$

and other obvious variants of this. Here we assume that  $t_1 \leq t_2 \leq \cdots \leq t_n$ .

The sequential proposition in (3.10) is to be interpreted (in a realist reading) as asserting that “ ‘The physical quantity  $A_1$  has a value that lies in  $\Delta_1$  at time  $t_1$ ’ and ‘the physical quantity  $A_2$  has a value that lies in  $\Delta_2$  at time  $t_2$ ’ and  $\cdots$  and ‘the physical quantity  $A_n$  has a value that lies in  $\Delta_n$  at time  $t_n$ ’ ”. Clearly what we have here is a type of *temporal* logic. Thus this would be an appropriate structure with which to discuss the ‘consistent histories’ interpretation of quantum theory, particularly in the, so-called, HPO (history projection formalism) [39]. In that context, (3.10) represents a, so-called, ‘homogeneous’ history.

From a general conceptual perspective, one might prefer to have an internal time object, rather than adding external time labels in the language. Indeed, in our later discussion of the higher-order language  $\mathcal{L}(S)$  we will strive to eliminate external entities. However, in the present case,  $\Delta \subseteq \mathbb{R}$  is already an ‘external’ (to the language) entity, as indeed is  $A \in \mathcal{Q}(S)$ , so there seems no particular objection to adding a time label too.

In the second approach, where there is only one time label, the representation  $\pi$  will map “ $A_t \varepsilon \Delta$ ” to a time-dependent element,  $\pi(A_t \varepsilon \Delta)$ , of the Heyting algebra,  $\mathfrak{H}$ ; one could say that this is a type of ‘Heisenberg picture’.

This suggests another option, which is to keep the language free of any time labels, but allow the *representation* to be time-dependent. In this case,

$\pi_t(A \varepsilon \Delta)$  is a time-dependent member of  $\mathfrak{H}$ .<sup>36</sup>

A different approach is to ascribe time dependence to the ‘truth objects’ in the theory: this corresponds to a type of Schrödinger picture. The concept of a truth object is discussed in detail in Section 6.

### 3.2.5 The Representation of $\mathcal{P}\mathcal{L}(S)$ in Classical Physics

Let us now look at the representation of  $\mathcal{P}\mathcal{L}(S)$  that corresponds to classical physics. In this case, the topos involved is just the category, **Sets**, of sets and functions between sets.

We will denote by  $\pi_{\text{cl}}$  the representation of  $\mathcal{P}\mathcal{L}(S)$  that describes the classical, Hamiltonian mechanics of a system,  $S$ , whose state-space is a symplectic (or Poisson) manifold  $\mathcal{S}$ . We denote by  $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$  the real-valued function<sup>37</sup> on  $\mathcal{S}$  that represents the physical quantity  $A$ .

Then the representation  $\pi_{\text{cl}}$  maps the primitive proposition “ $A \varepsilon \Delta$ ” to the subset of  $\mathcal{S}$  given by

$$\begin{aligned} \pi_{\text{cl}}(A \varepsilon \Delta) &:= \{s \in \mathcal{S} \mid \check{A}(s) \in \Delta\} \\ &= \check{A}^{-1}(\Delta). \end{aligned} \tag{3.11}$$

This representation can be extended to all the sentences in  $\mathcal{P}\mathcal{L}(S)$  with the aid of (3.5–3.8). Note that, since  $\Delta$  is a Borel subset of  $\mathbb{R}$ ,  $\check{A}^{-1}(\Delta)$  is a Borel subset of the state-space  $\mathcal{S}$ . Hence, in this case,  $\mathfrak{H}$  is equal to the Boolean algebra of all Borel subsets of  $\mathcal{S}$ .

We note that, for all (Borel) subsets  $\Delta_1, \Delta_2$  of  $\mathbb{R}$  we have

$$\check{A}^{-1}(\Delta_1) \cap \check{A}^{-1}(\Delta_2) = \check{A}^{-1}(\Delta_1 \cap \Delta_2) \tag{3.12}$$

$$\check{A}^{-1}(\Delta_1) \cup \check{A}^{-1}(\Delta_2) = \check{A}^{-1}(\Delta_1 \cup \Delta_2) \tag{3.13}$$

$$\neg \check{A}^{-1}(\Delta_1) = \check{A}^{-1}(\mathbb{R} \setminus \Delta_1) \tag{3.14}$$

and hence, in classical physics, all three conditions (3.1–3.3) that we discussed earlier can be added consistently to the language  $\mathcal{P}\mathcal{L}(S)$ .

Consider now the assignment of truth values to the propositions in this theory. This involves the idea of a ‘microstate’ which, in classical physics, is

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<sup>36</sup>Perhaps we should also consider the possibility that the Heyting algebra is time dependent, in which case  $\pi_t(A \varepsilon \Delta)$  is a member of  $\mathfrak{H}_t$ .

<sup>37</sup>In practice,  $\check{A}$  is required to be measurable, or smooth, depending on the type of physical quantity that  $A$  is. However, for the most part, these details of classical mechanics are not relevant to our discussions, and usually we will not characterise  $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$  beyond just saying that it is a function/map from  $\mathcal{S}$  to  $\mathbb{R}$ .

simply an element  $s$  of the state space  $\mathcal{S}$ . Each microstate  $s$  assigns to each primitive proposition “ $A \varepsilon \Delta$ ”, a truth value,  $\nu(A \varepsilon \Delta; s)$ , which lies in the set  $\{\text{false}, \text{true}\}$  (which we identify with  $\{0, 1\}$ ) and is defined as

$$\nu(A \varepsilon \Delta; s) := \begin{cases} 1 & \text{if } \check{A}(s) \in \Delta; \\ 0 & \text{otherwise} \end{cases} \quad (3.15)$$

for all  $s \in \mathcal{S}$ .

### 3.2.6 The Failure to Represent $\mathcal{P}\mathcal{L}(S)$ in Standard Quantum Theory.

The procedure above that works so easily for classical physics fails completely if one tries to apply it to standard quantum theory.

In quantum physics, a physical quantity  $A$  is represented by a self-adjoint operator  $\hat{A}$  on a Hilbert space  $\mathcal{H}$ , and the proposition “ $A \varepsilon \Delta$ ” is represented by the projection operator  $\hat{E}[A \in \Delta]$  which projects onto the subset  $\Delta$  of the spectrum of  $\hat{A}$ ; i.e.,

$$\pi(A \varepsilon \Delta) := \hat{E}[A \in \Delta]. \quad (3.16)$$

Of course, the set of all projection operators,  $\mathcal{P}(\mathcal{H})$ , in  $\mathcal{H}$  has a ‘logic’ of its own—the ‘quantum logic’<sup>38</sup> of the Hilbert space  $\mathcal{H}$ —but this is incompatible with the intuitionistic logic of the language  $\mathcal{P}\mathcal{L}(S)$ , and the representation (3.16).

Indeed, since the ‘logic’  $\mathcal{P}(\mathcal{H})$  is non-distributive, there will exist non-commuting operators  $\hat{A}, \hat{B}, \hat{C}$ , and Borel subsets  $\Delta_A, \Delta_B, \Delta_C$  of  $\mathbb{R}$  such that<sup>39</sup>

$$\begin{aligned} \hat{E}[A \in \Delta_A] \wedge \left( \hat{E}[B \in \Delta_B] \vee \hat{E}[C \in \Delta_C] \right) &\neq \\ \left( \hat{E}[A \in \Delta_A] \wedge \hat{E}[B \in \Delta_B] \right) \vee \left( \hat{E}[A \in \Delta_A] \wedge \hat{E}[C \in \Delta_C] \right) & \end{aligned} \quad (3.17)$$

while, on the other hand, the logical bi-implication

$$\alpha \wedge (\beta \vee \gamma) \Leftrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \quad (3.18)$$

can be deduced from the axioms of the language  $\mathcal{P}\mathcal{L}(S)$ .

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<sup>38</sup>For an excellent survey of quantum logic see [18]. This includes a discussion of a first-order axiomatisation of quantum logic, and with an associated sequent calculus. It is interesting to compare our work with what the authors of this paper have done. We hope to return to this at some time in the future.

<sup>39</sup>There is a well-known example that uses three rays in  $\mathbb{R}^2$ , so this phenomenon is not particularly exotic.

This failure of distributivity bars any naïve realist interpretation of quantum logic. If an instrumentalist interpretation is used instead, the spectral projectors  $\hat{E}[A \in \Delta]$  now represent propositions about what would happen *if* a measurement is made, not propositions about what is ‘actually the case’. And, of course, when a state is specified, this does not yield actual truth values but only the Born-rule probabilities of getting certain results.

## 4 A Higher-Order, Typed Language for Physics

### 4.1 The Basics of the Language $\mathcal{L}(S)$

We want now to consider the possibility of representing the physical quantities of a system by arrows in a topos other than **Sets**.

The physical meaning of such an arrow is not clear, *a priori*. Nor is it even clear *what* it is that is being represented in this way. However, what *is* clear is that in such a situation it is not correct to assume that the quantity-value object is necessarily the real-number object in the topos (assuming that there is one). Rather, the target-object,  $\mathcal{R}_S$ , has to be determined for each topos, and is therefore an important part of the ‘representation’.

A powerful technique for allowing the quantity-value object to be system-dependent is to add a symbol ‘ $\mathcal{R}$ ’ to the system language. Developing this line of thinking suggests that ‘ $\Sigma$ ’, too, should be added to the language, as should a set of symbols of the form ‘ $A : \Sigma \rightarrow \mathcal{R}$ ’, to be construed as ‘what it is’ (hopefully a physical quantity) that is represented by arrows in a topos. Similarly, there should be a symbol ‘ $\Omega$ ’, to act as the linguistic precursor to the sub-object classifier in the topos; in the topos **Sets**, this is just the set  $\{0, 1\}$ .

The clean way of doing all this is to construct a ‘local language’ [8]. Our basic assumption is that such a language,  $\mathcal{L}(S)$ , can be associated with each system  $S$ . A physical theory of  $S$  then corresponds to a representation of  $\mathcal{L}(S)$  in an appropriate topos.

**The symbols of  $\mathcal{L}(S)$ .** We first consider the minimal set of symbols needed to handle elementary physics. For more sophisticated theories in physics it will be necessary to change, or enlarge, this set of ‘ground-type’ symbols.

The symbols for the local language,  $\mathcal{L}(S)$ , are defined recursively as follows:

1. (a) The basic *type symbols* are  $1, \Omega, \Sigma, \mathcal{R}$ . The last two,  $\Sigma$  and  $\mathcal{R}$ , are known as *ground-type symbols*. They are the linguistic precursors of the state object, and quantity-value object, respectively.  
If  $T_1, T_2, \dots, T_n$ ,  $n \geq 1$ , are type symbols, then so is<sup>40</sup>  $T_1 \times T_2 \times \dots \times T_n$ .  
(b) If  $T$  is a type symbol, then so is  $PT$ .
2. (a) For each type symbol,  $T$ , there is associated a countable set of *variables of type  $T$* .  
(b) There is a special symbol  $*$ .
3. (a) To each pair  $(T_1, T_2)$  of type symbols there is associated a set,  $F_{\mathcal{L}(S)}(T_1, T_2)$ , of *function symbols*. Such a symbol,  $A$ , is said to have *signature*  $T_1 \rightarrow T_2$ ; this is indicated by writing  $A : T_1 \rightarrow T_2$ .  
(b) Some of these sets of function symbols may be empty. However, in our case, particular importance is attached to the set,  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ , of function symbols  $A : \Sigma \rightarrow \mathcal{R}$ , and we assume this set is non-empty.

The function symbols  $A : \Sigma \rightarrow \mathcal{R}$  represent the ‘physical quantities’ of the system, and hence  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  will depend on the system  $S$ . In fact, the only parts of the language that are system-dependent are these function symbols. The set  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  is the analogue of the set,  $\mathcal{Q}(S)$ , of physical quantities associated with the propositional language  $\mathcal{PL}(S)$ .

For example, if  $S_1$  is a point particle moving in one dimension, the set of physical quantities could be chosen to be  $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) = \{x, p, H\}$  which represent the position, momentum, and energy of the system. On the other hand, if  $S_2$  is a particle moving in three dimensions, we could have  $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) = \{x, y, z, p_x, p_y, p_z, H\}$  to allow for three-dimensional position and momentum (with respect to some given Euclidean coordinate system). Or, we could decide to add angular momentum too, to give the set  $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) = \{x, y, z, p_x, p_y, p_z, J_x, J_y, J_z, H\}$ . A still further extension would be to add the quantities  $\underline{x} \cdot \underline{n}$  and  $\underline{p} \cdot \underline{m}$  for all unit vectors  $\underline{n}$  and  $\underline{m}$ ; and so on.

Note that, as with the propositional language  $\mathcal{PL}(S)$ , the fact that a given system has a specific Hamiltonian<sup>41</sup>—expressed as a particular function of position and momentum coordinates—is not something that is to be coded

<sup>40</sup>By definition, if  $n = 0$  then  $T_1 \times T_2 \times \dots \times T_n := 1$ .

<sup>41</sup>It must be emphasised once more that the use of a local language is *not* restricted to standard, canonical systems in which the concept of a ‘Hamiltonian’ is meaningful. The

into the language: instead, such system dependence arises in the choice of *representation* of the language. This means that many different systems can have the same local language.

Finally, it should be emphasised that this list of symbols is minimal and one will certainly want to add more. One obvious, general, example is a type symbol  $\mathbb{N}$  that is to be interpreted as the linguistic analogue of the natural numbers. The language could then be augmented with the axioms of Peano arithmetic.

**The terms of  $\mathcal{L}(S)$ .** The next step is to enumerate the ‘terms’ in the language, together with their associated types [8, 52]:

1. (a) For each type symbol  $T$ , the variables of type  $T$  are terms of type  $T$ .  
 (b) The symbol  $*$  is a term of type 1.  
 (c) A term of type  $\Omega$  is called a *formula*; a formula with no free variables is called a *sentence*.
2. If  $A$  is function symbol with signature  $T_1 \rightarrow T_2$ , and  $t$  is a term of type  $T_1$ , then  $A(t)$  is term of type  $T_2$ .  
 In particular, if  $A : \Sigma \rightarrow \mathcal{R}$  is a physical quantity, and  $t$  is a term of type  $\Sigma$ , then  $A(t)$  is a term of type  $\mathcal{R}$ .
3. (a) If  $t_1, t_2, \dots, t_n$  are terms of type  $T_1, T_2, \dots, T_n$ , then  $\langle t_1, t_2, \dots, t_n \rangle$  is a term of type  $T_1 \times T_2 \times \dots \times T_n$ .  
 (b) If  $t$  is a term of type  $T_1 \times T_2 \times \dots \times T_n$ , and if  $1 \leq i \leq n$ , then  $(t)_i$  is a term of type  $T_i$ .
4. (a) If  $\omega$  is a term of type  $\Omega$ , and  $\tilde{x}$  is a variable of type  $T$ , then  $\{\tilde{x} \mid \omega\}$  is a term of type  $PT$ .  
 (b) If  $t_1, t_2$  are terms of the same type, then ‘ $t_1 = t_2$ ’ is a term of type  $\Omega$ .  
 (c) If  $t_1, t_2$  are terms of type  $T, PT$  respectively, then  $t_1 \in t_2$  is a term of type  $\Omega$ .

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scope of the linguistic ideas is *much* wider than that and the canonical systems are only an example. Indeed, our long-term interest is in the application of these ideas to quantum gravity where the local language is likely to be very different from that used here. However, we anticipate that the basic ideas will be the same.

Note that the logical operations are not included in the set of symbols. Instead, they can all be defined using what is already given. For example, (i)  $true := (* = *)$ ; and (ii) if  $\alpha$  and  $\beta$  are terms of type  $\Omega$ , then<sup>42</sup>  $\alpha \wedge \beta := (\langle \alpha, \beta \rangle = \langle true, true \rangle)$ . Thus, in terms of the original set of symbols, we have

$$\alpha \wedge \beta := (\langle \alpha, \beta \rangle = \langle * = *, * = * \rangle) \quad (4.1)$$

and so on.

**Terms of particular interest to us.** Let  $A$  be a physical quantity in the set  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ , and therefore a function symbol of signature  $\Sigma \rightarrow \mathcal{R}$ . In addition, let  $\tilde{\Delta}$  be a variable (and therefore a term) of type  $P\mathcal{R}$ ; and let  $\tilde{s}$  be a variable (and therefore a term) of type  $\Sigma$ . Then some terms of particular interest to us are the following:

1.  $A(\tilde{s})$  is a term of type  $\mathcal{R}$  with a free variable,  $\tilde{s}$ , of type  $\Sigma$ .
2. ' $A(\tilde{s}) \in \tilde{\Delta}$ ' is a term of type  $\Omega$  with free variables (i)  $\tilde{s}$  of type  $\Sigma$ ; and (ii)  $\tilde{\Delta}$  of type  $P\mathcal{R}$ .
3.  $\{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\}$  is a term of type  $P\Sigma$  with a free variable  $\tilde{\Delta}$  of type  $P\mathcal{R}$ .

As we shall see, ' $A(\tilde{s}) \in \tilde{\Delta}$ ' is an analogue of the primitive propositions " $A \varepsilon \Delta$ " in the propositional language  $\mathcal{P}\mathcal{L}(S)$ . However, there is a crucial difference. In  $\mathcal{P}\mathcal{L}(S)$ , the ' $\Delta$ ' in " $A \varepsilon \Delta$ " is a specific subset of the external (to the language) real line  $\mathbb{R}$ . On the other hand, in the local language  $\mathcal{L}(S)$ , the ' $\tilde{\Delta}$ ' in ' $A(\tilde{s}) \in \tilde{\Delta}$ ' is an *internal* variable within the language.

**Adding axioms to the language.** To make the language  $\mathcal{L}(S)$  into a deductive system we need to add a set of appropriate axioms and rules of inference. The former are expressed using *sequents*: defined as expressions of the form  $\Gamma : \alpha$  where  $\alpha$  is a formula (a term of type  $\Omega$ ) and  $\Gamma$  is a set of such formula. The intention is that ' $\Gamma : \alpha$ ' is to be read intuitively as "the collection of formula in  $\Gamma$  'imply'  $\alpha$ ". If  $\Gamma$  is empty we just write  $:\alpha$ .

The basic axioms include things like ' $\alpha : \alpha$ ' (tautology), and ' $:\tilde{t} \in \{\tilde{t} \mid$

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<sup>42</sup>The parentheses ( ) are not symbols in the language, they are just a way of grouping letters and sentences. The same remark applies to the inverted commas ''.

$\alpha\} \Leftrightarrow \alpha'$  (comprehension) where  $\tilde{t}$  is a variable of type  $T$ . These axioms<sup>43</sup> and the rules of inference (sophisticated analogues of *modus ponens*) give rise to a deductive system using intuitionistic logic. For the details see [8, 52].

For applications in physics we could, and presumably should, add extra axioms (in the form of sequents). For example, perhaps the quantity-value object should always be an abelian-group object, or at least a semi-group<sup>44</sup>? This can be coded into the language by adding the axioms for an abelian group structure for  $\mathcal{R}$ . This involves the following steps:

1. Add the following symbols:
  - (a) A ‘unit’ function symbol  $0 : 1 \rightarrow \mathcal{R}$ ; this will be the linguistic analogue of the unit element in an abelian group.
  - (b) An ‘addition’ function symbol  $+$  :  $\mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ .
  - (c) An ‘inverse’ function symbol  $-$  :  $\mathcal{R} \rightarrow \mathcal{R}$
2. Then add axioms like ‘ $\forall \tilde{r} ( + (\tilde{r}, 0(*)) = \tilde{r} )$ ’ where  $\tilde{r}$  is a variable of type  $\mathcal{R}$ , and so on.

For another example, consider a point particle moving in three dimensions, with the function symbols  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) = \{x, y, z, p_x, p_y, p_z, J_x, J_y, J_z, H\}$ . As  $\mathcal{L}(S)$  stands, there is no way to specify, for example, that ‘ $J_x = yp_z - zp_y$ ’. Such relations can only be implemented in a *representation* of the language. However, if this relation is felt to be ‘universal’ (i.e., if it is expected to hold in all physically-relevant representations) then it could be added to the language with the use of extra axioms.

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<sup>43</sup>The complete set is [8]:

Tautology:	$\alpha = \alpha$
Unity :	$\tilde{x}_1 = *$ where $\tilde{x}_1$ is a variable of type 1.
Equality:	$x = y, \alpha(\tilde{z}/x) : \alpha(\tilde{z}/y)$ . Here, $\alpha(\tilde{z}/x)$ is the term $\alpha$ with $\tilde{z}$ replaced by the term $x$ for each free occurrence of the variable $\tilde{z}$ . The terms $x$ and $y$ must be of the same type as $\tilde{z}$ .
Products:	$:(\langle x_1, \dots, x_n \rangle)_i = x_i$ $: x = \langle (x)_1, \dots, (x)_n \rangle$
Comprehension:	$:\tilde{t} \in \{\tilde{t} \mid \alpha\} \Leftrightarrow \alpha$

<sup>44</sup>One could go even further and add the axioms for real numbers. However, the example of quantum theory suggests that this is inappropriate: in general, the quantity-value object will *not* be the real-number object [23].



One of the delicate decisions that has to be made about  $\mathcal{L}(S)$  is what extra axioms to add to the base language. Too few, and the language lacks content; too many, and representations of potential physical significance are excluded. This is one of the places in the formalism where a degree of physical insight is necessary!

## 4.2 Representing $\mathcal{L}(S)$ in a Topos

The construction of a theory of the system  $S$  involves choosing a representation<sup>45</sup>/model,  $\phi$ , of the language  $\mathcal{L}(S)$  in a topos<sup>46</sup>  $\tau_\phi$ . The choice of both topos and representation depend on the theory-type being used.

For example, consider a system,  $S$ , that can be treated using both classical physics and quantum physics, such as a point particle moving in three dimensions. Then, for the application of the theory-type ‘classical physics’, in a representation denoted  $\sigma$ , the topos  $\tau_\sigma$  is **Sets**, and  $\Sigma$  is represented by the symplectic manifold  $\Sigma_\sigma := T^*\mathbb{R}^3$ ;  $\mathcal{R}$  is represented by the usual real numbers  $\mathbb{R}$ .

On the other hand, as we shall see in Section 5, for the application of the theory-type ‘quantum physics’,  $\tau_\phi$  is the topos, **Sets** <sup>$\mathcal{V}(\mathcal{H})^{\text{op}}$</sup> , of presheaves over the category<sup>47</sup>  $\mathcal{V}(\mathcal{H})$ , where  $\mathcal{H} \simeq L^2(\mathbb{R}^3, d^3x)$  is the Hilbert space of the system  $S$ . In this case,  $\Sigma$  is represented by  $\Sigma_\phi := \underline{\Sigma}$ , where  $\underline{\Sigma}$  is the spectral presheaf; this representation is discussed at length in Sections 5. For both theory types, the *details* of, for example, the Hamiltonian, are coded in the representation.

We now list the  $\tau_\phi$ -representation of the most significant symbols and terms in our language,  $\mathcal{L}(S)$  (we have picked out only the parts that are immediately relevant to our programme: for full details see [8, 52]).

1. (a) The ground type symbols  $\Sigma$  and  $\mathcal{R}$  are represented by objects  $\Sigma_\phi$  and  $\mathcal{R}_\phi$  in  $\tau_\phi$ . These are identified physically as the state object

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<sup>45</sup>The word ‘interpretation’ is often used in the mathematical literature, but we want to reserve that for use in discussions of interpretations of quantum theory, and the like.

<sup>46</sup>A more comprehensive notation is  $\tau_\phi(S)$ , which draws attention to the system  $S$  under discussion; similarly, the state object could be written as  $\Sigma_{\phi,S}$ , and so on. This extended notation is used in Section 11 where we are concerned with the relations between *different* systems, and then it is essential to indicate which system is meant. However, in the present article, only one system at a time is being considered, and so the truncated notation is fine.

<sup>47</sup>We recall that the objects in  $\mathcal{V}(\mathcal{H})$  are the unital, commutative von Neumann sub-algebras of the algebra,  $B(\mathcal{H})$ , of all bounded operators on  $\mathcal{H}$ . We will explain, and motivate, this later.

and quantity-value object, respectively.

- (b) The symbol  $\Omega$ , is represented by  $\Omega_\phi := \Omega_{\tau_\phi}$ , the sub-object classifier of the topos  $\tau_\phi$ .
- (c) The symbol  $1$ , is represented by  $1_\phi := 1_{\tau_\phi}$ , the terminal object in  $\tau_\phi$ .

2. For each type symbol  $PT$ , we have  $(PT)_\phi := PT_\phi$ , the power object of the object  $T_\phi$  in  $\tau_\phi$ .

In particular,  $(P\Sigma)_\phi = P\Sigma_\phi$  and  $(P\mathcal{R})_\phi = P\mathcal{R}_\phi$ .

3. Each function symbol  $A : \Sigma \rightarrow \mathcal{R}$  in  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  (i.e., each physical quantity) is represented by an arrow  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$  in  $\tau_\phi$ .

We will generally require the representation to be *faithful*: i.e., the map  $A \mapsto A_\phi$  is one-to-one.

4. A term of type  $\Omega$  of the form ‘ $A(\tilde{s}) \in \tilde{\Delta}$ ’ (which has free variables  $\tilde{s}, \tilde{\Delta}$  of type  $\Sigma$  and  $P\mathcal{R}$  respectively) is represented by an arrow  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi : \Sigma_\phi \times P\mathcal{R}_\phi \rightarrow \Omega_{\tau_\phi}$ . In detail, this arrow is

$$\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi = e_{\mathcal{R}_\phi} \circ \langle \llbracket A(\tilde{s}) \rrbracket_\phi, \llbracket \tilde{\Delta} \rrbracket_\phi \rangle \quad (4.2)$$

where  $e_{\mathcal{R}_\phi} : \mathcal{R}_\phi \times P\mathcal{R}_\phi \rightarrow \Omega_{\tau_\phi}$  is the usual evaluation map;  $\llbracket A(\tilde{s}) \rrbracket_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$  is the arrow  $A_\phi$ ; and  $\llbracket \tilde{\Delta} \rrbracket_\phi : P\mathcal{R}_\phi \rightarrow P\mathcal{R}_\phi$  is the identity.

Thus  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi$  is the chain of arrows:

$$\Sigma_\phi \times P\mathcal{R}_\phi \xrightarrow{A_\phi \times \text{id}} \mathcal{R}_\phi \times P\mathcal{R}_\phi \xrightarrow{e_{\mathcal{R}_\phi}} \Omega_{\tau_\phi}. \quad (4.3)$$

We see that the analogue of the ‘ $\Delta$ ’ used in the  $\mathcal{P}\mathcal{L}(S)$ -proposition “ $A \in \Delta$ ” is played by sub-objects of  $\mathcal{R}_\phi$  (i.e., global elements of  $P\mathcal{R}_\phi$ ) in the domain of the arrow in (4.3). These objects are, of course, representation-dependent (i.e., they depend on  $\phi$ ).

5. A term of type  $P\Sigma$  of the form  $\{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\}$  (which has a free variable  $\tilde{\Delta}$  of type  $P\mathcal{R}$ ) is represented by an arrow  $\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_\phi : P\mathcal{R}_\phi \rightarrow P\Sigma_\phi$ . This arrow is the power transpose<sup>48</sup> of  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi$ :

$$\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_\phi = \ulcorner \llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi \urcorner \quad (4.4)$$

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<sup>48</sup>One of the basic properties of a topos is that there is a one-to-one correspondence between arrows  $f : A \times B \rightarrow \Omega$  and arrows  $\ulcorner f \urcorner : B \rightarrow PA := \Omega^A$ . In general,  $\ulcorner f \urcorner$  is called the *power transpose* of  $f$ . If  $B \simeq 1$  then  $\ulcorner f \urcorner$  is known as the *name* of the arrow  $f : A \rightarrow \Omega$ .

6. A term,  $\omega$ , of type  $\Omega$  with no free variables is represented by a global element  $\llbracket \omega \rrbracket_\phi : 1_{\tau_\phi} \rightarrow \Omega_{\tau_\phi}$ . These will typically act as ‘truth values’ for propositions about the system.
7. Any axioms that have been added to the language are required to be represented by the arrow  $true : 1_{\tau_\phi} \rightarrow \Omega_{\tau_\phi}$ .

### 4.2.1 The Local Set Theory of a Topos.

We should emphasise that the decision to focus on the particular type of language that we have, is not an arbitrary one. Indeed, there is a deep connection between such languages and topos theory.

In this context, we first note that to any local language,  $\mathcal{L}$ , there is associated a ‘local set theory’. This involves defining an ‘ $\mathcal{L}$ -set’ to be a term  $X$  of power type (so that expressions of the form  $x \in X$  are meaningful) and with no free variables. Analogues of all the usual set operations can be defined on  $\mathcal{L}$ -sets. For example, if  $X, Y$  are  $\mathcal{L}$ -sets of type  $PT$ , one can define  $X \cap Y := \{\tilde{x} \mid \tilde{x} \in X \wedge \tilde{x} \in Y\}$  where  $\tilde{x}$  is a variable of type  $T$ .

Furthermore, each local language,  $\mathcal{L}$ , gives rise to an associated topos,  $\mathcal{C}(\mathcal{L})$ , whose objects are equivalence classes of  $\mathcal{L}$ -sets, where  $X \equiv Y$  is defined to mean that the equation  $X = Y$  (i.e., a term of type  $\Omega$  with no free variables) can be proved using the sequent calculus of the language with its axioms. From this perspective, a representation of the system-language  $\mathcal{L}(S)$  in a topos  $\tau$  is equivalent to a *functor* from the topos  $\mathcal{C}(\mathcal{L}(S))$  to  $\tau$ .

### 4.2.2 Theory Construction as a Translation of Languages

Conversely, for each topos  $\tau$  there is a local language,  $\mathcal{L}(\tau)$ , whose ground-type symbols are the objects of  $\tau$ , and whose function symbols are the arrows in  $\tau$ . It then follows that a representation of a local language,  $\mathcal{L}$ , in  $\tau$  is equivalent to a ‘translation’ of  $\mathcal{L}$  in  $\mathcal{L}(\tau)$ .

Thus constructing a theory of physics is equivalent to finding a suitable *translation* of the system language,  $\mathcal{L}(S)$ , to the language,  $\mathcal{L}(\tau)$ , of an appropriate topos  $\tau$ .

As we will see later, the idea of translating one local language into another plays a central role in the discussion of composite systems and sub-systems.

In the case of spoken languages, one can translate from, say, (i) English to German; or (ii) from English to Greek, and then from Greek to German. However, no matter how good the translators, these two ways of going from English to German will generally not agree. This is partly because the translation process is not unique, but also because each language possesses certain intrinsic features that simply do not admit of translation.

There is an interesting analogous question for the representation of the local languages  $\mathcal{L}(S)$ . Namely, suppose  $\phi_1 : \mathcal{L}(S) \rightarrow \mathcal{L}(\tau_{\phi_1})$  and  $\phi_2 : \mathcal{L}(S) \rightarrow \mathcal{L}(\tau_{\phi_2})$  are two different topos theories of the same system  $S$  (these could be, say, classical physics and quantum physics). The question is if/when will there be a translation  $\phi_{12} : \mathcal{L}(\tau_{\phi_1}) \rightarrow \mathcal{L}(\tau_{\phi_2})$  such that

$$\phi_2 = \phi_{12} \circ \phi_1 \tag{4.5}$$

In terms of the representation functors from the topos  $\mathcal{C}(\mathcal{L}(S))$  to the topoi  $\tau_{\phi_1}$  and  $\tau_{\phi_2}$ , the question is if there exists an interpolating functor from  $\tau_{\phi_1}$  to  $\tau_{\phi_2}$ .

In Section 12.2, we will introduce a certain category,  $\mathcal{M}(\mathbf{Sys})$ , whose objects are topoi and whose arrows are geometric morphisms between topoi. It would be natural to require the arrow from  $\tau_{\phi_1}$  to  $\tau_{\phi_2}$  (if it exists) to be an arrow in this category.

It is at this point that ‘geometric logic’ enters the scene (cf. Section 3.2.3). A formula in  $\mathcal{L}(S)$  is said to be *positive* if it does not contain the symbols<sup>49</sup>  $\Rightarrow$  or  $\forall$ . These conditions imply that  $\neg$  is also absent. In fact, a positive formula uses only  $\exists, \wedge$  and  $\vee$ . A disjunction can have an arbitrary index set, but a conjunction can have only a finite index set. A sentence of the form  $\forall x(\alpha \Rightarrow \beta)$  is said to be a *geometric implication* if both  $\alpha$  and  $\beta$  are positive. Then a *geometric logic* is one in which only geometric implications are present in the language.

The advantage of using just the geometric part of logic is that geometric implications are *preserved* under geometric morphisms. This makes it appropriate to ask for the existence of ‘geometric translations’  $\phi_{12} : \mathcal{L}(\tau_{\phi_1}) \rightarrow \mathcal{L}(\tau_{\phi_2})$ , as in (4.5), since these will preserve the logical structure of the language  $\mathcal{L}(S)$ .

The notion of ‘toinvariance’ introduced recently by Landsmann [53] can be interpreted within our structures as asserting that the translations  $\phi_{12} :$

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<sup>49</sup>Here, the formula  $\alpha \Rightarrow \beta$  is defined as  $\alpha \Rightarrow \beta := (\alpha \wedge \beta) = \alpha$ ;  $\forall$  is defined as  $\forall x\alpha := (\{x \mid \alpha\} = \{x \mid \text{true}\})$ ; where  $\text{true} := * = *$

$\mathcal{L}(\tau_{\phi_1}) \rightarrow \mathcal{L}(\tau_{\phi_2})$  should always exist; or, at least, they should under appropriate conditions. Of course, the significance of this depends on how much information about the system is reflected in the language  $\mathcal{L}(S)$  and how much in the individual representations.

For example, in the case of classical and quantum physics, one might go so far as to include information about the dynamics of the system within the local language  $\mathcal{L}(S)$ . If the topoi  $\phi_1$  and  $\phi_2$  are those for the classical and quantum physics of  $S$  respectively (so that  $\phi_1$  is **Sets** and  $\phi_2$  is  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ ), then an interpolating translation  $\phi_{12} : \mathcal{L}(\mathbf{Sets}) \rightarrow \mathcal{L}(\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}})$  would be a nice realisation of Landsmann’s long-term goal of regarding quantisation as some type of functorial operation.

Of course, introducing dynamics raises interesting questions about the status of the concept of ‘time’ (cf the discussion in Section 3.2.4). In particular, is time to be identified as an object in representing topos, or is it an external parameter, like the ‘ $\Delta$ ’ quantities in the propositional languages  $\mathcal{P}\mathcal{L}(S)$ ?

### 4.3 Classical Physics in the Local Language $\mathcal{L}(S)$

The quantum theory representation of  $\mathcal{L}(S)$  is studied in Section 5. Here we will look at the concrete form of the expressions above for the example of classical physics. In this case, for all systems  $S$ , and all classical representations,  $\sigma$ , the topos  $\tau_\sigma$  is **Sets**. This representation of  $\mathcal{L}(S)$  has the following ingredients:

1. (a) The ground-type symbol  $\Sigma$  is represented by a symplectic manifold,  $\Sigma_\sigma$ , that is the state-space for the system  $S$ .
- (b) The ground-type symbol  $\mathcal{R}$  is represented by the real line, i.e.,  $\mathcal{R}_\sigma := \mathbb{R}$ .
- (c) The type symbol  $P\Sigma$  is represented by the set,  $P\Sigma_\sigma$ , of all<sup>50</sup> subsets of the state space  $\Sigma_\sigma$ .

The type symbol  $P\mathcal{R}$  is represented by the set,  $P\mathbb{R}$ , of all subsets of  $\mathbb{R}$ .

2. (a) The type symbol  $\Omega$ , is represented by  $\Omega_{\mathbf{Sets}} := \{0, 1\}$ : the sub-object classifier in **Sets**.

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<sup>50</sup>To be super precise, we really need to use the collection  $P_{\text{Bor}}\Sigma_\sigma$  of all *Borel* subsets of  $\Sigma_\sigma$ .

- (b) The type symbol  $1$ , is represented by the singleton set: i.e.,  $1_{\mathbf{Sets}} = \{*\}$ , the terminal object in **Sets**.
3. Each function symbol  $A : \Sigma \rightarrow \mathcal{R}$ , and hence each physical quantity, is represented by a real-valued function,  $A_\sigma : \Sigma_\sigma \rightarrow \mathbb{R}$ , on the state space  $\Sigma_\sigma$ .
4. The term ‘ $A(\tilde{s}) \in \tilde{\Delta}$ ’ of type  $\Omega$  (where  $\tilde{s}$  and  $\tilde{\Delta}$  are free variables of type  $\Sigma$  and  $P\mathcal{R}$  respectively) is represented by the function  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\sigma : \Sigma_\sigma \times P\mathbb{R} \rightarrow \{0, 1\}$  that is defined by (c.f. (4.3))

$$\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\sigma(s, \Delta) = \begin{cases} 1 & \text{if } A_\sigma(s) \in \Delta; \\ 0 & \text{otherwise.} \end{cases} \quad (4.6)$$

for all  $(s, \Delta) \in \Sigma_\sigma \times P\mathbb{R}$ .

5. The term  $\{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\}$  of type  $P\Sigma$  (where  $\tilde{\Delta}$  is a free variable of type  $P\mathcal{R}$ ) is represented by the function  $\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_\sigma : P\mathbb{R} \rightarrow P\Sigma_\sigma$  that is defined by

$$\begin{aligned} \llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_\sigma(\Delta) &:= \{s \in \Sigma_\sigma \mid A_\sigma(s) \in \Delta\} \\ &= A_\sigma^{-1}(\Delta) \end{aligned} \quad (4.7)$$

for all  $\Delta \in P\mathbb{R}$ .

## 4.4 Adapting the Language $\mathcal{L}(S)$ to Other Types of Physical System

Our central contention in this work is that (i) each physical system,  $S$ , can be equipped with a local language,  $\mathcal{L}(S)$ ; and (ii) constructing an explicit theory of  $S$  in a particular theory-type is equivalent to finding a representation of  $\mathcal{L}(S)$  in a topos which may well be other than the topos of sets.

There are many situations in which the language is independent of the theory-type, and then, for a given system  $S$ , the different topos representations of  $\mathcal{L}(S)$  correspond to the application of the different theory-types to the same system  $S$ . We gave an example earlier of a point particle moving in three dimensions: the classical physics representation is in the topos **Sets**, but the quantum-theory representation is in the presheaf topos  $\mathbf{Sets}^{\mathcal{V}(L^2(\mathbb{R}^3, d^3x))}$ .

However, there are other situations where the relationship between the language and its representations is more complicated than this. In particular, there is the critical question about what features of the theory should go into

the language, and what into the representation. The first step in adding new features is to augment the set of ground-type symbols. This is because these represent the entities that are going to be of generic interest (such as a state object or quantity-value object). In doing this, extra axioms may also be introduced to encode the properties that the new objects are expected to possess in all representations of physical interest.

For example, suppose we want to use our formalism to discuss space-time physics: where does the information about the space-time go? If the subject is classical field theory in a curved space-time, then the topos  $\tau$  is **Sets**, and the space-time manifold is part of the *background* structure. This makes it natural to have the manifold assumed in the representation; i.e., the information about the space-time is in the representation.

Alternatively, one can add a new ground type symbol, ‘ $M$ ’, to the language, to serve as the linguistic progenitor of ‘space-time’; thus  $M$  would have the same theoretical status as the symbols  $\Sigma$  and  $\mathcal{R}$ . In this context, we recall the brief discussion in Section 2.2.1 about the use of the real numbers in modelling space and/or time, and the motivation this provides for representing space-time as an object in a topos, and whose sub-objects represent the fundamental ‘regions’.

If ‘ $M$ ’ is added to the language, a function symbol  $\psi : M \rightarrow \mathcal{R}$  is then the progenitor of a physical field. In a representation,  $\phi$ , the object  $M_\phi$  plays the role of ‘space-time’ in the topos  $\tau_\phi$ , and  $\psi_\phi : M_\phi \rightarrow \mathcal{R}_\phi$  is the representation of the field.

Of course, the language  $\mathcal{L}(S)$  says nothing about what sort of entity  $M_\phi$  is, except in so far as such information is encoded in extra axioms. For example, if the subject is classical field theory, then  $\tau_\phi = \mathbf{Sets}$ , and  $M_\phi$  would be a standard differentiable manifold. On the other hand, if the topos  $\tau_\phi$  admits ‘infinitesimals’, then  $M_\phi$  could be a manifold according to the language of synthetic differential geometry [51].

The same type of argument applies to the status of ‘time’ in a canonical theory. In particular, it would be possible to add a ground-type symbol,  $\mathcal{T}$ , so that, in any representation,  $\phi$ , the object  $\mathcal{T}_\phi$  in the topos  $\tau_\phi$  is the analogue of the ‘time-line’ for that theory. For standard physics in **Sets** we have  $\mathcal{T}_\phi = \mathbb{R}$ , but the form of  $\mathcal{T}_\phi$  in a more general topos,  $\tau_\phi$ , would be a rich subject for speculation.

The addition of a ‘time-type’ symbol,  $\mathcal{T}$ , to the language  $\mathcal{L}(S)$  is a prime example of a situation where one might want to add extra axioms. These could involve ordering properties, or algebraic properties like those of an abelian group, and so on. In any topos representation, these properties would

then be realised as the corresponding type of object in  $\tau_\phi$ . Thus abelian group axioms mean that  $\mathcal{T}_\phi$  is an abelian-group object in the topos  $\tau_\phi$ ; total-ordering axioms for the time-type  $\mathcal{T}$  mean that  $\mathcal{T}_\phi$  is a totally-ordered object in  $\tau_\phi$ , and so on.

As an interesting extension of this idea, one could have a space-time ground type symbol  $M$ , but then add the axioms for a partial ordering. In that case,  $M_\phi$  would be a poset-object in  $\tau_\phi$ , which could be interpreted physically as the  $\tau_\phi$ -analogue of a causal set [27].

## 5 Quantum Propositions as Sub-Objects of the Spectral Presheaf

### 5.1 Some Background Remarks

#### 5.1.1 The Kochen-Specker Theorem

The idea of representing quantum theory in a topos of presheaves stemmed originally [44] from a desire to acquire a new perspective on the Kochen-Specker theorem [50]. It will be helpful at this stage to review some of this older material.

A commonsense belief, and one apparently shared by Heidegger, is that at any given time any physical quantity *must* have a value even if we do not know what it is. In classical physics, this is not problematic since the underlying mathematical structure is geared precisely to realise it. Specifically, if  $\mathcal{S}$  is the state space of some classical system, and if the physical quantity  $A$  is represented by a real-valued function  $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$ , then the value  $V_s(A)$  of  $A$  in any state  $s \in \mathcal{S}$  is simply

$$V^s(A) = \check{A}(s). \quad (5.1)$$

Thus all physical quantities possess a value in any state. Furthermore, if  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a real-valued function, a new physical quantity  $h(A)$  can be defined by requiring the associated function  $h(A)$  to be

$$h(\check{A})(s) := h(\check{A}(s)) \quad (5.2)$$

for all  $s \in \mathcal{S}$ ; *i.e.*,  $h(\check{A}) := h \circ \check{A} : \mathcal{S} \rightarrow \mathbb{R}$ . Thus the physical quantity  $h(A)$  is *defined* by saying that its value in any state  $s$  is the result of applying the function  $h$  to the value of  $A$ ; hence, by definition, the values of the physical



quantities  $h(A)$  and  $A$  satisfy the ‘functional composition principle’

$$V^s(h(A)) = h(V^s(A)) \quad (5.3)$$

for all states  $s \in \mathcal{S}$ .

However, standard quantum theory precludes any such naive realist interpretation of the relation between formalism and physical world. And this obstruction comes from the mathematical formalism itself, in the guise of the famous Kochen-Specker theorem which asserts the impossibility of assigning values to all physical quantities whilst, at the same time, preserving the functional relations between them [50].

In a quantum theory, a physical quantity  $A$  is represented by a self-adjoint operator  $\hat{A}$  on the Hilbert space of the system, and the first thing one has to decide is whether to regard a valuation as a function of the physical quantities themselves, or on the operators that represent them. From a mathematical perspective, the latter strategy is preferable, and we shall therefore define a valuation to be a real-valued function  $V$  on the set of all bounded, self-adjoint operators, with the properties that : (i) the value  $V(\hat{A})$  of the physical quantity  $A$  represented by the operator  $\hat{A}$  belongs to the spectrum of  $\hat{A}$  (the so-called ‘value rule’); and (ii) the functional composition principle (or FUNC for short) holds:

$$V(\hat{B}) = h(V(\hat{A})) \quad (5.4)$$

for any pair of self-adjoint operators  $\hat{A}, \hat{B}$  such that  $\hat{B} = h(\hat{A})$  for some real-valued function  $h$ . If they existed, such valuations could be used to embed the set of self-adjoint operators in the commutative ring of real-valued functions on an underlying space of microstates, thereby laying the foundations for a hidden-variable interpretation of quantum theory.

Several important results follow from the definition of a valuation. For example, if  $\hat{A}_1$  and  $\hat{A}_2$  commute, it follows from the spectral theorem that there exists an operator  $\hat{C}$  and functions  $h_1$  and  $h_2$  such that  $\hat{A}_1 = h_1(\hat{C})$  and  $\hat{A}_2 = h_2(\hat{C})$ . It then follows from FUNC that

$$V(\hat{A}_1 + \hat{A}_2) = V(\hat{A}_1) + V(\hat{A}_2) \quad (5.5)$$

and

$$V(\hat{A}_1 \hat{A}_2) = V(\hat{A}_1)V(\hat{A}_2). \quad (5.6)$$

The defining equation (5.4) for a valuation makes sense whatever the nature of the spectrum  $\text{sp}(\hat{A})$  of the operator  $\hat{A}$ . However, if  $\text{sp}(\hat{A})$  contains a continuous part, one might doubt the physical meaning of assigning one of its elements as a value. To handle the more general case, we shall view

a valuation as primarily giving *truth-values* to *propositions* about the values of a physical quantity, rather than assigning a specific value to the quantity itself.

As in Section 3, the propositions concerned are of the type “ $A \varepsilon \Delta$ ”, which (in a realist reading) asserts that the value of the physical quantity  $A$  lies in the (Borel) subset  $\Delta$  of the spectrum  $\text{sp}(\hat{A})$  of the associated operator  $\hat{A}$ . This proposition is represented by the spectral projector  $\hat{E}[A \in \Delta]$ , which motivates studying the general mathematical problem of assigning truth-values to projection operators.

If  $\hat{P}$  is a projection operator, the identity  $\hat{P} = \hat{P}^2$  implies that  $V(\hat{P}) = V(\hat{P}^2) = (V(\hat{P}))^2$  (from (5.6)); and hence, necessarily,  $V(\hat{P}) = 0$  or  $1$ . Thus  $V$  defines a homomorphism from the Boolean algebra  $\{\hat{0}, \hat{1}, \hat{P}, \neg\hat{P} \equiv (\hat{1} - \hat{P})\}$  to the ‘false(0)-true(1)’ Boolean algebra  $\{0, 1\}$ . More generally, a valuation  $V$  induces a homomorphism  $\chi^V : W \rightarrow \{0, 1\}$  where  $W$  is any Boolean sub-algebra of the lattice  $\mathcal{P}(\mathcal{H})$  of projectors on  $\mathcal{H}$ . In particular,

$$\hat{\alpha} \preceq \hat{\beta} \text{ implies } \chi^V(\hat{\alpha}) \leq \chi^V(\hat{\beta}) \quad (5.7)$$

where ‘ $\hat{\alpha} \preceq \hat{\beta}$ ’ refers to the partial ordering in the lattice  $\mathcal{P}(\mathcal{H})$ , and ‘ $\chi^V(\hat{\alpha}) \leq \chi^V(\hat{\beta})$ ’ is the ordering in the Boolean algebra  $\{0, 1\}$ .

The Kochen-Specker theorem asserts that no global valuations exist if the dimension of the Hilbert space  $\mathcal{H}$  is greater than two. The obstructions to the existence of such valuations typically arise when trying to assign a single value to an operator  $\hat{C}$  that can be written as  $\hat{C} = g(\hat{A})$  and as  $\hat{C} = h(\hat{B})$  with  $[\hat{A}, \hat{B}] \neq 0$ .

The various interpretations of quantum theory that aspire to use ‘beables’, rather than ‘observables’, are all concerned in one way or another with addressing this issue. Inherent in such schemes is a type of ‘contextuality’ in which a value given to a physical quantity  $C$  cannot be part of a global assignment of values but must, instead, depend on some *context* in which  $C$  is to be considered. In practice, contextuality is endemic in any attempt to ascribe properties to quantities in a quantum theory. For example, as emphasized by Bell [9], in the situation where  $\hat{C} = g(\hat{A}) = h(\hat{B})$ , if the value of  $C$  is construed counterfactually as referring to what would be obtained *if* a measurement of  $A$  or of  $B$  is made—and with the value of  $C$  then being *defined* by applying to the result of the measurement the relation  $C = g(A)$ , or  $C = h(B)$ —then one can claim that the actual value obtained depends on whether the value of  $C$  is determined by measuring  $A$ , or by measuring  $B$ .

In the programme to be discussed here, the idea of a contextual valuation will be developed in a different direction from that of the existing modal

interpretations in which ‘reality’ is ascribed to only some commutative subset of physical quantities. In particular, rather than accepting such a limited domain of beables we shall propose a theory of ‘generalised’ valuations that are defined globally on *all* propositions about values of physical quantities. However, the price of global existence is that any given proposition may have only a ‘*generalised*’ truth-value. More precisely, (i) the truth-value of a proposition “ $A \varepsilon \Delta$ ” belongs to a logical structure that is larger than  $\{0, 1\}$ ; and (ii) these target-logics, and truth values, are context-dependent.

It is clear that the main task is to formulate mathematically the idea of a contextual, truth-value in such a way that the assignment of generalised truth-values is consistent with an appropriate analogue of the functional composition principle FUNC.

### 5.1.2 The Introduction of Coarse-Graining

In the original paper [44], this task is tackled using a type of ‘coarse-graining’ operation. The key idea is that, although in a given situation in quantum theory it may not be possible to declare a particular proposition “ $A \varepsilon \Delta$ ” to be true (or false), nevertheless there may be (Borel) functions  $f$  such that the associated propositions “ $f(A) \varepsilon f(\Delta)$ ” *can* be said to be true. This possibility arises for the following reason.

Let  $W_A$  denote the spectral algebra of the operator  $\hat{A}$  that represents a physical quantity  $A$ . Thus  $W_A$  is the Boolean algebra of projectors  $\hat{E}[A \in \Delta]$  that project onto the eigenspaces associated with the Borel subsets  $\Delta$  of the spectrum  $\text{sp}(\hat{A})$  of  $\hat{A}$ ; physically speaking,  $\hat{E}[A \varepsilon \Delta]$  represents the proposition “ $A \varepsilon \Delta$ ”. It follows from the spectral theorem that, for all Borel subsets  $J$  of the spectrum of  $f(\hat{A})$ , the spectral projector  $\hat{E}[f(A) \varepsilon J]$  for the operator  $f(\hat{A})$  is equal to the spectral projector  $\hat{E}[A \varepsilon f^{-1}(J)]$  for  $\hat{A}$ . In particular, if  $f(\Delta)$  is a Borel subset of  $\text{sp}(f(\hat{A}))$  then, since  $\Delta \subseteq f^{-1}(f(\Delta))$ , we have  $\hat{E}[A \varepsilon \Delta] \preceq \hat{E}[A \varepsilon f^{-1}(f(\Delta))]$ ; and hence

$$\hat{E}[A \varepsilon \Delta] \preceq \hat{E}[f(A) \varepsilon f(\Delta)]. \quad (5.8)$$

Physically, the inequality in (5.8) reflects the fact that the proposition “ $f(A) \varepsilon f(\Delta)$ ” is generally weaker than the proposition “ $A \varepsilon \Delta$ ” in the sense that the latter implies the former, but not necessarily vice versa. For example, the proposition “ $f(A) = f(a)$ ” is weaker than the original proposition “ $A = a$ ” if the function  $f$  is many-to-one and such that more than one eigenvalue of  $\hat{A}$  is mapped to the same eigenvalue of  $f(\hat{A})$ . In general, we shall say that “ $f(A) \varepsilon f(\Delta)$ ” is a *coarse-graining* of “ $A \varepsilon \Delta$ ”.

Now, if the proposition “ $A \varepsilon \Delta$ ” is evaluated as ‘true’ then, from (5.7) and (5.8), it follows that the weaker proposition “ $f(A) \varepsilon f(\Delta)$ ” is also evaluated as ‘true’.

This remark provokes the following observation. There may be situations in which, although the proposition “ $A \varepsilon \Delta$ ” cannot be said to be either true or false, the weaker proposition “ $f(A) \varepsilon f(\Delta)$ ” can. In particular, if the latter *can* be given the value ‘true’, then—by virtue of the remark above—it is natural to suppose that any further coarse-graining to give an operator  $g(f(\hat{A}))$  will yield a proposition “ $g(f(A)) \varepsilon g(f(\Delta))$ ” that will also be evaluated as ‘true’. Note that there may be more than one possible choice for the ‘initial’ function  $f$ , each of which can then be further coarse-grained in this way. This multi-branched picture of coarse-graining is one of the main justifications for our invocation of the topos-theoretic idea of a presheaf.

It transpires that the key remark above is the statement:

If “ $f(A) \varepsilon f(\Delta)$ ” is true, then so is “ $g(f(A)) \varepsilon g(f(\Delta))$ ” for any function  $g : \mathbb{R} \rightarrow \mathbb{R}$ .

This is key because the property thus asserted can be restated by saying that the collection of all functions  $f$  such that “ $f(A) \varepsilon f(\Delta)$ ” is ‘true’ is a *sieve*; and sieves are closely associated with global elements of the sub-object classifier in a category of presheaves.

To clarify this we start by defining a category  $\mathcal{O}$  whose objects are the bounded, self-adjoint operators on  $\mathcal{H}$ . For the sake of simplicity, we will assume for the moment that  $\mathcal{O}$  consists only of the operators whose spectrum is discrete. Then we say that there is a ‘morphism’ from  $\hat{B}$  to  $\hat{A}$  if there exists a Borel function (more precisely, an equivalence class of Borel functions)  $f : \text{sp}(\hat{A}) \rightarrow \mathbb{R}$  such that  $\hat{B} = f(\hat{A})$ , where  $\text{sp}(\hat{A})$  is the spectrum of  $\hat{A}$ . Any such function on  $\text{sp}(\hat{A})$  is unique (up to the equivalence relation), and hence there is at most one morphism between any two operators. If  $\hat{B} = f(\hat{A})$ , the corresponding morphism in the category  $\mathcal{O}$  will be denoted  $f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A}$ . It then becomes clear that the statement in the box above is equivalent to the statement that the collection of all functions  $f$  such that “ $f(A) \varepsilon f(\Delta)$ ” is ‘true’, is a sieve<sup>51</sup> on the object  $\hat{A}$  in the category  $\mathcal{O}$ .

This motivates very strongly looking at the topos category,  $\mathbf{Sets}^{\mathcal{O}^{\text{op}}}$  of contravariant<sup>52</sup>, set-valued functors on  $\mathcal{O}$ . Then, bearing in mind our dis-

<sup>51</sup>It is a matter of convention whether this is called a sieve or a co-sieve.

<sup>52</sup>Ab initio, we could just as well have looked at covariant functors, but with our definitions the contravariant ones are more natural.

cussion of values of physical quantities, it is rather natural to construct the following object in this topos:

**Definition 5.1** *The spectral presheaf on  $\mathcal{O}$  is the contravariant functor  $\underline{\Sigma} : \mathcal{O} \rightarrow \mathbf{Sets}$  defined as follows:*

1. *On objects:  $\underline{\Sigma}(\hat{A}) := \text{sp}(\hat{A})$ .*
2. *On morphisms: If  $f_{\mathcal{O}} : \hat{B} \rightarrow \hat{A}$ , so that  $\hat{B} = f(\hat{A})$ , then  $\underline{\Sigma}(f_{\mathcal{O}}) : \sigma(\hat{A}) \rightarrow \sigma(\hat{B})$  is defined by  $\underline{\Sigma}(f_{\mathcal{O}})(\lambda) := f(\lambda)$  for all  $\lambda \in \sigma(\hat{A})$ .*

Note that  $\underline{\Sigma}(f_{\mathcal{O}})$  is well-defined since, if  $\lambda \in \sigma(\hat{A})$ , then  $f(\lambda)$  is indeed an element of the spectrum of  $\hat{B}$ ; indeed, for these discrete-spectrum operators we have  $\sigma(f(\hat{A})) = f(\sigma(\hat{A}))$ .

The key remark now is the following. If  $\mathcal{C}$  is any category, a *global element*, of a contravariant functor  $\underline{X} : \mathcal{C} \rightarrow \mathbf{Sets}$  is defined to be a function  $\gamma$  that assigns to each object  $A$  in the category  $\mathcal{C}$  an element  $\gamma_A \in \underline{X}(A)$  in such a way that if  $f : B \rightarrow A$  then  $\underline{X}(f)(\gamma_A) = \gamma_B$  (see Appendix 2 for more details).

In the case of the spectral functor  $\underline{\Sigma}$ , a global element is therefore a function  $\gamma$  that assigns to each (bounded, discrete spectrum) self-adjoint operator  $\hat{A}$ , a real number  $\gamma_A \in \text{sp}(\hat{A})$  such that if  $\hat{B} = f(\hat{A})$  then  $f(\gamma_A) = \gamma_B$ . But this is precisely the condition FUNC in Eq. (5.4) for a valuation!

Thus, the Kochen-Specker theorem is equivalent to the statement that, if  $\dim \mathcal{H} > 2$ , the spectral presheaf  $\underline{\Sigma}$  has no global elements.

It was this observation that motivated the original suggestion by one of us (CJI) and his collaborators that quantum theory should be studied from the perspective of topos theory. However, as it stands, the discussion above works only for operators with a discrete spectrum. This is fine for finite-dimensional Hilbert spaces, but in an infinite-dimensional space operators can have continuous parts in their spectra, and then things get more complicated.

One powerful way of tackling this problem is to replace the category of operators with a category,  $\mathcal{V}(\mathcal{H})$ , whose objects are commutative von Neumann sub-algebras of the algebra  $B(\mathcal{H})$  of all bounded operators on  $\mathcal{H}$ . There is a close link with the category  $\mathcal{C}$  since each self-adjoint operator

generates a commutative von Neumann algebra, but using  $\mathcal{V}(\mathcal{H})$  rather than  $\mathcal{C}$  solves all the problems associated with continuous spectra [35].

Of course, this particular motivation for introducing  $\mathcal{V}(\mathcal{H})$  is purely mathematical, but there are also very good physics reasons for this step. As we have mentioned earlier, one approach to handling the implications of the Kochen-Specker theorem is to ‘reify’ only a subset of physical variables, as is done in the various ‘modal interpretations’. The topos-theoretic extension of this idea of ‘partial reification’, first proposed in [44, 45, 35, 13], is to build a structure in which all possible reifiable sets of physical variables are included on an equal footing. This involves constructing a category,  $\mathcal{C}$ , whose objects are collections of quantum observables that *can* be simultaneously reified because the corresponding self-adjoint operators *commute*. The application of this type of topos scheme to an actual modal interpretation is discussed in the recent paper by Nakayama [62]

From a physical perspective, the objects in the category  $\mathcal{C}$  can be viewed as *contexts* (or ‘world-views’, or ‘windows on reality’, or ‘classical snapshots’) from whose perspectives the quantum theory can be displayed. This is the physical motivation for using commutative von Neumann algebras.

In the normal, instrumentalist interpretation of quantum theory, a context is therefore a collection of physical variables that can be measured simultaneously. The physical significance of this contextual logic is discussed at length in [44, 45, 35, 13, 46] and [22, 23].

### 5.1.3 Alternatives to von Neumann Algebras

It should be remarked that  $\mathcal{V}(\mathcal{H})$  is not the only possible choice for the category of concepts. Another possibility is to construct a category whose objects are the Boolean sub-algebras of the non-distributive lattice of projection operators on the Hilbert space; more generally we could consider the Boolean sub-algebras of *any* non-distributive lattice. This option was discussed in [44].

Yet another possibility is to consider the abelian  $C^*$ -sub-algebras of the algebra  $B(\mathcal{H})$  of all bounded operators on  $\mathcal{H}$ . More generally, one could consider the abelian sub-algebras of *any*  $C^*$ -algebra; this is the option adopted by Heunen and Spitters [38] in their very interesting recent development of our scheme. One disadvantage of a  $C^*$ -algebra is that does not contain projectors, and if one wants to include them it is necessary to move to  $AW^*$ -algebras, which are the abstract analogue of the concrete von Neumann algebras that we employ. For each of these choices there is a corresponding spectral object,

and these different spectral objects are closely related.

It is clear that a similar procedure could be followed for *any* algebraic quantity  $\mathfrak{A}$  that has an ‘interesting’ collection of commutative sub-algebras. We will return to this remark in Section 14.1.4.

## 5.2 From Projections to Global Elements of the Outer Presheaf

### 5.2.1 The Definition of $\delta(\hat{P})_V$

The fundamental thesis of our work is that in constructing theories of physics one should seek representations of a formal language in a topos that may be other than **Sets**. We want now to study this idea closely in the context of the ‘toposification’ of standard quantum theory, with particular emphasis on a topos representation of propositions. Most ‘standard’ quantum systems (for example, one-dimensional motion with a Hamiltonian  $H = \frac{p^2}{2m} + V(x)$ ) are obtained by ‘quantising’ a classical system, and consequently the formal language is the same as it is for the classical system. Our immediate goal is to represent physical propositions with sub-objects of the spectral presheaf  $\underline{\Sigma}$ .

In this Section we concentrate on the propositional language  $\mathcal{P}\mathcal{L}(S)$  introduced in Section 3.2. Thus a key task is to find the map  $\pi_{\text{qt}} : \mathcal{P}\mathcal{L}(S)_0 \rightarrow \text{Sub}(\underline{\Sigma})$ , where the primitive propositions in  $\mathcal{P}\mathcal{L}(S)_0$  are of the form “ $A \varepsilon \Delta$ ”. As we shall see, this is where the critical concept of *daseinisation* arises: the procedure whereby a projector  $\hat{P}$  is transformed to a sub-object,  $\delta(\hat{P})$ , of the spectral presheaf,  $\underline{\Sigma}$ , in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  (the precise definition of  $\underline{\Sigma}$  is given in Section 5.3.1).

In standard quantum theory, a physical quantity is represented by a self-adjoint operator  $\hat{A}$  in the algebra,  $B(\mathcal{H})$ , of all bounded operators on  $\mathcal{H}$ . If  $\Delta \subseteq \mathbb{R}$  is a Borel subset, we know from the spectral theorem that the proposition “ $A \varepsilon \Delta$ ” is represented by<sup>53</sup> the projection operator  $\hat{E}[A \in \Delta]$  in  $B(\mathcal{H})$ . For typographical simplicity, for the rest of this Section,  $\hat{E}[A \in \Delta]$  will be denoted by  $\hat{P}$ .

We are going to consider the projection operator  $\hat{P}$  from the perspective of the ‘category of contexts’—a keystone of the topos approach to quantum theory. As we have remarked earlier, there are several possible choices for

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<sup>53</sup>Note, however, that the map from propositions to projections is not injective: two propositions “ $A \varepsilon \Delta_1$ ” and “ $B \varepsilon \Delta_2$ ” concerning two distinct physical quantities,  $A$  and  $B$ , can be represented by the same projector: i.e.,  $\hat{E}[A \in \Delta_1] = \hat{E}[B \in \Delta_2]$ .



this category most of which are considered in detail in the original papers [44, 45, 35, 13]. Here we have elected to use the category  $\mathcal{V}(\mathcal{H})$  of unital, abelian sub-algebras of  $B(\mathcal{H})$ . This partially-ordered set has a category structure in which (i) the objects are the abelian sub-algebras of  $B(\mathcal{H})$ ; and (ii) there is an arrow  $i_{V'V} : V' \rightarrow V$ , where  $V', V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ ,<sup>54</sup> if and only if  $V' \subseteq V$ . By definition, the trivial sub-algebra  $V_0 = \mathbb{C}1$  is not included in the objects of  $\mathcal{V}(\mathcal{H})$ . A context could also be called a ‘world-view’, a ‘classical snap-shot’, a ‘window on reality’, or even a *Weltanschauung*<sup>55</sup>; mathematicians often refer to it as a ‘stage of truth’.

The critical question is what can be said about the projector  $\hat{P}$  ‘from the perspective’ of a particular context  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ ? If  $\hat{P}$  belongs to  $V$  then a ‘full’ image of  $\hat{P}$  is obtained from this view-point, and there is nothing more to say. However, suppose the abelian sub-algebra  $V$  does *not* contain  $\hat{P}$ : what then?

We need to ‘approximate’  $\hat{P}$  from the perspective of  $V$ , and an important ingredient in our work is to define this as meaning the ‘smallest’ projection operator,  $\delta(\hat{P})_V$ , in  $V$  that is greater than, or equal to,  $\hat{P}$ :

$$\delta(\hat{P})_V := \bigwedge \{ \hat{\alpha} \in \mathcal{P}(V) \mid \hat{\alpha} \succeq \hat{P} \}. \quad (5.9)$$

where ‘ $\succeq$ ’ is the usual ordering of projection operators, and where  $\mathcal{P}(V)$  denotes the set of all projection operators in  $V$ .

To see what this means, let  $\hat{P}$  and  $\hat{Q}$  represent the propositions “ $A \varepsilon \Delta$ ” and “ $A \varepsilon \Delta'$ ” respectively with  $\Delta \subseteq \Delta'$ , so that  $\hat{P} \preceq \hat{Q}$ . Since we learn less about the value of  $A$  from the proposition “ $A \varepsilon \Delta'$ ” than from “ $A \varepsilon \Delta$ ”, the former proposition is said to be *weaker*. Clearly, the weaker proposition “ $A \varepsilon \Delta'$ ” is implied by the stronger proposition “ $A \varepsilon \Delta$ ”. The construction of  $\delta(\hat{P})_V$  as the smallest projection in  $V$  greater than or equal to  $\hat{P}$  thus gives the strongest proposition expressible in  $V$  that is implied by  $\hat{P}$  (although, if  $\hat{A} \notin V$ , the projection  $\delta(\hat{P})_V$  cannot usually be interpreted as a proposition *about*  $A$ ).<sup>56</sup> Note that if  $\hat{P}$  belongs to  $V$ , then  $\delta(\hat{P})_V = \hat{P}$ . The mapping  $\hat{P} \mapsto \delta(\hat{P})_V$  was originally introduced by de Groot in [32], who called it the ‘ $V$ -support’ of  $\hat{P}$ .

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<sup>54</sup>We denote by  $\text{Ob}(\mathcal{C})$  the collection of all objects in the category  $\mathcal{C}$ .

<sup>55</sup>‘*Weltanschauung*’ is a splendid German word. ‘*Welt*’ means world; ‘*schauen*’ is a verb and means to look, to view; ‘*anschauen*’ is to look at; and ‘-ung’ at the end of a word can make a noun from a verb. So it’s *Welt-an-schau-ung*.

<sup>56</sup>Note that the definition in (5.9) exploits the fact that the lattice  $\mathcal{P}(V)$  of projection operators in  $V$  is complete. This is the main reason why we chose von Neumann sub-algebras rather than  $C^*$ -algebras: the former contain enough projections, and their projection lattices are complete.



The key idea in this part of our scheme is that rather than thinking of a quantum proposition, “ $A \in \Delta$ ”, as being represented by the *single* projection operator  $\hat{E}[A \in \Delta]$ , instead we consider the entire *collection*  $\{\delta(\hat{E}[A \in \Delta])_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\}$  of projection operators, one for each context  $V$ . As we will see, the link with topos theory is that this collection of projectors is a global element of a certain presheaf.

This ‘certain’ presheaf is in fact the ‘outer’ presheaf, which is defined as follows:

**Definition 5.2** *The outer<sup>57</sup> presheaf  $\underline{Q}$  is defined over the category  $\mathcal{V}(\mathcal{H})$  as follows [44, 35]:*

- (i) *On objects  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ : We have  $\underline{Q}_V := \mathcal{P}(V)$*
- (ii) *On morphisms  $i_{V'V} : V' \subseteq V$ : The mapping  $\underline{Q}(i_{V'V}) : \underline{Q}_V \rightarrow \underline{Q}_{V'}$  is given by  $\underline{Q}(i_{V'V})(\hat{\alpha}) := \delta(\hat{\alpha})_{V'}$  for all  $\hat{\alpha} \in \mathcal{P}(V)$ .*

With this definition, it is clear that, for each projection operator  $\hat{P}$ , the assignment  $V \mapsto \delta(\hat{P})_V$  defines a global element of the presheaf  $\underline{Q}$ . Indeed, for each context  $V$ , we have the projector  $\delta(\hat{P})_V \in \mathcal{P}(V) = \underline{Q}_V$ , and if  $i_{V'V} : V' \subseteq V$ , then

$$\delta(\delta(\hat{P})_V)_{V'} = \bigwedge \{\hat{Q} \in \mathcal{P}(V') \mid \hat{Q} \succeq \delta(\hat{P})_V\} = \delta(\hat{P})_{V'} \quad (5.10)$$

and so the elements  $\delta(\hat{P})_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , are compatible with the structure of the outer presheaf. Thus we have a mapping

$$\begin{aligned} \delta : \mathcal{P}(\mathcal{H}) &\rightarrow \Gamma \underline{Q} \\ \hat{P} &\mapsto \{\delta(\hat{P})_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\} \end{aligned} \quad (5.11)$$

from the projectors in  $\mathcal{P}(\mathcal{H})$  to the global elements,  $\Gamma \underline{Q}$ , of the outer presheaf.<sup>58</sup>

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<sup>57</sup>In the original papers by CJI and collaborators, this was called the ‘coarse-graining’ presheaf, and was denoted  $\underline{G}$ . The reason for the change of nomenclature will become apparent later.

<sup>58</sup>Vis-a-vis our use of the language  $\mathcal{L}(S)$  a little further on, we should emphasise that the outer presheaf has no linguistic precursor, and in this sense, it has no fundamental status in the theory. In fact, we could avoid the outer presheaf altogether and always work directly with the spectral presheaf,  $\underline{\Sigma}$ , which, of course, *does* have a linguistic precursor. However, it is technically convenient to introduce the outer presheaf as an intermediate tool.

### 5.2.2 Properties of the Mapping $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \Gamma \underline{Q}$ .

Let us now note some properties of the map  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \Gamma \underline{Q}$  that are relevant to our overall scheme.

1. For all contexts  $V$ , we have  $\delta(\hat{0})_V = \hat{0}$ .

The null projector represents all propositions of the form “ $A \varepsilon \Delta$ ” with the property that  $\text{sp}(\hat{A}) \cap \Delta = \emptyset$ . These propositions are trivially false.

2. For all contexts  $V$ , we have  $\delta(\hat{1})_V = \hat{1}$ .

The unit operator  $\hat{1}$  represents all propositions of the form “ $A \varepsilon \Delta$ ” with the property that  $\text{sp}(\hat{A}) \cap \Delta = \text{sp}(\hat{A})$ . These propositions are trivially true.

3. There exist global elements of  $\underline{Q}$  that are *not* of the form  $\delta(\hat{P})$  for any projector  $\hat{P}$ . This phenomenon will be discussed later. However, if  $\gamma \in \Gamma \underline{Q}$  is of the form  $\delta(\hat{P})$  for some  $\hat{P}$ , then

$$\hat{P} = \bigwedge_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta(\hat{P})_V, \quad (5.12)$$

because  $\delta(\hat{P})_V \succeq \hat{P}$  for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , and  $\delta(\hat{P})_V = \hat{P}$  for any  $V$  that contains  $\hat{P}$ .

The next result is important as it means that ‘nothing is lost’ in mapping a projection operator  $\hat{P}$  to its associated global element,  $\delta(\hat{P})$ , of the presheaf  $\underline{Q}$ .

**Theorem 5.1** *The map  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \Gamma \underline{Q}$  is injective.*

This simply follows from (5.12): if  $\delta(\hat{P}) = \delta(\hat{Q})$  for two projections  $\hat{P}, \hat{Q}$ , then

$$\hat{P} = \bigwedge_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta(\hat{P})_V = \bigwedge_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta(\hat{Q})_V = \hat{Q}. \quad (5.13)$$

### 5.2.3 A Logical Structure for $\Gamma\underline{Q}$ ?

We have seen that the quantities  $\delta(\hat{P}) := \{\delta(\hat{P})_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\}$ ,  $\hat{P} \in \mathcal{P}(\mathcal{H})$ , are elements of  $\Gamma\underline{Q}$ , and if they are to represent quantum propositions, one might expect/hope that (i) these global elements of  $\underline{Q}$  form a Heyting algebra; and (ii) this algebra is related in some way to the Heyting algebra of sub-objects of  $\underline{\Sigma}$ . Let us see how far we can go in this direction.

Our first remark is that any two global elements  $\gamma_1, \gamma_2$  of  $\underline{Q}$  can be compared at each stage  $V$  in the sense of logical implication. More precisely, let  $\gamma_{1V} \in \mathcal{P}(V)$  denote the  $V$ 'th ‘component’ of  $\gamma_1$ , and ditto for  $\gamma_{2V}$ . Then we have the following result:

**Definition 5.3** *A partial ordering on  $\Gamma\underline{Q}$  can be constructed in a ‘local’ way (i.e., ‘local’ with respect to the objects in the category  $\mathcal{V}(\mathcal{H})$ ) by defining*

$$\gamma_1 \succeq \gamma_2 \text{ if, and only if, } \forall V \in \text{Ob}(\mathcal{V}(\mathcal{H})), \gamma_{1V} \succeq \gamma_{2V} \quad (5.14)$$

where the ordering on the right hand side of (5.14) is the usual ordering in the lattice of projectors  $\mathcal{P}(V)$ .

It is trivial to check that (5.14) defines a partial ordering on  $\Gamma\underline{Q}$ . Thus  $\Gamma\underline{Q}$  is a partially ordered set.

Note that if  $\hat{P}, \hat{Q}$  are projection operators, then it follows from (5.14) that

$$\delta(\hat{P}) \succeq \delta(\hat{Q}) \text{ if and only if } \hat{P} \succeq \hat{Q} \quad (5.15)$$

since  $\hat{P} \succeq \hat{Q}$  implies  $\delta(\hat{P})_V \succeq \delta(\hat{Q})_V$  for all contexts  $V$ .<sup>59</sup> Thus the mapping  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \Gamma\underline{Q}$  respects the partial order.

The next thing is to see if a logical ‘ $\vee$ ’-operation can be defined on  $\Gamma\underline{Q}$ . Once again, we try a ‘local’ definition:

**Theorem 5.2** *A ‘ $\vee$ ’-structure on  $\Gamma\underline{Q}$  can be defined locally by*

$$(\gamma_1 \vee \gamma_2)_V := \gamma_{1V} \vee \gamma_{2V} \quad (5.16)$$

for all  $\gamma_1, \gamma_2 \in \Gamma\underline{Q}$ , and for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ .

**Proof.** It is not instantly clear that (5.16) defines a global element of  $\underline{Q}$ . However, a key result in this direction is the following:

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<sup>59</sup>On the other hand, in general,  $\hat{P} \succ \hat{Q}$  does not imply  $\delta(\hat{P})_V \succ \delta(\hat{Q})_V$  but only  $\delta(\hat{P})_V \succeq \delta(\hat{Q})_V$ .

**Lemma 5.3** For each context  $V$ , and for all  $\hat{\alpha}, \hat{\beta} \in \mathcal{P}(V)$ , we have

$$\underline{Q}(i_{V'V})(\hat{\alpha} \vee \hat{\beta}) = \underline{Q}(i_{V'V})(\hat{\alpha}) \vee \underline{Q}(i_{V'V})(\hat{\beta}) \quad (5.17)$$

for all contexts  $V'$  such that  $V' \subseteq V$ .

The proof is a straightforward consequence of the definition of the presheaf  $\underline{Q}$ .

One immediate consequence is that (5.16) defines a global element<sup>60</sup> of  $\underline{Q}$ . Hence the theorem is proved. ■

It is also straightforward to show that, for any pair of projectors  $\hat{P}, \hat{Q} \in \mathcal{P}(\mathcal{H})$ , we have  $\delta(\hat{P} \vee \hat{Q})_V = \delta(\hat{P})_V \vee \delta(\hat{Q})_V$ , for all contexts  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . This means that, as elements of  $\Gamma \underline{Q}$ ,

$$\delta(\hat{P} \vee \hat{Q}) = \delta(\hat{P}) \vee \delta(\hat{Q}). \quad (5.18)$$

Thus the mapping  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \Gamma \underline{Q}$  preserves the logical ‘ $\vee$ ’ operation.

However, there is no analogous equation for the logical ‘ $\wedge$ ’-operation. The obvious local definition would be, for each context  $V$ ,

$$(\gamma_1 \wedge \gamma_2)_V := \gamma_{1V} \wedge \gamma_{2V} \quad (5.19)$$

but this does not define a global element of  $\underline{Q}$  since, unlike (5.17), for the  $\wedge$ -operation we have only

$$\underline{Q}(i_{V'V})(\hat{\alpha} \wedge \hat{\beta}) \preceq \underline{Q}(i_{V'V})(\hat{\alpha}) \wedge \underline{Q}(i_{V'V})(\hat{\beta}) \quad (5.20)$$

for all  $V' \subseteq V$ . As a consequence, for all  $V$ , we have only the inequality

$$\delta(\hat{P} \wedge \hat{Q})_V \preceq \delta(\hat{P})_V \wedge \delta(\hat{Q})_V \quad (5.21)$$

and hence

$$\delta(\hat{P} \wedge \hat{Q}) \preceq \delta(\hat{P}) \wedge \delta(\hat{Q}). \quad (5.22)$$

It is easy to find examples where the inequality is strict. For example, let  $\hat{P} \neq \hat{0}, \hat{1}$  and  $\hat{Q} = \hat{1} - \hat{P}$ . Then  $\hat{P} \wedge \hat{Q} = \hat{0}$  and hence  $\delta_V(\hat{P} \wedge \hat{Q}) = \hat{0}$ , while  $\delta(\hat{P})_V \wedge \delta(\hat{Q})_V$  can be strictly larger than  $\hat{0}$ , since  $\delta(\hat{P})_V \succeq \hat{P}$  and  $\delta(\hat{Q})_V \succeq \hat{Q}$ .

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<sup>60</sup>The existence of the  $\vee$ -operation on  $\Gamma \underline{Q}$  can be extended to  $\underline{Q}$  itself. More precisely, there is an arrow  $\vee : \underline{Q} \times \underline{Q} \rightarrow \underline{Q}$  where  $\underline{Q} \times \underline{Q}$  denotes the product presheaf over  $\mathcal{V}(\mathcal{H})$ , whose objects are  $(\underline{Q} \times \underline{Q})_V := \underline{Q}_V \times \underline{Q}_V$ . Then the arrow  $\vee : \underline{Q} \times \underline{Q} \rightarrow \underline{Q}$  is defined at any context  $V$  by  $\vee_V(\hat{\alpha}, \hat{\beta}) := \hat{\alpha} \vee \hat{\beta}$  for all  $\hat{\alpha}, \hat{\beta} \in \underline{Q}_V$ .

### 5.2.4 Hyper-Elements of $\Gamma\mathcal{Q}$ .

We have seen that the global elements of  $\mathcal{Q}$ , i.e., the elements of  $\Gamma\mathcal{Q}$ , can be equipped with a partial-ordering and a ‘ $\vee$ ’-operation, but attempts to define a ‘ $\wedge$ ’-operation in the same way fail because of the inequality in (5.21).

However, the form of (5.20–5.21) suggests the following procedure. Let us define a *hyper-element* of  $\mathcal{Q}$  to be an association, for each stage  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , of an element  $\gamma_V \in \mathcal{Q}_V$  with the property that

$$\gamma_{V'} \succeq \underline{Q}(i_{V'V})(\gamma_V) \quad (5.23)$$

for all  $V' \subseteq V$ . Clearly every element of  $\Gamma\mathcal{Q}$  is a hyper-element, but not conversely.

Now, if  $\gamma_1$  and  $\gamma_2$  are hyper-elements, we can define the operations ‘ $\vee$ ’ and ‘ $\wedge$ ’ locally as:

$$(\gamma_1 \vee \gamma_2)_V := \gamma_{1V} \vee \gamma_{2V} \quad (5.24)$$

$$(\gamma_1 \wedge \gamma_2)_V := \gamma_{1V} \wedge \gamma_{2V} \quad (5.25)$$

Because of (5.20) we have, for all  $V' \subseteq V$ ,

$$\underline{Q}(i_{V'V})((\gamma_1 \wedge \gamma_2)_V) = \underline{Q}(i_{V'V})(\gamma_{1V} \wedge \gamma_{2V}) \quad (5.26)$$

$$\leq \underline{Q}(i_{V'V})(\gamma_{1V}) \wedge \underline{Q}(i_{V'V})(\gamma_{2V}) \quad (5.27)$$

$$\leq \gamma_{1V'} \wedge \gamma_{2V'} \quad (5.28)$$

$$= (\gamma_1 \wedge \gamma_2)_{V'} \quad (5.29)$$

so that the hyper-element condition (5.23) is preserved.

The occurrence of a logical ‘ $\vee$ ’ and ‘ $\wedge$ ’ structure is encouraging, but it is not yet what we want. For one thing, there is no mention of a negation operation; and, anyway, this is not the expected algebra of sub-objects of a ‘state space’ object. To proceed further we must study more carefully the sub-objects of the spectral presheaf.

## 5.3 Daseinisation: Heidegger Encounters Physics

### 5.3.1 From Global Elements of $\mathcal{Q}$ to Sub-Objects of $\underline{\Sigma}$ .

The spectral presheaf,  $\underline{\Sigma}$ , played a central role in the earlier discussions of quantum theory from a topos perspective [44, 45, 35, 13]. Here is the formal definition.

**Definition 5.4** *The spectral presheaf,  $\underline{\Sigma}$ , is defined as the following functor from  $\mathcal{V}(\mathcal{H})^{\text{op}}$  to **Sets**:*

1. *On objects  $V$ :  $\underline{\Sigma}_V$  is the Gel'fand spectrum of the unital, abelian sub-algebra  $V$  of  $B(\mathcal{H})$ ; i.e., the set of all multiplicative linear functionals  $\lambda : V \rightarrow \mathbb{C}$  such that  $\langle \lambda, \hat{1} \rangle = 1$ .*
2. *On morphisms  $i_{V'V} : V' \subseteq V$ :  $\underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V \rightarrow \underline{\Sigma}_{V'}$  is defined by  $\underline{\Sigma}(i_{V'V})(\lambda) := \lambda|_{V'}$ ; i.e., the restriction of the functional  $\lambda : V \rightarrow \mathbb{C}$  to the sub-algebra  $V' \subseteq V$ .*

One central result of spectral theory is that  $\underline{\Sigma}_V$  has a topology that is compact and Hausdorff, and with respect to which the Gel'fand transforms<sup>61</sup> of the elements of  $V$  are continuous functions from  $\underline{\Sigma}_V$  to  $\mathbb{C}$ . This will be important in what follows [49].

The spectral presheaf plays a fundamental role in our research programme as applied to quantum theory. For example, it was shown in the earlier work that the Kochen-Specker theorem [50] is equivalent to the statement that  $\underline{\Sigma}$  has no global elements. However,  $\underline{\Sigma}$  *does* have sub-objects, and these are central to our scheme:

**Definition 5.5** *A sub-object  $\underline{S}$  of the spectral presheaf  $\underline{\Sigma}$  is a functor  $\underline{S} : \mathcal{V}(\mathcal{H})^{\text{op}} \rightarrow \mathbf{Sets}$  such that*

1.  *$\underline{S}_V$  is a subset of  $\underline{\Sigma}_V$  for all  $V$ .*
2. *If  $V' \subseteq V$ , then  $\underline{S}(i_{V'V}) : \underline{S}_V \rightarrow \underline{S}_{V'}$  is just the restriction  $\lambda \mapsto \lambda|_{V'}$  (i.e., the same as for  $\underline{\Sigma}$ ), applied to the elements  $\lambda \in \underline{S}_V \subseteq \underline{\Sigma}_V$ .*

This definition of a sub-object is standard. However, for our purposes we need something slightly different, namely concept of a ‘clopen’ sub-object. This is defined to be a sub-object  $\underline{S}$  of  $\underline{\Sigma}$  such that, for all  $V$ , the set  $\underline{S}_V$  is a *clopen*<sup>62</sup> subset of the compact, Hausdorff space  $\underline{\Sigma}_V$ . We denote by  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$  the set of all clopen sub-objects of  $\underline{\Sigma}$ . We will show later (in the Appendix) that, like  $\text{Sub}(\underline{\Sigma})$ , the set  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$  is a Heyting algebra. In Section 6.5 we show that there is an object  $P_{\text{cl}}\underline{\Sigma}$  whose global elements are precisely the clopen sub-objects of  $\underline{\Sigma}$ .

This interest in clopen sets is easy to explain. For, according to the Gel'fand spectral theory, a projection operator  $\hat{a} \in \mathcal{P}(V)$  corresponds to a

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<sup>61</sup>If  $\hat{A} \in V$ , the Gel'fand transform,  $\overline{A} : \underline{\Sigma}_V \rightarrow \mathbb{C}$ , of  $\hat{A}$  is defined by  $\overline{A}(\lambda) := \langle \lambda, \hat{A} \rangle$  for all  $\lambda \in \underline{\Sigma}_V$ .

<sup>62</sup>A ‘clopen’ subset of a topological space is one that is both open and closed.

unique clopen subset,  $S_{\hat{\alpha}}$  of the Gel'fand spectrum,  $\underline{\Sigma}_V$ . Furthermore, the Gel'fand transform  $\bar{\alpha} : \underline{\Sigma}_V \rightarrow \mathbb{C}$  of  $\hat{\alpha}$  takes the values  $0, 1$  only, since the spectrum of a projection operator is just  $\{0, 1\}$ .

It follows that  $\bar{\alpha}$  is the characteristic function of the subset,  $S_{\hat{\alpha}}$ , of  $\underline{\Sigma}_V$ , defined by

$$S_{\hat{\alpha}} := \{\lambda \in \underline{\Sigma}_V \mid \langle \lambda, \hat{\alpha} \rangle = 1\}. \quad (5.30)$$

The clopen nature of  $S_{\hat{\alpha}}$  follows from the fact that, by the spectral theory, the function  $\bar{\alpha} : \underline{\Sigma}_V \rightarrow \{0, 1\}$  is continuous.

In fact, there is a lattice isomorphism between the lattice  $\mathcal{P}(V)$  of projectors in  $V$  and the lattice  $\mathcal{CL}(\underline{\Sigma}_V)$  of clopen subsets of  $\underline{\Sigma}_V$ ,<sup>63</sup> given by

$$\hat{\alpha} \mapsto S_{\hat{\alpha}} := \{\lambda \in \underline{\Sigma}_V \mid \langle \lambda, \hat{\alpha} \rangle = 1\}. \quad (5.31)$$

Conversely, given a clopen subset  $S \in \mathcal{CL}(\underline{\Sigma}_V)$ , we get the corresponding projection  $\hat{\alpha}$  as the (inverse Gel'fand transform of the) characteristic function of  $S$ . Hence, each  $S \in \mathcal{CL}(\underline{\Sigma}_V)$  is of the form  $S = S_{\hat{\alpha}}$  for some  $\hat{\alpha} \in \mathcal{P}(V)$ .

Our claim is the following:

**Theorem 5.4** *For each projection operator  $\hat{P} \in \mathcal{P}(\mathcal{H})$ , the collection*

$$\underline{\delta(\hat{P})} := \{S_{\delta(\hat{P})_V} \subseteq \underline{\Sigma}_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\} \quad (5.32)$$

*forms a (clopen) sub-object of the spectral presheaf  $\underline{\Sigma}$ .*

**Proof.** To see this, let  $\lambda \in S_{\delta(\hat{P})_V}$ . Then if  $V'$  is some abelian sub-algebra of  $V$ , we have  $\delta(\hat{P})_{V'} = \bigwedge \{\hat{\alpha} \in \mathcal{P}(V') \mid \hat{\alpha} \succeq \delta(\hat{P})_V\} \succeq \delta(\hat{P})_V$ . Now let  $\hat{\alpha} := \delta(\hat{P})_{V'} - \delta(\hat{P})_V$ . Then  $\langle \lambda, \delta(\hat{P})_{V'} \rangle = \langle \lambda, \delta(\hat{P})_V \rangle + \langle \lambda, \hat{\alpha} \rangle = 1$ , since  $\langle \lambda, \delta(\hat{P})_V \rangle = 1$  and  $\langle \lambda, \hat{\alpha} \rangle \in \{0, 1\}$ . This shows that

$$\{\lambda|_{V'} \mid \lambda \in S_{\delta(\hat{P})_V}\} \subseteq S_{\delta(\hat{P})_{V'}}. \quad (5.33)$$

However, the left hand side of (5.33) is the subset  $\underline{Q}(i_{V'V})(S_{\delta(\hat{P})_V}) \subseteq \underline{\Sigma}_{V'}$  of the outer-presheaf restriction of elements in  $S_{\delta(\hat{P})_V}$  to  $\underline{\Sigma}_{V'}$ , and the restricted elements all lie in  $S_{\delta(\hat{P})_{V'}}$ . It follows that the collection of sets

$$\underline{\delta(\hat{P})} := \{S_{\delta(\hat{P})_V} \subseteq \underline{\Sigma}_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\} \quad (5.34)$$

<sup>63</sup>The lattice structure on  $\mathcal{CL}(\underline{\Sigma}_V)$  is defined as follows: if  $(U_i)_{i \in I}$  is an arbitrary family of clopen subsets of  $\underline{\Sigma}_V$ , then the *closure*  $\overline{\bigcup_{i \in I} U_i}$  is the maximum. The closure is necessary since the union of infinitely many closed sets need not be closed. The *interior*  $\text{int} \bigcap_{i \in I} U_i$  is the minimum of the family. One must take the interior since  $\bigcap_{i \in I} U_i$  is closed, but not necessarily open.

forms a (clopen) sub-object of the spectral presheaf  $\underline{\Sigma}$ . ■

By these means we have constructed a mapping

$$\begin{aligned} \delta : \mathcal{P}(\mathcal{H}) &\longrightarrow \text{Sub}_{\text{cl}}(\underline{\Sigma}) \\ \hat{P} &\mapsto \underline{\delta(\hat{P})} := \{S_{\delta(\hat{P})_V} \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\} \end{aligned} \quad (5.35)$$

which sends projection operators on  $\mathcal{H}$  to clopen sub-objects of  $\underline{\Sigma}$ . As a matter of notation, we will denote the clopen subset  $S_{\delta(\hat{P})_V} \subseteq \underline{\Sigma}_V$  as  $\underline{\delta(\hat{P})}_V$ . The notation  $\delta(\hat{P})_V$  refers to the element (i.e., projection operator) of  $\underline{Q}_V$  defined earlier.

### 5.3.2 The Definition of Daseinisation

As usual, the projection  $\hat{P}$  is regarded as representing a proposition about the quantum system. Thus  $\delta$  maps propositions about a quantum system to (clopen) sub-objects of the spectral presheaf. This is strikingly analogous to the situation in classical physics, in which propositions are represented by subsets of the classical state space.

**Definition 5.6** *The map  $\delta$  in (5.35) is a fundamental part of our constructions. We call it the daseinisation of  $\hat{P}$ . We shall use the same word to refer to the operation in (5.9) that relates to the outer presheaf.*

The expression ‘daseinisation’ comes from the German word *Dasein*, which plays a central role in Heidegger’s existential philosophy. Dasein translates to ‘existence’ or, in the very literal sense often stressed by Heidegger, to being-there-in-the-world<sup>64</sup>. Thus daseinisation ‘brings-a-quantum-property-into-existence’<sup>65</sup> by hurling it into the collection of all possible classical snapshots of the world provided by the category of contexts.

We will summarise here some useful properties of daseinisation.

1. The null projection  $\hat{0}$  is mapped to the empty sub-object of  $\underline{\Sigma}$ :

$$\underline{\delta(\hat{0})} = \{\emptyset_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\} \quad (5.36)$$

<sup>64</sup>The hyphens are *very* important.

<sup>65</sup>The hyphens are *very* important.



2. The identity projection  $\hat{1}$  is mapped to the unit sub-object of  $\underline{\Sigma}$ :

$$\underline{\delta(\hat{1})} = \{\underline{\Sigma}_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\} = \underline{\Sigma} \quad (5.37)$$

3. Since the daseinisation map  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \Gamma \underline{Q}$  is injective (see Section 5.2.2), and the mapping  $\Gamma \underline{Q} \rightarrow \Gamma(P_{\text{cl}} \underline{\Sigma})$  is injective (because there is a monic arrow  $\underline{Q} \rightarrow P_{\text{cl}} \underline{\Sigma}$  in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ ; see Section 6.5.2), it follows that the daseinisation map  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \Gamma(P_{\text{cl}} \underline{\Sigma}) \simeq \text{Sub}_{\text{cl}}(\underline{\Sigma})$  is also injective. Thus no information about the projector  $\hat{P}$  is lost when it is daseinised to become  $\underline{\delta(\hat{P})}$ .

## 5.4 The Heyting Algebra Structure on $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ .

The reason for daseinising projections is that the set,  $\text{Sub}(\underline{\Sigma})$ , of sub-objects of the spectral presheaf forms a *Heyting algebra*. Thus the idea is to find a map  $\pi_{\text{qt}} : \mathcal{P}\mathcal{L}(S)_0 \rightarrow \text{Sub}(\underline{\Sigma})$  and then extend it to all of  $\mathcal{P}\mathcal{L}(S)$  using the simple recursion ideas discussed in Section 3.2.2.

In our case, the act of daseinisation gives a map from the projection operators to the clopen sub-objects of  $\text{Sub}(\underline{\Sigma})$ , and therefore a map  $\pi_{\text{qt}} : \mathcal{P}\mathcal{L}(S)_0 \rightarrow \text{Sub}_{\text{cl}}(\underline{\Sigma})$  can be defined by

$$\pi_{\text{qt}}(A \varepsilon \Delta) := \underline{\delta(\hat{E}[A \in \Delta])} \quad (5.38)$$

However, to extend this definition to  $\mathcal{P}\mathcal{L}(S)$ , it is necessary to show that the set of clopen sub-objects,  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ , is a Heyting algebra. This is not completely obvious from the definition alone. However, it is true, and the proof is given in Theorem 16.1 in the Appendix.

In conclusion: daseinisation can be used to give a representation/model of the language  $\mathcal{P}\mathcal{L}(S)$  in the Heyting algebra  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ .<sup>66</sup>

## 5.5 Daseinisation and the Operations of Quantum Logic.

It is interesting to ask to what extent the map  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \text{Sub}_{\text{cl}}(\underline{\Sigma})$  respects the lattice structure on  $\mathcal{P}(\mathcal{H})$ . Of course, we know that it cannot be *completely* preserved since the quantum logic  $\mathcal{P}(\mathcal{H})$  is non-distributive, whereas  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$  is a Heyting algebra, and hence distributive.

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<sup>66</sup>Since the clopen subobjects of  $\underline{\Sigma}$  correspond bijectively to the global sections of the outer presheaf  $\underline{Q}$ , it is clear that  $\Gamma \underline{Q}$  too is a Heyting algebra.

We saw in Section 5.2.3 that, for the mapping  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \Gamma Q$ , we have

$$\delta(\hat{P} \vee \hat{Q})_V = \delta(\hat{P})_V \vee \delta(\hat{Q})_V, \quad (5.39)$$

$$\delta(\hat{P} \wedge \hat{Q})_V \preceq \delta(\hat{P})_V \wedge \delta(\hat{Q})_V \quad (5.40)$$

for all contexts  $V$  in  $\text{Ob}(\mathcal{V}(\mathcal{H}))$ .

The clopen subset of  $\underline{\Sigma}_V$  that corresponds to  $\delta(\hat{P})_V \vee \delta(\hat{Q})_V$  is  $S_{\delta(\hat{P})_V} \cup S_{\delta(\hat{Q})_V}$ . This implies that the daseinisation map  $\delta : \mathcal{P}(\mathcal{H}) \rightarrow \text{Sub}_{\text{cl}}(\underline{\Sigma})$  is a morphism of  $\vee$ -semi-lattices.

On the other hand,  $\delta(\hat{P})_V \wedge \delta(\hat{Q})_V$  corresponds to the subset  $S_{\delta(\hat{P})_V} \cap S_{\delta(\hat{Q})_V}$  of  $\underline{\Sigma}_V$ . Therefore, since  $S_{\delta(\hat{P} \wedge \hat{Q})_V} \subseteq S_{\delta(\hat{P})_V} \cap S_{\delta(\hat{Q})_V}$ , daseinisation is *not* a morphism of  $\wedge$ -semi-lattices. In summary, for all projectors  $\hat{P}, \hat{Q}$  we have

$$\underline{\delta(\hat{P} \vee \hat{Q})} = \underline{\delta(\hat{P})} \vee \underline{\delta(\hat{Q})} \quad (5.41)$$

$$\underline{\delta(\hat{P} \wedge \hat{Q})} \preceq \underline{\delta(\hat{P})} \wedge \underline{\delta(\hat{Q})} \quad (5.42)$$

where the logical connectives on the left hand side lie in the quantum logic  $\mathcal{P}(\mathcal{H})$ , and those on the right hand side lie in the Heyting algebra  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ , as do the symbols ‘=’ and ‘ $\preceq$ ’.

As remarked above, it is not surprising that (5.42) is not an equality. Indeed, the quantum logic  $\mathcal{P}(\mathcal{H})$  is non-distributive, whereas the Heyting algebra  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$  is distributive, and so it would be impossible for both (5.41) and (5.42) to be equalities. The inequality in (5.42) is the price that must be paid for liberating the projection operators from the shackles of quantum logic and transporting them to the existential world of Heyting algebras.

### 5.5.1 The Status of the Possible Axiom ‘ $A \varepsilon \Delta_1 \wedge A \varepsilon \Delta_2 \Leftrightarrow A \varepsilon \Delta_1 \cap \Delta_2$ ’

We have the representation in (5.38),  $\pi_{\text{qt}}(A \varepsilon \Delta) := \underline{\delta(\hat{E}[A \in \Delta])}$ , of the primitive propositions  $A \varepsilon \Delta$ , and, as explained in Section 3.2.2, this can be extended to compound sentences by making the obvious definitions:

$$(a) \quad \pi_{\text{qt}}(\alpha \vee \beta) := \pi_{\text{qt}}(\alpha) \vee \pi_{\text{qt}}(\beta) \quad (5.43)$$

$$(b) \quad \pi_{\text{qt}}(\alpha \wedge \beta) := \pi_{\text{qt}}(\alpha) \wedge \pi_{\text{qt}}(\beta) \quad (5.44)$$

$$(c) \quad \pi_{\text{qt}}(\neg \alpha) := \neg \pi_{\text{qt}}(\alpha) \quad (5.45)$$

$$(d) \quad \pi_{\text{qt}}(\alpha \Rightarrow \beta) := \pi_{\text{qt}}(\alpha) \Rightarrow \pi_{\text{qt}}(\beta) \quad (5.46)$$

As a result, we necessarily get a representation of the full language  $\mathcal{PL}(S)$

in the Heyting algebra  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ . However, we then find that:

$$\pi_{\text{qt}}(A \varepsilon \Delta_1 \wedge A \varepsilon \Delta_2) := \pi_{\text{qt}}(A \varepsilon \Delta_1) \wedge \pi_{\text{qt}}(A \varepsilon \Delta_2) \quad (5.47)$$

$$= \underline{\delta(\hat{E}[A \in \Delta_1])} \wedge \underline{\delta(\hat{E}[A \in \Delta_2])} \quad (5.48)$$

$$\preceq \underline{\delta(\hat{E}[A \in \Delta_1] \wedge \hat{E}[A \in \Delta_2])} \quad (5.49)$$

$$= \underline{\delta(\hat{E}[A \in \Delta_1 \cap \Delta_2])} \quad (5.50)$$

$$= \pi_{\text{qt}}(A \varepsilon \Delta_1 \cap \Delta_2) \quad (5.51)$$

where, (5.49) comes from (5.42), and in (5.50) we have used the property of spectral projectors that  $\hat{E}[A \in \Delta_1] \wedge \hat{E}[A \in \Delta_2] = \hat{E}[A \in \Delta_1 \cap \Delta_2]$ . Thus, although by definition,  $\pi_{\text{qt}}(A \varepsilon \Delta_1 \wedge A \varepsilon \Delta_2) = \pi_{\text{qt}}(A \varepsilon \Delta_1) \wedge \pi_{\text{qt}}(A \varepsilon \Delta_2)$ , we only have the inequality

$$\pi_{\text{qt}}(A \varepsilon \Delta_1 \cap \Delta_2) \preceq \pi_{\text{qt}}(A \varepsilon \Delta_1 \wedge A \varepsilon \Delta_2) \quad (5.52)$$

On the other hand, the same line of argument shows that

$$\pi_{\text{qt}}(A \varepsilon \Delta_1 \vee A \varepsilon \Delta_2) = \pi_{\text{qt}}(A \varepsilon \Delta_1 \cup \Delta_2) \quad (5.53)$$

Thus it would be consistent to add the axiom

$$A \varepsilon \Delta_1 \vee A \varepsilon \Delta_2 \Leftrightarrow A \varepsilon \Delta_1 \cup \Delta_2 \quad (5.54)$$

to the language  $\mathcal{P}\mathcal{L}(S)$ , but not

$$A \varepsilon \Delta_1 \wedge A \varepsilon \Delta_2 \Leftrightarrow A \varepsilon \Delta_1 \cap \Delta_2 \quad (5.55)$$

Of, course, both axioms are consistent with the representation of  $\mathcal{P}\mathcal{L}(S)$  in classical physics.

It should be emphasised that there is nothing wrong with this result: indeed, as stated above, it is the necessary price to be paid for forcing a non-distributive algebra to have a ‘representation’ in a Heyting algebra.

### 5.5.2 Inner Daseinisation and $\delta(\neg\hat{P})$ .

In the same spirit, one might ask about “ $\neg(A \varepsilon \Delta)$ ”. By definition, as in (3.8), we have  $\pi_{\text{qt}}(\neg(A \varepsilon \Delta)) := \neg\pi_{\text{qt}}(A \varepsilon \Delta) = \underline{\delta(\hat{E}[A \in \Delta])}$ . However, the question then is how, if at all, this is related to  $\underline{\delta(\hat{E}[A \in \mathbb{R}/\Delta])} = \underline{\delta(\neg\hat{E}[A \in \Delta])}$ , bearing in mind the axiom

$$\neg(A \varepsilon \Delta) \Leftrightarrow A \varepsilon \mathbb{R} \setminus \Delta \quad (5.56)$$

that can be added to the classical representation of  $\mathcal{P}\mathcal{L}(S)$ . Thus something needs to be said about  $\delta(-\hat{P})$ , where  $\neg\hat{P} = \hat{1} - \hat{P}$  is the negation operation in the quantum logic  $\mathcal{P}(\mathcal{H})$ .

To proceed further, we need to introduce another operation:

**Definition 5.7** *The inner daseinisation,  $\delta^i(\hat{P})$ , of  $\hat{P}$  is defined for each context  $V$  as*

$$\delta^i(\hat{P})_V := \bigvee \{ \hat{\beta} \in \mathcal{P}(V) \mid \hat{\beta} \preceq \hat{P} \}. \quad (5.57)$$

This should be contrasted with the definition of outer daseinisation in (5.9).

Thus  $\delta^i(\hat{P})_V$  is the best approximation that can be made to  $\hat{P}$  by taking the ‘largest’ projector in  $V$  that implies  $\hat{P}$ .

As with the other daseinisation construction, this operation was first introduced by de Groote in [32] where he called it the *core* of the projection operator  $\hat{P}$ . We prefer to use the phrase ‘inner daseinisation’, and then to refer to (5.9) as the ‘outer daseinisation’ operation on  $\hat{P}$ . The existing notation  $\delta(\hat{P})_V$  will be replaced with  $\delta^o(\hat{P})_V$  if there is any danger of confusing the two daseinisation operations.

With the aid of inner daseinisation, a new presheaf,  $\underline{I}$ , can be constructed as an exact analogue of the outer presheaf,  $\underline{Q}$ , defined in Section 5.2.1. Specifically:

**Definition 5.8** *The inner presheaf  $\underline{I}$  is defined over the category  $\mathcal{V}(\mathcal{H})$  as follows:*

- (i) *On objects  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ : We have  $\underline{I}_V := \mathcal{P}(V)$*
- (ii) *On morphisms  $i_{V'V} : V' \subseteq V$ : The mapping  $\underline{I}(i_{V'V}) : \underline{I}_V \rightarrow \underline{I}_{V'}$  is given by  $\underline{I}(i_{V'V})(\hat{\alpha}) := \delta^i(\hat{\alpha})_{V'}$  for all  $\hat{\alpha} \in \mathcal{P}(V)$ .*

It is easy to see that the collection  $\{\delta^i(\hat{P})_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\}$  of projection operators given by (5.57) is a global element of  $\underline{I}$ .

It is also straightforward to show that

$$\underline{Q}(i_{V'V})(-\hat{\alpha}) = \neg \underline{I}(i_{V'V})(\hat{\alpha}) \quad (5.58)$$

for all projectors  $\hat{\alpha}$  in  $V$ , and for all  $V' \subseteq V$ . It follows from (5.58) that

$$\delta^o(\neg\hat{P})_V = \hat{1} - \delta^i(\hat{P})_V \quad (5.59)$$

for all projectors  $\hat{P}$  and all contexts  $V$ .

It is clear from (5.58) that the negation operation on projectors defines a map  $\neg : \Gamma \underline{Q} \rightarrow \Gamma \underline{I}$ ,  $\gamma \mapsto \neg \gamma$ ; i.e., for all contexts  $V$ , we map  $\gamma(V) \mapsto \neg \gamma(V) := \hat{1} - \gamma(V)$ . Actually, one can go further than this and show that the presheaves  $\underline{Q}$  and  $\underline{I}$  are isomorphic in the category  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . This means that, in principle, we can always work with one presheaf only. However, for reasons of symmetry it is sometime useful to invoke both presheaves.

As with outer daseinisation, inner daseinisation can also be used to define a mapping from projection operators to sub-objects of the spectral presheaf. Specifically, if  $\hat{P}$  is a projection, for each  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  define

$$T_{\delta^i(\hat{P})_V} := \{\lambda \in \underline{\Sigma}_V \mid \langle \lambda, \delta^i(\hat{P})_V \rangle = 0\}. \quad (5.60)$$

It is easy to see that these subsets form a clopen subobject,  $\underline{\delta^i(\hat{P})}$ , of  $\underline{\Sigma}$ . It follows from (5.59) that  $T_{\delta^i(\hat{P})_V} = S_{\delta^o(\neg \hat{P})_V}$ .

### 5.5.3 Using Boolean Algebras as the Base Category

As we have mentioned several times already, the collection,  $\mathcal{V}(\mathcal{H})$ , of all commutative von Neumann sub-algebras of  $B(\mathcal{H})$  is not the only possible choice for the base category over which to construct presheaves. In fact, if we are only interested in the propositional language  $\mathcal{P}\mathcal{L}(S)$ , a somewhat simpler choice is the collection,  $\mathcal{B}l(\mathcal{H})$  of all Boolean sub-algebras of the non-distributive lattice,  $\mathcal{P}(\mathcal{H})$ , of projection operators on  $\mathcal{H}$ . More abstractly, for any non-distributive lattice  $\mathfrak{B}$ , one could use the category of Boolean sub-algebras of  $\mathfrak{B}$ . This possibility was raised in the original paper [44] but has not been used much thereafter. However, it does have some interesting features.

The analogue of the (von Neumann algebra) spectral presheaf,  $\underline{\Sigma}$ , is the so-called *dual* presheaf,  $\underline{D}$ :

**Definition 5.9** *The dual presheaf on  $\mathcal{B}l(\mathcal{H})$  is the contravariant functor  $\underline{D} : \mathcal{B}l(\mathcal{H}) \rightarrow \mathbf{Sets}$  defined as follows:*

1. *On objects in  $\mathcal{B}l(\mathcal{H})$ :  $\underline{D}(B)$  is the dual of  $B$ ; i.e., the set  $\text{Hom}(B, \{0, 1\})$  of all homomorphisms from the Boolean algebra  $B$  to the Boolean algebra  $\{0, 1\}$ .*
2. *On morphisms in  $\mathcal{B}l(\mathcal{H})$ : If  $i_{B_2 B_1} : B_2 \subseteq B_1$  then  $\underline{D}(i_{B_2 B_1}) : \underline{D}(B_1) \rightarrow \underline{D}(B_2)$  is defined by  $\underline{D}(i_{B_2 B_1})(\chi) := \chi|_{B_2}$ , where  $\chi|_{B_2}$  denotes the restriction of  $\chi \in \underline{D}(B_1)$  to the sub-algebra  $B_2 \subseteq B_1$ .*

A global element of the functor  $\underline{D} : \mathcal{Bl}(\mathcal{H})^{\text{op}} \rightarrow \mathbf{Set}$  is then a function  $\gamma$  that associates to each  $B \in \text{Ob}(\mathcal{Bl}(\mathcal{H}))$  an element  $\gamma_B$  of the dual of  $B$  such that if  $i_{B_2B_1} : B_2 \rightarrow B_1$  then  $\gamma_{B_1}|_{B_2} = \gamma_{B_2}$ ; thus, for all  $\hat{\alpha} \in B_2$ ,

$$\gamma_{B_2}(\hat{\alpha}) = \gamma_{B_1}(i_{B_2B_1}(\hat{\alpha})). \quad (5.61)$$

Since each projection operator,  $\hat{\alpha}$  belongs to at least one Boolean algebra (for example, the algebra  $\{\hat{0}, \hat{1}, \hat{\alpha}, \neg\hat{\alpha}\}$ ) it follows that a global element of the presheaf  $\underline{D}$  associates to each projection operator  $\hat{\alpha}$  a number  $V(\hat{\alpha})$  which is either 0 or 1, and is such that, if  $\hat{\alpha} \wedge \hat{\beta} = \hat{0}$ , then  $V(\hat{\alpha} \vee \hat{\beta}) = V(\hat{\alpha}) + V(\hat{\beta})$ . These types of valuation are often used in the proofs of the Kochen-Specker theorem that focus on the construction of specific counter-examples. In fact, it is easy to see the following:

The Kochen-Specker theorem is equivalent to the statement that, if  $\dim \mathcal{H} > 2$ , the dual presheaf  $\underline{D} : \mathcal{Bl}(\mathcal{H})^{\text{op}} \rightarrow \mathbf{Sets}$  has no global elements.

It is easy to apply the concept of ‘daseinisation’ to the topos  $\mathbf{Sets}^{\mathcal{Bl}(\mathcal{H})^{\text{op}}}$ . In the case of von Neumann algebras, the outer daseinisation of a projection operator  $\hat{P}$  was defined as (see (5.9))

$$\delta(\hat{P})_V := \bigwedge \{ \hat{\alpha} \in \mathcal{P}(V) \mid \hat{\alpha} \succeq \hat{P} \} \quad (5.62)$$

where  $\mathcal{P}(V)$  denotes the collection of all projection operators in the commutative von Neumann algebra  $V$ . In this form,  $\delta(\hat{P})$  appears as a global element of the outer presheaf  $\underline{Q}$ .

When using the base category,  $\mathcal{Bl}(\mathcal{H})$ , of Boolean sub-algebras of  $\mathcal{P}(\mathcal{H})$ , we define

$$\delta(\hat{P})_B := \bigwedge \{ \hat{\alpha} \in B \mid \hat{\alpha} \succeq \hat{P} \} \quad (5.63)$$

for each Boolean sub-algebra  $B$  of projection operators on  $\mathcal{H}$ . Clearly, the (outer) daseinisation,  $\delta(\hat{P})$ , is now a global element of the obvious  $\mathcal{Bl}(\mathcal{H})$ -analogue of the outer presheaf  $\underline{Q}$ . There are parallel remarks for the inner daseinisation and inner presheaf. The existence of these daseinisation operations means that the propositional language  $\mathcal{PL}(S)$  can be represented in the topos  $\mathbf{Sets}^{\mathcal{Bl}(\mathcal{H})^{\text{op}}}$  in a way that is closely analogous to that used above for the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

Note that (i) each Boolean algebra of projection operators  $B$  generates a commutative von Neumann algebra,  $B''$ , (the double commutant); and, conversely, (ii) to each von Neumann algebra  $V$  there is associated the Boolean

algebra  $\mathcal{P}(V)$  of the projection operators in  $V$ . This implies that the operation

$$\phi : \mathcal{Bl}(\mathcal{H}) \rightarrow \mathcal{V}(\mathcal{H}) \quad (5.64)$$

$$B \mapsto B'' \quad (5.65)$$

defines a full and faithful functor between the categories  $\mathcal{Bl}(\mathcal{H})$  and  $\mathcal{V}(\mathcal{H})$ . This functor can be used to pull-back the spectral presheaf,  $\underline{\Sigma}$ , in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  to the object  $\phi^*\underline{\Sigma} := \underline{\Sigma} \circ \phi$  in  $\mathbf{Sets}^{\mathcal{Bl}(\mathcal{H})^{\text{op}}}$ . This pull-back is closely related to the dual presheaf  $\underline{D}$ .

## 5.6 The Special Nature of Daseinised Projections

### 5.6.1 Daseinised Projections as Optimal Sub-Objects

We have shown how daseinisation leads to an interpretation/model of the language  $\mathcal{PL}(S)$  in the Heyting algebra  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ . In particular, any primitive proposition “ $A \varepsilon \Delta$ ” is represented by the clopen sub-object  $\underline{\delta}(\hat{E}[A \in \Delta])$ .

We have seen that, in general, the ‘and’,  $\underline{\delta}(\hat{P}) \wedge \underline{\delta}(\hat{Q})$ , of the daseinisation of two projection operators  $\hat{P}$  and  $\hat{Q}$ , is not itself of the form  $\underline{\delta}(\hat{R})$  for any projector  $\hat{R}$ . The same applies to the negation  $\neg \underline{\delta}(\hat{P})$ .

This raises the question of whether the sub-objects of  $\underline{\Sigma}$  that *are* of the form  $\underline{\delta}(\hat{P})$  can be characterised in a simple way. Rather interestingly, the answer is ‘yes’, as we will now see.

Let  $V', V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  be such that  $V' \subseteq V$ . As would be expected, there is a close connection between the restriction  $\underline{Q}(i_{V'V}) : \underline{Q}_V \rightarrow \underline{Q}_{V'}$ ,  $\underline{\delta}(\hat{P})_V \mapsto \underline{\delta}(\hat{P})_{V'}$ , of the outer presheaf, and the restriction  $\underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V \rightarrow \underline{\Sigma}_{V'}$ ,  $\lambda \mapsto \lambda|_{V'}$ , of the spectral presheaf. Indeed, if  $\hat{P} \in \mathcal{P}(\mathcal{H})$  is a projection operator, and  $S_{\delta(\hat{P})_V} \subseteq \underline{\Sigma}_V$  is defined as in (5.30), we have the following result:

$$S_{\underline{Q}(i_{V'V})(\delta(\hat{P})_V)} = \underline{\Sigma}(i_{V'V})(S_{\delta(\hat{P})_V}). \quad (5.66)$$

The proof is given in Theorem 16.2 in the Appendix

This result shows that the sub-objects  $\underline{\delta}(\hat{P}) = \{S_{\delta(\hat{P})_V} \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\}$  of  $\underline{\Sigma}$  are of a very special kind. Namely, they are such that the restrictions

$$\underline{\Sigma}(i_{V'V}) : S_{\delta(\hat{P})_V} \rightarrow S_{\delta(\hat{P})_{V'}} \quad (5.67)$$

are *surjective* mapping of sets.

For an arbitrary sub-object  $\underline{K}$  of  $\underline{\Sigma}$ , this will not be the case and  $\underline{\Sigma}(i_{V'V})$  only maps  $\underline{K}_V$  into  $\underline{K}_{V'}$ . Indeed, this is essentially the *definition* of a sub-object of a presheaf. Thus we see that the daseinised projections  $\underline{\delta}(\hat{P}) = \{S_{\delta(\hat{P})_V} \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\}$  are optimal in the following sense. As we go ‘down the line’ to smaller and smaller sub-algebras of a context  $V$ —for example, from  $V$  to  $V' \subseteq V$ , then to  $V'' \subseteq V'$  etc.—then the subsets  $S_{\delta(\hat{P})_{V'}}, S_{\delta(\hat{P})_{V''}}, \dots$  are as small as they can be; i.e.,  $S_{\delta(\hat{P})_{V'}}$  is the *smallest* subset of  $\underline{\Sigma}_{V'}$  such that  $\underline{\Sigma}(i_{V'V})(S_{\delta(\hat{P})_V}) \subseteq S_{\delta(\hat{P})_{V'}}$ , likewise  $S_{\delta(\hat{P})_{V''}}$  is the smallest subset of  $\underline{\Sigma}_{V''}$  such that  $\underline{\Sigma}(i_{V''V'})(S_{\delta(\hat{P})_{V'}}) \subseteq S_{\delta(\hat{P})_{V''}}$ , and so on.

It is also clear from this result that there are lots of sub-objects of  $\underline{\Sigma}$  that are *not* of the form  $\underline{\delta}(\hat{P})$  for any projector  $\hat{P} \in \mathcal{P}(\mathcal{H})$ .

These more general sub-objects of  $\underline{\Sigma}$  show up explicitly in the representation of the more sophisticated language  $\mathcal{L}(S)$ . This will be discussed thoroughly in Section 8 when we analyse the representation,  $\phi$ , of the language  $\mathcal{L}(S)$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . This involves constructing the quantity-value object  $\mathcal{R}_\phi$  (to be denoted  $\underline{\mathcal{R}}$ ), and then finding the representation of a function symbol  $A : \Sigma \rightarrow \mathcal{R}$  in  $\mathcal{L}(S)$ , in the form of a specific arrow  $\check{A} : \underline{\Sigma} \rightarrow \underline{\mathcal{R}}$  in the topos. The generic sub-objects of  $\underline{\Sigma}$  are then of the form  $\check{A}^{-1}(\underline{\Xi})$  for sub-objects  $\underline{\Xi}$  of  $\underline{\mathcal{R}}$ . This is an illuminating way of studying the sub-objects of  $\underline{\Sigma}$  that do not come from the propositional language  $\mathcal{P}\mathcal{L}(S)$ .

## 6 Truth Values in Topos Physics

### 6.1 The Mathematical Proposition “ $x \in K$ ”

So far we have concentrated on finding a Heyting-algebra representation of the propositions in quantum theory, but of course there is more to physics than that. We also want to know if/when a certain proposition is *true*: a question which, in physical theories, is normally answered by specifying a (*micro*) *state* of the system, or something that can play an analogous role.

In classical physics, the situation is straightforward (see Section 3.2.5). There, a proposition “ $A \varepsilon \Delta$ ” is represented by the subset  $\pi_{\text{cl}}(A \varepsilon \Delta) := \check{A}^{-1}(\Delta) \subseteq \mathcal{S}$  of the state space  $\mathcal{S}$ ; and then, the proposition is true in a state  $s$  if and only if  $s \in \check{A}^{-1}(\Delta)$ ; i.e., if and only if the (micro-) state  $s$  belongs to the subset,  $\pi_{\text{cl}}(A \varepsilon \Delta)$ , of  $\mathcal{S}$  that represents the proposition.

Thus, each state  $s$  assigns to any primitive proposition “ $A \varepsilon \Delta$ ”, a truth value,  $\nu(A \varepsilon \Delta; s)$ , which lies in the set {false, true} (which we identify with



$\{0, 1\}$ ) and is defined as

$$\nu(A \varepsilon \Delta; s) := \begin{cases} 1 & \text{if } s \in \pi_{\text{cl}}(A \varepsilon \Delta) := \check{A}^{-1}(\Delta); \\ 0 & \text{otherwise.} \end{cases} \quad (6.1)$$

However, the situation in quantum theory is very different. There, the spectral presheaf  $\underline{\Sigma}$ —which is the analogue of the classical state space  $\mathcal{S}$ —has no global elements at all. Our expectation is that this will be true in any topos-based theory that goes ‘beyond quantum theory’: i.e.,  $\Gamma \Sigma_\phi$  is empty; or, if  $\Sigma_\phi$  does have global elements, there are not enough of them to determine  $\Sigma_\phi$  as an object in the topos. In this circumstance, a new concept is required to replace the familiar idea of a ‘state of the system’. As we shall see, this involves the concept of a ‘truth object’, or ‘pseudo-state’.

In physics, the propositions of interest are of the form “ $A \varepsilon \Delta$ ”, which refers to the value of a *physical* quantity. However, in constructing a theory of physics, such physical propositions must first be translated into *mathematical* propositions. The concept of ‘truth’ is then studied in the context of the latter.

Let us start with set-theory based mathematics, where the most basic proposition is of the form “ $x \in K$ ”, where  $K$  is a subset of a set  $X$ , and  $x$  is an element of  $X$ . Then the truth value, denoted  $\nu(x \in K)$ , of the proposition “ $x \in K$ ” is

$$\nu(x \in K) = \begin{cases} 1 & \text{if } x \text{ belongs to } K; \\ 0 & \text{otherwise.} \end{cases} \quad (6.2)$$

Thus the proposition “ $x \in K$ ” is true if, and only if,  $x$  belongs to  $K$ . In other words,  $x \mapsto \nu(x \in K)$  is the characteristic function of the subset  $K$  of  $X$ ; cf. (17.1) in the Appendix.

This remark is the foundation of the assignment of truth values in classical physics. Specifically, if the state is  $s \in \mathcal{S}$ , the truth value,  $\nu(A \varepsilon \Delta; s)$ , of the *physical* proposition “ $A \varepsilon \Delta$ ” is defined to be the truth value of the *mathematical* proposition “ $\check{A}(s) \in \Delta$ ”; or, equivalently, of the mathematical proposition “ $s \in \check{A}^{-1}(\Delta)$ ”. Thus, using (6.2), we get, for all  $s \in \mathcal{S}$ ,

$$\nu(A \varepsilon \Delta; s) := \begin{cases} 1 & \text{if } s \text{ belongs to } \check{A}^{-1}(\Delta); \\ 0 & \text{otherwise.} \end{cases} \quad (6.3)$$

which reproduces (6.1).

We now consider the analogue of the above in a general topos  $\tau$ . Let  $X$  be an object in  $\tau$ , and let  $K$  be a sub-object of  $X$ . Then  $K$  is determined

by a characteristic arrow  $\chi_K : X \rightarrow \Omega_\tau$ , where  $\Omega_\tau$  is the sub-object classifier; equivalently, we have an arrow  $\ulcorner K \urcorner : 1_\tau \rightarrow PX$ .

Now suppose that  $x : 1_\tau \rightarrow X$  is a global element of  $X$ ; i.e.,  $x \in \Gamma X := \text{Hom}_\tau(1_\tau, X)$ . Then the truth value of the mathematical proposition “ $x \in K$ ” is defined to be

$$\nu(x \in K) := \chi_K \circ x \quad (6.4)$$

where  $\chi_K \circ x : 1_\tau \rightarrow \Omega_\tau$ . Thus  $\nu(x \in K)$  is an element of  $\Gamma\Omega_\tau$ ; i.e., it is a global element of the sub-object classifier  $\Omega_\tau$ .

The connection with the result (6.2) (in the topos **Sets**) can be seen by noting that, in (6.2), the characteristic function of the subset  $K \subseteq X$  is the function  $\chi_K : X \rightarrow \{0, 1\}$  such that  $\chi_K(x) = 1$  if  $x \in K$ , and  $\chi_K(x) = 0$  otherwise. It follows that (6.2) can be rewritten as

$$\nu(x \in K) = \chi_K(x) \quad (6.5)$$

$$= \chi_K \circ x \quad (6.6)$$

where in (6.6),  $x$  denotes the function  $x : \{*\} \rightarrow X$  that is defined by  $x(*) := x$ . The link with (6.4) is clear when one remembers that, in the topos **Sets**, the terminal object,  $1_{\mathbf{Sets}}$ , is just the singleton set  $\{*\}$ .

In quantum theory, the topos is  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , and so the objects are all presheaves. In particular, at each stage  $V$ , the sub-object classifier  $\underline{\Omega} := \Omega_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}$  is the set of sieves on  $V$ . In this case, if  $\underline{K}$  is a sub-object of  $\underline{X}$ , and  $x \in \Gamma\underline{X}$ , the explicit form for (6.6) is the sieve

$$\nu(x \in \underline{K})_V := \{V' \subseteq V \mid x_{V'} \in \underline{K}_{V'}\} \quad (6.7)$$

at each stage  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . In other words, at each stage/context  $V$ , the truth value of the mathematical proposition “ $x \in \underline{K}$ ” is defined to be all those stages  $V' \subseteq V$  ‘down the line’ such that the ‘component’,  $x_{V'}$  of  $x$  at that stage is an element of the component,  $\underline{K}_{V'} \subseteq \underline{X}_{V'}$ , of  $\underline{K}$  at that stage.

The definitions (6.4) and (6.7) play a central role in constructing truth values in our quantum topos scheme. However, as  $\underline{\Sigma}$  has no global elements, these truth values cannot be derived from some expression  $\nu(s \in \underline{K})$  with  $s : 1_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}} \rightarrow \underline{\Sigma}$ . Therefore, we must proceed differently, as will become clear by the end of the following Section.

However, before we do so, let us make one final remark concerning (6.2). Namely, in normal set theory the proposition “ $x \in K$ ” is true if, and only if,

$$\{x\} \subseteq K \quad (6.8)$$

i.e., if and only if the set  $\{x\}$  is a subset of  $K$ . The transition from the proposition “ $x \in K$ ” to the proposition “ $\{x\} \subseteq K$ ” is seemingly trivial, but in a topos other than sets it takes on a new significance. In particular, as we shall see shortly, although the spectral presheaf,  $\underline{\Sigma}$ , has no global elements, it does have certain ‘minimal’ sub-objects that are as ‘close’ as one can get to a global element, and then the topos analogue of (6.8) is very important.

## 6.2 Truth Objects

### 6.2.1 Linguistic Aspects of Truth Objects.

To understand how ‘truth values’ of physical propositions arise we return again to our earlier discussion of local languages. In this Section we will employ the local language  $\mathcal{L}(S)$  rather than the propositional language,  $\mathcal{P}\mathcal{L}(S)$ , that was used earlier in this article.

Thus, let  $\mathcal{L}(S)$  be the local language for a system  $S$ . This is a typed language whose minimal set of ground-type symbols is  $\Sigma$  and  $\mathcal{R}$ . In addition, there is a non-empty set,  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ , of function symbols  $A : \Sigma \rightarrow \mathcal{R}$  that correspond to the physical quantities of  $S$ .

Now consider a representation,  $\phi$ , of  $\mathcal{L}(S)$  in a topos  $\tau_\phi$ . As discussed earlier, the propositional aspects of the language  $\mathcal{L}(S)$  are captured in the term ‘ $A(\tilde{s}) \in \tilde{\Delta}$ ’ of type  $\Omega$ , where  $\tilde{s}$  and  $\tilde{\Delta}$  are variables of type  $\Sigma$  and  $P\mathcal{R}$  respectively [21]. In a topos representation,  $\phi$ , the representation,  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi$ , of the term ‘ $A(\tilde{s}) \in \tilde{\Delta}$ ’ is given by the chain of arrows<sup>67</sup>[8] (cf (4.3))

$$\Sigma_\phi \times P\mathcal{R}_\phi \xrightarrow{A_\phi \times \text{id}} \mathcal{R}_\phi \times P\mathcal{R}_\phi \xrightarrow{e_{\mathcal{R}_\phi}} \Omega_{\tau_\phi} \quad (6.9)$$

in the topos  $\tau_\phi$ . Then, if  $\ulcorner \Xi \urcorner : 1_{\tau_\phi} \rightarrow P\mathcal{R}_\phi$  is the name of a sub-object,  $\Xi$ , of the quantity-value object  $\mathcal{R}_\phi$ , we get the chain

$$\Sigma_\phi \simeq \Sigma_\phi \times 1_{\tau_\phi} \xrightarrow{\text{id} \times \ulcorner \Xi \urcorner} \Sigma_\phi \times P\mathcal{R}_\phi \xrightarrow{A_\phi \times \text{id}} \mathcal{R}_\phi \times P\mathcal{R}_\phi \xrightarrow{e_{\mathcal{R}_\phi}} \Omega_{\tau_\phi}. \quad (6.10)$$

which is the characteristic arrow of the sub-object of  $\Sigma_\phi$  that represents the physical proposition “ $A \varepsilon \Xi$ ”.

Equivalently, we can use the term,  $\{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\}$ , which has a free variable  $\tilde{\Delta}$  of type  $P\mathcal{R}$  and is of type  $P\Sigma$ . This term is represented by the arrow  $\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_\phi : P\mathcal{R}_\phi \rightarrow P\Sigma_\phi$ , which is the power transpose of  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi$  (cf (4.4)):

$$\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_\phi = \ulcorner \llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi \urcorner \quad (6.11)$$

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<sup>67</sup>In (6.9),  $e_{\mathcal{R}_\phi} : \mathcal{R}_\phi \times P\mathcal{R}_\phi \rightarrow \Omega_{\tau_\phi}$  is the evaluation arrow associated with the power object  $P\mathcal{R}_\phi$ .

The proposition “ $A \varepsilon \Xi$ ” is then represented by the arrow  $\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_{\phi} \circ \lceil \Xi \rceil : 1_{\tau_{\phi}} \rightarrow P\Sigma_{\phi}$ ; this is the name of the sub-object of  $\Sigma_{\phi}$  that represents “ $A \varepsilon \Xi$ ”.

We note an important difference from the analogous situation for the language  $\mathcal{P}\mathcal{L}(S)$ . In propositions of the type “ $A \varepsilon \Delta$ ”, the symbol ‘ $\Delta$ ’ is a specific subset of  $\mathbb{R}$  and is hence *external* to the language. In particular, it is *independent* of the representation of  $\mathcal{P}\mathcal{L}(S)$ . However, in the case of  $\mathcal{L}(S)$ , the variable  $\tilde{\Delta}$  is *internal* to the language, and the quantity  $\Xi$  in the proposition “ $A \varepsilon \Xi$ ” is a sub-object of  $\mathcal{R}_{\phi}$  in a *specific* topos representation,  $\phi$ , of  $\mathcal{L}(S)$ .

So, this is how physical propositions are represented mathematically. But how are truth values to be assigned to these propositions? In the topos  $\tau_{\phi}$  a truth value is an element of the Heyting algebra  $\Gamma\Omega_{\tau_{\phi}}$ . Thus the challenge is to assign a global element of  $\Omega_{\tau_{\phi}}$  to each proposition associated with the representation of the term  $\{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\}$  of type  $P\Sigma$ ; (or, equivalently, the representation of the term ‘ $A(\tilde{s}) \in \tilde{\Delta}$ ’).

Let us first pose this question at a linguistic level. In a representation  $\phi$ , an element of  $\Gamma\Omega_{\tau_{\phi}}$  is associated with a representation of a term of type  $\Omega$  with no free variables. Hence the question can be rephrased as asking how a term,  $t$ , in  $\mathcal{L}(S)$  of type  $P\Sigma$  can be ‘converted’ into a term of type  $\Omega$ ? At this stage, we are happy to have free variables, in which case the desired term will be represented by an arrow in  $\tau_{\phi}$  whose co-domain is  $\Omega_{\tau_{\phi}}$ , but whose domain is other than  $1_{\tau_{\phi}}$ . This would be an intermediate stage to obtaining a global element of  $\Omega_{\tau_{\phi}}$ .

In the context of the language  $\mathcal{L}(S)$  there are three obvious ways of ‘converting’ the term  $t$  of type  $P\Sigma$  to a term of type  $\Omega$ :

1. Choose a term,  $s$ , of type  $\Sigma$ ; then the term ‘ $s \in t$ ’ is of type  $\Omega$ . We will call this the ‘micro-state’ option.
2. Choose a term,  $\mathbb{T}$ , of type  $PP\Sigma$ ; then the term ‘ $t \in \mathbb{T}$ ’ is of type  $\Omega$ . We shall refer to this as the ‘truth-object’ option.
3. Choose a term,  $\mathfrak{w}$ , of type  $P\Sigma$ ; then the term ‘ $\mathfrak{w} \subseteq t$ ’ is of type  $\Omega$ .<sup>68</sup> For reasons that will become clear later we shall refer to this as the ‘pseudo-state’ option.

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<sup>68</sup>In general, if  $t$  and  $s$  are set-like terms (i.e., terms of power type,  $PX$ , say), then ‘ $t \subseteq s$ ’ is defined as the term ‘ $\forall \tilde{x} \in t(\tilde{x} \in s)$ ’; here,  $\tilde{x}$  is a variable of type  $X$ .

### 6.2.2 The Micro-State Option

In regard to the first option, the simplest example of a term of type  $\Sigma$  is a variable  $\tilde{s}_1$  of type  $\Sigma$ . Then, the term ‘ $\tilde{s}_1 \in \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\}$ ’ is of type  $\Omega$  with the free variables  $\tilde{s}_1$  and  $\tilde{\Delta}$  of type  $\Sigma$  and  $P\mathcal{R}$  respectively. However, the axiom of comprehension in  $\mathcal{L}(S)$  says that

$$\tilde{s}_1 \in \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \Leftrightarrow A(\tilde{s}_1) \in \tilde{\Delta} \quad (6.12)$$

and so we are back with the term ‘ $A(\tilde{s}) \in \tilde{\Delta}$ ’, which is of type  $\Omega$  and with the free variable  $\tilde{s}$  of type  $\Sigma$ .

As stated above, the  $\phi$ -representation,  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi$ , of ‘ $A(\tilde{s}) \in \tilde{\Delta}$ ’ is the chain of arrows in (6.9). Now, suppose the representation,  $\phi$ , is such that there exist global elements,  $s : 1_{\tau_\phi} \rightarrow \Sigma_\phi$ , of  $\Sigma_\phi$ . Then each such element can be regarded as a ‘(micro)-state’ of the system in that topos representation. Furthermore, let  $\ulcorner \Xi \urcorner : 1_{\tau_\phi} \rightarrow P\mathcal{R}_\phi$  be the name of a sub-object,  $\Xi$ , of the quantity-value object  $\mathcal{R}_\phi$ . Then, by the basic property of the product  $\Sigma_\phi \times P\mathcal{R}_\phi$ , there is an arrow  $\langle s, \ulcorner \Xi \urcorner \rangle : 1_{\tau_\phi} \rightarrow \Sigma_\phi \times P\mathcal{R}_\phi$ . This can be combined with the arrow  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi : \Sigma_\phi \times P\mathcal{R}_\phi \rightarrow \Omega_{\tau_\phi}$  to give the arrow

$$\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi \circ \langle s, \ulcorner \Xi \urcorner \rangle : 1_{\tau_\phi} \longrightarrow \Omega_{\tau_\phi} \quad (6.13)$$

This is the desired global element of  $\Omega_{\tau_\phi}$ .

In other words, when the ‘state of the system’ is  $s \in \Gamma\Sigma_\phi$ , the ‘truth value’ of the proposition “ $A \varepsilon \Xi$ ” is the global element of  $\Omega_{\tau_\phi}$  given by the arrow  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\phi \circ \langle s, \ulcorner \Xi \urcorner \rangle : 1_{\tau_\phi} \rightarrow \Omega_{\tau_\phi}$ .

This is the procedure that is adopted in classical physics when a truth value is assigned to propositions by specifying a micro-state,  $s \in \Sigma_\sigma$ , where  $\Sigma_\sigma$  is the classical state space in the representation  $\sigma$  of  $\mathcal{L}(S)$ . Specifically, for all  $s \in \Sigma_\sigma$ , the truth value of the proposition “ $A \varepsilon \Delta$ ” as given by (6.13) is (c.f. (6.1))

$$\nu(A \varepsilon \Delta; s) = \llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\sigma(s, \Delta) = \begin{cases} 1 & \text{if } A_\sigma(s) \in \Delta; \\ 0 & \text{otherwise.} \end{cases} \quad (6.14)$$

where  $\llbracket A(\tilde{s}) \in \tilde{\Delta} \rrbracket_\sigma : \Sigma_\sigma \times P\mathbb{R} \rightarrow \Omega_{\tau_\sigma} \simeq \{0, 1\}$ . Thus we recover the earlier result (6.3).

### 6.2.3 The Truth Object Option.

By hindsight, we know that the option to use global elements of  $\Sigma_\phi$  is not available in the quantum case. For there the state object,  $\underline{\Sigma}$ , is the spectral

presheaf, and this has no global elements by virtue of the Kochen-Specker theorem. The absence of global elements of the state object  $\Sigma_\phi$  could well be true in many other topos models of physics (particularly those that go ‘beyond quantum theory’), and therefore an alternative general strategy is needed to that employing micro-states  $\lceil s \rceil : 1_{\tau_\phi} \rightarrow \Sigma_\phi$ .

This takes us to the second possibility: namely, to introduce a term,  $\mathbb{T}$ , of type  $PP\Sigma$ , and then work with the term ‘ $\{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \in \mathbb{T}$ ’, which is of type  $\Omega$ , and has whatever free variables are contained in  $\mathbb{T}$ , plus the variable  $\tilde{\Delta}$  of type  $PR$ .

The simplest choice is to let the term of type  $PP\Sigma$  be a variable,  $\tilde{\mathbb{T}}$ , of type  $PP\Sigma$ , in which case the term ‘ $\{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \in \tilde{\mathbb{T}}$ ’ has variables  $\tilde{\Delta}$  and  $\tilde{\mathbb{T}}$  of type  $PR$  and  $PP\Sigma$  respectively. Therefore, in a topos representation it is represented by an arrow  $\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \in \tilde{\mathbb{T}} \rrbracket_\phi : PR_\phi \times P(P\Sigma_\phi) \rightarrow \Omega_{\tau_\phi}$ . In detail (see [8]) we have that

$$\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \in \tilde{\mathbb{T}} \rrbracket_\phi = e_{P\Sigma_\phi} \circ \llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_\phi \times \llbracket \tilde{\mathbb{T}} \rrbracket_\phi \quad (6.15)$$

where  $e_{P\Sigma_\phi} : P\Sigma_\phi \times P(P\Sigma_\phi) \rightarrow \Omega_{\tau_\phi}$  is the usual evaluation arrow. In using this expression we need the  $\phi$ -representatives:

$$\llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \rrbracket_\phi : PR_\phi \rightarrow P\Sigma_\phi \quad (6.16)$$

$$\llbracket \tilde{\mathbb{T}} \rrbracket_\phi : P(P\Sigma_\phi) \xrightarrow{\text{id}} P(P\Sigma_\phi) \quad (6.17)$$

Finally, let  $\langle \lceil \Xi \rceil, \lceil \mathbb{T} \rceil \rangle$  be a pair of global elements in  $PR_\phi$  and  $P(P\Sigma_\phi)$  respectively, so that  $\lceil \Xi \rceil : 1_{\tau_\phi} \rightarrow PR_\phi$  and  $\lceil \mathbb{T} \rceil : 1_{\tau_\phi} \rightarrow P(P\Sigma_\phi)$ . Thus,  $\lceil \mathbb{T} \rceil$  is the name of a ‘truth object’,  $\mathbb{T}$ , in  $\tau_\phi$ . Then, for the physical proposition “ $A \varepsilon \Xi$ ”, we have the truth value

$$\nu(A \varepsilon \Xi; \mathbb{T}) = \llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \in \tilde{\mathbb{T}} \rrbracket_\phi \circ \langle \lceil \Xi \rceil, \lceil \mathbb{T} \rceil \rangle : 1_{\tau_\phi} \rightarrow \Omega_{\tau_\phi} \quad (6.18)$$

where  $\langle \lceil \Xi \rceil, \lceil \mathbb{T} \rceil \rangle : 1_{\tau_\phi} \rightarrow PR_\phi \times P(P\Sigma_\phi)$ .

**A small generalisation:** Slightly more generally, if  $\tilde{J}$  and  $\tilde{\mathbb{T}}$  are variables of type  $P\Sigma$  and  $P(P\Sigma)$  respectively, the term of interest is ‘ $\tilde{J} \in \tilde{\mathbb{T}}$ ’. In the representation,  $\phi$ , of  $\mathcal{L}(S)$ , this term maps to an arrow  $\llbracket \tilde{J} \in \tilde{\mathbb{T}} \rrbracket_\phi : P\Sigma_\phi \times P(P\Sigma_\phi) \rightarrow \Omega_{\tau_\phi}$ . Here,  $\llbracket \tilde{J} \in \tilde{\mathbb{T}} \rrbracket_\phi = e_{P\Sigma_\phi} \circ \llbracket \tilde{J} \rrbracket_\phi \times \llbracket \tilde{\mathbb{T}} \rrbracket_\phi$  where  $\llbracket \tilde{J} \rrbracket_\phi : P\Sigma_\phi \xrightarrow{\text{id}} P\Sigma_\phi$  and  $\llbracket \tilde{\mathbb{T}} \rrbracket_\phi : P(P\Sigma_\phi) \xrightarrow{\text{id}} P(P\Sigma_\phi)$ . Let  $\lceil J \rceil, \lceil \mathbb{T} \rceil$  be global elements of  $P\Sigma_\phi$  and  $P(P\Sigma_\phi)$  respectively, so that  $\lceil J \rceil : 1_{\tau_\phi} \rightarrow P\Sigma_\phi$  and  $\lceil \mathbb{T} \rceil : 1_{\tau_\phi} \rightarrow P(P\Sigma_\phi)$ . Then the truth of the (mathematical) proposition

“ $J \in \mathbb{T}$ ” is

$$\begin{aligned} \nu(J \in \mathbb{T}) &= \llbracket \tilde{J} \in \tilde{\mathbb{T}} \rrbracket_\phi \circ \langle \ulcorner J \urcorner, \ulcorner \mathbb{T} \urcorner \rangle \\ &= e_{P\Sigma_\phi} \circ \langle \ulcorner J \urcorner, \ulcorner \mathbb{T} \urcorner \rangle : 1_{\tau_\phi} \rightarrow \Omega_{\tau_\phi} \end{aligned} \quad (6.19)$$

#### 6.2.4 The Example of Classical Physics.

If classical physics is studied this way, the general formalism simplifies, and the term ‘ $\{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \in \tilde{\mathbb{T}}$ ’ is represented by the function  $\nu(A \varepsilon \Delta; \mathbb{T}) := \llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \in \tilde{\mathbb{T}} \rrbracket_\sigma : P\mathbb{R} \times P(P\Sigma_\sigma) \rightarrow \Omega_{\mathbf{Sets}} \simeq \{0, 1\}$  defined by

$$\begin{aligned} \nu(A \varepsilon \Delta; \mathbb{T}) := \llbracket \{\tilde{s} \mid A(\tilde{s}) \in \tilde{\Delta}\} \in \tilde{\mathbb{T}} \rrbracket_\sigma(\Delta, \mathbb{T}) &= \begin{cases} 1 & \text{if } \{s \in \Sigma_\sigma \mid A_\sigma(s) \in \Delta\} \in \mathbb{T}; \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } A_\sigma^{-1}(\Delta) \in \mathbb{T}; \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (6.20)$$

for all  $\mathbb{T} \in P(P\Sigma_\sigma)$ . We can clearly see the sense in which the truth object  $\mathbb{T}$  is playing the role of a state. Note that the result (6.20) of classical physics is a special case of (6.18).

To recover the usual truth values given in (6.14), an appropriate truth object,  $\mathbb{T}^s$ , must be associated with each micro-state  $s \in \Sigma_\sigma$ . The correct choice is

$$\mathbb{T}^s := \{J \subseteq \Sigma_\sigma \mid s \in J\} \quad (6.21)$$

for each  $s \in \Sigma_\sigma$ . It is clear that  $s \in A_\sigma^{-1}(\Delta)$  (or, equivalently,  $A_\sigma(s) \in \Delta$ ) if, and only if,  $A_\sigma^{-1}(\Delta) \in \mathbb{T}^s$ . Hence (6.20) can be rewritten as

$$\nu(A \varepsilon \Delta; \mathbb{T}^s) = \begin{cases} 1 & \text{if } s \in A_\sigma^{-1}(\Delta); \\ 0 & \text{otherwise.} \end{cases} \quad (6.22)$$

which reproduces (6.14) once  $\nu(A \varepsilon \Delta; s)$  is identified with  $\nu(A \varepsilon \Delta; \mathbb{T}^s)$ .

## 6.3 Truth Objects in Quantum Theory

### 6.3.1 Preliminary Remarks

We can now start to discuss the application of these ideas to quantum theory. In order to use (6.18) (or (6.19)) we need to construct concrete truth objects,  $\underline{\mathbb{T}}$ , in the topos  $\tau_\phi := \mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{op}}$ . Thus the presheaf  $\underline{\mathbb{T}}$  is a sub-object of  $P\underline{\Sigma}$ ; equivalently,  $\ulcorner \underline{\mathbb{T}} \urcorner : 1_{\tau_\phi} \rightarrow P(P\underline{\Sigma})$ .

However, we have to keep in mind the need to restrict to *clopen* sub-objects of  $\underline{\Sigma}$ . In particular, we must show that there is a well-defined presheaf  $P_{\text{cl}}\underline{\Sigma}$  such that

$$\text{Sub}_{\text{cl}}(\underline{\Sigma}) \simeq \Gamma(P_{\text{cl}}\underline{\Sigma}) \quad (6.23)$$

We will prove this in Section 6.5. Given (6.23) and  $\underline{J} \in \text{Sub}_{\text{cl}}(\underline{\Sigma})$ , it is then clear that a truth object,  $\underline{\mathbb{T}}$ , actually has to be a sub-object of  $P_{\text{cl}}\underline{\Sigma}$  in order that the valuation  $\nu(\underline{J} \in \underline{\mathbb{T}})$  in (6.19) is meaningful.

This truth value,  $\nu(\underline{J} \in \underline{\mathbb{T}})$ , is a global element of  $\underline{\Omega}$ , and in the topos of presheaves,  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , we have (see (6.7))

$$\nu(\underline{J} \in \underline{\mathbb{T}})_V := \{V' \subseteq V \mid \underline{J}_{V'} \in \underline{\mathbb{T}}_{V'}\} \quad (6.24)$$

for each context  $V$ .

There are various examples of the presheaf  $\underline{J}$  that are of interest to us. In particular, let  $\underline{J} = \underline{\delta(\hat{P})}$  for some projector  $\hat{P}$ . Then, using the propositional language  $\mathcal{P}\mathcal{L}(S)$  introduced earlier, the ‘truth’ of the proposition represented by  $\hat{P}$  (for example, “ $A \varepsilon \Delta$ ”) is

$$\nu(\underline{\delta(\hat{P})} \in \underline{\mathbb{T}})_V = \{V' \subseteq V \mid \underline{\delta(\hat{P})}_{V'} \in \underline{\mathbb{T}}_{V'}\} \quad (6.25)$$

for all stages  $V$ .

When using the local language  $\mathcal{L}(S)$ , an important class of examples of the sub-object  $\underline{J}$  of  $\underline{\Sigma}$  are of the form  $A_{\phi}^{-1}(\underline{\Xi})$ , for some sub-object  $\underline{\Xi}$  of  $\underline{\mathcal{R}}$ . This will yield the truth value,  $\nu(A \varepsilon \underline{\Xi}; \underline{\mathbb{T}})$ , in (6.18). However, to discuss this further requires the representation of function symbols  $A : \Sigma \rightarrow \mathcal{R}$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , and this is deferred until Section 7.

### 6.3.2 The Truth Objects $\underline{\mathbb{T}}^{|\psi\rangle}$ .

The definition of truth objects in quantum theory was studied in the original papers [44, 45, 35, 13]. It was shown there that to each quantum state  $|\psi\rangle \in \mathcal{H}$ , there corresponds a truth object,  $\underline{\mathbb{T}}^{|\psi\rangle}$ , which was defined as the following sub-object of the outer presheaf,  $\underline{Q}$ :

$$\begin{aligned} \underline{\mathbb{T}}_V^{|\psi\rangle} &:= \{\hat{\alpha} \in \underline{Q}_V \mid \text{Prob}(\hat{\alpha}; |\psi\rangle) = 1\} \\ &= \{\hat{\alpha} \in \underline{Q}_V \mid \langle \psi | \hat{\alpha} | \psi \rangle = 1\} \end{aligned} \quad (6.26)$$

for all stages  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . Here,  $\text{Prob}(\hat{\alpha}; |\psi\rangle)$  is the usual expression for the probability that the proposition represented by the projector  $\hat{\alpha}$  is true, given that the quantum state is the (normalised) vector  $|\psi\rangle$ .



It is easy to see that (6.26) defines a genuine sub-object  $\underline{\mathbb{T}}^{|\psi\rangle} = \{\underline{\mathbb{T}}_V^{|\psi\rangle} \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\}$  of  $\underline{Q}$ . Indeed, if  $\hat{\beta} \succeq \hat{\alpha}$ , then  $\langle \psi | \hat{\beta} | \psi \rangle \geq \langle \psi | \hat{\alpha} | \psi \rangle$ , and therefore, if  $V' \subseteq V$  and  $\hat{\alpha} \in \underline{Q}_V$ , then  $\langle \psi | \underline{Q}(i_{V'V})(\hat{\alpha}) | \psi \rangle \geq \langle \psi | \hat{\alpha} | \psi \rangle$ . In particular, if  $\langle \psi | \hat{\alpha} | \psi \rangle = 1$  then  $\langle \psi | \underline{Q}(i_{V'V})(\hat{\alpha}) | \psi \rangle = 1$ .

The next step is to define the presheaf  $P_{\text{cl}}\underline{\Sigma}$ , and show that there is a monic arrow  $\underline{Q} \rightarrow P_{\text{cl}}\underline{\Sigma}$ , so that  $\underline{Q}$  is a sub-object of  $P_{\text{cl}}\underline{\Sigma}$ . Then, since  $\underline{\mathbb{T}}^{|\psi\rangle}$  is a sub-object of  $\underline{Q}$ , and  $\underline{Q}$  is a sub-object of  $P_{\text{cl}}\underline{\Sigma}$ , it follows that  $\underline{\mathbb{T}}^{|\psi\rangle}$  is a sub-object of  $P_{\text{cl}}\underline{\Sigma}$ , as required. The discussion of the construction of  $P_{\text{cl}}\underline{\Sigma}$  is deferred to Section 6.5 so as not to break the flow of the presentation.

With this definition of  $\underline{\mathbb{T}}^{|\psi\rangle}$ , the truth value, (6.25), for the propositional language  $\mathcal{P}\mathcal{L}(S)$  becomes

$$\nu(\underline{\delta(\hat{P})} \in \underline{\mathbb{T}}^{|\psi\rangle})_V = \{V' \subseteq V \mid \langle \psi | \delta(\hat{P})_{V'} | \psi \rangle = 1\} \quad (6.27)$$

It is easy to see that the definition of a truth object in (6.26) can be extended to a mixed state with a density-matrix operator  $\hat{\rho}$ : simply replace the definition in (6.26) with

$$\begin{aligned} \underline{\mathbb{T}}_V^{\hat{\rho}} &:= \{\hat{\alpha} \in \underline{Q}_V \mid \text{Prob}(\hat{\alpha}; \rho) = 1\} \\ &= \{\hat{\alpha} \in \underline{Q}_V \mid \text{tr}(\hat{\rho}\hat{\alpha}) = 1\} \end{aligned} \quad (6.28)$$

However there is an important difference between the truth object associated with a vector state,  $|\psi\rangle$ , and the one associated with a density matrix,  $\rho$ . In the vector case, it is easy to see that the mapping  $|\psi\rangle \rightarrow \underline{\mathbb{T}}^{|\psi\rangle}$  is one-to-one (up to a phase factor on  $|\psi\rangle$ ) so that, in principle, the state  $|\psi\rangle$  can be *recovered* from  $\underline{\mathbb{T}}^{|\psi\rangle}$  (up to a phase-factor). On the other hand, there are simple counterexamples which show that, in general, the density matrix,  $\rho$  *cannot* be recovered from  $\underline{\mathbb{T}}^{\hat{\rho}}$ .

In a sense, this should not surprise us. The analogue of a density matrix in classical physics is a probability measure  $\mu$  defined on the classical state space  $\mathcal{S}$ . Individual microstates  $s \in \mathcal{S}$  are in one-to-one correspondence with probability measures of the form  $\mu_s$  defined by  $\mu_s(J) = 1$  if  $s \in J$ ,  $\mu_s(J) = 0$  if  $s \notin J$ .

However, one of the main claims of our programme is that any theory can be made to ‘look like’ classical physics in the appropriate topos. This suggests that, in the topos version of quantum theory, a density matrix should be represented by some sort of measure on the state object  $\underline{\Sigma}$  in the topos  $\tau_\phi$ ; and this should relate in some way to an ‘integral’ of ‘vector truth objects’. The recent work by Heunen and Spitters provides the mathematical basis for such a construction [38]. We shall return to some of their ideas later.

## 6.4 The Pseudo-state Option

### 6.4.1 Some Background Remarks

We turn now to the third way mentioned above whereby a term,  $t$ , of type  $P\Sigma$  in  $\mathcal{L}(S)$  can be ‘converted’ to a term of type  $\Omega$ . Namely, choose a term,  $\mathfrak{w}$ , of type  $P\Sigma$  and then use ‘ $\mathfrak{w} \subseteq t$ ’. As we shall see, this idea is easy to implement in the case of quantum theory and leads to an alternative way of thinking about truth objects.

Let us start by considering once more the case of classical physics. There, for each microstate  $s$  in the symplectic state manifold  $\Sigma_\sigma$ , there is an associated truth object,  $\mathbb{T}^s$ , defined by  $\mathbb{T}^s := \{J \subseteq \Sigma_\sigma \mid s \in J\}$ , as in (6.21). It is clear that the state  $s$  can be uniquely recovered from the collection of sets  $\mathbb{T}^s$  as

$$s = \bigcap \{J \subseteq \Sigma_\sigma \mid s \in J\} \quad (6.29)$$

Note that (6.29) implies that  $\mathbb{T}^s$  is an *ultrafilter* of subsets of  $\Sigma_\sigma$ <sup>69</sup>. As we shall shortly see, there is an intriguing analogue of this property for the quantum truth objects.

The analogue of (6.29) in the case of quantum theory is rather interesting. Now, of course, there are no microstates, but we do have the truth objects defined in (6.26), one for each vector state  $|\psi\rangle \in \mathcal{H}$ . To proceed further we note that  $\langle\psi|\hat{\alpha}|\psi\rangle = 1$  if and only if  $|\psi\rangle\langle\psi| \preceq \hat{\alpha}$ . Thus  $\mathbb{T}^{|\psi\rangle}$  can be rewritten as

$$\mathbb{T}_V^{|\psi\rangle} := \{\hat{\alpha} \in \underline{Q}_V \mid |\psi\rangle\langle\psi| \preceq \hat{\alpha}\} \quad (6.30)$$

for each stage  $V$ . Note that, as defined in (6.30),  $\mathbb{T}^{|\psi\rangle}$  is a sub-object of  $\underline{Q}$ ; i.e., it is defined in terms of projection operators. However, as will be shown in Section 6.5.2, there is a monic arrow  $\underline{Q} \rightarrow P_{\text{cl}}\underline{\Sigma}$ , and by using this arrow,  $\mathbb{T}^{|\psi\rangle}$  can be regarded as a sub-object of  $P_{\text{cl}}\underline{\Sigma}$ ; hence  $\Gamma\mathbb{T}^{|\psi\rangle}$  is a collection of clopen sub-objects of  $\underline{\Sigma}$ . In this form, the definition of  $\mathbb{T}^{|\psi\rangle}$  involves clopen subsets of the spectral sets  $\underline{\Sigma}_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ .

It is clear from (6.30) that, for each  $V$ ,  $\mathbb{T}_V^{|\psi\rangle}$  is a *filter* of projection operators in  $\underline{Q}_V \simeq \mathcal{P}(V)$ ; equivalently, it is a filter of clopen sub-sets of  $\underline{\Sigma}_V$ .

These ordering properties are associated with the following observation.

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<sup>69</sup>Let  $\mathbb{L}$  be a lattice with zero element 0. A subset  $F \subset \mathbb{L}$  is a ‘filter base’ if (i)  $0 \notin F$  and (ii) for all  $a, b \in F$ , there is some  $c \in F$  such that  $c \leq a \wedge b$ . A subset  $D \subset \mathbb{L}$  is called a ‘(proper) dual ideal’ or a ‘filter’ if (i)  $0 \notin D$ , (ii) for all  $a, b \in D$ ,  $a \wedge b \in D$  and (iii)  $a \in D$  and  $b > a$  implies  $b \in D$ . A maximal dual ideal/filter  $F$  in a complemented, *distributive* lattice  $\mathbb{L}$  is called an ‘ultrafilter’. It has the property that for all  $a \in \mathbb{L}$ , either  $a \in F$  or  $a' \in F$ , where  $a'$  is the complement of  $a$ .

If  $|\psi\rangle$  is any vector state, we can collect together all the projection operators that are ‘larger’ or equal to  $|\psi\rangle\langle\psi|$  and define:

$$T^{|\psi\rangle} := \{\hat{\alpha} \in \mathcal{P}(\mathcal{H}) \mid |\psi\rangle\langle\psi| \preceq \hat{\alpha}\} \quad (6.31)$$

It is clear that, for all stages/contexts  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , we have

$$\underline{\mathbb{T}}_V^{|\psi\rangle} = T^{|\psi\rangle} \cap V \quad (6.32)$$

Thus the presheaf  $\underline{\mathbb{T}}^{|\psi\rangle}$  is obtained by ‘localising’  $T^{|\psi\rangle}$  at each context  $V$ .

The significance of this localisation property is that  $T^{|\psi\rangle}$  is a *maximal* (proper) filter in the non-distributive lattice,  $\mathcal{P}(\mathcal{H})$ , of all projection operators on  $\mathcal{H}$ . Such maximal filters in the projection lattices of von Neumann algebras were extensively discussed by de Groote [33] who called them ‘quasi-points’. In particular,  $T^{|\psi\rangle}$  is a, so-called, ‘atomic’ quasi-point in  $\mathcal{P}(\mathcal{H})$ . Every pure state  $|\psi\rangle$  gives rise to an atomic quasi-point,  $T^{|\psi\rangle}$ , and vice versa. We will return to these entities in Section 8.4.

#### 6.4.2 Using Pseudo-States in Lieu of Truth Objects

The equation (6.29) from classical physics suggests that, in the quantum case, we look at the set-valued function on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$  defined by

$$V \mapsto \bigwedge \{\hat{\alpha} \in \underline{\mathbb{T}}_V^{|\psi\rangle}\} = \bigwedge \{\hat{\alpha} \in \underline{Q}_V \mid |\psi\rangle\langle\psi| \preceq \hat{\alpha}\} \quad (6.33)$$

where we have used (6.30) as the definition of  $\underline{\mathbb{T}}^{|\psi\rangle}$ . It is easy to check that this is a global element of  $\underline{Q}$ ; in fact, the right hand side of (6.33) is nothing but the outer daseinisation  $\delta(|\psi\rangle\langle\psi|)$  of the projection operator  $|\psi\rangle\langle\psi|$ . Evidently, the quantity

$$\mathfrak{w}^{|\psi\rangle} := \delta(|\psi\rangle\langle\psi|) = V \mapsto \bigwedge \{\hat{\alpha} \in \underline{Q}_V \mid |\psi\rangle\langle\psi| \preceq \hat{\alpha}\} \quad (6.34)$$

is of considerable interest. We shall refer to it as a ‘pseudo-state’ for reasons that appear below.

Note that  $\mathfrak{w}^{|\psi\rangle}$  is defined by (6.34) as an element of  $\Gamma\underline{Q}$ . However, because of the monic  $\underline{Q} \rightarrow P_{\text{cl}}\underline{\Sigma}$  we can also regard  $\mathfrak{w}^{|\psi\rangle}$  as an element of  $\Gamma(P_{\text{cl}}\underline{\Sigma}) \simeq \text{Sub}_{\text{cl}}(\underline{\Sigma})$ . The corresponding (clopen) sub-object of  $\underline{\Sigma}$  will be denoted  $\underline{\mathfrak{w}}^{|\psi\rangle} := \underline{\delta(|\psi\rangle\langle\psi|)}$ .

We know that the map  $|\psi\rangle \mapsto \underline{\mathbb{T}}^{|\psi\rangle}$  is injective. What can be said about the map  $|\psi\rangle \mapsto \underline{\mathfrak{w}}^{|\psi\rangle}$ ? In this context, we note that  $\underline{\mathbb{T}}^{|\psi\rangle}$  is readily recoverable from  $\mathfrak{w}^{|\psi\rangle} \in \Gamma\underline{Q}$  as

$$\underline{\mathbb{T}}_V^{|\psi\rangle} = \{\hat{\alpha} \in \underline{Q}_V \mid \hat{\alpha} \succeq \mathfrak{w}_V^{|\psi\rangle}\} \quad (6.35)$$

for all contexts  $V$ . From these relations it follows that  $|\psi\rangle \mapsto \underline{\mathfrak{w}}^{|\psi\rangle}$  is injective.

Note that, (6.35) essentially follows from the fact that, for each  $V$ , the collection,  $\underline{\mathbb{T}}_V^{|\psi\rangle}$  of projectors in  $\underline{Q}_V$  is an *upper set* (in fact, as remarked earlier, it is a filter). In this respect, the projectors/clopen subsets  $\underline{\mathbb{T}}_V^{|\psi\rangle}$  behave like the filter of *clopen neighbourhoods* of a subset in a topological space. This remark translates globally to the relation of the collection,  $\Gamma\underline{\mathbb{T}}^{|\psi\rangle}$ , of sub-objects of  $\underline{\Sigma}$  to the specific sub-object  $\underline{\mathfrak{w}}^{|\psi\rangle}$ .

It follows that there is a one-to-one correspondence between truth objects,  $\underline{\mathbb{T}}^{|\psi\rangle}$ , and pseudo-states,  $\underline{\mathfrak{w}}^{|\psi\rangle}$ . However, the former is (a representation of) a term of type  $P(P\Sigma)$ , whereas the latter is of type  $P\Sigma$ . So how is this reflected in the assignment of generalised truth values?

Note first that, from the definition of  $\underline{\mathfrak{w}}^{|\psi\rangle}$ , it follows that if  $\hat{\alpha} \in \underline{\mathbb{T}}_V^{|\psi\rangle}$  then  $\hat{\alpha} \succeq \underline{\mathfrak{w}}_V^{|\psi\rangle}$ . On the other hand, from (6.35) we have that if  $\hat{\alpha} \succeq \underline{\mathfrak{w}}_V^{|\psi\rangle}$  then  $\hat{\alpha} \in \underline{\mathbb{T}}_V^{|\psi\rangle}$ . Thus we have the simple, but important, result:

$$\hat{\alpha} \in \underline{\mathbb{T}}_V^{|\psi\rangle} \text{ if, and only if } \hat{\alpha} \succeq \underline{\mathfrak{w}}_V^{|\psi\rangle} \quad (6.36)$$

In particular, for any projector  $\hat{P}$  we have  $\delta(\hat{P})_V \in \underline{\mathbb{T}}_V^{|\psi\rangle}$  if, and only if  $\delta(\hat{P})_V \succeq \underline{\mathfrak{w}}_V^{|\psi\rangle}$ .

In terms of sub-objects of  $\underline{\Sigma}$ , we have  $\delta(\hat{P})_V \succeq \underline{\mathfrak{w}}_V^{|\psi\rangle}$  if and only if  $\underline{\delta(\hat{P})}_V \supseteq \underline{\mathfrak{w}}_V^{|\psi\rangle}$ . Hence, (6.36) can be rewritten as

$$\delta(\hat{P})_V \in \underline{\mathbb{T}}_V^{|\psi\rangle} \text{ if, and only if } \underline{\delta(\hat{P})}_V \supseteq \underline{\mathfrak{w}}_V \quad (6.37)$$

and so (6.25) can be written as

$$\nu(\underline{\delta(\hat{P})} \in \underline{\mathbb{T}})_V = \{V' \subseteq V \mid \underline{\delta(\hat{P})}_{V'} \supseteq \underline{\mathfrak{w}}_{V'}^{|\psi\rangle}\} \quad (6.38)$$

However, the right hand side of (6.38) is just the topos truth value,  $\nu(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})})$ . It follows that

$$\underline{\delta(\hat{P})} \in \underline{\mathbb{T}}^{|\psi\rangle} \text{ is equivalent to } \underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})} \quad (6.39)$$

and hence we can use the generalised truth values  $\nu(\underline{\delta(\hat{P})} \in \underline{\mathbb{T}}^{|\psi\rangle})$  or  $\nu(\underline{\mathfrak{w}}^{|\psi\rangle} \subseteq \underline{\delta(\hat{P})})$  interchangeably.

Thus, if desired, a truth object in quantum theory can be regarded as a sub-object of  $\underline{\Sigma}$ , rather than a sub-object of  $P\underline{\Sigma}$ . In a sense, these sub-objects,  $\underline{\mathfrak{w}}^{|\psi\rangle}$ , of  $\underline{\Sigma}$  are the ‘closest’ we can get to global elements of  $\underline{\Sigma}$ . This

is why we call them ‘pseudo-states’. However, note that a pseudo-state is not a *minimal* element of the Heyting algebra  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$  since these will include stalks that are empty sets, something that is not possible for a pseudo-state.

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### 6.4.3 Linguistic Implications

The result (6.39) is very suggestive for a more general development. In our existing treatment, in the formal language  $\mathcal{L}(S)$  we have concentrated on propositions of the form “ $\tilde{J} \in \tilde{\mathbb{T}}$ ” which, in a representation  $\phi$ , maps to the arrow  $\llbracket \tilde{J} \in \tilde{\mathbb{T}} \rrbracket_{\phi} : P\Sigma_{\phi} \times P(P\Sigma_{\phi}) \rightarrow \Omega_{\tau_{\phi}}$ . Here  $\tilde{J}$  and  $\tilde{\mathbb{T}}$  are variables of type  $P\Sigma$  and  $P(P\Sigma)$  respectively.

What is suggested by the discussion above is that we could equally focus on terms of the form “ $\tilde{\mathfrak{w}} \subseteq \tilde{J}$ ”, where both  $\tilde{\mathfrak{w}}$  and  $\tilde{J}$  are variables of type  $P\Sigma$ .

Note that, in general, the  $\phi$ -representation of such a term is of the form

$$\llbracket \tilde{\mathfrak{w}} \subseteq \tilde{J} \rrbracket_{\phi} : P\Sigma_{\phi} \times P\Sigma_{\phi} \rightarrow \Omega_{\tau_{\phi}} \quad (6.40)$$

where the ‘first slot’ on the right hand side of the pairing in (6.40) is a truth-object (in pseudo-state form), and the second correspond to a proposition represented by a sub-object of  $\Sigma_{\phi}$ .

However, this raises the rather obvious question “What *is* a pseudo-state?”. More precisely, we would like to know a generic set of characteristic properties of those sub-objects of  $\Sigma_{\phi}$  that can be regarded as ‘pseudo-states’. A first step would be to answer this question in the case of quantum theory. In particular, are there any quantum pseudo-states that are *not* of the form  $\underline{\mathfrak{m}}^{|\psi\rangle}$  for some vector  $|\psi\rangle \in \mathcal{H}$ ?

In this context the localisation property expressed by (6.31) is rather suggestive. In the case that  $\mathcal{H}$  has infinite dimension, de Groot has shown that there exist quasi-points in  $\mathcal{P}(\mathcal{H})$  that are not of the form  $T^{|\psi\rangle}$  for some  $|\psi\rangle \in \mathcal{H}$  [33].<sup>71</sup> If  $T$  is any such quasi-point, (6.31) suggests strongly that we

<sup>70</sup>Note that the sub-objects  $\underline{\mathfrak{m}}^{|\psi\rangle}$  do not have any global elements since any such would give a global element of  $\underline{\Sigma}$  and, of course, there are none. Thus if one is seeking examples of presheaves with no global elements, the collection  $\underline{\mathfrak{m}}^{|\psi\rangle}$ ,  $|\psi\rangle \in \mathcal{H}$ , afford many such.

<sup>71</sup>However, he has also shown that, in an appropriate topology, the set of all atomic quasi-points is *dense* in the set of all quasi-points. Of course, none of these intriguing

define an associated presheaf,  $\underline{T}$ , by

$$\underline{T}_V = T \cap V \quad (6.41)$$

for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . This construction seems natural enough from a mathematical perspective, but we are not yet clear of the physical significance of the existence of such ‘quasi truth-objects’. The same applies to the associated ‘quasi pseudo-state’,  $\underline{\mathfrak{m}}^T$ , defined by

$$\underline{\mathfrak{m}}_V^T := \bigwedge \{\hat{\alpha} \in \underline{T}_V\} = \bigwedge \{\hat{\alpha} \in T \cap V\} \quad (6.42)$$

#### 6.4.4 Time-Dependence and the Truth Object.

As emphasised at the end of Section 3.2, the question of time dependence depends on the theory-type being considered. The structure of the language  $\mathcal{L}(S)$  that has been used so far is such that the time variable lies outside the language. In this situation, the time dependence of the system can be implemented in several ways.

For example, we can make the truth object time dependent, giving a family of truth objects,  $t \mapsto \underline{\mathbb{T}}^t$ ,  $t \in \mathbb{R}$ . In the case of classical physics, with the truth objects  $\mathbb{T}^s$ ,  $s \in \Sigma_\sigma$ , the time evolution comes from the time dependence,  $t \mapsto s_t$ , of the microstate in accordance with the classical equations of motion. This gives the family  $t \mapsto \mathbb{T}^{s_t}$  of truth objects.

Something very similar happens in quantum theory, and we acquire a family,  $t \mapsto \underline{\mathbb{T}}^{|\psi\rangle_t}$ , of truth objects, where the states  $|\psi\rangle_t$  satisfy the usual time-dependent Schrödinger equation. Thus both classical and quantum truth objects belong to a ‘Schrödinger picture’ of time evolution. Of course, there is a pseudo-state analogue of this in which we get a one-parameter family,  $t \mapsto \underline{\mathfrak{m}}^{|\psi\rangle_t}$ , of clopen sub-objects of  $\underline{\Sigma}$ .

It is also possible to construct a ‘Heisenberg picture’ where the truth object is constant but the physical quantities and associated propositions are time dependent. We will return to this in Section 10 when we discuss the use of unitary operators.

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structures arise in a finite-dimensional Hilbert space in anything other than a trivial way. So, in that sense, it is unlikely that they will play any fundamental role in explicating the topos representation of quantum theory.

## 6.5 The Presheaf $P_{\text{cl}}(\underline{\Sigma})$ .

### 6.5.1 The Definition of $P_{\text{cl}}(\underline{\Sigma})$ .

We must now show that there really is a presheaf  $P_{\text{cl}}\underline{\Sigma}$ .

The easiest way of defining  $P_{\text{cl}}\underline{\Sigma}$  is to start with the concrete expression for the normal power object  $P\underline{\Sigma}$  [29]. First, if  $\underline{F}$  is any presheaf over  $\mathcal{V}(\mathcal{H})$ , define the *restriction* of  $\underline{F}$  to  $V$  to be the functor  $\underline{F}\downarrow V$  from the category<sup>72</sup>  $\downarrow V$  to **Sets** that assigns to each  $V_1 \subseteq V$ , the set  $\underline{F}_{V_1}$ , and with the obvious induced presheaf maps.

Then, at each stage  $V$ ,  $P\underline{\Sigma}_V$  is the set of natural transformations from  $\underline{\Sigma}\downarrow V$  to  $\underline{\Omega}\downarrow V$ . These are in one-to-one correspondence with families of maps  $\sigma := \{\sigma_{V_1} : \underline{\Sigma}_{V_1} \rightarrow \underline{\Omega}_{V_1} \mid V_1 \subseteq V\}$ , with the following commutative diagram for all  $V_2 \subseteq V_1 \subseteq V$ :<sup>73</sup>

$$\begin{array}{ccc} \underline{\Sigma}_{V_1} & \xrightarrow{\sigma_{V_1}} & \underline{\Omega}_{V_1} \\ \downarrow \underline{\Sigma}(i_{V_2 V_1}) & & \downarrow \underline{\Omega}(i_{V_2 V_1}) \\ \underline{\Sigma}_{V_2} & \xrightarrow{\sigma_{V_2}} & \underline{\Omega}_{V_2} \end{array}$$

The presheaf maps are defined by

$$P\underline{\Sigma}(i_{V_1 V}) : P\underline{\Sigma}_V \rightarrow P\underline{\Sigma}_{V_1} \quad (6.43)$$

$$\sigma \mapsto \{\sigma_{V_2} \mid V_2 \subseteq V_1\} \quad (6.44)$$

and the evaluation arrow  $\text{ev} : P\underline{\Sigma} \times \underline{\Sigma} \rightarrow \underline{\Omega}$ , has the form, at each stage  $V$ :

$$\text{ev}_V : P\underline{\Sigma}_V \times \underline{\Sigma}_V \rightarrow \underline{\Omega}_V \quad (6.45)$$

$$(\sigma, \lambda) \mapsto \sigma_V(\lambda) \quad (6.46)$$

Moreover, in general, given a map  $\chi : \underline{\Sigma}_V \rightarrow \underline{\Omega}_V$ , the subset of  $\underline{\Sigma}_V$  associated with the corresponding sub-object is  $\chi^{-1}(1)$ , where 1 is the unit ('truth') in the Heyting algebra  $\underline{\Omega}_V$ .

This suggests strongly that an object,  $P_{\text{cl}}\underline{\Sigma}$ , in **Sets** <sup>$\mathcal{V}(\mathcal{H})^{\text{op}}$</sup>  can be defined using the same definition of  $P\underline{\Sigma}$  as above, except that the family of maps

<sup>72</sup>The notation  $\downarrow V$  means the partially-ordered set of all sub-algebras  $V' \subseteq V$ .

<sup>73</sup>Note that any sub-object,  $\underline{J}$  of  $\underline{\Sigma}$ , gives rise to such a natural transformation from  $\underline{\Sigma}\downarrow V$  to  $\underline{\Omega}\downarrow V$  for all stages  $V$ . Namely, for all  $V_1 \subseteq V$ ,  $\sigma_{V_1} : \underline{\Sigma}_{V_1} \rightarrow \underline{\Omega}_{V_1}$  is defined to be the characteristic arrow  $\chi_{\underline{J}V_1} : \underline{\Sigma}_{V_1} \rightarrow \underline{\Omega}_{V_1}$  of the sub-object  $\underline{J}$  of  $\underline{\Sigma}$ .

$\sigma := \{\sigma_{V_1} : \underline{\Sigma}_{V_1} \rightarrow \underline{\Omega}_{V_1} \mid V_1 \subseteq V\}$  must be such that, for all  $V_1 \subseteq V$ ,  $\sigma_{V_1}^{-1}(1)$  is a *clopen* subset of the (extremely disconnected) Hausdorff space  $\underline{\Sigma}_{V_1}$ . It is straightforward to check that such a restriction is consistent, and that  $\text{Sub}_{\text{cl}}(\underline{\Sigma}) \simeq \Gamma(P_{\text{cl}}\underline{\Sigma})$  as required.

### 6.5.2 The Monic Arrow From $\underline{Q}$ to $P_{\text{cl}}(\underline{\Sigma})$ .

We define  $\iota : \underline{Q} \times \underline{\Sigma} \rightarrow \underline{\Omega}$ , with the power transpose  $\ulcorner \iota \urcorner : \underline{Q} \rightarrow P_{\text{cl}}\underline{\Sigma}$ , as follows. First recall that in any topos,  $\tau$  there is a bijection  $\text{Hom}_{\tau}(A, C^B) \simeq \text{Hom}_{\tau}(A \times B, C)$ , and hence, in particular, (using  $P\underline{\Sigma} = \underline{\Omega}^{\underline{\Sigma}}$ )

$$\text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{Q}, P\underline{\Sigma}) \simeq \text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{Q} \times \underline{\Sigma}, \underline{\Omega}). \quad (6.47)$$

Now let  $\hat{\alpha} \in \mathcal{P}(V)$ , and let  $S_{\hat{\alpha}} := \{\lambda \in \underline{\Sigma}_V \mid \langle \lambda, \hat{\alpha} \rangle = 1\}$  be the clopen subset of  $\underline{\Sigma}_V$  that corresponds to the projector  $\hat{\alpha}$  via the spectral theorem; see (5.30). Then we define  $\iota : \underline{Q} \times \underline{\Sigma} \rightarrow \underline{\Omega}$  at stage  $V$  by

$$\iota_V(\hat{\alpha}, \lambda) := \{V' \subseteq V \mid \underline{\Sigma}(i_{V'V})(\lambda) \in S_{\underline{Q}(i_{V'V})(\hat{\alpha})}\} \quad (6.48)$$

for all  $(\hat{\alpha}, \lambda) \in \underline{Q}_V \times \underline{\Sigma}_V$ .

On the other hand, the basic result relating coarse-graining to subsets of  $\underline{\Sigma}$  is

$$S_{\underline{Q}(i_{V'V})(\delta(\hat{\alpha})_V)} = \underline{\Sigma}(i_{V'V})(S_{\delta(\hat{\alpha})_V}) \quad (6.49)$$

for all  $V' \subseteq V$  and for all  $\hat{\alpha} \in \underline{Q}_V$ . It follows that

$$\iota_V(\hat{\alpha}, \lambda) := \{V' \subseteq V \mid \underline{\Sigma}(i_{V'V})(\lambda) \in \underline{\Sigma}(i_{V'V})(S_{\hat{\alpha}})\} \quad (6.50)$$

for all  $(\hat{\alpha}, \lambda) \in \underline{Q}_V \times \underline{\Sigma}_V$ . In this form it is clear that  $\iota_V(\hat{\alpha}, \lambda)$  is indeed a *sieve* on  $V$ ; i.e., an element of  $\underline{\Omega}_V$ .

The next step is to show that the collection of maps  $\iota_V : \underline{Q}_V \times \underline{\Sigma}_V \rightarrow \underline{\Omega}_V$  defined in (6.48) constitutes a natural transformation from the object  $\underline{Q} \times \underline{\Sigma}$  to the object  $\underline{\Omega}$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . This involves chasing around a few commutative squares, and we will spare the reader the ordeal. There is some subtlety, since we really want to deal with  $\text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{Q}, P_{\text{cl}}\underline{\Sigma})$ , not  $\text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{Q}, P\underline{\Sigma})$ ; but all works in the end.

To prove that  $\ulcorner \iota \urcorner : \underline{Q} \rightarrow P_{\text{cl}}\underline{\Sigma}$  is monic, it suffices to show that the map  $\ulcorner \iota \urcorner_V : \underline{Q}_V \rightarrow P_{\text{cl}}\underline{\Sigma}_V$  is injective at all stages  $V$ . This is a straightforward exercise and the details will not be given here.

The conclusion of this exercise is that, since  $\ulcorner \iota \urcorner : \underline{Q} \rightarrow P_{\text{cl}}\underline{\Sigma}$  is monic, the truth sub-objects  $\underline{\mathbb{T}}^{|\psi\rangle}$  of  $\underline{Q}$  can also be regarded as sub-objects of  $P_{\text{cl}}\underline{\Sigma}$ , and hence the truth value assignment in (6.25) is well-defined.



Finally then, for any given quantum state  $|\psi\rangle$  the basic proposition “ $A \varepsilon \Delta$ ” can be assigned a generalised truth value  $\nu(A \varepsilon \Delta; |\psi\rangle)$  in  $\Gamma\Omega$ , where  $\tau := \mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  is the topos of presheaves over  $\mathcal{V}(\mathcal{H})$ . This is defined at each stage/context  $V$  as

$$\begin{aligned} \nu(A \varepsilon \Delta; |\psi\rangle)_V &:= \nu(\underline{\delta(\hat{E}[A \in \Delta])} \in \mathbb{T}^{|\psi\rangle})_V \\ &= \{V' \subseteq V \mid \underline{\delta(\hat{E}[A \in \Delta])}_{V'} \in \mathbb{T}_{V'}^{|\psi\rangle}\} \end{aligned} \quad (6.51)$$

## 6.6 Yet Another Perspective on the K-S Theorem

In classical physics, the pseudo-state  $\mathfrak{w}^s \subseteq \mathcal{S}$  associated with the microstate  $s \in \mathcal{S}$  is just  $\mathfrak{w}^s := \{s\}$ . This gives the diagram

$$\begin{array}{ccc} \mathfrak{w}^s & \longrightarrow & \mathcal{S} \\ & & \downarrow \pi \\ \{*\} & \xrightarrow{\ulcorner \mathfrak{w}^s \urcorner} & P\mathcal{S} \end{array} \quad (6.52)$$

where  $\ulcorner \mathfrak{w}^s \urcorner(*) := \{s\}$  and  $\pi$  is the canonical map

$$\begin{aligned} \pi : \mathcal{S} &\longrightarrow P\mathcal{S} \\ s &\mapsto \{s\} \end{aligned} \quad (6.53)$$

The singleton  $\{*\}$  is the terminal object in the category,  $\mathbf{Sets}$ , of sets, and the subset embedding  $\mathfrak{w}^s \rightarrow \mathcal{S}$  in (6.52) is the categorical pull-back by  $\pi$  of the monic  $\ulcorner \mathfrak{w}^s \urcorner : \{*\} \rightarrow P\mathcal{S}$ .

In the quantum case, the analogue of the diagram (6.52) is

$$\begin{array}{ccc} \underline{\mathfrak{w}}^{|\psi\rangle} & \longrightarrow & \underline{\Sigma} \\ & & \downarrow \pi \\ \underline{\mathbb{1}} & \xrightarrow{\ulcorner \underline{\mathfrak{w}}^{|\psi\rangle} \urcorner} & P\underline{\Sigma} \end{array} \quad (6.54)$$

where the arrow  $\pi : \underline{\Sigma} \rightarrow P\underline{\Sigma}$  has yet to be defined. To proceed further, let us first return to the set-theory map

$$\begin{aligned} X &\rightarrow PX \\ x &\mapsto \{x\} \end{aligned} \quad (6.55)$$

where  $X$  is any set.

We can think of (6.55) as the power transpose,  $\ulcorner \beta \urcorner : X \rightarrow PX$ , of the map  $\beta : X \times X \rightarrow \{0, 1\}$  defined by

$$\beta(x, y) := \begin{cases} 1 & \text{if } x = y; \\ 0 & \text{otherwise.} \end{cases} \quad (6.56)$$

In our topos case, the obvious definition for the arrow  $\pi : \underline{\Sigma} \rightarrow P\underline{\Sigma}$  is the power transpose  $\ulcorner \beta \urcorner : \underline{\Sigma} \rightarrow P\underline{\Sigma}$ , of the arrow  $\beta : \underline{\Sigma} \times \underline{\Sigma} \rightarrow \underline{\Omega}$ , defined by

$$\beta_V(\lambda_1, \lambda_2) := \{V' \subseteq V \mid \lambda_1|_{V'} = \lambda_2|_{V'}\} \quad (6.57)$$

for all stages  $V$ . Note that, in linguistic terms, the arrow defined in (6.57) is just the representation in the quantum topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , of the term ' $\tilde{\sigma}_1 = \tilde{\sigma}_2$ ', where  $\tilde{\sigma}_1$  and  $\tilde{\sigma}_2$  are terms of type  $\Sigma$ ; i.e.,  $\llbracket \tilde{\sigma}_1 = \tilde{\sigma}_2 \rrbracket : \underline{\Sigma} \times \underline{\Sigma} \rightarrow \underline{\Omega}$ .

With this definition of  $\pi$ , the diagram in (6.54) becomes meaningful: in particular the monic  $\underline{\mathbf{w}}^{|\psi\rangle} \hookrightarrow \underline{\Sigma}$  is the categorical pull-back by  $\pi$  of the monic  $\ulcorner \underline{\mathbf{w}}^{|\psi\rangle} \urcorner : \underline{\mathbf{1}} \rightarrow P\underline{\Sigma}$ .

There is, however, a significant difference between (6.54) and its classical analogue (6.52). In the latter case, the function  $\ulcorner \mathbf{w}^{s\uparrow} \urcorner : \{*\} \rightarrow P\mathcal{S}$  can be 'lifted' to a function  $\ulcorner \mathbf{w}^{s\uparrow} \urcorner : \{*\} \rightarrow \mathcal{S}$  to give a commutative diagram: i.e., such that

$$\pi \circ \ulcorner \mathbf{w}^{s\uparrow} \urcorner = \ulcorner \mathbf{w}^{s\uparrow} \urcorner. \quad (6.58)$$

Indeed, simply define

$$\ulcorner \mathbf{w}^{s\uparrow} \urcorner(*) := s \quad (6.59)$$

However, in the quantum case there can be no 'lift'  $\ulcorner \underline{\mathbf{w}}^{|\psi\rangle} \urcorner : \underline{\mathbf{1}} \rightarrow \underline{\Sigma}$ , as this would correspond to a global element of the spectral presheaf  $\underline{\Sigma}$ , and of course there are none. Thus, from this perspective, the Kochen-Specker theorem can be understood as asserting the existence of an *obstruction* to lifting the arrow  $\ulcorner \underline{\mathbf{w}}^{|\psi\rangle} \urcorner : \underline{\mathbf{1}} \rightarrow P\underline{\Sigma}$ .

Lifting problems of the type

$$\begin{array}{ccc} & & A \\ & & \downarrow \pi \\ C & \xrightarrow{\phi} & B \end{array} \quad (6.60)$$

occur in many places in mathematics. A special, but very well-known, example of (6.60) arises when trying to construct cross-sections of a non-trivial principle fiber bundle  $\pi : P \rightarrow M$ . In diagrammatic terms we have

$$\begin{array}{ccc}
 & P & \\
 & \downarrow \pi & \\
 M & \xrightarrow{\text{id}} & M
 \end{array} \tag{6.61}$$

A cross-section of this bundle corresponds to a lifting of the map  $\text{id} : M \rightarrow M$ .

The obstructions to lifting  $\text{id} : M \rightarrow M$  through  $\pi$  can be studied in various ways. One technique is to decompose the bundle  $\pi : P \rightarrow M$  into a series of interpolating fibrations  $P \rightarrow P_1 \rightarrow P_2 \rightarrow \dots \rightarrow M$  where each fibration  $P_i \rightarrow P_{i+1}$  has the special property that the fiber is a particular Eilenberg-McLane space (this is known as a ‘Postnikov tower’). One then studies the sequential lifting of the function  $\text{id} : M \rightarrow M$ , i.e., first try to lift it through the fibration  $P_1 \rightarrow M$ ; if that is successful try to lift it through  $P_2 \rightarrow P_1$ ; and so on. Potential obstructions to performing these liftings appear as elements of the cohomology groups  $H^k(M; \pi^{k-1}(F))$ ,  $k = 1, 2, \dots$ , where  $F$  is the fiber of the bundle.

We have long felt that it should be possible to describe the non-existence of global elements of  $\underline{\Sigma}$  (i.e., the Kochen-Specker theorem) in some cohomological way, and the remark above suggests one possibility. Namely, perhaps there is some analogue of a ‘Postnikov factorisation’ for the arrow  $\underline{\pi} : \underline{\Sigma} \rightarrow P\underline{\Sigma}$  that could give a cohomological description of the obstructions to a global element of  $\underline{\Sigma}$ , i.e., to the lifting of a pseudo-state  $\ulcorner \underline{\mathbf{w}}^{|\psi} \urcorner : 1 \rightarrow P\underline{\Sigma}$  through the arrow  $\underline{\pi} : \underline{\Sigma} \rightarrow P\underline{\Sigma}$  to give an arrow  $\underline{\mathbf{1}} \rightarrow \underline{\Sigma}$ .

Related to this is the question of if there is a ‘pseudo-state object’,  $\underline{\mathbb{W}}$ , with the defining property that  $\Gamma \underline{\mathbb{W}}$  is equal to the set of all pseudo-states. Of course, to do this properly requires a definition of a pseudo-state that goes beyond the specific constructions of the objects  $\underline{\mathbf{w}}^{|\psi}$ ,  $|\psi\rangle \in \mathcal{H}$ . In particular, are there pseudo-states that are *not* of the form  $\underline{\mathbf{w}}^{|\psi}$ ?

If such an object,  $\underline{\mathbb{W}}$  can be found then  $\underline{\mathbb{W}}$  will be a sub-object of  $P\underline{\Sigma}$ , and in the diagram in (6.54) one could then look to replace  $P\underline{\Sigma}$  with  $\underline{\mathbb{W}}$ .

## 7 The de Groot Presheaves of Physical Quantities

### 7.1 Background Remarks

Our task now is to consider the representation of the local language,  $\mathcal{L}(S)$ , in the case of quantum theory. We assume that the relevant topos is the same as that used for the propositional language  $\mathcal{P}\mathcal{L}(S)$ , i.e.,  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , but the emphasis is very different.

From a physics perspective, the key symbols in  $\mathcal{L}(S)$  are (i) the ground-type symbols,  $\Sigma$  and  $\mathcal{R}$ —the linguistic precursors of the state object and the quantity-value object respectively—and (ii) the function symbols  $A : \Sigma \rightarrow \mathcal{R}$ , which are the precursors of physical quantities. In the quantum-theory representation,  $\phi$ , of  $\mathcal{L}(S)$ , the representation,  $\Sigma_\phi$ , of  $\Sigma$  is defined to be the spectral presheaf  $\underline{\Sigma}$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

The critical question is to find the object,  $\mathcal{R}_\phi$  (provisionally denoted as a presheaf  $\underline{\mathcal{R}}$ ), in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  that represents  $\mathcal{R}$ , and is hence the quantity-value object. One might anticipate that  $\underline{\mathcal{R}}$  is just the real-number object in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , but that turns out to be quite wrong, and the right answer cannot just be guessed. In fact, the correct choice for  $\underline{\mathcal{R}}$  is found indirectly by considering a related question: namely, how to represent each function symbol  $A : \Sigma \rightarrow \mathcal{R}$ , with a concrete arrow  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$  in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , i.e., with a natural transformation  $\check{A} : \underline{\Sigma} \rightarrow \underline{\mathcal{R}}$  between the presheaves  $\underline{\Sigma}$  and  $\underline{\mathcal{R}}$ .

Critical to this task are the daseinisation operations on projection operators that were defined earlier as (5.9) and (5.57), and which are repeated here for convenience:

**Definition 7.1** *If  $\hat{P}$  is a projection operator, and  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  is any context/stage, we define:*

1. The ‘outer daseinisation’ operation is

$$\delta^o(\hat{P})_V := \bigwedge \{ \hat{\alpha} \in \mathcal{P}(V) \mid \hat{P} \preceq \hat{\alpha} \}. \quad (7.1)$$

where ‘ $\preceq$ ’ denotes the usual ordering of projection operators, and where  $\mathcal{P}(V)$  is the set of all projection operators in  $V$ .

2. Similarly, the ‘inner daseinisation’ operation is defined in the context  $V$  as (c.f. (5.57))

$$\delta^i(\hat{P})_V := \bigvee \{ \hat{\beta} \in \mathcal{P}(V) \mid \hat{\beta} \preceq \hat{P} \}. \quad (7.2)$$

Thus  $\delta^o(\hat{P})_V$  is the best approximation to  $\hat{P}$  in  $V$  from ‘above’, being the smallest projection in  $V$  that is larger than or equal to  $\hat{P}$ . Similarly,  $\delta^i(\hat{P})_V$  is the best approximation to  $\hat{P}$  from ‘below’, being the largest projection in  $V$  that is smaller than or equal to  $\hat{P}$ .

In Section 6.5, we showed that the outer presheaf is a sub-object of the power object  $P_{\text{cl}}\underline{\Sigma}$  (in the category  $\mathbf{Sets}^{V(\mathcal{H})^{\text{op}}}$ ), and hence that the global element  $\delta^o(\hat{P})$  of  $\underline{Q}$  determines a (clopen) sub-object,  $\underline{\delta^o(\hat{P})}$ , of the spectral presheaf  $\underline{\Sigma}$ . By these means, the quantum logic of the lattice  $\mathcal{P}(\mathcal{H})$  is mapped into the Heyting algebra of the set,  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ , of clopen sub-objects of  $\underline{\Sigma}$ .

Our task now is to perform the second stage of the programme: namely (i) identify the quantity-value presheaf,  $\underline{\mathcal{R}}$ ; and (ii) show that any physical quantity can be represented by an arrow from  $\underline{\Sigma}$  to  $\underline{\mathcal{R}}$ .

## 7.2 The Daseinisation of an Arbitrary Self-Adjoint Operator

### 7.2.1 Spectral Families and Spectral Order

We now want to extend the daseinisation operations from projections to arbitrary (bounded) self-adjoint operators. To this end, consider first a bounded, self-adjoint operator,  $\hat{A}$ , whose spectrum is purely discrete. Then the spectral theorem can be used to write  $\hat{A} = \sum_{i=1}^{\infty} a_i \hat{P}_i$  where  $a_1, a_2, \dots$  are the eigenvalues of  $\hat{A}$ , and  $\hat{P}_1, \hat{P}_2, \dots$  are the spectral projection operators onto the corresponding eigenspaces.

A construction that comes immediately to mind is to use the daseinisation operation on projections to define

$$\delta^o(\hat{A})_V := \sum_{i=1}^{\infty} a_i \delta^o(\hat{P}_i)_V \quad (7.3)$$

for each stage  $V$ . However, this procedure is rather unnatural. For one thing, the projections,  $\hat{P}_i$ ,  $i = 1, 2, \dots$  form a complete orthonormal set:

$$\sum_{i=1}^{\infty} \hat{P}_i = \hat{1}, \quad (7.4)$$

$$\hat{P}_i \hat{P}_j = \delta_{ij} \hat{P}_i, \quad (7.5)$$

whereas, in general, the collection of daseinised projections,  $\delta^o(\hat{P}_i)_V$ ,  $i = 1, 2, \dots$  will not satisfy either of these conditions. In addition, it is hard

to see how the expression  $\delta^o(\hat{A})_V := \sum_{i=1}^{\infty} a_i \delta^o(\hat{P}_i)_V$  can be generalised to operators,  $\hat{A}$ , with a continuous spectrum.

The answer to this conundrum lies in the work of de Groot. He realised that although it is not useful to daseinise the spectral projections of an operator  $\hat{A}$ , it *is* possible to daseinise the *spectral family* of  $\hat{A}$  [32].

**Spectral families.** We first recall that a *spectral family* is a family of projection operators  $\hat{E}_\lambda$ ,  $\lambda \in \mathbb{R}$ , with the following properties:

1. If  $\lambda_2 \leq \lambda_1$  then  $\hat{E}_{\lambda_2} \preceq \hat{E}_{\lambda_1}$ .
2. The net  $\lambda \mapsto \hat{E}_\lambda$  of projection operators in the lattice  $\mathcal{P}(\mathcal{H})$  is bounded above by  $\hat{1}$ , and below by  $\hat{0}$ . In fact,

$$\lim_{\lambda \rightarrow \infty} \hat{E}_\lambda = \hat{1}, \quad (7.6)$$

$$\lim_{\lambda \rightarrow -\infty} \hat{E}_\lambda = \hat{0}. \quad (7.7)$$

3. The map  $\lambda \mapsto \hat{E}_\lambda$  is right-continuous:<sup>74</sup>

$$\bigwedge_{\epsilon \downarrow 0} \hat{E}_{\lambda+\epsilon} = \hat{E}_\lambda \quad (7.8)$$

for all  $\lambda \in \mathbb{R}$ .

The spectral theorem asserts that for any self-adjoint operator  $\hat{A}$ , there exists a spectral family,  $\lambda \mapsto \hat{E}_\lambda^A$ , such that

$$\hat{A} = \int_{\mathbb{R}} \lambda d\hat{E}_\lambda^A \quad (7.9)$$

We are only concerned with bounded operators, and so the (weak Stieljes) integral in (7.9) is really over the bounded spectrum of  $\hat{A}$  which, of course, is a compact subset of  $\mathbb{R}$ . Conversely, given a bounded spectral family  $\{\hat{E}_\lambda\}_{\lambda \in \mathbb{R}}$ ,<sup>75</sup> there is a bounded self-adjoint operator  $\hat{A}$  such that  $\hat{A} = \int_{\mathbb{R}} \lambda d\hat{E}_\lambda$ .

<sup>74</sup>It is a matter of convention whether one chooses right-continuous or left-continuous.

<sup>75</sup>I.e., there are  $a, b \in \mathbb{R}$  such that  $\hat{E}_\lambda = \hat{0}$  for all  $\lambda \leq a$  and  $\hat{E}_\lambda = \hat{1}$  for all  $\lambda \geq b$ .

**The spectral order.** A key element for our work is the so-called *spectral order* that was introduced in [63].<sup>76</sup> It is defined as follows. Let  $\hat{A}$  and  $\hat{B}$  be (bounded) self-adjoint operators with spectral families  $\{\hat{E}_\lambda^A\}_{\lambda \in \mathbb{R}}$  and  $\{\hat{E}_\lambda^B\}_{\lambda \in \mathbb{R}}$ , respectively. Then define:

$$\hat{A} \preceq_s \hat{B} \text{ if and only if } \hat{E}_\lambda^B \preceq \hat{E}_\lambda^A \text{ for all } \lambda \in \mathbb{R}. \quad (7.10)$$

It is easy to see that (7.10) defines a genuine partial ordering on  $B(\mathcal{H})_{\text{sa}}$  (the self-adjoint operators in  $B(\mathcal{H})$ ). In fact,  $B(\mathcal{H})_{\text{sa}}$  is a ‘boundedly complete’ lattice with respect to the spectral order, i.e., each bounded set  $S$  of self-adjoint operators has a minimum  $\bigwedge S \in B(\mathcal{H})_{\text{sa}}$  and a maximum  $\bigvee S \in B(\mathcal{H})_{\text{sa}}$  with respect to this order.

If  $\hat{P}, \hat{Q}$  are projections, then

$$\hat{P} \preceq_s \hat{Q} \text{ if and only if } \hat{P} \preceq \hat{Q}, \quad (7.11)$$

so the spectral order coincides with the usual partial order on  $\mathcal{P}(\mathcal{H})$ . To ensure this, the ‘reverse’ relation in (7.10) is necessary, since the spectral family of a projection  $\hat{P}$  is given by

$$E_\lambda^{\hat{P}} = \begin{cases} \hat{0} & \text{if } \lambda < 0 \\ \hat{1} - \hat{P} & \text{if } 0 \leq \lambda < 1 \\ \hat{1} & \text{if } \lambda \geq 1. \end{cases} \quad (7.12)$$

If  $\hat{A}, \hat{B}$  are self-adjoint operators such that (i) either  $\hat{A}$  or  $\hat{B}$  is a projection, or (ii)  $[\hat{A}, \hat{B}] = \hat{0}$ , then  $\hat{A} \preceq_s \hat{B}$  if and only if  $\hat{A} \preceq \hat{B}$ . Here ‘ $\preceq$ ’ denotes the usual ordering on  $B(\mathcal{H})_{\text{sa}}$ .<sup>77</sup>

Moreover, if  $\hat{A}, \hat{B}$  are arbitrary self-adjoint operators, then  $\hat{A} \preceq_s \hat{B}$  implies  $\hat{A} \preceq \hat{B}$ , but not vice versa in general. Thus the spectral order is a partial order on  $B(\mathcal{H})_{\text{sa}}$  that is coarser than the usual one.

## 7.2.2 Daseinisation of Self-Adjoint Operators.

De Groote’s crucial observation was the following. Let  $\lambda \mapsto \hat{E}_\lambda$  be a spectral family in  $\mathcal{P}(\mathcal{H})$  (or, equivalently, a self-adjoint operator  $\hat{A}$ ). Then, for each stage  $V$ , the following maps:

$$\lambda \mapsto \bigwedge_{\mu > \lambda} \delta^o(\hat{E}_\mu)_V \quad (7.13)$$

$$\lambda \mapsto \delta^i(\hat{E}_\lambda)_V \quad (7.14)$$

<sup>76</sup>The spectral order was later reinvented by de Groote, see [31].

<sup>77</sup>The ‘usual’ ordering is  $\hat{A} \preceq \hat{B}$  if  $\langle \psi | \hat{A} | \psi \rangle \leq \langle \psi | \hat{B} | \psi \rangle$  for all vectors  $|\psi\rangle \in \mathcal{H}$ .

also define spectral families.<sup>78</sup> These spectral families lie in  $\mathcal{P}(V)$  and hence, by the spectral theorem, define self-adjoint operators in  $V$ . This leads to the definition of the two daseinisations of an arbitrary self-adjoint operator:

**Definition 7.2** *Let  $\hat{A}$  be an arbitrary self-adjoint operator. Then the outer and inner daseinisations of  $\hat{A}$  are defined at each stage  $V$  as:*

$$\delta^o(\hat{A})_V := \int_{\mathbb{R}} \lambda d(\delta_V^i(\hat{E}_\lambda^A)), \quad (7.15)$$

$$\delta^i(\hat{A})_V := \int_{\mathbb{R}} \lambda d\left(\bigwedge_{\mu>\lambda} \delta_V^o(\hat{E}_\mu^A)\right), \quad (7.16)$$

respectively.

Note that for all  $\lambda \in \mathbb{R}$ , and for all stages  $V$ , we have

$$\delta^i(\hat{E}_\lambda)_V \preceq \bigwedge_{\mu>\lambda} \delta^o(\hat{E}_\mu)_V \quad (7.17)$$

and hence, for all  $V$ ,

$$\delta^i(\hat{A})_V \preceq_s \delta^o(\hat{A})_V. \quad (7.18)$$

This explains why the ‘ $i$ ’ and ‘ $o$ ’ superscripts in (7.15–7.16) are defined the way round that they are.

Both outer daseinisation (7.15) and inner daseinisation (7.16) can be used to ‘adapt’ a self-adjoint operator  $\hat{A}$  to contexts  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  that do not contain  $\hat{A}$ . (On the other hand, if  $\hat{A} \in V$ , then  $\delta^o(\hat{A})_V = \delta^i(\hat{A})_V = \hat{A}$ .)

### 7.2.3 Properties of Daseinisation.

We will now list some useful properties of daseinisation.

1. It is clear that the outer, and inner, daseinisation operations can be extended to situations where the self-adjoint operator  $\hat{A}$  does not belong to  $B(\mathcal{H})_{\text{sa}}$ , or where  $V$  is not an *abelian* sub-algebra of  $B(\mathcal{H})$ . Specifically, let  $\mathcal{N}$  be an arbitrary von Neumann algebra, and let  $\mathcal{S} \subset \mathcal{N}$  be a proper von Neumann sub-algebra such that  $\hat{1}_{\mathcal{N}} = \hat{1}_{\mathcal{S}} = \hat{1}$ . Then outer and inner

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<sup>78</sup>The reason (7.13) and (7.14) have a different form is that  $\lambda \mapsto \delta^i(\hat{E}_\lambda)_V$  is right continuous whereas  $\lambda \mapsto \delta^o(\hat{E}_\lambda)_V$  is not. On the other hand, the family  $\lambda \mapsto \bigwedge_{\mu>\lambda} \delta^o(\hat{E}_\mu)_V$  is right continuous.



daseinisation can be defined as the mappings

$$\begin{aligned} \delta^o : \mathcal{N}_{\text{sa}} &\rightarrow \mathcal{S}_{\text{sa}} \\ \hat{A} &\mapsto \int_{\mathbb{R}} \lambda d(\delta_{\mathcal{S}}^i(\hat{E}_{\lambda}^A)), \end{aligned} \quad (7.19)$$

$$\begin{aligned} \delta^i : \mathcal{N}_{\text{sa}} &\rightarrow \mathcal{S}_{\text{sa}} \\ \hat{A} &\mapsto \int_{\mathbb{R}} \lambda d\left(\bigwedge_{\mu > \lambda} \delta_{\mathcal{S}}^o(\hat{E}_{\mu}^A)\right). \end{aligned} \quad (7.20)$$

A particular case is  $\mathcal{N} = V$  and  $\mathcal{S} = V'$  for two contexts  $V, V'$  such that  $V' \subset V$ . Hence, a self-adjoint operator can be restricted from one context to a sub-context.

For the moment, we will let  $\mathcal{N}$  be an arbitrary von Neumann algebra, with  $\mathcal{S} \subset \mathcal{N}$ .

**2.** By construction,

$$\delta^o(\hat{A})_{\mathcal{S}} = \bigwedge \{\hat{B} \in \mathcal{S}_{\text{sa}} \mid \hat{B} \succeq_s \hat{A}\}, \quad (7.21)$$

where the minimum is taken with respect to the spectral order; i.e.,  $\delta^o(\hat{A})_{\mathcal{S}}$  is the smallest self-adjoint operator in  $\mathcal{S}$  that is spectrally larger than (or equal to)  $\hat{A}$ . This implies  $\delta^o(\hat{A})_{\mathcal{S}} \succeq \hat{A}$  in the usual order. Likewise,

$$\delta^i(\hat{A})_{\mathcal{S}} = \bigvee \{\hat{B} \in \mathcal{S}_{\text{sa}} \mid \hat{B} \preceq_s \hat{A}\}, \quad (7.22)$$

so  $\delta^i(\hat{A})_{\mathcal{S}}$  is the largest self-adjoint operator in  $\mathcal{S}$  spectrally smaller than (or equal to)  $\hat{A}$ , which implies  $\delta^i(\hat{A})_{\mathcal{S}} \preceq \hat{A}$ .

**3.** In general, neither  $\delta^o(\hat{A})_{\mathcal{S}}$  nor  $\delta^i(\hat{A})_{\mathcal{S}}$  can be written as Borel functions of the operator  $\hat{A}$ , since daseinisation changes the elements of the spectral family, while a function merely ‘shuffles them around’.

**4.** Let  $\hat{A} \in \mathcal{N}$  be self-adjoint. The spectrum,  $\text{sp}(\hat{A})$ , consists of all  $\lambda \in \mathbb{R}$  such that the spectral family  $\{\hat{E}_{\lambda}^A\}_{\lambda \in \mathbb{R}}$  is non-constant on any neighbourhood of  $\lambda$ . By definition, outer daseinisation of  $\hat{A}$  acts on the spectral family of  $\hat{A}$  by sending  $\hat{E}_{\lambda}^A$  to  $\hat{E}_{\lambda}^{\delta^o(\hat{A})_{\mathcal{S}}} = \delta^i(\hat{E}_{\lambda}^A)_{\mathcal{S}}$ . If  $\{\hat{E}_{\lambda}^A\}_{\lambda \in \mathbb{R}}$  is constant on some neighbourhood of  $\lambda$ , then the spectral family  $\{\hat{E}_{\lambda}^{\delta^o(\hat{A})_{\mathcal{S}}}\}_{\lambda \in \mathbb{R}}$  of  $\delta^o(\hat{A})_{\mathcal{S}}$  is also constant on this neighbourhood. This shows that

$$\text{sp}(\delta^o(\hat{A})_{\mathcal{S}}) \subseteq \text{sp}(\hat{A}) \quad (7.23)$$

for all self-adjoint operators  $\hat{A} \in \mathcal{N}_{\text{sa}}$  and all von Neumann sub-algebras  $\mathcal{S}$ . Analogous arguments apply to inner daseinisation.

Heuristically, this result implies that the spectrum of the operator  $\delta^o(\hat{A})_{\mathcal{S}}$  is more degenerate than that of  $\hat{A}$ ; i.e., the effect of daseinisation is to ‘collapse’ eigenvalues.

**5.** Outer and inner daseinisation are both non-linear mappings. We will show this for projections explicitly. For example, let  $\hat{Q} := \hat{1} - \hat{P}$ . Then  $\delta^o(\hat{Q} + \hat{P})_{\mathcal{S}} = \delta^o(\hat{1})_{\mathcal{S}} = \hat{1}$ , while  $\delta^o(\hat{1} - \hat{P})_{\mathcal{S}} \succ \hat{1} - \hat{P}$  and  $\delta^o(\hat{P})_{\mathcal{S}} \succ \hat{P}$  in general, so  $\delta^o(\hat{1} - \hat{P})_{\mathcal{S}} + \delta^o(\hat{P})_{\mathcal{S}}$  is the sum of two non-orthogonal projections in general (and hence not equal to  $\hat{1}$ ). For inner daseinisation, we have  $\delta^i(\hat{1} - \hat{P})_{\mathcal{S}} \prec \hat{1} - \hat{P}$  and  $\delta^i(\hat{P})_{\mathcal{S}} \prec \hat{P}$  in general, so  $\delta^i(\hat{1} - \hat{P})_{\mathcal{S}} + \delta^i(\hat{P})_{\mathcal{S}} \prec \hat{1} = \delta^i(\hat{1} - \hat{P} + \hat{P})_{\mathcal{S}}$  in general.

**6.** If  $a \geq 0$ , then  $\delta^o(a\hat{A})_{\mathcal{S}} = a\delta^o(\hat{A})_{\mathcal{S}}$  and  $\delta^i(a\hat{A})_{\mathcal{S}} = a\delta^i(\hat{A})_{\mathcal{S}}$ . If  $a < 0$ , then  $\delta^o(a\hat{A})_{\mathcal{S}} = a\delta^i(\hat{A})_{\mathcal{S}}$  and  $\delta^i(a\hat{A})_{\mathcal{S}} = a\delta^o(\hat{A})_{\mathcal{S}}$ . This is due the behaviour of spectral families under the mapping  $\hat{A} \mapsto -\hat{A}$ .

**7.** Let  $\hat{A}$  be a self-adjoint operator, and let  $\hat{E}[A \leq \lambda] = \hat{E}_{\lambda}^A$  be an element of the spectral family of  $\hat{A}$ . From (7.15) we get

$$\hat{E}[\delta_{\mathcal{S}}^o(A) \leq \lambda] = \delta_{\mathcal{S}}^i(\hat{E}[A \leq \lambda]) \quad (7.24)$$

and then

$$\hat{E}[\delta^o(\hat{A})_{\mathcal{S}} > \lambda] = \hat{1} - \hat{E}[\delta^o(\hat{A})_{\mathcal{S}} \leq \lambda] \quad (7.25)$$

$$= \hat{1} - \delta_{\mathcal{S}}^i(\hat{E}[A \leq \lambda]) \quad (7.26)$$

$$= \delta_{\mathcal{S}}^o(\hat{1} - \hat{E}[A \leq \lambda]) \quad (7.27)$$

where we have used the general result that, for any projection  $\hat{P}$ , we have  $\hat{1} - \delta^i(\hat{P})_{\mathcal{S}} = \delta_{\mathcal{S}}^o(\hat{1} - \hat{P})$ . Then, (7.27) gives

$$\hat{E}[\delta^o(\hat{A})_{\mathcal{S}} > \lambda] = \delta^o(\hat{E}[A > \lambda])_{\mathcal{S}}. \quad (7.28)$$

## 7.2.4 The de Grootte Presheaves

We know that  $V \mapsto \delta^o(\hat{P})_V$  and  $V \mapsto \delta^i(\hat{P})_V$  are global elements of the outer presheaf,  $\underline{\mathcal{Q}}$ , and inner presheaf,  $\underline{\mathcal{I}}$ , respectively. Using the daseinisation operation for self-adjoint operators, it is straightforward to construct analogous presheaves for which  $V \mapsto \delta^o(\hat{A})_V$  and  $V \mapsto \delta^i(\hat{A})_V$  are global elements. One of these presheaves was briefly considered in [32]. We call these the ‘de Grootte presheaves’ in recognition of the importance of de Grootte’s work.

**Definition 7.3** *The outer de Grootte presheaf,  $\underline{\mathcal{Q}}$ , is defined as follows:*

- (i) On objects  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ : We define  $\underline{\mathbb{Q}}_V := V_{\text{sa}}$ , the collection of self-adjoint members of  $V$ .
- (ii) On morphisms  $i_{V'V} : V' \subseteq V$  : The mapping  $\underline{\mathbb{Q}}(i_{V'V}) : \underline{\mathbb{Q}}_V \rightarrow \underline{\mathbb{Q}}_{V'}$  is given by

$$\underline{\mathbb{Q}}(i_{V'V})(\hat{A}) := \delta^\circ(\hat{A})_{V'} \quad (7.29)$$

$$= \int_{\mathbb{R}} \lambda d(\delta^i(\hat{E}_\lambda^A)_{V'}) \quad (7.30)$$

$$= \int_{\mathbb{R}} \lambda d(\underline{I}(i_{V'V})(\hat{E}_\lambda^A)) \quad (7.31)$$

for all  $\hat{A} \in \underline{\mathbb{Q}}_V$ .

Here we used the fact that the restriction mapping  $\underline{I}(i_{V'V})$  of the inner presheaf  $\underline{I}$  is the inner daseinisation of projections  $\delta^i : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$ .

**Definition 7.4** The inner de Grootte presheaf,  $\underline{\mathbb{I}}$ , is defined as follows:

- (i) On objects  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ : We define  $\underline{\mathbb{I}}_V := V_{\text{sa}}$ , the collection of self-adjoint members of  $V$ .
- (ii) On morphisms  $i_{V'V} : V' \subseteq V$  : The mapping  $\underline{\mathbb{I}}(i_{V'V}) : \underline{\mathbb{I}}_V \rightarrow \underline{\mathbb{I}}_{V'}$  is given by

$$\underline{\mathbb{I}}(i_{V'V})(\hat{A}) := \delta^i(\hat{A})_{V'} \quad (7.32)$$

$$= \int_{\mathbb{R}} \lambda d\left(\bigwedge_{\mu > \lambda} (\delta^\circ(\hat{E}_\mu^A)_{V'})\right) \quad (7.33)$$

$$= \int_{\mathbb{R}} \lambda d\left(\bigwedge_{\mu > \lambda} (\underline{Q}(i_{V'V})(\hat{E}_\mu^A))\right) \quad (7.34)$$

for all  $\hat{A} \in \underline{\mathbb{Q}}_V$  (where  $\underline{Q}(i_{V'V}) = \delta^\circ : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$ ).

It is now clear that, by construction,  $\delta^\circ(\hat{A}) := V \mapsto \delta^\circ(\hat{A})_V$  is a global element of  $\underline{\mathbb{Q}}$ , and  $\delta^i(\hat{A}) := V \mapsto \delta^i(\hat{A})_V$  is a global element of  $\underline{\mathbb{I}}$ .

De Grootte found an example of an element of  $\Gamma \underline{\mathbb{Q}}$  that is *not* of the form  $\delta^\circ(\hat{A})$  (as mentioned in [32]). The same example can be used to show that there are global elements of the outer presheaf  $\underline{Q}$  that are not of the form  $\delta^\circ(\hat{P})$  for any projection  $\hat{P} \in \mathcal{P}(\mathcal{H})$ .

On the other hand, we have:

**Theorem 7.1** *The mapping*

$$\delta^i : B(\mathcal{H})_{\text{sa}} \rightarrow \Gamma \underline{\mathbb{I}} \quad (7.35)$$

$$\hat{A} \mapsto \delta^i(\hat{A}) \quad (7.36)$$

from self-adjoint operators in  $B(\mathcal{H})$  to global sections of the outer de Groot presheaf is injective. Likewise,

$$\delta^o : B(\mathcal{H})_{\text{sa}} \rightarrow \Gamma \underline{\mathbb{O}} \quad (7.37)$$

$$\hat{A} \mapsto \delta^o(\hat{A}) \quad (7.38)$$

is injective.

**Proof.** By construction,  $\hat{A} \geq_s \delta^i(\hat{A})_V$  for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . Since  $\hat{A}$  is contained in at least one context, so

$$\hat{A} = \bigvee_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta^i(\hat{A})_V, \quad (7.39)$$

where the maximum is taken with respect to the spectral order. If  $\delta^i(\hat{A}) = \delta^i(\hat{B})$ , then we have

$$\hat{A} = \bigvee_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta^i(\hat{A})_V = \bigvee_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta^i(\hat{B})_V = \hat{B}. \quad (7.40)$$

Analogously,  $\hat{A} \leq_s \delta^o(\hat{A})_V$  for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , so

$$\hat{A} = \bigwedge_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta^o(\hat{A})_V. \quad (7.41)$$

If  $\delta^o(\hat{A}) = \delta^o(\hat{B})$ , then we have

$$\hat{A} = \bigwedge_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta^o(\hat{A})_V = \bigwedge_{V \in \text{Ob}(\mathcal{V}(\mathcal{H}))} \delta^o(\hat{B})_V = \hat{B}. \quad (7.42)$$

■

The same argument also holds more generally for arbitrary von Neumann algebras, not just  $B(\mathcal{H})$ .

## 8 The Presheaves $\underline{\text{sp}}(\hat{A})^{\succeq}$ , $\underline{\mathbb{R}}^{\succeq}$ and $\underline{\mathbb{R}}^{\leftrightarrow}$

### 8.1 Background to the Quantity-Value Presheaf $\underline{\mathcal{R}}$ .

Our goal now is to construct a ‘quantity-value’ presheaf  $\underline{\mathcal{R}}$  with the property that inner and/or outer daseinisation of a self-adjoint operator  $\hat{A}$  can be used to define an arrow, i.e., a natural transformation, from  $\underline{\Sigma}$  to  $\underline{\mathcal{R}}$ .<sup>79</sup>

The arrow corresponding to a self-adjoint operator  $\hat{A} \in B(\mathcal{H})$  is denoted for now by  $\check{A} : \underline{\Sigma} \rightarrow \underline{\mathcal{R}}$ . At each stage  $V$ , we need a mapping

$$\check{A}_V : \underline{\Sigma}_V \rightarrow \underline{\mathcal{R}}_V \quad (8.1)$$

$$\lambda \mapsto \check{A}_V(\lambda), \quad (8.2)$$

and we make the basic assumption that this mapping is given by evaluation. More precisely,  $\lambda \in \underline{\Sigma}_V$  is a spectral element<sup>80</sup> of  $V$  and hence can be evaluated on operators lying in  $V$ . And, while  $\hat{A}$  will generally not lie in  $V$ , both the inner daseinisation  $\delta^i(\hat{A})_V$  and the outer daseinisation  $\delta^o(\hat{A})_V$  do.

Let us start by considering the operators  $\delta^o(\hat{A})_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . Each of these is a self-adjoint operator in the commutative von Neumann algebra  $V$ , and hence, by the spectral theorem, can be represented by a function, (the Gel’fand transform<sup>81</sup>)  $\overline{\delta^o(\hat{A})_V} : \underline{\Sigma}_V \rightarrow \text{sp}(\delta^o(\hat{A})_V)$ , with values in the spectrum  $\text{sp}(\delta^o(\hat{A})_V)$  of the self-adjoint operator  $\delta^o(\hat{A})_V$ . Since the spectrum of a self-adjoint operator is a subset of  $\mathbb{R}$ , we can also write  $\overline{\delta^o(\hat{A})_V} : \underline{\Sigma}_V \rightarrow \mathbb{R}$ . The question now is whether the collection of maps  $\overline{\delta^o(\hat{A})_V} : \underline{\Sigma}_V \rightarrow \mathbb{R}$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , can be regarded as an arrow from  $\underline{\Sigma}$  to some presheaf  $\underline{\mathcal{R}}$ .

To answer this we need to see how these operators behave as we go ‘down a chain’ of sub-algebras  $V' \subseteq V$ . The first remark is that if  $V' \subseteq V$  then  $\delta^o(\hat{A})_{V'} \succeq \delta^o(\hat{A})_V$ . When applied to the Gel’fand transforms, this leads to the equation

$$\overline{\delta^o(\hat{A})_{V'}}(\lambda|_{V'}) \geq \overline{\delta^o(\hat{A})_V}(\lambda) \quad (8.3)$$

for all  $\lambda \in \underline{\Sigma}_V$ , where  $\lambda|_{V'}$  denotes the restriction of the spectral element  $\lambda \in \underline{\Sigma}_V$  to the sub-algebra  $V' \subseteq V$ . However, the definition of the spectral

<sup>79</sup>In fact, we will define several closely related presheaves that can serve as a quantity-value object.

<sup>80</sup>A ‘spectral element’,  $\lambda \in \underline{\Sigma}_V$  of  $V$ , is a multiplicative, linear functional  $\lambda : V \rightarrow \mathbb{C}$  with  $\langle \lambda, \hat{1} \rangle = 1$ , see also Def. 5.4.

<sup>81</sup>This use of the ‘overline’ symbol for the Gel’fand transform should not be confused with our later use of the same symbol to indicate a co-presheaf.

presheaf is such that  $\lambda|_{V'} = \underline{\Sigma}(i_{V'V})(\lambda)$ , and hence (8.3) can be rewritten as

$$\overline{\delta^o(\hat{A})_{V'}}(\underline{\Sigma}(i_{V'V})(\lambda)) \geq \overline{\delta^o(\hat{A})_V}(\lambda) \quad (8.4)$$

for all  $\lambda \in \underline{\Sigma}_V$ .

It is a standard result that the Dedekind real number object,  $\underline{\mathbb{R}}$ , in a presheaf topos  $\mathbf{Sets}^{\mathcal{C}^{op}}$  is the *constant* functor from  $\mathcal{C}^{op}$  to  $\mathbb{R}$  [56]. It follows that the family of Gel'fand transforms,  $\overline{\delta^o(\hat{A})_V}$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , of the daseinised operators  $\delta^o(\hat{A})_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , does *not* define an arrow from  $\underline{\Sigma}$  to  $\underline{\mathbb{R}}$ , as this would require an equality in (8.4), which is not true. Thus the quantity-value presheaf,  $\underline{\mathcal{R}}$ , in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{op}}$  is *not* the real-number object  $\underline{\mathbb{R}}$ , although clearly  $\underline{\mathcal{R}}$  has *something* to do with the real numbers. We must take into account the growth of these real numbers as we go from  $V$  to smaller sub-algebras  $V'$ . Similarly, if we consider inner daseinisation, we get a series of falling real numbers.

The presheaf,  $\underline{\mathcal{R}}$ , that we will choose, and which will be denoted by  $\underline{\mathbb{R}}^{\leftrightarrow}$ , incorporates both aspects (growing and falling real numbers).

## 8.2 Definition of the Presheaves $\underline{\text{sp}}(\hat{A})^{\succeq}$ , $\underline{\mathbb{R}}^{\succeq}$ and $\underline{\mathbb{R}}^{\leftrightarrow}$ .

The inapplicability of the real-number object  $\underline{\mathbb{R}}$  may seem strange at first,<sup>82</sup> but actually it is not that surprising. Because of the Kochen-Specker theorem, we do not expect to be able to assign (constant) real numbers as values of physical quantities, at least not globally. Instead, we draw on some recent results of M. Jackson [47], obtained as part of his extensive study of measure theory on a topos of presheaves. Here, we use a single construction in Jackson's thesis: the presheaf of 'order-preserving functions' over a partially ordered set—in our case,  $\mathcal{V}(\mathcal{H})$ . In fact, we will need both order-reversing and order-preserving functions.

**Definition 8.1** *Let  $(\mathcal{Q}, \preceq)$  and  $(\mathcal{P}, \preceq)$  be partially ordered sets. A function*

$$\mu : \mathcal{Q} \rightarrow \mathcal{P} \quad (8.5)$$

*is order-preserving if  $q_1 \preceq q_2$  implies  $\mu(q_1) \preceq \mu(q_2)$  for all  $q_1, q_2 \in \mathcal{Q}$ . It is order-reversing if  $q_1 \preceq q_2$  implies  $\mu(q_1) \succeq \mu(q_2)$ . We denote by  $\mathcal{OP}(\mathcal{Q}, \mathcal{P})$  the set of order-preserving functions  $\mu : \mathcal{Q} \rightarrow \mathcal{P}$ , and by  $\mathcal{OR}(\mathcal{Q}, \mathcal{P})$  the set of order-reversing functions.*

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<sup>82</sup>Indeed, it puzzled us for a while!

We note that if  $\mu$  is order-preserving, then  $-\mu$  is order-reversing, and vice versa.

Adapting Jackson's definitions slightly, if  $\mathcal{P}$  is any partially-ordered set, we have the following.

**Definition 8.2** *The  $\mathcal{P}$ -valued presheaf,  $\underline{\mathcal{P}}^{\succeq}$ , of order-reversing functions over  $\mathcal{V}(\mathcal{H})$  is defined as follows:*

(i) *On objects  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ :*

$$\underline{\mathcal{P}}_V^{\succeq} := \{\mu : \downarrow V \rightarrow \mathcal{P} \mid \mu \in \mathcal{OR}(\downarrow V, \mathcal{P})\} \quad (8.6)$$

*where  $\downarrow V \subset \text{Ob}(\mathcal{V}(\mathcal{H}))$  is the set of all von Neumann sub-algebras of  $V$ .*

(ii) *On morphisms  $i_{V'V} : V' \subseteq V$  : The mapping  $\underline{\mathcal{P}}^{\succeq}(i_{V'V}) : \underline{\mathcal{P}}_{V'}^{\succeq} \rightarrow \underline{\mathcal{P}}_V^{\succeq}$  is given by*

$$\underline{\mathcal{P}}^{\succeq}(i_{V'V})(\mu) := \mu|_{\downarrow V'} \quad (8.7)$$

*where  $\mu|_{\downarrow V'}$  denotes the restriction of the function  $\mu$  to  $\downarrow V' \subseteq \downarrow V$ .*

Jackson uses order-preserving functions with  $\mathcal{P} := [0, \infty)$  (the non-negative reals), with the usual order  $\leq$ .

Clearly, there is an analogous definition of the  $\mathcal{P}$ -valued presheaf,  $\underline{\mathcal{P}}^{\preceq}$ , of order-preserving functions from  $\downarrow V$  to  $\mathcal{P}$ . It can be shown that  $\underline{\mathcal{P}}^{\succeq}$  and  $\underline{\mathcal{P}}^{\preceq}$  are isomorphic objects in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

Let us first consider  $\underline{\mathcal{P}}^{\succeq}$ . For us, the key examples for the partially ordered set  $\mathcal{P}$  are (i)  $\mathbb{R}$ , the real numbers with the usual order  $\leq$ , and (ii)  $\text{sp}(\hat{A}) \subset \mathbb{R}$ , the spectrum of some bounded self-adjoint operator  $\hat{A}$ , with the order  $\leq$  inherited from  $\mathbb{R}$ . Clearly, the associated presheaf  $\underline{\text{sp}(\hat{A})}^{\succeq}$  is a sub-object of the presheaf  $\underline{\mathbb{R}}^{\succeq}$ .

Now let  $\hat{A} \in B(\mathcal{H})_{\text{sa}}$ , and let  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . Then to each  $\lambda \in \underline{\Sigma}_V$  there is associated the function

$$\check{\delta}^o(\hat{A})_V(\lambda) : \downarrow V \rightarrow \text{sp}(\hat{A}), \quad (8.8)$$

given by

$$\left( \check{\delta}^o(\hat{A})_V(\lambda) \right) (V') := \overline{\delta^o(\hat{A})_{V'}(\underline{\Sigma}(i_{V'V})(\lambda))} \quad (8.9)$$

$$= \overline{\delta^o(\hat{A})_{V'}(\lambda|_{V'})} \quad (8.10)$$

$$= \langle \lambda|_{V'}, \delta^o(\hat{A})_{V'} \rangle \quad (8.11)$$

$$= \langle \lambda, \delta^o(\hat{A})_{V'} \rangle \quad (8.12)$$

for all  $V' \subseteq V$ . We note that as  $V'$  becomes smaller,  $\delta^\circ(\hat{A})_{V'}$  becomes larger (or stays the same) in the spectral order, and hence in the usual order on operators. Therefore,  $\check{\delta}^\circ(\hat{A})_V(\lambda) : \downarrow V \rightarrow \text{sp}(\hat{A})$  is an *order-reversing function*, for each  $\lambda \in \underline{\Sigma}_V$ .

It is worth noting that daseinisation of  $\hat{A}$ , i.e., the approximation of the self-adjoint operator  $\hat{A}$  in the *spectral* order, allows to define a function  $\check{\delta}^\circ(\hat{A})_V(\lambda)$  (for each  $\lambda \in \underline{\Sigma}_V$ ) with values in the spectrum of  $\hat{A}$ , since we have  $\text{sp}(\delta^\circ(\hat{A})_V) \subseteq \text{sp}(\hat{A})$ , see (7.23). If we had chosen an approximation in the usual *linear* order on  $B(\mathcal{H})_{\text{sa}}$ , then the approximated operators would not have a spectrum that is contained in  $\text{sp}(\hat{A})$  in general.

Let

$$\check{\delta}^\circ(\hat{A})_V : \underline{\Sigma}_V \rightarrow \underline{\text{sp}(\hat{A})}_V^{\check{\leq}} \quad (8.13)$$

$$\lambda \mapsto \check{\delta}^\circ(\hat{A})_V(\lambda) \quad (8.14)$$

denote the set of order-reversing functions from  $\downarrow V$  to  $\text{sp}(\hat{A})$  obtained in this way. We then have the following, fundamental, result which can be regarded as a type of ‘non-commutative’ spectral theorem in which each bounded, self-adjoint operator  $\hat{A}$  is mapped to an arrow from  $\underline{\Sigma}$  to  $\underline{\mathbb{R}}^{\check{\leq}}$ :

**Theorem 8.1** *The mappings  $\check{\delta}^\circ(\hat{A})_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , are the components of a natural transformation/arrow  $\check{\delta}^\circ(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\text{sp}(\hat{A})}^{\check{\leq}}$ .*

**Proof.** We only have to prove that, whenever  $V' \subset V$ , the diagram

$$\begin{array}{ccc} \underline{\Sigma}_V & \xrightarrow{\check{\delta}^\circ(\hat{A})_V} & \underline{\text{sp}(\hat{A})}_V^{\check{\leq}} \\ \downarrow & & \downarrow \\ \underline{\Sigma}_{V'} & \xrightarrow{\check{\delta}^\circ(\hat{A})_{V'}} & \underline{\text{sp}(\hat{A})}_{V'}^{\check{\leq}} \end{array}$$

commutes. Here, the vertical arrows are the restrictions of the relevant presheaves from the stage  $V$  to  $V' \subseteq V$ .

In fact, the commutativity of the diagram follows directly from the definitions. For each  $\lambda \in \underline{\Sigma}_V$ , the composition of the upper arrow and the right vertical arrow gives

$$(\check{\delta}^\circ(\hat{A})_V(\lambda))|_{V'} = \check{\delta}^\circ(\hat{A})_{V'}(\lambda|_{V'}), \quad (8.15)$$



which is the same function that we get by first restricting  $\lambda$  from  $\underline{\Sigma}_V$  to  $\underline{\Sigma}_{V'}$  and then applying  $\check{\delta}^o(\hat{A})_{V'}$ . ■

In this way, to each physical quantity  $\hat{A}$  in quantum theory there is assigned a natural transformation  $\check{\delta}^o(\hat{A})$  from the state object  $\underline{\Sigma}$  to the presheaf  $\underline{\text{sp}}(\hat{A})^{\check{\succeq}}$ . Since  $\underline{\text{sp}}(\hat{A})^{\check{\succeq}}$  is a sub-object of  $\underline{\mathbb{R}}^{\check{\succeq}}$  for each  $\hat{A}$ ,  $\check{\delta}^o(\hat{A})$  can also be seen as a natural transformation/arrow from  $\underline{\Sigma}$  to  $\underline{\mathbb{R}}^{\check{\succeq}}$ . Hence the presheaf  $\underline{\mathbb{R}}^{\check{\succeq}}$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  is one candidate for the quantity-value object of quantum theory. Note that it follows from Theorem 7.1 that the mapping

$$\theta : B(\mathcal{H})_{\text{sa}} \rightarrow \text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{\Sigma}, \underline{\mathbb{R}}^{\check{\succeq}}) \quad (8.16)$$

$$\hat{A} \mapsto \check{\delta}^o(\hat{A}) \quad (8.17)$$

is injective. <sup>83</sup>

If  $S$  denotes our quantum system, then, on the level of the formal language  $\mathcal{L}(S)$ , we expect the mapping  $A \rightarrow \hat{A}$  to be injective, where  $A$  is a function symbol of signature  $\Sigma \rightarrow \mathcal{R}$ . It follows that we have obtained a faithful representation of these function symbols by arrows  $\check{\delta}^o(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\check{\succeq}}$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

Similarly, there is an order-preserving function

$$\check{\delta}^i(\hat{A})_V(\lambda) : \downarrow V \rightarrow \text{sp}(\hat{A}), \quad (8.18)$$

that is defined for all  $V' \subseteq V$  by

$$\left( \check{\delta}^i(\hat{A})_V(\lambda) \right) (V') = \overline{\delta^i(\hat{A})_{V'}(\underline{\Sigma}(i_{V'V})(\lambda))} \quad (8.19)$$

$$= \langle \lambda, \delta^i(\hat{A})_{V'} \rangle. \quad (8.20)$$

Since  $\delta^i(\hat{A})_{V'}$  becomes smaller (or stays the same) as  $V'$  gets smaller,  $\check{\delta}^i(\hat{A})_V(\lambda)$  indeed is an *order-preserving* function from  $\downarrow V$  to  $\text{sp}(\hat{A})$  for each  $\lambda \in \underline{\Sigma}_V$ . Again, approximation in the spectral order (in this case from below) allows us to define a function with values in  $\text{sp}(\hat{A})$ , which would not be possible when using the linear order.

Clearly, we can use the functions  $\check{\delta}^i(\hat{A})_V(\lambda)$ ,  $\lambda \in \underline{\Sigma}_V$ , to define a natural transformation  $\check{\delta}^i(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\check{\preceq}}$  from the spectral presheaf,  $\underline{\Sigma}$ , to the presheaf

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<sup>83</sup>Interestingly, these results all carry over to an arbitrary von Neumann algebra  $\mathcal{N} \subseteq B(\mathcal{H})$ . In this way, the formalism is flexible enough to adapt to situations where we have symmetries (which can be described mathematically by a von Neumann algebra  $\mathcal{N}$  that has a non-trivial commutant) and super-selection rules (which corresponds to  $\mathcal{N}$  having a non-trivial centre).

$\underline{\mathbb{R}}^{\preceq}$  of real-valued, order-preserving functions on  $\downarrow V$ . The components of  $\check{\delta}^i(\hat{A})$  are

$$\check{\delta}^i(\hat{A})_V : \underline{\Sigma}_V \rightarrow \underline{\text{sp}}(\hat{A})^{\preceq}_V \quad (8.21)$$

$$\lambda \mapsto \check{\delta}^i(\hat{A})_V(\lambda). \quad (8.22)$$

It follows from Theorem 7.1 that the mapping from self-adjoint operators to natural transformations  $\check{\delta}^i(\hat{A})$  is injective.

The functions obtained from inner and outer daseinisation can be combined to give yet another presheaf, and one that will be particularly useful for the physical interpretation of these constructions. The general definition is the following.

**Definition 8.3** *Let  $\mathcal{P}$  be a partially-ordered set. The  $\mathcal{P}$ -valued presheaf,  $\underline{\mathcal{P}}^{\leftrightarrow}$ , of order-preserving and order-reversing functions on  $\mathcal{V}(\mathcal{H})$  is defined as follows:*

(i) *On objects  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ :*

$$\underline{\mathcal{P}}^{\leftrightarrow}_V := \{(\mu, \nu) \mid \mu \in \mathcal{OP}(\downarrow V, \mathcal{P}), \nu \in \mathcal{OR}(\downarrow V, \mathcal{P}), \mu \leq \nu\} \quad (8.23)$$

where  $\downarrow V \subset \text{Ob}(\mathcal{V}(\mathcal{H}))$  is the set of all sub-algebras  $V'$  of  $V$ . Note that we introduce the condition  $\mu \leq \nu$ , i.e., for all  $V' \in \downarrow V$  we demand  $\mu(V') \leq \nu(V')$ .

(ii) *On morphisms  $i_{V'V} : V' \subseteq V$ :*

$$\underline{\mathcal{P}}^{\leftrightarrow}(i_{V'V}) : \underline{\mathcal{P}}^{\leftrightarrow}_V \longrightarrow \underline{\mathcal{P}}^{\leftrightarrow}_{V'} \quad (8.24)$$

$$(\mu, \nu) \longmapsto (\mu|_{V'}, \nu|_{V'}), \quad (8.25)$$

where  $\mu|_{V'}$  denotes the restriction of  $\mu$  to  $\downarrow V' \subseteq \downarrow V$ , and analogously for  $\nu|_{V'}$ .

Note that since we have the condition  $\mu \leq \nu$  in (i), the presheaf  $\underline{\mathcal{P}}^{\leftrightarrow}$  is not simply the product of the presheaves  $\underline{\mathcal{P}}^{\succeq}$  and  $\underline{\mathcal{P}}^{\preceq}$ .

As we will discuss shortly, the presheaf,  $\underline{\mathbb{R}}^{\leftrightarrow}$ , of order-preserving and order-reversing, real-valued functions is closely related to the ‘ $k$ -extension’ of the presheaf  $\underline{\mathbb{R}}^{\succeq}$  (see the Appendix for details of the  $k$ -extension procedure).

Now let

$$\check{\delta}(\hat{A})_V := \left( \check{\delta}^i(\hat{A})_V(\cdot), \check{\delta}^o(\hat{A})_V(\cdot) \right) : \underline{\Sigma}_V \rightarrow \underline{\mathbb{R}}^{\leftrightarrow}_V \quad (8.26)$$

denote the set of all pairs of order-preserving and order-reversing functions from  $\downarrow V$  to  $\mathbb{R}$  that can be obtained from inner and outer daseinisation. It is easy to see that we have the following result:

**Theorem 8.2** *The mappings  $\check{\delta}(\hat{A})_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , are the components of a natural transformation  $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\leftrightarrow}$ .*

Again from Theorem 7.1, the mapping from self-adjoint operators to natural transformations,  $\hat{A} \rightarrow \check{\delta}(\hat{A})$ , is injective.

Since  $\check{\delta}^i(\hat{A})_V(\lambda) \leq \check{\delta}^o(\hat{A})_V(\lambda)$  for all  $\lambda \in \underline{\Sigma}_V$ , we can interpret each pair  $(\check{\delta}^i(\hat{A})_V(\lambda), \check{\delta}^o(\hat{A})_V(\lambda))$  of values as an *interval*, which gives a first hint at the physical interpretation.

### 8.3 Inner and Outer Daseinisation from Functions on Filters

There is a close relationship between inner and outer daseinisation, and certain functions on the filters in the projection lattice  $\mathcal{P}(\mathcal{H})$  of  $B(\mathcal{H})$ . We give a summary of these results here: details can be found in de Groote's work [32, 34], the article [20], and a forthcoming paper [26]. This subsection serves as a preparation for the physical interpretation of the arrows  $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\leftrightarrow}$ .

**Filter bases, filters and ultrafilters.** We first need some basic definitions. Let  $\mathbb{L}$  be a lattice with zero element 0. A subset  $f$  of  $\mathbb{L}$  is called a *filter base* if (i)  $0 \notin f$  and (ii) for all  $a, b \in f$ , there is a  $c \in f$  such that  $c \leq a \wedge b$ .

A subset  $F$  of a lattice  $\mathbb{L}$  with zero element 0 is a (proper) *filter* (or *dual ideal*) if (i)  $0 \notin F$ , (ii)  $a, b \in F$  implies  $a \wedge b \in F$  and (iii)  $a \in F$  and  $b \geq a$  imply  $b \in F$ . In other words, a filter is an upper set in the lattice  $\mathbb{L}$  that is closed under finite minima.

By Zorn's lemma, every filter is contained in a maximal filter. Obviously, such a maximal filter is also a maximal filter base.

Let  $\mathbb{L}'$  be a sublattice of  $\mathbb{L}$  (with common 0), and let  $F'$  be a filter in  $\mathbb{L}'$ . Then  $F'$ , seen as a subset of  $\mathbb{L}$ , is a filter base in  $\mathbb{L}$ . The smallest filter in  $\mathbb{L}$  that contains  $F'$  is the *cone over  $F'$  in  $\mathbb{L}$* :

$$\mathcal{C}_{\mathbb{L}}(F') := \{b \in \mathbb{L} \mid \exists a \in F' : a \leq b\}. \quad (8.27)$$

This is nothing but the upper set  $\uparrow F'$  of  $F'$  in  $\mathbb{L}$ .

In our applications,  $\mathbb{L}$  typically is the lattice  $\mathcal{P}(\mathcal{H})$  of projections in  $B(\mathcal{H})$ , and  $\mathbb{L}'$  is the lattice  $\mathcal{P}(V)$  of projections in an abelian sub-algebra  $V$ .

If  $\mathbb{L}$  is a *Boolean* lattice, i.e., if it is a distributive lattice with minimal element 0 and maximal element 1, and a complement (negation)  $\neg : \mathbb{L} \rightarrow \mathbb{L}$

such that  $a \vee \neg a = 1$  for all  $a \in \mathbb{L}$ , then we define an *ultrafilter*  $\tilde{F}$  to be a maximal filter in  $\mathbb{L}$ . An ultrafilter  $\tilde{F}$  is characterised by the following property: for all  $a \in \mathbb{L}$ , either  $a \in \tilde{F}$  or  $\neg a \in \tilde{F}$ . This can easily be seen: we have  $a \vee \neg a = 1$  by definition. Let us assume that  $\tilde{F}$  is an ultrafilter and  $a \notin \tilde{F}$ . This means that there is some  $b \in \tilde{F}$  such that  $b \wedge a = 0$ . Using distributivity of the lattice  $\mathbb{L}$ , we get

$$b = b \wedge (a \vee \neg a) = (b \wedge a) \vee (b \wedge \neg a) = b \wedge \neg a, \quad (8.28)$$

so  $b \leq \neg a$ . Since  $b \in \tilde{F}$  and  $\tilde{F}$  is a filter, this implies  $\neg a \in \tilde{F}$ . Conversely, if  $\neg a \in \tilde{F}$ , we obtain  $a \in \tilde{F}$ .

The projection lattice  $\mathcal{P}(V)$  of an abelian von Neumann algebra  $V$  is a Boolean lattice. The maximal element is the identity operator  $\hat{1}$  and, as we saw earlier, the complement of a projection is given as  $\neg \hat{\alpha} = \hat{1} - \hat{\alpha}$ . Each ultrafilter  $\tilde{F}$  in  $\mathcal{P}(V)$  hence contains either  $\hat{\alpha}$  or  $\hat{1} - \hat{\alpha}$  for all  $\hat{\alpha} \in \mathcal{P}(V)$ .

**Spectral elements and ultrafilters.** Let  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , and let  $\lambda \in \underline{\Sigma}_V$  be a spectral element of the von Neumann algebra  $V$ . This means that  $\lambda$  is a multiplicative state of  $V$ . For all projections  $\hat{\alpha} \in \mathcal{P}(V)$ , we have

$$\langle \lambda, \hat{\alpha} \rangle = \langle \lambda, \hat{\alpha}^2 \rangle = \langle \lambda, \hat{\alpha} \rangle \langle \lambda, \hat{\alpha} \rangle, \quad (8.29)$$

and so  $\langle \lambda, \hat{\alpha} \rangle \in \{0, 1\}$ . Moreover,  $\langle \lambda, \hat{0} \rangle = 0$ ,  $\langle \lambda, \hat{1} \rangle = 1$ , and if  $\langle \lambda, \hat{\alpha} \rangle = 0$ , then  $\langle \lambda, \hat{1} - \hat{\alpha} \rangle = 1$  (since  $\langle \lambda, \hat{\alpha} \rangle + \langle \lambda, \hat{1} - \hat{\alpha} \rangle = \langle \lambda, \hat{1} \rangle$ ). Hence, for each  $\hat{\alpha} \in \mathcal{P}(V)$  we have either  $\langle \lambda, \hat{\alpha} \rangle = 1$  or  $\langle \lambda, \hat{1} - \hat{\alpha} \rangle = 1$ . This shows that the family

$$F_\lambda := \{\hat{\alpha} \in \mathcal{P}(V) \mid \langle \lambda, \hat{\alpha} \rangle = 1\} \quad (8.30)$$

of projections is an ultrafilter in  $\mathcal{P}(V)$ . Conversely, each  $\lambda \in \underline{\Sigma}_V$  is uniquely determined by the set  $\{\langle \lambda, \hat{\alpha} \rangle \mid \hat{\alpha} \in \mathcal{P}(V)\}$  and hence by an ultrafilter in  $\mathcal{P}(V)$ . This shows that there is a bijection between the set  $\mathcal{Q}(V)$  of ultrafilters in  $\mathcal{P}(V)$  and the Gel'fand spectrum  $\underline{\Sigma}_V$ .

**Observable and antonymous functions.** Let  $\mathcal{N}$  be a von Neumann algebra, and let  $\mathcal{F}(\mathcal{N})$  be the set of filters in the projection lattice  $\mathcal{P}(\mathcal{N})$  of  $\mathcal{N}$ . De Groote has shown [34] that to each self-adjoint operator  $\hat{A} \in \mathcal{N}$ , there corresponds a, so-called, ‘observable function’  $f_{\hat{A}} : \mathcal{F}(\mathcal{N}) \rightarrow \text{sp}(\hat{A})$ . If  $\mathcal{N}$  is abelian,  $\mathcal{N} = V$ , then  $f_{\hat{A}}|_{\mathcal{Q}(V)}$  is just the Gel'fand transform of  $\hat{A}$ . However, it is striking that  $f_{\hat{A}}$  can be defined even if  $\mathcal{N}$  is *non*-abelian; for us, the important example is  $\mathcal{N} = B(\mathcal{H})$ .

If  $\{\hat{E}_\mu^A\}_{\mu \in \mathbb{R}}$  is the spectral family of  $\hat{A}$ , then  $f_{\hat{A}}$  is defined as

$$\begin{aligned} f_{\hat{A}} : \mathcal{F}(\mathcal{N}) &\rightarrow \text{sp}(\hat{A}) \\ F &\mapsto \inf\{\mu \in \mathbb{R} \mid \hat{E}_\mu^A \in F\}. \end{aligned} \quad (8.31)$$

Conversely, given a bounded function  $f : \mathcal{D}(\mathcal{H}) \rightarrow \mathbb{R}$  with certain properties, one can find a unique self-adjoint operator  $\hat{A} \in B(\mathcal{H})$  such that  $f = f_{\hat{A}}$ .

It can be shown that each observable function is completely determined by its restriction to the space of maximal filters [34]. Let  $\mathcal{Q}(\mathcal{N})$  denote the space of maximal filters in  $\mathcal{P}(\mathcal{N})$ . The sets

$$\mathcal{Q}_{\hat{P}}(\mathcal{N}) := \{F \in \mathcal{Q}(\mathcal{N}) \mid \hat{P} \in F\}, \quad \hat{P} \in \mathcal{P}(\mathcal{N}), \quad (8.32)$$

form the base of a totally disconnected topology on  $\mathcal{Q}(\mathcal{N})$ . Following de Groot, this space is called the *Stone spectrum* of  $\mathcal{N}$ . If  $\mathcal{N}$  is abelian,  $\mathcal{N} = V$ , then, upon the identification of maximal filters (which are ultrafilters) in  $\mathcal{P}(V)$  and spectral elements in  $\underline{\Sigma}_V$ , the Stone spectrum  $\mathcal{Q}(V)$  is the Gel'fand spectrum  $\underline{\Sigma}_V$  of  $V$ .

This shows that for an arbitrary von Neumann algebra  $\mathcal{N}$ , the Stone spectrum  $\mathcal{Q}(\mathcal{N})$  is a generalisation of the Gel'fand spectrum (the latter is only defined for abelian algebras). The observable function  $f_{\hat{A}}$  is a generalisation of the Gel'fand transform of  $\hat{A}$ .

We want to show that the observable function  $f_{\delta^\circ(\hat{V})_A}$  of the outer daseinisation of  $\hat{A}$  to  $V$  can be expressed by the observable function  $f_{\hat{A}}$  of  $\hat{A}$  directly. Since this works for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , we obtain a nice encoding of all the functions  $f_{\delta^\circ(\hat{A})_V}$  and hence of the self-adjoint operators  $\delta^\circ(\hat{A})_V$ . The result (already shown in [32]) is that, for all stages  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  and all filters  $F$  in  $\mathcal{F}(V)$ ,

$$f_{\delta^\circ(\hat{A})_V}(F) = f_{\hat{A}}(\mathcal{C}_{B(\mathcal{H})}(F)). \quad (8.33)$$

We want to give an elementary proof of this. We need

**Lemma 8.3** *Let  $\mathcal{N}$  be a von Neumann algebra,  $\mathcal{S}$  a von Neumann sub-algebra of  $\mathcal{N}$ , and let  $\delta_S^i : \mathcal{P}(\mathcal{N}) \rightarrow \mathcal{P}(\mathcal{S})$  be the inner daseinisation map on projections. Then, for all filters  $F \in \mathcal{F}(\mathcal{S})$ ,*

$$(\delta_S^i)^{-1}(F) = \mathcal{C}_{\mathcal{N}}(F). \quad (8.34)$$

**Proof.** If  $\hat{Q} \in F \subset \mathcal{P}(\mathcal{S})$ , then  $(\delta_S^i)^{-1}(\hat{Q}) = \{\hat{P} \in \mathcal{P}(\mathcal{N}) \mid \delta^i(\hat{P})_S = \hat{Q}\}$ . Let  $\hat{P} \in \mathcal{P}(\mathcal{N})$  be such that there is a  $\hat{Q} \in F$  with  $\hat{Q} \leq \hat{P}$ , i.e.,  $\hat{P} \in \mathcal{C}_{\mathcal{N}}(F)$ .

Then  $\delta^i(\hat{P})_{\mathcal{S}} \geq \hat{Q}$ , which implies  $\delta^i(\hat{P})_{\mathcal{S}} \in F$ , since  $F$  is a filter in  $\mathcal{P}(\mathcal{S})$ . This shows that  $\mathcal{C}_{\mathcal{N}}(F) \subseteq (\delta_{\mathcal{S}}^i)^{-1}(F)$ . Now let  $\hat{P} \in \mathcal{P}(\mathcal{N})$  be such that there is no  $\hat{Q} \in F$  with  $\hat{Q} \leq \hat{P}$ . Since  $\delta^i(\hat{P})_{\mathcal{S}} \leq \hat{P}$ , there also is no  $\hat{Q} \in F$  with  $\hat{Q} \leq \delta^i(\hat{P})_{\mathcal{S}}$ , so  $\hat{P} \notin (\delta_{\mathcal{S}}^i)^{-1}(F)$ . This shows that  $(\delta_{\mathcal{S}}^i)^{-1}(F) \subseteq \mathcal{C}_{\mathcal{N}}(F)$ . ■

We now can prove

**Theorem 8.4** *Let  $\hat{A} \in \mathcal{N}_{sa}$ . For all von Neumann sub-algebras  $\mathcal{S} \subseteq \mathcal{N}$  and all filters  $F \in \mathcal{F}(\mathcal{S})$ , we have*

$$f_{\delta^o(\hat{A})_{\mathcal{S}}}(F) = f_{\hat{A}}(\mathcal{C}_{\mathcal{N}}(F)). \quad (8.35)$$

**Proof.** We have

$$\begin{aligned} f_{\delta^o(\hat{A})_{\mathcal{S}}(F)} &= \inf\{\lambda \in \mathbb{R} \mid \hat{E}_{\lambda}^{\delta^o(\hat{A})_{\mathcal{S}}} \in F\} \\ &= \inf\{\lambda \in \mathbb{R} \mid \delta^i(\hat{E}_{\lambda}^A)_{\mathcal{S}} \in F\} \\ &= \inf\{\lambda \in \mathbb{R} \mid \hat{E}_{\lambda}^A \in (\delta_{\mathcal{S}}^i)^{-1}(F)\} \\ &= \inf\{\lambda \in \mathbb{R} \mid \hat{E}_{\lambda}^A \in \mathcal{C}_{\mathcal{N}}(F)\} \\ &= f_{\hat{A}}(\mathcal{C}_{\mathcal{N}}(F)). \end{aligned}$$

The second equality is the definition of outer daseinisation (on the level of spectral projections, see (7.14)). In the penultimate step, we used Lemma 8.3. ■

This clearly implies (8.33). We saw above that to each  $\lambda \in \underline{\Sigma}_V$  there corresponds a unique ultrafilter  $F_{\lambda} \in \mathcal{Q}(V)$ . Since  $\delta^o(\hat{A})_V \in V_{sa}$ , the observable function  $f_{\delta^o(\hat{A})_V}$  is the Gel'fand transform of  $\delta^o(\hat{A})_V$ , and so, upon identifying the ultrafilter  $F_{\lambda}$  with the spectral element  $\lambda$ , we have

$$f_{\delta^o(\hat{A})_V}(F_{\lambda}) = \overline{\delta^o(\hat{A})_V(\lambda)} = \langle \lambda, \delta^o(\hat{A})_V \rangle. \quad (8.36)$$

From (8.33) we have

$$\langle \lambda, \delta^o(\hat{A})_V \rangle = f_{\delta^o(\hat{A})_V}(F_{\lambda}) = f_{\hat{A}}(\mathcal{C}_{B(\mathcal{H})}(F_{\lambda})) \quad (8.37)$$

for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  and for all  $\lambda \in \underline{\Sigma}_V$ . In this sense, the observable function  $f_{\hat{A}}$  encodes all the outer daseinisations  $\delta^o(\hat{A})_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , of  $\hat{A}$ .

There is also a function,  $g_{\hat{A}}$ , on the filters in  $\mathcal{P}(\mathcal{H})$  that encodes all the inner daseinisations  $\delta^i(\hat{A})_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . This function is given for an arbitrary von Neumann algebra  $\mathcal{N}$  by

$$g_{\hat{A}} : \mathcal{F}(\mathcal{N}) \rightarrow \text{sp}(\hat{A}) \quad (8.38)$$

$$F \mapsto \sup\{\lambda \in \mathbb{R} \mid \hat{1} - \hat{E}_\lambda^A \in F\} \quad (8.39)$$

and is called the *antonymous function* of  $\hat{A}$  [20]. If  $\mathcal{N}$  is abelian, then  $g_{\hat{A}}|_{\mathcal{Q}(V)}$  is the Gel'fand transform of  $\hat{A}$  and coincides with  $f_{\hat{A}}$  on the space  $\mathcal{Q}(V)$  of maximal filters, i.e., ultrafilters in  $\mathcal{P}(V)$ . As functions on  $\mathcal{F}(V)$ ,  $f_{\hat{A}}$  and  $g_{\hat{A}}$  are different also in the abelian case. For an arbitrary von Neumann algebra  $\mathcal{N}$ , the antonymous function  $g_{\hat{A}}$  is another generalisation of the Gel'fand transform of  $\hat{A}$ .

There is a close relationship between observable and antonymous functions [34, 20]: for all von Neumann algebras  $\mathcal{N}$  and all self-adjoint operators  $\hat{A} \in \mathcal{N}_{\text{sa}}$ , it holds that

$$-f_{\hat{A}} = g_{-\hat{A}}. \quad (8.40)$$

There is a relation analogous to (8.33) for antonymous functions: for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  and all filters  $F$  in  $\mathcal{F}(V)$ ,

$$g_{\delta^i(\hat{A})_V}(F) = g_{\hat{A}}(\mathcal{C}_{B(\mathcal{H})}(F)). \quad (8.41)$$

This follows from

**Theorem 8.5** *Let  $\hat{A} \in \mathcal{N}_{\text{sa}}$ . For all von Neumann sub-algebras  $\mathcal{S} \subseteq \mathcal{N}$  and all filters  $F \in \mathcal{F}(\mathcal{S})$ , we have*

$$g_{\delta^i(\hat{A})_{\mathcal{S}}}(F) = g_{\hat{A}}(\mathcal{C}_{\mathcal{N}}(F)). \quad (8.42)$$

**Proof.** We have

$$\begin{aligned} g_{\delta^i(\hat{A})_{\mathcal{S}}}(F) &= \sup\{\lambda \in \mathbb{R} \mid \hat{1} - \hat{E}_\lambda^{\delta^i(\hat{A})_{\mathcal{S}}} \in F\} \\ &= \sup\{\lambda \in \mathbb{R} \mid \hat{1} - \bigwedge_{\mu > \lambda} \delta^o(\hat{E}_\mu^A)_{\mathcal{S}} \in F\} \\ &= \sup\{\lambda \in \mathbb{R} \mid \hat{1} - \delta^o(\hat{E}_\lambda^A)_{\mathcal{S}} \in F\} \\ &= \sup\{\lambda \in \mathbb{R} \mid \delta^i(\hat{1} - \hat{E}_\lambda^A)_{\mathcal{S}} \in F\} \\ &= \sup\{\lambda \in \mathbb{R} \mid \hat{1} - \hat{E}_\lambda^A \in (\delta_{\mathcal{S}}^i)^{-1}(F)\} \\ &= \sup\{\lambda \in \mathbb{R} \mid \hat{1} - \hat{E}_\lambda^A \in \mathcal{C}_{\mathcal{N}}(F)\} \\ &= g_{\hat{A}}(\mathcal{C}_{\mathcal{N}}(F)), \end{aligned}$$

where in the penultimate step we used Lemma 8.3. ■

Let  $\lambda \in \underline{\Sigma}_V$ , and let  $F_\lambda \in \mathcal{Q}(V)$  be the corresponding ultrafilter. Since  $\delta^i(\hat{A})_V \in V$ , the anonymous function  $g_{\delta^i(\hat{A})_V}$  is the Gel'fand transform of  $\delta^i(\hat{A})_V$ , and we have

$$g_{\delta^i(\hat{A})_V}(F_\lambda) = \overline{\delta^i(\hat{A})_V(\lambda)} = \langle \lambda, \delta^i(\hat{A})_V \rangle. \quad (8.43)$$

From (8.41), we get

$$\langle \lambda, \delta^i(\hat{A})_V \rangle = g_{\delta^i(\hat{A})_V}(F_\lambda) = g_{\hat{A}}(\mathcal{C}_{B(\mathcal{H})}(F_\lambda)) \quad (8.44)$$

for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  and all  $\lambda \in \underline{\Sigma}_V$ . Thus the anonymous function  $g_{\hat{A}}$  encodes all the inner daseinisations  $\delta^i(\hat{A})_V$ ,  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , of  $\hat{A}$ .

## 8.4 A Physical Interpretation of the Arrow $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$

Let  $|\psi\rangle \in \mathcal{H}$  be a unit vector in the Hilbert space of the quantum system. The expectation value of a self-adjoint operator  $\hat{A} \in B(\mathcal{H})$  in the state  $|\psi\rangle$  is given by

$$\langle \psi | \hat{A} | \psi \rangle = \int_{-\|\hat{A}\|}^{\|\hat{A}\|} \lambda d\langle \psi | \hat{E}_\lambda^{\hat{A}} | \psi \rangle. \quad (8.45)$$

In the discussion of truth objects in section 6, we introduced the maximal filter  $T^{|\psi\rangle}$  in  $\mathcal{P}(\mathcal{H})$ ,<sup>84</sup> given by (cf. (6.31))

$$T^{|\psi\rangle} := \{ \hat{\alpha} \in \mathcal{P}(\mathcal{H}) \mid \hat{\alpha} \succeq |\psi\rangle\langle\psi| \}, \quad (8.46)$$

where  $|\psi\rangle\langle\psi|$  is the projection onto the one-dimensional subspace of  $\mathcal{H}$  generated by  $|\psi\rangle$ . As shown in [20], the expectation value  $\langle \psi | \hat{A} | \psi \rangle$  can be written as

$$\langle \psi | \hat{A} | \psi \rangle = \int_{g_{\hat{A}}(T^{|\psi\rangle})}^{f_{\hat{A}}(T^{|\psi\rangle})} \lambda d\langle \psi | \hat{E}_\lambda^{\hat{A}} | \psi \rangle. \quad (8.47)$$

In an instrumentalist interpretation,<sup>85</sup> one would interpret  $g_{\hat{A}}(T^{|\psi\rangle})$ , resp.  $f_{\hat{A}}(T^{|\psi\rangle})$ , as the smallest, resp. largest, possible result of a measurement of

<sup>84</sup>Since  $\mathcal{P}(\mathcal{H})$  is not distributive,  $T^{|\psi\rangle}$  is not an ultrafilter; i.e., there are projections  $\hat{P} \in \mathcal{P}(\mathcal{H})$  such that neither  $\hat{P} \in T^{|\psi\rangle}$  nor  $\hat{1} - \hat{P} \in T^{|\psi\rangle}$ .

<sup>85</sup>Which we avoid in general, of course!



the physical quantity  $A$  when the state is  $|\psi\rangle$ . If  $|\psi\rangle$  is an eigenstate of  $\hat{A}$ , then  $\langle\psi|\hat{A}|\psi\rangle$  is an eigenvalue of  $\hat{A}$ , and in this case,  $\langle\psi|\hat{A}|\psi\rangle \in \text{sp}(\hat{A})$ ; moreover,

$$\langle\psi|\hat{A}|\psi\rangle = g_{\hat{A}}(T^{|\psi\rangle}) = f_{\hat{A}}(T^{|\psi\rangle}). \quad (8.48)$$

If  $|\psi\rangle$  is not an eigenstate of  $\hat{A}$ , then

$$g_{\hat{A}}(T^{|\psi\rangle}) < \langle\psi|\hat{A}|\psi\rangle < f_{\hat{A}}(T^{|\psi\rangle}). \quad (8.49)$$

For details, see [20].

Let  $V$  be an abelian sub-algebra of  $B(\mathcal{H})$  such that  $\underline{\Sigma}_V$  contains the spectral element,  $\lambda^{|\psi\rangle}$ , associated with  $|\psi\rangle$ .<sup>86</sup> The corresponding ultrafilter in  $\mathcal{P}(V)$  consists of those projections  $\hat{\alpha} \in \mathcal{P}(V)$  such that  $\hat{\alpha} \succeq |\psi\rangle\langle\psi|$ . This is just the evaluation,  $\mathbb{T}_V^{|\psi\rangle}$ , at stage  $V$  of our truth object,  $\mathbb{T}^{|\psi\rangle}$ ; see (6.30).

Hence the cone  $\mathcal{C}(\mathbb{T}_V^{|\psi\rangle}) := \mathcal{C}_{B(\mathcal{H})}(\mathbb{T}_V^{|\psi\rangle})$  consists of all projections  $\hat{R} \in \mathcal{P}(\mathcal{H})$  such that  $\hat{R} \succeq |\psi\rangle\langle\psi|$ ; and so, for all stages  $V$  such that  $|\psi\rangle\langle\psi| \in \mathcal{P}(V)$  we have

$$\mathcal{C}(\mathbb{T}_V^{|\psi\rangle}) = T^{|\psi\rangle}. \quad (8.50)$$

This allows us to write the expectation value as

$$\langle\psi|\hat{A}|\psi\rangle = \int_{g_{\hat{A}}(\mathcal{C}(\mathbb{T}_V^{|\psi\rangle}))}^{f_{\hat{A}}(\mathcal{C}(\mathbb{T}_V^{|\psi\rangle}))} \lambda d\langle\psi|\hat{E}_\lambda^A|\psi\rangle \quad (8.51)$$

$$= \int_{g_{\delta^i(\hat{A})_V}(\mathbb{T}_V^{|\psi\rangle})}^{f_{\delta^o(\hat{A})_V}(\mathbb{T}_V^{|\psi\rangle})} \lambda d\langle\psi|\hat{E}_\lambda^A|\psi\rangle \quad (8.52)$$

for these stages  $V$ .

Equations (8.36) and (8.43) show that  $f_{\delta^o(\hat{A})_V}(\mathbb{T}_V^{|\psi\rangle}) = \langle\psi|\delta^o(\hat{A})_V|\psi\rangle$  and  $g_{\delta^i(\hat{A})_V}(\mathbb{T}_V^{|\psi\rangle}) = \langle\psi|\delta^i(\hat{A})_V|\psi\rangle$ . In the language of instrumentalism, for stages  $V$  for which  $\lambda^{|\psi\rangle} \in \underline{\Sigma}_V$ , the value  $\langle\psi|\delta^i(\hat{A})_V|\psi\rangle \in \text{sp}(\hat{A})$  is the smallest possible measurement result for  $\hat{A}$  in the quantum state  $|\psi\rangle$ ; and  $\langle\psi|\delta^o(\hat{A})_V|\psi\rangle \in \text{sp}(\hat{A})$  is the largest possible result.

These results depend on the fact that we use (inner and outer) daseinisation, i.e., approximations in the spectral, not the linear order.

If  $\lambda \in \underline{\Sigma}_V$  is not of the form  $\lambda = \lambda^{|\psi\rangle}$ , for some  $|\psi\rangle \in \mathcal{H}$ , then the cone  $\mathcal{C}(F_\lambda)$  over the ultrafilter  $F_\lambda$  corresponding to  $\lambda$  cannot be identified with a

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<sup>86</sup>This is the element defined by  $\lambda^{|\psi\rangle}(\hat{A}) := \langle\psi|\hat{A}|\psi\rangle$  for all  $\hat{A} \in V$ . It is characterised by the fact that  $\lambda^{|\psi\rangle}(|\psi\rangle\langle\psi|) = 1$  and  $\lambda^{|\psi\rangle}(\hat{Q}) = 0$  for all  $\hat{Q} \in \mathcal{P}(V)$  such that  $\hat{Q}|\psi\rangle\langle\psi| = \hat{0}$ . We have  $\lambda^{|\psi\rangle} \in \underline{\Sigma}_V$  if and only if  $|\psi\rangle\langle\psi| \in \mathcal{P}(V)$ .

vector in  $\mathcal{H}$ . Nevertheless, the quantity  $\mathcal{C}(F_\lambda)$  is well-defined, and (8.33) and (8.41) hold. If we go from  $V$  to a sub-algebra  $V' \subseteq V$ , then  $\delta^i(\hat{A})_{V'} \preceq \delta^i(\hat{A})_V$  and  $\delta^o(\hat{A})_{V'} \succeq \delta^o(\hat{A})_V$ , hence

$$\langle \lambda, \delta^i(\hat{A})_{V'} \rangle \leq \langle \lambda, \delta^i(\hat{A})_V \rangle, \quad (8.53)$$

$$\langle \lambda, \delta^o(\hat{A})_{V'} \rangle \geq \langle \lambda, \delta^o(\hat{A})_V \rangle \quad (8.54)$$

for all  $\lambda \in \underline{\Sigma}_V$ .

We can interpret the function

$$\check{\delta}(\hat{A})_V : \underline{\Sigma}_V \rightarrow \underline{\mathbb{R}}_{\leftarrow V} \quad (8.55)$$

$$\lambda \mapsto \check{\delta}(\hat{A})_V(\lambda) = \left( \delta^i(\hat{A})_V(\lambda), \delta^o(\hat{A})_V(\lambda) \right) \quad (8.56)$$

as giving the ‘spread’ or ‘range’ of the physical quantity  $A$  at stages  $V' \subseteq V$ . Each element  $\lambda \in \underline{\Sigma}_V$  gives its own ‘spread’  $\check{\delta}(\hat{A})_V(\lambda) : \downarrow V \rightarrow \text{sp}(\hat{A}) \times \text{sp}(\hat{A})$ . The intuitive idea is that at stage  $V$ , given a point  $\lambda \in \underline{\Sigma}_V$ , the physical quantity  $A$  ‘spreads over’ the subset of the spectrum,  $\text{sp}(\hat{A})$ , of  $\hat{A}$  given by the closed interval of  $\text{sp}(\hat{A}) \subset \mathbb{R}$  defined by

$$[\check{\delta}^i(\hat{A})_V(\lambda)(V), \check{\delta}^o(\hat{A})_V(\lambda)(V)] \cap \text{sp}(\hat{A}) = [\langle \lambda, \delta^i(\hat{A})_V \rangle, \langle \lambda, \delta^o(\hat{A})_V \rangle] \cap \text{sp}(\hat{A}), \quad (8.57)$$

For a proper sub-algebra  $V' \subset V$ , the spreading is over the (potentially larger) subset

$$[\check{\delta}^i(\hat{A})_V(\lambda)(V'), \check{\delta}^o(\hat{A})_V(\lambda)(V')] \cap \text{sp}(\hat{A}) = [\langle \lambda, \delta^i(\hat{A})_{V'} \rangle, \langle \lambda, \delta^o(\hat{A})_{V'} \rangle] \cap \text{sp}(\hat{A}). \quad (8.58)$$

All this is local in the sense that these expressions are defined at a stage  $V$  and for sub-algebras,  $V'$ , of  $V$ , where  $\lambda \in \underline{\Sigma}_V$ . No similar global construction or interpretation is possible, since the spectral presheaf  $\underline{\Sigma}$  has no global elements, i.e., no points (while the *set*  $\underline{\Sigma}_V$  does have points).

As we go down to smaller sub-algebras  $V' \subseteq V$ , the spread gets larger. This comes from the fact that  $\hat{A}$  has to be adapted more and more as we go to smaller sub-algebras  $V'$ . More precisely,  $\hat{A}$  is approximated from below by  $\delta^i(\hat{A})_{V'} \in V'$  and from above by  $\delta^o(\hat{A})_{V'} \in V'$ . This approximation gets coarser as  $V'$  gets smaller, which basically means that  $V'$  contains less and less projections.

It should be remarked that  $\check{\delta}(\hat{A})$  does not assign actual values to the physical quantity  $A$ , but rather the possible *range* of such values; and these are independent of any state  $|\psi\rangle$ . This is analogous to the classical case where physical quantities are represented by real-valued functions on state space. The range of possible values is state-independent, but the actual value possessed by a physical quantity *does* depend on the state of the system.

**The quantity-value presheaf  $\mathbb{R}^{\leftrightarrow}$  as the interval domain.** Spitters and Heunen observed [38] that the presheaf  $\mathbb{R}^{\leftrightarrow}$  is the *interval domain* in out topos. This object has mainly been considered in theoretical computer science [28] and can be used to systematically encode situations where real numbers are only known—or can only be defined—up to a certain degree of accuracy. Approximation processes can be well described using the mathematics of domain theory. Clearly, this has close relations to our physical situation, where the real numbers are spectral values of self-adjoint operators and coarse-graining (or rather the inverse process of fine-graining) can be understood as a process of approximation.

## 8.5 The value of a physical quantity in a quantum state

We now want to discuss how physical quantities, represented by natural transformations  $\check{\delta}(\hat{A})$ , acquire ‘values’ in a given quantum state. Of course, this is not as straightforward as in the classical case, since from the Kochen-Specker theorem, we know that physical quantities do not have real numbers as their values. As we saw, this is related to the fact that there are no microstates, i.e., the spectral presheaf has no global elements.

In classical physics, a physical quantity  $A$  is represented by a function  $\check{A} : \mathcal{S} \rightarrow \mathbb{R}$  from the state space  $\mathcal{S}$  to the real numbers. A point  $s \in \mathcal{S}$  is a microstate, and the physical quantity  $A$  has the value  $\check{A}(s)$  in this state.

We want to mimic this as closely as possible in the quantum case. In order to do so, we take a pseudo-state

$$\mathfrak{w}^{|\psi\rangle} := \delta(|\psi\rangle\langle\psi|) = V \mapsto \bigwedge \{\hat{\alpha} \in \underline{Q}_V \mid |\psi\rangle\langle\psi| \preceq \hat{\alpha}\} \quad (8.59)$$

(see (6.34)) and consider it as a sub-object of  $\underline{\Sigma}$ . This means that at each stage  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , we consider the set

$$\mathfrak{w}_V^{|\psi\rangle} := \{\lambda \in \underline{\Sigma}_V \mid \langle \lambda, \delta^o(|\psi\rangle\langle\psi|)_V \rangle = 1\} \subseteq \underline{\Sigma}_V. \quad (8.60)$$

Of course, the sub-object of  $\underline{\Sigma}$  that we get simply is  $\underline{\delta^o}(|\psi\rangle\langle\psi|)$ . Sub-objects of this kind are as close to microstates as we can get, see the discussion in section 6.3 and [25]. We can then form the composition

$$\mathfrak{w}^{|\psi\rangle} \rightarrow \underline{\Sigma} \xrightarrow{\check{\delta}(\hat{A})} \mathbb{R}^{\leftrightarrow}, \quad (8.61)$$

which is also denoted by  $\check{\delta}(\hat{A})(\mathfrak{w}^{|\psi\rangle})$ . One can think of this arrow as being the ‘value’ of the physical quantity  $A$  in the state described by  $\mathfrak{w}^{|\psi\rangle}$ .

The first question is if we actually obtain a sub-object of  $\underline{\mathbb{R}}^{\leftrightarrow}$  in this way. Let  $V, V' \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ ,  $V' \subseteq V$ . We have to show that

$$\underline{\mathbb{R}}^{\leftrightarrow}(i_{V'V})(\check{\delta}(\hat{A})_V(\underline{\mathfrak{w}}_V^{|\psi\rangle})) \subseteq \check{\delta}(\hat{A})_{V'}(\underline{\mathfrak{w}}_{V'}^{|\psi\rangle}). \quad (8.62)$$

Let  $\lambda \in \underline{\mathfrak{w}}_V^{|\psi\rangle}$ , then

$$\underline{\mathbb{R}}^{\leftrightarrow}(i_{V'V})(\check{\delta}(\hat{A})_V(\lambda)) = (\check{\delta}(\hat{A})_V(\lambda))|_{V'} = \check{\delta}(\hat{A})_{V'}(\lambda|_{V'}). \quad (8.63)$$

By definition, we have

$$\underline{\mathfrak{w}}_{V'}^{|\psi\rangle} = \underline{\Sigma}(i_{V'V})(\underline{\mathfrak{w}}_V^{|\psi\rangle}) = \{\lambda|_{V'} \mid \lambda \in \underline{\mathfrak{w}}_V^{|\psi\rangle}\}, \quad (8.64)$$

that is, *every*  $\lambda' \in \underline{\mathfrak{w}}_{V'}^{|\psi\rangle}$  is given as the restriction of some  $\lambda \in \underline{\mathfrak{w}}_V^{|\psi\rangle}$ . This implies that we even obtain the equality

$$\underline{\mathbb{R}}^{\leftrightarrow}(i_{V'V})(\check{\delta}(\hat{A})_V(\underline{\mathfrak{w}}_V^{|\psi\rangle})) = \check{\delta}(\hat{A})_{V'}(\underline{\mathfrak{w}}_{V'}^{|\psi\rangle}), \quad (8.65)$$

so  $\check{\delta}(\hat{A})(\underline{\mathfrak{w}}^{|\psi\rangle})$  is indeed a sub-object of  $\underline{\mathbb{R}}^{\leftrightarrow}$ .

**Values as pairs of functions and eigenvalues.** At each stage  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , we have pairs of order-preserving and order-reversing functions  $\check{\delta}(\hat{A})(\lambda)$ , one function for each  $\lambda \in \underline{\mathfrak{w}}_V^{|\psi\rangle}$ . If  $|\psi\rangle$  is an eigenstate of  $\hat{A}$  and  $V$  is an abelian sub-algebra that contains  $\hat{A}$ , then  $\delta^i(\hat{A})_V = \delta^o(\hat{A})_V = \hat{A}$ . Moreover,  $\underline{\mathfrak{w}}_V^{|\psi\rangle}$  contains the single element  $\lambda_{|\psi\rangle\langle\psi|} \in \underline{\Sigma}_V$ , which is the pure state that assigns 1 to  $|\psi\rangle\langle\psi|$  and 0 to all projections in  $\mathcal{P}(V)$  orthogonal to  $|\psi\rangle\langle\psi|$ .

Evaluating  $\check{\delta}(\hat{A})(\underline{\mathfrak{w}}^{|\psi\rangle})$  at  $V$  hence gives a pair, consisting of an order-preserving function  $\check{\delta}^i(\hat{A})_V(\lambda_{|\psi\rangle\langle\psi|}) : \downarrow V \rightarrow \text{sp}(\hat{A})$  and an order-reversing function  $\check{\delta}^o(\hat{A})_V(\lambda_{|\psi\rangle\langle\psi|}) : \downarrow V \rightarrow \text{sp}(\hat{A})$ :

$$\check{\delta}(\hat{A})_V(\underline{\mathfrak{w}}_V^{|\psi\rangle}) = (\check{\delta}^i(\hat{A})_V(\lambda_{|\psi\rangle\langle\psi|}), \check{\delta}^o(\hat{A})_V(\lambda_{|\psi\rangle\langle\psi|})). \quad (8.66)$$

The value of both functions at stage  $V$  is  $\overline{\hat{A}}(\lambda_{|\psi\rangle\langle\psi|}) = \langle \lambda_{|\psi\rangle\langle\psi|}, \hat{A} \rangle$ , which is the eigenvalue of  $\hat{A}$  in the state  $|\psi\rangle$ . In this sense, we get back the ordinary eigenvalue of  $\hat{A}$  when the system is in the eigenstate  $\psi$ .

**A simple example.** We consider the value of the self-adjoint projection operator  $|\psi\rangle\langle\psi|$ , seen as (the representative of) a physical quantity, in the (pseudo-)state  $\underline{\mathfrak{w}}^{|\psi\rangle}$ . We remark that  $\text{sp}(|\psi\rangle\langle\psi|) = \{0, 1\}$ . By definition,

$$\check{\delta}(|\psi\rangle\langle\psi|)_V(\lambda) = (\check{\delta}^i(|\psi\rangle\langle\psi|)_V(\lambda), \check{\delta}^o(|\psi\rangle\langle\psi|)_V(\lambda)) \quad (8.67)$$

for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  and all  $\lambda \in \underline{\Sigma}_V$ . In particular, the function

$$\check{\delta}^o(|\psi\rangle\langle\psi|)_V(\lambda) : \downarrow V \rightarrow \{0, 1\} \quad (8.68)$$

is given as (see (8.12), for all  $V' \subseteq V$ ,

$$\check{\delta}^o(|\psi\rangle\langle\psi|)_V(\lambda)(V') = \langle \lambda, \delta^o(|\psi\rangle\langle\psi|)_{V'} \rangle. \quad (8.69)$$

If  $\lambda \in \underline{\mathbf{m}}_V^{|\psi\rangle}$ , then  $\langle \lambda, \delta^o(|\psi\rangle\langle\psi|)_V \rangle = 1$ , see (8.60). Hence, for all  $\lambda \in \underline{\mathbf{m}}_V^{|\psi\rangle}$ , we obtain, for all  $V' \subseteq V$ ,

$$\check{\delta}^o(|\psi\rangle\langle\psi|)_V(\lambda)(V') = \langle \lambda, \delta^o(|\psi\rangle\langle\psi|)_{V'} \rangle = 1. \quad (8.70)$$

If we denote the constant function on  $\downarrow V$  with value 1 as  $1_{\downarrow V}$ , then we can write

$$\check{\delta}(|\psi\rangle\langle\psi|)_V(\lambda) = (\check{\delta}^i(|\psi\rangle\langle\psi|)_V(\lambda), 1_{\downarrow V}) \quad (8.71)$$

for all  $V$  and all  $\lambda \in \underline{\mathbf{m}}_V^{|\psi\rangle}$ . The constant function  $1_{\downarrow V}$  trivially is an order-reversing function from  $\downarrow V$  to  $\text{sp}(|\psi\rangle\langle\psi|)$ . We now consider the function

$$\check{\delta}^i(|\psi\rangle\langle\psi|)_V(\lambda) : \downarrow V \rightarrow \{0, 1\}. \quad (8.72)$$

It is given as (see (8.20), for all  $V' \subseteq V$ ,

$$\check{\delta}^i(|\psi\rangle\langle\psi|)_V(\lambda)(V') = \langle \lambda, \delta^i(|\psi\rangle\langle\psi|)_{V'} \rangle. \quad (8.73)$$

If  $|\psi\rangle\langle\psi| \in \mathcal{P}(V')$ , then, for all  $\lambda \in \underline{\mathbf{m}}_{V'}^{|\psi\rangle}$ , we have  $\langle \lambda, \delta^i(|\psi\rangle\langle\psi|)_{V'} \rangle = \langle \lambda, |\psi\rangle\langle\psi| \rangle = 1$ . If  $|\psi\rangle\langle\psi| \notin \mathcal{P}(V')$ , then  $\delta^i(|\psi\rangle\langle\psi|)_{V'} = \hat{0}$ , since  $\delta^i(|\psi\rangle\langle\psi|)_V \preceq |\psi\rangle\langle\psi|$  and  $|\psi\rangle\langle\psi|$  is a projection onto a one-dimensional subspace, so  $\delta^i(|\psi\rangle\langle\psi|)_{V'}$  must project onto the zero-dimensional subspace.

Thus we get, for all  $V$ , for all  $\lambda \in \underline{\mathbf{m}}_V^{|\psi\rangle}$  and all  $V' \subseteq V$ :

$$\check{\delta}^i(|\psi\rangle\langle\psi|)_V(\lambda)(V') = \begin{cases} 1 & \text{if } |\psi\rangle\langle\psi| \in V' \\ 0 & \text{if } |\psi\rangle\langle\psi| \notin V' \end{cases} \quad (8.74)$$

Summing up, we have completely described the ‘value’  $\check{\delta}(|\psi\rangle\langle\psi|)(\underline{\mathbf{m}}^{|\psi\rangle})$  of the physical quantity described by  $|\psi\rangle\langle\psi|$  in the pseudo-state given by  $\underline{\mathbf{m}}^{|\psi\rangle}$ .

There is an immediate generalisation of one part of this result: Consider an arbitrary non-zero projection  $\hat{P} \in \mathcal{V}(\mathcal{H})$ ,<sup>87</sup> the corresponding sub-object  $\underline{\delta}^o(\hat{P})$  of  $\underline{\Sigma}$  obtained from outer daseinisation, and the sub-object  $\check{\delta}(\hat{P})(\underline{\delta}^o(\hat{P}))$

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<sup>87</sup>Of course, if  $\hat{P}$  is not a projection onto a one-dimensional subspace, then it cannot be identified with a state.

of  $\underline{\mathbb{R}}^{\leftarrow}$ . For all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  and all  $\lambda \in \underline{\delta^o(\hat{P})}_V$ , a completely analogous argument to the one given above shows that for the order-reversing functions  $\check{\delta}^o(\hat{P})_V(\lambda) : \downarrow V \rightarrow \{0, 1\}$ , we always obtain the constant function  $1_{\downarrow V}$ .

The behaviour of the order-preserving functions  $\check{\delta}^i(\hat{P})_V(\lambda) : \downarrow V \rightarrow \{0, 1\}$  is more complicated than in the case that  $\hat{P}$  projects onto a one-dimensional subspace. In general,  $\hat{P} \notin V'$  does not imply  $\check{\delta}^i(\hat{P})_{V'}(\lambda)(V') = 0$  for  $\lambda \in \underline{\delta^o(\hat{P})}_V$ , so the analogue of (8.74) does not hold in general.

## 8.6 Properties of $\underline{\mathbb{R}}^{\leftarrow}$ .

From the perspective of our overall programme, Theorem 8.2 is a key result and shows that  $\underline{\mathbb{R}}^{\leftarrow}$  is a possible choice for the quantity-value object for quantum theory. To explore this further, we start by noting some elementary properties of the presheaf  $\underline{\mathbb{R}}^{\leftarrow}$ . Analogous arguments apply to the presheaves  $\underline{\mathbb{R}}^{\leq}$  and  $\underline{\mathbb{R}}^{\geq}$ .

1. The presheaf  $\underline{\mathbb{R}}^{\leftarrow}$  has global elements: namely, pairs of order-preserving and order-reversing functions on the partially-ordered set  $\text{Ob}(\mathcal{V}(\mathcal{H}))$  of objects in the category  $\mathcal{V}(\mathcal{H})$ ; i.e., pairs of functions  $(\mu, \nu) : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \mathbb{R}$  such that:

$$\forall V_1, V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H})), V_2 \subseteq V_1 : \mu(V_2) \leq \mu(V_1), \nu(V_2) \geq \nu(V_1). \quad (8.75)$$

2. (a) Elements of  $\Gamma \underline{\mathbb{R}}^{\leftarrow}$  can be added: i.e., if  $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \Gamma \underline{\mathbb{R}}^{\leftarrow}$ , define  $(\mu_1, \nu_1) + (\mu_2, \nu_2)$  at each stage  $V$  by

$$((\mu_1, \nu_1) + (\mu_2, \nu_2))(V') := (\mu_1(V') + \mu_2(V'), \nu_1(V') + \nu_2(V')) \quad (8.76)$$

for all  $V' \subseteq V$ . Note that if  $V_2 \subseteq V_1 \subseteq V$ , then  $\mu_1(V_2) \leq \mu_1(V_1)$  and  $\mu_2(V_2) \leq \mu_2(V_1)$ , and so  $\mu_1(V_2) + \mu_2(V_2) \leq \mu_1(V_1) + \mu_2(V_1)$ . Likewise,  $\nu_1(V_2) + \nu_2(V_2) \geq \nu_1(V_1) + \nu_2(V_1)$ . Thus the definition of  $(\mu_1, \nu_1) + (\mu_2, \nu_2)$  in (8.76) makes sense. Obviously, addition is commutative and associative.

- (b) However, it is *not* possible to define ‘ $(\mu_1, \nu_1) - (\mu_2, \nu_2)$ ’ in this way since the difference between two order-preserving functions may not be order-preserving, nor need the difference of two order-reversing functions be order-reversing. This problem is addressed in Section 9.
- (c) A zero/unit element can be defined for the additive structure on  $\Gamma \underline{\mathbb{R}}^{\leftarrow}$  as  $0(V) := (0, 0)$  for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , where  $(0, 0)$  de-

notes a pair of two copies of the function that is constantly 0 on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$ .

It follows from (a) and (c) that  $\Gamma\mathbb{R}^{\leftrightarrow}$  is a commutative monoid (i.e., a semi-group with a unit).

The commutative monoid structure for  $\Gamma\mathbb{R}^{\leftrightarrow}$  is a reflection of the stronger fact that  $\mathbb{R}^{\leftrightarrow}$  is a *commutative-monoid* object in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . Specifically, there is an arrow

$$+ : \mathbb{R}^{\leftrightarrow} \times \mathbb{R}^{\leftrightarrow} \rightarrow \mathbb{R}^{\leftrightarrow}, \quad (8.77)$$

$$+_V((\mu_1, \nu_1), (\mu_2, \nu_2)) := (\mu_1 + \mu_2, \nu_1 + \nu_2) \quad (8.78)$$

for all  $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \mathbb{R}^{\leftrightarrow}_V$ , and for all stages  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . Here,  $(\mu_1 + \mu_2, \nu_1 + \nu_2)$  denotes the real-valued function on  $\downarrow V$  defined by

$$(\mu_1 + \mu_2, \nu_1 + \nu_2)(V') := (\mu_1(V') + \mu_2(V'), \nu_1(V') + \nu_2(V')) \quad (8.79)$$

for all  $V' \subseteq V$ .

3. The real numbers,  $\mathbb{R}$ , form a ring, and so it is natural to see if a multiplicative structure can be put on  $\Gamma\mathbb{R}^{\leftrightarrow}$ . The obvious ‘definition’ would be, for all  $V$ ,

$$(\mu_1, \nu_1)(\mu_2, \nu_2)(V) := (\mu_1(V)\mu_2(V), \nu_1(V)\nu_2(V)) \quad (8.80)$$

for  $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \Gamma\mathbb{R}^{\leftrightarrow}$ . However, this fails because the right hand side of (8.80) may not be a pair consisting of an order-preserving and an order-reversing function. This problem arises, for example, if  $\nu_1(V)$  and  $\nu_2(V)$  become negative: then, as  $V$  gets smaller, the product  $\nu_1(V)\nu_2(V)$  gets larger and thus defines an *order-preserving* function.

## 8.7 The Representation of Propositions From Inverse Images

In Section 3.2, we introduced a simple propositional language,  $\mathcal{PL}(S)$ , for each system  $S$ , and discussed its representations for the case of classical physics. Then, in Section 5 we analysed the, far more complicated, quantum-theoretical representation of this language in the set of clopen subsets of the spectral presheaf,  $\underline{\Sigma}$ , in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . This gives a representation of the primitive propositions “ $A \varepsilon \Delta$ ” as sub-objects of  $\underline{\Sigma}$ :

$$\pi_{\text{qt}}(A \varepsilon \Delta) := \underline{\delta^o(\hat{E}[A \in \Delta])} \quad (8.81)$$

where ‘ $\delta^\circ$ ’ is the (outer) daseinisation operation, and  $\hat{E}[A \in \Delta]$  is the spectral projection corresponding to the subset  $\Delta \cap \text{sp}(\hat{A})$  of the spectrum,  $\text{sp}(\hat{A})$ , of the self-adjoint operator  $\hat{A}$ .

We now want to remark briefly on the nature, and representation, of propositions using the ‘local’ language  $\mathcal{L}(S)$ .

In any classical representation,  $\sigma$ , of  $\mathcal{L}(S)$  in **Sets**, the representation,  $\mathcal{R}_\sigma$ , of the quantity-value symbol  $\mathcal{R}$  is always just the real numbers  $\mathbb{R}$ . Therefore, it is simple to take a subset  $\Delta \subseteq \mathbb{R}$  of  $\mathbb{R}$ , and construct the propositions “ $A \varepsilon \Delta$ ”. In fact, if  $A_\sigma : \Sigma_\sigma \rightarrow \mathbb{R}$  is the representation of the function symbol  $A$  with signature  $\Sigma \rightarrow \mathcal{R}$ , then  $A_\sigma^{-1}(\Delta)$  is a subset of the symplectic manifold  $\Sigma_\sigma$  (the representation of the ground type  $\Sigma$ ). This subset,  $A_\sigma^{-1}(\Delta) \subseteq \Sigma_\sigma$ , represents the proposition “ $A \varepsilon \Delta$ ” in the Boolean algebra of all (Borel) subsets of  $\Sigma_\sigma$ .

We should consider the analogue of these steps in the representation,  $\phi$ , of the same language,  $\mathcal{L}(S)$ , in the topos  $\tau_\phi := \mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . In fact, the issues to be discussed apply to a representation in *any* topos.

We first note that if  $\Xi$  is a sub-object of  $\mathcal{R}_\phi$ , and if  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$ , then there is an associated sub-object of  $\Sigma_\phi$ , denoted  $A_\phi^{-1}(\Xi)$ . Specifically, if  $\chi_\Xi : \mathcal{R}_\phi \rightarrow \Omega_{\tau_\phi}$  is the characteristic arrow of the sub-object  $\Xi$ , then  $A_\phi^{-1}(\Xi)$  is defined to be the sub-object of  $\Sigma_\phi$  whose characteristic arrow is  $\chi_\Xi \circ A_\phi : \Sigma_\phi \rightarrow \Omega_{\tau_\phi}$ . These sub-objects are analogues of the subsets,  $A_\sigma^{-1}(\Delta)$ , of the classical state space  $\Sigma_\sigma$ : as such, they can represent propositions. In this spirit, we could denote by “ $A \varepsilon \Xi$ ” the proposition which the sub-object  $A_\phi^{-1}(\Xi)$  represents, although, of course, it would be a mistake to interpret “ $A \varepsilon \Xi$ ” as asserting that the value of something lies in something else: in a general topos, there are no such values.

In the case of quantum theory, the arrows  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$  are of the form  $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^\leftrightarrow}$  where  $\mathcal{R}_\phi := \underline{\mathbb{R}^\leftrightarrow}$ . It follows that the propositions in our  $\mathcal{L}(S)$ -theory are represented by the sub-objects  $\check{\delta}(\hat{A})^{-1}(\underline{\Xi})$  of  $\underline{\Sigma}$ , where  $\underline{\Xi}$  is a sub-object of  $\underline{\mathbb{R}^\leftrightarrow}$ .

To interpret such propositions, note first that in the  $\mathcal{P}\mathcal{L}(S)$ -propositions “ $A \varepsilon \Delta$ ”, the range ‘ $\Delta$ ’ belongs to the world that is external to the language. Consequently, the *meaning* of  $\Delta$  is given independently of  $\mathcal{P}\mathcal{L}(S)$ . This ‘externally interpreted’  $\Delta$  is then inserted into the quantum representation of  $\mathcal{P}\mathcal{L}(S)$  via the daseinisation of propositions discussed in Section 5.

However, the situation is very different for the  $\mathcal{L}(S)$ -propositions “ $A \varepsilon \Xi$ ”. Here, the quantity ‘ $\Xi$ ’ belongs to the particular topos  $\tau_\phi$ , and hence it is representation dependent. The implication is that the ‘meaning’ of “ $A \varepsilon \Xi$ ”



can only be discussed from ‘within the topos’ using the internal language that is associated with  $\tau_\phi$ , which, we recall, carries the translation of  $\mathcal{L}(S)$  given by the topos-representation  $\phi$ .

From a conceptual perspective, this situation is ‘relational’, with the meanings of the various propositions being determined by their relations to each other as formulated in the internal language of the topos. Concomitantly, the meaning of ‘truth’ cannot be understood using the *correspondence theory* (much favoured by instrumentalists) for there is nothing external to which a proposition can ‘correspond’. Instead, what is needed is more like a *coherence* theory of truth in which a whole body of propositions is considered together [30]. This is a fascinating subject, but further discussion must be deferred to later work.

## 8.8 The relation between the formal languages $\mathcal{L}(S)$ and $\mathcal{P}\mathcal{L}(S)$

In the propositional language  $\mathcal{P}\mathcal{L}(S)$ , we have symbols “ $A \varepsilon \Delta$ ” representing primitive propositions. In the quantum case, such a primitive proposition is represented by the outer daseinisation  $\underline{\delta^o(\hat{P})}$  of the projection corresponding to the proposition. (The spectral theorem gives the link between propositions and projections.)

We now want to show that the sub-objects of  $\underline{\Sigma}$  of the form  $\underline{\delta^o(\hat{P})}$  for  $\hat{P} \in \mathcal{P}(\mathcal{H})$  also are part of the language  $\mathcal{L}(S)$ . More precisely, we will show that  $\underline{\delta^o(\hat{P})}$  can be obtained as the inverse image of a certain sub-object of  $\underline{\mathbb{R}^{\leftrightarrow}}$ .

The sub-object of  $\underline{\mathbb{R}^{\leftrightarrow}}$  that we consider is  $\check{\delta}(\hat{P})(\underline{\delta^o(\hat{P})})$ . We take the inverse image of this sub-object under the natural transformation  $\check{\delta}(\hat{P}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$ . This means that we assume that the language  $\mathcal{L}(S)$  contains a function symbol  $P : \Sigma \rightarrow \mathcal{R}$  that is represented by the natural transformation  $\check{\delta}(\hat{P})$ .

One more remark: although it may look as if we put in from the start the sub-object  $\underline{\delta^o(\hat{P})}$  that we want to construct, this is not the case: we can take the inverse of an *arbitrary* sub-object of  $\underline{\mathbb{R}^{\leftrightarrow}}$ , and we happen to choose  $\check{\delta}(\hat{P})(\underline{\delta^o(\hat{P})})$ . Forming the inverse image of a sub-object of  $\underline{\mathbb{R}^{\leftrightarrow}}$  under the natural transformation  $\check{\delta}(\hat{P})$  is analogous to taking the inverse image  $f^{-1}\{r\}$  of some real value  $r$  under some real-valued function  $f$ . The real value  $r$  can be given as the value  $r = f(x)$  of the function at some element  $x$  of its domain. This does not imply that  $f^{-1}\{r\} = \{x\}$ : the inverse image may

contain more elements than just  $\{x\}$ . Likewise, we have to discuss whether the inverse image  $\check{\delta}(\hat{P})^{-1}(\check{\delta}(\hat{P})(\underline{\delta^o(\hat{P})}))$  equals  $\underline{\delta^o(\hat{P})}$  or is some larger sub-object of  $\underline{\Sigma}$ .

We start with the case that  $\hat{P} = |\psi\rangle\langle\psi|$ , i.e.,  $\hat{P}$  is the projection onto a one-dimensional subspace.

**Theorem 8.6** *The inverse image  $\check{\delta}(|\psi\rangle\langle\psi|)^{-1}(\check{\delta}(|\psi\rangle\langle\psi|)(\underline{\mathfrak{m}}^{|\psi\rangle})$  is  $\underline{\mathfrak{m}}^{|\psi\rangle}$ .*

**Proof.** Let  $\underline{\mathcal{S}} := \check{\delta}(|\psi\rangle\langle\psi|)^{-1}(\check{\delta}(|\psi\rangle\langle\psi|)(\underline{\mathfrak{m}}^{|\psi\rangle}))$ . In any case, we have

$$\underline{\mathcal{S}} \supseteq \underline{\mathfrak{m}}^{|\psi\rangle}. \quad (8.82)$$

Let us assume that the inclusion is proper. Then there exists some  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  such that

$$\underline{\mathcal{S}}(V) \supset \underline{\mathfrak{m}}_V^{|\psi\rangle} = \{\lambda \in \underline{\Sigma}_V \mid \langle \lambda, \delta^o(|\psi\rangle\langle\psi|)_V \rangle = 1\}, \quad (8.83)$$

which is equivalent to the existence of some  $\lambda_0 \in \underline{\mathcal{S}}_V$  such that

$$\langle \lambda_0, \delta^o(|\psi\rangle\langle\psi|)_V \rangle = 0. \quad (8.84)$$

By definition, we have  $\underline{\mathcal{S}}(V) = \{\lambda \in \underline{\Sigma}_V \mid \check{\delta}(|\psi\rangle\langle\psi|)_V(\lambda) \in \check{\delta}(|\psi\rangle\langle\psi|)_V(\underline{\mathfrak{m}}^{|\psi\rangle})\}$ . For all  $\tilde{\lambda} \in \underline{\mathfrak{m}}_V^{|\psi\rangle}$ , it holds that

$$\langle \tilde{\lambda}, \delta^o(|\psi\rangle\langle\psi|)_V \rangle = 1, \quad (8.85)$$

see (8.70). This implies that for all  $\lambda \in \underline{\mathcal{S}}(V)$ , we must have  $\langle \lambda, \delta^o(|\psi\rangle\langle\psi|)_V \rangle = 1$ , which contradicts (8.84). Hence there cannot be a proper inclusion  $\underline{\mathcal{S}} \supset \underline{\mathfrak{m}}^{|\psi\rangle}$ , and we rather have equality

$$\underline{\mathcal{S}}(V) = \underline{\mathfrak{m}}_V^{|\psi\rangle} \quad (8.86)$$

for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . ■

The proof is based on the fact that the order-reversing functions of the form  $\check{\delta}^o(|\psi\rangle\langle\psi|)_V(\lambda) : \downarrow V \rightarrow \{0, 1\}$ , where  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  and  $\lambda \in \underline{\mathfrak{m}}_V^{|\psi\rangle}$ , are constant functions  $1_{\downarrow V}$ . The remark at the end of Section 8.5 shows that this holds more generally for arbitrary non-zero projections  $\hat{P}$ . Hence we obtain:

**Corollary 8.7** *The inverse image  $\check{\delta}(\hat{P})^{-1}(\check{\delta}(\hat{P})(\underline{\delta^o(\hat{P})}))$  is  $\underline{\delta^o(\hat{P})}$ .*

## 9 Extending the Quantity-Value Presheaf to an Abelian Group Object

### 9.1 Preliminary Remarks

We have shown how each self-adjoint operator,  $\hat{A}$ , on the Hilbert space  $\mathcal{H}$  gives rise to an arrow  $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . Thus, in the topos representation,  $\phi$ , of  $\mathcal{L}(S)$  for the theory-type ‘quantum theory’, the arrow  $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\leftrightarrow}}$  is one possible choice<sup>88</sup> for the representation,  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$ , of the function symbol,  $A : \Sigma \rightarrow \mathcal{R}$ .

This implies that the quantity-value object,  $\mathcal{R}_\phi$ , is the presheaf,  $\underline{\mathbb{R}^{\leftrightarrow}}$ . However, although such an identification is possible, it does impose certain restrictions on the formalism. These stem from the fact that  $\underline{\mathbb{R}^{\leftrightarrow}}$  is only a *monoid*-object in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , and  $\Gamma \underline{\mathbb{R}^{\leftrightarrow}}$  is only a monoid, whereas the real numbers of standard physics are an abelian group; indeed, they are a commutative ring.

In standard classical physics,  $\text{Hom}_{\mathbf{Sets}}(\Sigma_\sigma, \mathbb{R})$  is the set of real-valued functions on the manifold  $\Sigma_\sigma$ ; as such, it possesses the structure of a commutative ring. On the other hand, the set of arrows  $\text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{\Sigma}, \underline{\mathbb{R}^{\leftrightarrow}})$  has only the structure of an additive monoid. This additive structure is defined locally in the following way. Let  $\alpha, \beta \in \text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{\Sigma}, \underline{\mathbb{R}^{\leftrightarrow}})$ . At each stage  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ ,  $\alpha_V$  is a pair  $(\mu_{1,V}, \nu_{1,V})$ , consisting of a function  $\mu_{1,V}$  from  $\underline{\Sigma}_V$  to  $\underline{\mathbb{R}^{\succeq}_V}$ , and a function  $\nu_{1,V}$  from  $\underline{\Sigma}_V$  to  $\underline{\mathbb{R}^{\preceq}_V}$ . For each  $\lambda \in \underline{\Sigma}_V$ , one has an order-preserving function  $\mu_{1,V}(\lambda) : \downarrow V \rightarrow \mathbb{R}$ , and an order-reversing function  $\nu_{1,V}(\lambda) : \downarrow V \rightarrow \mathbb{R}$ . We use the notation  $\alpha_V(\lambda) := (\mu_{1,V}(\lambda), \nu_{1,V}(\lambda))$ .

Similarly,  $\beta$  is given at each stage  $V$  by a pair of functions  $(\mu_{2,V}, \nu_{2,V})$ , and for all  $\lambda \in \underline{\Sigma}_V$ , we write  $\beta_V(\lambda) := (\mu_{2,V}(\lambda), \nu_{2,V}(\lambda))$

We define, for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , and all  $\lambda \in \underline{\Sigma}_V$ , (c.f. (8.79))

$$(\alpha + \beta)_V(\lambda) = ((\mu_{1,V}(\lambda), \nu_{1,V}(\lambda)) + (\mu_{2,V}(\lambda), \nu_{2,V}(\lambda))) \quad (9.1)$$

$$:= (\mu_{1,V}(\lambda) + \mu_{2,V}(\lambda), \nu_{1,V}(\lambda) + \nu_{2,V}(\lambda)) \quad (9.2)$$

$$= \alpha_V(\lambda) + \beta_V(\lambda), \quad (9.3)$$

It is clear that  $(\alpha + \beta)_V(\lambda)$  is a pair consisting of an order-preserving and an order-reversing function for all  $V$  and all  $\lambda \in \underline{\Sigma}_V$ , so that  $\alpha + \beta$  is well defined.<sup>89</sup>

<sup>88</sup>Another choice is to use the presheaf  $\underline{\mathbb{R}^{\succeq}}$  as the quantity-value object, or the isomorphic presheaf  $\underline{\mathbb{R}^{\preceq}}$ .

<sup>89</sup>To avoid confusion we should emphasise that, in general, the sum  $\delta(\hat{A}) + \delta(\hat{B})$  is *not* equal to  $\delta(\hat{A} + \hat{B})$ .

Arguably, the fact that  $\text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{\Sigma}, \underline{\mathbb{R}^{\leftrightarrow}})$  is only a monoid<sup>90</sup> is a weakness in so far as we are trying to make quantum theory ‘look’ as much like classical physics as possible. Of course, in more obscure applications such as Planck-length quantum gravity, the nature of the quantity-value object is very much open for debate. But when applied to regular physics, we might like our formalism to look more like classical physics than the monoid-only structure of  $\text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}}(\underline{\Sigma}, \underline{\mathbb{R}^{\leftrightarrow}})$ .

The need for a subtraction, i.e. some sort of abelian group structure on  $\underline{\mathbb{R}^{\leftrightarrow}}$ , brings to mind the well-known Grothendieck  $k$ -construction that is much used in algebraic topology and other branches of pure mathematics. This gives a way of ‘extending’ an abelian semi-group to become an abelian group, and this technique can be adapted to the present situation. The goal is to construct a ‘Grothendieck completion’,  $k(\underline{\mathbb{R}^{\leftrightarrow}})$ , of  $\underline{\mathbb{R}^{\leftrightarrow}}$  that is an abelian-group object in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .<sup>91</sup>

Of course, we can apply the  $k$ -construction also to the presheaf  $\underline{\mathbb{R}^{\succeq}}$  (or  $\underline{\mathbb{R}^{\preceq}}$ , if we like). This comes with an extra advantage: it is then possible to define the square of an arrow  $\check{\delta}^o(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}^{\succeq}}$ , as is shown in the Appendix. Hence, given arrows  $\check{\delta}^o(\hat{A})$  and  $\check{\delta}^o(\hat{A}^2)$ , we can define an ‘intrinsic dispersion’:<sup>92</sup>

$$\nabla(\hat{A}) := \check{\delta}^o(\hat{A}^2) - \check{\delta}^o(\hat{A})^2. \quad (9.4)$$

Since the whole  $k$ -construction is quite complicated and is not used in this paper beyond the present section, we have decided to put all the relevant definitions into the Appendix where it can be read at leisure by anyone who is interested.

Interestingly, there is a close relation between  $\underline{\mathbb{R}^{\leftrightarrow}}$  and  $k(\underline{\mathbb{R}^{\succeq}})$ , as shown in the next subsection.

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<sup>90</sup>An internal version of this result would show that the exponential object  $\underline{\mathbb{R}^{\leftrightarrow}}^{\underline{\Sigma}}$  is a monoid object in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . This could well be true, but we have not studied it in detail.

<sup>91</sup>Ideally, we might like  $k(\underline{\mathbb{R}^{\succeq}})$  or  $k(\underline{\mathbb{R}^{\preceq}})$  to be a commutative-ring object, but this is not true.

<sup>92</sup>The notation used here is potentially a little misleading. We have not given any meaning to ‘ $A^2$ ’ in the language  $\mathcal{L}(S)$ ; i.e., in its current form, the language does not give meaning to the square of a function symbol. Therefore, when we write  $\check{\delta}^o(\hat{A}^2)$  this must be understood as being the Gel’fand transform of the outer daseinisation of the operator  $\hat{A}^2$ .

## 9.2 The relation between $\underline{\mathbb{R}}^{\leftrightarrow}$ and $k(\underline{\mathbb{R}}^{\rightleftharpoons})$ .

In Section 8.2, we considered the presheaf  $\underline{\mathbb{R}}^{\leftrightarrow}$  of order-preserving and order-reversing functions as a possible quantity-value object. The advantage of this presheaf is the symmetric utilisation of inner and outer daseinisation, and the associated physical interpretation of arrows from  $\underline{\Sigma}$  to  $\underline{\mathbb{R}}^{\leftrightarrow}$ .

It transpires that  $\underline{\mathbb{R}}^{\leftrightarrow}$  is closely related to  $k(\underline{\mathbb{R}}^{\rightleftharpoons})$ . Namely, for each  $V$ , we can define an equivalence relation  $\equiv$  on  $\underline{\mathbb{R}}^{\leftrightarrow}_V$  by

$$(\mu_1, \nu_1) \equiv (\mu_2, \nu_2) \text{ iff } \mu_1 + \nu_1 = \mu_2 + \nu_2. \quad (9.5)$$

Then  $\underline{\mathbb{R}}^{\leftrightarrow}/\equiv$  is isomorphic to  $k(\underline{\mathbb{R}}^{\rightleftharpoons})$  under the mapping

$$[\mu, \nu] \mapsto [\nu, -\mu] \in k(\underline{\mathbb{R}}^{\rightleftharpoons})_V \quad (9.6)$$

for all  $V$  and all  $[\mu, \nu] \in (\underline{\mathbb{R}}^{\leftrightarrow}/\equiv)_V$ .<sup>93</sup>

However, there is a difference between the arrows that represent physical quantities. The arrow  $[\check{\delta}^o(\hat{A})] : \underline{\Sigma} \rightarrow k(\underline{\mathbb{R}}^{\rightleftharpoons})$  is given by first sending  $\hat{A} \in B(\mathcal{H})_{\text{sa}}$  to  $\check{\delta}^o(\hat{A})$  and then taking  $k$ -equivalence classes—a construction that only involves outer daseinisation. On the other hand, there is an arrow  $[\check{\delta}(\hat{A})] : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\leftrightarrow}/\equiv$ , given by first sending  $\hat{A}$  to  $\check{\delta}(\hat{A})$  and then taking the equivalence classes defined in (9.5). This involves both inner and outer daseinisation.

We can show that  $[\check{\delta}^o(\hat{A})]$  uniquely determines  $\hat{A}$  as follows: Let

$$[\check{\delta}^o(\hat{A})] : \underline{\Sigma} \rightarrow k(\underline{\mathbb{R}}^{\rightleftharpoons}) \quad (9.7)$$

denote the natural transformation from the spectral presheaf to the abelian group-object  $k(\underline{\mathbb{R}}^{\rightleftharpoons})$ , given by first sending  $\hat{A}$  to  $\check{\delta}^o(\hat{A})$  and then taking the  $k$ -equivalence classes at each stage  $V$ . The monoid  $\underline{\mathbb{R}}^{\rightleftharpoons}$  is embedded into  $k(\underline{\mathbb{R}}^{\rightleftharpoons})$  by sending  $\nu \in \underline{\mathbb{R}}^{\rightleftharpoons}_V$  to  $[\nu, 0] \in k(\underline{\mathbb{R}}^{\rightleftharpoons})_V$  for all  $V$ , which implies that  $\hat{A}$  is also uniquely determined by  $[\check{\delta}^o(\hat{A})]$ .<sup>94</sup> We note that, currently, it is an open question if  $[\check{\delta}(\hat{A})]$  also fixes  $\hat{A}$  uniquely.

We now have constructed several presheaves that are abelian group objects within  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , namely  $k(\underline{\mathbb{R}}^{\leftrightarrow})$ ,  $k(\underline{\mathbb{R}}^{\rightleftharpoons})$  and  $\underline{\mathbb{R}}^{\leftrightarrow}/\equiv$ . The latter two are isomorphic presheaves, as we have shown. All three presheaves can serve

<sup>93</sup>This identification also explains formula (9.5), which may look odd at first sight. Recall that  $[\mu, \nu] \in (\underline{\mathbb{R}}^{\leftrightarrow}/\equiv)_V$  means that  $\mu$  is order-preserving and  $\nu$  is order-reversing.

<sup>94</sup>In an analogous manner, one can show that the arrows  $\check{\delta}^i(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\rightleftharpoons}$  and  $[\check{\delta}^i(\hat{A})] : \underline{\Sigma} \rightarrow k(\underline{\mathbb{R}}^{\rightleftharpoons})$  uniquely determine  $\hat{A}$ , and that the arrow  $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\leftrightarrow}$  also uniquely determines  $\hat{A}$ .

as the quantity-value presheaf if one wants to have an abelian-group object for this purpose. Intuitively, if the quantity-value object is only an abelian-monoid object like  $\underline{\mathbb{R}}^{\leftarrow}$ , then the ‘values’ can only be added, while in the case of an abelian-group object, they can be added and subtracted.

### 9.3 Algebraic properties of the potential quantity-value presheaves

As matters stand, we have several possible choices for the quantity-value presheaf, which is the representation for quantum theory of the symbol  $\mathcal{R}$  of the formal language  $\mathcal{L}(S)$  that describes our physical system. In this sub-section, we want to compare the algebraic properties of these various presheaves. In particular, we will consider the presheaves  $\underline{\mathbb{R}}^{\succeq}$ ,  $k(\underline{\mathbb{R}}^{\succeq})$ ,  $\underline{\mathbb{R}}^{\leftarrow}$  and  $k(\underline{\mathbb{R}}^{\leftarrow})$ .<sup>95</sup>

**Global elements.** We first note that all these presheaves have global elements. For example, a global element of  $\underline{\mathbb{R}}^{\succeq}$  is given by an order-reversing function  $\nu : \mathcal{V}(\mathcal{H}) \rightarrow \mathbb{R}$ . As remarked in the Appendix, we have  $\Gamma k(\underline{\mathbb{R}}^{\succeq}) \simeq k(\Gamma \underline{\mathbb{R}}^{\succeq})$ . Global elements of  $\underline{\mathbb{R}}^{\leftarrow}$  are pairs  $(\mu, \nu)$  consisting of an order-preserving function  $\mu : \mathcal{V}(\mathcal{H}) \rightarrow \mathbb{R}$  and an order-reversing function  $\nu : \mathcal{V}(\mathcal{H}) \rightarrow \mathbb{R}$ . Finally, it is easy to show that  $\Gamma k(\underline{\mathbb{R}}^{\leftarrow}) \simeq k(\Gamma \underline{\mathbb{R}}^{\leftarrow})$ .

**The real number object as a sub-object.** In a presheaf topos  $\mathbf{Sets}^{\mathcal{C}^{op}}$ , the Dedekind real number object  $\underline{\mathbb{R}}$  is the constant functor from  $\mathcal{C}^{op}$  to  $\mathbb{R}$ . The presheaf  $\underline{\mathbb{R}}$  is an internal field object (see e.g. [48]).

The presheaf  $\underline{\mathbb{R}}^{\succeq}$  contains the constant presheaf  $\underline{\mathbb{R}}$  as a sub-object: let  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  and  $r \in \underline{\mathbb{R}}_V \simeq \mathbb{R}$ . Then the function  $c_{r,V} : \downarrow V \rightarrow \mathbb{R}$  that has the constant value  $r$  is an element of  $\underline{\mathbb{R}}^{\succeq}_V$  since it is an order-reversing function. Moreover, the global sections of  $\underline{\mathbb{R}}$  are given by constant functions  $r : \mathcal{V}(\mathcal{H}) \rightarrow \mathbb{R}$ , and such functions are also global sections of  $\underline{\mathbb{R}}^{\succeq}$ .

The presheaf  $\underline{\mathbb{R}}^{\succeq}$  can be seen as a sub-object of  $k(\underline{\mathbb{R}}^{\succeq})$ : let  $V \in \mathcal{V}(\mathcal{H})$  and  $\nu \in \underline{\mathbb{R}}^{\succeq}_V$ , then  $[\nu, 0] \in k(\underline{\mathbb{R}}^{\succeq})_V$ . Thus the real number object  $\underline{\mathbb{R}}$  is also a sub-object of  $k(\underline{\mathbb{R}}^{\succeq})$ .

A real number  $r \in \underline{\mathbb{R}}_V$  defines the pair  $(c_{r,V}, c_{r,V})$  consisting of two copies of the constant function  $c_{r,V} : \downarrow V \rightarrow \mathbb{R}$ . Since  $c_{r,V}$  is both order-preserving and order-reversing,  $(c_{r,V}, c_{r,V})$  is an element of  $\underline{\mathbb{R}}^{\leftarrow}_V$  and hence  $\underline{\mathbb{R}}$  is a sub-

<sup>95</sup>The presheaf  $\underline{\mathbb{R}}^{\preceq}$  is isomorphic to  $\underline{\mathbb{R}}^{\succeq}$  and hence will not be considered separately.

object of  $\underline{\mathbb{R}}^{\leftrightarrow}$ . Since  $\underline{\mathbb{R}}^{\leftrightarrow}$  is a sub-object of  $k(\underline{\mathbb{R}}^{\leftrightarrow})$ , the latter presheaf also contains the real number object  $\underline{\mathbb{R}}$  as a sub-object.

**Multiplying with real numbers and vector space structure.** Let  $c_r \in \Gamma \underline{\mathbb{R}}$  be the constant function on  $\mathcal{V}(\mathcal{H})$  with value  $r$ . The global element  $c_r$  of  $\underline{\mathbb{R}}$  defines locally, at each  $V \in \mathcal{V}(\mathcal{H})$ , a constant function  $c_{r,V} : \downarrow V \rightarrow \underline{\mathbb{R}}$ . We want to consider if, and how, multiplication with these constant functions is defined in the various presheaves. We will call this ‘multiplying with a real number’.

Let  $\mu \in \underline{\mathbb{R}}^{\succeq}_V$ . For all  $V' \in \downarrow V$ , we define the product

$$(c_{r,V}\mu)(V') := c_{r,V}(V')\mu(V') = r\mu(V'). \quad (9.8)$$

If  $r \geq 0$ , then  $r\mu : \downarrow V \rightarrow \underline{\mathbb{R}}$  is an order-reversing function again. However, if  $r < 0$ , then  $r\mu$  is order-preserving and hence not an element of  $\underline{\mathbb{R}}^{\succeq}_V$ . This shows that for the presheaf  $\underline{\mathbb{R}}^{\succeq}$  only multiplication by non-negative real numbers is well-defined.

However, if we consider  $k(\underline{\mathbb{R}}^{\succeq})$  then multiplication with an arbitrary real number is well-defined. For simplicity, we first consider  $r = -1$ , i.e., negation. Let  $[\nu, \kappa] \in \underline{\mathbb{R}}^{\succeq}_V$ , then, for all  $V' \in \downarrow V$ ,

$$(c_{-1,V}[\nu, \kappa])(V') := c_{-1,V}(V')[\nu(V'), \kappa(V')] \quad (9.9)$$

$$= -[\nu(V'), \kappa(V')] \quad (9.10)$$

$$= [\kappa(V'), \nu(V')], \quad (9.11)$$

so we have

$$c_{-1,V}[\nu, \kappa] = -[\nu, \kappa] = [\kappa, \nu]. \quad (9.12)$$

This multiplication with the real number  $-1$  is, of course, defined in such a way that it fits in with the additive group structure on  $k(\underline{\mathbb{R}}^{\succeq})$ .

It follows that multiplying an element  $[\nu, \kappa]$  of  $k(\underline{\mathbb{R}}^{\succeq})_V$  with an arbitrary real number  $r$  can be defined as

$$c_{r,V}[\nu, \kappa] := \begin{cases} [c_{r,V}\nu, c_{r,V}\kappa] = [r\nu, r\kappa] & \text{if } r \geq 0 \\ -[c_{-r,V}\nu, c_{-r,V}\kappa] = [-r\kappa, -r\nu] & \text{if } r < 0. \end{cases} \quad (9.13)$$

**Remark 9.1** *In this way, the group object  $k(\underline{\mathbb{R}}^{\succeq})$  in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  becomes a vector space object, with the field object  $\underline{\mathbb{R}}$  as the scalars.*

Interestingly, one can define multiplication with arbitrary real numbers also for  $\underline{\mathbb{R}}^{\leftrightarrow}$ , although this presheaf is not a group object in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . Let

$(\mu, \nu) \in \underline{\mathbb{R}}^{\leftrightarrow}_V$ , so that  $\mu : \downarrow V \rightarrow \mathbb{R}$  is an order-preserving function and  $\nu : \downarrow V \rightarrow \mathbb{R}$  is order-reversing. Let  $r$  be an arbitrary real number. We define

$$c_{r,V}(\mu, \nu) := \begin{cases} (c_{r,V}\mu, c_{r,V}\nu) = (r\mu, r\nu) & \text{if } r \geq 0 \\ (c_{r,V}\nu, c_{r,V}\mu) = (r\nu, r\mu) & \text{if } r < 0. \end{cases} \quad (9.14)$$

This is well-defined since if  $\mu$  is order-preserving, then  $-\mu$  is order-reversing, and if  $\nu$  is order-reversing, then  $-\nu$  is order-preserving. For  $r = -1$ , we obtain

$$c_{-1,V}(\mu, \nu) = -(\mu, \nu) = (-\nu, -\mu). \quad (9.15)$$

But this does *not* mean that  $-(\mu, \nu)$  is an additive inverse of  $(\mu, \nu)$ . Such inverses do not exist in  $\underline{\mathbb{R}}^{\leftrightarrow}$ , since it is not a group object. Rather, we get

$$(\mu, \nu) + (-(\mu, \nu)) = (\mu, \nu) + (-\nu, -\mu) = (\mu - \nu, \nu - \mu). \quad (9.16)$$

If, for all  $V' \in \downarrow V$ , we interpret the absolute value  $|(\mu - \nu)(V')|$  as a measure of uncertainty as given by the pair  $(\mu, \nu)$  at stage  $V'$ , then we see from (9.16) that adding  $(\mu, \nu)$  and  $(-\nu, -\mu)$  gives a pair  $(\mu - \nu, \nu - \mu) \in \underline{\mathbb{R}}^{\leftrightarrow}_V$  concentrated around  $(0, 0)$ , but with an uncertainty twice as large (for all stages  $V'$ ). We call  $-(\mu, \nu) = (-\nu, -\mu)$  the *pseudo-inverse* of  $(\mu, \nu) \in \underline{\mathbb{R}}^{\leftrightarrow}$ .

More generally, we can define a second monoid structure (besides addition) on  $\underline{\mathbb{R}}^{\leftrightarrow}$ , called *pseudo-subtraction* and given by

$$(\mu_1, \nu_1) - (\mu_2, \nu_2) := (\mu_1, \nu_1) + (-\nu_2, -\mu_2) = (\mu_1 - \nu_2, \nu_1 - \mu_2). \quad (9.17)$$

This operation has a neutral element, namely  $(c_{0,V}, c_{0,V})$ , for all stages  $V \in \mathcal{V}(\mathcal{H})$ , which of course is also the neutral element for addition. In this sense,  $\underline{\mathbb{R}}^{\leftrightarrow}$  is close to being a group object. Taking equivalence classes as described in (9.5) makes  $\underline{\mathbb{R}}^{\leftrightarrow}$  into a group object,  $\underline{\mathbb{R}}^{\leftrightarrow}/\cong$ , isomorphic to  $k(\underline{\mathbb{R}}^{\pm})$ .

Since multiplication with arbitrary real numbers is well-defined, the presheaf  $\underline{\mathbb{R}}^{\leftrightarrow}$  is ‘almost a vector space object’ over  $\underline{\mathbb{R}}$ .

Elements of  $k(\underline{\mathbb{R}}^{\leftrightarrow})_V$  are of the form  $[(\mu_1, \nu_1), (\mu_2, \nu_2)]$ . Multiplication with an arbitrary real number  $r$  is defined in the following way:

$$c_{r,V}[(\mu_1, \nu_1), (\mu_2, \nu_2)] := \begin{cases} [(r\mu_1, r\nu_1), (r\mu_2, r\nu_2)] & \text{if } r \geq 0 \\ [(-r\mu_2, -r\nu_2), (-r\mu_1, -r\nu_1)] & \text{if } r < 0. \end{cases} \quad (9.18)$$

The additive group structure on  $k(\underline{\mathbb{R}}^{\leftrightarrow})$  implies

$$-[(\mu_1, \nu_1), (\mu_2, \nu_2)] = [(\mu_2, \nu_2), (\mu_1, \nu_1)], \quad (9.19)$$

so the multiplication with the real number  $-1$  fits in with the group structure. On the other hand, this negation is completely different from the negation on  $\underline{\mathbb{R}}^{\leftrightarrow}$  (where  $-(\mu, \nu) = (-\nu, -\mu)$  for all  $(\mu, \nu) \in \underline{\mathbb{R}}^{\leftrightarrow}_V$  and all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ ).



**Remark 9.2** *The presheaf  $k(\underline{\mathbb{R}}^{\leftarrow})$  is a vector space object in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , with  $\underline{\mathbb{R}}$  as the scalars.*

## 10 The Role of Unitary Operators

### 10.1 The Daseinisation of Unitary Operators

Unitary operators play an important role in the formulation of quantum theory, and we need to understand the analogue of this in our topos formalism.

Unitary operators arise in the context of both ‘covariance’ and ‘invariance’. In elementary quantum theory, the ‘covariance’ aspect comes the fact that if we have made the associations

$$\text{Physical state} \mapsto \text{state vector } |\psi\rangle \in \mathcal{H}$$

$$\text{Physical observable } A \mapsto \text{self-adjoint operator } \hat{A} \text{ acting on } \mathcal{H}$$

then the same physical predictions will be obtained if the following associations are used instead

$$\text{Physical state} \mapsto \text{state vector } \hat{U}|\psi\rangle \in \mathcal{H} \tag{10.1}$$

$$\text{Physical observable } A \mapsto \text{self-adjoint operator } \hat{U}\hat{A}\hat{U}^{-1} \text{ acting on } \mathcal{H}$$

for any unitary operator  $\hat{U}$ . Thus the mathematical representatives of physical quantities are defined only up to arbitrary transformations of the type above. In non-relativistic quantum theory, this leads to the canonical commutation relations; the angular-momentum commutator algebra; and the unitary time displacement operator. Similar considerations in relativistic quantum theory involve the Poincaré group.

The ‘invariance’ aspect of unitary operators arises when the operator commutes with the Hamiltonian, giving rise to conserved quantities.

**Daseinisation of unitary operators.** As a side remark, we first consider the question if daseinisation can be applied to a unitary operator  $\hat{U}$ . The answer is clearly ‘yes’, via the spectral representation:

$$\hat{U} = \int_{\mathbb{R}} e^{i\lambda} d\hat{E}_{\lambda}^{\hat{U}} \tag{10.2}$$

where  $\lambda \mapsto E_{\lambda}^{\hat{U}}$  is the spectral family for  $\hat{U}$ . Then, in analogy with (7.15–7.16) we have the following:

**Definition 10.1** *The outer daseinisation,  $\delta^o(\hat{U})$ , resp. the inner daseinisation,  $\delta^i(\hat{U})$ , of a unitary operator  $\hat{U}$  are defined as follows:*

$$\delta^o(\hat{U})_V := \int_{\mathbb{R}} e^{i\lambda} d(\delta_V^i(\hat{E}_\lambda^U)), \quad (10.3)$$

$$\delta^i(\hat{U})_V := \int_{\mathbb{R}} e^{i\lambda} d\left(\bigwedge_{\mu>\lambda} \delta_V^o(\hat{E}_\mu^U)\right), \quad (10.4)$$

at each stage  $V$ .

To interpret these entities<sup>96</sup> we need to introduce a new presheaf defined as follows.

**Definition 10.2** *The outer, unitary de Groote presheaf,  $\underline{\mathbb{U}}$ , is defined by:*

(i) *On objects  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ :  $\underline{\mathbb{U}}_V := V_{\text{un}}$ , the collection of unitary operators in  $V$ .*

(ii) *On morphisms  $i_{V'V} : V' \subseteq V$ : The mapping  $\underline{\mathbb{U}}(i_{V'V}) : \underline{\mathbb{U}}_V \rightarrow \underline{\mathbb{U}}_{V'}$  is given by*

$$\underline{\mathbb{U}}(i_{V'V})(\hat{\alpha}) := \delta^o(\hat{\alpha})_{V'} \quad (10.5)$$

$$= \int_{\mathbb{R}} e^{i\lambda} d(\delta^i(\hat{E}_\lambda^\alpha)_{V'}) \quad (10.6)$$

$$= \int_{\mathbb{R}} e^{i\lambda} d(\underline{\mathbb{I}}(i_{V'V})(\hat{E}_\lambda^\alpha)) \quad (10.7)$$

for all  $\hat{\alpha} \in \underline{\mathbb{U}}_V$ .

Clearly, (i) there is an analogous definition of an ‘inner’, unitary de Groote presheaf; and (ii) the map  $V \mapsto \delta^o(\hat{U})_V$  defines a global element of  $\underline{\mathbb{U}}$ .

This definition has the interesting consequence that, at each stage  $V$ ,

$$\delta^o(e^{i\hat{A}})_V = e^{i\delta^o(\hat{A})_V} \quad (10.8)$$

A particular example of this construction is the one-parameter family of unitary operators,  $t \mapsto e^{it\hat{H}}$ , where  $\hat{H}$  is the Hamiltonian of the system.

Of course, in our case everything commutes. Thus suppose  $g \mapsto \hat{U}_g$  is a representation of a Lie group  $G$  on the Hilbert space  $\mathcal{H}$ . Then these operators

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<sup>96</sup>It would be possible to ‘complexify’ the presheaf  $k(\mathbb{R}^\succeq)$  in order to represent unitary operators as arrows from  $\underline{\Sigma}$  to  $\mathbb{C}k(\mathbb{R}^\succeq)$ . Similar remarks apply to the presheaf  $\underline{\mathbb{R}}^{\leftrightarrow}$ . However, there is no obvious physical use for this procedure.

can be daseinised to give the map  $g \mapsto \delta^\circ(\hat{U}_g)$ , but generally this is not a representation of  $G$  (or of its Lie algebra) since, at each stage  $V$  we have

$$\delta^\circ(\hat{U}_{g_1})_V \delta^\circ(\hat{U}_{g_2})_V = \delta^\circ(\hat{U}_{g_2})_V \delta^\circ(\hat{U}_{g_1})_V \quad (10.9)$$

for all  $g_1, g_2 \in G$ . Clearly, there is an analogous result for inner daseinisation.

## 10.2 Unitary Operators and Arrows in $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

### 10.2.1 The Definition of $\ell_{\hat{U}} : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \text{Ob}(\mathcal{V}(\mathcal{H}))$

In classical physics, the analogue of unitary operators are ‘canonical transformations’; i.e., symplectic diffeomorphisms from the state space  $\mathcal{S}$  to itself. This suggests that should try to associate arrows in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  with each unitary operator  $\hat{U}$ .

Thus we want to see if unitary operators can act on the objects in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . In fact, if  $\mathcal{U}(\mathcal{H})$  denotes the group of all unitary operators in  $\mathcal{H}$ , we would like to find a *realisation* of  $\mathcal{U}(\mathcal{H})$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

As a first step, if  $\hat{U} \in \mathcal{U}(\mathcal{H})$  and  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  is an abelian von Neumann sub-algebra of  $B(\mathcal{H})$ , let us define

$$\ell_{\hat{U}}(V) := \{\hat{U}\hat{A}\hat{U}^{-1} \mid \hat{A} \in V\}. \quad (10.10)$$

It is clear that  $\ell_{\hat{U}}(V)$  is a unital, abelian algebra of operators, and that self-adjoint operators are mapped into self-adjoint operators. Furthermore, the map  $\hat{A} \mapsto \hat{U}\hat{A}\hat{U}^{-1}$  is continuous in the weak-operator topology, and hence, if  $\{\hat{A}_i\}_{i \in I}$  is a weakly-convergent net of operators in  $V$ , then  $\{\hat{U}\hat{A}_i\hat{U}^{-1}\}_{i \in I}$  is a weakly-convergent net of operators in  $\ell_{\hat{U}}(V)$ , and vice versa.

It follows that  $\ell_{\hat{U}}(V)$  is an abelian von Neumann algebra (i.e., it is weakly closed), and hence  $\ell_{\hat{U}}$  can be viewed as a map  $\ell_{\hat{U}} : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \text{Ob}(\mathcal{V}(\mathcal{H}))$ . We note the following:

1. Clearly, for all  $\hat{U}_1, \hat{U}_2 \in \mathcal{U}(\mathcal{H})$ ,

$$\ell_{\hat{U}_1} \circ \ell_{\hat{U}_2} = \ell_{\hat{U}_1 \hat{U}_2} \quad (10.11)$$

Thus  $\hat{U} \mapsto \ell_{\hat{U}}$  is a realisation of the group  $\mathcal{U}(\mathcal{H})$  as a group of transformations of  $\text{Ob}(\mathcal{V}(\mathcal{H}))$ .

2. For all  $U \in \mathcal{U}(\mathcal{H})$ ,  $V$  and  $\ell_{\hat{U}}(V)$  are *isomorphic* sub-algebras of  $B(\mathcal{H})$ , and  $\ell_{\hat{U}}^{-1} = \ell_{\hat{U}^{-1}}$ .

3. If  $V' \subseteq V$ , then, for all  $\hat{U} \in \mathcal{U}(\mathcal{H})$ ,

$$\ell_{\hat{U}}(V') \subseteq \ell_{\hat{U}}(V). \quad (10.12)$$

Hence, each transformation  $\ell_{\hat{U}}$  preserves the partial-ordering of the poset category  $\mathcal{V}(\mathcal{H})$ .

From this it follows that each  $\ell_{\hat{U}} : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \text{Ob}(\mathcal{V}(\mathcal{H}))$  is a *functor* from the category  $\mathcal{V}(\mathcal{H})$  to itself.

4. One consequence of the order-preserving property of  $\ell_{\hat{U}}$  is as follows. Let  $S$  be a sieve of arrows on  $V$ , i.e., a collection of sub-algebras of  $V$  with the property that if  $V' \in S$ , then, for all  $V'' \subseteq V'$  we have  $V'' \in S$ . Then

$$\ell_{\hat{U}}(S) := \{\ell_{\hat{U}}(V') \mid V' \in S\} \quad (10.13)$$

is a sieve of arrows on  $\ell_{\hat{U}}(V)$ .<sup>97</sup>

### 10.2.2 The Effect of $\ell_{\hat{U}}$ on Daseinisation

We recall that if  $\hat{P}$  is any projection, then the (outer) daseinisation,  $\delta^\circ(\hat{P})_V$ , of  $\hat{P}$  at stage  $V$  is ((5.9))

$$\delta^\circ(\hat{P})_V := \bigwedge \{\hat{Q} \in \mathcal{P}(V) \mid \hat{Q} \succeq \hat{P}\} \quad (10.14)$$

where we have resorted once more to using the propositional language  $\mathcal{PL}(S)$ . Thus

$$\begin{aligned} \hat{U}\delta^\circ(\hat{P})_V\hat{U}^{-1} &= \hat{U} \bigwedge \{\hat{Q} \in \mathcal{P}(V) \mid \hat{Q} \succeq \hat{P}\} \hat{U}^{-1} \\ &= \bigwedge \{\hat{U}\hat{Q}\hat{U}^{-1} \in \mathcal{P}(\ell_{\hat{U}}(V)) \mid \hat{Q} \succeq \hat{P}\} \\ &= \bigwedge \{\hat{U}\hat{Q}\hat{U}^{-1} \in \mathcal{P}(\ell_{\hat{U}}(V)) \mid \hat{U}\hat{Q}\hat{U}^{-1} \succeq \hat{U}\hat{P}\hat{U}^{-1}\} \\ &= \delta^\circ(\hat{U}\hat{P}\hat{U}^{-1})_{\ell_{\hat{U}}(V)} \end{aligned} \quad (10.15)$$

where we used the fact that the map  $\hat{Q} \mapsto \hat{U}\hat{Q}\hat{U}^{-1}$  is weakly continuous.

Thus we have the important result

$$\hat{U}\delta^\circ(\hat{P})_V\hat{U}^{-1} = \delta^\circ(\hat{U}\hat{P}\hat{U}^{-1})_{\ell_{\hat{U}}(V)} \quad (10.16)$$

for all unitary operators  $\hat{U}$ , and for all stages  $V$ . There is an analogous result for inner daseinisation.

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<sup>97</sup>In the partially ordered set  $\mathcal{V}(\mathcal{H})$ , an arrow from  $V'$  to  $V$  can be identified with the sub-algebra  $V' \subseteq V$ , since there is exactly one arrow from  $V'$  to  $V$ .

Equation (10.16) can be applied to the de Groote presheaf  $\underline{\mathcal{Q}}$  to give

$$\hat{U}\delta^o(\hat{A})_V\hat{U}^{-1} = \delta^o(\hat{U}\hat{A}\hat{U}^{-1})_{\ell_{\hat{U}}(V)} \quad (10.17)$$

for unitary operators  $\hat{U}$ , and all stages  $V$ .

We recall that the truth sub-object,  $\underline{\mathbb{T}}^{|\psi\rangle}$ , of the outer presheaf,  $\underline{\mathcal{Q}}$ , is defined at each stage  $V$  by (cf (5.57))

$$\begin{aligned} \underline{\mathbb{T}}_V^{|\psi\rangle} &:= \{\hat{\alpha} \in \underline{\mathcal{Q}}_V \mid \text{Prob}(\hat{\alpha}; |\psi\rangle) = 1\} \\ &= \{\hat{\alpha} \in \underline{\mathcal{Q}}_V \mid \langle \psi | \hat{\alpha} | \psi \rangle = 1\} \end{aligned} \quad (10.18)$$

The neo-realist, physical interpretation of  $\underline{\mathbb{T}}^{|\psi\rangle}$  is that the ‘truth’ of the proposition represented by  $\hat{P}$  is

$$\nu(\delta^o(\hat{P}) \in \underline{\mathbb{T}}^{|\psi\rangle})_V := \{V' \subseteq V \mid \delta^o(\hat{P})_{V'} \in \underline{\mathbb{T}}_{V'}^{|\psi\rangle}\} \quad (10.19)$$

$$= \{V' \subseteq V \mid \langle \psi | \delta^o(\hat{P})_{V'} | \psi \rangle = 1\} \quad (10.20)$$

for all stages  $V$ . We then get

$$\begin{aligned} \ell_{\hat{U}}(\nu(\delta^o(\hat{P}) \in \underline{\mathbb{T}}^{|\psi\rangle})_V) & \\ = \ell_{\hat{U}}\{V' \subseteq V \mid \langle \psi | \delta^o(\hat{P})_{V'} | \psi \rangle = 1\} & \quad (10.21) \\ = \{\ell_{\hat{U}}(V') \subseteq \ell_{\hat{U}}(V) \mid \langle \psi | \delta^o(\hat{P})_{V'} | \psi \rangle = 1\} & \quad (10.22) \end{aligned}$$

$$= \{\ell_{\hat{U}}(V') \subseteq \ell_{\hat{U}}(V) \mid \langle \psi | \hat{U}^{-1}\hat{U}\delta^o(\hat{P})_{V'}\hat{U}^{-1}\hat{U} | \psi \rangle = 1\} \quad (10.23)$$

$$= \{\ell_{\hat{U}}(V') \subseteq \ell_{\hat{U}}(V) \mid \langle \psi | \hat{U}^{-1}\delta^o(\hat{U}\hat{P}\hat{U}^{-1})_{\ell_{\hat{U}}(V)}\hat{U} | \psi \rangle = 1\} \quad (10.24)$$

$$= \nu(\delta^o(\hat{U}\hat{P}\hat{U}^{-1}) \in \underline{\mathbb{T}}^{\hat{U}|\psi\rangle})_{\ell_{\hat{U}}(V)}. \quad (10.25)$$

Thus we get the important result

$$\nu(\delta^o(\hat{U}\hat{P}\hat{U}^{-1}) \in \underline{\mathbb{T}}^{\hat{U}|\psi\rangle})_{\ell_{\hat{U}}(V)} = \ell_{\hat{U}}(\nu(\delta^o(\hat{P}) \in \underline{\mathbb{T}}^{|\psi\rangle})_V). \quad (10.26)$$

This can be viewed as the topos analogue of the statement in (10.1) about the invariance of the results of quantum theory under the transformations  $|\psi\rangle \mapsto \hat{U}|\psi\rangle$ ,  $\hat{A} \mapsto \hat{U}\hat{A}\hat{U}^{-1}$ . Of course, there is a pseudo-state analogue of all these expressions involving the sub-objects  $\underline{\mathbf{m}}^{|\psi\rangle}$ ,  $|\psi\rangle \in \mathcal{H}$ .

### 10.2.3 The $\hat{U}$ -twisted Presheaf

Let us return once more to the definition (10.10) of the functor  $\ell_{\hat{U}} : \mathcal{V}(\mathcal{H}) \rightarrow \mathcal{V}(\mathcal{H})$ . As we shall see later, any such functor induces a ‘geometric morphism’

from  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  to  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . The exact definition is not needed here: it suffices to remark that part of this geometric morphism is an arrow  $\ell_{\hat{U}}^* : \mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  defined by

$$\underline{F} \mapsto \ell_{\hat{U}}^* \underline{F} := \underline{F} \circ \ell_{\hat{U}}. \quad (10.27)$$

Note that, if  $\hat{U}_1, \hat{U}_2 \in \mathcal{U}(\mathcal{H})$  then, for all presheaves  $\underline{F}$ ,

$$\begin{aligned} \ell_{\hat{U}_2}^* (\ell_{\hat{U}_1}^* \underline{F}) &= (\ell_{\hat{U}_1}^* \underline{F}) \circ \ell_{\hat{U}_2} = (\underline{F} \circ \ell_{\hat{U}_1}) \circ \ell_{\hat{U}_2} \\ &= \underline{F} \circ (\ell_{\hat{U}_1} \circ \ell_{\hat{U}_2}) = \underline{F} \circ \ell_{\hat{U}_1 \hat{U}_2} \\ &= \ell_{\hat{U}_1 \hat{U}_2}^* \underline{F}. \end{aligned} \quad (10.28)$$

Since this is true for all functors  $\underline{F}$  in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , we deduce that

$$\ell_{\hat{U}_2}^* \circ \ell_{\hat{U}_1}^* = \ell_{\hat{U}_1 \hat{U}_2}^* \quad (10.29)$$

and hence the map  $\hat{U} \mapsto \ell_{\hat{U}}^*$  is an (anti-)representation of the group  $\mathcal{U}(\mathcal{H})$  by arrows in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

Of particular interest to us are the presheaves  $\ell_{\hat{U}}^* \underline{\Sigma}$  and  $\ell_{\hat{U}}^* k(\underline{\mathbb{R}}^{\leq})$ . We denote them by  $\underline{\Sigma}^{\hat{U}}$  and  $k(\underline{\mathbb{R}}^{\leq})^{\hat{U}}$  respectively and say that they are ‘ $\hat{U}$ -twisted’.

**Theorem 10.1** *For each  $\hat{U} \in \mathcal{U}(\mathcal{H})$ , there is a natural isomorphism  $\iota : \underline{\Sigma} \rightarrow \underline{\Sigma}^{\hat{U}}$  as given in the following diagram*

$$\begin{array}{ccc} \underline{\Sigma}_V & \xrightarrow{\iota_V^{\hat{U}}} & \underline{\Sigma}_V^{\hat{U}} \\ \underline{\Sigma}(i_{V'V}) \downarrow & & \downarrow \underline{\Sigma}^{\hat{U}}(i_{V'V}) \\ \underline{\Sigma}_{V'} & \xrightarrow{\iota_{V'}^{\hat{U}}} & \underline{\Sigma}_{V'}^{\hat{U}} \end{array}$$

where, at each stage  $V$ ,

$$(\iota_V^{\hat{U}}(\lambda))(\hat{A}) := \langle \lambda, \hat{U}^{-1} \hat{A} \hat{U} \rangle \quad (10.30)$$

for all  $\lambda \in \underline{\Sigma}_V$ , and all  $\hat{A} \in V_{\text{sa}}$ .

The proof, which just involves chasing round the diagram above using the basic definitions, is not included here.

Even simpler is the following theorem:

**Theorem 10.2** For each  $\hat{U} \in \mathcal{U}(\mathcal{H})$ , there is a natural isomorphism  $\kappa^{\hat{U}} : \underline{\mathbb{R}}^{\succeq} \rightarrow (\underline{\mathbb{R}}^{\succeq})^{\hat{U}}$  whose components  $\kappa_V^{\hat{U}} : \underline{\mathbb{R}}^{\succeq}_V \rightarrow (\underline{\mathbb{R}}^{\succeq})_V^{\hat{U}}$  are given by

$$\kappa_V^{\hat{U}}(\mu)(\ell_{\hat{U}}(V')) := \mu(V') \quad (10.31)$$

for all  $V' \subseteq V$ .

Here, we recall  $\mu \in \underline{\mathbb{R}}^{\succeq}_V$  is a function  $\mu : \downarrow V \rightarrow \mathbb{R}$  such that if  $V_2 \subseteq V_1 \subseteq V$  then  $\mu(V_2) \geq \mu(V_1)$ , i.e., an order-reversing function. In (10.31) we have used the fact that there is a bijection between the sets  $\downarrow \ell_{\hat{U}}(V)$  and  $\downarrow V$ .

Finally,

**Theorem 10.3** We have the following commutative diagram:

$$\begin{array}{ccc} \underline{\Sigma} & \xrightarrow{\iota^{\hat{U}}} & \underline{\Sigma}^{\hat{U}} \\ \downarrow \check{\delta}(\hat{A}) & & \downarrow \check{\delta}(\hat{U}^{-1}\hat{A}\hat{U}) \\ \underline{\mathbb{R}}^{\succeq} & \xrightarrow{\kappa^{\hat{U}}} & \underline{\mathbb{R}}^{\succeq \hat{U}} \end{array}$$

**The analogue of unitary operators for a general topos.** It is interesting to reflect on the analogue of the above constructions for a general topos. It soon becomes clear that, once again, we encounter the antithetical concepts of ‘internal’ and ‘external’.

For example, in the discussion above, the unitary operators and the group  $\mathcal{U}(\mathcal{H})$  lie outside the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  and enter directly from the underlying, standard quantum formalism. As such, they are external to both the languages  $\mathcal{P}\mathcal{L}(S)$  and  $\mathcal{L}(S)$ . We anticipate that notions of ‘covariance’ and ‘symmetry’ have applications well beyond those in classical physics and quantum physics. However, at the very least, in a general topos one would presumably replace the external  $\mathcal{U}(\mathcal{H})$  with an internal group object in the topos concerned. And, of course, the notion of ‘symmetry’ is closely related to the concept to time, and time development, which opens up a Pandora’s box of possible speculation. These issues are important, and await further development.

# 11 The Category of Systems

## 11.1 Background Remarks

We now return to the more general aspects of our theory, and study its application to a *collection* of systems, each one of which may be associated with a different topos. For example, if  $S_1, S_2$  is a pair of systems, with associated topoi  $\tau(S_1)$  and  $\tau(S_2)$ , and if  $S_1$  is a sub-system of  $S_2$ , then we wish to consider how  $\tau(S_1)$  is related to  $\tau(S_2)$ . Similarly, if a composite system is formed from a pair of systems  $S_1, S_2$ , what relations are there between the topos of the composite system and the topoi of the constituent parts?

Of course, in one sense, there is only one true ‘system’, and that is the universe as a whole. Concomitantly, there is just one local language, and one topos. However, in practice, the science community divides the universe conceptually into portions that are sufficiently simple to be amenable to theoretical and/or empirical discussion. Of course, this division is not unique, but it must be such that the coupling between portions is weak enough that, to a good approximation, their theoretical models can be studied in isolation from each other. Such an essentially isolated<sup>98</sup> portion of the universe is called a ‘sub-system’. By an abuse of language, sub-systems of the universe are usually called ‘systems’ (so that the universe as a whole is one super-system), and then we can talk about ‘sub-systems’ of these systems; or ‘composites’ of them; or sub-systems of the composite systems, and so on.

In practice, references by physicists to systems and sub-systems<sup>99</sup> do not generally signify *actual* sub-systems of the real universe but rather idealisations of possible systems. This is what a physics lecturer means when he or she starts a lecture by saying “Consider a point particle moving in three dimensions.....”.

To develop these ideas further we need mathematical control over the systems of interest, and their interrelations. To this end, we start by focussing on some collection, **Sys**, of physical systems to which a particular theory-type is deemed to be applicable. For example, we could consider a collection of systems that are to be discussed using the methodology of classical physics;

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<sup>98</sup>The ideal monad has no windows.

<sup>99</sup>The word ‘sub-system’ does not only mean a collection of objects that is spatially localised. One could also consider sub-systems of field systems by focussing on a just a few modes of the fields as is done, for example, in the Robertson-Walker model for cosmology. Another possibility would be to use fields localised in some fixed space, or space-time region provided that this is consistent with the dynamics.



or systems to be discussed using standard quantum theory; or whatever. For completeness, we require that every sub-system of a system in **Sys** is itself a member of **Sys**, as is every composite of members of **Sys**.

We shall assume that the systems in **Sys** are all associated with local languages of the type discussed earlier, and that they all have the *same* set of ground symbols which, for the purposes of the present discussion, we take to be just  $\Sigma$  and  $\mathcal{R}$ . It follows that the languages  $\mathcal{L}(S)$ ,  $S \in \mathbf{Sys}$ , differ from each other only in the set of function symbols  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ ; i.e., the set of *physical quantities*.

As a simple example of the system-dependence of the set of function symbols let system  $S_1$  be a point particle moving in one dimension, and let the set of physical quantities be  $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) = \{x, p, H\}$ . In the language  $\mathcal{L}(S_1)$ , these function-symbols represent the position, momentum, and energy of the system respectively. On the other hand, if  $S_2$  is a particle moving in three dimensions, then in the language  $\mathcal{L}(S_2)$  we could have  $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) = \{x, y, z, p_x, p_y, p_z, H\}$  to allow for three-dimensional position and momentum. Or, we could decide to add angular momentum as well, to give the set  $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) = \{x, y, z, p_x, p_y, p_z, J_x, J_y, J_z, H\}$ .

## 11.2 The Category **Sys**

### 11.2.1 The Arrows and Translations for the Disjoint Sum $S_1 \sqcup S_2$ .

The use of local languages is central to our overall topos scheme, and therefore we need to understand, in particular, (i) the relation between the languages  $\mathcal{L}(S_1)$  and  $\mathcal{L}(S_2)$  if  $S_1$  is a sub-system of  $S_2$ ; and (ii) the relation between  $\mathcal{L}(S_1)$ ,  $\mathcal{L}(S_2)$  and  $\mathcal{L}(S_1 \diamond S_2)$ , where  $S_1 \diamond S_2$  denotes the composite of systems  $S_1$  and  $S_2$ .

These discussions can be made more precise by regarding **Sys** as a category whose objects are the systems.<sup>100</sup> The arrows in **Sys** need to cover two basic types of relation: (i) that between  $S_1$  and  $S_2$  if  $S_1$  is a ‘sub-system’ of  $S_2$ ; and (ii) that between a composite system,  $S_1 \diamond S_2$ , and its constituent systems,  $S_1$  and  $S_2$ .

This may seem straightforward but, in fact, care is needed since although the idea of a ‘sub-system’ seems intuitively clear, it is hard to give a physically acceptable definition that is universal. However, some insight into this idea can be gained by considering its meaning in classical physics. This is very

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<sup>100</sup>To control the size of **Sys** we assume that the collection of objects/systems is a *set* rather than a more general class.

relevant for the general scheme since one of our main goals is to make all theories ‘look’ like classical physics in the appropriate topos.

To this end, let  $S_1$  and  $S_2$  be classical systems whose state spaces are the symplectic manifolds  $\mathcal{S}_1$  and  $\mathcal{S}_2$  respectively. If  $S_1$  is deemed to be a sub-system of  $S_2$ , it is natural to require that  $\mathcal{S}_1$  is a *sub-manifold* of  $\mathcal{S}_2$ , i.e.,  $\mathcal{S}_1 \subseteq \mathcal{S}_2$ . However, this condition cannot be used as a *definition* of a ‘sub-system’ since the converse may not be true: i.e., if  $\mathcal{S}_1 \subseteq \mathcal{S}_2$ , this does not necessarily mean that, from a physical perspective,  $S_1$  could, or would, be said to be a sub-system of  $S_2$ .<sup>101</sup>

On the other hand, there are situations where being a sub-manifold clearly *does* imply being a physical sub-system. For example, suppose the state space  $\mathcal{S}$  of a system  $S$  is a disconnected manifold with two components  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , so that  $\mathcal{S}$  is the disjoint union,  $\mathcal{S}_1 \amalg \mathcal{S}_2$ , of the sub-manifolds  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Then it seems physically appropriate to say that the system  $S$  itself is disconnected, and to write  $S = S_1 \sqcup S_2$  where the symplectic manifolds that represent the sub-systems  $S_1$  and  $S_2$  are  $\mathcal{S}_1$  and  $\mathcal{S}_2$  respectively.

One reason why it is reasonable to call  $S_1$  and  $S_2$  ‘sub-systems’ in this particular situation is that any continuous dynamical evolution of a state point in  $\mathcal{S} \simeq \mathcal{S}_1 \amalg \mathcal{S}_2$  will always lie in either one component or the other. This suggests that perhaps, in general, a necessary condition for a sub-manifold  $\mathcal{S}_1 \subseteq \mathcal{S}_2$  to represent a physical sub-system is that the dynamics of the system  $S_2$  must be such that  $\mathcal{S}_1$  is mapped into itself under the dynamical evolution on  $\mathcal{S}_2$ ; in other words,  $\mathcal{S}_1$  is a *dynamically-invariant* sub-manifold of  $\mathcal{S}_2$ . This correlates with the idea mentioned earlier that sub-systems are weakly-coupled with each other.

However, such a dynamical restriction is not something that should be coded into the languages,  $\mathcal{L}(S_1)$  and  $\mathcal{L}(S_2)$ : rather, the dynamics is to be associated with the *representation* of these languages in the appropriate topos.

Still, this caveat does not apply to the disjoint sum  $S_1 \sqcup S_2$  of two systems  $S_1, S_2$ , and we will assume that, in general, (i.e., not just in classical physics) it is legitimate to think of  $S_1$  and  $S_2$  as being sub-systems of  $S_1 \sqcup S_2$ ; something that we indicate by defining arrows  $i_1 : S_1 \rightarrow S_1 \sqcup S_2$ , and  $i_2 : S_2 \rightarrow S_1 \sqcup S_2$  in **Sys**.

To proceed further it is important to understand the connection between the putative arrows in the category **Sys**, and the ‘translations’ of the as-

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<sup>101</sup>For example, consider the diagonal sub-manifold  $\Delta(\mathcal{S}) \subset \mathcal{S} \times \mathcal{S}$  of the symplectic manifold  $\mathcal{S} \times \mathcal{S}$  that represents the composite  $S \diamond S$  of two copies of a system  $S$ . Evidently, the states in  $\Delta(\mathcal{S})$  correspond to the situation in which both copies of  $S$  ‘march together’. It is doubtful if this would be recognised physically as a sub-system.

sociated languages. The first step is to consider what can be said about the relation between  $\mathcal{L}(S_1 \sqcup S_2)$ , and  $\mathcal{L}(S_1)$  and  $\mathcal{L}(S_2)$ . All three languages share the same ground-type symbols, and so what we are concerned with is the relation between the function symbols of signature  $\Sigma \rightarrow \mathcal{R}$  in these languages.

By considering what is meant intuitively by the disjoint sum, it seems plausible that each physical quantity for the system  $S_1 \sqcup S_2$  produces a physical quantity for  $S_1$ , and another one for  $S_2$ . Conversely, specifying a pair of physical quantities—one for  $S_1$  and one for  $S_2$ —gives a physical quantity for  $S_1 \sqcup S_2$ . In other words,

$$F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R}) \simeq F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) \quad (11.1)$$

However, it is important not to be too dogmatic about statements of this type since in non-classical theories new possibilities can arise that are counter to intuition.

Associated with (11.1) are the maps  $\mathcal{L}(i_1) : F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$  and  $\mathcal{L}(i_2) : F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R})$ , defined as the projection maps of the product. In the theory of local languages, these transformations are essentially *translations* [8] of  $\mathcal{L}(S_1 \sqcup S_2)$  in  $\mathcal{L}(S_1)$  and  $\mathcal{L}(S_2)$  respectively; a situation that we denote  $\mathcal{L}(i_1) : \mathcal{L}(S_1 \sqcup S_2) \rightarrow \mathcal{L}(S_1)$ , and  $\mathcal{L}(i_2) : \mathcal{L}(S_1 \sqcup S_2) \rightarrow \mathcal{L}(S_2)$ .

To be more precise, these operations are translations if, taking  $\mathcal{L}(i_1)$  as the explanatory example, the map  $\mathcal{L}(i_1) : F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$  is supplemented with the following map from the ground symbols of  $\mathcal{L}(S_1 \sqcup S_2)$  to those of  $\mathcal{L}(S_1)$ :

$$\mathcal{L}(i_1)(\Sigma) := \Sigma, \quad (11.2)$$

$$\mathcal{L}(i_1)(\mathcal{R}) := \mathcal{R}, \quad (11.3)$$

$$\mathcal{L}(i_1)(1) := 1, \quad (11.4)$$

$$\mathcal{L}(i_1)(\Omega) := \Omega. \quad (11.5)$$

Such a translation map is then extended to all type symbols using the definitions

$$\mathcal{L}(i_1)(T_1 \times T_2 \times \cdots \times T_n) = \mathcal{L}(i_1)(T_1) \times \mathcal{L}(i_1)(T_2) \times \cdots \times \mathcal{L}(i_1)(T_n) \quad (11.6)$$

$$\mathcal{L}(i_1)(PT) = P[\mathcal{L}(i_1)(T)] \quad (11.7)$$

for all finite  $n$  and all type symbols  $T, T_1, T_2, \dots, T_n$ . This, in turn, can be extended inductively to all terms in the language. Thus, in our case, the translations act trivially on all the type symbols.

**Arrows in  $\mathbf{Sys}$  are translations.** Motivated by this argument we now turn everything around and, in general, *define* an arrow  $j : S_1 \rightarrow S$  in the category  $\mathbf{Sys}$  to mean that there is some *physically meaningful* way of transforming the physical quantities in  $S$  to physical quantities in  $S_1$ . If, for any pair of systems  $S_1, S$  there is more than one such transformation, then there will be more than one arrow from  $S_1$  to  $S$ .

To make this more precise, let  $\mathbf{Loc}$  denote the collection of all (small<sup>102</sup>) local languages. This is a category whose objects are the local languages, and whose arrows are translations between languages. Then our basic assumption is that the association  $S \mapsto \mathcal{L}(S)$  is a covariant functor from  $\mathbf{Sys}$  to  $\mathbf{Loc}^{\text{op}}$ , which we denote as  $\mathcal{L} : \mathbf{Sys} \rightarrow \mathbf{Loc}^{\text{op}}$ .

Note that the combination of a pair of arrows in  $\mathbf{Sys}$  exists in so far as the associated translations can be combined.

### 11.2.2 The Arrows and Translations for the Composite System $S_1 \diamond S_2$ .

Let us now consider the composition  $S_1 \diamond S_2$  of a pair of systems. In the case of classical physics, if  $\mathcal{S}_1$  and  $\mathcal{S}_2$  are the symplectic manifolds that represent the systems  $S_1$  and  $S_2$  respectively, then the manifold that represents the composite system is the cartesian product  $\mathcal{S}_1 \times \mathcal{S}_2$ . This is distinguished by the existence of the two projection functions  $\text{pr}_1 : \mathcal{S}_1 \times \mathcal{S}_2 \rightarrow \mathcal{S}_1$  and  $\text{pr}_2 : \mathcal{S}_1 \times \mathcal{S}_2 \rightarrow \mathcal{S}_2$ .

It seems reasonable to impose the same type of structure on  $\mathbf{Sys}$ : i.e., to require there to be arrows  $p_1 : S_1 \diamond S_2 \rightarrow S_1$  and  $p_2 : S_1 \diamond S_2 \rightarrow S_2$  in  $\mathbf{Sys}$ . However, bearing in mind the definition above, these arrows  $p_1, p_2$  exist if, and only if, there are corresponding translations  $\mathcal{L}(p_1) : \mathcal{L}(S_1) \rightarrow \mathcal{L}(S_1 \diamond S_2)$ , and  $\mathcal{L}(p_2) : \mathcal{L}(S_2) \rightarrow \mathcal{L}(S_1 \diamond S_2)$ . But there *are* such translations: for if  $A_1$  is a physical quantity for system  $S_1$ , then  $\mathcal{L}(p_1)(A_1)$  can be defined as that same physical quantity, but now regarded as pertaining to the combined system  $S_1 \diamond S_2$ ; and analogously for system  $S_2$ .<sup>103</sup> We shall denote this translated quantity,  $\mathcal{L}(p_1)(A_1)$ , by  $A_1 \diamond 1$ .

Note that we do *not* postulate any simple relation between  $F_{\mathcal{L}(S_1 \diamond S_2)}(\Sigma, \mathcal{R})$  and  $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$  and  $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R})$ ; i.e., there is no analogue of (11.1) for

<sup>102</sup>This means that the collection of symbols is a set, not a more general class.

<sup>103</sup>For example, if  $A$  is the energy of particle 1, then we can talk about this energy in the combination of a pair of particles. Of course, in—for example—classical physics there is no reason why the energy of particle 1 should be *conserved* in the composite system, but that, dynamical, question is a different matter.

combinations of systems.

The definitions above of the basic arrows suggest that we might also want to impose the following conditions:

1. The arrows  $i_1 : S_1 \rightarrow S_1 \sqcup S_2$ , and  $i_2 : S_2 \rightarrow S_1 \sqcup S_2$  are *monic* in **Sys**.
2. The arrows  $p_1 : S_1 \diamond S_2 \rightarrow S_1$  and  $p_2 : S_1 \diamond S_2 \rightarrow S_2$  are *epic* arrows in **Sys**.

However, we do *not* require that  $S_1 \sqcup S_2$  and  $S_1 \diamond S_2$  are the co-product and product, respectively, of  $S_1$  and  $S_2$  in the category **Sys**.

### 11.2.3 The Concept of ‘Isomorphic’ Systems.

We also need to decide what it means to say that two systems  $S_1$  and  $S_2$  are *isomorphic*, to be denoted  $S_1 \simeq S_2$ . As with the concept of sub-system, the notion of isomorphism is to some extent a matter of definition rather than obvious physical structure, albeit with the expectation that isomorphic systems in **Sys** will correspond to isomorphic local languages, and be represented by isomorphic mathematical objects in any concrete realisation of the axioms: for example, by isomorphic symplectic manifolds in classical physics.

To a considerable extent, the physical meaning of ‘isomorphism’ depends on whether one is dealing with actual physical systems, or idealisations of them. For example, an electron confined in a box in Cambridge is presumably isomorphic to one confined in the same type of box in London, although they are not the same physical system. On the other hand, when a lecturer says “Consider an electron trapped in a box....”, he/she is referring to an idealised system.

One could, perhaps, say that an idealised system is an *equivalence class* (under isomorphisms) of real systems, but even working only with idealisations does not entirely remove the need for the concept of isomorphism.

For example, in classical mechanics, consider the (idealised) system  $S$  of a point particle moving in a box, and let  $1$  denote the ‘trivial system’ that consists of just a single point with no internal or external degrees of freedom. Now consider the system  $S \diamond 1$ . In classical mechanics this is represented by the symplectic manifold  $\mathcal{S} \times \{*\}$ , where  $\{*\}$  is a single point, regarded as a zero-dimensional manifold. However,  $\mathcal{S} \times \{*\}$  is isomorphic to the manifold  $\mathcal{S}$ , and it is clear physically that the system  $S \diamond 1$  is isomorphic to the system  $S$ . On the other hand, one cannot say that  $S \diamond 1$  is literally *equal* to  $S$ , so the concept of ‘isomorphism’ needs to be maintained.

One thing that *is* clear is that if  $S_1 \simeq S_2$  then  $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \simeq F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R})$ , and if any other non-empty sets of function symbols are present, then they too must be isomorphic.

Note that when introducing a trivial system,  $1$ , it necessary to specify its local language,  $\mathcal{L}(1)$ . The set of function symbols  $F_{\mathcal{L}(1)}(\Sigma, \mathcal{R})$  is not completely empty since, in classical physics, one does have a preferred physical quantity, which is just the number  $1$ . If one asks what is meant in general by the ‘number  $1$ ’ the answer is not trivial since, in the reals  $\mathbb{R}$ , the number  $1$  is the multiplicative identity. It would be possible to add the existence of such a unit to the axioms for  $\mathcal{R}$  but this would involve introducing a multiplicative structure and we do not know if there might be physically interesting topos representations that do not have this feature.

For the moment then, we will say that the trivial system has just a single physical quantity, which in classical physics translates to the number  $1$ . More generally, for the language  $\mathcal{L}(1)$  we specify that  $F_{\mathcal{L}(1)}(\Sigma, \mathcal{R}) := \{I\}$ , i.e.,  $F_{\mathcal{L}(1)}(\Sigma, \mathcal{R})$  has just a single element,  $I$ , say. Furthermore, we add the axiom

$$: \forall \tilde{s}_1 \forall \tilde{s}_2, I(\tilde{s}_1) = I(\tilde{s}_2), \quad (11.8)$$

where  $\tilde{s}_1$  and  $\tilde{s}_2$  are variables of type  $\Sigma$ . In fact, it seems natural to add such a trivial quantity to the language  $\mathcal{L}(S)$  for *any* system  $S$ , and from now on we will assume that this has been done.

A related issue is that, in classical physics, if  $A$  is a physical quantity, then so is  $rA$  for any  $r \in \mathbb{R}$ . This is because the set of classical quantities  $A_\sigma : \Sigma_\sigma \rightarrow \mathcal{R}_\sigma \simeq \mathbb{R}$  forms a *ring* whose structure derives from the ring structure of  $\mathbb{R}$ . It would be possible to add ring axioms for  $\mathcal{R}$  to the language  $\mathcal{L}(S)$ , but this is too strong, not least because, as shown earlier, it fails in quantum theory. Clearly, the general question of axioms for  $\mathcal{R}$  needs more thought: a task for later work.

If desired, an ‘*empty*’ system,  $0$ , can be added too, with  $F_{\mathcal{L}(0)}(\Sigma, \mathcal{R}) := \emptyset$ . This, so called, ‘*pure language*’,  $\mathcal{L}(0)$ , is an initial object in the category **Loc**.

#### 11.2.4 An Axiomatic Formulation of the Category **Sys**

Let us now summarise, and clarify, our list of axioms for a category **Sys**:

1. The collection **Sys** is a small category whose objects are the systems of interest (or, if desired, isomorphism classes of such systems) and whose arrows are defined as above.

Thus the fundamental property of an arrow  $j : S_1 \rightarrow S$  in **Sys** is that it induces, and is essentially *defined by*, a translation  $\mathcal{L}(j) : \mathcal{L}(S_1) \rightarrow \mathcal{L}(S)$

$\mathcal{L}(S_1)$ . Physically, this corresponds to the physical quantities for system  $S$  being ‘pulled-back’ to give physical quantities for system  $S_1$ .

Arrows of particular interest are those associated with ‘sub-systems’ and ‘composite systems’, as discussed above.

2. The axioms for a category are satisfied because:

(a) Physically, the ability to form composites of arrows follows from the concept of ‘pulling-back’ physical quantities. From a mathematical perspective, if  $j : S_1 \rightarrow S_2$  and  $k : S_2 \rightarrow S_3$ , then the translations give functions  $\mathcal{L}(j) : F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$  and  $\mathcal{L}(k) : F_{\mathcal{L}(S_3)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R})$ . Then clearly  $\mathcal{L}(j) \circ \mathcal{L}(k) : F_{\mathcal{L}(S_3)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ , and this can thought of as the translation corresponding to the arrow  $k \circ j : S_1 \rightarrow S_3$ .

The associativity of the law of arrow combination can be proved in a similar way.

(b) We add by hand a special arrow  $\text{id}_S : S \rightarrow S$  which is defined to correspond to the translation  $\mathcal{L}(\text{id}_S)$  that is given by the identity map on  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ . Clearly,  $\text{id}_S : S \rightarrow S$  acts as an identity morphism should.

3. For any pair of systems  $S_1, S_2$ , there is a *disjoint sum*, denoted  $S_1 \sqcup S_2$ . The disjoint sum has the following properties:

(a) For all systems  $S_1, S_2, S_3$  in **Sys**:

$$(S_1 \sqcup S_2) \sqcup S_3 \simeq S_1 \sqcup (S_2 \sqcup S_3). \quad (11.9)$$

(b) For all systems  $S_1, S_2$  in **Sys**:

$$S_1 \sqcup S_2 \simeq S_2 \sqcup S_1. \quad (11.10)$$

(c) There are arrows in **Sys**:

$$i_1 : S_1 \rightarrow S_1 \sqcup S_2 \quad \text{and} \quad i_2 : S_2 \rightarrow S_1 \sqcup S_2 \quad (11.11)$$

that are associated with translations in the sense discussed in Section 11.2.1. These are associated with the decomposition

$$F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R}) \simeq F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}). \quad (11.12)$$

We assume that if  $S_1, S_2$  belong to **Sys**, then **Sys** also contains  $S_1 \sqcup S_2$ .

4. For any given pair of systems  $S_1, S_2$ , there is a *composite* system in  $\mathbf{Sys}$ , denoted<sup>104</sup>  $S_1 \diamond S_2$ , with the following properties:

(a) For all systems  $S_1, S_2, S_3$  in  $\mathbf{Sys}$ :

$$(S_1 \diamond S_2) \diamond S_3 \simeq S_1 \diamond (S_2 \diamond S_3). \quad (11.13)$$

(b) For all systems  $S_1, S_2$  in  $\mathbf{Sys}$ :

$$S_1 \diamond S_2 \simeq S_2 \diamond S_1. \quad (11.14)$$

(c) There are arrows in  $\mathbf{Sys}$ :

$$p_1 : S_1 \diamond S_2 \rightarrow S_1 \text{ and } p_2 : S_1 \diamond S_2 \rightarrow S_2 \quad (11.15)$$

that are associated with translations in the sense discussed in Section 11.2.2.

We assume that if  $S_1, S_2$  belong to  $\mathbf{Sys}$ , then  $\mathbf{Sys}$  also contains the composite system  $S_1 \diamond S_2$ .

5. It seems physically reasonable to add the axiom

$$(S_1 \sqcup S_2) \diamond S \simeq (S_1 \diamond S) \sqcup (S_2 \diamond S) \quad (11.16)$$

for all systems  $S_1, S_2, S$ . However, physical intuition can be a dangerous thing, and so, as with most of these axioms, we are not dogmatic, and feel free to change them as new insights emerge.

6. There is a trivial system,  $1$ , such that for all systems  $S$ , we have

$$S \diamond 1 \simeq S \simeq 1 \diamond S \quad (11.17)$$

7. It may be convenient to postulate an ‘empty system’,  $0$ , with the properties

$$S \diamond 0 \simeq 0 \diamond S \simeq 0 \quad (11.18)$$

$$S \sqcup 0 \simeq 0 \sqcup S \simeq S \quad (11.19)$$

for all systems  $S$ .

Within the meaning given to arrows in  $\mathbf{Sys}$ ,  $0$  is a *terminal object* in  $\mathbf{Sys}$ . This is because the empty set of function symbols of signature  $\Sigma \rightarrow \mathcal{R}$  is a subset of any other set of function symbols of this signature.

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<sup>104</sup>The product operation in a monoidal category is often written ‘ $\otimes$ ’. However, a different symbol has been used here to avoid confusion with existing usages in physics of the tensor product sign ‘ $\otimes$ ’.



It might seem tempting to postulate that composition laws are well-behaved with respect to arrows. Namely, if  $j : S_1 \rightarrow S_2$ , then, for any  $S$ , there is an arrow  $S_1 \diamond S \rightarrow S_2 \diamond S$  and an arrow  $S_1 \sqcup S \rightarrow S_2 \sqcup S$ .<sup>105</sup>

In the case of the disjoint sum, such an arrow can be easily constructed using (11.12). First split the function symbols in  $F_{\mathcal{L}(S_1 \sqcup S)}(\Sigma, \mathcal{R})$  into  $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  and the function symbols in  $F_{\mathcal{L}(S_2 \sqcup S)}(\Sigma, \mathcal{R})$  into  $F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ . Since there is an arrow  $j : S_1 \rightarrow S_2$ , there is a translation  $\mathcal{L}(j) : \mathcal{L}(S_2) \rightarrow \mathcal{L}(S_1)$ , given by a mapping  $\mathcal{L}(j) : F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ . Of course, then there is also a mapping  $\mathcal{L}(j) \times \mathcal{L}(\text{id}_S) : F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ , i.e., a translation between  $\mathcal{L}(S_2 \sqcup S)$  and  $\mathcal{L}(S_1 \sqcup S)$ . Since we assume that there is an arrow in **Sys** whenever there is a translation (in the opposite direction), there is indeed an arrow  $S_1 \sqcup S \rightarrow S_2 \sqcup S$ .

In the case of the composition, however, this would require a translation  $\mathcal{L}(S_2 \diamond S) \rightarrow \mathcal{L}(S_1 \diamond S)$ , and this cannot be done in general since we have no *prima facie* information about the set of function symbols  $F_{\mathcal{L}(S_2 \diamond S)}(\Sigma, \mathcal{R})$ . However, if we restrict the arrows in **Sys** to be those associated with subsystems, combination of systems, and compositions of such arrows, then it is easy to see that the required translations exist (the proof of this makes essential use of (11.16)).

If we make this restriction of arrows, then the axioms (11.14), (11.17–11.20), mean that, essentially, **Sys** has the structure of a *symmetric monoidal*<sup>106</sup> category in which the monoidal product operation is ‘ $\diamond$ ’, and the left and right unit object is 1. There is also a monoidal structure associated with the disjoint sum ‘ $\sqcup$ ’, with 0 as the unit object.

We say ‘essentially’ because in order to comply with all the axioms of a monoidal category, **Sys** must satisfy certain additional, so-called, ‘coherence’ axioms. However, from a physical perspective these are very plausible statements about (i) how the unit object 1 intertwines with the  $\diamond$ -operation; how the null object intertwines with the  $\sqcup$ -operation; and (iii) certain properties of quadruple products (and disjoint sums) of systems.

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<sup>105</sup>A more accurate way of capturing this idea is to say that the operation  $\mathbf{Sys} \times \mathbf{Sys} \rightarrow \mathbf{Sys}$  in which

$$\langle S_1, S_2 \rangle \mapsto S_1 \diamond S_2 \tag{11.20}$$

is a *bi-functor* from  $\mathbf{Sys} \times \mathbf{Sys}$  to  $\mathbf{Sys}$ . Ditto for the operation in which  $\langle S_1, S_2 \rangle \mapsto S_1 \sqcup S_2$ .

<sup>106</sup>In the actual definition of a monoidal category the two isomorphisms in (11.17) are separated from each other, whereas we have identified them. Further more, these isomorphism are required to be natural. This seems a correct thing to require in our case, too.

**A simple example of a category  $\mathbf{Sys}$ .** It might be helpful at this point to give a simple example of a category  $\mathbf{Sys}$ . To that end, let  $S$  denote a point particle that moves in three dimensions, and let us suppose that  $S$  has no sub-systems other than the trivial system 1. Then  $S \diamond S$  is defined to be a pair of particles moving in three dimensions, and so on. Thus the objects in our category are  $1, S, S \diamond S, \dots, S \diamond S \diamond \dots S \dots$  where the ‘ $\diamond$ ’ operation is formed any finite number of times.

At this stage, the only arrows are those that are associated with the constituents of a composite system. However, we could contemplate adding to the systems the disjoint sum  $S \sqcup (S \diamond S)$  which is a system that is either one particle or two particles (but, of course, not both at the same time). And, clearly, we could extend this to  $S \sqcup (S \diamond S) \sqcup (S \diamond S \diamond S)$ , and so on. Each of these disjoint sums comes with its own arrows, as explained above.

Note that this particular category of systems has the property that it can be treated using either classical physics or quantum theory.

### 11.3 Representations of $\mathbf{Sys}$ in Topoi

We assume that all the systems in  $\mathbf{Sys}$  are to be treated with the same theory type. We also assume that systems in  $\mathbf{Sys}$  with the *same* language are to be represented in the same topos. Then we define:<sup>107</sup>

**Definition 11.1** *A topos realisation of  $\mathbf{Sys}$  is an association,  $\phi$ , to each system  $S$  in  $\mathbf{Sys}$ , of a triple  $\phi(S) = \langle \rho_{\phi,S}, \mathcal{L}(S), \tau_{\phi}(S) \rangle$  where:*

- (i)  $\tau_{\phi}(S)$  is the topos in which the theory-type applied to system  $S$  is to be realised.
- (ii)  $\mathcal{L}(S)$  is the local language in  $\mathbf{Loc}$  that is associated with  $S$ . This is not dependent on the realisation  $\phi$ .
- (iii)  $\rho_{\phi,S}$  is a representation of the local language  $\mathcal{L}(S)$  in the topos  $\tau_{\phi}(S)$ . As a more descriptive piece of notation we write  $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_{\phi}(S)$ . The key part of this representation is the map

$$\rho_{\phi,S} : F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow \text{Hom}_{\tau_{\phi}(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}) \quad (11.21)$$

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<sup>107</sup>As emphasised already, the association  $S \mapsto \mathcal{L}(S)$  is generally not one-to-one: i.e., many systems may share the same language. Thus, when we come discuss the representation of the language  $\mathcal{L}(S)$  in a topos, the extra information about the system  $S$  is used in fixing the representation.

where  $\Sigma_{\phi,S}$  and  $\mathcal{R}_{\phi,S}$  are the state object and quantity-value object, respectively, of the representation  $\phi$  in the topos  $\tau_\phi(S)$ . As a convenient piece of notation we write  $A_{\phi,S} := \rho_{\phi,S}(A)$  for all  $A \in F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ .

This definition is only partial; the possibility of extending it will be discussed shortly.

Now, if  $j : S_1 \rightarrow S$  is an arrow in **Sys**, then there is a translation arrow  $\mathcal{L}(j) : \mathcal{L}(S) \rightarrow \mathcal{L}(S_1)$ . Thus we have the beginnings of a commutative diagram

$$\begin{array}{ccc}
 S_1 & \xrightarrow{\phi} & \langle \rho_{\phi,S_1}, \mathcal{L}(S_1), \tau_\phi(S_1) \rangle \\
 \downarrow j & & \uparrow ? \times \mathcal{L}(j) \times ? \\
 S & \xrightarrow{\phi} & \langle \rho_{\phi,S}, \mathcal{L}(S), \tau_\phi(S) \rangle
 \end{array} \tag{11.22}$$

However, to be useful, the arrow on the right hand side of this diagram should refer to some relation between (i) the topoi  $\tau_\phi(S_1)$  and  $\tau_\phi(S)$ ; and (ii) the realisations  $\rho_{\phi,S_1} : \mathcal{L}(S_1) \rightsquigarrow \tau_\phi(S_1)$  and  $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$ : this is the significance of the two ‘?’ symbols in the arrow written ‘ $? \times \mathcal{L}(j) \times ?$ ’.

Indeed, as things stand, Definition 11.1 says nothing about relations between the topoi representations of different systems in **Sys**. We are particularly interested in the situation where there are two different systems  $S_1$  and  $S$  with an arrow  $j : S_1 \rightarrow S$  in **Sys**.

We know that the arrow  $j$  is associated with a translation  $\mathcal{L}(j) : \mathcal{L}(S) \rightarrow \mathcal{L}(S_1)$ , and an attractive possibility, therefore, would be to seek, or postulate, a ‘covering’ map  $\phi(\mathcal{L}(j)) : \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}) \rightarrow \text{Hom}_{\tau_\phi(S_1)}(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1})$  to be construed as a topos representation of the translation  $\mathcal{L}(j) : \mathcal{L}(S) \rightarrow \mathcal{L}(S_1)$ , and hence of the arrow  $j : S_1 \rightarrow S$  in **Sys**.

This raises the questions of what properties these ‘translation representations’ should possess in order to justify saying that they ‘cover’ the translations. A minimal requirement is that if  $k : S_2 \rightarrow S_1$  and  $j : S_1 \rightarrow S$ , then the map  $\phi(\mathcal{L}(j \circ k)) : \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}) \rightarrow \text{Hom}_{\tau_\phi(S_2)}(\Sigma_{\phi,S_2}, \mathcal{R}_{\phi,S_2})$  factorises as

$$\phi(\mathcal{L}(j \circ k)) = \phi(\mathcal{L}(k)) \circ \phi(\mathcal{L}(j)). \tag{11.23}$$

We also require that

$$\phi(\mathcal{L}(\text{id}_S)) = \text{id} : \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}) \rightarrow \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}) \tag{11.24}$$

for all systems  $S$ .

The conditions (11.23) and (11.24) seem eminently plausible, and they are not particularly strong. A far more restrictive axiom would be to require the following diagram to commute:

$$\begin{array}{ccc}
 F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) & \xrightarrow{\rho_{\phi, S}} & \text{Hom}_{\tau_{\phi}(S)}(\Sigma_{\phi, S}, \mathcal{R}_{\phi, S}) \\
 \mathcal{L}(j) \downarrow & & \downarrow \phi(\mathcal{L}(j)) \\
 F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) & \xrightarrow{\rho_{\phi, S_1}} & \text{Hom}_{\tau_{\phi}(S_1)}(\Sigma_{\phi, S_1}, \mathcal{R}_{\phi, S_1})
 \end{array} \tag{11.25}$$

At first sight, this requirement seems very appealing. However, caution is needed when postulating ‘axioms’ for a theoretical structure in physics. It is easy to get captivated by the underlying mathematics and to assume, erroneously, that what is mathematically elegant is necessarily true in the physical theory.

The translation  $\phi(\mathcal{L}(j))$  maps an arrow from  $\Sigma_{\phi, S}$  to  $\mathcal{R}_{\phi, S}$  to an arrow from  $\Sigma_{\phi, S_1}$  to  $\mathcal{R}_{\phi, S_1}$ . Intuitively, if  $\Sigma_{\phi, S_1}$  is a ‘much larger’ object than  $\Sigma_{\phi, S}$  (although since they lie in different topoi, no direct comparison is available), the translation can only be ‘faithful’ on some part of  $\Sigma_{\phi, S_1}$  that can be identified with (the ‘image’ of)  $\Sigma_{\phi, S}$ . A concrete example of this will show up in the treatment of composite quantum systems, see Subsection 13.3. As one might expect, a form of entanglement plays a role here.

## 11.4 Classical Physics in This Form

### 11.4.1 The Rules so Far.

Constructing maps  $\phi(\mathcal{L}(j)) : \text{Hom}_{\tau_{\phi}(S)}(\Sigma_{\phi, S}, \mathcal{R}_{\phi, S}) \rightarrow \text{Hom}_{\tau_{\phi}(S_1)}(\Sigma_{\phi, S_1}, \mathcal{R}_{\phi, S_1})$  is likely to be complicated when  $\tau_{\phi}(S)$  and  $\tau_{\phi}(S_1)$  are different topoi, and so we begin with the example of classical physics, where the topos is always **Sets**.

In general, we are interested in the relation(s) between the representations  $\rho_{\phi, S_1} : \mathcal{L}(S_1) \rightsquigarrow \tau_{\phi}(S_1)$  and  $\rho_{\phi, S} : \mathcal{L}(S) \rightsquigarrow \tau_{\phi}(S)$  that is associated with an arrow  $j : S_1 \rightarrow S$  in **Sys**. In classical physics, we only have to study the relation between the representations  $\rho_{\sigma, S_1} : \mathcal{L}(S_1) \rightsquigarrow \mathbf{Sets}$  and  $\rho_{\sigma, S} : \mathcal{L}(S) \rightsquigarrow \mathbf{Sets}$ .

Let us summarise what we have said so far (with  $\sigma$  denoting the **Sets**-realisation of classical physics):

1. For any system  $S$  in **Sys**, a representation  $\rho_{\sigma,S} : \mathcal{L}(S) \rightsquigarrow \mathbf{Sets}$  consists of the following ingredients.
  - (a) The ground symbol  $\Sigma$  is represented by a symplectic manifold,  $\Sigma_{\sigma,S} := \rho_{\sigma,S}(\Sigma)$ , that serves as the classical state space.
  - (b) For all systems  $S$ , the ground symbol  $\mathcal{R}$  is represented by the real numbers  $\mathbb{R}$ , i.e.,  $\mathcal{R}_{\sigma,S} = \mathbb{R}$ , where  $\mathcal{R}_{\sigma,S} := \rho_{\sigma,S}(\mathcal{R})$ .
  - (c) Each function symbol  $A : \Sigma \rightarrow \mathcal{R}$  in  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  is represented by a function  $A_{\sigma,S} = \rho_{\sigma,S}(A) : \Sigma_{\sigma,S} \rightarrow \mathbb{R}$  in the set of functions<sup>108</sup>  $C(\Sigma_{\sigma,S}, \mathbb{R})$ .
2. The trivial system is mapped to a singleton set  $\{*\}$  (viewed as a zero-dimensional symplectic manifold):

$$\Sigma_{\sigma,1} := \{*\}. \quad (11.26)$$

The empty system is represented by the empty set:

$$\Sigma_{\sigma,0} := \emptyset. \quad (11.27)$$

3. Propositions about the system  $S$  are represented by (Borel) subsets of the state space  $\Sigma_{\sigma,S}$ .
4. The composite system  $S_1 \diamond S_2$  is represented by the Cartesian product  $\Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}$ ; i.e.,

$$\Sigma_{\sigma,S_1 \diamond S_2} \simeq \Sigma_{\sigma,S_1} \times \Sigma_{\sigma,S_2}. \quad (11.28)$$

The disjoint sum  $S_1 \sqcup S_2$  is represented by the disjoint union  $\Sigma_{\sigma,S_1} \coprod \Sigma_{\sigma,S_2}$ ; i.e.,

$$\Sigma_{\sigma,S_1 \sqcup S_2} \simeq \Sigma_{\sigma,S_1} \coprod \Sigma_{\sigma,S_2}. \quad (11.29)$$

5. Let  $j : S_1 \rightarrow S$  be an arrow in **Sys**. Then
  - (a) There is a translation map  $\mathcal{L}(j) : F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ .
  - (b) There is a symplectic function  $\sigma(j) : \Sigma_{\sigma,S_1} \rightarrow \Sigma_{\sigma,S}$  from the symplectic manifold  $\Sigma_{\sigma,S_1}$  to the symplectic manifold  $\Sigma_{\sigma,S}$ .

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<sup>108</sup>In practice, these functions are required to be measurable with respect to the Borel structures on the symplectic manifold  $\Sigma_{\sigma}$  and  $\mathbb{R}$ . Many of the functions will also be smooth, but we will not go into such details here.

The existence of this function  $\sigma(j) : \Sigma_{\sigma, S_1} \rightarrow \Sigma_{\sigma, S}$  follows directly from the properties of sub-systems and composite systems in classical physics. It is discussed in detail below in Section (11.4.2). As we shall see, it underpins the classical realisation of our axioms.

These properties of the arrows stem from the fact that the linguistic function symbols in  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  are represented by real-valued functions in  $C(\Sigma_{\sigma, S}, \mathbb{R})$ . Thus we can write  $\rho_{\sigma, S} : F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow C(\Sigma_{\sigma, S}, \mathbb{R})$ , and similarly  $\rho_{\sigma, S_1} : F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \rightarrow C(\Sigma_{\sigma, S_1}, \mathbb{R})$ . The diagram in (11.25) now becomes

$$\begin{array}{ccc}
 F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) & \xrightarrow{\rho_{\sigma, S}} & C(\Sigma_{\sigma, S}, \mathbb{R}) \\
 \mathcal{L}(j) \downarrow & & \downarrow \sigma(\mathcal{L}(j)) \\
 F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) & \xrightarrow{\rho_{\sigma, S_1}} & C(\Sigma_{\sigma, S_1}, \mathbb{R})
 \end{array} \tag{11.30}$$

and, therefore, the question of interest is if there is a ‘translation representation’ function  $\sigma(\mathcal{L}(j)) : C(\Sigma_{\sigma, S}, \mathbb{R}) \rightarrow C(\Sigma_{\sigma, S_1}, \mathbb{R})$  so that this diagram commutes.

Now, as stated above, a physical quantity,  $A$ , for the system  $S$  is represented in classical physics by a real-valued function  $A_{\sigma, S} = \rho_{\sigma, S}(A) : \Sigma_{\sigma, S} \rightarrow \mathbb{R}$ . Similarly, the representation of  $\mathcal{L}(j)(A)$  for  $S_1$  is given by a function  $A_{\sigma, S_1} := \rho_{\sigma, S_1}(A) : \Sigma_{\sigma, S_1} \rightarrow \mathbb{R}$ . However, in this classical case we also have the function  $\sigma(j) : \Sigma_{\sigma, S_1} \rightarrow \Sigma_{\sigma, S}$ , and it is clear that we can use it to define  $[\rho_{\sigma, S_1}(\mathcal{L}(j)(A))](s) := \rho_{\sigma, S}(A)(\sigma(j)(s))$  for all  $s \in \Sigma_{\sigma, S_1}$ . In other words

$$\rho_{\sigma, S_1}(\mathcal{L}(j)(A)) = \rho_{\sigma, S}(A) \circ \sigma(j) \tag{11.31}$$

or, in simpler notation

$$((\mathcal{L}(j)(A))_{\sigma, S_1}) = A_{\sigma, S} \circ \sigma(j). \tag{11.32}$$

But then it is clear that a translation-representation function  $\sigma(\mathcal{L}(j)) : C(\Sigma_{\sigma, S}, \mathbb{R}) \rightarrow C(\Sigma_{\sigma, S_1}, \mathbb{R})$  with the desired property of making (11.30) commute can be defined by

$$\sigma(\mathcal{L}(j))(f) := f \circ \sigma(j) \tag{11.33}$$

for all  $f \in C(\Sigma_{\sigma, S}, \mathbb{R})$ ; i.e., the function  $\sigma(\mathcal{L}(j))(f) : \Sigma_{\sigma, S_1} \rightarrow \mathbb{R}$  is the usual pull-back of the function  $f : \Sigma_{\sigma, S} \rightarrow \mathbb{R}$  by the function  $\sigma(j) : \Sigma_{\sigma, S_1} \rightarrow \Sigma_{\sigma, S}$ .

Thus, in the case of classical physics, the commutative diagram in (11.22) can be completed to give

$$\begin{array}{ccc}
S_1 & \xrightarrow{\sigma} & \langle \rho_{\sigma, S_1}, \mathcal{L}(S_1), \mathbf{Sets} \rangle \\
j \downarrow & & \uparrow \sigma(\mathcal{L}(j)) \times \mathcal{L}(j) \times \text{id} \\
S & \xrightarrow{\sigma} & \langle \rho_{\sigma, S}, \mathcal{L}(S), \mathbf{Sets} \rangle
\end{array} \tag{11.34}$$

#### 11.4.2 Details of the Translation Representation.

**The translation representation for a disjoint sum of classical systems.** We first consider arrows of the form

$$S_1 \xrightarrow{i_1} S_1 \sqcup S_2 \xleftarrow{i_2} S_2 \tag{11.35}$$

from the components  $S_1, S_2$  to the disjoint sum  $S_1 \sqcup S_2$ . The systems  $S_1, S_2$  and  $S_1 \sqcup S_2$  have symplectic manifolds  $\Sigma_{\sigma, S_1}, \Sigma_{\sigma, S_2}$  and  $\Sigma_{\sigma, S_1 \sqcup S_2} = \Sigma_{\sigma, S_1} \amalg \Sigma_{\sigma, S_2}$ . We write  $i := i_1$ .

Let  $S$  be a classical system. We assume that the function symbols  $A \in F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  in the language  $\mathcal{L}(S)$  are in bijective correspondence with an appropriate subset of the functions  $A_{\sigma, S} \in C(\Sigma_{\sigma, S}, \mathbb{R})$ .<sup>109</sup>

There is an obvious translation representation. For if  $A \in F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R})$ , then since  $\Sigma_{\sigma, S_1 \sqcup S_2} = \Sigma_{\sigma, S_1} \amalg \Sigma_{\sigma, S_2}$ , the associated function  $A_{\sigma, S_1 \sqcup S_2} : \Sigma_{\sigma, S_1 \sqcup S_2} \rightarrow \mathbb{R}$  is given by a pair of functions  $A_1 \in C(\Sigma_{\sigma, S_1}, \mathbb{R})$  and  $A_2 \in C(\Sigma_{\sigma, S_2}, \mathbb{R})$ ; we write  $A_{\sigma, S_1 \sqcup S_2} = \langle A_1, A_2 \rangle$ . It is natural to demand that the translation representation  $\sigma(\mathcal{L}(i))(A_{\sigma, S_1 \sqcup S_2})$  is  $A_1$ . Note that what is essentially being discussed here is the classical-physics representation of the relation (11.1).

The canonical choice for  $\sigma(i)$  is

$$\sigma(i) : \Sigma_{\sigma, S_1} \rightarrow \Sigma_{\sigma, S_1 \sqcup S_2} = \Sigma_{\sigma, S_1} \amalg \Sigma_{\sigma, S_2} \tag{11.36}$$

$$s_1 \mapsto s_1. \tag{11.37}$$

Then the pull-back along  $\sigma(i)$ ,

$$\sigma(i)^* : C(\Sigma_{\sigma, S_1 \sqcup S_2}, \mathbb{R}) \rightarrow C(\Sigma_{\sigma, S_1}, \mathbb{R}) \tag{11.38}$$

$$A_{\sigma, S_1 \sqcup S_2} \mapsto A_{\sigma, S_1 \sqcup S_2} \circ \sigma(i), \tag{11.39}$$

<sup>109</sup>Depending on the setting, one can assume that  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  contains function symbols corresponding bijectively to measurable, continuous or smooth functions.

maps (or ‘translates’) the topos representative  $A_{\sigma, S_1 \sqcup S_2} = \langle A_1, A_2 \rangle$  of the function symbol  $A \in F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R})$  to a real-valued function  $A_{\sigma, S_1 \sqcup S_2} \circ \sigma(i)$  on  $\Sigma_{\sigma, S_1}$ . This function is clearly equal to  $A_1$ .

**The translation in the case of a composite classical system.** We now consider arrows in **Sys** of the form

$$S_1 \xleftarrow{p_1} S_1 \diamond S_2 \xrightarrow{p_2} S_2 \quad (11.40)$$

from the composite classical system  $S_1 \diamond S_2$  to the constituent systems  $S_1$  and  $S_2$ . Here,  $p_1$  signals that  $S_1$  is a constituent of the composite system  $S_1 \diamond S_2$ , likewise  $p_2$ . The systems  $S_1$ ,  $S_2$  and  $S_1 \diamond S_2$  have symplectic manifolds  $\Sigma_{\sigma, S_1}$ ,  $\Sigma_{\sigma, S_2}$  and  $\Sigma_{\sigma, S_1 \diamond S_2} = \Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2}$ , respectively; i.e., the state space of the composite system  $S_1 \diamond S_2$  is the cartesian product of the state spaces of the components. For typographical simplicity in what follows we denote  $p := p_1$ .

There is a canonical translation  $\mathcal{L}(p)$  between the languages  $\mathcal{L}(S_1)$  and  $\mathcal{L}(S_1 \diamond S_2)$  whose representation is the following. Namely, if  $A$  is in  $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ , then the corresponding function  $A_{\sigma, S_1} \in C(\Sigma_{\sigma, S_1}, \mathbb{R})$  is translated to a function  $\sigma(\mathcal{L}(p))(A_{\sigma, S_1}) \in C(\Sigma_{\sigma, S_1 \diamond S_2}, \mathbb{R})$  such that

$$\sigma(\mathcal{L}(p))(A_{\sigma, S_1})(s_1, s_2) = A_{\sigma, S_1}(s_1) \quad (11.41)$$

for all  $(s_1, s_2) \in \Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2}$ .

This natural translation representation is based on the fact that, for the symplectic manifold  $\Sigma_{\sigma, S_1 \diamond S_2} = \Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2}$ , each point  $s \in \Sigma_{\sigma, S_1 \diamond S_2}$  can be identified with a pair,  $(s_1, s_2)$ , of points  $s_1 \in \Sigma_{\sigma, S_1}$  and  $s_2 \in \Sigma_{\sigma, S_2}$ . This is possible since the cartesian product  $\Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2}$  is a product in the categorial sense and hence has projections  $\Sigma_{\sigma, S_1} \leftarrow \Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2} \rightarrow \Sigma_{\sigma, S_2}$ . Then the translation representation of functions is constructed in a straightforward manner. Thus, let

$$\begin{aligned} \sigma(p) : \Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2} &\rightarrow \Sigma_{\sigma, S_1} \\ (s_1, s_2) &\mapsto s_1 \end{aligned} \quad (11.42)$$

be the canonical projection. Then, if  $A_{\sigma, S_1} \in C(\Sigma_{\sigma, S_1}, \mathbb{R})$ , the function

$$A_{\sigma, S_1} \circ \sigma(p) \in C(\Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2}, \mathbb{R}) \quad (11.43)$$

is such that, for all  $(s_1, s_2) \in \Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2}$ ,

$$A_{\sigma, S_1} \circ \sigma(p)(s_1, s_2) = A_{\sigma, S_1}(s_1). \quad (11.44)$$



Thus we can define

$$\sigma(\mathcal{L}(p))(A_{\sigma, S_1}) := A_{\sigma, S_1} \circ \sigma(p). \quad (11.45)$$

Clearly,  $\sigma(\mathcal{L}(p))(A_{\sigma, S_1})$  can be seen as the representation of the function symbol  $A \diamond 1 \in F_{\mathcal{L}(S_1 \diamond S_2)}(\Sigma, \mathcal{R})$ .

## 12 Theories of Physics in a General Topos

### 12.1 The Pull-Back Operations

#### 12.1.1 The Pull-Back of Physical Quantities.

Motivated by the above, let us try now to see what can be said about the scheme in general. Basically, what is involved is the topos representation of translations of languages. To be more precise, let  $j : S_1 \rightarrow S$  be an arrow in **Sys**, so that there is a translation  $\mathcal{L}(j) : \mathcal{L}(S) \rightarrow \mathcal{L}(S_1)$  defined by the translation function  $\mathcal{L}(j) : F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ . Now suppose that the systems  $S$  and  $S_1$  are represented in the topoi  $\tau_\phi(S)$  and  $\tau_\phi(S_1)$  respectively. Then, in these representations, the function symbols of signature  $\Sigma \rightarrow \mathcal{R}$  in  $\mathcal{L}(S)$  and  $\mathcal{L}(S_1)$  are represented by elements of  $\text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi, S}, \mathcal{R}_{\phi, S})$  and  $\text{Hom}_{\tau_\phi(S_1)}(\Sigma_{\phi, S_1}, \mathcal{R}_{\phi, S_1})$  respectively.

Our task is to find a function

$$\phi(\mathcal{L}(j)) : \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi, S}, \mathcal{R}_{\phi, S}) \rightarrow \text{Hom}_{\tau_\phi(S_1)}(\Sigma_{\phi, S_1}, \mathcal{R}_{\phi, S_1}) \quad (12.1)$$

that can be construed as the topos representation of the translation  $\mathcal{L}(j) : \mathcal{L}(S) \rightarrow \mathcal{L}(S_1)$ , and hence of the arrow  $j : S_1 \rightarrow S$  in **Sys**. We are particularly interested in seeing if  $\phi(\mathcal{L}(j))$  can be chosen so that the following diagram, (see (11.25)) commutes:

$$\begin{array}{ccc} F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) & \xrightarrow{\rho_{\phi, S}} & \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi, S}, \mathcal{R}_{\phi, S}) \\ \mathcal{L}(j) \downarrow & & \downarrow \phi(\mathcal{L}(j)) \\ F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) & \xrightarrow{\rho_{\phi, S_1}} & \text{Hom}_{\tau_\phi(S_1)}(\Sigma_{\phi, S_1}, \mathcal{R}_{\phi, S_1}) \end{array} \quad (12.2)$$

However, as has been emphasised already, it is not clear that one *should* expect to find a function  $\phi(\mathcal{L}(j)) : \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi, S}, \mathcal{R}_{\phi, S}) \rightarrow \text{Hom}_{\tau_\phi(S_1)}(\Sigma_{\phi, S_1}, \mathcal{R}_{\phi, S_1})$

with this property. The existence and/or properties of such a function will be dependent on the theory-type, and it seems unlikely that much can be said in general about the diagram (12.2). Nevertheless, let us see how far we *can* get in discussing the existence of such a function in general.

Thus, if  $\mu \in \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S})$ , the critical question is if there is some ‘natural’ way whereby this arrow can be ‘pulled-back’ to give an element  $\phi(\mathcal{L}(j))(\mu) \in \text{Hom}_{\tau_\phi(S_1)}(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1})$ .

The first pertinent remark is that  $\mu$  is an arrow in the topos  $\tau_\phi(S)$ , whereas the sought-for pull-back will be an arrow in the topos  $\tau_\phi(S_1)$ , and so we need a mechanism for getting from one topos to the other (this problem, of course, does not arise in classical physics since the topos of every representation is always **Sets**).

The obvious way of implementing this change of topos is via some *functor*,  $\tau_\phi(j)$  from  $\tau_\phi(S)$  to  $\tau_\phi(S_1)$ . Indeed, given such a functor, an arrow  $\mu : \Sigma_{\phi,S} \rightarrow \mathcal{R}_{\phi,S}$  in  $\tau_\phi(S)$  is transformed to the arrow

$$\tau_\phi(j)(\mu) : \tau_\phi(j)(\Sigma_{\phi,S}) \rightarrow \tau_\phi(j)(\mathcal{R}_{\phi,S}) \quad (12.3)$$

in  $\tau_\phi(S_1)$ .

To convert this to an arrow from  $\Sigma_{\phi,S_1}$  to  $\mathcal{R}_{\phi,S_1}$ , we need to supplement (12.3) with a pair of arrows  $\phi(j), \beta_\phi(j)$  in  $\tau_\phi(S_1)$  to get the diagram:

$$\begin{array}{ccc} \tau_\phi(j)(\Sigma_{\phi,S}) & \xrightarrow{\tau_\phi(j)(\mu)} & \tau_\phi(j)(\mathcal{R}_{\phi,S}) \\ \phi(j) \uparrow & & \downarrow \beta_\phi(j) \\ \Sigma_{\phi,S_1} & & \mathcal{R}_{\phi,S_1} \end{array} \quad (12.4)$$

The pull-back,  $\phi(\mathcal{L}(j))(\mu) \in \text{Hom}_{\tau_\phi(S_1)}(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1})$ , with respect to these choices can then be defined as

$$\phi(\mathcal{L}(j))(\mu) := \beta_\phi(j) \circ \tau_\phi(j)(\mu) \circ \phi(j). \quad (12.5)$$

It follows that a key part of the construction of a topos representation,  $\phi$ , of **Sys** will be to specify the functor  $\tau_\phi(j)$  from  $\tau_\phi(S)$  to  $\tau_\phi(S_1)$ , and the arrows  $\phi(j) : \Sigma_{\phi,S_1} \rightarrow \tau_\phi(j)(\Sigma_{\phi,S})$  and  $\beta_\phi(j) : \tau_\phi(j)(\mathcal{R}_{\phi,S}) \rightarrow \mathcal{R}_{\phi,S_1}$  in the topos  $\tau_\phi(S_1)$ . These need to be defined in such a way as to be consistent with a chain of arrows  $S_2 \rightarrow S_1 \rightarrow S$ .

When applied to the representative  $A_{\phi,S} : \Sigma_{\phi,S} \rightarrow \mathcal{R}_{\phi,S}$  of a physical quantity  $A \in F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ , the diagram (12.4) becomes (augmented with the

upper half)

$$\begin{array}{ccc}
\Sigma_{\phi,S} & \xrightarrow{A_{\phi,S}} & \mathcal{R}_{\phi,S} \\
\tau_{\phi}(j) \downarrow & & \downarrow \tau_{\phi}(j) \\
\tau_{\phi}(j)(\Sigma_{\phi,S}) & \xrightarrow{\tau_{\phi}(j)(A_{\phi,S})} & \tau_{\phi}(j)(\mathcal{R}_{\phi,S}) \\
\phi(j) \uparrow & & \downarrow \beta_{\phi}(j) \\
\Sigma_{\phi,S_1} & \xrightarrow{\phi(\mathcal{L}(j))(A_{\phi,S})} & \mathcal{R}_{\phi,S_1}
\end{array} \tag{12.6}$$

The commutativity of (12.2) would then require

$$\phi(\mathcal{L}(j))(A_{\phi,S}) = (\mathcal{L}(j)A)_{\phi,S_1} \tag{12.7}$$

or, in a more expanded notation,

$$\phi(\mathcal{L}(j)) \circ \rho_{\phi,S} = \rho_{\phi,S_1} \circ \mathcal{L}(j), \tag{12.8}$$

where both the left hand side and the right hand side of (12.8) are mappings from  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  to  $\text{Hom}_{\tau_{\phi}(S_1)}(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1})$ .

Note that the analogous diagram in classical physics is simply

$$\begin{array}{ccc}
\Sigma_{\sigma,S} & \xrightarrow{A_{\sigma,S}} & \mathbb{R} \\
\sigma(j) \uparrow & & \downarrow \text{id} \\
\Sigma_{\sigma,S_1} & \xrightarrow{\sigma(\mathcal{L}(j))(A_{\sigma,S})} & \mathbb{R}
\end{array} \tag{12.9}$$

and the commutativity/pull-back condition (12.7) becomes

$$\sigma(\mathcal{L}(j))(A_{\sigma,S}) = (\mathcal{L}(j)A)_{\sigma,S_1} \tag{12.10}$$

which is satisfied by virtue of (11.33).

It is clear from the above that the arrow  $\phi(j) : \Sigma_{\phi,S_1} \rightarrow \tau_{\phi}(j)(\Sigma_{\phi,S})$  can be viewed as the topos analogue of the map  $\sigma(j) : \Sigma_{\sigma,S_1} \rightarrow \Sigma_{\sigma,S}$  that arises in classical physics whenever there is an arrow  $j : S_1 \rightarrow S$ .

### 12.1.2 The Pull-Back of Propositions.

More insight can be gained into the nature of the triple  $\langle \tau_{\phi}(j), \phi(j), \beta_{\phi}(j) \rangle$  by considering the analogous operation for propositions. First, consider an

arrow  $j : S_1 \rightarrow S$  in **Sys** in classical physics. Associated with this there is (i) a translation  $\mathcal{L}(j) : \mathcal{L}(S) \rightarrow \mathcal{L}(S_1)$ ; (ii) an associated translation mapping  $\mathcal{L}(j) : F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ ; and (iii) a symplectic function  $\sigma(j) : \Sigma_{\sigma, S_1} \rightarrow \Sigma_{\sigma, S}$ .

Let  $K$  be a (Borel) subset of the state space,  $\Sigma_{\sigma, S}$ ; hence  $K$  represents a proposition about the system  $S$ . Then  $\sigma(j)^*(K) := \sigma(j)^{-1}(K)$  is a subset of  $\Sigma_{\sigma, S_1}$  and, as such, represents a proposition about the system  $S_1$ . We say that  $\sigma(j)^*(K)$  is the *pull-back* to  $\Sigma_{\sigma, S_1}$  of the  $S$ -proposition represented by  $K$ . The existence of such pull-backs is part of the consistency of the representation of propositions in classical mechanics, and it is important to understand what the analogue of this is in our topos scheme.

Consider the general case with the two systems  $S_1, S$  as above. Then let  $K$  be a proposition, represented as a sub-object of  $\Sigma_{\phi, S}$ , with a monic arrow  $i_K : K \hookrightarrow \Sigma_{\phi, S}$ . The question now is if the triple  $\langle \tau_\phi(j), \phi(j), \beta_\phi(j) \rangle$  can be used to pull  $K$  back to give a proposition in  $\tau(S_1)$ , i.e., a sub-object of  $\Sigma_{\phi, S_1}$ ?

The first requirement is that the functor  $\tau_\phi(j) : \tau_\phi(S) \rightarrow \tau_\phi(S_1)$  should *preserve monics*. In this case, the monic arrow  $i_K : K \hookrightarrow \Sigma_{\phi, S}$  in  $\tau_\phi(S)$  is transformed to the monic arrow

$$\tau_\phi(j)(i_K) : \tau_\phi(j)(K) \hookrightarrow \tau_\phi(j)(\Sigma_{\phi, S}) \quad (12.11)$$

in  $\tau_\phi(S_1)$ ; thus  $\tau_\phi(j)(K)$  is a sub-object of  $\tau_\phi(j)(\Sigma_{\phi, S})$  in  $\tau_\phi(S_1)$ . It is a property of a topos that the pull-back of a monic arrow is monic ; i.e., if  $M \hookrightarrow Y$  is monic, and if  $\psi : X \rightarrow Y$ , then  $\psi^{-1}(M)$  is a sub-object of  $X$ . Therefore, in the case of interest, the monic arrow  $\tau_\phi(j)(i_K) : \tau_\phi(j)(K) \hookrightarrow \tau_\phi(j)(\Sigma_{\phi, S})$  can be pulled back along  $\phi(j) : \Sigma_{\phi, S_1} \rightarrow \tau_\phi(j)(\Sigma_{\phi, S})$  (see diagram (12.6)) to give the monic  $\phi(j)^{-1}(\tau_\phi(j)(K)) \subseteq \Sigma_{\phi, S_1}$ . This is a candidate for the pull-back of the proposition represented by the sub-object  $K \subseteq \Sigma_{\phi, S}$ .

In conclusion, propositions can be pulled-back provided that the functor  $\tau_\phi(j) : \tau_\phi(S) \rightarrow \tau_\phi(S_1)$  preserves monics. A sufficient way of satisfying this requirement is for  $\tau_\phi(j)$  to be left-exact. However, this raises the question of “where do left-exact functors come from?”.

### 12.1.3 The Idea of a Geometric Morphism.

It transpires that there *is* a natural source of left-exact functors, via the idea of a *geometric morphism*. This fundamental concept in topos theory is defined as follows [56].

**Definition 12.1** A geometric morphism  $\phi : \mathcal{F} \rightarrow \mathcal{E}$  between topoi  $\mathcal{F}$  and  $\mathcal{E}$  is a pair of functors  $\phi^* : \mathcal{E} \rightarrow \mathcal{F}$  and  $\phi_* : \mathcal{F} \rightarrow \mathcal{E}$  such that

(i)  $\phi^* \dashv \phi_*$ , i.e.,  $\phi^*$  is left adjoint to  $\phi_*$ ;

(ii)  $\phi^*$  is left exact, i.e., it preserves all finite limits.

The morphism  $\phi^* : \mathcal{E} \rightarrow \mathcal{F}$  is called the inverse image part of the geometric morphism  $\varphi$ ;  $\phi_* : \mathcal{F} \rightarrow \mathcal{E}$  is called the direct image part.

Geometric morphisms are very important because they are the topos equivalent of continuous functions. More precisely, if  $X$  and  $Y$  are topological spaces, then any continuous function  $f : X \rightarrow Y$  induces a geometric morphism between the topoi  $\text{Sh}(X)$  and  $\text{Sh}(Y)$  of sheaves on  $X$  and  $Y$  respectively. In practice, just as the arrows in the category of topological spaces are continuous functions, so in any category whose objects are topoi, the arrows are normally defined to be geometric morphisms. In our case, as we shall shortly see, all the examples of left-exact functors that arise in the quantum case do, in fact, come from geometric morphisms. For these reasons, from now on we will postulate that any arrows between our topoi arise from geometric morphisms.

One central property of a geometric morphism is that it preserves expressions written in terms of geometric logic. This greatly enhances the attractiveness of assuming from the outset that the internal logic of the system languages,  $\mathcal{L}(S)$ , is restricted to the sub-logic afforded by geometric logic.

*En passant*, another key result for us is the following theorem ([56] p359):

**Theorem 12.1** *If  $\varphi : \mathcal{C} \rightarrow \mathcal{D}$  is a functor between categories  $\mathcal{C}$  and  $\mathcal{D}$ , then it induces a geometric morphism (also denoted  $\varphi$ )*

$$\varphi : \mathbf{Sets}^{\mathcal{C}^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{D}^{\text{op}}} \quad (12.12)$$

for which the functor  $\varphi^* : \mathbf{Sets}^{\mathcal{D}^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{C}^{\text{op}}}$  takes a functor  $\underline{F} : \mathcal{D} \rightarrow \mathbf{Sets}$  to the functor

$$\varphi^*(\underline{F}) := \underline{F} \circ \varphi^{\text{op}} \quad (12.13)$$

from  $\mathcal{C}$  to  $\mathbf{Sets}$ .

In addition,  $\varphi^*$  has a left adjoint  $\varphi_!$ ; i.e.,  $\varphi_! \dashv \varphi^*$ .

We will use this important theorem in several crucial places.

## 12.2 The Topos Rules for Theories of Physics

We will now present our general rules for using topos theory in the mathematical representation of physical systems and their theories.

**Definition 12.2** *The category  $\mathcal{M}(\mathbf{Sys})$  is the following:*

1. *The objects of  $\mathcal{M}(\mathbf{Sys})$  are the topoi that are to be used in representing the systems in  $\mathbf{Sys}$ .*
2. *The arrows from  $\tau_1$  to  $\tau_2$  are defined to be the geometric morphisms from  $\tau_2$  to  $\tau_1$ . Thus the inverse part,  $\varphi^*$ , of an arrow  $\varphi^* : \tau_1 \rightarrow \tau_2$  is a left-exact functor from  $\tau_1$  to  $\tau_2$ .*

**Definition 12.3** *The rules for using topos theory are as follows:*

1. A *topos realisation*,  $\phi$ , of  $\mathbf{Sys}$  in  $\mathcal{M}(\mathbf{Sys})$  is an assignment, to each system  $S$  in  $\mathbf{Sys}$ , of a triple  $\phi(S) = \langle \rho_{\phi,S}, \mathcal{L}(S), \tau_\phi(S) \rangle$  where:
  - (a)  $\tau_\phi(S)$  is the topos in  $\mathcal{M}(\mathbf{Sys})$  in which the physical theory of system  $S$  is to be realised.
  - (b)  $\mathcal{L}(S)$  is the local language that is associated with  $S$ . This is independent of the realisation,  $\phi$ , of  $\mathbf{Sys}$  in  $\mathcal{M}(\mathbf{Sys})$ .
  - (c)  $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$  is a representation of the local language  $\mathcal{L}(S)$  in the topos  $\tau_\phi(S)$ .
  - (d) In addition, for each arrow  $j : S_1 \rightarrow S$  in  $\mathbf{Sys}$  there is a triple  $\langle \tau_\phi(j), \phi(j), \beta_\phi(j) \rangle$  that interpolates between  $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$  and  $\rho_{\phi,S_1} : \mathcal{L}(S_1) \rightsquigarrow \tau_\phi(S_1)$ ; for details see below.
2.
  - (a) The representations,  $\rho_{\phi,S}(\Sigma)$  and  $\rho_{\phi,S}(\mathcal{R})$ , of the ground symbols  $\Sigma$  and  $\mathcal{R}$  in  $\mathcal{L}(S)$  are denoted  $\Sigma_{\phi,S}$  and  $\mathcal{R}_{\phi,S}$ , respectively. They are known as the ‘state object’ and ‘quantity-value object’ in  $\tau_\phi(S)$ .
  - (b) The representation by  $\rho_{\phi,S}$  of each function symbol  $A : \Sigma \rightarrow \mathcal{R}$  of the system  $S$  is an arrow,  $\rho_{\phi,S}(A) : \Sigma_{\phi,S} \rightarrow \mathcal{R}_{\phi,S}$  in  $\tau_\phi(S)$ ; we will usually denote this arrow as  $A_{\phi,S} : \Sigma_{\phi,S} \rightarrow \mathcal{R}_{\phi,S}$ .
  - (c) Propositions about the system  $S$  are represented by sub-objects of  $\Sigma_{\phi,S}$ . These will typically be of the form  $A_{\phi,S}^{-1}(\Xi)$ , where  $\Xi$  is a sub-object of  $\mathcal{R}_{\phi,S}$ .<sup>110</sup>
3. Generally, there are no ‘microstates’ for the system  $S$ ; i.e., no global elements (arrows  $1 \rightarrow \Sigma_{\phi,S}$ ) of the state object  $\Sigma_{\phi,S}$ ; or, if there are any, they may not be enough to determine  $\Sigma_{\phi,S}$  as an object in  $\tau_\phi(S)$ .

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<sup>110</sup>Here,  $A_{\phi,S}^{-1}(\Xi)$  denotes the sub-object of  $\Sigma_{\phi,S}$  whose characteristic arrow is  $\chi_\Xi \circ A_{\phi,S} : \Sigma_{\phi,S} \rightarrow \Omega_{\tau_\phi(S)}$ , where  $\chi_\Xi : \mathcal{R}_{\phi,S} \rightarrow \Omega_{\tau_\phi(S)}$  is the characteristic arrow of the sub-object  $\Xi$ .

Instead, the role of a state is played by a ‘truth sub-object’  $\mathbb{T}$  of  $P\Sigma_{\phi,S}$ .<sup>111</sup> If  $J \in \text{Sub}(\Sigma_{\phi,S}) \simeq \Gamma(P\Sigma_{\phi,S})$ , the ‘truth of the proposition represented by  $J$ ’ is defined to be

$$\nu(J \in \mathbb{T}) = \llbracket \tilde{J} \in \tilde{\mathbb{T}} \rrbracket_{\phi} \circ \langle \ulcorner J \urcorner, \ulcorner \mathbb{T} \urcorner \rangle \quad (12.14)$$

See Section 6.2 for full information on the idea of a ‘truth object’. Alternatively, one may use pseudo-states rather than truth objects, in which case the relevant truth values are of the form  $\nu(\mathfrak{w} \subseteq J)$ .

4. There is a ‘unit object’  $1_{\mathcal{M}(\mathbf{Sys})}$  in  $\mathcal{M}(\mathbf{Sys})$  such that if  $1_{\mathbf{Sys}}$  denotes the trivial system in  $\mathbf{Sys}$  then, for all topos realisations  $\phi$ ,

$$\tau_{\phi}(1_{\mathbf{Sys}}) = 1_{\mathcal{M}(\mathbf{Sys})}. \quad (12.15)$$

Motivated by the results for quantum theory (see Section 13.2), we postulate that the unit object  $1_{\mathcal{M}(\mathbf{Sys})}$  in  $\mathcal{M}(\mathbf{Sys})$  is the category of sets:

$$1_{\mathcal{M}(\mathbf{Sys})} = \mathbf{Sets}. \quad (12.16)$$

5. To each arrow  $j : S_1 \rightarrow S$  in  $\mathbf{Sys}$ , we have the following:

- (a) There is a translation  $\mathcal{L}(j) : \mathcal{L}(S) \rightarrow \mathcal{L}(S_1)$ . This is specified by a map between function symbols:  $\mathcal{L}(j) : F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ .
- (b) With the translation  $\mathcal{L}(j) : F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$  there is associated a corresponding function

$$\phi(\mathcal{L}(j)) : \text{Hom}_{\tau_{\phi}(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}) \rightarrow \text{Hom}_{\tau_{\phi}(S_1)}(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}). \quad (12.17)$$

These may, or may not, fit together in the commutative diagram:

$$\begin{array}{ccc} F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) & \xrightarrow{\rho_{\phi,S}} & \text{Hom}_{\tau_{\phi}(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}) \\ \mathcal{L}(j) \downarrow & & \downarrow \phi(\mathcal{L}(j)) \\ F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) & \xrightarrow{\rho_{\phi,S_1}} & \text{Hom}_{\tau_{\phi}(S_1)}(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1}) \end{array} \quad (12.18)$$

<sup>111</sup>In classical physics, the truth object corresponding to a micro-state  $s$  is the collection of all propositions that are true in the state  $s$ .

(c) The function  $\phi(\mathcal{L}(j)) : \text{Hom}_{\tau_\phi(S)}(\Sigma_{\phi,S}, \mathcal{R}_{\phi,S}) \rightarrow \text{Hom}_{\tau_\phi(S_1)}(\Sigma_{\phi,S_1}, \mathcal{R}_{\phi,S_1})$  is built from the following ingredients. For each topos realisation  $\phi$ , there is a triple  $\langle \nu_\phi(j), \phi(j), \beta_\phi(j) \rangle$  where:

(i)  $\nu_\phi(j) : \tau_\phi(S_1) \rightarrow \tau_\phi(S)$  is a geometric morphism; i.e., an arrow in the category  $\mathcal{M}(\mathbf{Sys})$  (thus  $\nu_\phi(j)^* : \tau_\phi(S) \rightarrow \tau_\phi(S_1)$  is left exact).

**N.B.** To simplify the notation a little we will denote  $\nu_\phi(j)^*$  by  $\tau_\phi(j)$ . This is sensible in so far as, for the most part, only the inverse part of  $\nu_\phi(j)$  will be used in our constructions.

(ii)  $\phi(j) : \Sigma_{\phi,S_1} \rightarrow \tau_\phi(j)(\Sigma_{\phi,S})$  is an arrow in the topos  $\tau_\phi(S_1)$ .

(iii)  $\beta_\phi(j) : \tau_\phi(j)(\mathcal{R}_{\phi,S}) \rightarrow \mathcal{R}_{\phi,S_1}$  is an arrow in the topos  $\tau_\phi(S_1)$ .

These fit together in the diagram

$$\begin{array}{ccc}
 \Sigma_{\phi,S} & \xrightarrow{A_{\phi,S}} & \mathcal{R}_{\phi,S} \\
 \tau_\phi(j) \downarrow & & \downarrow \tau_\phi(j) \\
 \tau_\phi(j)(\Sigma_{\phi,S}) & \xrightarrow{\tau_\phi(j)(A_{\phi,S})} & \tau_\phi(j)(\mathcal{R}_{\phi,S}) \\
 \phi(j) \uparrow & & \downarrow \beta_\phi(j) \\
 \Sigma_{\phi,S_1} & \xrightarrow{\phi(\mathcal{L}(j))(A_{\phi,S})} & \mathcal{R}_{\phi,S_1}
 \end{array} \tag{12.19}$$

The arrows  $\phi(j)$  and  $\beta_\phi(j)$  should behave appropriately under composition of arrows in  $\mathbf{Sys}$ .

The commutativity of the diagram (12.18) is equivalent to the relation

$$\phi(\mathcal{L}(j))(A_{\phi,S}) = [\mathcal{L}(j)(A)]_{\phi,S_1} \tag{12.20}$$

for all  $A \in F_{\mathcal{L}(\phi,S)}(\Sigma, \mathcal{R})$ . As we keep emphasising, the satisfaction or otherwise of this relation will depend on the theory-type and, possibly, the representation  $\phi$ .

(d) If a proposition in  $\tau_\phi(S)$  is represented by the monic arrow,  $K \hookrightarrow \Sigma_{\phi,S}$ , the ‘pull-back’ of this proposition to  $\tau_\phi(S_1)$  is defined to be  $\phi(j)^{-1}(\tau_\phi(j)(K)) \subseteq \Sigma_{\phi,S_1}$ .

6. (a) If  $S_1$  is a sub-system of  $S$ , with an associated arrow  $i : S_1 \rightarrow S$  in  $\mathbf{Sys}$  then, in the diagram in (12.19), the arrow  $\phi(j) : \Sigma_{\phi,S_1} \rightarrow \tau_\phi(j)(\Sigma_{\phi,S})$  is a monic arrow in  $\tau_\phi(S_1)$ .



In other words,  $\Sigma_{\phi, S_1}$  is a sub-object of  $\tau_\phi(j)(\Sigma_{\phi, S})$ , which is denoted

$$\Sigma_{\phi, S_1} \subseteq \tau_\phi(j)(\Sigma_{\phi, S}). \quad (12.21)$$

We may also want to conjecture

$$\mathcal{R}_{\phi, S_1} \simeq \tau_\phi(j)(\mathcal{R}_{\phi, S}). \quad (12.22)$$

- (b) Another possible conjecture is the following: if  $j : S_1 \rightarrow S$  is an epic arrow in **Sys**, then, in the diagram in (12.19), the arrow  $\phi(j) : \Sigma_{\phi, S_1} \rightarrow \tau_\phi(j)(\Sigma_{\phi, S})$  is an epic arrow in  $\tau_\phi(S_1)$ .

In particular, for the epic arrow  $p_1 : S_1 \diamond S_2 \rightarrow S_1$ , the arrow  $\phi(p_1) : \Sigma_{\phi, S_1 \diamond S_2} \rightarrow \tau_\phi(\Sigma_{\phi, S_1})$  is an epic arrow in the topos  $\tau_\phi(S_1 \diamond S_2)$ .

One should not read Rule 2. above as implying that the choice of the state object and quantity-value object are *unique* for any given system  $S$ . These objects would at best be selected only up to isomorphism in the topos  $\tau(S)$ . Such morphisms in the  $\tau(S)$ <sup>112</sup> can be expected to play a key role in developing the topos analogue of the important idea of a *symmetry*, or *covariance* transformation of the theory.

In the example of classical physics, for all systems we have  $\tau(S) = \mathbf{Sets}$  and  $\Sigma_{\sigma, S}$  is a symplectic manifold, and the collection of all symplectic manifolds is a category. It would be elegant if we could assert that, in general, for a given theory-type the possible state objects in a given topos  $\tau$  form the objects of an *internal* category in  $\tau$ . However, to make such a statement would require a general theory of state-objects and, at the moment, we do not have such a thing.

From a more conceptual viewpoint we note that the ‘similarity’ of our axioms to those of standard classical physics is reflected in the fact that (i) physical quantities are represented by arrows  $A_{\phi, S} : \Sigma_{\phi, S} \rightarrow \mathcal{R}_{\phi, S}$ ; (ii) propositions are represented by sub-objects of  $\Sigma_{\phi, S}$ ; and (iii) propositions are assigned truth values. Thus any theory satisfying these axioms ‘looks’ like classical physics, and has an associated neo-realist interpretation.

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<sup>112</sup>Care is needed not to confuse morphisms in the topos  $\tau(S)$  with morphisms in the category  $\mathcal{M}(\mathbf{Sys})$  of topoi. An arrow from the object  $\tau(S)$  to itself in the category  $\mathcal{M}(\mathbf{Sys})$  is a geometric morphism in the topos  $\tau(S)$ . However, not every arrow in  $\tau(S)$  need arise in this way, and an important role can be expected to be played by arrows of this second type. A good example is when  $\tau(S)$  is the category of sets, **Sets**. Typically,  $\tau_\phi(j) : \mathbf{Sets} \rightarrow \mathbf{Sets}$  is the identity, but there are many morphisms from an object  $O$  in **Sets** to itself: they are just the functions from  $O$  to  $O$ .

## 13 The General Scheme applied to Quantum Theory

### 13.1 Background Remarks

We now want to study the extent to which our ‘rules’ apply to the topos representation of quantum theory.

For a quantum system with (separable) Hilbert space  $\mathcal{H}$ , the appropriate topos (what we earlier called  $\tau_\phi(S)$ ) is  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ : the category of presheaves over the category (actually, partially-ordered set)  $\mathcal{V}(\mathcal{H})$  of unital, abelian von Neumann sub-algebras of the algebra,  $B(\mathcal{H})$ , of bounded operators on  $\mathcal{H}$ .

A particularly important object in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  is the *spectral presheaf*  $\underline{\Sigma}$ , where, for each  $V$ ,  $\underline{\Sigma}_V$  is defined to be the Gel’fand spectrum of the abelian algebra  $V$ . The sub-objects of  $\underline{\Sigma}$  can be identified as the topos representations of propositions, just as the subsets of  $\mathcal{S}$  represent propositions in classical physics.

In Sections 8 and 9, several closely related choices for a quantity-value object  $\mathcal{R}_\phi$  in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  were discussed. In order to keep the notation simpler, we concentrate here on the presheaf  $\underline{\mathbb{R}}^\succeq$  of real-valued, order-reversing functions. All results hold analogously if the presheaf  $\underline{\mathbb{R}}^\leftrightarrow$  (which we actually prefer for giving a better physical interpretation) is used.<sup>113</sup>

Hence, physical quantities  $A : \Sigma \rightarrow \mathcal{R}$ , which correspond to self-adjoint operators  $\hat{A}$ , are represented by natural transformations/arrows  $\check{\delta}^\circ(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^\succeq$ . The mapping  $\hat{A} \mapsto \check{\delta}^\circ(\hat{A})$  is injective. For brevity, we write  $\check{\delta}(\hat{A}) := \check{\delta}^\circ(\hat{A})$ .<sup>114</sup>

### 13.2 The Translation Representation for a Disjoint Sum of Quantum Systems

Let  $\mathbf{Sys}$  be a category whose objects are systems that can be treated using quantum theory. Let  $\mathcal{L}(S)$  be the local language of a system  $S$  in  $\mathbf{Sys}$  whose quantum Hilbert space is denoted  $\mathcal{H}_S$ . We assume that to each function symbol,  $A : \Sigma \rightarrow \mathcal{R}$ , in  $\mathcal{L}(S)$  there is associated a self-adjoint operator

<sup>113</sup>Since the construction of the arrows  $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^\leftrightarrow$  involves both inner and outer daseinisation, we would have double work with the notation, which we avoid here.

<sup>114</sup>Note that this is *not* the same as the convention used earlier, where  $\check{\delta}(\hat{A})$  denoted a different natural transformation!

$\hat{A} \in \mathcal{B}(\mathcal{H}_S)$ ,<sup>115</sup> and that the map

$$F_{\mathcal{L}(S)}(\Sigma, \mathcal{R}) \rightarrow B(\mathcal{H})_{\text{sa}} \quad (13.1)$$

$$A \mapsto \hat{A} \quad (13.2)$$

is injective (but not necessarily surjective, as we will see in the case of a disjoint sum of quantum systems).

We consider first arrows of the form

$$S_1 \xrightarrow{i_1} S_1 \sqcup S_2 \xleftarrow{i_2} S_2 \quad (13.3)$$

from the components  $S_1, S_2$  to a disjoint sum  $S_1 \sqcup S_2$ ; for convenience we write  $i := i_1$ . The systems  $S_1, S_2$  and  $S_1 \sqcup S_2$  have the Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$  and  $\mathcal{H}_1 \oplus \mathcal{H}_2$ , respectively.

As always, the translation  $\mathcal{L}(i)$  goes in the opposite direction to the arrow  $i$ , so

$$\mathcal{L}(i) : F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R}) \rightarrow F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}). \quad (13.4)$$

Then our first step is find an ‘operator translation’ from the relevant self-adjoint operators in  $\mathcal{H}_1 \oplus \mathcal{H}_2$  to those in  $\mathcal{H}_1$ ,

To do this, let  $A$  be a function symbol in  $F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R})$ . In Section 11.2.1, we argued that  $F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R}) \simeq F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R}) \times F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R})$  (as in (11.1)), and hence we introduce the notation  $A = \langle A_1, A_2 \rangle$ , where  $A_1 \in F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$  and  $A_2 \in F_{\mathcal{L}(S_2)}(\Sigma, \mathcal{R})$ . It is then natural to assume that the quantisation scheme is such that the operator,  $\hat{A}$ , on  $\mathcal{H}_1 \oplus \mathcal{H}_2$  can be decomposed as  $\hat{A} = \hat{A}_1 \oplus \hat{A}_2$ , where the operators  $\hat{A}_1$  and  $\hat{A}_2$  are defined on  $\mathcal{H}_1$  and  $\mathcal{H}_2$  respectively, and correspond to the function symbols  $A_1$  and  $A_2$ .<sup>116</sup> Then the obvious operator translation is  $\hat{A} \mapsto \hat{A}_1 \in \mathcal{B}(\mathcal{H}_1)_{\text{sa}}$ .

We now consider the general rules in the Definition 12.2 and see to what extent they apply in the example of quantum theory.

**1.** As we have stated several times, the topos  $\tau_\phi(S)$  associated with a quantum system  $S$  is

$$\tau_\phi(S) = \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_S)^{\text{op}}}. \quad (13.5)$$

Thus (i) the objects of the category  $\mathcal{M}(\mathbf{Sys})$  are topoi of the form  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_S)^{\text{op}}}$ ,  $S \in \text{Ob}(\mathbf{Sys})$ ; and (ii) the arrows between two topoi are defined to be geometric morphisms. In particular, to each arrow  $j : S_1 \rightarrow S$  in  $\mathbf{Sys}$  there

<sup>115</sup>More specifically, one could postulate that the elements of  $F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$  are associated with self-adjoint operators in some unital von Neumann sub-algebra of  $\mathcal{B}(\mathcal{H}_S)$ .

<sup>116</sup>It should be noted that our scheme does not use all the self-adjoint operators on the direct sum  $\mathcal{H}_1 \oplus \mathcal{H}_2$ : only the ‘block diagonal’ operators of the form  $\hat{A} = \hat{A}_1 \oplus \hat{A}_2$  arise.

must correspond a geometric morphism  $\nu_\phi(j) : \tau_\phi(S_1) \rightarrow \tau_\phi(S)$  with associated left-exact functor  $\tau_\phi(j) := \nu_\phi(j)^* : \tau_\phi(S) \rightarrow \tau_\phi(S_1)$ . Of course, the existence of these functors in the quantum case has yet to be shown.

**2.** The realisation  $\rho_{\phi,S} : \mathcal{L}(S) \rightsquigarrow \tau_\phi(S)$  of the language  $\mathcal{L}(S)$  in the topos  $\tau_\phi(S)$  is given as follows. First, we define the state object  $\Sigma_{\phi,S}$  to be the spectral presheaf,  $\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_S)}$ , over  $\mathcal{V}(\mathcal{H}_S)$ , the context category of  $\mathcal{B}(\mathcal{H}_S)$ . To keep the notation brief, we will denote<sup>117</sup>  $\underline{\Sigma}^{\mathcal{V}(\mathcal{H}_S)}$  as  $\underline{\Sigma}^{\mathcal{H}_S}$ .

Furthermore, we define the quantity-value object,  $\mathcal{R}_{\phi,S}$ , to be the presheaf  $\underline{\mathbb{R}}^{\geq \mathcal{H}_S}$  that was defined in Section 8 . Finally, we define

$$A_{\phi,S} := \check{\delta}(\hat{A}), \quad (13.6)$$

for all  $A \in F_{\mathcal{L}(S)}(\Sigma, \mathcal{R})$ . Here  $\check{\delta}(\hat{A}) : \underline{\Sigma}^{\mathcal{H}_S} \rightarrow \underline{\mathbb{R}}^{\geq \mathcal{H}_S}$  is constructed using the Gel'fand transforms of the (outer) daseinisation of  $\hat{A}$ , for details see later.

**3.** The truth object  $\underline{\mathbb{T}}^{|\psi\rangle}$  corresponding to a pure state  $|\psi\rangle$  was discussed in Section 6.3. Alternatively, we have the pseudo-state  $\underline{\mathbf{m}}^{|\psi\rangle}$ .

**4.** Let  $\mathcal{H} = \mathbb{C}$  be the one-dimensional Hilbert space, corresponding to the trivial quantum system 1. There is exactly one abelian sub-algebra of  $\mathcal{B}(\mathbb{C}) \simeq \mathbb{C}$ , namely  $\mathbb{C}$  itself. This leads to

$$\tau_\phi(1_{\text{sys}}) = \mathbf{Sets}^{\{*\}} \simeq \mathbf{Sets} = 1_{\mathcal{M}(\text{sys})}. \quad (13.7)$$

**5.** Let  $A \in F_{\mathcal{L}(S_1 \sqcup S_2)}(\Sigma, \mathcal{R})$  be a function symbol for the system  $S_1 \sqcup S_2$ . Then, as discussed above,  $A$  is of the form  $A = \langle A_1, A_2 \rangle$  (compare equation (11.1)), which corresponds to a self-adjoint operator  $\hat{A}_1 \oplus \hat{A}_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)_{\text{sa}}$ . The topos representation of  $A$  is the natural transformation  $\check{\delta}(\langle A_1, A_2 \rangle) : \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \rightarrow \underline{\mathbb{R}}^{\geq \mathcal{H}_1 \oplus \mathcal{H}_2}$ , which is defined at each stage  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$  as

$$\begin{aligned} \check{\delta}(\langle A_1, A_2 \rangle)_V : \underline{\Sigma}_V^{\mathcal{H}_1 \oplus \mathcal{H}_2} &\rightarrow \underline{\mathbb{R}}_V^{\geq \mathcal{H}_1 \oplus \mathcal{H}_2} \\ \lambda &\mapsto \{V' \mapsto \langle \lambda|_{V'}, \delta(\hat{A}_1 \oplus \hat{A}_2)_{V'} \rangle \mid V' \subseteq V\} \end{aligned} \quad (13.8)$$

where the right hand side (13.8) denotes an order-reversing function.

We will need the following:

**Lemma 13.1** *Let  $\hat{A}_1 \oplus \hat{A}_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)_{\text{sa}}$ , and let  $V = V_1 \oplus V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$  such that  $V_1 \in \text{Ob}(\mathcal{V}(\mathcal{H}_1))$  and  $V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}_2))$ . Then*

$$\delta(\hat{A}_1 \oplus \hat{A}_2)_V = \delta(\hat{A}_1)_{V_1} \oplus \delta(\hat{A}_2)_{V_2}. \quad (13.9)$$

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<sup>117</sup>Presheaves are always denoted by symbols that are underlined.

**Proof.** Every projection  $\hat{Q} \in V$  is of the form  $\hat{Q} = \hat{Q}_1 \oplus \hat{Q}_2$  for unique projections  $\hat{Q}_1 \in \mathcal{P}(\mathcal{H}_1)$  and  $\hat{Q}_2 \in \mathcal{P}(\mathcal{H}_2)$ . Let  $\hat{P} \in \mathcal{P}(\mathcal{H})$  be of the form  $\hat{P} = \hat{P}_1 \oplus \hat{P}_2$  such that  $\hat{P}_1 \in \mathcal{P}(\mathcal{H}_1)$  and  $\hat{P}_2 \in \mathcal{P}(\mathcal{H}_2)$ . The largest projection in  $V$  smaller than or equal to  $\hat{P}$ , i.e., the inner daseinisation of  $\hat{P}$  to  $V$ , is

$$\delta^i(\hat{P})_V = \hat{Q}_1 \oplus \hat{Q}_2, \quad (13.10)$$

where  $\hat{Q}_1 \in \mathcal{P}(V_1)$  is the largest projection in  $V_1$  smaller than or equal to  $\hat{P}_1$ , and  $\hat{Q}_2 \in \mathcal{P}(V_2)$  is the largest projection in  $V_2$  smaller than or equal to  $\hat{P}_2$ , so

$$\delta^i(\hat{P})_V = \delta(\hat{P}_1)_{V_1} \oplus \delta(\hat{P}_2)_{V_2}. \quad (13.11)$$

This implies  $\delta(\hat{A} \oplus \hat{B})_V = \delta(\hat{A})_{V_1} \oplus \delta(\hat{B})_{V_2}$ , since (outer) daseinisation of a self-adjoint operator just means inner daseinisation of the projections in its spectral family, and all the projections in the spectral family of  $\hat{A} \oplus \hat{B}$  are of the form  $\hat{P} = \hat{P}_1 \oplus \hat{P}_2$ . ■

As discussed in Section 12, in order to mimic the construction that we have in the classical case, we need to pull back the arrow/natural transformation  $\check{\delta}(\langle A_1, A_2 \rangle) : \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \rightarrow \underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \oplus \mathcal{H}_2}$  to obtain an arrow from  $\underline{\Sigma}^{\mathcal{H}_1}$  to  $\underline{\mathbb{R}}^{\succeq \mathcal{H}_1}$ . Since we decided that the translation on the level of operators sends  $\hat{A}_1 \oplus \hat{A}_2$  to  $\hat{A}_1$ , we expect that this arrow from  $\underline{\Sigma}^{\mathcal{H}_1}$  to  $\underline{\mathbb{R}}^{\succeq \mathcal{H}_1}$  is  $\check{\delta}(\hat{A}_1)$ . We will now show how this works.

The presheaves  $\underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$  and  $\underline{\Sigma}^{\mathcal{H}_1}$  lie in different topoi, and in order to ‘transform’ between them we need a (left-exact) functor from the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2)^{\text{op}}}$  to the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\text{op}}}$ : this is the functor  $\tau_\phi(j) : \tau_\phi(S) \rightarrow \tau_\phi(S_1)$  in (12.19). One natural place to look for such a functor is as the inverse-image part of a geometric morphism from  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\text{op}}}$  to  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2)^{\text{op}}}$ . According to Theorem 12.1, one source of such a geometric morphism,  $\mu$ , is a functor

$$m : \mathcal{V}(\mathcal{H}_1) \rightarrow \mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2), \quad (13.12)$$

and the obvious choice for this is

$$m(V) := V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2} \quad (13.13)$$

for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}_1))$ . This function from  $\text{Ob}(\mathcal{V}(\mathcal{H}_1))$  to  $\text{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$  is clearly order preserving, and hence  $m$  is a genuine functor.

Let  $\mu : \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2)^{\text{op}}}$  denote the geometric morphism induced by  $m$ . The inverse-image functor of  $\mu$  is given by

$$\mu^* : \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2)^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\text{op}}} \quad (13.14)$$

$$\underline{F} \mapsto \underline{F} \circ m^{\text{op}}. \quad (13.15)$$

This means that, for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}_1))$ , we have

$$(\mu^* \underline{F}^{\mathcal{H}_1 \oplus \mathcal{H}_2})_V = \underline{F}_{m(V)}^{\mathcal{H}_1 \oplus \mathcal{H}_2} = \underline{F}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}. \quad (13.16)$$

For example, for the spectral presheaf we get

$$(\mu^* \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2})_V = \underline{\Sigma}_{m(V)}^{\mathcal{H}_1 \oplus \mathcal{H}_2} = \underline{\Sigma}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}. \quad (13.17)$$

This is the functor that is denoted  $\tau_\phi(j) : \tau_\phi(S_1) \rightarrow \tau_\phi(S)$  in (12.19).

We next need to find an arrow  $\phi(i) : \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \mu^* \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$  that is the analogue of the arrow  $\phi(j) : \Sigma_{\phi, S_1} \rightarrow \tau_\phi(j)(3\Sigma_{\phi, S})$  in (12.19).

For each  $V$ , the set  $(\mu^* \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2})_V = \underline{\Sigma}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$  contains two types of spectral elements  $\lambda$ : the first type are those  $\lambda$  such that  $\langle \lambda, \hat{0}_{\mathcal{H}_1} \oplus \hat{1}_{\mathcal{H}_2} \rangle = 0$ . Then, clearly, there is some  $\tilde{\lambda} \in \underline{\Sigma}_V^{\mathcal{H}_1}$  such that  $\langle \tilde{\lambda}, \hat{A} \rangle = \langle \lambda, \hat{A} \oplus \hat{0}_{\mathcal{H}_2} \rangle = \langle \lambda, \hat{A} \oplus \hat{1}_{\mathcal{H}_2} \rangle$  for all  $\hat{A} \in V_{\text{sa}}$ . The second type of spectral elements  $\lambda \in \underline{\Sigma}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$  are such that  $\langle \lambda, \hat{0}_{\mathcal{H}_1} \oplus \hat{1}_{\mathcal{H}_2} \rangle = 1$ . In fact, there is exactly one such  $\lambda$ , and we denote it by  $\lambda_0$ . This shows that  $\underline{\Sigma}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \simeq \underline{\Sigma}_V^{\mathcal{H}_1} \cup \{\lambda_0\}$ . Accordingly, at each stage  $V$ , the mapping  $\phi(i)$  sends each  $\tilde{\lambda} \in \underline{\Sigma}_V^{\mathcal{H}_1}$  to the corresponding  $\lambda \in \underline{\Sigma}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\mathcal{H}_1 \oplus \mathcal{H}_2}$ .

The presheaf  $\underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \oplus \mathcal{H}_2}$  is given at each stage  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$  as the order-reversing functions  $\nu : \downarrow W \rightarrow \mathbb{R}$ , where  $\downarrow W$  denotes the set of unital, abelian von Neumann sub-algebras of  $W$ . Let  $W = V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}$ . Clearly, there is a bijection between the sets  $\downarrow W \subset \text{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$  and  $\downarrow V \subset \text{Ob}(\mathcal{V}(\mathcal{H}))$ . We can thus identify

$$(\mu^* \underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \oplus \mathcal{H}_2})_V = \underline{\mathbb{R}}_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\succeq \mathcal{H}_1 \oplus \mathcal{H}_2} \simeq \underline{\mathbb{R}}_V^{\succeq \mathcal{H}_1} \quad (13.18)$$

for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . This gives an isomorphism  $\beta_\phi(i) : \mu^* \underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \oplus \mathcal{H}_2} \rightarrow \underline{\mathbb{R}}^{\succeq \mathcal{H}_1}$ , which corresponds to the arrow  $\beta_\phi(j) : \tau_\phi(j)(\mathcal{R}_{\phi, S}) \rightarrow \mathcal{R}_{\phi, S_1}$  in (12.19).

Now consider the arrow  $\check{\delta}(\langle A_1, A_2 \rangle) : \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \rightarrow \underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \oplus \mathcal{H}_2}$ . This is the analogue of the arrow  $A_{\phi, S} : \Sigma_{\phi, S} \rightarrow \mathcal{R}_{\phi, S}$  in (12.19). At each stage  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$ , this arrow is given by the (outer) daseinisation  $\delta(\hat{A}_1 \oplus \hat{A}_2)_{W'}$  for all  $W' \in \downarrow W$ . According to Lemma 13.1, we have

$$\delta(\hat{A}_1 \oplus \hat{A}_2)_{V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2}} = \delta(\hat{A}_1)_V \oplus \delta(A_2)_{\mathbb{C}\hat{1}_{\mathcal{H}_2}} = \delta(\hat{A}_1)_V \oplus \max(\text{sp}(\hat{A}_2))\hat{1}_{\mathcal{H}_2} \quad (13.19)$$

for all  $V \oplus \mathbb{C}\hat{1}_{\mathcal{H}_2} \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \oplus \mathcal{H}_2))$ . This makes clear how the arrow

$$\mu^*(\check{\delta}(\langle A_1, A_2 \rangle)) : \mu^* \underline{\Sigma}^{\mathcal{H}_1 \oplus \mathcal{H}_2} \rightarrow \mu^* \underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \oplus \mathcal{H}_2} \quad (13.20)$$

is defined. Our conjectured pull-back/translation representation is

$$\phi(\mathcal{L}(i))(\check{\delta}(\langle A_1, A_2 \rangle)) := \beta_\phi(i) \circ \mu^*(\check{\delta}(\langle A_1, A_2 \rangle)) \circ \phi(i) : \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \underline{\mathbb{R}}^{\mathcal{H}_1}. \quad (13.21)$$

Using the definitions of  $\phi(i)$  and  $\beta_\phi(i)$ , it becomes clear that

$$\beta_\phi(i) \circ \mu^*(\check{\delta}(\langle A_1, A_2 \rangle)) \circ \phi(i) = \check{\delta}(\hat{A}_1). \quad (13.22)$$

Hence, the commutativity condition in (12.20) is satisfied for arrows in **Sys** of the form  $i_{1,2} : S_{1,2} \rightarrow S_1 \sqcup S_2$ .

### 13.3 The Translation Representation for Composite Quantum Systems

We now consider arrows in **Sys** of the form

$$S_1 \xleftarrow{p_1} S_1 \diamond S_2 \xrightarrow{p_2} S_1, \quad (13.23)$$

where the quantum systems  $S_1$ ,  $S_2$  and  $S_1 \diamond S_2$  have the Hilbert spaces  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{H}_1 \otimes \mathcal{H}_2$ , respectively.<sup>118</sup>

The canonical translation<sup>119</sup>  $\mathcal{L}(p_1)$  between the languages  $\mathcal{L}(S_1)$  and  $\mathcal{L}(S_1 \diamond S_2)$  (see Section 11.2.2) is such that if  $A_1$  is a function symbol in  $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$ , then the corresponding operator  $\hat{A}_1 \in \mathcal{B}(\mathcal{H}_1)_{\text{sa}}$  will be ‘translated’ to the operator  $\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ . By assumption, this corresponds to the function symbol  $A_1 \diamond 1$  in  $F_{\mathcal{L}(S_1 \diamond S_2)}(\Sigma, \mathcal{R})$ .

#### 13.3.1 Operator Entanglement and Translations.

We should be cautious about what to expect from this translation when we represent a physical quantity  $A : \Sigma \rightarrow \mathcal{R}$  in  $F_{\mathcal{L}(S_1)}(\Sigma, \mathcal{R})$  by an arrow between presheaves, since there are no canonical projections

$$\mathcal{H}_1 \leftarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_2, \quad (13.24)$$

and hence no canonical projections

$$\underline{\Sigma}^{\mathcal{H}_1} \leftarrow \underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2} \rightarrow \underline{\Sigma}^{\mathcal{H}_2} \quad (13.25)$$

<sup>118</sup>As usual, the composite system  $S_1 \diamond S_2$  has as its Hilbert space the tensor product of the Hilbert spaces of the components.

<sup>119</sup>As discussed in Section 11.2.2, this translation,  $\mathcal{L}(p_1)$ , transforms a physical quantity  $A_1$  of system  $S_1$  into a physical quantity  $A_1 \diamond 1$ , which is the ‘same’ physical quantity but now seen as a part of the composite system  $S_1 \diamond S_2$ . The symbol 1 is the trivial physical quantity: it is represented by the operator  $\hat{1}_{\mathcal{H}_2}$ .

from the spectral presheaf of the composite system to the spectral presheaves of the components.<sup>120</sup>

This is the point where a form of *entanglement* enters the picture. The spectral presheaf  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  is a presheaf over the context category  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  of  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . Clearly, the context category  $\mathcal{V}(\mathcal{H}_1)$  can be embedded into  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  by the mapping  $V_1 \mapsto V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$ , and likewise  $\mathcal{V}(\mathcal{H}_2)$  can be embedded into  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ . But not every  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$  is of the form  $V_1 \otimes V_2$ .

This comes from the fact that not all vectors in  $\mathcal{H}_1 \otimes \mathcal{H}_2$  are of the form  $\psi_1 \otimes \psi_2$ , hence not all projections in  $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  are of the form  $\hat{P}_{\psi_1} \otimes \hat{P}_{\psi_2}$ , which in turn implies that not all  $W \in \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  are of the form  $V_1 \otimes V_2$ . There are more contexts, or world-views, available in  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  than those coming from  $\mathcal{V}(\mathcal{H}_1)$  and  $\mathcal{V}(\mathcal{H}_2)$ . We call this ‘operator entanglement’.

The topos representative of  $\hat{A}_1$  is  $\check{\delta}(\hat{A}_1) : \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \underline{\mathbb{R}}^{\geq \mathcal{H}_1}$ , and the representative of  $\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2}$  is  $\check{\delta}(A_1 \hat{\diamond} 1) : \underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2} \rightarrow \underline{\mathbb{R}}^{\geq \mathcal{H}_1 \otimes \mathcal{H}_2}$ . At sub-algebras  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$  which are *not* of the form  $W = V_1 \otimes V_2$  for any  $V_1 \in \text{Ob}(\mathcal{V}(\mathcal{H}_1))$  and  $V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}_2))$ , the daseinised operator  $\delta(\hat{A}_1|_W \otimes \hat{1}_{\mathcal{H}_2}) \in W_{\text{sa}}$  will not be of the form  $\delta(\hat{A}_1)_{V_1} \otimes \delta(\hat{A}_1)_{V_2}$ .<sup>121</sup> On the other hand, it is easy to see that  $\delta(\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2})_W = \delta(\hat{A}_1)_{V_1} \otimes \hat{1}_{\mathcal{H}_2}$  if  $W = V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$ .

Given a physical quantity  $A_1$ , represented by the arrow  $\check{\delta}(\hat{A}_1) : \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \underline{\mathbb{R}}^{\geq \mathcal{H}_1}$ , we can (at best) expect that the translation of this arrow into an arrow from  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  to  $\underline{\mathbb{R}}^{\geq \mathcal{H}_1 \otimes \mathcal{H}_2}$  coincides with the arrow  $\check{\delta}(A_1 \hat{\diamond} 1)$  on the ‘image’ of  $\underline{\Sigma}^{\mathcal{H}_1}$  in  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ . This image will be constructed below using a certain geometric morphism. As one might expect, the image of  $\underline{\Sigma}^{\mathcal{H}_1}$  is a presheaf  $\underline{P}$  on  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  such that  $\underline{P}_{V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}} \simeq \underline{\Sigma}_{V_1}^{\mathcal{H}_1}$  for all  $V_1 \in \mathcal{V}(\mathcal{H}_1)$ , i.e., the presheaf  $\underline{P}$  can be identified with  $\underline{\Sigma}^{\mathcal{H}_1}$  exactly on the image of  $\mathcal{V}(\mathcal{H}_1)$  in  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  under the embedding  $V_1 \mapsto V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$ . At these stages, the translation of  $\check{\delta}(\hat{A}_1)$  will coincide with  $\check{\delta}(A_1 \hat{\diamond} 1)$ . At other stages  $W \in \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ , the translation cannot be expected to be the same natural transformation as  $\check{\delta}(A \hat{\diamond} 1)$  in general.

<sup>120</sup>On the other hand, in the classical case, there *are* canonical projections

$$\Sigma_{\sigma, S_1} \leftarrow \Sigma_{\sigma, S_1 \diamond S_2} \rightarrow \Sigma_{\sigma, S_2} \quad (13.26)$$

because the symplectic manifold  $\Sigma_{\sigma, S_1 \diamond S_2}$  that represents the composite system is the cartesian product  $\Sigma_{\sigma, S_1 \diamond S_2} = \Sigma_{\sigma, S_1} \times \Sigma_{\sigma, S_2}$ , which is a product in the categorical sense and hence comes with canonical projections.

<sup>121</sup>Currently, it is even an open question if  $\delta(\hat{A}_1|_W \otimes \hat{1}_{\mathcal{H}_2}) = \delta(\hat{A}_1)_{V_1} \otimes \hat{1}_{\mathcal{H}_2}$  if  $W = V_1 \otimes V_2$  for a non-trivial algebra  $V_2$ .



### 13.3.2 A Geometrical Morphism and a Possible Translation

The most natural approach to a translation is the following. Let  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ , and define  $V_W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1))$  to be the *largest* sub-algebra of  $\mathcal{B}(\mathcal{H}_1)$  such that  $V_W \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$  is a sub-algebra of  $W$ . Depending on  $W$ ,  $V_W$  may, or may not, be the trivial sub-algebra  $\mathbb{C}\hat{1}_{\mathcal{H}_1}$ . We note that if  $W' \subseteq W$ , then

$$V_{W'} \subseteq V_W, \quad (13.27)$$

but  $W' \subset W$  only implies  $V_{W'} \subseteq V_W$ .

The trivial algebra  $\mathbb{C}\hat{1}_{\mathcal{H}_1}$  is not an object in the category  $\mathcal{V}(\mathcal{H}_1)$ . This is why we introduce the ‘augmented context category’  $\mathcal{V}(\mathcal{H}_1)_*$ , whose objects are those of  $\mathcal{V}(\mathcal{H}_1)$  united with  $\mathbb{C}\hat{1}_{\mathcal{H}_1}$ , and with the obvious morphisms ( $\mathbb{C}\hat{1}_{\mathcal{H}_1}$  is a sub-algebra of all  $V \in \mathcal{V}(\mathcal{H}_1)$ ).

Then there is a functor  $n : \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow \mathcal{V}(\mathcal{H}_1)_*$ , defined as follows. On objects,

$$\begin{aligned} n : \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)) &\rightarrow \text{Ob}(\mathcal{V}(\mathcal{H}_1)_*) \\ W &\mapsto V_W, \end{aligned} \quad (13.28)$$

and if  $i_{W'W} : W' \rightarrow W$  is an arrow in  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ , we define  $n(i_{W'W}) := i_{V_{W'}, V_W}$  (an arrow in  $\mathcal{V}(\mathcal{H}_1)_*$ ); if  $V_{W'} = V_W$ , then  $i_{V_{W'}, V_W}$  is the identity arrow  $\text{id}_{V_W}$ .

Now let

$$\nu : \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}} \rightarrow \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1)_*)^{\text{op}}} \quad (13.29)$$

denote the geometric morphism induced by  $\pi$ . Then the (left-exact) inverse-image functor

$$\nu^* : \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1)_*)^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}} \quad (13.30)$$

acts on a presheaf  $\underline{F} \in \mathbf{Sets}^{(\mathcal{V}(\mathcal{H}_1)_*)^{\text{op}}}$  in the following way. For all  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ , we have

$$(\nu^* \underline{F})_W = \underline{F}_{n^{\text{op}}(W)} = \underline{F}_{V_W} \quad (13.31)$$

and

$$(\nu^* \underline{F})(i_{W'W}) = \underline{F}(i_{V_{W'}, V_W}) \quad (13.32)$$

for all arrows  $i_{W'W}$  in the category  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ .<sup>122</sup>

<sup>122</sup>We remark, although will not prove it here, that the inverse-image presheaf  $\nu^* \underline{F}$  coincides with the direct image presheaf  $\phi_* \underline{F}$  of  $\underline{F}$  constructed from the geometric morphism

In particular, for all  $W \in \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ , we have

$$(\nu^* \underline{\Sigma}^{\mathcal{H}_1})_W = \underline{\Sigma}_{V_W}^{\mathcal{H}_1}, \quad (13.34)$$

$$(\nu^* \underline{\mathbb{R}}^{\succeq \mathcal{H}_1})_W = \underline{\mathbb{R}}_{V_W}^{\succeq \mathcal{H}_1}. \quad (13.35)$$

Since  $V_W$  can be  $\mathbb{C}\hat{1}_{\mathcal{H}_1}$ , we have to extend the definition of the spectral presheaf  $\underline{\Sigma}^{\mathcal{H}_1}$  and the quantity-value presheaf  $\underline{\mathbb{R}}^{\succeq \mathcal{H}_1}$  such that they become presheaves over  $\mathcal{V}(\mathcal{H}_1)_*$  (and not just  $\mathcal{V}(\mathcal{H}_1)$ ). This can be done in a straightforward way: the Gel'fand spectrum  $\underline{\Sigma}_{\mathbb{C}\hat{1}_{\mathcal{H}_1}}$  of  $\mathbb{C}\hat{1}_{\mathcal{H}_1}$  consists of the single spectral element  $\lambda_1$  such that  $\langle \lambda_1, \hat{1}_{\mathcal{H}_1} \rangle = 1$ . Moreover,  $\mathbb{C}\hat{1}_{\mathcal{H}_1}$  has no sub-algebras, so the order-reversing functions on this algebra correspond bijectively to the real numbers  $\mathbb{R}$ .

Using these equations, we see that the arrow  $\check{\delta}(\hat{A}_1) : \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \underline{\mathbb{R}}^{\succeq \mathcal{H}_1}$  that corresponds to the self-adjoint operator  $\hat{A}_1 \in \mathcal{B}(\mathcal{H}_1)_{\text{sa}}$  gives rise to the arrow

$$\nu^*(\check{\delta}(\hat{A}_1)) : \nu^* \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \nu^* \underline{\mathbb{R}}^{\succeq \mathcal{H}_1}. \quad (13.36)$$

In terms of our earlier notation, the functor  $\tau_\phi(p_1) : \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1)^{\text{op}}} \rightarrow \mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}}$  is  $\nu^*$ , and the arrow in (13.36) is the arrow  $\tau_\phi(j)(A_{\phi,S}) : \tau_\phi(j)(\Sigma_{\phi,S}) \rightarrow \tau_\phi(j)(\mathcal{R}_{\phi,S})$  in (12.19) with  $j : S_1 \rightarrow S$  being replaced by  $p : S_1 \diamond S_2 \rightarrow S_1$ , which is the arrow in  $\mathbf{Sys}$  whose translation representation we are trying to construct.

The next arrow we need is the one denoted  $\beta_\phi(j) : \tau_\phi(j)(\mathcal{R}_{\phi,S}) \rightarrow \mathcal{R}_{\phi,S_1}$  in (12.19). In the present case, we define  $\beta_\phi(p) : \nu^* \underline{\mathbb{R}}^{\succeq \mathcal{H}_1} \rightarrow \underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \otimes \mathcal{H}_2}$  as follows. Let  $\alpha \in (\nu^* \underline{\mathbb{R}}^{\succeq \mathcal{H}_1})_W \simeq \underline{\mathbb{R}}_{V_W}^{\succeq \mathcal{H}_1}$  be an order-reversing real-valued function on  $\downarrow V_W$ . Then we define an order-reversing function  $\beta_\phi(p)(\alpha) \in \underline{\mathbb{R}}_{W'}^{\succeq \mathcal{H}_1 \otimes \mathcal{H}_2}$  as follows. For all  $W' \subseteq W$ , let

$$[\beta_\phi(p)(\alpha)](W') := \alpha(V_{W'}) \quad (13.37)$$

which, by virtue of (13.27), is an order-reversing function and hence a member of  $\underline{\mathbb{R}}_{W'}^{\succeq \mathcal{H}_1 \otimes \mathcal{H}_2}$ .

We also need an arrow in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}}$  from  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  to  $\nu^* \underline{\Sigma}^{\mathcal{H}_1}$ , where  $\nu^* \underline{\Sigma}^{\mathcal{H}_1}$  is defined in (13.34). This is the arrow denoted  $\phi(j) : \Sigma_{\phi,S_1} \rightarrow \tau_\phi(j)(\Sigma_{\phi,S})$  in (12.19).

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$\phi$  induced by the functor

$$\begin{aligned} \kappa : \mathcal{V}(\mathcal{H}_1) &\rightarrow \mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2) \\ V &\mapsto V \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}. \end{aligned} \quad (13.33)$$

Of course, the inverse image presheaf  $\beta^* \underline{F}$  is much easier to construct.

The obvious choice is to restrict  $\lambda \in \underline{\Sigma}_W^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  to the sub-algebra  $V_W \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2} \subseteq W$ , and to identify  $V_W \otimes \mathbb{C}\hat{1}_{\mathcal{H}_1} \simeq V_W \otimes \hat{1}_{\mathcal{H}_1} \simeq V_W$  as von Neumann algebras, which gives  $\underline{\Sigma}_{V_W \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}}^{\mathcal{H}_1 \otimes \mathcal{H}_2} \simeq \underline{\Sigma}_{V_W}^{\mathcal{H}_1}$ . Let

$$\begin{aligned} \phi(p)_W : \underline{\Sigma}_W^{\mathcal{H}_1 \otimes \mathcal{H}_2} &\rightarrow \underline{\Sigma}_{V_W}^{\mathcal{H}_1} \\ \lambda &\mapsto \lambda|_{V_W} \end{aligned} \quad (13.38)$$

denote this arrow at stage  $W$ . Then

$$\beta_\phi(p) \circ \nu^*(\check{\delta}(\hat{A}_1)) \circ \phi(p) : \underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2} \rightarrow \underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \otimes \mathcal{H}_2} \quad (13.39)$$

is a natural transformation which is defined for all  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$  and all  $\lambda \in W$  by

$$\left( \beta_\phi(p) \circ \nu^*(\check{\delta}(\hat{A}_1)) \circ \phi(p) \right)_W (\lambda) = \nu^*(\check{\delta}(\hat{A}))(\lambda|_{V_W}) \quad (13.40)$$

$$= \{V' \mapsto \langle \lambda|_{V'}, \delta(\hat{A})_{V'} \rangle \mid V' \subseteq V_W\} \quad (13.41)$$

This is clearly an order-reversing real-valued function on the set  $\downarrow W$  of sub-algebras of  $W$ , i.e., it is an element of  $\underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \otimes \mathcal{H}_2}$ . We define  $\beta_\phi(p) \circ \nu^*(\check{\delta}(\hat{A}_1)) \circ \phi(p)$  to be the translation representation,  $\phi(\mathcal{L}(p))(\check{\delta}(\hat{A}_1))$  of  $\check{\delta}(\hat{A}_1)$  for the composite system.

Note that, by construction, for each  $W$ , the arrow  $(\beta_\phi(p) \circ \nu^*(\check{\delta}(\hat{A}_1)) \circ \phi(p))_W$  corresponds to the self-adjoint operator  $\delta(\hat{A}_1)_{V_W} \otimes \hat{1}_{\mathcal{H}_2} \in W_{\text{sa}}$ , since

$$\langle \lambda|_{V_W}, \delta(\hat{A}_1)_{V_W} \rangle = \langle \lambda, \delta(\hat{A}_1)_{V_W} \otimes \hat{1}_{\mathcal{H}_2} \rangle \quad (13.42)$$

for all  $\lambda \in \underline{\Sigma}_W^{\mathcal{H}_1 \otimes \mathcal{H}_2}$ .

**Comments on these results.** This is about as far as we can get with the arrows associated with the composite of two quantum systems. The results above can be summarised in the equation

$$\phi(\mathcal{L}(p))(\check{\delta}(\hat{A}_1))_W = \check{\delta}(A_1)_{V_W} \otimes \hat{1}_{\mathcal{H}_2} \quad (13.43)$$

for all contexts  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ . If  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$  is of the form  $W = V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$ , i.e., if  $W$  is in the image of the embedding of  $\mathcal{V}(\mathcal{H}_1)$  into  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ , then  $V_W = V_1$  and the translation formula gives just what one expects: the arrow  $\check{\delta}(\hat{A}_1)$  is translated into the arrow  $\check{\delta}(A_1 \hat{\diamond} 1)$  at these stages, since  $\delta(\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2})_{V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}} = \delta(\hat{A}_1)_{V_1} \otimes \hat{1}_{\mathcal{H}_2}$ .<sup>123</sup>

<sup>123</sup>To be precise, both the translation  $\phi(\mathcal{L}(p))(\check{\delta}(\hat{A}_1))_W$ , given by (13.43), and  $\check{\delta}(A \hat{\diamond} 1)_W$  are mappings from  $\underline{\Sigma}_W^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  to  $\underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \otimes \mathcal{H}_2}$ . Each  $\lambda \in \underline{\Sigma}_W^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  is mapped to an order-reversing function on  $\downarrow W$ . The mappings  $\phi(\mathcal{L}(p))(\check{\delta}(\hat{A}_1))_W$  and  $\check{\delta}(A \hat{\diamond} 1)_W$  coincide at all  $W' \in \downarrow W$  that are of the form  $W' = V' \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$ .

If  $W \in \text{Ob}(\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2))$  is not of the form  $W = V_1 \otimes \mathbb{C}\hat{1}_{\mathcal{H}_2}$ , then it is relatively easy to show that

$$\delta(\hat{A}_1 \otimes \hat{1}_{\mathcal{H}_2})_W \neq \delta(\hat{A}_1)_{V_W} \otimes \hat{1}_{\mathcal{H}_2} \quad (13.44)$$

in general. Hence

$$\phi(\mathcal{L}(p))(\check{\delta}(\hat{A}_1)) \neq \check{\delta}(A_1 \diamond 1), \quad (13.45)$$

whereas, intuitively, one might have expected equality. Thus the ‘commutativity’ condition (12.7) is not satisfied.

In fact, there appears to be *no* operator  $\hat{B} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  such that  $\phi(\mathcal{L}(p))(\check{\delta}(\hat{A}_1)) = \check{\delta}(\hat{B})$ . Thus the quantity,  $\beta_\phi(p) \circ \nu^*(\check{\delta}(\hat{A}_1)) \circ \phi(p)$ , that is our conjectured pull-back, is an arrow in  $\text{Hom}_{\mathbf{Sets}^{\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)^{\text{op}}}}(\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}, \underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \otimes \mathcal{H}_2})$  that is not of the form  $A_{\phi, S_1 \diamond S_2}$  for any physical quantity  $A \in F_{\mathcal{L}(S_1 \diamond S_2)}(\Sigma, \mathcal{R})$ .

Our current understanding is that this translation is ‘as good as possible’: the arrow  $\check{\delta}(\hat{A}_1) : \underline{\Sigma}^{\mathcal{H}_1} \rightarrow \underline{\mathbb{R}}^{\succeq \mathcal{H}_1}$  is translated into an arrow from  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  to  $\underline{\mathbb{R}}^{\succeq \mathcal{H}_1 \otimes \mathcal{H}_2}$  that coincides with  $\check{\delta}(\hat{A}_1)$  on those part of  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  that can be identified with  $\underline{\Sigma}^{\mathcal{H}_1}$ . But  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  is much larger, and it is not simply a product of  $\underline{\Sigma}^{\mathcal{H}_1}$  and  $\underline{\Sigma}^{\mathcal{H}_2}$ . The context category  $\mathcal{V}(\mathcal{H}_1 \otimes \mathcal{H}_2)$  underlying  $\underline{\Sigma}^{\mathcal{H}_1 \otimes \mathcal{H}_2}$  is much richer than a simple product of  $\mathcal{V}(\mathcal{H}_1)$  and  $\mathcal{V}(\mathcal{H}_2)$ . This is due to a kind of operator entanglement. A translation can at best give a faithful picture of an arrow, but it cannot possibly ‘know’ about the more complicated contextual structure of the larger category.

Clearly, both technical and interpretational work remain to be done.

## 14 Characteristic Properties of $\Sigma_\phi$ , $\mathcal{R}_\phi$ and $\mathbb{T}/\mathfrak{m}$

### 14.1 The State Object $\Sigma_\phi$

A major motivation for our work is the desire to find mathematical structures with whose aid genuinely new types of theory can be constructed. Consequently, however fascinating the ‘toposification’ of quantum theory may be, this particular theory should not be allowed to divert us too much from the main goal. However, it is also important to see if any general lessons can be learnt from what has been done so far. This is likely to be crucial in the construction of new theories.

In developing the topos version of quantum theory we have constructed concrete objects in the topos to function as the state object and quantity-

value object. We have also seen how each quantum vector state gives a precise truth object, or ‘pseudo-state’.

The challenging question now is what, if anything, can be said in general about these key ingredients in our scheme. Thus, ideally, we would be able to specify characteristic properties for  $\Sigma_\phi$ ,  $\mathcal{R}_\phi$ , and the truth objects/pseudo-states. A related problem is to understand if there is an object,  $\mathbb{W}_\phi$ , of all truth objects/pseudo-states, and, if so, what are its defining properties. Any such universal properties could be coded into the structure of the language,  $\mathcal{L}(S)$ , of the system, hence ensuing that they are present in all topos representations of  $S$ . In particular, should a symbol  $\mathbb{W}$  be added to  $\mathcal{L}(S)$  as the linguistic precursor of an object of pseudo-states?

So far, we only know of two explicit examples of physically-relevant topos representations of a system language,  $\mathcal{L}(S)$ : (i) the representation of classical physics in **Sets**; and (ii) the representation of quantum physics in topoi of the form  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . This does provide much to go on when it comes to speculating on characteristic properties of the key objects  $\Sigma_\phi$  and  $\mathcal{R}_\phi$ . From this perspective, it would be helpful if there is an alternative way of finding the quantum objects  $\underline{\Sigma}$  and  $\underline{\mathbb{R}}^{\leftarrow}$  in addition to the one provided by the approach that we have adopted. Fortunately, this has been done recently by Heunen and Spitters [38]; as we shall see in Section 14.1.2, this does provide more insight into a possible generic structure for  $\Sigma_\phi$ .

### 14.1.1 An Analogue of a Symplectic Structure or Cotangent Bundle?

Let us start with the state object  $\Sigma_\phi$ . In classical physics, this is a symplectic manifold; in quantum theory it is the spectral presheaf  $\underline{\Sigma}$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . Does this suggest any properties for  $\Sigma_\phi$  in general?

One possibility is that the state-object,  $\Sigma_\phi$ , has some sort of ‘symplectic structure’. If taken literally, this phrase suggests synthetic differential geometry (SDG): a theory that is based on the existence in certain topoi (not **Sets**) of genuine ‘infinitesimals’. However, this seems unlikely for the quantum topoi  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  and we would probably need to extend these topoi considerably in order to incorporate SDG. Thus when we say “...some sort of symplectic structure”, the phrase ‘some sort’ has to be construed rather broadly.

We suspect that, with this caveat, the state object  $\underline{\Sigma}$  may have such a structure, particularly for those quantum systems that come from quantising a given classical system. However, at the moment this is still a conjecture.

We are currently studying systems whose classical state space is the cotangent bundle,  $T^*Q$ , of a configuration space  $Q$ . We think that the quantum analogue of this space is a certain presheaf,  $M_Q$ , that is associated with the *maximal* commutative sub-algebra,  $M_Q \in \overline{\text{Ob}}(\mathcal{V}(\mathcal{H}))$ , generated by the smooth, real-valued functions on  $Q$ . This is currently work in progress.

But even if the quantum state-object *does* have a remnant ‘symplectic structure’, it is debatable if this should be axiomatised in general. Symplectic structures arise in classical physics because the underlying equations of motion are *second-order* in the ‘configuration’ variables  $q$ , and hence first-order in the pair  $(q, p)$ , where  $p$  are the ‘momentum variables’.

However if, say, Newton’s equations of gravity had been third-order in  $q$ , this would lead to triples  $(q, p, a)$  ( $a$  are ‘acceleration’ variables) and symplectic structure would not be appropriate.

#### 14.1.2 $\Sigma_\phi$ as a Spectral Object: the Work of Heunen and Spitters

Another way of understanding the state object  $\Sigma_\phi$  is suggested by the recent work of Heunen and Spitters [38]. They start with a non-commutative  $C^*$ -algebra,  $\mathcal{A}$ , of observables in some ‘ambient topos’,  $\mathfrak{S}$ —in our case, this is **Sets**—and then proceed with the following steps:

1. Construct the poset category<sup>124</sup>  $\mathcal{V}(\mathcal{A})$  of commutative sub-algebras of  $\mathcal{A}$ .
2. Construct the topos,  $\mathfrak{S}^{\mathcal{V}(\mathcal{A})}$  of covariant functors (i.e., *co-presheaves*) on the category/poset  $\mathcal{V}(\mathcal{A})$ .<sup>125</sup>
3. Construct the ‘tautological’ co-presheaf  $\overline{\mathcal{A}}$  in which  $\overline{\mathcal{A}}(V) := V$  for each commutative sub-algebra,  $V$ , of  $\mathcal{A}$ . Then if  $i_{V_1 V_2} : V_1 \subseteq V_2$ , the associated arrow  $\overline{\mathcal{A}}(i_{V_1 V_2}) : \overline{\mathcal{A}}(V_1) \rightarrow \overline{\mathcal{A}}(V_2)$  is just the inclusion map of  $\overline{\mathcal{A}}(V_1)$  in  $\overline{\mathcal{A}}(V_2)$ .
4. They show that  $\overline{\mathcal{A}}$  has the structure of a commutative (pre<sup>126</sup>)  $C^*$ -algebra inside the topos

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<sup>124</sup>This notation has been chosen to suggest more clearly the analogues with our topos constructions that use the base category  $\mathcal{V}(\mathcal{H})$ . It is not that used by Heunen and Spitters.

<sup>125</sup>They affirm that the operation  $\mathcal{A} \mapsto \mathfrak{S}^{\mathcal{V}(\mathcal{A})}$  defines a functor from the category of  $C^*$ -algebras in  $\mathfrak{S}$  to the category of elementary topoi and geometric morphisms.

<sup>126</sup>The ‘pre’ refers to the fact that the algebra is defined using only the co-presheaf,  $\overline{\mathbb{Q}}$ , of complexified rationals. This co-presheaf is constructed from the natural-number object,  $\overline{\mathbb{N}}$ , only. This restriction is necessary because it is not possible to define the *norm* of a  $C^*$ -algebra in purely constructive terms.

5. Using a recent, very important, result of Banachowski and Mulvey [7], Heunen and Spitters show that the spectrum,  $\overline{\Sigma}$ , of the commutative algebra  $\overline{\mathcal{A}}$  can be computed internally, and that it has the structure of an internal locale in  $\mathfrak{S}^{\mathcal{V}(\mathcal{A})}$ .
6. They then show that, in the case of quantum theory,  $\overline{\Sigma}$  is essentially our spectral object,  $\underline{\Sigma}$ , but viewed as a co-presheaf of *locales*, rather than as a presheaf of topological spaces.

Thus Heunen and Spitters differ from us in that (i) they work in a general ambient topos  $\mathfrak{S}$ , whereas we use **Sets**; (ii) they use  $C^*$ -algebras rather than von Neumann algebras<sup>127</sup>; and (iii) they use covariant rather than contravariant functors.

The fact that they recover what is (essentially) our spectral presheaf is striking. Amongst other things, it suggests a possible axiomatisation of the state object,  $\Sigma_\phi$ . Namely, we could require that in *any* topos representation,  $\phi$ , the state object is (i) the spectrum of some internal, commutative (pre)  $C^*$ -algebra; and (ii) the spectrum has the structure of an internal locale in the topos  $\tau_\phi$ .

It is not currently clear whether or not it makes physical sense to always require  $\Sigma_\phi$  to be the spectrum of an internal algebra. However, even in the contrary case it still makes sense to explore the possibility that  $\Sigma_\phi$  has the ‘topological’ property of being an internal locale. This opens up many possibilities, including that of constructing the (internal) topos,  $\text{Sh}(\Sigma_\phi)$ , of sheaves over  $\Sigma_\phi$ .

### 14.1.3 Using Boolean Algebras as the Base Category

As remarked earlier, there are several possible choices for the base category over which the set-valued functors are defined. Most of our work has been based on the category,  $\mathcal{V}(\mathcal{H})$ , of commutative von Neumann sub-algebras of  $B(\mathcal{H})$ . As indicated above, the Heunen-Spitters constructions use the category of commutative  $C^*$ -algebras. More abstractly, one can start with any  $AW^*$ -algebra or  $C^*$ -algebra.

However, as discussed briefly in Section 5.5.3, another possible choice is the category,  $\mathcal{Bl}(\mathcal{H})$ , of all Boolean sub-algebras of the lattice of projection

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<sup>127</sup>One problem with  $C^*$ -algebras is that they very often do not contain enough projectors; and, of course, these are the entities that represent propositions. This obliges Heunen and Spitters to move from a  $C^*$ -algebra to a  $AW^*$ -algebra, which is just an abstract version of a von Neumann algebra.

operators on  $\mathcal{H}$ . The ensuing topos,  $\mathbf{Sets}^{\mathcal{Bl}(\mathcal{H})^{\text{op}}}$ , or  $\mathbf{Sets}^{\mathcal{Bl}(\mathcal{H})}$ , is interesting in its own right, but particularly so when combined with the ideas of Heunen and Spitters. As applied to the category  $\mathcal{Bl}(\mathcal{H})$ , their work suggests that we first construct the tautological co-presheaf  $\overline{\mathcal{Bl}(\mathcal{H})}$  which associates to each  $B \in \text{Ob}(\mathcal{Bl}(\mathcal{H}))$ , the Boolean algebra  $B$ . Viewed internally in the topos  $\mathbf{Sets}^{\mathcal{Bl}(\mathcal{H})}$ , this co-presheaf is a Boolean-algebra object. We conjecture that the spectrum of  $\overline{\mathcal{Bl}(\mathcal{H})}$  can be obtained in a constructive way using the internal logic of  $\mathbf{Sets}^{\mathcal{Bl}(\mathcal{H})}$ . If so, it seems clear that, after using the locale trick of [38], this spectrum will essentially be the same as our dual presheaf  $\underline{D}$ .

Thus, in this approach, the state object is the spectrum of an internal *Boolean-algebra*, and daseinisation maps the projection operators in  $\mathcal{H}$  into elements of this algebra. This reinforces still further our claim that quantum theory looks like classical physics in an appropriate topos. This raises some fascinating possibilities. For example, we make the following:

**Conjecture:** The subject of quantum computation is equivalent to the study of ‘classical’ computation in the quantum topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

#### 14.1.4 Application to Other Branches of Algebra

It is clear that the scheme discussed above could fruitfully be extended to various branches of algebra. Thus, if  $\mathfrak{A}$  is any algebraic structure<sup>128</sup>, we can consider the category  $\mathcal{V}(\mathfrak{A})$  whose objects are the commutative sub-algebras of  $\mathfrak{A}$ , and whose arrows are algebra embeddings (or, slightly more generally, monomorphisms). One can then consider the topos,  $\mathbf{Sets}^{\mathcal{V}(\mathfrak{A})^{\text{op}}}$ , of all set-valued, contravariant functors on  $\mathcal{V}(\mathfrak{A})$ ; alternatively, one might look at the topos,  $\mathbf{Sets}^{\mathcal{V}(\mathfrak{A})}$ , of covariant functors.

For this structure to be mathematically interesting it is necessary that the abelian sub-objects of  $\mathfrak{A}$  have a well-defined spectral structure. For example, let  $\mathfrak{A}$  be any locally-compact topological group. Then the *spectrum* of any commutative (locally-compact) subgroup  $A$  is just the Pontryagin dual of  $A$ , which is itself a locally-compact, commutative group. The *spectral presheaf* of  $\mathfrak{A}$  can then be defined as the object,  $\Sigma_{\mathfrak{A}}$ , in  $\mathbf{Sets}^{\mathcal{V}(\mathfrak{A})^{\text{op}}}$  that is constructed in the obvious way (i.e., analogous to the way in which  $\underline{\Sigma}$  was constructed) from this collection of Pontryagin duals.

<sup>128</sup>We are assuming that the ambient topos is  $\mathbf{Sets}$ , but other choices could be considered.



We conjecture that a careful analysis would show that, for at least some structures of this type:

1. There is a ‘tautological’ object,  $\overline{\mathfrak{A}}$ , in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathfrak{A})}$  that is associated with the category  $\mathcal{V}(\mathfrak{A})$ .
2. Viewed internally, this tautological object is a commutative algebra.
3. This object has a spectrum that can be constructed internally, and is essentially the spectral presheaf,  $\Sigma_{\mathfrak{A}}$ , of  $\mathfrak{A}$ .

It seems clear that, in general, the spectral presheaf,  $\Sigma_{\mathfrak{A}}$ , is a potential candidate as the basis for non-commutative spectral theory.

#### 14.1.5 The Partial Existence of Points of $\Sigma_{\phi}$

One of the many intriguing features of topos theory is that it makes sense to talk about entities that only ‘partially exist’. One can only speculate on what would have been Heidegger’s reaction had he been told that the answer to “What is a thing?” is “Something that partially exists”. However, in the realm of topos theory the notion of ‘partial existence’ lies easily with the concept of propositions that are only ‘partly true’.

A particularly interesting example is the existence, or otherwise, of ‘points’ (i.e., global elements) of the state object  $\Sigma_{\phi}$ . If  $\Sigma_{\phi}$  has no global elements (as is the case for the quantum spectral presheaf,  $\underline{\Sigma}$ ) it may still have ‘partial elements’. A partial element is defined to be an arrow  $\xi : U \rightarrow \Sigma_{\phi}$  where the object  $U$  in the topos  $\tau_{\phi}$  is a sub-object of the terminal object  $1_{\tau_{\phi}}$ . Thus there is a monic  $U \hookrightarrow 1_{\tau_{\phi}}$  with the property that the arrow  $\xi : U \rightarrow \Sigma_{\phi}$  cannot be extended to an arrow  $1_{\tau_{\phi}} \rightarrow \Sigma_{\phi}$ . Studying the obstruction to such extensions could be another route to finding a cohomological expression of the Kochen-Specker theorem.

Pedagogically, it is attractive to say that the non-existence of a global element of  $\underline{\Sigma}$  is analogous to the non-existence of a cross-section of the familiar ‘double-circle’, helical covering of a single circle,  $S^1$ . This principal  $\mathbb{Z}_2$ -bundle over  $S^1$  is non-trivial, and hence has no cross-sections.

However, *local* cross-sections do exist, these being defined as sections of the bundle restricted to any open subset of the base space  $S^1$ . In fact, this bundle is *locally trivial*; i.e., each point  $s \in S^1$  has a neighbourhood  $U_s$  such that the restriction of the bundle to  $U_s$  is trivial, and hence sections of the bundle restricted to  $U_s$  exist.

There is an analogue of local triviality in the topos quantum theory where  $\tau_\phi = \mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . Thus, let  $V$  be any object in  $\mathcal{V}(\mathcal{H})$  and define  $\downarrow V := \{V_1 \in \text{Ob}(\mathcal{V}(\mathcal{H})) \mid V_1 \subseteq V\}$ . Then  $\downarrow V$  is like a ‘neighbourhood’ of  $V$ ; indeed, that is precisely what it is if the poset  $\text{Ob}(\mathcal{V}(\mathcal{H}))$  is equipped with the topology generated by the lower sets. Furthermore, given any presheaf  $\underline{F}$  in  $\tau_\phi$ , the restriction,  $\underline{F}\downarrow V$ , to  $V$ , can be defined as in Section 6.5. It is easy to see that, for all stages  $V$ , the presheaf  $\underline{F}\downarrow V$  *does* have global elements. In this sense, every presheaf in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  is ‘locally trivial’. Furthermore, to each  $V$  there is associated a sub-object  $\underline{U}^V$  of  $\underline{1}$  such that each global element of  $\underline{F}\downarrow V$  corresponds to a partial element of  $\underline{F}$ .

Thus, for the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , there is a precise sense in which the spectral presheaf has ‘local elements’, or ‘points that partially exist’. However, it is not clear to what extent such an assertion can, or should, be made for a general topos  $\tau_\phi$ . Certainly, for any presheaf topos,  $\mathbf{Sets}^{C^{\text{op}}}$ , one can talk about ‘localising’ with respect to the objects in the base category  $C$ , but the situation for a more general topos is less clear.

#### 14.1.6 The Work of Corbett *et al*

Another interesting question is whether these different ways of seeing the state-object relate at all to the work of Corbett and his collaborators [3, 16].

For some time Corbett has been studying what he calls ‘quantum’ real numbers, or ‘qr-numbers’, as another way of obtaining a ‘realist’ interpretation of quantum theory. The first step is to take the space of states,  $\mathcal{E}_S$ , of a quantum system (where a state is viewed as a positive linear functional on an appropriate  $C^*$ -algebra,  $\mathcal{A}$ ) and equip it with the weakest topology such that the functions  $\hat{A} \mapsto \text{tr}(\hat{A}\hat{\rho})$  are continuous for all states  $\hat{\rho} \in \mathcal{E}_S$ . Then a ‘qr-number’ is defined as a global element of the sheaf of germs of continuous real-valued functions on  $\mathcal{E}_S$ . Put another way, a qr-number is a (Dedekind) real number in the topos,  $\text{Sh}(\mathcal{E}_S)$ , of sheaves over  $\mathcal{E}_S$ . The fundamental physical postulate is then:

1. The ‘numerical values’ of a physical quantity,  $A$ , are given by the qr-numbers  $a_Q(U) := \text{tr}(\hat{A}\hat{\rho})_{\hat{\rho} \in U}$  where  $U$  is an open subset of  $\mathcal{E}_S$ .
2. Every physical quantity has a qr-number value at all times.
3. Every physical quantity has an open subset of  $\mathcal{E}_S$  associated with it at all times. This is the extent to which the quantity can be said to ‘exist’.

Evidently this theory also is ‘contextual’, with the contexts now being identified with the open sets of  $\mathcal{E}_S$ .

There seems a possible link between these ideas and the work of Heunen and Spitter. The latter construct the (internal in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})}$ ) topos,  $\text{Sh}(\overline{\Sigma})$ , of sheaves over the locale  $\overline{\Sigma}$  and then show that each bounded self-adjoint operator in  $\mathcal{H}$  is represented by an analogue of an ‘interval domain’<sup>129</sup> in this topos. This is their analogue of our non-commutative spectral theory in which  $\hat{A}$  is represented by an arrow  $\delta^\circ(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^\succeq$  (or an arrow from  $\underline{\Sigma}$  to  $\underline{\mathbb{R}}^\leftarrow$ , if one prefers that choice of quantity-value object).

It would be interesting to see if there is any relation between the real numbers in the Corbett presheaf,  $\text{Sh}(\mathcal{E}_S)$  and the interval domains in the Heunen-Spitters presheaf  $\text{Sh}(\overline{\Sigma})$ . Roughly speaking, we can say that Corbett *et al* assign exact values to physical quantities by making the state ‘fuzzy’, whereas we (and Heunen & Spitter) keep the state sharp, but ascribe ‘fuzzy’ values to physical quantities. Clearly, there are some interesting questions here for further research.

## 14.2 The Quantity-Value Object $\mathcal{R}_\phi$

Let us turn now to the quantity-value object  $\mathcal{R}_\phi$ . This plays a key role in the representation of any physical quantity,  $A$ , by an arrow  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$ . In so far as a ‘thing’ is a bundle of properties, these properties refer to values of physical quantities, and so the nature of these ‘values’ is of central importance.

We anticipate that  $\mathcal{R}_\phi$  has many global elements  $1_{\tau_\phi} \rightarrow \mathcal{R}_\phi$ , and these can be interpreted as the possible ‘values’ for physical quantities. If  $\Sigma_\phi$  also has global elements/microstates  $s : 1_{\tau_\phi} \rightarrow \Sigma_\phi$ , then these combine with any arrow  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$  to give global elements of  $\mathcal{R}_\phi$ . It seems reasonable to refer to the element,  $A_\phi \circ s : 1_{\tau_\phi} \rightarrow \mathcal{R}_\phi$  as the ‘value’ of  $A$  when the microstate is  $s$ . However, our expectation is that, in general,  $\Sigma_\phi$  may well have no global elements, in which case the interpretation of  $A_\phi : \Sigma_\phi \rightarrow \mathcal{R}_\phi$  in terms of values is somewhat subtler. This has to be done internally using the language  $\mathcal{L}(\tau_\phi)$  associated with the topos  $\tau_\phi$ : the overall logical structure is a nice example of a ‘coherence’ theory of truth [30].

As far as axiomatic properties of  $\mathcal{R}_\phi$  are concerned, the minimal requirement is presumably that it should have some ordering property that arises in all topos representations of the system  $S$ . This universal property could be coded into the internal language,  $\mathcal{L}(S)$ , of  $S$ . This implements our intuitive

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<sup>129</sup>This is rather like a Dedekind real number except that the overlap axiom is missing.

feeling that, in so far as the concept of ‘value’ has any meaning, it must be possible to say that the value of one quantity is ‘larger’ (or ‘smaller’) than that of another. It seems reasonable to expect this relation to be transitive, but that is about all. In particular, we see no reason to suppose that this relation will always correspond to a *total* ordering: perhaps there are pairs of physical quantities whose ‘values’ simply cannot be compared at all. Thus, tentatively, we can augment  $\mathcal{L}(S)$  with the axioms for a poset structure on  $\mathcal{R}$ .

Beyond this simple ordering property, it becomes less clear what to assume about the quantity-value object. The example of quantum theory shows that it is wrong to automatically equate  $\mathcal{R}_\phi$  with the real-number object  $\mathbb{R}_\phi$  in the topos  $\tau_\phi$ . Indeed, we believe that is almost always the case.

However, this makes it harder to know what to assume of  $\mathcal{R}_\phi$ . The quantum case shows that  $\mathcal{R}_\phi$  may have considerably fewer algebraic properties than the real-number object  $\mathbb{R}_\phi$ . On a more ‘topological’ front it is attractive to assume that  $\mathcal{R}_\phi$  is an internal locale in the topos  $\tau_\phi$ . However, one should be cautious when conjecturing about  $\mathcal{R}_\phi$  since our discussion of various possible quantity-value objects in quantum theory depended closely on the specific details of the spectral structure in this topos.

A more general perspective is given by the work of Heunen and Spitters [38]. We recall that their starting point is a (non-commutative)  $C^*$ -algebra,  $\mathcal{A}$  in an ambient topos  $\mathfrak{S}$ . Then they construct the topos of co-presheaves,  $\mathfrak{S}^{\mathcal{V}(\mathcal{A})}$ , and show that  $\overline{\mathcal{A}}$  is an internal, pre  $C^*$ -algebra in this topos. Finally, they construct the spectrum,  $\overline{\Sigma}$ , of  $\overline{\mathcal{A}}$  and show that it is an internal locale.

Having shown that  $\overline{\Sigma}$  has the structure of a locale, it is rather natural to consider the (internal) topos of sheaves,  $\text{Sh}(\overline{\Sigma})$ , over  $\overline{\Sigma}$ , and then construct the ‘interval-domain’ object in this topos<sup>130</sup>. In the case of quantum theory, they show that this is related to what we have called  $\mathbb{R}^{\leftrightarrow}$ .

This approach might be a useful tool when looking for ways of axiomatising  $\mathcal{R}_\phi$ . Thus, if in any topos representation  $\phi$ , we assume that the state object  $\Sigma_\phi$  is an internal local in  $\tau_\phi$ , we can construct the internal topos  $\text{Sh}(\Sigma_\phi)$  and consider its interval-domain number object. It remains to be seen if this has any generic use in practice.

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<sup>130</sup>If  $X$  is any topological space it is well-known that the real numbers in the topos  $\text{Sh}(X)$  are in one-to-one correspondence with elements of the space,  $C(X, \mathbb{R})$ , of continuous, real-valued functions on  $X$

### 14.3 The Truth Objects $\mathbb{T}$ , or Pseudo-State Object $\mathbb{W}_\phi$

The truth objects in a topos representation are certain sub-objects of  $P\Sigma_\phi$ . Their construction will be very theory-dependent, as are the pseudo-states, and the pseudo-state object,  $\mathbb{W}_\phi$ , if there is one. Each proposition about the physical system is represented by a sub-object  $J \subseteq \Sigma_\phi$ , and given a truth-object  $\mathbb{T} \subseteq P\Sigma_\phi$ , the generalised truth value of the proposition is  $\llbracket J \in \mathbb{T} \rrbracket \in \Gamma\Omega_\phi$ ; in terms of pseudo-states,  $\mathfrak{w}$ , the generalised truth values are of the form  $\llbracket \mathfrak{w} \subseteq J \rrbracket \in \Gamma\Omega_\phi$ .

The key properties of the quantum truth objects,  $\underline{\mathbb{T}}^{|\psi\rangle}$ , (or pseudo-states  $\underline{\mathfrak{w}}^{|\psi\rangle}$ ) can easily be emulated if one is dealing with a more general base category,  $\mathcal{C}$ , so that the topos concerned is  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ . However, it is not clear what, if anything, can be said about the structure of truth objects/pseudo-states in a more generic topos representation.

An attractive possibility is that there is a general analogue of (6.54) in the form

$$\begin{array}{ccc}
 \mathfrak{w} & \longrightarrow & \Sigma_\phi \\
 & & \downarrow \pi_\phi \\
 1_{\tau_\phi} & \xrightarrow{\ulcorner \mathfrak{w} \urcorner} & P\Sigma_\phi
 \end{array} \tag{14.1}$$

and that obstructions to the existence of global elements of the state-object  $\Sigma_\phi$  can be studied with the aid of this diagram. If there is a pseudo-state object  $\mathbb{W}_\phi$  then this could be a natural replacement for  $P\Sigma_\phi$  in this diagram.

## 15 Conclusion

In this long article we have developed the idea that, for any given theory-type (classical physics, quantum physics, DI-physics,...) the theory of a particular physical system,  $S$ , is to be constructed in the framework of a certain, system-dependent, topos. The central idea is that a local language,  $\mathcal{L}(S)$ , is attached to each system  $S$ , and that the application of a given theory-type to  $S$  is equivalent to finding a representation,  $\phi$ , of  $\mathcal{L}(S)$  in a topos  $\tau_\phi(S)$ ; this is equivalent to finding a translation of  $\mathcal{L}(S)$  into the internal language associated with  $\tau_\phi(S)$ ; or a functor to  $\tau_\phi(S)$  from the topos associated with  $\mathcal{L}(S)$ .

Physical quantities are represented by arrows in the topos from the state

object  $\Sigma_{\phi,S}$  to the quantity-value object  $\mathcal{R}_{\phi,S}$ , and propositions are represented by sub-objects of the state object. The idea of a ‘truth sub-object’ of  $P\Sigma_{\phi,S}$  (or a ‘pseudo-state’ sub-object of  $\Sigma_{\phi,S}$ ) then leads to a neo-realist interpretation of propositions in which each proposition is assigned a truth value that is a global element of the sub-object classifier  $\Omega_{\tau_\phi(S)}$ . In general, neo-realist statements about the world/system  $S$  are to be expressed in the internal language of the topos  $\tau_\phi(S)$ . Underlying this is the intuitionistic, deductive logic provided by the local language  $\mathcal{L}(S)$ .

These axioms are based on ideas from the topos representation of quantum theory, which we have discussed in depth. Here, the topos involved is  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ : the topos of presheaves over the base category  $\mathcal{V}(\mathcal{H})$  of commutative, von Neumann sub-algebras of the algebra,  $B(\mathcal{H})$ , of all bounded operators on the quantum Hilbert space,  $\mathcal{H}$ . Each such sub-algebra can be viewed as a context in which the theory can be viewed from a classical perspective. Thus a context can be described as a ‘classical snap-shot’, or ‘window on reality’, or ‘world-view’/weltanschauung. Mathematically, a context is a ‘stage of truth’: a concept that goes back to Kripke’s use of a presheaf topos as a model of his intuitionistic view of time and process.

We have shown how the process of ‘daseinisation’ maps projection operators (and hence equivalent classes of propositions) into sub-objects of the state-object  $\underline{\Sigma}$ . We have also shown how this can be extended to an arbitrary, bounded self-adjoint operator,  $\hat{A}$ . This produces an arrow  $\check{A} : \underline{\Sigma} \rightarrow \underline{\mathcal{R}}$  where the minimal choice for the quantity-value object,  $\underline{\mathcal{R}}$ , is the object  $\underline{\mathbb{R}}^{\succeq}$ . We have also argued that, from a physical perspective, it is more attractive to choose  $\underline{\mathbb{R}}^{\leftrightarrow}$  as the quantity-value presheaf. The significance of these results is enhanced considerably by the alternative, Heunen & Spitters derivation of  $\underline{\Sigma}$  as the spectrum of an internal, commutative algebra. These, and related, results all encourage the idea that quantum theory can be viewed as classical theory but in a topos other than the topos of sets,  $\mathbf{Sets}$ .

Every classical system uses the same topos,  $\mathbf{Sets}$ . However, in general, the topos will be system dependent as, for example, is the case with the quantum topoi of the form  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , where  $\mathcal{H}$  is the Hilbert space of the system. This leads to the problem of understanding how the topoi for a class of systems behave under the action of taking a sub-system, or combining a pair of systems to give a single composite system. We have presented a set of axioms that capture the general ideas we are trying to develop. Of course, these axioms are not cast in stone, and are still partly ‘experimental’ in nature. However, we have shown that classical physics exactly fits our suggested scheme, and that quantum physics ‘almost’ does: ‘almost’ because of the issues concerning the translation representation of the arrows associated

with compositions of systems that were discussed in Section 13.3.

An important challenge for future work is to show that our general topos scheme can be used to develop genuinely new theories of physics, not just to rewrite old ones in a new language. Of particular interest is the problem with which we motivated the scheme in the first place: namely, to find tools for constructing theories that go beyond quantum theory and which do not use Hilbert spaces, path integrals, or any of the other familiar entities in which the continuum real and/or complex numbers play a fundamental role.

As we have discussed, the topoi for quantum systems are of the form  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , and hence embody contextual logic in a fundamental way. One way of going ‘beyond’ quantum theory, while escaping the *a priori* imposition of continuum concepts, is to use presheaves over a more general ‘category of contexts’,  $\mathcal{C}$ , i.e., develop the theory in the topos  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ . Such a structure embodies contextual, multi-valued logic in an intrinsic way, and in that sense might be said to encapsulate one of the fundamental insights of quantum theory. However, and unlike in quantum theory, there is no obligation to use the real or complex numbers in the construction of the category  $\mathcal{C}$ .

Indeed, early on in this work we noted that real numbers arise in theories of physics in three different (but related) ways: (i) as the values of physical quantities; (ii) as the values of probabilities; and (iii) as a fundamental ingredient in models of space and time. The first of these is now subsumed into the quantity-value object  $\mathcal{R}_\phi$ , and which now has no *a priori* relation to the real number object in  $\tau_\phi$ . The second source of real numbers has gone completely since we no longer have probabilities of propositions but rather generalised truth values whose values lie in  $\Gamma\Omega_{\tau_\phi}$ . The third source is also no long binding since models of space and time in a topos could depend on many things: for example, infinitesimals.

Of course, although true, these remarks do not of themselves give a concrete example of a theory that is ‘beyond quantum theory’. On the other hand, these ideas certainly point in a novel direction, and one at which, almost certainly, we would not have arrived if the challenge to ‘go beyond quantum theory’ had been construed only in terms of trying to generalise Hilbert spaces, path integrals, and the like.

From a more general perspective, other types of topoi are possible realms for the construction of physical theories. One simple, but mathematically rich example arises from the theory of  $M$ -sets. Here,  $M$  is a monoid and, like all monoids, can be viewed as a category with a single object, and whose arrows are the elements of  $M$ . Thought of as a category, a monoid is ‘complementary’ to a partially-ordered set. In a monoid, there is only one object, but plenty of



arrows from that object to itself; whereas in a partially-ordered set there are plenty of objects, but at most one arrow between any pair of objects. Thus a partially-ordered set is the most economical category with which to capture the concept of ‘contextual logic’. On the other hand, the logic associated with a monoid is non-contextual as there is only one object in the category.

It is easy to see that a functor from  $M$  to **Sets** is just an ‘ $M$ -set’: i.e., a set on which  $M$  acts as a monoid of transformations. An arrow between two such  $M$ -sets is an equivariant map between them. In physicists’ language, one would say that the topos  $\mathbf{Sets}^M$ —usually denoted  $BM$ —is the category of the ‘non-linear realisations’ of  $M$ .

The sub-object classifier,  $\Omega_{BM}$ , in  $BM$  is the collection of left ideals in  $M$ ; hence, many of the important constructions in the topos can be handled using the language of algebra. The topos  $BM$  is one of the simplest to define and work with and, for that reason, it is a popular source of examples in texts on topos theory. It would be intriguing to experiment with constructing model theories of physics using one of these simple topoi. One possible use of  $M$ -sets is discussed in [43] in the context of reduction of the state vector, but there will surely be others.

**Is there ‘un gros topos’?** It is clear that there are many other topics for future research. A question that is of particular interest is if there is a *single* topos within which all systems of a given theory-type can be discussed. For example, in the case of quantum theory the relevant topoi are of the form  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , where  $\mathcal{H}$  is a Hilbert space, and the question is whether all such topoi can be gathered together to form a single topos (what Grothendieck termed ‘un gros topos’) within which all quantum systems can be discussed.

There are well-known examples of such constructions in the mathematical literature. For example, the category,  $\text{Sh}(X)$ , of sheaves on a topological space  $X$  is a topos, and there are collections  $\mathbf{T}$  of topological spaces which form a Grothendieck site, so that the topos  $\text{Sh}(\mathbf{T})$  can be constructed. A particular object in  $\text{Sh}(\mathbf{T})$  will then be a sheaf over  $\mathbf{T}$  whose stalk over any object  $X$  in  $\mathbf{T}$  will be the topos  $\text{Sh}(X)$ .

For our purposes, the ideal situation would be if the various categories of systems, **Sys**, can be chosen in such a way that  $\mathcal{M}(\mathbf{Sys})$  is a site. Then the topos of sheaves,  $\text{Sh}(\mathcal{M}(\mathbf{Sys}))$ , over this site would provide a common topos in which all systems of this theory type—i.e., the objects of **Sys**—can be discussed. We do not know if this is possible, and it is a natural subject for future study.



**Some more speculative lines of future research.** At a conceptual level, one motivating desire for the entire research programme was to find a formalism that would always give some sort of ‘realist’ interpretation, even in the case of quantum theory which is normally presented in an instrumentalist way. But this particular example raises an interesting point because the neo-realist interpretation takes place in the topos  $\mathbf{Sets}^{\vee(\mathcal{H})^{\text{op}}}$ , whereas the instrumentalist interpretation works in the familiar topos  $\mathbf{Sets}$  of sets, and one might wonder how universal is the use of a pair of topoi in this way.

Another, related, issue concerns the representation of the  $\mathcal{P}\mathcal{L}(S)$ -propositions of the form “ $A \varepsilon \Delta$ ” discussed in Section 5. This serves as a bridge between the ‘external’ world of a background spatial structure, and the internal world of the topos. This link is not present with the  $\mathcal{L}(S)$  language whose propositions are purely internal terms of type  $\Omega$  of the form ‘ $A(\tilde{s}) \in \tilde{\Delta}$ ’. In a topos representation,  $\phi$ , of  $\mathcal{L}(S)$ , these become propositions of the form ‘ $A \in \Xi$ ’, where  $\Xi$  is a sub-object of  $\mathcal{R}_\phi$ .

In general, if we have an example of our axioms working neo-realistically in a topos  $\tau$ , one might wonder if there is an ‘instrumentalist’ interpretation of the same theory in a different topos,  $\tau_i$ , say? Of course, the word ‘instrumentalism’ is used metaphorically here, and any serious consideration of such a pair  $(\tau, \tau_i)$  would require a lot of very careful thought.

However, if a pair  $(\tau, \tau_i)$  does exist, the question then arises of whether there is a *categorical* way of linking the neo-realist and instrumentalist interpretations: for example, via a functor  $I : \tau \rightarrow \tau_i$ . If so, is this related to some analogue of the daseinisation operation that produced the representation of the  $\mathcal{P}\mathcal{L}(S)$ -propositions, “ $A \varepsilon \Delta$ ” in quantum theory? Care is needed in discussing such issues since informal set theory is used as a meta-language in constructing a topos, and one has to be careful not to confuse this with the existence, or otherwise, of an ‘instrumentalist’ interpretation of any given representation.

If such a functor,  $I : \tau \rightarrow \tau_i$ , did exist then one could speculate on the possibility of finding an ‘interpolating chain’ of functors

$$\tau \rightarrow \tau^1 \rightarrow \tau^2 \rightarrow \dots \rightarrow \tau^n \rightarrow \tau_i \quad (15.1)$$

which could be interpreted conceptually as corresponding to an interpolation between the philosophical views of realism and instrumentalism!

Even more speculatively one might wonder if “one person’s realism is another person’s instrumentalism”. More precisely, given a pair  $(\tau, \tau_i)$  in the sense above, could there be cases in which the topos  $\tau$  carries a neo-realist interpretation of a theory with respect to an instrumentalist interpretation in

$\tau_i$ , whilst being the carrier of an instrumentalist interpretation with respect to the neo-realism of a ‘higher’ topos; and so on? For example, is there some theory whose ‘instrumentalist manifestation’ takes place in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ ?

On the other hand, one might want to say that ‘instrumentalist’ interpretations always take place in the world of classical set theory, so that  $\tau_i$  should always be chosen to be  $\mathbf{Sets}$ . In any event, it would be interesting to study the quantum case more closely to see if there are any categorial relations between the formulation in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$  and the instrumentalism interpretation in  $\mathbf{Sets}$ . It can be anticipated that the action of daseinisation will play an important role here.

All this is, perhaps<sup>131</sup>, rather speculative but there is a more obvious situation in which a double-topos structure will be necessary, irrespective of philosophical musings on instrumentalism. This is if one wants to discuss the ‘classical limit’ of some topos theory. In this case this limit will exist in the topos,  $\mathbf{Sets}$ , and this must be used in addition to the topos of the basic theory. A good example of this, of course, is the topos of quantum theory discussed in this article. If, *pace* Landsmann, one thinks of ‘quantisation’ as a functor from  $\mathbf{Sets}$  to  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ , then the classical limit will perhaps involve a functor going in the opposite direction.

**Implications for quantum gravity.** A serious claim stemming from our work is that a successful theory of quantum gravity should be constructed in some topos  $\mathcal{U}$ —the ‘topos of the universe’—that is *not* the topos of sets. All entities of physical interest will be represented in this topos, including models for space-time (if there are any at a fundamental level in quantum gravity) and, if relevant, loops, membranes etc. as well as incorporating the anticipated generalisation of quantum theory.

Such a theory of quantum gravity will have a neo-realist interpretation in the topos  $\mathcal{U}$ , and hence would be particularly useful in the context of quantum cosmology. However, in practice, physicists divide the world up into smaller, more easily handled, chunks, and each of them would correspond to what earlier we have called a ‘system’ and, correspondingly, would have its own topos. Thus  $\mathcal{U}$  is something like the ‘gros topos’ of the theory, and would combine together the individual ‘sub-systems’ in a categorial way. Of course, it is most unlikely that there is any preferred way of dividing the universe up into bite-sized chunks, but this is not problematic as the ensuing relativism is naturally incorporated into the idea of a Grothendieck site.

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<sup>131</sup>To be honest, the ‘perhaps’ should really be replaced by ‘highly’.

## 16 Appendix 1: Some Theorems and Constructions Used in the Main Text

### 16.1 Results on Clopen Subobjects of $\underline{\Sigma}$ .

**Theorem 16.1** *The collection,  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ , of all clopen sub-objects of  $\underline{\Sigma}$  is a Heyting algebra.*

**Proof.** First recall how a Heyting algebra structure is placed on the set,  $\text{Sub}(\underline{\Sigma})$ , of all sub-objects of  $\underline{\Sigma}$ .

**The ‘ $\vee$ ’- and ‘ $\wedge$ ’-operations.** Let  $\underline{S}, \underline{T}$  be two sub-objects of  $\underline{\Sigma}$ . Then the ‘ $\vee$ ’ and ‘ $\wedge$ ’ operations are defined by

$$(\underline{S} \vee \underline{T})_V := \underline{S}_V \cup \underline{T}_V \quad (16.1)$$

$$(\underline{S} \wedge \underline{T})_V := \underline{S}_V \cap \underline{T}_V \quad (16.2)$$

for all contexts  $V$ . It is easy to see that if  $\underline{S}$  and  $\underline{T}$  are clopen sub-objects of  $\underline{\Sigma}$ , then so are  $\underline{S} \vee \underline{T}$  and  $\underline{S} \wedge \underline{T}$ .

**The zero and unit elements.** The zero element in the Heyting algebra  $\text{Sub}(\underline{\Sigma})$  is the empty sub-object  $\underline{0} := \{\emptyset_V \mid V \in \text{Ob}(\mathcal{V}(\mathcal{H}))\}$ , where  $\emptyset_V$  is the empty subset of  $\underline{\Sigma}_V$ . The unit element in  $\text{Sub}(\underline{\Sigma})$  is  $\underline{\Sigma}$ . It is clear that both  $\underline{0}$  and  $\underline{\Sigma}$  are clopen sub-objects of  $\underline{\Sigma}$ .

**The ‘ $\Rightarrow$ ’-operation.** The most interesting part is the definition of the implication  $\underline{S} \Rightarrow \underline{T}$ . For all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , it is given by

$$\begin{aligned} (\underline{S} \Rightarrow \underline{T})_V &:= \{\lambda \in \underline{\Sigma}_V \mid \forall V' \subseteq V, \text{ if} \\ &\quad \underline{\Sigma}(i_{V'V})(\lambda) \in \underline{S}_{V'}, \text{ then } \underline{\Sigma}(i_{V'V})(\lambda) \in \underline{T}_{V'}\} \end{aligned} \quad (16.3)$$

$$\begin{aligned} &= \{\lambda \in \underline{\Sigma}_V \mid \forall V' \subseteq V, \text{ if} \\ &\quad \lambda|_{V'} \in \underline{S}_{V'}, \text{ then } \lambda|_{V'} \in \underline{T}_{V'}\}. \end{aligned} \quad (16.4)$$

Since  $\neg \underline{S} := \underline{S} \Rightarrow \underline{0}$ , the expression for negation follows from the above as

$$(\neg \underline{S})_V = \{\lambda \in \underline{\Sigma}_V \mid \forall V' \subseteq V, \underline{\Sigma}(i_{V'V})(\lambda) \notin \underline{S}_{V'}\} \quad (16.5)$$

$$= \{\lambda \in \underline{\Sigma}_V \mid \forall V' \subseteq V, \lambda|_{V'} \notin \underline{S}_{V'}\}. \quad (16.6)$$

We rewrite the formula for negation as

$$(\neg \underline{S})_V = \bigcap_{V' \subseteq V} \{ \lambda \in \underline{\Sigma}_V \mid \lambda|_{V'} \in \underline{S}_{V'}^c \} \quad (16.7)$$

where  $\underline{S}_{V'}^c$  denotes the complement of  $\underline{S}_{V'}$  in  $\underline{\Sigma}_{V'}$ . Clearly,  $\underline{S}_{V'}^c$  is clopen in  $\underline{\Sigma}_{V'}$  since  $\underline{S}_{V'}$  is clopen. Since the restriction  $\underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V \rightarrow \underline{\Sigma}_{V'}$  is continuous and surjective<sup>132</sup>, it is easy to see that the inverse image  $\underline{\Sigma}(i_{V'V})^{-1}(\underline{S}_{V'}^c)$  is clopen in  $\underline{\Sigma}_V$ . Clearly,

$$\underline{\Sigma}(i_{V'V})^{-1}(\underline{S}_{V'}^c) = \{ \lambda \in \underline{\Sigma}_V \mid \lambda|_{V'} \in \underline{S}_{V'}^c \} \quad (16.8)$$

and so, from (16.7) we have

$$(\neg \underline{S})_V = \bigcap_{V' \subseteq V} \underline{\Sigma}(i_{V'V})^{-1}(\underline{S}_{V'}^c) \quad (16.9)$$

The problem is that we want  $(\neg \underline{S})_V$  to be a *clopen* subset of  $\underline{\Sigma}_V$ . Now the right hand side of (16.9) is the intersection of a family, parameterised by  $\{V' \mid V' \subseteq V\}$ , of clopen sets. Such an intersection is always closed, but it is only guaranteed to be open if  $\{V' \mid V' \subseteq V\}$  is a finite set, which of course may not be the case.

If  $V'' \subseteq V'$  and  $\lambda|_{V''} \in \underline{S}_{V''}^c$ , then  $\lambda|_{V'} \in \underline{S}_{V'}^c$ . Indeed, if we had  $\lambda|_{V'} \in \underline{S}_{V'}$ , then  $(\lambda|_{V'})|_{V''} = \lambda|_{V''} \in \underline{S}_{V''}$  by the definition of a sub-object, so we would have a contradiction. This implies  $\underline{\Sigma}(i_{V''V})^{-1}(\underline{S}_{V''}^c) \subseteq \underline{\Sigma}(i_{V'V})^{-1}(\underline{S}_{V'}^c)$ , and hence the right hand side of (16.9) is a decreasing net of clopen subsets of  $\underline{\Sigma}_V$  which converges to something, which we take as the subset of  $\underline{\Sigma}_V$  that is to be  $(\neg \underline{S})_V$ .

Here we have used the fact that the set of clopen subsets of  $\underline{\Sigma}_V$  is a complete lattice, where the minimum of a family  $(U_i)_{i \in I}$  of clopen subsets is defined as the interior of  $\bigcap_{i \in I} U_i$ . This leads us to define

$$(\neg \underline{S})_V := \text{int} \bigcap_{V' \subseteq V} \underline{\Sigma}(i_{V'V})^{-1}(\underline{S}_{V'}^c) \quad (16.10)$$

$$= \text{int} \bigcap_{V' \subseteq V} \{ \lambda \in \underline{\Sigma}_V \mid \lambda|_{V'} \in (\underline{S}_{V'}^c) \} \quad (16.11)$$

as the negation in  $\text{Sub}_{\text{cl}}(\underline{\Sigma})$ . This modified definition guarantees that  $\neg \underline{S}$  is a *clopen* sub-object. A straightforward extension of this method gives a consistent definition of  $\underline{S} \Rightarrow \underline{T}$ .

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<sup>132</sup>See proof of Theorem 16.2 below.

This concludes the proof of the theorem. ■

The following theorem shows the relation between the restriction mappings of the outer presheaf  $\underline{Q}$  and those of the spectral presheaf  $\underline{\Sigma}$ . We basically follow de Groote's proof of Prop. 3.22 in [33] and show that this result, which uses quite a different terminology, actually gives the desired relation.

**Theorem 16.2** *Let  $V, V' \in \text{Ob}(\mathcal{V}(\mathcal{H}))$  such that  $V' \subset V$ . Then*

$$S_{\underline{Q}(i_{V'V})(\delta^\circ(\hat{P})_V)} = \underline{\Sigma}(i_{V'V})(S_{\delta^\circ(\hat{P})_V}). \quad (16.12)$$

**Proof.** First of all, to simplify notation, we can replace  $\delta^\circ(\hat{P})_V$  by  $\hat{P}$  (which amounts to the assumption that  $\hat{P} \in \mathcal{P}(V)$ . This does not play a rôle for the current argument). By definition,  $\underline{Q}(i_{V'V})(\hat{P}) = \delta^\circ(\hat{P})_{V'}$ , so we have to show that  $S_{\delta^\circ(\hat{P})_{V'}} = \underline{\Sigma}(i_{V'V})(S_{\hat{P}})$  holds.

If  $\lambda \in S_{\hat{P}}$ , then  $\lambda(\hat{P}) = 1$ , which implies  $\lambda(\hat{Q}) = 1$  for all  $\hat{Q} \geq \hat{P}$ . In particular,  $\lambda(\delta^\circ(\hat{P})_{V'}) = 1$ , so  $\underline{\Sigma}(i_{V'V})(\lambda) = \lambda|_{V'} \in S_{\delta^\circ(\hat{P})_{V'}}$ . This shows that  $\underline{\Sigma}(i_{V'V})(S_{\hat{P}}) \subseteq S_{\delta^\circ(\hat{P})_{V'}}$ .

To show the converse inclusion, let  $\lambda' \in S_{\delta^\circ(\hat{P})_{V'}}$ , which means that  $\lambda'(\delta^\circ(\hat{P})_{V'}) = 1$ . We have  $\hat{P} \in \underline{Q}(i_{V'V})^{-1}(\delta^\circ(\hat{P})_{V'})$ . Let

$$F_{\lambda'} := \{\hat{Q} \in \mathcal{P}(V') \mid \lambda'(\hat{Q}) = 1\} = \lambda'^{-1}(1) \cap \mathcal{P}(V'). \quad (16.13)$$

As shown in section 8.3,  $F_{\lambda'}$  is an ultrafilter in the projection lattice  $\mathcal{P}(V')$ .<sup>133</sup> The idea is to show that  $F_{\lambda'} \cup \hat{P}$  is a filter base in  $\mathcal{P}(V)$  that can be extended to an ultrafilter, which corresponds to an element of the Gel'fand spectrum of  $V$ .

Let us assume that  $F_{\lambda'} \cup \hat{P}$  is not a filter base in  $\mathcal{P}(V)$ . Then there exists some  $\hat{Q} \in F_{\lambda'}$  such that

$$\hat{Q} \wedge \hat{P} = \hat{Q}\hat{P} = \hat{0}, \quad (16.14)$$

which implies  $\hat{P} \leq \hat{1} - \hat{Q}$ , so

$$\underline{Q}(i_{V'V})(\hat{P}) = \delta^\circ(\hat{P})_{V'} \leq \underline{Q}(i_{V'V})(\hat{1} - \hat{Q}) = \hat{1} - \hat{Q} \quad (16.15)$$

and hence we get the contradiction

$$1 = \lambda'(\delta^\circ(\hat{P})_{V'}) \leq \lambda'(\hat{1} - \hat{Q}) = 0. \quad (16.16)$$

<sup>133</sup>In general, each ultrafilter  $F$  in the projection lattice of an abelian von Neumann algebra  $V$  corresponds to a unique element  $\lambda_F$  of the Gel'fand spectrum of  $V$ . The ultrafilter is the collection of all those projections that are mapped to 1 by  $\lambda$ , i.e.,  $F = \lambda_F^{-1}(1) \cap \mathcal{P}(V)$ .

By Zorn's lemma, the filter base  $F_{\lambda'} \cup \hat{P}$  is contained in some (not necessarily unique) maximal filter base in  $\mathcal{P}(V)$ . Such a maximal filter base is an ultrafilter and thus corresponds to an element  $\lambda$  of the Gel'fand spectrum  $\underline{\Sigma}_V$  of  $V$ . Since  $\hat{P}$  is contained in the ultrafilter, we have  $\lambda(\hat{P}) = 1$ , so  $\lambda \in \mathcal{S}_{\hat{P}}$ . By construction,  $\underline{\Sigma}(i_{V'V})(\lambda) = \lambda|_{V'} = \lambda' \in S_{\delta^\circ(\hat{P})_{V'}}$ , the element of  $\underline{\Sigma}_{V'}$  we started from. This shows that  $S_{\delta^\circ(\hat{P})_{V'}} \subseteq \underline{\Sigma}(i_{V'V})(\mathcal{S}_{\hat{P}})$ , and we obtain

$$S_{\delta^\circ(\hat{P})_{V'}} = \underline{\Sigma}(i_{V'V})(\mathcal{S}_{\hat{P}}). \quad (16.17)$$

It is well-known that every state  $\lambda' \in \underline{\Sigma}_{V'}$  is of the form  $\lambda' = \underline{\Sigma}(i_{V'V})(\lambda) = \lambda|_{V'}$  for some  $\lambda \in \underline{\Sigma}_V$ . This implies

$$\underline{\Sigma}(i_{V'V})^{-1}(S_{\delta^\circ(\hat{P})_{V'}}) = S_{\delta^\circ(\hat{P})_{V'}} \subseteq \underline{\Sigma}_V. \quad (16.18)$$

Note that on the right hand side,  $S_{\delta^\circ(\hat{P})_{V'}}$  (and not  $\mathcal{S}_{\hat{P}}$ , which is a smaller set in general) shows up. ■

De Groote has shown in [33] that for any unital abelian von Neumann algebra  $V$ , the clopen sets  $S_{\hat{Q}}$ ,  $\hat{Q} \in \mathcal{P}(V)$ , form a base of the Gel'fand topology on  $\underline{\Sigma}_V$ . Formulas (16.17) and (16.18) hence show that the restriction mappings

$$\begin{aligned} \underline{\Sigma}(i_{V'V}) : \underline{\Sigma}_V &\rightarrow \underline{\Sigma}_{V'} \\ \lambda &\mapsto \lambda|_{V'} \end{aligned}$$

of the spectral presheaf are open and continuous. Using continuity, it is easy to see that  $\underline{\Sigma}(i_{V'V})$  is also closed: let  $C \subseteq \underline{\Sigma}_V$  be a closed subset. Since  $\underline{\Sigma}_V$  is compact,  $C$  is compact, and since  $\underline{\Sigma}(i_{V'V})$  is continuous,  $\underline{\Sigma}(i_{V'V})(C) \subseteq \underline{\Sigma}_{V'}$  is compact, too. However,  $\underline{\Sigma}_{V'}$  is Hausdorff, and so  $\underline{\Sigma}(i_{V'V})(C)$  is closed in  $\underline{\Sigma}_{V'}$ .

## 16.2 The Grothendieck $k$ -Construction for an Abelian Monoid

Let us briefly review the Grothendieck construction for an abelian monoid  $M$ .

**Definition 16.1** *A group completion of  $M$  is an abelian group  $k(M)$  together with a monoid map  $\theta : M \rightarrow k(M)$  that is universal. Namely, given any monoid morphism  $\phi : M \rightarrow G$ , where  $G$  is an abelian group, there exists a unique group morphism  $\phi' : k(M) \rightarrow G$  such that  $\phi$  factors through  $\phi'$ ; i.e., we have the commutative diagram*

$$\begin{array}{ccc}
M & \xrightarrow{\phi} & G \\
& \searrow \theta & \nearrow \phi' \\
& & k(M)
\end{array}$$

with  $\phi = \phi' \circ \theta$ .

It is easy to see that any such  $k(M)$  is unique up to isomorphism.

To prove existence, first take the set of all pairs  $(a, b) \in M \times M$ , each of which is to be thought of heuristically as  $a - b$ . Then, note that if inverses existed in  $M$ , we would have  $a - b = c - d$  if and only if  $a + d = c + b$ . This suggests defining an equivalence relation on  $M \times M$  in the following way:

$$(a, b) \equiv (c, d) \text{ iff } \exists g \in M \text{ such that } a + d + g = b + c + g. \quad (16.19)$$

**Definition 16.2** *The Grothendieck completion of an abelian monoid  $M$  is the pair  $(k(M), \theta)$  defined as follows:*

(i)  $k(M)$  is the set of equivalence classes  $[a, b]$ , where the equivalence relation is defined in (16.19). A group law on  $k(M)$  is defined by

$$(i) \quad [a, b] + [c, d] := [a + c, b + d], \quad (16.20)$$

$$(ii) \quad 0_{k(M)} := [0_M, 0_M], \quad (16.21)$$

$$(iii) \quad -[a, b] := [b, a], \quad (16.22)$$

where  $0_M$  is the unit in the abelian monoid  $M$ .

(ii) The map  $\theta : M \rightarrow k(M)$  is defined by

$$\theta(a) := [a, 0] \quad (16.23)$$

for all  $a \in M$ .

It is straightforward to show that (i) these definitions are independent of the representative elements in the equivalence classes; (ii) the axioms for a group are satisfied; and (iii) the map  $\theta$  is universal in the sense mentioned above.

It is also clear that  $k$  is a *functor* from the category of abelian monoids to the category of abelian groups. For, if  $f : M_1 \rightarrow M_2$  is a morphism between abelian monoids, define  $k(f) : k(M_1) \rightarrow k(M_2)$  by  $k(f)[a, b] := [f(a), f(b)]$  for all  $a, b \in M_1$ .

### 16.3 Functions of Bounded Variation and $\Gamma\mathbb{R}^{\leq}$

These techniques will now be applied to the set,  $\Gamma\mathbb{R}^{\leq}$ , of global elements of  $\mathbb{R}^{\leq}$ . We could equally well consider  $\Gamma\mathbb{R}^{\leftrightarrow}$  and its  $k$ -extension, but this would just make the notation more complex, so in this and the following subsections, we will mainly concentrate on  $\Gamma\mathbb{R}^{\leq}$  (resp.  $\mathbb{R}^{\leq}$ ). The results can easily be extended to  $\Gamma\mathbb{R}^{\leftrightarrow}$  (resp.  $\mathbb{R}^{\leftrightarrow}$ ).

It was discussed in Section 8.2 how global elements of  $\mathbb{R}^{\leftrightarrow}$  are in one-to-one correspondence with pairs  $(\mu, \nu)$  consisting of an order-preserving and an order-reversing function on the category  $\mathcal{V}(\mathcal{H})$ ; i.e., with functions  $\mu : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \mathbb{R}$  such that, for all  $V_1, V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , if  $V_2 \subseteq V_1$  then  $\mu(V_2) \leq \mu(V_1)$  and  $\nu : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \mathbb{R}$  such that, for all  $V_1, V_2 \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , if  $V_2 \subseteq V_1$  then  $\nu(V_2) \geq \nu(V_1)$ ; see (8.75). The monoid law on  $\Gamma\mathbb{R}^{\leftrightarrow}$  is given by (8.79).

Clearly, global elements of  $\mathbb{R}^{\leq}$  are given by order-reversing functions  $\nu : \mathcal{V}(\mathcal{H}) \rightarrow \mathbb{R}$ , and  $\Gamma\mathbb{R}^{\leq}$  is an abelian monoid in the obvious way. Hence the Grothendieck construction can be applied to give an abelian group  $k(\Gamma\mathbb{R}^{\leq})$ . This is defined to be the set of equivalence classes  $[\nu, \kappa]$  where  $\nu, \kappa \in \Gamma\mathbb{R}^{\leq}$ , and where  $(\nu_1, \kappa_1) \equiv (\nu_2, \kappa_2)$  if, and only if, there exists  $\alpha \in \Gamma\mathbb{R}^{\leq}$ , such that

$$\nu_1 + \kappa_2 + \alpha = \kappa_1 + \nu_2 + \alpha \quad (16.24)$$

Since  $\Gamma\mathbb{R}^{\leq}$  has a cancellation law, we have  $(\nu_1, \kappa_1) \equiv (\nu_2, \kappa_2)$  if, and only if,

$$\nu_1 + \kappa_2 = \kappa_1 + \nu_2. \quad (16.25)$$

Intuitively, we can think of  $[\nu, \kappa]$  as being ‘ $\nu - \kappa$ ’, and embed  $\Gamma\mathbb{R}^{\leq}$  in  $k(\Gamma\mathbb{R}^{\leq})$  by  $\nu \mapsto [\nu, 0]$ . However,  $\nu, \kappa$  are  $\mathbb{R}$ -valued functions on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$  and hence, in this case, the expression ‘ $\nu - \kappa$ ’ also has a *literal* meaning: i.e., as the function  $(\nu - \kappa)(V) := \nu(V) - \kappa(V)$  for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ .

This is not just a coincidence of notation. Indeed, let  $F(\text{Ob}(\mathcal{V}(\mathcal{H})), \mathbb{R})$  denote the set of all real-valued functions on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$ . Then we can construct the map,

$$\begin{aligned} j : k(\Gamma\mathbb{R}^{\leq}) &\rightarrow F(\text{Ob}(\mathcal{V}(\mathcal{H})), \mathbb{R}) \\ [\nu, \kappa] &\mapsto \nu - \kappa \end{aligned} \quad (16.26)$$

which is well-defined on equivalence classes.

It is easy to see that the map in (16.26) is injective. This raises the question of the *image* in  $F(\text{Ob}(\mathcal{V}(\mathcal{H})), \mathbb{R})$  of the map  $j$ : i.e., what types of real-valued function on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$  can be written as the difference between two order-reversing functions?



For functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , it is a standard result that a function can be written as the difference between two monotonic functions if, and only if, it has bounded variation. The natural conjecture is that a similar result applies here. To show this, we proceed as follows.

Let  $f : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \mathbb{R}$  be a real-valued function on the set of objects in the category  $\mathcal{V}(\mathcal{H})$ . At each  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ , consider a finite chain

$$C := \{V_0, V_1, V_2, \dots, V_{n-1}, V \mid V_0 \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset V\} \quad (16.27)$$

of proper subsets, and define the *variation* of  $f$  on this chain to be

$$V_f(C) := \sum_{j=1}^n |f(V_j) - f(V_{j-1})| \quad (16.28)$$

where we set  $V_n := V$ . Now take the supremum of  $V_f(C)$  for all such chains  $C$ . If this is finite, we say that  $f$  has a *bounded variation* and define

$$I_f(V) := \sup_C V_f(C) \quad (16.29)$$

Then it is clear that (i)  $V \mapsto I_f(V)$  is an order-preserving function on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$ ; (ii)  $f - I_f$  is an order-reversing function on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$ ; and (iii)  $-I_f$  is an order-reversing function on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$ . Thus, any function,  $f$ , of bounded variation can be written as

$$f \equiv (f - I_f) - (-I_f) \quad (16.30)$$

which is the difference of two order-reversing functions; i.e.,  $f$  can be expressed as the difference of two elements of  $\Gamma \underline{\mathbb{R}}^{\leq}$ .

Conversely, it is a straightforward modification of the proof for functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ , to show that if  $f : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \mathbb{R}$  is the difference of two order-reversing functions, then  $f$  is of bounded variation. The conclusion is that  $k(\Gamma \underline{\mathbb{R}}^{\leq})$  is in bijective correspondence with the set,  $\text{BV}(\text{Ob}(\mathcal{V}(\mathcal{H})), \mathbb{R})$ , of functions  $f : \text{Ob}(\mathcal{V}(\mathcal{H})) \rightarrow \mathbb{R}$  of bounded variation.

## 16.4 Taking Squares in $k(\Gamma \underline{\mathbb{R}}^{\leq})$

We can now think of  $k(\Gamma \underline{\mathbb{R}}^{\leq})$  in two ways: (i) as the set of equivalence classes  $[\nu, \kappa]$ , of elements  $\nu, \kappa \in \Gamma \underline{\mathbb{R}}^{\leq}$ ; and (ii) as the set,  $\text{BV}(\text{Ob}(\mathcal{V}(\mathcal{H})), \mathbb{R})$ , of differences  $\nu - \kappa$  of such elements.

As expected,  $\text{BV}(\text{Ob}(\mathcal{V}(\mathcal{H})), \mathbb{R})$  is an abelian group. Indeed: suppose  $\alpha = \nu_1 - \kappa_1$  and  $\beta = \nu_2 - \kappa_2$  with  $\nu_1, \nu_2, \kappa_1, \kappa_2 \in \Gamma\mathbb{R}^{\succeq}$ , then

$$\alpha + \beta = (\nu_1 + \nu_2) - (\kappa_1 + \kappa_2) \quad (16.31)$$

Hence  $\alpha + \beta$  belongs to  $\text{BV}(\text{Ob}(\mathcal{V}(\mathcal{H})), \mathbb{R})$  since  $\nu_1 + \nu_2$  and  $\kappa_1 + \kappa_2$  belong to  $\Gamma\mathbb{R}^{\succeq}$ .

**The definition of  $[\nu, 0]^2$ .** We will now show how to take the square of elements of  $k(\Gamma\mathbb{R}^{\succeq})$  that are of the form  $[\nu, 0]$ . Clearly,  $\nu^2$  is well-defined as a function on  $\text{Ob}(\mathcal{V}(\mathcal{H}))$ , but it may not belong to  $\Gamma\mathbb{R}^{\succeq}$ . Indeed, if  $\nu(V) < 0$  for any  $V$ , then the function  $V \mapsto \nu^2(V)$  can get smaller as  $V$  gets smaller, so it is order-preserving instead of order-reversing.

This suggests the following strategy. First, define functions  $\nu_+$  and  $\nu_-$  by

$$\nu_+(V) := \begin{cases} \nu(V) & \text{if } \nu(V) \geq 0 \\ 0 & \text{if } \nu(V) < 0 \end{cases} \quad (16.32)$$

and

$$\nu_-(V) := \begin{cases} 0 & \text{if } \nu(V) \geq 0 \\ \nu(V) & \text{if } \nu(V) < 0. \end{cases} \quad (16.33)$$

Clearly,  $\nu(V) = \nu_+(V) + \nu_-(V)$  for all  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ . Also, for all  $V$ ,  $\nu_+(V)\nu_-(V) = 0$ , and hence

$$\nu(V)^2 = \nu_+(V)^2 + \nu_-(V)^2 \quad (16.34)$$

However, (i) the function  $V \mapsto \nu_+(V)^2$  is order-reversing; and (ii) the function  $V \mapsto \nu_-(V)^2$  is order-preserving. But then  $V \mapsto -\nu_-(V)^2$  is order-reversing. Hence, by rewriting (16.34) as

$$\nu(V)^2 = \nu_+(V)^2 - (-\nu_-(V)^2) \quad (16.35)$$

we see that the function  $V \mapsto \nu^2(V) := \nu(V)^2$  is an element of  $\text{BV}(\text{Ob}(\mathcal{V}(\mathcal{H})), \mathbb{R})$ .

In terms of  $k(\Gamma\mathbb{R}^{\succeq})$ , we can define

$$[\nu, 0]^2 := [\nu_+^2, -\nu_-^2] \quad (16.36)$$

which belongs to  $k(\Gamma\mathbb{R}^{\succeq})$ . Hence, although there exist  $\nu \in \Gamma\mathbb{R}^{\succeq}$  that have no square in  $\Gamma\mathbb{R}^{\succeq}$ , such global elements of  $\mathbb{R}^{\succeq}$  do have squares in the  $k$ -completion,  $k(\Gamma\mathbb{R}^{\succeq})$ . On the level of functions of bounded variation, we have shown that the square of a monotonic (order-reversing) function is a function of bounded variation.

On the other hand, we cannot take the square of an arbitrary element  $[\nu, \kappa] \in \Gamma \underline{\mathbb{R}}^{\leq}$ , since the square of a function of bounded variation need not be a function of bounded variation.<sup>134</sup>

## 16.5 The Object $k(\underline{\mathbb{R}}^{\leq})$ in the Topos $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

### 16.5.1 The Definition of $k(\underline{\mathbb{R}}^{\leq})$ .

The next step is to translate these results about the set  $k(\Gamma \underline{\mathbb{R}}^{\leq})$  into the construction of an object  $k(\underline{\mathbb{R}}^{\leq})$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . We anticipate that, if this can be done, then  $k(\Gamma \underline{\mathbb{R}}^{\leq}) \simeq \Gamma k(\underline{\mathbb{R}}^{\leq})$ .

As was discussed in Section (8.2), the presheaf  $\underline{\mathbb{R}}^{\leq}$  is defined at each stage  $V$  by

$$\underline{\mathbb{R}}^{\leq}_V := \{\nu : \downarrow V \rightarrow \mathbb{R} \mid \nu \in \mathcal{OR}(\downarrow V, \mathbb{R})\}. \quad (16.37)$$

If  $i_{V'V} : V' \subseteq V$ , then the presheaf map from  $\underline{\mathbb{R}}^{\leq}_V$  to  $\underline{\mathbb{R}}^{\leq}_{V'}$  is just the restriction of the order-reversing functions from  $\downarrow V$  to  $\downarrow V'$ .

The first step in constructing  $k(\underline{\mathbb{R}}^{\leq})$  is to define an equivalence relation on pairs of functions,  $\nu, \kappa \in \underline{\mathbb{R}}^{\leq}_V$ , for each stage  $V$ , by saying that  $(\nu_1, \kappa_1) \equiv (\nu_2, \kappa_2)$  if, and only, there exists  $\alpha \in \underline{\mathbb{R}}^{\leq}_V$  such that

$$\nu_1(V') + \kappa_2(V') + \alpha(V') = \kappa_1(V') + \nu_2(V') + \alpha(V') \quad (16.38)$$

for all  $V' \subseteq V$ .

**Definition 16.3** *The presheaf  $k(\underline{\mathbb{R}}^{\leq})$  is defined over the category  $\mathcal{V}(\mathcal{H})$  in the following way.*

(i) *On objects  $V \in \text{Ob}(\mathcal{V}(\mathcal{H}))$ :*

$$k(\underline{\mathbb{R}}^{\leq})_V := \{[\nu, \kappa] \mid \nu, \kappa \in \mathcal{OR}(\downarrow V, \mathbb{R})\}, \quad (16.39)$$

*where  $[\nu, \kappa]$  denotes the  $k$ -equivalence class of  $(\nu, \kappa)$ .*

(ii) *On morphisms  $i_{V'V} : V' \subseteq V$ : The arrow  $k(\underline{\mathbb{R}}^{\leq})(i_{V'V}) : k(\underline{\mathbb{R}}^{\leq})_V \rightarrow k(\underline{\mathbb{R}}^{\leq})_{V'}$  is given by  $(k(\underline{\mathbb{R}}^{\leq})(i_{V'V}))([\nu, \kappa]) := [\nu|_{V'}, \kappa|_{V'}]$  for all  $[\nu, \kappa] \in k(\underline{\mathbb{R}}^{\leq})_V$ .*

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<sup>134</sup>We have to consider functions like  $(\nu_+ + \nu_- - (\kappa_+ + \kappa_-))^2$ , which contains terms of the form  $\nu_+ \kappa_-$  and  $\nu_- \kappa_+$ : in general, these are neither order-preserving nor order-reversing.

It is straightforward to show that  $k(\underline{\mathbb{R}}^{\check{>}})$  is an abelian group-object in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . In particular, an arrow  $+ : k(\underline{\mathbb{R}}^{\check{>}}) \times k(\underline{\mathbb{R}}^{\check{>}}) \rightarrow k(\underline{\mathbb{R}}^{\check{>}})$  is defined at each stage  $V$  by

$$+_V([\nu_1, \kappa_1], [\nu_2, \kappa_2]) := [\nu_1 + \nu_2, \kappa_1 + \kappa_2] \quad (16.40)$$

for all  $([\nu_1, \kappa_1], [\nu_2, \kappa_2]) \in k(\underline{\mathbb{R}}^{\check{>}})_V \times k(\underline{\mathbb{R}}^{\check{>}})_V$ . It is easy to see that (i)  $\Gamma k(\underline{\mathbb{R}}^{\check{>}}) \simeq k(\Gamma \underline{\mathbb{R}}^{\check{>}})$ ; and (ii)  $\underline{\mathbb{R}}^{\check{>}}$  is a sub-object of  $k(\underline{\mathbb{R}}^{\check{>}})$  in the topos  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ .

### 16.5.2 The Presheaf $k(\underline{\mathbb{R}}^{\check{>}})$ as the Quantity-Value Object.

We can now identify  $k(\underline{\mathbb{R}}^{\check{>}})$  as a possible quantity-value object in  $\mathbf{Sets}^{\mathcal{V}(\mathcal{H})^{\text{op}}}$ . To each bounded, self-adjoint operator  $\hat{A}$ , there is an arrow  $[\check{\delta}^o(\hat{A})] : \underline{\Sigma} \rightarrow k(\underline{\mathbb{R}}^{\check{>}})$ , given by first sending  $\hat{A} \in B(\mathcal{H})_{\text{sa}}$  to  $\check{\delta}^o(\hat{A})$  and then taking  $k$ -equivalence classes. More precisely, one takes the monic  $\iota : \underline{\mathbb{R}}^{\check{>}} \hookrightarrow k(\underline{\mathbb{R}}^{\check{>}})$  and then constructs  $\iota \circ \check{\delta}^o(\hat{A}) : \underline{\Sigma} \rightarrow k(\underline{\mathbb{R}}^{\check{>}})$ .

Since, for each stage  $V$ , the elements in the image of  $[\check{\delta}^o(\hat{A})]_V = (\iota \circ \check{\delta}^o(\hat{A}))_V$  are of the form  $[\nu, 0]$ ,  $\nu \in \underline{\mathbb{R}}^{\check{>}}_V$ , their square is well-defined. From a physical perspective, the use of  $k(\underline{\mathbb{R}}^{\check{>}})$  rather than  $\underline{\mathbb{R}}^{\check{>}}$  renders possible the definition of things like the ‘intrinsic dispersion’,  $\nabla(\hat{A}) := \check{\delta}^o(\hat{A}^2) - \check{\delta}^o(\hat{A})^2$ ; see (9.4).

### 16.5.3 The square of an arrow $[\check{\delta}^o(\hat{A})]$ .

An arrow  $[\check{\delta}^o(\hat{A})] : \underline{\Sigma} \rightarrow k(\underline{\mathbb{R}}^{\check{>}})$  is constructed by first forming the outer daseinisation  $\check{\delta}^o(\hat{A})$  of  $\hat{A}$ , which is an arrow from  $\underline{\Sigma}$  to  $\underline{\mathbb{R}}^{\check{>}}$ , and then composing with the monic arrow from  $\underline{\mathbb{R}}^{\check{>}}$  to  $k(\underline{\mathbb{R}}^{\check{>}})$ . Since only outer daseinisation is used, for each  $V \in \mathcal{V}(\mathcal{H})$  and each  $\lambda \in \underline{\Sigma}_V$  one obtains an element of  $k(\underline{\mathbb{R}}^{\check{>}})_V$  of the form  $[\delta^o(\hat{A})_V(\lambda), 0]$ . We saw how to take the square of these functions, and applying this to all  $\lambda \in \underline{\Sigma}_V$  and all  $V \in \mathcal{V}(\mathcal{H})$ , we get the square  $[\check{\delta}^o(\hat{A})]^2$  of the arrow  $[\check{\delta}^o(\hat{A})]$ .

If we consider an arrow of the form  $\check{\delta}(\hat{A}) : \underline{\Sigma} \rightarrow \underline{\mathbb{R}}^{\check{<}}$ , then the construction involves both inner and outer daseinisation, see Theorem 8.2. For each  $V$  and each  $\lambda \in \underline{\Sigma}_V$ , we obtain a pair of functions  $(\delta^i(\hat{A})_V(\lambda), \delta^o(\hat{A})_V(\lambda))$ , which are both not constantly 0 in general. There is no canonical way to take the square of these in  $\underline{\mathbb{R}}^{\check{<}}_V$ . Going to the  $k$ -extension  $k(\underline{\mathbb{R}}^{\check{<}})$  of  $\underline{\mathbb{R}}^{\check{<}}$  does not improve the situation, so we cannot define the square of an arrow  $\check{\delta}(\hat{A})$  (or  $[\check{\delta}(\hat{A})]$  in general).

## 17 Appendix 2: A Short Introduction to the Relevant Parts of Topos Theory

### 17.1 What is a Topos?

It is impossible to give here more than the briefest of introductions to topos theory. At the danger of being highly imprecise, we restrict ourselves to mentioning some aspects of this well-developed mathematical theory and give a number of pointers to the literature. The aim merely is to give a very rough idea of the structure and internal logic of a topos.

There are a number of excellent textbooks on topos theory, and the reader should consult at least one of them. We found the following books useful: [54, 29, 56, 48, 8, 52].

Topos theory is a remarkably rich branch of mathematics which can be approached from a variety of different viewpoints. The basic area of mathematics is category theory; where, we recall, a category consists of a collection of *objects* and a collection of *morphisms* (or *arrows*).

In the special case of the category of sets, the objects are sets, and a morphism is a function between a pair of sets. In general, each morphism  $f$  in a category is associated with a pair of objects<sup>135</sup>, known as its ‘domain’ and ‘codomain’, and is written as  $f : B \rightarrow A$  where  $B$  and  $A$  are the domain and codomain respectively. Note that this arrow notation is used even if  $f$  is not a function in the normal set-theoretic sense. A key ingredient in the definition of a category is that if  $f : B \rightarrow A$  and  $g : C \rightarrow B$  (i.e., the codomain of  $g$  is equal to the domain of  $f$ ) then  $f$  and  $g$  can be ‘composed’ to give an arrow  $f \circ g : C \rightarrow A$ ; in the case of the category of sets, this is just the usual composition of functions.

A simple example of a category is given by any partially-ordered set (‘poset’)  $\mathcal{C}$ : (i) the objects are defined to be the elements of  $\mathcal{C}$ ; and (ii) if  $p, q \in \mathcal{C}$ , a morphism from  $p$  to  $q$  is defined to exist if, and only if,  $p \preceq q$  in the poset structure. Thus, in a poset regarded as a category, there is at most one morphism between any pair of objects  $p, q \in \mathcal{C}$ ; if it exists, we shall write this morphism as  $i_{pq} : p \rightarrow q$ . This example is important for us in form of the ‘category of contexts’,  $\mathcal{V}(\mathcal{H})$ , in quantum theory. The objects in  $\mathcal{V}(\mathcal{H})$  are the commutative, unital<sup>136</sup> von Neumann sub-algebras of the algebra,  $B(\mathcal{H})$ ,

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<sup>135</sup>The collection of all objects in category,  $\mathcal{C}$ , is denoted  $\text{Ob}(\mathcal{C})$ . The collection of arrows from  $B$  to  $A$  is denoted  $\text{Hom}_{\mathcal{C}}(B, A)$ . We will only be interested in ‘small’ categories in which both these collections are sets (rather than the, more general, classes.)

<sup>136</sup>‘Unital’ means that all these algebras contain the identity operator  $\hat{1} \in B(\mathcal{H})$ .

of all bounded operators on the Hilbert space  $\mathcal{H}$ .

**Topoi as mathematical universes.** Every (elementary) topos  $\tau$  can be seen as a *mathematical universe*. As a category, a topos  $\tau$  possesses a number of structures that generalise constructions that are possible in the category, **Sets**, of sets and functions.<sup>137</sup> Namely, in **Sets**, we can construct new sets from given ones in several ways. Specifically, let  $S, T$  be two sets, then we can form the cartesian product  $S \times T$ , the disjoint union  $S \amalg T$  and the exponential  $S^T$ —the set of all functions from  $T$  to  $S$ .

These constructions turn out to be fundamental, and they can all be phrased in an abstract, categorical manner, where they are called the ‘product’, ‘co-product’ and ‘exponential’, respectively. By definition, in a topos  $\tau$ , these operations always exist. The first and second of these properties are called ‘finite completeness’ and ‘finite co-completeness’, respectively.

One consequence of the existence of finite limits is that each topos,  $\tau$ , has a *terminal object*, denoted by  $1_\tau$ . This is characterised by the property that for any object  $A$  in the topos  $\tau$ , there exists exactly one arrow from  $A$  to  $1_\tau$ . In **Sets**, any one-element set  $1 = \{*\}$  is terminal.<sup>138</sup>

Of course, **Sets** is a topos, too, and it is precisely the topos which usually plays the rôle of our mathematical universe, since we construct our mathematical objects starting from sets and functions between them. As a slogan, we have: a topos  $\tau$  is a category with ‘certain crucial’ properties that are similar to those in **Sets**. A very nice and gentle introduction to these aspects of topos theory is the book [54]. Other good sources are [29, 55].

In order to ‘do mathematics’, one must also have a logic, including a deductive system. Each topos comes equipped with an *internal logic*, which is of *intuitionistic* type. We will now very briefly sketch the main characteristics of intuitionistic logic and the mathematical structures in a topos that realise this logic.

**The sub-object classifier.** Let  $X$  be a set, and let  $P(X)$  be the power set of  $X$ ; i.e., the set of subsets of  $X$ . Given a subset  $K \in P(X)$ , one can ask for each point  $x \in X$  whether or not it lies in  $K$ . Thus there is the *characteristic*

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<sup>137</sup>More precisely, *small* sets and functions between them. Small means that we do not have proper classes. One must take care in these foundational issues to avoid problems like Russell’s paradox.

<sup>138</sup>Like many categorical constructions, the terminal object is fixed only up to isomorphism: all one-element sets are isomorphic to each other, and any of them can serve as a terminal object. Nonetheless, one speaks of *the* terminal object.

function  $\chi_K : X \rightarrow \{0, 1\}$  of  $K$ , which is defined as

$$\chi_K(x) := \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases} \quad (17.1)$$

for all  $x \in X$ ; cf. (6.2). The two-element set  $\{0, 1\}$  plays the rôle of a set of *truth-values* for propositions (of the form “ $x \in K$ ”). Clearly, 1 corresponds to ‘true’, 0 corresponds to ‘false’, and there are no other possibilities. This is an argument about sets, so it takes place in, and uses the logic of, the topos **Sets** of sets and functions. **Sets** is a *Boolean topos*, in which the familiar two-valued logic and the axiom (\*) hold. (This does not contradict the fact that the internal logic of topoi is intuitionistic, since Boolean logic is a special case of intuitionistic logic.)

In an arbitrary topos,  $\tau$ , there is a special object  $\Omega_\tau$ , called the *sub-object classifier*, that takes the rôle of the set  $\{0, 1\} \simeq \{\text{false}, \text{true}\}$  of truth-values. Let  $B$  be an object in the topos, and let  $A$  be a sub-object of  $B$ . This means that there is a monic  $A \rightarrow B$ ,<sup>139</sup> (this is the categorical generalisation of the inclusion of a subset  $K$  into a larger set  $X$ ). As in the case of **Sets**, we can also characterise  $A$  as a sub-object of  $B$  by an arrow from  $B$  to the sub-object classifier  $\Omega_\tau$ ; in **Sets**, this arrow is the characteristic function  $\chi_K : X \rightarrow \{0, 1\}$  of (17.1). Intuitively, this ‘characteristic arrow’ from  $B$  to  $\Omega_\tau$  describes how  $A$  ‘lies in’  $B$ . The textbook definition is:

**Definition 17.1** *In a category  $\tau$  with finite limits, a sub-object classifier is an object  $\Omega_\tau$ , together with a monic  $\text{true} : 1_\tau \rightarrow \Omega_\tau$ , such that to every monic  $m : A \rightarrow B$  in  $\tau$  there is a unique arrow  $\chi_A : B \rightarrow \Omega_\tau$  which, with the given monic, forms a pullback square*

$$\begin{array}{ccc} A & \xrightarrow{\quad} & 1_\tau \\ m \downarrow & & \downarrow \text{true} \\ B & \xrightarrow{\chi_A} & \Omega_\tau \end{array}$$

In **Sets**, the arrow  $\text{true} : 1 \rightarrow \{0, 1\}$  is given by  $\text{true}(*) = 1$ . In general, the sub-object classifier,  $\Omega_\tau$ , need not be a set, since it is an object in the topos  $\tau$ , and the objects of  $\tau$  need not be sets. Nonetheless, there is an

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<sup>139</sup>A *monic* is the categorical version of an injective function. In the topos **Sets**, monics exactly are injective functions.

abstract notion of *elements* (or *points*) in category theory that we can use. Then the elements of  $\Omega_\tau$  are the truth-values available in the internal logic of our topos  $\tau$ , just like ‘false’ and ‘true’, the elements of  $\{\text{false}, \text{true}\}$ , are the truth-values available in the topos **Sets**.

To understand the abstract notion of elements, let us consider sets for a moment. Let  $1 = \{*\}$  be a one-element set, the terminal object in **Sets**. Let  $S$  be a set and consider an arrow  $e$  from  $1$  to  $S$ . Clearly, (i)  $e(*) \in S$  is an element of  $S$ ; and (ii) the set of all functions from  $1$  to  $S$  corresponds exactly to the set of all elements of  $S$ .

This idea can be generalised to any category that has a terminal object  $1$ . More precisely, an *element* of an object  $A$  is defined to be an arrow from  $1$  to  $A$  in the category. For example, in the definition of the sub-object classifier the arrow ‘true :  $1_\tau \rightarrow \Omega_\tau$ ’ is an element of  $\Omega_\tau$ . It may happen that an object  $A$  has no elements, i.e., there are no arrows  $1_\tau \rightarrow A$ . It is common to consider arrows from subobjects  $U$  of  $A$  to  $A$  as *generalised elements*.

As mentioned above, the elements of the sub-object classifier, understood as the arrows  $1_\tau \rightarrow \Omega_\tau$ , are the truth-values. Moreover, the set of these arrows forms a Heyting algebra (see, for example, section 8.3 in [29]). This is how (the algebraic representation of) intuitionistic logic manifests itself in a topos. Another, closely related fact is that the set,  $\text{Sub}(A)$ , of sub-objects of any object  $A$  in a topos forms a Heyting algebra.

**The definition of a topos.** Let us pull together these various remarks and list the most important properties of a topos,  $\tau$ , for our purposes:

1. There is a terminal object  $1_\tau$  in  $\tau$ . Thus, given any object  $A$  in the topos, there is a unique arrow  $A \rightarrow 1_\tau$ .

For any object  $A$  in the topos, an arrow  $1_\tau \rightarrow A$  is called a *global element* of  $A$ . The set of all global elements of  $A$  is denoted  $\Gamma A$ .

Given  $A, B \in \text{Ob}(\tau)$ , there is a product  $A \times B$  in  $\tau$ . In fact, a topos always has *pull-backs*, and the product is just a special case of this.<sup>140</sup>

2. There is an initial object  $0_\tau$  in  $\tau$ . This means that given any object  $A$  in the topos, there is a unique arrow  $0_\tau \rightarrow A$ .

Given  $A, B \in \text{Ob}(\tau)$ , there is a co-product  $A \sqcup B$  in  $\tau$ . In fact, a topos always has *push-outs*, and the co-product is just a special case of this.<sup>141</sup>

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<sup>140</sup>The conditions in 1. above are equivalent to saying that  $\tau$  is finitely complete.

<sup>141</sup>The conditions in 2. above are equivalent to saying that  $\tau$  is finitely co-complete.



3. There is *exponentiation*: i.e., given objects  $A, B$  in  $\tau$  we can form an object  $A^B$ , which is the topos analogue of the set of functions from  $B$  to  $A$  in set theory. The definitive property of exponentiation is that, given any object  $C$ , there is an isomorphism

$$\mathrm{Hom}_\tau(C, A^B) \simeq \mathrm{Hom}_\tau(C \times B, A) \quad (17.2)$$

that is natural in  $A$  and  $C$ ; i.e., it is ‘well-behaved’ under morphisms of the objects involved.

4. There is a sub-object classifier  $\Omega_\tau$ .

## 17.2 Presheaves on a Poset

To illustrate the main ideas, we will first give a few definitions from the theory of presheaves on a partially ordered set (or ‘poset’); in the case of quantum theory, this poset is the space of ‘contexts’ in which propositions are asserted. We shall then use these ideas to motivate the definition of a presheaf on a general category. Only the briefest of treatments is given here, and the reader is referred to the standard literature for more information [29, 56].

A *presheaf* (also known as a *varying set*)  $\underline{X}$  on a poset  $\mathcal{C}$  is a function that assigns to each  $p \in \mathcal{C}$ , a set  $\underline{X}_p$ ; and to each pair  $p \preceq q$  (i.e.,  $i_{pq} : p \rightarrow q$ ), a map  $\underline{X}_{qp} : \underline{X}_q \rightarrow \underline{X}_p$  such that (i)  $\underline{X}_{pp} : \underline{X}_p \rightarrow \underline{X}_p$  is the identity map  $\mathrm{id}_{\underline{X}_p}$  on  $\underline{X}_p$ , and (ii) whenever  $p \preceq q \preceq r$ , the composite map  $\underline{X}_r \xrightarrow{\underline{X}_{rq}} \underline{X}_q \xrightarrow{\underline{X}_{qp}} \underline{X}_p$  is equal to  $\underline{X}_r \xrightarrow{\underline{X}_{rp}} \underline{X}_p$ , i.e.,

$$\underline{X}_{rp} = \underline{X}_{qp} \circ \underline{X}_{rq}. \quad (17.3)$$

The notation  $\underline{X}_{qp}$  is shorthand for the more cumbersome  $\underline{X}(i_{pq})$ ; see below in the definition of a functor.

An *arrow*, or *natural transformation*  $\eta : \underline{X} \rightarrow \underline{Y}$  between two presheaves  $\underline{X}, \underline{Y}$  on  $\mathcal{C}$  is a family of maps  $\eta_p : \underline{X}_p \rightarrow \underline{Y}_p$ ,  $p \in \mathcal{C}$ , that satisfy the intertwining conditions

$$\eta_p \circ \underline{X}_{qp} = \underline{Y}_{qp} \circ \eta_q \quad (17.4)$$

whenever  $p \preceq q$ . This is equivalent to the commutative diagram

$$\begin{array}{ccc}
 \underline{X}_q & \xrightarrow{\underline{X}_{qp}} & \underline{X}_p \\
 \eta_q \downarrow & & \downarrow \eta_p \\
 \underline{Y}_q & \xrightarrow{\underline{Y}_{qp}} & \underline{Y}_p
 \end{array} \tag{17.5}$$

It follows from these basic definitions, that a sub-object of a presheaf  $\underline{X}$  is a presheaf  $\underline{K}$ , with an arrow  $i : \underline{K} \rightarrow \underline{X}$  such that (i)  $\underline{K}_p \subseteq \underline{X}_p$  for all  $p \in \mathcal{C}$ ; and (ii) for all  $p \preceq q$ , the map  $K_{qp} : \underline{K}_q \rightarrow \underline{K}_p$  is the restriction of  $\underline{X}_{qp} : \underline{X}_q \rightarrow \underline{X}_p$  to the subset  $\underline{K}_q \subseteq \underline{X}_q$ . This is shown in the commutative diagram

$$\begin{array}{ccc}
 \underline{K}_q & \xrightarrow{\underline{K}_{qp}} & \underline{K}_p \\
 \downarrow & & \downarrow \\
 \underline{X}_q & \xrightarrow{\underline{X}_{qp}} & \underline{X}_p
 \end{array} \tag{17.6}$$

where the vertical arrows are subset inclusions.

The collection of all presheaves on a poset  $\mathcal{C}$  forms a category, denoted  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ . The arrows/morphisms between presheaves in this category the arrows (natural transformations) defined above.

### 17.3 Presheaves on a General Category

The ideas sketched above admit an immediate generalization to the theory of presheaves on an arbitrary ‘small’ category  $\mathcal{C}$  (the qualification ‘small’ means that the collection of objects is a genuine set, as is the collection of all arrows/morphisms between any pair of objects). To make the necessary definition we first need the idea of a ‘functor’:

**The idea of a functor.** A central concept is that of a ‘functor’ between a pair of categories  $\mathcal{C}$  and  $\mathcal{D}$ . Broadly speaking, this is an arrow-preserving function from one category to the other. The precise definition is as follows.

**Definition 17.2** 1. A covariant functor  $F$  from a category  $\mathcal{C}$  to a category  $\mathcal{D}$  is a function that assigns

- (a) to each  $\mathcal{C}$ -object  $A$ , a  $\mathcal{D}$ -object  $F_A$ ;
- (b) to each  $\mathcal{C}$ -morphism  $f : B \rightarrow A$ , a  $\mathcal{D}$ -morphism  $F(f) : F_B \rightarrow F_A$  such that  $F(\text{id}_A) = \text{id}_{F_A}$ ; and, if  $g : C \rightarrow B$ , and  $f : B \rightarrow A$  then

$$F(f \circ g) = F(f) \circ F(g). \quad (17.7)$$

2. A contravariant functor  $X$  from a category  $\mathcal{C}$  to a category  $\mathcal{D}$  is a function that assigns

- (a) to each  $\mathcal{C}$ -object  $A$ , a  $\mathcal{D}$ -object  $X_A$ ;
- (b) to each  $\mathcal{C}$ -morphism  $f : B \rightarrow A$ , a  $\mathcal{D}$ -morphism  $X(f) : X_A \rightarrow X_B$  such that  $X(\text{id}_A) = \text{id}_{X_A}$ ; and, if  $g : C \rightarrow B$ , and  $f : B \rightarrow A$  then

$$X(f \circ g) = X(g) \circ X(f). \quad (17.8)$$

The connection with the idea of a presheaf on a poset is straightforward. As mentioned above, a poset  $\mathcal{C}$  can be regarded as a category in its own right, and it is clear that a presheaf on the poset  $\mathcal{C}$  is the same thing as a contravariant functor  $\underline{X}$  from the category  $\mathcal{C}$  to the category **Sets** of normal sets. Equivalently, it is a covariant functor from the ‘opposite’ category<sup>142</sup>  $\mathcal{C}^{\text{op}}$  to **Sets**. Clearly, (17.3) corresponds to the contravariant condition (17.8). Note that mathematicians usually call the objects in  $\mathcal{C}$  ‘stages of truth’, or just ‘stages’. For us they are ‘contexts’, ‘classical snap-shops’, or ‘world views’.

**Presheaves on an arbitrary category  $\mathcal{C}$ .** These remarks motivate the definition of a presheaf on an arbitrary small category  $\mathcal{C}$ : namely, a *presheaf* on  $\mathcal{C}$  is a covariant functor<sup>143</sup>  $\underline{X} : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Sets}$  from  $\mathcal{C}^{\text{op}}$  to the category of sets. Equivalently, a presheaf is a contravariant functor from  $\mathcal{C}$  to the category of sets.

We want to make the collection of presheaves on  $\mathcal{C}$  into a category, and therefore we need to define what is meant by a ‘morphism’ between two presheaves  $\underline{X}$  and  $\underline{Y}$ . The intuitive idea is that such a morphism from  $\underline{X}$  to

<sup>142</sup>The ‘opposite’ of a category  $\mathcal{C}$  is a category, denoted  $\mathcal{C}^{\text{op}}$ , whose objects are the same as those of  $\mathcal{C}$ , and whose morphisms are defined to be the opposite of those of  $\mathcal{C}$ ; i.e., a morphism  $f : A \rightarrow B$  in  $\mathcal{C}^{\text{op}}$  is said to exist if, and only if, there is a morphism  $f : B \rightarrow A$  in  $\mathcal{C}$ .

<sup>143</sup>Throughout this series of papers, a presheaf is indicated by a letter that is underlined.

$\underline{Y}$  must give a ‘picture’ of  $\underline{X}$  within  $\underline{Y}$ . Formally, such a morphism is defined to be a *natural transformation*  $N : \underline{X} \rightarrow \underline{Y}$ , by which is meant a family of maps (called the *components* of  $N$ )  $N_A : \underline{X}_A \rightarrow \underline{Y}_A$ ,  $A \in \text{Ob}(\mathcal{C})$ , such that if  $f : B \rightarrow A$  is a morphism in  $\mathcal{C}$ , then the composite map  $\underline{X}_A \xrightarrow{N_A} \underline{Y}_A \xrightarrow{\underline{Y}(f)} \underline{Y}_B$  is equal to  $\underline{X}_A \xrightarrow{\underline{X}(f)} \underline{X}_B \xrightarrow{N_B} \underline{Y}_B$ . In other words, we have the commutative diagram

$$\begin{array}{ccc}
 \underline{X}_A & \xrightarrow{\underline{X}(f)} & \underline{X}_B \\
 N_A \downarrow & & \downarrow N_B \\
 \underline{Y}_A & \xrightarrow{\underline{Y}(f)} & \underline{Y}_B
 \end{array} \tag{17.9}$$

of which (17.5) is clearly a special case. The category of presheaves on  $\mathcal{C}$  equipped with these morphisms is denoted  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ .

The idea of a sub-object generalizes in an obvious way. Thus we say that  $\underline{K}$  is a *sub-object* of  $\underline{X}$  if there is a morphism in the category of presheaves (i.e., a natural transformation)  $\iota : \underline{K} \rightarrow \underline{X}$  with the property that, for each  $A$ , the component map  $\iota_A : \underline{K}_A \rightarrow \underline{X}_A$  is a subset embedding, i.e.,  $\underline{K}_A \subseteq \underline{X}_A$ . Thus, if  $f : B \rightarrow A$  is any morphism in  $\mathcal{C}$ , we get the analogue of the commutative diagram (17.6):

$$\begin{array}{ccc}
 \underline{K}_A & \xrightarrow{\underline{K}(f)} & \underline{K}_B \\
 \downarrow & & \downarrow \\
 \underline{X}_A & \xrightarrow{\underline{X}(f)} & \underline{X}_B
 \end{array} \tag{17.10}$$

where, once again, the vertical arrows are subset inclusions.

The category of presheaves on  $\mathcal{C}$ ,  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ , forms a topos. We do not need the full definition of a topos; but we do need the idea, mentioned in Section 17.2, that a topos has a sub-object classifier  $\Omega$ , to which we now turn.

**Sieves and the sub-object classifier  $\underline{\Omega}$ .** Among the key concepts in presheaf theory is that of a ‘sieve’, which plays a central role in the construction of the sub-object classifier in the topos of presheaves on a category  $\mathcal{C}$ .

A *sieve* on an object  $A$  in  $\mathcal{C}$  is defined to be a collection  $S$  of morphisms  $f : B \rightarrow A$  in  $\mathcal{C}$  with the property that if  $f : B \rightarrow A$  belongs to  $S$ , and if  $g : C \rightarrow B$  is any morphism with co-domain  $B$ , then  $f \circ g : C \rightarrow A$  also belongs to  $S$ . In the simple case where  $\mathcal{C}$  is a poset, a sieve on  $p \in \mathcal{C}$  is any subset  $S$  of  $\mathcal{C}$  such that if  $r \in S$  then (i)  $r \preceq p$ , and (ii)  $r' \in S$  for all  $r' \preceq r$ ; in other words, a sieve is nothing but a *lower set* in the poset.

The presheaf  $\underline{\Omega} : \mathcal{C} \rightarrow \mathbf{Sets}$  is now defined as follows. If  $A$  is an object in  $\mathcal{C}$ , then  $\underline{\Omega}_A$  is defined to be the set of all sieves on  $A$ ; and if  $f : B \rightarrow A$ , then  $\underline{\Omega}(f) : \underline{\Omega}_A \rightarrow \underline{\Omega}_B$  is defined as

$$\underline{\Omega}(f)(S) := \{h : C \rightarrow B \mid f \circ h \in S\} \quad (17.11)$$

for all  $S \in \underline{\Omega}_A$ ; the sieve  $\underline{\Omega}(f)(S)$  is often written as  $f^*(S)$ , and is known as the *pull-back* to  $B$  of the sieve  $S$  on  $A$  by the morphism  $f : B \rightarrow A$ .

It should be noted that if  $S$  is a sieve on  $A$ , and if  $f : B \rightarrow A$  belongs to  $S$ , then from the defining property of a sieve we have

$$f^*(S) := \{h : C \rightarrow B \mid f \circ h \in S\} = \{h : C \rightarrow B\} =: \downarrow B \quad (17.12)$$

where  $\downarrow B$  denotes the *principal* sieve on  $B$ , defined to be the set of all morphisms in  $\mathcal{C}$  whose codomain is  $B$ .

If  $\mathcal{C}$  is a poset, the pull-back operation corresponds to a family of maps  $\underline{\Omega}_{qp} : \underline{\Omega}_q \rightarrow \underline{\Omega}_p$  (where  $\underline{\Omega}_p$  denotes the set of all sieves/lower sets on  $p$  in the poset) defined by  $\underline{\Omega}_{qp} = \underline{\Omega}(i_{pq})$  if  $i_{pq} : p \rightarrow q$  (*i.e.*,  $p \preceq q$ ). It is straightforward to check that if  $S \in \underline{\Omega}_q$ , then

$$\underline{\Omega}_{qp}(S) := \downarrow p \cap S \quad (17.13)$$

where  $\downarrow p := \{r \in \mathcal{C} \mid r \preceq p\}$ .

A crucial property of sieves is that the set  $\underline{\Omega}_A$  of sieves on  $A$  has the structure of a Heyting algebra. Specifically, the unit element  $1_{\underline{\Omega}_A}$  in  $\underline{\Omega}_A$  is the principal sieve  $\downarrow A$ , and the null element  $0_{\underline{\Omega}_A}$  is the empty sieve  $\emptyset$ . The partial ordering in  $\underline{\Omega}_A$  is defined by  $S_1 \preceq S_2$  if, and only if,  $S_1 \subseteq S_2$ ; and the logical connectives are defined as:

$$S_1 \wedge S_2 := S_1 \cap S_2 \quad (17.14)$$

$$S_1 \vee S_2 := S_1 \cup S_2 \quad (17.15)$$

$$S_1 \Rightarrow S_2 := \{f : B \rightarrow A \mid \forall g : C \rightarrow B \text{ if } f \circ g \in S_1 \text{ then } f \circ g \in S_2\} \quad (17.16)$$

As in any Heyting algebra, the negation of an element  $S$  (called the *pseudo-complement* of  $S$ ) is defined as  $\neg S := S \Rightarrow 0$ ; so that

$$\neg S := \{f : B \rightarrow A \mid \text{for all } g : C \rightarrow B, f \circ g \notin S\}. \quad (17.17)$$

It can be shown that the presheaf  $\underline{\Omega}$  is a sub-object classifier for the topos  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ . That is to say, sub-objects of any object  $\underline{X}$  in this topos (*i.e.*, any presheaf on  $\mathcal{C}$ ) are in one-to-one correspondence with morphisms  $\chi : \underline{X} \rightarrow \underline{\Omega}$ . This works as follows. First, let  $\underline{K}$  be a sub-object of  $\underline{X}$  with an associated characteristic arrow  $\chi_{\underline{K}} : \underline{X} \rightarrow \underline{\Omega}$ . Then, at any stage  $A$  in  $\mathcal{C}$ , the ‘components’ of this arrow,  $\chi_{\underline{K}A} : \underline{X}_A \rightarrow \underline{\Omega}_A$ , are defined as

$$\chi_{\underline{K}A}(x) := \{f : B \rightarrow A \mid \underline{X}(f)(x) \in \underline{K}_B\} \quad (17.18)$$

for all  $x \in \underline{X}_A$ . That the right hand side of (17.18) actually *is* a sieve on  $A$  follows from the defining properties of a sub-object.

Thus, in each ‘branch’ of the category  $\mathcal{C}$  going ‘down’ from the stage  $A$ ,  $\chi_{\underline{K}A}(x)$  picks out the first member  $B$  in that branch for which  $\underline{X}(f)(x)$  lies in the subset  $\underline{K}_B$ , and the commutative diagram (17.10) then guarantees that  $\underline{X}(h \circ f)(x)$  will lie in  $\underline{K}_C$  for all  $h : C \rightarrow B$ . Thus each stage  $A$  in  $\mathcal{C}$  serves as a possible context for an assignment to each  $x \in \underline{X}_A$  of a generalised truth value—a sieve belonging to the Heyting algebra  $\underline{\Omega}_A$ . This is the sense in which contextual, generalised truth values arise naturally in a topos of presheaves.

There is a converse to (17.18): namely, each morphism  $\chi : \underline{X} \rightarrow \underline{\Omega}$  (*i.e.*, a natural transformation between the presheaves  $\underline{X}$  and  $\underline{\Omega}$ ) defines a sub-object  $\underline{K}^\chi$  of  $\underline{X}$  via

$$\underline{K}_A^\chi := \chi_A^{-1}\{1_{\underline{\Omega}_A}\}. \quad (17.19)$$

at each stage  $A$ .

**Global elements of a presheaf.** For the category of presheaves on  $\mathcal{C}$ , a terminal object  $\underline{1} : \mathcal{C} \rightarrow \mathbf{Sets}$  can be defined by  $\underline{1}_A := \{*\}$  at all stages  $A$  in  $\mathcal{C}$ ; if  $f : B \rightarrow A$  is a morphism in  $\mathcal{C}$  then  $\underline{1}(f) : \{*\} \rightarrow \{*\}$  is defined to be the map  $* \mapsto *$ . This is indeed a terminal object since, for any presheaf  $\underline{X}$ , we can define a unique natural transformation  $N : \underline{X} \rightarrow \underline{1}$  whose components  $N_A : \underline{X}_A \rightarrow \underline{1}_A = \{*\}$  are the constant maps  $x \mapsto *$  for all  $x \in \underline{X}_A$ .

As a morphism  $\gamma : \underline{1} \rightarrow \underline{X}$  in the topos  $\mathbf{Sets}^{\mathcal{C}^{\text{op}}}$ , a global element corresponds to a choice of an element  $\gamma_A \in \underline{X}_A$  for each stage  $A$  in  $\mathcal{C}$ , such that, if  $f : B \rightarrow A$ , the ‘matching condition’

$$\underline{X}(f)(\gamma_A) = \gamma_B \quad (17.20)$$

is satisfied.

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