# The Dirac equation in general relativity 

# A guide for calculations 

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#### Abstract

In these informal lecture notes we outline different approaches used in doing calculations involving the Dirac equation in curved spacetime. We have tried to clarify the subject by carefully pointing out the various conventions used and by including several examples from textbooks and the existing literature. In addition some basic material has been included in the appendices. It is our hope that graduate students and other researchers will find these notes useful.


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## 1 The spinorial covariant derivative

### 1.1 The Fock-Ivanenko coefficients

In the calculations that follow we will specify our metric signature as needed. We adopt the convention that upper case latin indices run over $(0,1,2,3)$. Also we shall adopt Planck units so that, $c=G=\hbar=1$. A review of the gamma matrix representations and the Dirac equation in Minkowski spacetime is given in Appendices B and D.

Fermion fields are described by spinors $\psi(x)$, and in order to accommodate spinors in general relativity we need the tetrad formalism. The tetrad formalism is briefly reviewed in Appendix A, where apart from the standard material we have included some additional material relevant to spinors [1].

From the basic tetrad expression, Eq. (A.4) in Appendix A, namely,

$$
\begin{equation*}
\eta_{A B}=e_{A}{ }^{\alpha} e_{B}^{\beta} g_{\alpha \beta}, \tag{1.1}
\end{equation*}
$$

we see that $\eta_{00}=e_{0}{ }^{\alpha} e_{0}{ }^{\beta} g_{\alpha \beta}$, and thus by definition the tetrad vector $e_{0}$ is a velocity field at least momentarily tangent to a timelike path. This is what Schutz [2] refers to as the "momentarily comoving reference frame" (MCRF) and it is in this sense that our choice of a tetrad vector set, $e_{A}{ }^{\alpha}$, determines the frame we shall refer to as the the reference frame for the Dirac particle, or the particle frame for short.

For example in reference [3], Sec. VI, Parker considers a freely falling hydrogenic atom, with its nucleus on the geodesic (an approximation), and constructs approximate Fermi coordinates along the chosen geodesic. The corresponding tetrad is referred to as the proper frame. In general the choice of a tetrad set may be dictated for reasons of convenience and one should read the comments in Remark 9 and keep in mind the fact that from a given tetrad one can obtain an infinity of tetrads related to each other by local Lorentz transformations (see Sec. 3.2).

In general relativity the spinors, $\psi(x)$, are sections of the spinor bundle. We limit ourselves to presenting the bare essentials required for calculations, and on clarifying the different sign conventions related to the definition of the spinorial covariant derivative, the spinor affine connection, $\Gamma_{\mu}$, and Fock-Ivanenko coefficients $\Gamma_{C}$.

Each component of a spinor transforms as a scalar function under general coordinate transformations, so this kind of transformation is straightforward. However, the transformation of spinors under tetrad rotations requires additional formalism.

If we change from an initial set of tetrad vector fields, $h_{A}$, to another set, $e_{A}$, then the new tetrad vectors can be expressed as linear combinations of the old as shown below

$$
\begin{equation*}
e_{A}{ }^{\mu}=\Lambda_{A}{ }^{B} h_{B}{ }^{\mu}, \tag{1.2}
\end{equation*}
$$

We show in Appendix A. 4 that $\Lambda$ is a Lorentz matrix. So in the context of general relativity the Lorentz group is the group of tetrad rotations [1]. We also remark that the $\Lambda$ matrices are in general spacetime-dependent and we refer to them as local Lorentz transformations.

In order to write the Dirac equation in general relativity, we also need to introduce the spacetime dependent matrices $\bar{\gamma}^{\alpha}(x)$. The $\bar{\gamma}^{\alpha}$ matrices are related to the constant special relativity gamma matrices, $\gamma^{A}$, by the relation

$$
\begin{equation*}
\bar{\gamma}^{\alpha}(x):=e_{A}^{\alpha}(x) \gamma^{A} \tag{1.3}
\end{equation*}
$$

Using Eq. (A.4) we can now relate the anti-commutators below,

$$
\begin{align*}
& \left\{\gamma^{A}, \gamma^{B}\right\}=\varepsilon 2 \eta^{A B} I  \tag{1.4}\\
& \left\{\bar{\gamma}^{\alpha}(x), \bar{\gamma}^{\beta}(x)\right\}=\varepsilon 2 g^{\alpha \beta} I \tag{1.5}
\end{align*}
$$

where $\varepsilon= \pm 1$. We note that the matrices in Eq. (1.23) anticommute for $\alpha \neq \beta$, only if the metric is diagonal.

A spinor, $\psi$, may be defined as a quantity that transforms as

$$
\begin{equation*}
\tilde{\psi}_{e}=L \psi_{h} \tag{1.6}
\end{equation*}
$$

where $L=L(x)$ is the spacetime-dependent spinor representative of a tetrad rotation $\Lambda=\Lambda(x)$ [1]. We initially follow the sign conventions of references [4] - [6], and although all these references use the metric signature $(+,-,-,-)$, we shall maintain, wherever possible, greater generality.

The derivative of a spinor does not transform like a spinor since

$$
\begin{equation*}
\tilde{\psi}_{\mu}=L \psi,_{\mu}+L,_{\mu} \psi \tag{1.7}
\end{equation*}
$$

Therefore we define the covariant derivative of a spinor by the expression,

$$
\begin{equation*}
D_{\mu} \psi=I \psi{ }_{\mu}+\Gamma_{\mu} \psi \tag{1.8}
\end{equation*}
$$

with the spinor affine connection, $\Gamma_{\mu}$ to be determined. The connection $\Gamma_{\mu}$ is a matrix, actually four matrices, that is, $\left(\Gamma_{\mu}\right)_{a}{ }^{b}$. We require that,

$$
\begin{equation*}
\tilde{D}_{\mu} \tilde{\psi}=L D_{\mu} \psi \tag{1.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{D}_{\mu} \tilde{\psi}=I \tilde{\psi}_{\mu}+\tilde{\Gamma}_{\mu} \tilde{\psi} \tag{1.10}
\end{equation*}
$$

Eq. (1.9) is satisfied if we let

$$
\begin{equation*}
\tilde{\Gamma}_{\mu}=L \Gamma_{\mu} L^{-1}-L,{ }_{\mu} L^{-1}, \tag{1.11}
\end{equation*}
$$

since then

$$
\begin{align*}
\tilde{D}_{\mu} \tilde{\psi} & =\partial_{\mu}(L \psi)+\tilde{\Gamma}_{\mu} \tilde{\psi}  \tag{1.12}\\
& =\left(L,{ }_{\mu}\right) \psi+L \psi,{ }_{\mu}+L \Gamma_{\mu} \psi-\left(L,{ }_{\mu}\right) \psi  \tag{1.13}\\
& =L\left(I \psi{ }_{\mu}+\Gamma_{\mu} \psi\right) \tag{1.14}
\end{align*}
$$

With a slight abuse of notation we write the spinor covariant derivative acting on a spinor $\psi(x)$ as

$$
\begin{equation*}
D_{\mu} \psi=\left(I \partial_{\mu}+\Gamma_{\mu}\right) \psi:=\left(\partial_{\mu}+\Gamma_{\mu}\right) \psi \tag{1.15}
\end{equation*}
$$

where we may omit the identity matrix factor $I$ in the second part of Eq. (1.15). We now proceed to deduce the expression for $\Gamma_{\mu}$.

Under the assumption that the operator $D_{\mu}$ is a connection and therefore a derivation (i.e., satisfies the product rule for tensor products), it may be extended as an operator on a matrix-valued field $M$, [6], [7]. By writing $M$ as a linear combination of tensor products of vectors with co-vectors, a calculation shows that

$$
\begin{equation*}
D_{\mu} M=\nabla_{\mu} M+\left[\Gamma_{\mu}, M\right] . \tag{1.16}
\end{equation*}
$$

In particular, if $M=I$, then $D_{\mu} M=0$. We now impose the additional requirement that the derivative, $D_{\mu}$, is metric compatible, i.e.,

$$
\begin{equation*}
D_{\mu} g^{\alpha \beta} I=0 \tag{1.17}
\end{equation*}
$$

where $g^{\alpha \beta}$ in this expression is understood to be a scalar (the element of a matrix) rather than a matrix. Recalling Eq. (1.23),

$$
\begin{equation*}
\varepsilon 2 g^{\alpha \beta} I=\left\{\bar{\gamma}^{\alpha}(x), \bar{\gamma}^{\beta}(x)\right\}, \tag{1.18}
\end{equation*}
$$

we see that Eq. (1.17) is equivalent to

$$
\begin{equation*}
D_{\mu}\left(\left\{\bar{\gamma}^{\alpha}(x), \bar{\gamma}^{\beta}(x)\right\}\right)=0, \tag{1.19}
\end{equation*}
$$

and a sufficient condition for the above equation is

$$
\begin{equation*}
D_{\mu} \bar{\gamma}^{\nu}(x)=0 . \tag{1.20}
\end{equation*}
$$

The operator $D_{\mu}$ of Eq. (1.16) acting on $\bar{\gamma}^{\nu}$ is,

$$
\begin{equation*}
D_{\mu} \bar{\gamma}^{\nu}=\nabla_{\mu} \bar{\gamma}^{\nu}+\left[\Gamma_{\mu}, \bar{\gamma}^{\nu}\right] . \tag{1.21}
\end{equation*}
$$

Thus Eq. (1.20) is

$$
\begin{equation*}
D_{\mu} \bar{\gamma}^{\nu}=\bar{\gamma}_{, \mu}^{\nu}+\Gamma_{\lambda \mu}^{\nu} \bar{\gamma}^{\lambda}+\Gamma_{\mu} \bar{\gamma}^{\nu}-\bar{\gamma}^{\nu} \Gamma_{\mu}=0 . \tag{1.22}
\end{equation*}
$$

We refer the reader to Appendix C, for further details of the effect of signature choices on the $\Gamma_{\mu}$ and the Dirac equation.

Remark 1. Reference [1], Eq. (13.27), for example, has a (-) sign in front of the commutator in Eq. (1.21) thus, effectively, changing the sign of $\Gamma_{\mu}$. This is compensated for by writing $\left(\partial_{\mu}-\Gamma_{\mu}\right)$ in Eq. (1.15).

We now introduce the spin connection (coefficients) $\omega^{A}{ }_{B \mu}$ by the relation below (e.g., see [8], pp.222-224, and [9], p. 487),

$$
\begin{gather*}
\omega_{B \mu}^{A}:=-e_{B}^{\nu}\left(\partial_{\mu} e_{\nu}^{A}-\Gamma_{\mu \nu}^{\lambda} e_{\lambda}^{A}\right)  \tag{1.23}\\
\omega_{A B \mu}=e_{A \beta} \nabla_{\mu} e_{B}^{\beta}=g_{\beta \alpha} e_{A}^{\alpha} \nabla_{\mu} e_{B}^{\beta}=\eta_{A C} e^{C}{ }_{\beta} \nabla_{\mu} e_{B}^{\beta} . \tag{1.24}
\end{gather*}
$$

We see from the second (or third) equality in Eq. (1.24) that the metric signature will affect the signs of the $\omega_{A B \mu}$.

Exercise 1. Obtain Eq. (1.23) using Eq (1.24) and the result of Proposition 1 below. (Hint: Pay attention to the ordering of the indices $A$ and $B$ in Eqs. (1.23) and (1.24).)

Proposition 1. The $\omega_{A B \mu}$ are antisymmetric in $A$ and $B$.
Proof: Recall Eq. (A.4), then we have that (cf. [9], p.489),

$$
\begin{align*}
\nabla_{\mu} \eta_{A B}=e_{B}^{\beta}\left(\nabla_{\mu} e_{A}^{\alpha}\right) g_{\alpha \beta}+e_{A}^{\alpha}\left(\nabla_{\mu} e_{B}^{\beta}\right) g_{\alpha \beta} & =0 \\
e_{B \alpha}\left(\nabla_{\mu} e_{A}^{\alpha}\right)+e_{A \beta}\left(\nabla_{\mu} e_{B}^{\beta}\right) & =0 \\
\omega_{B A \mu}+\omega_{A B \mu} & =0 \tag{1.25}
\end{align*}
$$

Remark 2. The $\omega_{A B \mu}$ are also written as $\omega_{\mu A B}$, [8]. Lord refers to $\nabla_{\mu} e_{B}{ }^{\beta}$ as the Ricci rotation coefficients while in our nomenclature the Ricci rotation coefficients are given by Eq. (1.45), namely their components along the tetrad field $e_{C}{ }^{\mu}$. Our point here is that the terminology varies a little in the literature and care is required.

Using Eq. (1.3) and (1.4) in Eq. (1.21), one can show that the $\Gamma_{\mu}$ below satisfies Eq. (1.22) and hence (1.17),

$$
\begin{equation*}
\Gamma_{\mu}=\frac{\varepsilon}{4} \omega_{A B \mu} \gamma^{A} \gamma^{B}=\frac{\varepsilon}{2} \omega_{A B \mu} \Sigma^{A B} \tag{1.26}
\end{equation*}
$$

where

$$
\begin{equation*}
\Sigma^{A B}=\frac{1}{4}\left[\gamma^{A}, \gamma^{B}\right] \tag{1.27}
\end{equation*}
$$

We have included an $\varepsilon$ factor in the expression for $\Gamma_{\mu}$ in order take into account the choice made in Eq. (1.4). The factor of $1 / 4$ in $\Gamma_{\mu}$, Eq. (1.26), compensates for the factor of 2 in Eq. (1.4) and thus is dimension independent. The reader may consult Fursaev and Vassilevich [10], p. 16 for a different line of reasoning.

Remark 3. We caution the reader that Parker and Toms [8] define a $\Gamma_{\mu}$ on $p$. 145 which is our Eq. (1.26) with $\varepsilon=-1$ and a $B_{\mu}$ on $p$. 228, which is our Eq. (1.26) with $\varepsilon=+1$. No problem arises since the corresponding Dirac equations also differ in the appropriate sign.

The Fock-Ivanenko coefficients, $\Gamma_{C}$, are given by

$$
\begin{equation*}
\Gamma_{C}=e_{C}{ }^{\mu} \Gamma_{\mu} \tag{1.28}
\end{equation*}
$$

thus we may write

$$
\begin{equation*}
D_{C} \psi=\left(e_{C}+\Gamma_{C}\right) \psi \tag{1.29}
\end{equation*}
$$

Finally, for a free spin $1 / 2$ particle of mass $m$ we have the Dirac equation in curved spacetime,

$$
\begin{equation*}
i \gamma^{C} D_{C} \psi-m \psi=0 \tag{1.30}
\end{equation*}
$$

Using Eqs. (1.3), (1.15), and (1.28), we may also write Eq. (1.30) in the form

$$
\begin{align*}
i \bar{\gamma}^{\mu}\left(\partial_{\mu}+\Gamma_{\mu}\right) \psi-m \psi & =0  \tag{1.31}\\
i \bar{\gamma}^{\mu} D_{\mu} \psi-m \psi & =0 \tag{1.32}
\end{align*}
$$

Remark 4. We note that the $e_{C}=e_{C}{ }^{\gamma} \partial_{\gamma}$, in Eqs. (1.29), (1.30), is regarded as a differential operator, thus for our four component spinor $\psi$, we have, after re-inserting the identity matrix $I$,

$$
e_{C} I \psi=\left(\begin{array}{c}
e_{C}^{\gamma} \partial_{\gamma} \psi_{1}  \tag{1.33}\\
e_{C}^{\gamma} \partial_{\gamma} \psi_{2} \\
e_{C}^{\gamma} \partial_{\gamma} \psi_{3} \\
e_{C}^{\gamma} \partial_{\gamma} \psi_{4}
\end{array}\right)
$$

Proposition 2. If the tetrad $e_{B}{ }^{\beta}$ in Eq. (1.24) is parallel along a path with tangent $e_{0}{ }^{\mu}$, then it is a Fermi tetrad, (see Eq. (A.15)) and it turns out that the Fock-Ivanenko coefficient $\Gamma_{0}=0$.

Proof: Using Eqs. (1.24), (1.26), we obtain the expression below for Eq. (1.28),

$$
\begin{equation*}
\Gamma_{C}=\frac{\varepsilon}{4} e_{A \beta}\left(e_{C}{ }^{\mu} \nabla_{\mu} e_{B}^{\beta}\right) \gamma^{A} \gamma^{B} \tag{1.34}
\end{equation*}
$$

thus if $e_{B}{ }^{\beta}$ is parallel, the terms in parentheses vanish for $C=0$.
Remark 5. Conversely, if $\Gamma_{0} \neq 0$, then $e_{B}{ }^{\beta}$ is not parallel. However, it may happen that $\Gamma_{0}=0$, while $e_{B}{ }^{\beta}$ is not parallel.

### 1.2 The Ricci rotation coefficient approach

We now give an alternative, and possibly more efficient, way of calculating the Fock-Ivanenko coefficients. We remark again that the terminology and definitions of some the quantities below vary in the literature and one has to be very careful. We work in the tetrad frame and define the structure coefficients (or structure constants), $C^{D}{ }_{A B}$, by the relations below, [11], [12], [13].

$$
\begin{equation*}
d e^{D}=-\frac{1}{2} C_{A B}^{D} e^{A} \wedge e^{B} \tag{1.35}
\end{equation*}
$$

An equivalent expression is

$$
\begin{equation*}
\left[e_{A}, e_{B}\right]^{\gamma}=e_{A}^{\alpha} \partial_{\alpha} e_{B}^{\gamma}-e_{B}^{\beta} \partial_{\beta} e_{A}^{\gamma}=C_{A B}^{D} e_{D}^{\gamma}, \tag{1.36}
\end{equation*}
$$

while a most convenient expression is

$$
\begin{equation*}
C_{A B}^{D}=\left(e_{\alpha, \beta}^{D}-e_{\beta, \alpha}^{D}\right) e_{A}^{\alpha} e_{B}^{\beta} \tag{1.37}
\end{equation*}
$$

We derive below Eq. (1.37) from Eq. (1.36). We begin by multiplying both sides of Eq. (1.36) by $e^{C}{ }_{\gamma}$ and using Eq. (A.2). Thus

$$
\begin{equation*}
e_{A}^{\alpha}\left(\partial_{\alpha} e_{B}^{\gamma}\right) e_{\gamma}^{C}-e_{B}^{\beta}\left(\partial_{\beta} e_{A}^{\gamma}\right) e_{\gamma}^{C}=C_{A B}^{D} \delta_{D}^{C}=C_{A B}^{C} \tag{1.38}
\end{equation*}
$$

We also have the identities

$$
\begin{align*}
& \partial_{\alpha}\left(e_{B}{ }^{\gamma} e^{C}{ }_{\gamma}\right)=\partial_{\alpha} \delta_{B}^{C}=\left(\partial_{\alpha} e_{B}{ }^{\gamma}\right) e^{C}{ }_{\gamma}+e_{B}{ }^{\gamma} \partial_{\alpha} e^{C}{ }_{\gamma}=0,  \tag{1.39}\\
& \partial_{\beta}\left(e_{A}{ }^{\gamma} e^{C}{ }_{\gamma}\right)=\partial_{\beta} \delta_{A}^{C}=\left(\partial_{\beta} e_{A}{ }^{\gamma}\right) e^{C}{ }_{\gamma}+e_{A}{ }^{\gamma} \partial_{\beta} e^{C}{ }_{\gamma}=0 . \tag{1.40}
\end{align*}
$$

Therefore Eq. (1.38) is

$$
\begin{equation*}
-e_{A}^{\alpha} e_{B}^{\gamma} e_{\gamma, \alpha}^{C}+e_{B}^{\beta} e_{A}^{\gamma} e_{\gamma, \beta}^{C}=C_{A B}^{C}, \tag{1.41}
\end{equation*}
$$

which, after relabeling the dummy indices, reduces to Eq. (1.37).
Clearly

$$
\begin{equation*}
C_{A B}^{D}=-C_{B A}^{D} \tag{1.42}
\end{equation*}
$$

and we have,

$$
\begin{align*}
& C_{A B C}=\eta_{A D} C^{D}{ }_{B C},  \tag{1.43}\\
& C_{A B C}=-C_{A C B} . \tag{1.44}
\end{align*}
$$

Finally we give two expressions for the Ricci rotation coefficients, $\Gamma_{A B C}$. The $\Gamma_{A B C}$ are related to the $\omega_{A B \mu}$, defined in Eq. (1.24), by the relation Eq. (1.45) below, where the metric signature affects the $\omega_{A B \mu}$ and hence the $\Gamma_{A B C}$.

$$
\begin{equation*}
\Gamma_{A B C}=\omega_{A B \mu} e_{C}{ }^{\mu} \tag{1.45}
\end{equation*}
$$

Another expression for the $\Gamma_{A B C}$, which is written here specifically for the metric signature $(+,-,-,-)$ is

$$
\begin{equation*}
\Gamma_{A B C}=-\frac{1}{2}\left(C_{A B C}+C_{B C A}-C_{C A B}\right) \tag{1.46}
\end{equation*}
$$

Eqs. (1.45) and (1.46) agree with the definitions in ref. [13], Eqs. (253), p. 37 , and (272), p. 39. For metric signature $(-,+,+,+)$ one has to change the overall sign in Eq. (1.46), in addition note that the $C$ 's in Eq. (1.36) are the negatives of the $C$ 's defined in [14], Eq. (2.11) (see also Remark 15). From Eq. (1.45) we see that

$$
\begin{equation*}
\Gamma_{A B C}=-\Gamma_{B A C} \tag{1.47}
\end{equation*}
$$

It is important to keep in mind the difference in index antisymmetry in Eqs. (1.44) and (1.47). Finally, using Eqs. (1.26), (1.28), and(1.45), we may now express the Fock-Ivanenko coefficients in terms of the $\Gamma_{A B C}$,

$$
\begin{equation*}
\Gamma_{C}=\frac{\varepsilon}{4} \Gamma_{A B C} \gamma^{A} \gamma^{B} \tag{1.48}
\end{equation*}
$$

We have used as references, [11], [12], and [14]. (Ref. [11] has some misprints in Sec. 11.4.) If one were to adopt the definitions and terminology of Soleng [14], one would have the advantage of being able to use the Mathematica package CARTAN to calculate all of these quantities (symbolically) by computer.

Remark 6. Apart from being tetrad-dependent, it is clear from the above derivation, that the sign of the Ricci rotation coefficients, $\Gamma_{A B C}$, will depend on the metric signature since $C_{A B C}=\eta_{A D} C^{D}{ }_{B C}$ (we add that certain authors, e.g., [12], define their spin connection Eq. (1.24) with the opposite sign). Furthermore, it is after Eq. (1.48), that is, when we write Eq. (1.30), that we have to choose a metric compatible representation of the $\gamma$ matrices.

### 1.3 The electromagnetic interaction

As mentioned above, the requirement Eq. (1.20) is sufficient but not necessary. In general one may add a (possibly complex) [3] vector multiple of the unit matrix to the solution, Eq. (1.26). In this way we may generalize the $\Gamma_{\mu}$ 's for the case where an arbitrary electromagnetic potential $A_{\mu}$ is present [3], [15]. We simply make the replacements:

$$
\begin{align*}
\Gamma_{\mu} & \rightarrow \Gamma_{\mu}+i q A_{\mu} I  \tag{1.49}\\
D_{\mu} & \rightarrow D_{\mu}+i q A_{\mu} I \tag{1.50}
\end{align*}
$$

where $q$ is the charge of the particle described by $\psi$. Thus Eq. (1.32) is now generalized to

$$
\begin{align*}
i \bar{\gamma}^{\mu} D_{\mu} \psi-m \psi & =0  \tag{1.51}\\
i \gamma^{C}\left(e_{C}+\frac{1}{4} \Gamma_{A B C} \gamma^{A} \gamma^{B}+i q A_{C}\right) \psi-m \psi & =0 \tag{1.52}
\end{align*}
$$

where

$$
\begin{equation*}
\bar{\gamma}^{\mu} A_{\mu}=e_{C}{ }^{\mu} \gamma^{C} A_{\mu}=\gamma^{C} e_{C}{ }^{\mu} A_{\mu}=\gamma^{C} A_{C} \tag{1.53}
\end{equation*}
$$

This is consistent with the so-called minimal coupling procedure. One can easily deduce the correctness of this term by considering the Minkowski limit (e.g., see [16], pp. 64-67). For example, in the case of the hydrogenic atom, $q=-e$, $e>0$, and in the standard notation the components $A_{C}$ of electromagnetic potential due to the proton are

$$
\begin{equation*}
A=\left(A_{0}, A_{1}, A_{2}, A_{3}\right)=\left(\frac{Z e}{r}, 0,0,0\right) \tag{1.54}
\end{equation*}
$$

so that $i q A_{0}=-i \frac{Z e^{2}}{r}$.
Of course in curved spacetime one has to use appropriate Maxwell's equations. We refer the reader again to [3], Sec. VII, or [12].

### 1.4 The Newman-Penrose formalism

In this section we give a short introduction to the Newman-Penrose formalism [17], [18]. Apart from the Newman-Penrose paper, we have found useful the exposition in the following texts [19], [20], and [21]. In addition the software package Cartan, [14], may be used with the N-P formalism for considerably faster calculations. However one has to be careful because, as usual, there are differences in some definitions and conventions among these references.

In the Newman-Penrose formalism the calculations are done using a complex null tetrad. One straightforward way to construct a complex null tetrad for a given metric, is to choose a set, $e_{A}$, of orthonormal tetrad vector fields (as discussed in Appendix A). These satisfy

$$
\begin{equation*}
\eta_{A B}=e_{A}^{\alpha} e_{B}^{\beta} g_{\alpha \beta} \tag{1.55}
\end{equation*}
$$

Then we define the complex null tetrad $l, n, m, \bar{m}$, below [22]:

$$
\begin{align*}
& \lambda_{1}=l=\frac{1}{\sqrt{2}}\left(e_{0}+e_{3}\right),  \tag{1.56}\\
& \lambda_{2}=n=\frac{1}{\sqrt{2}}\left(e_{0}-e_{3}\right),  \tag{1.57}\\
& \lambda_{3}=m=\frac{1}{\sqrt{2}}\left(e_{1}+i e_{2}\right),  \tag{1.58}\\
& \lambda_{4}=\bar{m}=\frac{1}{\sqrt{2}}\left(e_{1}-i e_{2}\right) . \tag{1.59}
\end{align*}
$$

Note that $\bar{m}=m^{*}$, so that, in general, we will use $A^{*}$ for the complex conjugate of $A$. Using Eq. (1.63) we can show that the null tetrad vectors $l, n, m, \bar{m}$, of Eqs. (1.56) - (1.59), satisfy the relations

$$
\begin{align*}
& l^{\mu} l_{\mu}=n^{\mu} n_{\mu}=\bar{m}^{\mu} \bar{m}_{\mu}=m^{\mu} m_{\mu}=0  \tag{1.60}\\
& l^{\mu} m_{\mu}=n^{\mu} m_{\mu}=l^{\mu} \bar{m}_{\mu}=n^{\mu} \bar{m}_{\mu}=0  \tag{1.61}\\
& l^{\mu} n_{\mu}=+1, m^{\mu} \bar{m}_{\mu}=-1 \tag{1.62}
\end{align*}
$$

The frame field metric components, $\zeta_{A B}$, for the $\lambda_{A}$ of Eqs. (1.56) - (1.59), are given by

$$
\begin{equation*}
\zeta_{A B}=\lambda_{A}{ }^{\alpha} \lambda_{B}{ }^{\beta} g_{\alpha \beta}=\lambda_{A}{ }^{\alpha} \lambda_{B \alpha}, \tag{1.63}
\end{equation*}
$$

where $A, B=1,2,3,4$. We find that

$$
\left(\zeta^{A B}\right)=\left(\zeta_{A B}\right)=\left(\begin{array}{cccc}
0 & 1 & 0 & 0  \tag{1.64}\\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{array}\right)
$$

Exercise 2. Use Eq. (1.63) to derive Eq. (1.64).
We also have the relations

$$
\begin{align*}
g^{\alpha \beta} & =\zeta^{A B} \lambda_{A}{ }^{\alpha} \lambda_{B}^{\beta}  \tag{1.65}\\
\zeta^{C B} \zeta_{A B} & =\lambda_{A}{ }^{\alpha} \zeta^{C B} \lambda_{B \alpha}=\lambda_{A}{ }^{\alpha} \lambda^{C}{ }_{\alpha}=\delta_{A}^{C} \tag{1.66}
\end{align*}
$$

In order to conform to the notation in the NP formalism literature, we shall change slightly the notation for the Ricci rotation coefficients given by Eq. (1.46), and write $\Gamma_{A B C}=\gamma_{A B C}$ (with $A, B, C=1,2,3,4$ where 4 is the time label). The collection of equations below is very important and handy:

$$
\begin{align*}
\lambda_{\alpha}^{A} & =g_{\alpha \beta} \zeta^{A B} \lambda_{B}^{\beta}  \tag{1.67}\\
\gamma_{B C}^{D} & =\lambda_{B}{ }^{\beta} \lambda_{C}{ }^{\alpha} \nabla_{\alpha} \lambda^{D}{ }_{\beta}  \tag{1.68}\\
\gamma_{A B C} & =\zeta_{A D} \gamma^{D}{ }_{B C} \tag{1.69}
\end{align*}
$$

and the equivalents to Eqs. (1.37), (1.42), and (1.46),

$$
\begin{align*}
C_{B C}^{D} & =\left(\lambda_{\alpha, \beta}^{D}-\lambda_{\beta, \alpha}^{D}\right) \lambda_{B}^{\alpha} \lambda_{C}^{\beta}  \tag{1.70}\\
C_{A B C} & =\zeta_{A D} C_{B C}^{D}  \tag{1.71}\\
\gamma_{A B C} & =-\frac{1}{2}\left(C_{A B C}+C_{B C A}-C_{C A B}\right) . \tag{1.72}
\end{align*}
$$

Remark 7. It is clear from Eqs. (1.56) - (1.59) that if the tetrad, $e_{A}$, is a Fermi tetrad, then the null tetrad, $\lambda_{B}$, is parallelly transported along the chosen congruence of timelike geodesics.

Remark 8. A choice of a null tetrad, $\lambda_{B}$, is equivalent to a choice of an orthonormal tetrad, $e_{A}$, since we can solve Eqs. (1.56) - (1.59), to express the $e_{A}$ in terms of the $\lambda_{B}$, [22], thus,

$$
\begin{align*}
& e_{0}=\frac{1}{\sqrt{2}}(l+n),  \tag{1.73}\\
& e_{1}=\frac{1}{\sqrt{2}}(m+\bar{m}),  \tag{1.74}\\
& e_{2}=\frac{-i}{\sqrt{2}}(m-\bar{m}),  \tag{1.75}\\
& e_{3}=\frac{1}{\sqrt{2}}(l-n) \tag{1.76}
\end{align*}
$$

Remark 9. From Eq. (1.73) we may deduce the properties of the observer frame tetrad by finding the acceleration,

$$
\begin{equation*}
a=\nabla_{e_{0}} e_{0} \tag{1.77}
\end{equation*}
$$

Moreover, evaluating, $\nabla_{e_{0}} e_{A}, A=0,1,2,3$, will tell us whether the tetrad is a Fermi tetrad or not (see Eq. (A.15)).

Exercise 3. Starting with Eq. (1.65), show that

$$
\begin{equation*}
g_{\alpha \beta}=n_{\alpha} l_{\beta}+l_{\alpha} n_{\beta}-\bar{m}_{\alpha} m_{\beta}-m_{\alpha} \bar{m}_{\beta} \tag{1.78}
\end{equation*}
$$

The null tetrad $l, n, m, \bar{m}$, is not uniquely defined by Eqs. (1.56)-(1.59). Without changing the direction of the field $l$, we may rescale it by an arbitrary factor $A$, where $A$ is a non-vanishing real function. Thus

$$
\begin{equation*}
l^{\prime \alpha}=A l^{\alpha} \tag{1.79}
\end{equation*}
$$

This amounts to a reparametrization of the curves tangent to $l$. The vectors $m$ and $\bar{m}$ may be rotated in their plane by an arbitrary angle $\phi$; moreover, their scalar products with $l$, do not change when a multiple of $l$ is added to them. Thus $m, \bar{m}$, are defined up to the transformations

$$
\begin{equation*}
m^{\prime \alpha}=e^{i \phi} m^{\alpha}+B l^{\alpha} \tag{1.80}
\end{equation*}
$$

where $\phi$ is a real function and $B$ is a complex function. The remaining vector $n$, may be changed by a fixed multiple of $l$ and a fixed multiple of a fixed vector in the $m, \bar{m}$ plane, so finally, we have [20]

$$
\begin{equation*}
n^{\alpha}=\frac{1}{A}\left(n^{\alpha}+B^{*} e^{i \phi} m^{\alpha}+B e^{-i \phi} \bar{m}^{\alpha}+B B^{*} l^{\alpha}\right) \tag{1.81}
\end{equation*}
$$

We can express these transformations as three classes of transformations [23],

$$
\begin{align*}
& l^{\prime}=l, m^{\prime}=m+B l, n^{\prime}=n+B^{*} m+B \bar{m}+B B^{*} l, \quad \text { (null rotation), }  \tag{1.82}\\
& n^{\prime}=n, m^{\prime}=m+B l, l^{\prime}=l+B^{*} m+B \bar{m}+B B^{*} n, \quad \text { (null rotation), }  \tag{1.83}\\
& l^{\prime}=A l, m^{\prime}=e^{i \phi} m, n^{\prime}=A^{-1} n, \quad \text { (boost and orthogonal rotation) } \tag{1.84}
\end{align*}
$$

We shall use the standard notation below, to designate the null tetrad vectors as directional derivatives

$$
\begin{equation*}
\lambda_{1}=l=D, \quad \lambda_{2}=n=\Delta, \quad \lambda_{3}=m=\delta, \quad \lambda_{4}=\bar{m}=\delta^{*} \tag{1.85}
\end{equation*}
$$

The above is expressed very clearly in reference [22]. "The role of the (vector) covariant derivative operator $\nabla_{\alpha}$ is taken over in the NP formalism by four scalar operators:"

$$
\begin{equation*}
D=l^{\alpha} \nabla_{\alpha}, \quad \Delta=n^{\alpha} \nabla_{\alpha}, \quad \delta=m^{\alpha} \nabla_{\alpha}, \quad \delta^{*}=\bar{m}^{\alpha} \nabla_{\alpha} \tag{1.86}
\end{equation*}
$$

Thus, e.g., for any scalar function $f$ we write

$$
\begin{equation*}
D f=l^{\alpha} f_{, \alpha}, \quad \Delta f=n^{\alpha} f_{, \alpha}, \quad \delta f=m^{\alpha} f_{, \alpha}, \quad \delta^{*} f=\bar{m}^{\alpha} f_{, \alpha} \tag{1.87}
\end{equation*}
$$

We give below the twelve so-called spin coefficients in terms of the Ricci rotation coefficients.

$$
\begin{align*}
& \kappa=\gamma_{311}, \quad \rho=\gamma_{314}, \quad \varepsilon=\frac{1}{2}\left(\gamma_{211}+\gamma_{341}\right) \\
& \sigma=\gamma_{313}, \quad \mu=\gamma_{243}, \quad \gamma=\frac{1}{2}\left(\gamma_{212}+\gamma_{342}\right)  \tag{1.88}\\
& \lambda=\gamma_{244}, \quad \tau=\gamma_{312}, \quad \alpha=\frac{1}{2}\left(\gamma_{214}+\gamma_{344}\right) \\
& \nu=\gamma_{242}, \quad \pi=\gamma_{241}, \quad \beta=\frac{1}{2}\left(\gamma_{213}+\gamma_{343}\right)
\end{align*}
$$

Remark 10. One could choose the functions, $A, B, \phi$, in Eqs. (1.79), (1.80), and (1.81), so as to maximize the number of vanishing the spin coefficients, thus possibly making the Dirac equation easier to solve. However it may not be easy to determine what observer frame the new tetrad corresponds to (see Remark 8).

We finally write the Dirac equation in the N-P formalism [13], [24], [25], [26].

$$
\begin{array}{r}
\left(\Delta+\mu^{*}-\gamma^{*}\right) G_{1}-\left(\delta^{*}+\beta^{*}-\tau^{*}\right) G_{2}=i \mu_{*} F_{1} \\
\left(D+\varepsilon^{*}-\rho^{*}\right) G_{2}-\left(\delta+\pi^{*}-\alpha^{*}\right) G_{1}=i \mu_{*} F_{2} \\
(D+\varepsilon-\rho) F_{1}+\left(\delta^{*}+\pi-\alpha\right) F_{2}=i \mu_{*} G_{1}  \tag{1.89}\\
(\Delta+\mu-\gamma) F_{2}+(\delta+\beta-\tau) F_{1}=i \mu_{*} G_{2}
\end{array}
$$

where $\mu_{*} \sqrt{2}=m=$ the mass of the particle. Eqs. (1.89) are in the chiral representation given by Eqs. (B.11), see ref. [27].

## 2 Examples

### 2.1 Schwarzschild spacetime N-P

In this section we compare the two approaches, N-P and F-I. Chandrasekhar uses the metric signature $(+,-,-,-)$ and so we write the Schwarzschild metric,

$$
\begin{equation*}
d s^{2}=\left(1-\frac{2 M}{r}\right) d t^{2}-\frac{d r^{2}}{\left(1-\frac{2 M}{r}\right)}-r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right) \tag{2.1}
\end{equation*}
$$

We adopt essentially Chandrasekhar's null tetrad cf., [13], Eq. (281) p. 134, where $l=\left(l^{t}, l^{r}, l^{\theta}, l^{\phi}\right)$, etc.

$$
\begin{align*}
l & =\frac{1}{\sqrt{2}}\left(\frac{1}{X}, 1,0,0\right)  \tag{2.2}\\
n & =\frac{1}{\sqrt{2}}(1,-X, 0,0)  \tag{2.3}\\
m & =\frac{1}{\sqrt{2}}\left(0,0, \frac{1}{r}, \frac{i}{r \sin \theta}\right),  \tag{2.4}\\
\bar{m} & =\frac{1}{\sqrt{2}}\left(0,0, \frac{1}{r}, \frac{-i}{r \sin \theta}\right), \tag{2.5}
\end{align*}
$$

where

$$
\begin{equation*}
X=1-\frac{2 M}{r} \tag{2.6}
\end{equation*}
$$

We find the spin coefficients below.

$$
\begin{align*}
& \alpha=-\frac{\cot \theta}{2 \sqrt{2} r}  \tag{2.7}\\
& \beta=\frac{\cot \theta}{2 \sqrt{2} r}  \tag{2.8}\\
& \gamma=\frac{M}{\sqrt{2} r^{2}}  \tag{2.9}\\
& \mu=-\frac{X}{\sqrt{2} r}  \tag{2.10}\\
& \rho=-\frac{1}{\sqrt{2} r} \tag{2.11}
\end{align*}
$$

Now we use Eqs. (1.87) and (1.89) to write the Dirac equations. We write below only the first two of the four equations for those who wish to check their results

$$
\begin{array}{r}
\partial_{t} G_{1}-X \partial_{r} G_{1}+\frac{M-r}{r^{2}} G_{1}-\frac{1}{r} \partial_{\theta} G_{2}+\frac{i}{r \sin \theta} \partial_{\phi} G_{2}- \\
\frac{\cot \theta}{2 r} G_{2}=i m F_{1} \tag{2.12}
\end{array}
$$

$$
\begin{array}{r}
\frac{1}{X} \partial_{t} G_{2}+\partial_{r} G_{2}+\frac{1}{r} G_{2}-\frac{1}{r} \partial_{\theta} G_{1}-\frac{i}{r \sin \theta} \partial_{\phi} G_{1}- \\
\frac{\cot \theta}{2 r} G_{1}=i m F_{2} \tag{2.13}
\end{array}
$$

where the Dirac wavefunction is

$$
\psi=\left(\begin{array}{c}
F_{1}  \tag{2.14}\\
F_{2} \\
G_{1} \\
G_{2}
\end{array}\right)
$$

We remark that the details of these calculations can be carried using the software package Cartan.

To do the calculations following the Fock-Ivanenko approach, we recall Remark 8 and use Eqs. (1.73) - (1.76), to obtain the orthogonal tetrad corresponding to the null tetrad given by Eqs. (2.2) - (2.5). We have

$$
\begin{align*}
& e_{0}=\left(\frac{1+X}{2 X}, \frac{1-X}{2}, 0,0\right)  \tag{2.15}\\
& e_{1}=\left(0,0, \frac{1}{r}, 0\right)  \tag{2.16}\\
& e_{2}=\left(0,0,0, \frac{1}{r \sin \theta}\right)  \tag{2.17}\\
& e_{3}=\left(\frac{1-X}{2 X}, \frac{1+X}{2}, 0,0\right) \tag{2.18}
\end{align*}
$$

where $e_{A}=\left(e_{A}^{t}, e_{A}^{r}, e_{A}{ }^{\theta}, e_{A}{ }^{\phi}\right)$, and $X$ is given by Eq. (2.6). Recalling Remark 9 , we find that $a=\nabla_{e_{0}} e_{0} \neq 0$, so this is not a freely falling particle frame. Finally the Dirac equations obtained with this approach are, of course, identical to Eqs. (2.12), (2.13), etc.

### 2.2 Schwarzschild spacetime F-I

In this example we calculate the Fock-Ivanenko coefficients for the Schwarzschild metric, using the tetrad below and write the resulting Dirac equations. This calculation may be found in Ryder [11] although there are several misprints there. We use Ryder's conventions, i.e., the metric signature below, the standard representation of the $\gamma$ matrices Eqs. (B.8), (B.9), and $\varepsilon=-1$.

The Schwarzschild metric is

$$
\begin{equation*}
d s^{2}=-\left(1-\frac{2 M}{r}\right) d t^{2}+\frac{d r^{2}}{\left(1-\frac{2 M}{r}\right)}+r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right) \tag{2.19}
\end{equation*}
$$

and we choose the orthonormal 1-forms and corresponding vectors below which
satisfy $g_{\alpha \beta}=\eta_{A B} e^{A}{ }_{\alpha} e^{B}{ }_{\beta}$, (see Appendix A.1).

$$
\begin{array}{ll}
e^{0}=\left(1-\frac{2 M}{r}\right)^{\frac{1}{2}} d t, & e_{0}=\left(1-\frac{2 M}{r}\right)^{-\frac{1}{2}} \partial_{t} \\
e^{1}=\left(1-\frac{2 M}{r}\right)^{-\frac{1}{2}} d r, & e_{1}=\left(1-\frac{2 M}{r}\right)^{\frac{1}{2}} \partial_{r} \\
e^{2}=r d \theta, & e_{2}=\frac{1}{r} \partial_{\theta} \\
e^{3}=r \sin \theta d \phi, & e_{3}=\frac{1}{r \sin \theta} \partial_{\phi} \tag{2.23}
\end{array}
$$

Equations (2.20) - (2.23) are the tetrad 1-forms and vectors of an observer with $\dot{r}=0, \dot{\theta}=0, \dot{\phi}=0$. Using Eq. (1.24) we find the nonvanishing spin connection coefficients,

$$
\begin{align*}
& \omega_{10 t}=\frac{M}{r^{2}}  \tag{2.24}\\
& \omega_{21 \theta}=\left(1-\frac{2 M}{r}\right)^{\frac{1}{2}}  \tag{2.25}\\
& \omega_{31 \phi}=\sin \theta\left(1-\frac{2 M}{r}\right)^{\frac{1}{2}}  \tag{2.26}\\
& \omega_{32 \phi}=\cos \theta \tag{2.27}
\end{align*}
$$

We now use Eqs. (1.26) and (1.28), to obtain the Fock-Ivanenko coefficients $\Gamma_{C}$.

$$
\begin{align*}
& \Gamma_{0}=\frac{M}{2 r^{2}}\left(1-\frac{2 M}{r}\right)^{-\frac{1}{2}} \gamma^{0} \gamma^{1}  \tag{2.28}\\
& \Gamma_{1}=0  \tag{2.29}\\
& \Gamma_{2}=\frac{1}{2 r}\left(1-\frac{2 M}{r}\right)^{\frac{1}{2}} \gamma^{1} \gamma^{2}  \tag{2.30}\\
& \Gamma_{3}=\frac{1}{2 r}\left(1-\frac{2 M}{r}\right)^{\frac{1}{2}} \gamma^{1} \gamma^{3} \frac{\cot \theta}{2 r} \gamma^{2} \gamma^{3} . \tag{2.31}
\end{align*}
$$

The Dirac equation, (1.28) and (1.30), is

$$
\begin{align*}
i\left[\gamma^{0}\left(e_{0}^{t} \partial_{t}+\Gamma_{0}\right)\right. & +\gamma^{1} e_{1}^{r} \partial_{r}+\gamma^{2}\left(e_{2}^{\theta} \partial_{\theta}+\Gamma_{2}\right) \\
& \left.+\gamma^{3}\left(e_{3}^{\phi} \partial_{\phi}+\Gamma_{3}\right)\right] \psi-m \psi=0 \tag{2.32}
\end{align*}
$$

From Eq. (2.28) we see that the term $\gamma^{0} \Gamma_{0}$ appearing in Eq. (2.32) simplifies to

$$
\begin{equation*}
\gamma^{0} \Gamma_{0}=\frac{M}{2 r^{2}}\left(1-\frac{2 M}{r}\right)^{-\frac{1}{2}} \gamma^{1} \tag{2.33}
\end{equation*}
$$

and so on.

### 2.3 Nonfactorizable metric

In this example we shall write the Dirac equation in the spacetime considered by Hounkonnou and Mendy [28]

$$
\begin{equation*}
d s^{2}=-d t^{2}+a^{2}(t)\left(d x^{2}+b^{2}(x)\left[d y^{2}+c^{2}(y) d z^{2}\right]\right) \tag{2.34}
\end{equation*}
$$

We remark that the de Sitter universe metric and the usual Friedman-Lemaître-Robertson-Walker metric of standard cosmology, for each curvature parameter $k$ separately, are special cases of the above form.

As in the example of Sec. 2.2 we again choose the tetrad for an observer with $\dot{x}=0, \dot{y}=0, \dot{z}=0$. The 1 -forms and corresponding vectors are given below.

$$
\begin{array}{ll}
e^{0}=d t, & e_{0}=\partial_{t} \\
e^{1}=a(t) d x, & e_{1}=\frac{1}{a(t)} \partial_{x}, \\
e^{2}=a(t) b(x) d y, & e_{2}=\frac{1}{a(t) b(x)} \partial_{y} \\
e^{3}=a(t) b(x) c(y) d z, & e_{3}=\frac{1}{a(t) b(x) c(y)} \partial_{z} \tag{2.38}
\end{array}
$$

Using Eq. (1.24) we find the nonvanishing spin connection coefficients,

$$
\begin{array}{ll}
\omega_{10 x}=a, t, & \omega_{20 y}=b a,_{t}, \\
\omega_{21 y}=b,_{x}  \tag{2.40}\\
\omega_{30 z}=b c a,_{t}, & \omega_{31 z}=c b,_{x},
\end{array} \omega_{32 z}=c,_{z} .
$$

Again we use Eqs. (1.26) and (1.28), to obtain the the Fock-Ivanenko coefficients $\Gamma_{C}$.

$$
\begin{align*}
& \Gamma_{0}=0  \tag{2.41}\\
& \Gamma_{1}=-\frac{a, t}{2 a} \gamma^{0} \gamma^{1}  \tag{2.42}\\
& \Gamma_{2}=-\frac{a, t}{2 a} \gamma^{0} \gamma^{2}-\frac{b,_{x}}{2 a b} \gamma^{1} \gamma^{2}  \tag{2.43}\\
& \Gamma_{3}=-\frac{a, t}{2 a} \gamma^{0} \gamma^{3}-\frac{b,_{x}}{2 a b} \gamma^{1} \gamma^{3}-\frac{c,,_{y}}{2 a b c} \gamma^{2} \gamma^{3} \tag{2.44}
\end{align*}
$$

Note that Hounkonnou and Mendy in ref. [28], define their $\Gamma_{\mu}$ with opposite sign from the one adopted here, Eq. (1.26), and in the first part of their paper they effectively multiply their $\gamma$ matrices by $(-i)$. Thus using Eq. (C.11) and the relations (B.19), we write the resulting Dirac equation as,

$$
\begin{align*}
\gamma^{C}\left(e_{C}+\Gamma_{C}\right) \psi+m \psi= & {\left[\gamma^{0}\left(\partial_{t}+\frac{3 a, t}{2 a}\right)+\gamma^{1}\left(\frac{1}{a} \partial_{x}+\frac{b,_{x}}{a b}\right)+\right.} \\
& \left.\gamma^{2}\left(\frac{1}{a b} \partial_{y}+\frac{c,_{y}}{2 a b c}\right)+\gamma^{3} \frac{1}{a b c} \partial_{z}\right] \psi+m \psi=0 \tag{2.45}
\end{align*}
$$

A further simplification is achieved if we let

$$
\begin{equation*}
\psi \equiv a^{-\frac{3}{2}} b^{-1} c^{-\frac{1}{2}} \Psi \tag{2.46}
\end{equation*}
$$

A short calculation shows that we may now rewrite the Dirac equation in the simplified form

$$
\begin{equation*}
\left[\gamma^{0} \partial_{t}+\frac{1}{a} \gamma^{1} \partial_{x}+\frac{1}{a b} \gamma^{2} \partial_{y}+\frac{1}{a b c} \gamma^{3} \partial_{z}\right] \Psi+m \Psi=0 . \tag{2.47}
\end{equation*}
$$

At some point in ref. [28] the authors specifically adopt the Jauch-Rohrlich representation of the $\gamma$ matrices discussed in Sec. B.5. Further clarification may be obtained by reviewing Sec. B.3.

## 2.4 de Sitter spacetime, Fermi coordinates

In this section we shall consider the Dirac equation in the de Sitter universe using exact (global) Fermi coordinates $\left(x^{0}, x^{1}, x^{2}, x^{3}\right)$ with respect to the reference observer $(\tau, 0,0,0)$. We shall first write the de Sitter metric in the standard coordinates used in ref. [29], (but we adopt a different notation from the one used there in order to avoid confusion).

$$
\begin{equation*}
d s^{2}=-d\left(y^{0}\right)^{2}+e^{2 a y^{0}} \delta_{i j} d y^{i} d y^{j} \tag{2.48}
\end{equation*}
$$

A Fermi tetrad field on the set of geodesics $\gamma(\tau)=\left(\tau, y^{1}{ }_{0}, y^{2}{ }_{0}, y^{3}{ }_{0}\right)$, i.e., $y^{0}=\tau$ and $y^{i}=$ const., is

$$
\begin{equation*}
\lambda_{0}=\partial_{y^{0}}, \quad \lambda_{I}=e^{-a \tau} \partial_{y^{I}}, \quad I=(1,2,3) \tag{2.49}
\end{equation*}
$$

We transform the metric of Eq. (2.48) to the metric in global Fermi coordinates using the transformation derived in [29], [30], relating the $y^{\mu}$ to the Fermi $x^{\mu}$,

$$
\begin{align*}
e^{a y^{0}} & =e^{a x^{0}} \cos (a \rho)  \tag{2.50}\\
y^{i} & =e^{-a x^{0}} \frac{\tan (a \rho)}{a \rho} x^{i} \tag{2.51}
\end{align*}
$$

where $\rho=\sqrt{\delta_{i j} x^{i} x^{j}}, a=\sqrt{\Lambda / 3}$, and $0 \leq \rho<\pi /(2 a)$.
The resulting metric is,

$$
\begin{equation*}
d s^{2}=-\cos ^{2}(a \rho) d\left(x^{0}\right)^{2}+\left[\frac{x^{i} x^{j}}{\rho^{2}}+\frac{\sin ^{2}(a \rho)}{a^{2} \rho^{2}}\left(\delta_{i j}-\frac{x^{i} x^{j}}{\rho^{2}}\right)\right] d x^{i} d x^{j} \tag{2.52}
\end{equation*}
$$

We may obtain a set of Fermi tetrad 1-form field in Fermi coordinates by transforming the 1-forms corresponding to the vectors of Eq. (2.49), using the transformation Eqs. (2.50), (2.51). The Fermi tetrad field obtained is complicated because the set of geodesics corresponding to the set $\gamma(\tau)$ has lost its original simplicity in the $x^{\mu}$ coordinates. We can find the inverse of the transformation Eqs. (2.50), (2.51), which we shall refer to as $F$, thus

$$
\begin{align*}
& x^{0}=\left(\frac{1}{a}\right) \ln \left(\frac{e^{a y^{0}}}{\left.\sqrt{1-a^{2} e^{2 a y^{0} R^{2}}}\right)}\right. \text {, }  \tag{2.53}\\
& x^{i}=\frac{\arccos \sqrt{1-a^{2} e^{2 a y^{0}} R^{2}}}{a R} y^{i} \tag{2.54}
\end{align*}
$$

where $R=\sqrt{\delta_{i j} y^{i} y^{j}}$. We know that if $F: M \rightarrow N$ is an isometry and $\gamma$ is a geodesic in $M$, then $F \circ \gamma$ is a geodesic in $N$. Therefore the set of geodesics, $\gamma(\tau)$, are now given by (recall that $y^{0}=\tau$ )

$$
\begin{equation*}
F \circ \gamma=\left(x^{0}(\tau), x^{1}(\tau), x^{2}(\tau), x^{3}(\tau)\right) \tag{2.55}
\end{equation*}
$$

The above complications do not prevent us from carrying on with our calculations for the Dirac equation. We give the set of tetrad 1-forms we obtained below.

$$
\begin{align*}
e^{0} & =d x^{0}-f(\rho)\left(x^{1} d x^{1}+x^{2} d x^{2}+x^{3} d x^{3}\right),  \tag{2.56}\\
e^{1} & =-x^{1} h(\rho) d x^{0} \\
& +\frac{a \rho\left(x^{1}\right)^{2} \sec (a \rho)+\left[\left(x^{2}\right)^{2}+\left(x^{3}\right)^{2}\right] \sin (a \rho)}{a \rho^{3}} d x^{1} \\
& +x^{1} x^{2} p(\rho) d x^{2}+x^{1} x^{3} p(\rho) d x^{3},  \tag{2.57}\\
e^{2} & =-x^{2} h(\rho) d x^{0}+x^{1} x^{2} p(\rho) d x^{1} \\
& +\frac{a \rho\left(x^{2}\right)^{2} \sec (a \rho)+\left[\left(x^{1}\right)^{2}+\left(x^{3}\right)^{2}\right] \sin (a \rho)}{a \rho^{3}} d x^{2} \\
& +x^{2} x^{3} p(\rho) d x^{3},  \tag{2.58}\\
e^{3} & =-x^{3} h(\rho) d x^{0}+x^{1} x^{3} p(\rho) d x^{1}+x^{2} x^{3} p(\rho) d x^{2} \\
& +\frac{a \rho\left(x^{3}\right)^{2} \sec (a \rho)+\left[\left(x^{1}\right)^{2}+\left(x^{2}\right)^{2}\right] \sin (a \rho)}{a \rho^{3}} d x^{3}, \tag{2.59}
\end{align*}
$$

where

$$
\begin{align*}
& f(\rho)=\frac{\tan (a \rho)}{\rho},  \tag{2.60}\\
& h(\rho)=\frac{\sin (a \rho)}{\rho},  \tag{2.61}\\
& p(\rho)=\left(\frac{\sec (a \rho)}{\rho^{2}}-\frac{\sin (a \rho)}{a \rho^{3}}\right),  \tag{2.62}\\
& q(\rho)=\left(-\frac{a \csc (a \rho)}{\rho}+\frac{\sec (a \rho)}{\rho^{2}}\right) . \tag{2.63}
\end{align*}
$$

The corresponding tetrad vectors are,

$$
\begin{align*}
e_{0} & =\sec ^{2}(a \rho) \partial_{x^{0}}+f(\rho)\left(x^{1} \partial_{x^{1}}+x^{2} \partial_{x^{2}}+x^{3} \partial_{x^{3}}\right)  \tag{2.64}\\
e_{1} & =x^{1} f(\rho) \sec (a \rho) \partial_{x^{0}} \\
& +\frac{a \rho\left[\left(x^{2}\right)^{2}+\left(x^{3}\right)^{2}\right] \csc (a \rho)+\left(x^{1}\right)^{2} \sec (a \rho)}{\rho^{2}} \partial_{x^{1}} \\
& +q(\rho)\left(x^{1} x^{2} \partial_{x^{2}}+x^{1} x^{3} \partial_{x^{3}}\right),  \tag{2.65}\\
e_{2} & =x^{2} f(\rho) \sec (a \rho) \partial_{x^{0}}+x^{1} x^{2} q(\rho) \partial_{x^{1}} \\
& +\frac{a \rho\left[\left(x^{1}\right)^{2}+\left(x^{3}\right)^{2}\right] \csc (a \rho)+\left(x^{2}\right)^{2} \sec (a \rho)}{\rho^{2}} \partial_{x^{2}} \\
& +x^{2} x^{3} q(\rho) \partial_{x^{3}},  \tag{2.66}\\
e_{3} & =x^{3} f(\rho) \sec (a \rho) \partial_{x^{0}}+q(\rho)\left(x^{1} x^{3} \partial_{x^{1}}+x^{2} x^{3} \partial_{x^{2}}\right) \\
& +\frac{a \rho\left[\left(x^{1}\right)^{2}+\left(x^{2}\right)^{2}\right] \csc (a \rho)+\left(x^{3}\right)^{2} \sec (a \rho)}{\rho^{2}} \partial_{x^{3}} . \tag{2.67}
\end{align*}
$$

We now use the Mathematica package Cartan [14] to obtain the nonvanishing Ricci rotation coefficients, $\Gamma_{A B C}$, of Eqs. (1.45), (1.46), (but see comment below (1.46)). Then the Fock-Ivanenko coefficients, $\Gamma_{C}$, of Eq. (1.48). We have

$$
\begin{equation*}
\Gamma_{101}=\Gamma_{202}=\Gamma_{303}=a \tag{2.68}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma_{0}=0, \quad \Gamma_{A}=-\frac{a}{2} \gamma^{0} \gamma^{A}, \quad A=1,2,3 \tag{2.69}
\end{equation*}
$$

Using Eqs. (2.69), we write the Dirac equation (C.9),

$$
\begin{gather*}
{\left[\gamma^{0} e_{0}+\gamma^{1}\left(e_{1}-\frac{a}{2} \gamma^{0} \gamma^{1}\right)+\gamma^{2}\left(e_{2}-\frac{a}{2} \gamma^{0} \gamma^{2}\right)+\right.} \\
\left.\gamma^{3}\left(e_{3}-\frac{a}{2} \gamma^{0} \gamma^{3}\right)\right] \psi-m \psi=0 \tag{2.70}
\end{gather*}
$$

We note that because of our $(-,+,+,+$,$) signature, the \gamma$ matrices will satisfy Eq. (B.19) and the Dirac equation, simplifies to

$$
\begin{equation*}
\left[\gamma^{A} e_{A}+\gamma^{0}\left(e_{0}+\frac{3 a}{2}\right)\right] \psi-m \psi=0, \quad A=1,2,3 \tag{2.71}
\end{equation*}
$$

Equation (2.71) looks deceptively simple, but the complications arise from the expressions for the $e_{A}$, therefore it may be better to work in the original coordinates, Eq. (2.48) with the original tetrad Eq. (2.49).

## 3 The Dirac equation in (1+1) GR

### 3.1 Introduction to ( $1+1$ )

We shall adopt the metric signature $(+,-)$. In $(1+1)$ general relativity the Dirac equation simplifies and may be written as follows [31], [32], [33].

$$
\begin{equation*}
\left[i \gamma^{A} e_{A}{ }^{\mu} \partial_{\mu}+\frac{i}{2} \gamma^{A} \frac{1}{\sqrt{-g}} \partial_{\mu}\left(\sqrt{-g} e_{A}^{\mu}\right)-m I_{2}\right] \psi=0 \tag{3.1}
\end{equation*}
$$

where the zweibein vector label $A$ runs over 0,1 , and for the spinor we write

$$
\begin{equation*}
\psi=\binom{\psi_{1}}{\psi_{2}} \tag{3.2}
\end{equation*}
$$

In what follows we will further restrict ourselves to the chiral (Weyl) representation of the Dirac $\gamma$ matrices, specifically we choose [31]

$$
\gamma^{0}=\left(\begin{array}{ll}
0 & 1  \tag{3.3}\\
1 & 0
\end{array}\right), \quad \gamma^{1}=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)
$$

Thus

$$
\begin{equation*}
\left(\gamma^{0}\right)^{2}=I_{2}, \quad\left(\gamma^{1}\right)^{2}=-I_{2} \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{\gamma^{A}, \gamma^{B}\right\}=2 \eta^{A B} I_{2} \tag{3.5}
\end{equation*}
$$

We also define the matrix

$$
\gamma^{5}:=\gamma^{0} \gamma^{1}=\left(\begin{array}{cc}
1 & 0  \tag{3.6}\\
0 & -1
\end{array}\right)
$$

One great advantage of the chiral representation is the ease of decoupling of Eq. (3.1). Of course the spinor wave function components of Eq. (3.2) are now eigenstates of the operator $\gamma^{5}$, so we may write

$$
\begin{equation*}
\psi=\binom{\psi_{(+)}}{\psi_{(-)}} \tag{3.7}
\end{equation*}
$$

with the eigenvalues $\gamma^{5} \psi_{(+)}=+\psi_{(+)}$, and, $\gamma^{5} \psi_{(-)}=-\psi_{(-)}$.

### 3.2 The Dirac equation in the Milne universe

We shall consider solutions of the Dirac equation in the Milne universe in two different charts: (a) in standard comoving coordinates $(t, x)$, in which case the metric is

$$
\begin{equation*}
d s^{2}=d t^{2}-a_{0}^{2} t^{2} d x^{2} \tag{3.8}
\end{equation*}
$$

and (b) in exact Fermi coordinates ( $\tau, \rho$ ), in which case the Milne universe is the interior of the forward lightcone of Minkowski spacetime, [34], [35] thus

$$
\begin{equation*}
d s^{2}=d \tau^{2}-d \rho^{2} \tag{3.9}
\end{equation*}
$$

where $\tau>|\rho|$.
(a) We use the default zweibein

$$
\begin{equation*}
\bar{e}_{0}=\partial_{t}, \quad \bar{e}_{1}=\frac{1}{a_{0} t} \partial_{x} \tag{3.10}
\end{equation*}
$$

Since the metric in Eq. (3.8) does not depend on $x$ the corresponding canonical momentum $p_{x}$ is a constant both in classical and quantum mechanics. We take advantage of this fact and write the 2-component spinor $\psi$ as

$$
\begin{equation*}
\psi(t, x)=\binom{\psi_{1}}{\psi_{2}}=e^{-i p_{x} x}\binom{f_{1}(t)}{f_{2}(t)} \tag{3.11}
\end{equation*}
$$

where $p_{x}$ is the 1 -form of the particle's momentum.
One finds that the only nonvanishing term from the second set of terms in Eq. (3.1) is

$$
\begin{equation*}
\frac{i}{2} \gamma^{0} \frac{1}{a_{0} t}\left[\partial_{t}\left(a_{0} t \bar{e}_{0}^{t}\right)\right]=\frac{i}{2 t} \gamma^{0} . \tag{3.12}
\end{equation*}
$$

Thus Eq. (3.1) reduces to

$$
\begin{equation*}
\left[i \gamma^{0} \partial_{t}+\frac{i}{a_{0} t} \gamma^{1} \partial_{x}+\frac{i}{2 t} \gamma^{0}-m I_{2}\right] \psi=0 \tag{3.13}
\end{equation*}
$$

Now we substitute Eqs. (3.3) and (3.11) in Eq. (3.13) and obtain the coupled equations,

$$
\begin{align*}
& f_{1}=\frac{1}{m}\left(i \partial_{t}-\frac{p_{x}}{a_{0} t}+\frac{i}{2 t}\right) f_{2}  \tag{3.14}\\
& f_{2}=\frac{1}{m}\left(i \partial_{t}+\frac{p_{x}}{a_{0} t}+\frac{i}{2 t}\right) f_{1} \tag{3.15}
\end{align*}
$$

Finally, decoupling Eqs. (3.14) and (3.15), we obtain

$$
\begin{equation*}
t^{2} f_{1}^{\prime \prime}+t f_{1}^{\prime}+\left[m^{2} t^{2}-\left(\frac{1}{2}-\frac{i p_{x}}{a_{0}}\right)^{2}\right] f_{1}=0 \tag{3.16}
\end{equation*}
$$

where the primes denote derivatives with respect to $t$. The solution of this equation is given below in terms of the Bessel functions $J_{\nu}$ and $Y_{\nu}$ of the first and second kind respectively,

$$
\begin{equation*}
f_{1}(t)=A J_{\nu}(m t)+B Y_{\nu}(m t) \tag{3.17}
\end{equation*}
$$

where $A$ and $B$ are arbitrary (complex) constants and

$$
\begin{equation*}
\nu=\frac{1}{2}-\frac{i p_{x}}{a_{0}} . \tag{3.18}
\end{equation*}
$$

Using Eq. (3.15), we find that

$$
\begin{equation*}
f_{2}(t)=i A J_{\nu-1}(m t)+i B Y_{\nu-1}(m t) \tag{3.19}
\end{equation*}
$$

(b) Now we transform the solution to the exact Fermi coordinates $(\tau, \rho)$. The transformation and its inverse is given by [34]

$$
\begin{align*}
& t=\sqrt{\tau^{2}-\rho^{2}}  \tag{3.20}\\
& x=\left(\frac{1}{a_{0}}\right) \tanh ^{-1}\left(\frac{\rho}{\tau}\right),  \tag{3.21}\\
& \tau=t \cosh \left(a_{0} x\right)  \tag{3.22}\\
& \rho=t \sinh \left(a_{0} x\right) \tag{3.23}
\end{align*}
$$

Under the above coordinate transformation the original tetrad 1-form fields,

$$
\begin{equation*}
\bar{e}^{0}=d t, \quad \bar{e}^{1}=a_{0} t d x \tag{3.24}
\end{equation*}
$$

transform into the 1-form fields $h^{A}$ below

$$
\begin{align*}
& h^{0}=\frac{\tau}{\sqrt{\tau^{2}-\rho^{2}}} d \tau-\frac{\rho}{\sqrt{\tau^{2}-\rho^{2}}} d \rho  \tag{3.25}\\
& h^{1}=\frac{-\rho}{\sqrt{\tau^{2}-\rho^{2}}} d \tau+\frac{\tau}{\sqrt{\tau^{2}-\rho^{2}}} d \rho \tag{3.26}
\end{align*}
$$

which, of course, satisfy the relation

$$
\begin{equation*}
h_{\alpha}^{A} h^{B}{ }_{\beta} \eta^{\alpha \beta}=\eta^{A B}, \tag{3.27}
\end{equation*}
$$

where the upper case latin indices run over 0,1 , while the greek indices run over $\tau, \rho$. We shall write $\psi_{h}(\tau, \rho)$ for the $\psi(t, x)$ of Eq. (3.11) transformed using Eqs. (3.20) and (3.21). Thus

$$
\begin{equation*}
\psi_{h}(\tau, \rho)=e^{-i p_{x}\left(\frac{1}{a_{0}}\right) \tanh ^{-1}\left(\frac{\rho}{\tau}\right)}\binom{f_{1}\left(\sqrt{\tau^{2}-\rho^{2}}\right)}{f_{2}\left(\sqrt{\tau^{2}-\rho^{2}}\right)} \tag{3.28}
\end{equation*}
$$

We will now perform a local Lorentz transformation, $\Lambda$, which will transform the zweibein 1-form fields $h^{A}$ into the canonical zweibein for the metric (3.9),

$$
\begin{equation*}
e^{0}=d \tau, \quad e^{1}=d \rho \tag{3.29}
\end{equation*}
$$

The transformation $\Lambda$ is given by

$$
\begin{equation*}
e^{A}{ }_{\alpha}=\Lambda^{A}{ }_{B} h^{B}{ }_{\alpha} . \tag{3.30}
\end{equation*}
$$

We find

$$
\Lambda=\left(\begin{array}{ll}
\Lambda^{0}{ }_{0} & \Lambda^{0}{ }_{1}  \tag{3.31}\\
\Lambda_{0}^{1} & \Lambda_{1}^{1}
\end{array}\right)=\frac{1}{\sqrt{\tau^{2}-\rho^{2}}}\left(\begin{array}{cc}
\tau & \rho \\
\rho & \tau
\end{array}\right)
$$

We can calculate the matrix $L$ used in Eq. (1.6) following the prescription given in Appendix $B$ of [36],

$$
L=\left(\begin{array}{cc}
\left(\frac{\tau+\rho}{\tau-\rho}\right)^{1 / 4} & 0  \tag{3.32}\\
0 & \left(\frac{\tau-\rho}{\tau+\rho}\right)^{1 / 4}
\end{array}\right)
$$

The solutions using the zweibein set $e_{A}$ Eq. (3.29) is

$$
\begin{equation*}
\psi_{e}(\tau, \rho)=L \psi_{h}(\tau, \rho) \tag{3.33}
\end{equation*}
$$

where $\psi_{h}(\tau, \rho)$ is given by Eq. (3.28). We have then,

$$
\begin{equation*}
\psi_{e}(\tau, \rho)=e^{-i p_{x} \tanh ^{-1}\left(\frac{\rho}{\tau}\right)}\binom{\left(\frac{\tau+\rho}{\tau-\rho}\right)^{1 / 4}\left(A J_{\nu}(z)+B Y_{\nu}(z)\right)}{i\left(\frac{\tau-\rho}{\tau+\rho}\right)^{1 / 4}\left(A J_{\nu-1}(z)+B Y_{\nu-1}(z)\right)} \tag{3.34}
\end{equation*}
$$

where

$$
\begin{align*}
\nu & =\frac{1}{2}-\frac{i p_{x}}{a_{0}}  \tag{3.35}\\
z & =m \sqrt{\tau^{2}-\rho^{2}} \tag{3.36}
\end{align*}
$$

The wavefunction $\psi_{e}(\tau, \rho)$, satisfies Eq. (3.1) which now reduces to the usual Minkowski spacetime Dirac equation, namely,

$$
\begin{equation*}
\left(i \gamma^{A} \partial_{A}-m I_{2}\right) \psi_{e}=0 \tag{3.37}
\end{equation*}
$$

In order to show that $\psi_{e}(\tau, \rho)$ satisfies Eq. (3.37), one has to use the Bessel function identity,

$$
\begin{equation*}
C_{\nu-1}(z)+C_{\nu+1}(z)=\frac{2 \nu}{z} C_{\nu}(z) \tag{3.38}
\end{equation*}
$$

where $C_{\nu}(z)$ denotes either of the Bessel functions $J_{\nu}(z), Y_{\nu}(z)$.
(c) In this subsection we shall consider the normalization integral [16], p. 69. This integral is referred to as the "probability integral" in [37]. Thus in Fermi coordinates with the canonical zweibein we have,

$$
\begin{equation*}
\left(\psi_{e} \mid \psi_{e}\right)=\int_{\Sigma} \bar{\psi}_{e} \gamma^{0} \psi_{e} d \rho \tag{3.39}
\end{equation*}
$$

In our case the spacelike hypersurface $\Sigma$ is the usual $\left(\tau_{0}, \rho\right), \tau_{0}>0$, hyperplane. Using Eq. (3.4) we have

$$
\begin{equation*}
\left(\psi_{e} \mid \psi_{e}\right)=\int_{-\tau_{0}}^{\tau_{0}} \psi_{e}^{\dagger} \psi_{e} d \rho \tag{3.40}
\end{equation*}
$$

For the remainder of this section we will write $\bar{a}$ to denote the complex conjugate of $a$. We shall make use of the fact that

$$
\begin{equation*}
\overline{C_{\nu}(z)}=C_{\bar{\nu}}(\bar{z}) \tag{3.41}
\end{equation*}
$$

where again $C_{\nu}(z)$ denotes either of the Bessel functions $J_{\nu}(z), Y_{\nu}(z)$. Since in our case $z$ is real we have that

$$
\begin{equation*}
\overline{C_{\nu}(z)}=C_{\bar{\nu}}(z) \tag{3.42}
\end{equation*}
$$

In order to check the behavior of the integrand in Eq. (3.40) at the endpoints of integration, we use the limiting forms of the Bessel functions when $\nu$ is fixed and $z \sim 0$. Using Abramowitz and Stegun's Eqs. (9.1.2), p. 358 and (9.1.7), p.360, [38], one easily deduces that for $\nu \neq$ negative integer,

$$
\begin{align*}
& Y_{\nu}(z)=\frac{J_{\nu}(z) \cos (\nu \pi)-J_{-\nu}(z)}{\sin (\nu \pi)}  \tag{3.43}\\
& J_{\nu}(z) \sim\left(\frac{z}{2}\right)^{\nu} \frac{1}{\Gamma(\nu+1)}  \tag{3.44}\\
& Y_{\nu}(z) \sim-\left(\frac{z}{2}\right)^{-\nu} \frac{\Gamma(\nu)}{\pi}, \quad \operatorname{Re}(\nu)>0  \tag{3.45}\\
& Y_{\nu}(z) \sim-\left(\frac{z}{2}\right)^{\nu} \frac{\cot (\nu \pi)}{\Gamma(\nu+1)}, \quad \operatorname{Re}(\nu)<0 \tag{3.46}
\end{align*}
$$

Using Eqs. (3.44)-(3.46) in the integrand of Eq. (3.40), we see that some terms blow up at the endpoints like $\sim 1 /\left(\tau_{0}-\rho\right)$ as the integration variable $\rho \rightarrow \tau_{0}$, regardless of the value of $p_{x}$. We can eliminate these terms by setting $B=B_{1}+i B_{2}=0$, where $B_{1}, B_{2} \in$ Reals. Thus the solution $\psi_{e}$ of Eq. (3.34) reduces to

$$
\psi_{e}(\tau, \rho)=A e^{-i p_{x} \tanh ^{-1}\left(\frac{\rho}{\tau}\right)}\left(\begin{array}{cc}
\left(\frac{\tau+\rho}{\tau-\rho}\right)^{1 / 4} & J_{\nu}(z)  \tag{3.47}\\
i\left(\frac{\tau-\rho}{\tau+\rho}\right)^{1 / 4} & J_{\nu-1}(z)
\end{array}\right)
$$

where $A$ is a complex normalization constant.

## 4 Scalar product

### 4.1 Conservation of $j$ in SR

The probability current density for a Dirac field is given by

$$
\begin{equation*}
j^{A}=\bar{\psi} \gamma^{A} \psi \tag{4.1}
\end{equation*}
$$

where the adjoint spinor is $\bar{\psi}=\psi^{\dagger} \gamma^{0}$. The probability current density $j^{A}$ transforms like a 4 -vector under a Lorentz transformation, $\Lambda$, so that,

$$
\begin{equation*}
j^{\prime A}=\Lambda^{A}{ }_{B} j^{B} \tag{4.2}
\end{equation*}
$$

moreover $j^{A}$ is conserved, that is,

$$
\begin{equation*}
\partial_{A} j^{A}=0 \tag{4.3}
\end{equation*}
$$

First we will prove Eq. (4.2). We know that under a tetrad rotation (local Lorentz transformation) $\Lambda$,

$$
\begin{equation*}
\psi^{\prime}=L \psi \quad \Rightarrow \quad \psi^{\prime \dagger}=\psi^{\dagger} L^{\dagger} \tag{4.4}
\end{equation*}
$$

where $L$ and $\Lambda$ are related as in (1.6). Thus, using Eqs. (A.37) and (A.38), we have that

$$
\begin{align*}
j^{\prime A} & =\psi^{\prime \dagger} \gamma^{0} \gamma^{A} \psi^{\prime}  \tag{4.5}\\
& =\psi^{\dagger} L^{\dagger} \gamma^{0} \gamma^{A} L \psi  \tag{4.6}\\
& =\psi^{\dagger} \gamma^{0}\left(L^{-1} \gamma^{A} L\right) \psi,  \tag{4.7}\\
& =\psi^{\dagger} \gamma^{0} \Lambda^{A}{ }_{B} \gamma^{B} \psi  \tag{4.8}\\
& =\Lambda^{A}{ }_{B} \psi^{\dagger} \gamma^{0} \gamma^{B} \psi,  \tag{4.9}\\
& =\Lambda^{A}{ }_{B} j^{B} . \tag{4.10}
\end{align*}
$$

Next in order to show that the current is conserved it will be useful to write the expression for the adjoint of the Dirac equation. We begin with the usual special relativity Dirac equation,

$$
\begin{equation*}
i \gamma^{A} \partial_{A} \psi-m \psi=i \not \partial \psi-m \psi=0 \tag{4.11}
\end{equation*}
$$

Then the adjoint is obtained as follows:

$$
\begin{align*}
\left(i \gamma^{A} \partial_{A} \psi-m \psi\right)^{\dagger} & =0  \tag{4.12}\\
-i \partial_{A} \psi^{\dagger}\left(\gamma^{A}\right)^{\dagger}-m \psi^{\dagger} & =0  \tag{4.13}\\
i \partial_{A} \psi^{\dagger} \gamma^{0} \gamma^{A} \gamma^{0}+m \psi^{\dagger} & =0  \tag{4.14}\\
i\left(\partial_{A} \bar{\psi}\right) \gamma^{A}+m \bar{\psi} & =0 \tag{4.15}
\end{align*}
$$

where we used the relations (B.4) and (B.12). Thus we shall use the notation

$$
\begin{align*}
\bar{\psi} \overleftarrow{\phi} & =\partial_{A} \bar{\psi} \gamma^{A}  \tag{4.16}\\
\vec{\partial} \psi & =\partial_{A} \psi \gamma^{A} \tag{4.17}
\end{align*}
$$

So we have the shorthand Feynman slash notation for the Dirac equation and its adjoint:

$$
\begin{align*}
& (i \overrightarrow{\not \partial}-m I) \psi=0  \tag{4.18}\\
& \bar{\psi}(i \overleftarrow{\not \partial}+m I)=0 \tag{4.19}
\end{align*}
$$

Multiplying Eq. (4.18) on the left with $\bar{\psi}$, and Eq. (4.19) on the right with $\psi$ and adding, we obtain

$$
\begin{equation*}
\bar{\psi}(\overleftarrow{\not \partial}+\overrightarrow{\not \partial}) \psi \equiv \partial_{A}\left(\bar{\psi} \gamma^{A} \psi\right)=0 \tag{4.20}
\end{equation*}
$$

which completes the proof of the conservation Eq. (4.3).
It follows from Eq. (4.3) that

$$
\begin{equation*}
\frac{d}{d t} \int_{V} j^{0} d^{3} x=-\int_{V} \partial_{K} j^{K} d^{3} x=-\int_{\partial V} j^{K} d S_{K}=0, \quad K=(1,2,3) \tag{4.21}
\end{equation*}
$$

In Eq. (4.21) we have used Gauss' theorem where $\partial V$ is the boundary of the volume $V$, so that we may write $j^{K} d S_{K}=j^{1} d x^{2} \wedge d x^{3}+j^{2} d x^{3} \wedge d x^{1}+j^{3} d x^{1} \wedge$ $d x^{2}$. The last step of Eq. (4.21) is valid for infinite volumes, with surface at infinity, provided $\psi$ vanishes sufficiently fast there. Eq. (4.21) is an expression of conservation of (total) probability in time.

### 4.2 The current density in GR

The probability current density in general relativity (curved spacetime) is given by

$$
\begin{equation*}
j^{\alpha}=\bar{\psi} \bar{\gamma}^{\alpha}(x) \psi \tag{4.22}
\end{equation*}
$$

where $\psi$ is a solution of Eq. (1.31), the $\bar{\gamma}^{\alpha}(x)$ are given by Eq. (1.3) and $\bar{\psi}=\psi^{\dagger} \gamma^{0}$. The curved spacetime proof of Eq. (4.3) is given in [3] and [8], p. 145 , and follows similar steps as the above derivation except that partial derivatives become covariant derivatives and, of course, one has to use Eq. (1.31) instead of the Minkowski spacetime Dirac equation. A discussion of the generalization of Eq. (4.21) is given in Appendix E in ref. [9]. We also refer the reader to our Proposition 3 below.

### 4.3 The Scalar product in SR

We now define the usual scalar product (see for example ref. [16], p. 69)

$$
\begin{equation*}
(\phi \mid \psi)=\int_{\Sigma} \bar{\phi} \gamma^{0} \psi d^{3} x . \tag{4.23}
\end{equation*}
$$

In the case where our spacelike hypersurface $\Sigma$ is not the usual $\left(t_{0}, x, y, z\right)$ hyperplane, Eq. (4.23) generalizes to

$$
\begin{equation*}
(\phi \mid \psi)=\int_{\Sigma} \bar{\phi} \gamma^{A} n_{A} \psi d \Sigma, \tag{4.24}
\end{equation*}
$$

where $n$ is the future-directed normal to $\Sigma$, and $d \Sigma$ is the invariant "volume element" on $\Sigma$. The probability integral $(\psi \mid \psi)$ is then given by

$$
\begin{equation*}
(\psi \mid \psi)=\int_{\Sigma} j^{A} n_{A} d \Sigma \tag{4.25}
\end{equation*}
$$

In special relativity one usually chooses $\Sigma$ to be the $t=0$ hyperplane.

### 4.4 Scalar product in GR

We follow ref. [37] and define the scalar product

$$
\begin{equation*}
(\phi \mid \psi)=\int_{\Sigma} \bar{\phi} \bar{\gamma}^{\alpha}(x) n_{\alpha} \psi d \Sigma \tag{4.26}
\end{equation*}
$$

where $\bar{\phi}=\phi^{\dagger} \gamma^{0}$, and the $\bar{\gamma}^{\alpha}(x)$ are given by Eq. (1.3). The vector $n$ is the future-directed normal to the spacelike Cauchy hypersurface $\Sigma$, and $d \Sigma$ is the invariant "volume element" on $\Sigma$. Using Eq. (4.22) we have that the probability integral $(\psi \mid \psi)$ is given by

$$
\begin{equation*}
(\psi \mid \psi)=\int_{\Sigma} j^{\alpha} n_{\alpha} d \Sigma \tag{4.27}
\end{equation*}
$$

We briefly comment on Parker's definitions [3], [39]. Parker defines the current of Eq. (4.22) with a minus sign in front. This is necessary because he has chosen a representation where $\left(\gamma^{0}\right)^{2}=-I$ and metric the signature $(-,+,+,+)$ (see also his argument regarding the positive definiteness of $(\psi \mid \psi)$ around his Eq. (3.5)). In addition he defines the scalar product

$$
\begin{equation*}
(\phi \mid \psi)=-\int_{\Sigma} \bar{\phi} \bar{\gamma}^{0}(x) \psi \sqrt{-g} d^{3} x \tag{4.28}
\end{equation*}
$$

where the integration is over a constant $x^{0}$ Cauchy hypersurface $\left(d^{3} x=d \Sigma_{t}\right)$. Parker is essentially using the lapse and shift formulation [40], where the metric is written in the $(3+1)$ decomposition (in his signature)

$$
\begin{equation*}
d s^{2}=-N^{2} d t^{2}+h_{\alpha \beta}\left(d x^{\alpha}+N^{\alpha} d t\right)\left(d x^{\beta}+N^{\beta} d t\right) . \tag{4.29}
\end{equation*}
$$

In Eq. (4.29) $N$ and $N^{\alpha}$ are the lapse and shift functions respectively, $h_{\alpha \beta}$ is the induced metric on $\Sigma_{t}$ and one can show that

$$
\begin{equation*}
\sqrt{-g}=N \sqrt{h} \tag{4.30}
\end{equation*}
$$

Therefore definitions (4.26) and (4.28) agree.
We would now like to prove Proposition 3 below.

## Proposition 3.

$$
\begin{equation*}
D_{\mu}\left(\bar{\phi} \bar{\gamma}^{\mu} \psi\right)=0 \tag{4.31}
\end{equation*}
$$

The proof of this proposition is simple provided one has certain preliminary results available. So we first derive the required results.

We give the derivation implied in ref. [4], Eq. (21), in order to find the expression for $D_{\mu} \bar{\psi}$, where $\psi$ is a solution of the Dirac equation. We use the fact that $\bar{\psi} \psi$ is a 0 -form field, therefore

$$
\begin{align*}
D_{\mu}(\bar{\psi} \psi) & =\left(D_{\mu} \bar{\psi}\right) \psi+\bar{\psi} D_{\mu} \psi  \tag{4.32}\\
& =\left(D_{\mu} \bar{\psi}\right) \psi+\bar{\psi}\left(I \partial_{\mu}+\Gamma_{\mu}\right) \psi  \tag{4.33}\\
& \equiv\left(I \partial_{\mu} \bar{\psi}+\mathbb{G}_{\mu} \bar{\psi}\right) \psi+\bar{\psi} \partial_{\mu} \psi+\bar{\psi} \Gamma_{\mu} \psi  \tag{4.34}\\
& =\left(\partial_{\mu} \bar{\psi}\right) \psi+\bar{\psi} \partial_{\mu} \psi \tag{4.35}
\end{align*}
$$

Eqs. (4.34) and (4.35) imply that

$$
\begin{align*}
\left(\mathbb{G}_{\mu} \bar{\psi}\right) \psi+\bar{\psi} \Gamma_{\mu} \psi & =0  \tag{4.36}\\
\left(\mathbb{G}_{\mu} \bar{\psi}+\bar{\psi} \Gamma_{\mu}\right) \psi & =0  \tag{4.37}\\
\mathbb{G}_{\mu} \bar{\psi}+\bar{\psi} \Gamma_{\mu} & =0 \tag{4.38}
\end{align*}
$$

Thus

$$
\begin{equation*}
\mathbb{G}_{\mu} \bar{\psi}=-\bar{\psi} \Gamma_{\mu} \tag{4.39}
\end{equation*}
$$

so, finally, using Eqs. (4.33), (4.34), and (4.39) we may write

$$
\begin{equation*}
D_{\mu} \bar{\psi}=I \partial_{\mu} \bar{\psi}-\bar{\psi} \Gamma_{\mu} \tag{4.40}
\end{equation*}
$$

Now we would like to find the adjoint of the Dirac equation in curved spacetime, i.e.,

$$
\begin{equation*}
\left(\bar{\gamma}^{\mu} D_{\mu} \psi+i m \psi\right)^{\dagger}=0 \tag{4.41}
\end{equation*}
$$

The derivation requires three steps. The first step to note that

$$
\begin{equation*}
\left(\bar{\gamma}^{\mu}\right)^{\dagger}=e_{A}{ }^{\mu}\left(\gamma^{A}\right)^{\dagger}=e_{A}^{\mu} \gamma^{0} \gamma^{A} \gamma^{0}=\gamma^{0} \bar{\gamma}^{\mu} \gamma^{0} \tag{4.42}
\end{equation*}
$$

The second step is to recall the expression for $\Gamma_{\mu}$ and obtain the result below

$$
\begin{align*}
\Gamma_{\mu}^{\dagger} & =\frac{1}{4} \omega_{A B \mu}\left(\gamma^{A} \gamma^{B}\right)^{\dagger}  \tag{4.43}\\
& =\frac{1}{4} \omega_{A B \mu}\left(\gamma^{B}\right)^{\dagger}\left(\gamma^{A}\right)^{\dagger}  \tag{4.44}\\
& =\frac{1}{4} \omega_{A B \mu}\left(\gamma^{0} \gamma^{B} \gamma^{0} \gamma^{0} \gamma^{A} \gamma^{0}\right)  \tag{4.45}\\
& =\frac{1}{4} \omega_{A B \mu}\left(\gamma^{0} \gamma^{B} \gamma^{A} \gamma^{0}\right)  \tag{4.46}\\
& =-\frac{1}{4} \omega_{A B \mu}\left(\gamma^{0} \gamma^{A} \gamma^{B} \gamma^{0}\right)  \tag{4.47}\\
& =-\gamma^{0} \Gamma_{\mu} \gamma^{0} \tag{4.48}
\end{align*}
$$

In the third and final step we make use of Eqs. (4.42) and (4.48) and re-write Eq. (4.41) as follows,

$$
\begin{align*}
\left(D_{\mu} \psi\right)^{\dagger}\left(\bar{\gamma}^{\mu}\right)^{\dagger}-i m \psi^{\dagger} & =0,  \tag{4.49}\\
{\left[\left(I \partial_{\mu}+\Gamma_{\mu}\right) \psi\right]^{\dagger}\left(\bar{\gamma}^{\mu}\right)^{\dagger}-i m \psi^{\dagger} } & =0,  \tag{4.50}\\
\left(\partial_{\mu} \psi^{\dagger}+\psi^{\dagger} \Gamma_{\mu}^{\dagger}\right)\left(\bar{\gamma}^{\mu}\right)^{\dagger}-i m \psi^{\dagger} & =0,  \tag{4.51}\\
\left(\partial_{\mu} \psi^{\dagger}\right) \gamma^{0} \bar{\gamma}^{\mu} \gamma^{0}-\psi^{\dagger} \gamma^{0} \Gamma_{\mu} \gamma^{0} \gamma^{0} \bar{\gamma}^{\mu} \gamma^{0}-i m \psi^{\dagger} & =0,  \tag{4.52}\\
\left(\partial_{\mu} \bar{\psi}\right) \bar{\gamma}^{\mu} \gamma^{0}-\bar{\psi} \Gamma_{\mu} \bar{\gamma}^{\mu} \gamma^{0}-i m \psi^{\dagger} & =0,  \tag{4.53}\\
\left(\partial_{\mu} \bar{\psi}\right) \bar{\gamma}^{\mu}-\bar{\psi} \Gamma_{\mu} \bar{\gamma}^{\mu}-i m \bar{\psi} & =0,  \tag{4.54}\\
\left(I \partial_{\mu} \bar{\psi}-\bar{\psi} \Gamma_{\mu}\right) \bar{\gamma}^{\mu}-i m \bar{\psi} & =0,  \tag{4.55}\\
\left(D_{\mu} \bar{\psi}\right) \bar{\gamma}^{\mu}-i m \bar{\psi} & =0 \tag{4.56}
\end{align*}
$$

Finally, we return to the proof of Proposition 3. We assume that $\phi$ and $\psi$ satisfy the Dirac Eq. (1.32) and its adjoint Eq. (4.56), namely,

$$
\begin{equation*}
i \bar{\gamma}^{\mu} D_{\mu} \psi-m \psi=0, \quad i\left(D_{\mu} \bar{\psi}\right) \bar{\gamma}^{\mu}+m \bar{\psi}=0 \tag{4.57}
\end{equation*}
$$

from which it follows that,

$$
\begin{equation*}
\bar{\gamma}^{\mu} D_{\mu} \psi=-i m \psi, \quad\left(D_{\mu} \bar{\psi}\right) \bar{\gamma}^{\mu}=i m \bar{\psi} \tag{4.58}
\end{equation*}
$$

The proof of Proposition 3 now follows by applying the Leibniz rule to Eq. (4.31) and using Eqs. (4.58) and (1.20).

Proposition 4. The scalar product, $(\phi \mid \psi)$, Eq. (4.28), is conserved, i.e.,

$$
\begin{equation*}
\frac{d}{d t}(\phi \mid \psi)=0 \tag{4.59}
\end{equation*}
$$

where $t=x^{0}$, and provided $\phi$ and $\psi$ satisfy the Dirac Eq. (1.30) and its adjoint.
Proposition 4 follows from Proposition 3 provided, as stated in ref. [3], "we assume that $\phi$ and $\psi$, vanish sufficiently rapidly at spatial infinity or obey suitable boundary conditions in a closed universe, so that the spatial components of Eq. (4.31) give vanishing contributions upon integration and the various products are well defined" (c.f. Eq. (4.21)). For more details refer to [3] and [39].

### 4.5 Example. The closed FRW universe

In this section we go over the example from Finster and Reintjes, ref. [37]. We consider the closed FRW universe whose line element, in conformal coordinates, is (this is in lapse and shift form, Eq. (4.29))

$$
\begin{equation*}
d s^{2}=S(\eta)^{2}\left(d \eta^{2}-d \chi^{2}-f(\chi)^{2}\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right)\right) \tag{4.60}
\end{equation*}
$$

In the metric (4.60) $\eta$ is the conformal time, $\chi$ is the radial coordinate, and $\theta \in(0, \pi), \phi \in[0,2 \pi)$, are the angular coordinates. The scale function $S(\eta)$ depends on the type of matter under consideration. In the present example $S(\eta)$ is left unspecified and is an arbitrary positive function. We have

$$
f(\chi)=\left\{\begin{array}{lll}
\sin (\chi), & \text { closed universe, } & \chi \in(0, \pi)  \tag{4.61}\\
\sinh (\chi), & \text { open universe, } & \chi>0 \\
\chi, & \text { flat universe }, & \chi>0
\end{array}\right.
$$

We choose the tetrad vectors in the Cartesian gauge (see Sec. A.3). In this example these tetrad vectors correspond to a class of static observers on the timelike path $\sigma(s)=\left(\int_{0}^{s} \frac{d s}{S(\eta(s))}, \chi_{0}, \theta_{0}, \phi_{0}\right)$. One can show that $\nabla_{e_{0}} e_{A}=0$, for
all $A$ and any well-behaved $S(\eta)$, so the tetrad $e_{A}$ is a Fermi tetrad field, Eq. (A.15).

$$
\begin{align*}
& e_{0}=\frac{1}{S(\eta)} \partial_{\eta},  \tag{4.62}\\
& e_{1}=\frac{\sin \theta \cos \phi}{S(\eta)} \partial_{\chi}+\frac{\cos \theta \cos \phi}{S(\eta) f(\chi)} \partial_{\theta}-\frac{\sin \phi}{S(\eta) f(\chi) \sin \theta} \partial_{\phi},  \tag{4.63}\\
& e_{2}=\frac{\sin \theta \sin \phi}{S(\eta)} \partial_{\chi}+\frac{\cos \theta \sin \phi}{S(\eta) f(\chi)} \partial_{\theta}+\frac{\cos \phi}{S(\eta) f(\chi) \sin \theta} \partial_{\phi},  \tag{4.64}\\
& e_{3}=\frac{\cos \theta}{S(\eta)} \partial_{\chi}-\frac{\sin \theta}{S(\eta) f(\chi)} \partial_{\theta}, \tag{4.65}
\end{align*}
$$

and using Eq. (1.3) we obtain the spacetime-dependent gamma matrices,

$$
\begin{align*}
& \bar{\gamma}^{\eta}=\frac{1}{S(\eta)} \gamma^{0}  \tag{4.66}\\
& \bar{\gamma}^{\chi}=\frac{1}{S(\eta)}\left(\sin \theta \cos \phi \gamma^{1}+\sin \theta \sin \phi \gamma^{2}+\cos \theta \gamma^{3}\right)  \tag{4.67}\\
& \bar{\gamma}^{\theta}=\frac{1}{S(\eta) f(\chi)}\left(\cos \theta \cos \phi \gamma^{1}+\cos \theta \sin \phi \gamma^{2}-\sin \theta \gamma^{3}\right)  \tag{4.68}\\
& \bar{\gamma}^{\phi}=\frac{1}{S(\eta) f(\chi) \sin \theta}\left(-\sin \phi \gamma^{1}+\cos \phi \gamma^{2}\right) \tag{4.69}
\end{align*}
$$

In Eqs. (4.66) - (4.69) the $\gamma^{A}$ are the constant $\gamma$ matrices in the standard representation.

We write the Dirac equation as in [37]

$$
\begin{equation*}
\left[i \bar{\gamma}^{\eta}\left(\partial_{\eta}+\frac{3}{2} \frac{\dot{S}}{S}\right)+i \bar{\gamma}^{\chi}\left(\partial_{\chi}+\frac{f^{\prime}-1}{f}\right)+i \bar{\gamma}^{\theta} \partial_{\theta}+i \bar{\gamma}^{\phi} \partial_{\phi}-m\right] \Psi=0 \tag{4.70}
\end{equation*}
$$

where $\dot{S}$ is the derivative with respect to $\eta$, and $f^{\prime}$ is the derivative with respect to $\chi$. Note that $3 \dot{S} /(2 S) \neq \Gamma_{\eta}$, etc. What happens here is similar to what happened in deriving Eq. (2.71).

At this point Finster and Reintjes restrict themselves to the closed universe case, $f(\chi)=\sin (\chi)$, and assume a solution of the form

$$
\begin{equation*}
\Psi(\eta, \chi, \theta, \phi)=\frac{1}{S(\eta)^{\frac{3}{2}}}\binom{h_{1}(\eta) \psi_{\lambda}(\chi, \theta, \phi)}{h_{2}(\eta) \tilde{\psi}_{\lambda}(\chi, \theta, \phi)} \tag{4.71}
\end{equation*}
$$

We shall examine the probability integral, Eq. (4.27), using the above solution of the Dirac equation in a closed FRW universe. We choose $\Sigma$ to be a slice of constant conformal time $\eta$, then the future-directed normal 1-form $n$ has components (cf. Eq. (4.62)),

$$
\begin{equation*}
\left(n_{\eta}, n_{\chi}, n_{\theta}, n_{\phi}\right)=(S(\eta), 0,0,0) \tag{4.72}
\end{equation*}
$$

Using Eqs. (4.27) and (4.66), we have

$$
\begin{align*}
(\Psi \mid \Psi) & =\int_{\Sigma} \bar{\Psi} \bar{\gamma}^{\alpha} \Psi n_{\alpha} d \Sigma  \tag{4.73}\\
& =\int_{\Sigma} \bar{\Psi} \bar{\gamma}^{\eta} \Psi n_{\eta} d \Sigma  \tag{4.74}\\
& =\int_{\Sigma} \Psi^{\dagger} \Psi d \Sigma \tag{4.75}
\end{align*}
$$

In going from Eq. (4.73) to Eq. (4.75), we used Eqs. (4.66) and (4.72). Also

$$
\begin{equation*}
d \Sigma=\sqrt{\left|g_{\Sigma}\right|} d \chi d \theta d \phi=S^{3}(\eta) d \mu_{S^{3}} \tag{4.76}
\end{equation*}
$$

In Eq. (4.76), $g_{\Sigma}$ is the determinant of the induced metric on $\Sigma$, and

$$
\begin{equation*}
d \mu_{S^{3}}=\sin ^{2}(\chi) \sin (\theta) d \chi d \theta d \phi \tag{4.77}
\end{equation*}
$$

is the volume element on the unit sphere $S^{3}$ in hyperspherical coordinates. So substituting Eq. (4.71) in Eq. (4.73), we obtain

$$
\begin{align*}
(\Psi \mid \Psi) & =\int_{\Sigma} \Psi^{\dagger} \Psi d \Sigma  \tag{4.78}\\
& =\left|h_{1}\right|^{2} \int_{S^{3}}\left|\psi_{\lambda}\right|^{2} d \mu_{S^{3}}+\left|h_{2}\right|^{2} \int_{S^{3}}\left|\tilde{\psi}_{\lambda}\right|^{2} d \mu_{S^{3}}  \tag{4.79}\\
& =\left|h_{1}\right|^{2}+\left|h_{2}\right|^{2} \tag{4.80}
\end{align*}
$$

where we have followed the normalization of ref. [37] (cf. Eq. (D.60)). It follows from Eq. (4.59) that the probability integral is constant in time.

## Appendices

## A Tetrads

## A. 1 The tetrad formalism

A tetrad is a set of four linearly independent vectors that can be defined at each point in a (semi -) Riemannian spacetime. Here we give a summary of useful relations for tetrad fields. Good detailed discussions can be found in several texts see, for example, appendix J of [9]. We have the following basic relations that determine the vector fields $e_{A}{ }^{\alpha}$ or the 1-forms (covector fields) $e^{A}{ }_{\alpha}$, (we may use the notation, $e^{A}=e^{A}{ }_{\alpha} d x^{\alpha}$ and $e_{A}=e_{A}{ }^{\alpha} \partial_{\alpha}$ ). The tetrads by definition satisfy the relations (see [9], Eq. (J.3))

$$
\begin{gather*}
e_{\alpha}^{A} e_{A}^{\beta}=\delta_{\alpha}{ }^{\beta}  \tag{A.1}\\
e_{\alpha}^{A} e_{B}^{\alpha}=\delta_{B}^{A} \tag{A.2}
\end{gather*}
$$

The choice of the tetrad field determines the metric through Eq. (A.3) below.

$$
\begin{align*}
& g_{\alpha \beta}=e_{\alpha}^{A} e_{\beta}^{B} \eta_{A B},  \tag{A.3}\\
& \eta_{A B}=e_{A}^{\alpha} e_{B}^{\beta} g_{\alpha \beta} \tag{A.4}
\end{align*}
$$

where $\eta_{A B}$ is the Minkowski spacetime metric in Cartesian coordinates. We shall always assume that the velocity vector field, $e_{0}$, is tangent to a congruence of timelike paths and thus the tetrads are moving along these paths. The reader should also read the comments in Sec. 1.1 below Eq. (1.3), and in Sec. 1.4 below Eq. (1.76).

Under coordinate transformations, greek indices are treated as tensor indices, while latin indices are merely labels (thus the $e_{A}{ }^{\alpha}$ represent four different vector fields). Equations (A.4) are also a statement of the orthonormality of the vectors $e_{A}{ }^{\alpha}$. The tetrad components may be determined using the Eqs. (A.3) or (A.4).

Remark 11. It is easy to convince oneself that relabeling the subscripts (or superscripts) of the $e_{A}{ }^{\alpha}$ in a consistent way, does not affect the relation (A.4). However, problems may arise, if one is careless with relabeling and reordering variables while using a symbolic manipulation software.

Although in these notes we have considered spacetimes with dimensionality of two or four, in general, if $n$ is the dimensionality of the manifold, Eqs. (A.4) are a set of $\left(\frac{1}{2}\right) n(n+1)$ equations for the $n^{2}$ unknown components of the vielbein $e_{A}{ }^{\alpha}$. Therefore $\left(\frac{1}{2}\right) n(n-1)$ components can be freely chosen or determined by extra conditions.

Exercise 4. It is a simple exercise to show that Eqs. (A.2) and (A.3), imply Eq. (A.4), while Eqs. (A.1) and (A.4), imply Eq. (A.3).

We have the following rules for raising and lowering indices,

$$
\begin{align*}
& e_{A \alpha}=g_{\alpha \beta} e_{A}^{\beta},  \tag{A.5}\\
& e_{\alpha}^{A}=\eta^{A B} e_{B \alpha} . \tag{A.6}
\end{align*}
$$

The components of tensors in the tetrad frame are given by relations such as the ones below

$$
\begin{gather*}
V^{A}=e^{A}{ }_{\alpha} V^{\alpha},  \tag{A.7}\\
T_{B}^{A}=e^{A}{ }_{\alpha} e_{B}{ }^{\beta} T^{\alpha}{ }_{\beta}, \tag{A.8}
\end{gather*}
$$

and so on. Note that in Eq. (A.7) we are taking the product of a vector $V$ with $n 1$-forms $e^{A}{ }_{\alpha}$, as a result, we are replacing the vector $V$ with $n$ scalars $V^{A}$. Likewise in Eq. (A.8), we are replacing the $\binom{1}{1}$ tensor T with $n^{2}$ scalars $T_{B}^{A}$ [41].

We can obtain the tensor components in the global chart from the "components" in the tetrad frame using relations like the one below

$$
\begin{equation*}
V^{\alpha}=e_{A}{ }^{\alpha} V^{A}=e^{A \alpha} V_{A} \tag{A.9}
\end{equation*}
$$

Using the above relations we can show that $U_{\mu} V^{\mu}=U_{A} V^{A}$.

$$
\begin{align*}
g_{\mu \nu} U^{\mu} V^{\nu} & =\eta_{A B} e^{A}{ }_{\mu} e^{B}{ }_{\nu} U^{\mu} V^{\nu},  \tag{A.10}\\
g_{\mu \nu} U^{\mu} V^{\nu} & =\eta_{A B} U^{A} V^{B},  \tag{A.11}\\
U_{\mu} V^{\mu} & =U_{A} V^{A} . \tag{A.12}
\end{align*}
$$

## A. 2 Fermi tetrad fields

A Fermi tetrad field must satisfy some special conditions. As usual the tetrad field, satisfies Eq. (A.3), etc., but in the Fermi case, the velocity vector field, $e_{0}$, is tangent to a congruence of timelike geodesics, $\sigma(\tau)$, parametrized by the proper time $\tau$. Thus

$$
\begin{equation*}
e_{0}=\frac{d \sigma(\tau)}{d \tau} \tag{A.13}
\end{equation*}
$$

and therefore,

$$
\begin{equation*}
\nabla_{e_{0}} e_{0}=0 \tag{A.14}
\end{equation*}
$$

A Fermi tetrad field must satisfy the equations

$$
\begin{equation*}
\nabla_{e_{0}} e_{A}=0, \quad A=0,1,2,3 \tag{A.15}
\end{equation*}
$$

so that all the tetrad vectors are parallelly transported along the chosen congruence of timelike geodesics.
(Recall that if $\nabla_{u} u \neq 0$ but $\nabla_{u} v=0$, then $v$ is parallel transported but not on a geodesic).

Remark 12. Given a Fermi tetrad enables one to obtain approximate Fermi coordinates by the well-known process given in [42]. Examples of how to obtain exact Fermi coordinates, in cases where this is possible, were given in [29], [30], [43], [44].

## A. 3 The Cartesian gauge tetrad

A tetrad field referred to as the "Cartesian gauge" was introduced by Brill and Wheeler [45], and has been found useful by many authors [46] - [51]). An example of the Cartesian gauge tetrad used in the FRW universe was given above, Eqs. (4.62) - (4.65). Here we first discuss it in the simplest case namely flat Lorentz spacetime. Consider the standard tetrad 1-forms in (Cartesian) Minkowski spacetime, namely,

$$
\begin{array}{ll}
e^{0}=d t, & e^{1}=d x \\
e^{2}=d y, & e^{3}=d z \tag{A.16}
\end{array}
$$

Then, if we transform to spherical coordinates $(t, r, \theta, \phi)$, the above 1-forms transform into the 1-form tetrad below

$$
\begin{align*}
& \omega^{0}=d t  \tag{A.17}\\
& \omega^{1}=\sin \theta \cos \phi d r+r \cos \theta \cos \phi d \theta-r \sin \theta \sin \phi d \phi  \tag{A.18}\\
& \omega^{2}=\sin \theta \sin \phi d r+r \cos \theta \sin \phi d \theta+r \sin \theta \cos \phi d \phi  \tag{A.19}\\
& \omega^{3}=\cos \theta d r-r \sin \theta d \theta \tag{A.20}
\end{align*}
$$

For the case of the flat spacetime the Fock-Ivanenko coefficients vanish with this tetrad in spherical coordinates. The metric is the usual spherical coordinate metric,

$$
\begin{equation*}
d s^{2}=d t^{2}-d r^{2}-r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right) \tag{A.21}
\end{equation*}
$$

Note that the default 1-form tetrad for the metric of Eq. (A.21), namely,

$$
\begin{align*}
& \omega^{0}=d t  \tag{A.22}\\
& \omega^{1}=d r  \tag{A.23}\\
& \omega^{2}=r d \theta  \tag{А.24}\\
& \omega^{3}=r \sin \theta d \phi \tag{A.25}
\end{align*}
$$

is "rotated" (see Appendix A.4) with respect to the original Cartesian tetrad Eq. (A.16) and the Fock-Ivanenko coefficients no longer vanish.

Exercise 5. (a) Write the Dirac equation using the tetrad, (A.17) - (A.20), for the metric (A.21), in the chiral representation Eq. (B.11). (b) Transform the chiral plane wave solutions Eq. (D.43) to spherical coordinates and show that they satisfy the Dirac equation obtained in part (a).

## A. 4 Vielbeins, spinors and Lorentz matrices

In this section we present the results without proofs and we refer the reader to the proofs given in [1]. We mention for the sake of clarity that if $F \in G$, where $G$ is a group of coordinate transformations, then we write

$$
\begin{equation*}
\bar{x}=F x . \tag{A.26}
\end{equation*}
$$

Thus in general for a scalar function, $\phi(x)$, we have

$$
\begin{equation*}
\phi(x)=\bar{\phi}(\bar{x}) . \tag{A.27}
\end{equation*}
$$

If in a coordinate system $\left(x^{0}, x^{i}\right)$, we change from an initial chosen vielbein ${ }^{3}$ set, $h_{A}$, to another set, $e_{A}$, then the new vielbein vectors can be expressed as linear combinations of the old ${ }^{4}$,

$$
\begin{equation*}
e_{A}{ }^{\mu}=\Lambda_{A}{ }^{B} h_{B}{ }^{\mu} . \tag{A.28}
\end{equation*}
$$

However, both vielbein sets must satisfy Eq. (A.4), i.e.,

$$
\begin{align*}
& \eta_{A B}=h_{A}{ }^{\alpha} h_{B}{ }^{\beta} g_{\alpha \beta},  \tag{A.29}\\
& \eta_{A D}=e_{A}{ }^{\mu} e_{D}{ }^{\nu} g_{\mu \nu} . \tag{A.30}
\end{align*}
$$

Substituting Eq. (A.28) in Eq. (A.29), we obtain

$$
\begin{align*}
\Lambda_{A}{ }^{B} h_{B}{ }^{\mu} \Lambda_{D}{ }^{C} h_{C}{ }^{\nu} g_{\mu \nu} & =\eta_{A D},  \tag{A.31}\\
\Lambda_{A}^{B} \Lambda_{D}{ }^{C} \eta_{B C} & =\eta_{A D},  \tag{A.32}\\
\Lambda^{T} \eta \Lambda & =\eta, \tag{A.33}
\end{align*}
$$

where $\Lambda^{T}$ is the transpose of $\Lambda$. From Eq. (A.33) we have that $\operatorname{det} \Lambda= \pm 1$. Thus it follows then from Eq. (A.32) that $\Lambda_{A}{ }^{B}$ is a Lorentz matrix. So in the

[^1]context of general relativity the Lorentz group is the group of vielbein rotations [1] p. 143. We also remark that the $\Lambda$ matrices will in general be spacetimedependent.

Under coordinate transformations spinors, $\psi$, behave like scalars so that, [1] p. 147,

$$
\begin{equation*}
\binom{\bar{\phi}(\bar{x})}{\bar{\chi}(\bar{x})}=\binom{\phi(x)}{\chi(x)} . \tag{A.34}
\end{equation*}
$$

However when a vielbein $h_{B}$, is rotated by $\Lambda$ as in Eq. (A.28), then

$$
\begin{equation*}
\psi_{e}=L \psi_{h}, \tag{A.35}
\end{equation*}
$$

where $L$ is a (spacetime-dependent) spinor representative of a vielbein rotation $\Lambda$, [1] pp. 76, 147,

$$
L=\left(\begin{array}{cc}
S & 0  \tag{А.36}\\
0 & \left(S^{\dagger}\right)^{-1}
\end{array}\right),
$$

with $\operatorname{det}(L)=1$, [52], that satisfies the relations, [1] p. 147,

$$
\begin{equation*}
L^{-1} \gamma^{A} L=\Lambda_{B}^{A} \gamma^{B} \tag{А.37}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma^{0} L^{\dagger} \gamma^{0}=L^{-1} \tag{A.38}
\end{equation*}
$$

Given in [8], Eq. (5.396), p.246. For a derivation of Eq. (A.37) see, e.g., [53].

## B The gamma matrices

## B. 1 General summary

The $2 \times 2$ Pauli spin matrices are

$$
\sigma^{1}=\left(\begin{array}{cc}
0 & 1  \tag{B.1}\\
1 & 0
\end{array}\right), \quad \sigma^{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), \quad \sigma^{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

For a free spin $1 / 2$ particle of mass $m$ we write the Dirac equation in Minkowski spacetime as

$$
\begin{equation*}
i \gamma^{A} \partial_{A} \psi-m \psi=0 \tag{B.2}
\end{equation*}
$$

where $\psi$ is a 4 -component (contravariant) spinor and the $4 \times 4 \gamma$ (constant) matrices ${ }^{5}$, satisfy the anticommutation relation

$$
\begin{equation*}
\left\{\gamma^{A}, \gamma^{B}\right\}=\varepsilon 2 \eta^{A B} I \tag{B.3}
\end{equation*}
$$

where $\varepsilon= \pm 1$, and the Hermiticity conditions

$$
\begin{equation*}
\left(\gamma^{A}\right)^{\dagger}=\gamma^{0} \gamma^{A} \gamma^{0} \tag{B.4}
\end{equation*}
$$

[^2]We raise and lower the indices using the metric $\eta$, e.g., $\gamma^{A}=\eta^{A B} \gamma_{B}$.
With the exception of the Jauch-Rohrlich representation below, the other three representations are given in a form so that they satisfy the relation

$$
\begin{equation*}
\left\{\gamma^{A}, \gamma^{B}\right\}=2 \eta^{A B} I \tag{B.5}
\end{equation*}
$$

with signature convention $(+,-,-,-)$. We also define the matrix

$$
\begin{equation*}
\gamma^{5}:=i \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3} \tag{B.6}
\end{equation*}
$$

which satisfies the representation and signature independent relations,

$$
\begin{equation*}
\left\{\gamma^{A}, \gamma^{5}\right\}=0, \quad\left(\gamma^{5}\right)^{2}=I_{4}, \quad\left(\gamma^{5}\right)^{\dagger}=\gamma^{5} \tag{B.7}
\end{equation*}
$$

Remark 13. The sign choice in Eq. (B.3) depends on the metric sign convention and the representation of the $\gamma$ matrices. There are several commonly used representations each with its own advantages. One can avoid the $(\varepsilon=-1)$ choice in Eq. (B.3) by multiplying the $\gamma$ matrices with $\pm i$, (e.g., both [3] and [55] multiply by $-i$ ).

Remark 14. We also point out that the (-) sign in front of the mass $m$ in the Dirac equation (B.2), can be changed to a $(+$ ) by multiplying the Dirac equation (from the left) by $\gamma^{5}$. One finds that the spinor $\gamma^{5} \psi$ obeys the Dirac equation with the opposite sign in the mass term. This is true also in curved spacetime, see (1.30).

## B. 2 The standard or Dirac-Pauli representation

In the standard or Dirac-Pauli representation, (or the Bjorken-Drell representation) we have

$$
\gamma^{0}=\left(\begin{array}{cc}
I_{2} & 0  \tag{B.8}\\
0 & -I_{2}
\end{array}\right), \quad \gamma^{K}=\left(\begin{array}{cc}
0 & \sigma^{K} \\
-\sigma^{K} & 0
\end{array}\right), \quad K=(1,2,3)
$$

It is easy to verify that,

$$
\begin{equation*}
\left(\gamma^{0}\right)^{2}=I, \quad\left(\gamma^{K}\right)^{2}=-I \tag{B.9}
\end{equation*}
$$

We also give below the Dirac $\beta$ and $\alpha^{K}$ matrices,

$$
\beta=\left(\begin{array}{cc}
I_{2} & 0  \tag{B.10}\\
0 & -I_{2}
\end{array}\right), \quad \alpha^{K}=\left(\begin{array}{cc}
0 & \sigma^{K} \\
\sigma^{K} & 0
\end{array}\right), \quad K=(1,2,3),
$$

that is, $\gamma^{0}=\beta, \gamma^{K}=\beta \alpha^{K}$.

## B. 3 The chiral or Weyl representation

In the chiral or Weyl representation, there are two possible choices for $\gamma^{0}$, we choose

$$
\gamma^{0}=\left(\begin{array}{cc}
0 & -I_{2}  \tag{B.11}\\
-I_{2} & 0
\end{array}\right), \quad \gamma^{K}=\left(\begin{array}{cc}
0 & \sigma^{K} \\
-\sigma^{K} & 0
\end{array}\right), \quad K=(1,2,3)
$$

and

$$
\begin{equation*}
\left(\gamma^{0}\right)^{2}=I, \quad\left(\gamma^{K}\right)^{2}=-I \tag{B.12}
\end{equation*}
$$

Another option, in the chiral representation, is to choose the negative of the above $\gamma^{0}$, in which case $\gamma^{5}$ changes sign, unless one defines it as the negative of Eq. (B.13). Some authors define the chiral $\gamma$ matrices by multiplying all of the $\gamma^{\prime}$ 's in Eq. (B.11) by $(-1)$, then the $\gamma^{5}$ does not change sign. In any of the above-mentioned chiral representations the $\gamma^{5}$ given by Eq. (B.13) is equal to

$$
\gamma^{5}= \pm\left(\begin{array}{cc}
I_{2} & 0  \tag{B.13}\\
0 & -I_{2}
\end{array}\right)
$$

(see also Sec. D.4).

## B. 4 The Majorana representation

In the Majorana representation the $\gamma$ matrices are imaginary and the spinors are real.

$$
\begin{align*}
& \gamma^{0}=\left(\begin{array}{cc}
0 & \sigma^{2} \\
\sigma^{2} & 0
\end{array}\right), \quad \gamma^{1}=\left(\begin{array}{cc}
i \sigma^{3} & 0 \\
0 & i \sigma^{3}
\end{array}\right),  \tag{B.14}\\
& \gamma^{2}=\left(\begin{array}{cc}
0 & -\sigma^{2} \\
\sigma^{2} & 0
\end{array}\right), \quad \gamma^{3}=\left(\begin{array}{cc}
-i \sigma^{1} & 0 \\
0 & -i \sigma^{1}
\end{array}\right), \tag{B.15}
\end{align*}
$$

and

$$
\begin{equation*}
\left(\gamma^{0}\right)^{2}=I, \quad\left(\gamma^{K}\right)^{2}=-I \tag{B.16}
\end{equation*}
$$

## B. 5 The Jauch-Rohrlich representation

In the Jauch-Rorhlich representation [56], we have

$$
\gamma^{0}=-i\left(\begin{array}{cc}
I_{2} & 0  \tag{B.17}\\
0 & -I_{2}
\end{array}\right), \quad \gamma^{K}=\left(\begin{array}{cc}
0 & \sigma^{K} \\
\sigma^{K} & 0
\end{array}\right), \quad K=(1,2,3)
$$

in fact form Eq. (B.10) we have that,

$$
\begin{equation*}
\gamma^{0}=-i \beta, \quad \gamma^{K}=\alpha^{K} \tag{B.18}
\end{equation*}
$$

thus,

$$
\begin{equation*}
\left(\gamma^{0}\right)^{2}=-I, \quad\left(\gamma^{K}\right)^{2}=I \tag{B.19}
\end{equation*}
$$

and we satisfy

$$
\begin{equation*}
\left\{\gamma^{A}, \gamma^{B}\right\}=2 \eta^{A B} I \tag{B.20}
\end{equation*}
$$

with signature $(-,+,+,+)$.
Jauch and Rorhlich define $\gamma^{5} \equiv \gamma_{5} \equiv \gamma^{0} \gamma^{1} \gamma^{2} \gamma^{3}$, so again $\gamma^{5}$ satisfies Eqs. (B.7). Furthermore, instead of Eq. (B.2), we now have

$$
\begin{equation*}
\gamma^{A} \partial_{A} \psi+m \psi=0 \tag{B.21}
\end{equation*}
$$

## C Metric signatures, the FI coefficients, etc.

For easy reference we begin by recalling our definitions of the spin connection coefficients, $\omega_{A B \mu}$, the spinor affine connection, $\Gamma_{\mu}$, the Fock-Ivanenko coefficients, $\Gamma_{C}$, and the anticommutation relations of the $\gamma$ matrices.

$$
\begin{align*}
& \omega_{A B \mu}=g_{\beta \alpha} e_{A}^{\alpha} \nabla_{\mu} e_{B}^{\beta},  \tag{C.1}\\
& \Gamma_{\mu}=\frac{\varepsilon}{4} \omega_{A B \mu} \gamma^{A} \gamma^{B}  \tag{C.2}\\
& \Gamma_{C}=e_{C}{ }^{\mu} \Gamma_{\mu}  \tag{C.3}\\
& \left\{\gamma^{A}, \gamma^{B}\right\}=\varepsilon 2 \eta^{A B} I \tag{C.4}
\end{align*}
$$

## C. 1 Signature (-2)

It is clear that the sign of the $\omega_{A B \mu}$ coefficients depends on the signature because of the $g_{\beta \alpha}$ factor in Eq. (C.1). We now let $\varepsilon=+1$ in Eqs. (C.2), (C.4) and use any $\gamma$ matrix representation whose matrices satisfy

$$
\begin{equation*}
\left(\gamma^{0}\right)^{2}=I, \quad\left(\gamma^{K}\right)^{2}=-I \tag{C.5}
\end{equation*}
$$

and consequently Eq. (C.4). We then obtain $\Gamma_{\mu}$ and $\Gamma_{C}$ and we may write the Dirac equation as

$$
\begin{equation*}
i \gamma^{C}\left(e_{C}+\Gamma_{C}\right) \psi-m \psi=0 \tag{C.6}
\end{equation*}
$$

Remark 15. Following the above assumptions and steps in the software package Cartan, one will find that the resulting $\Gamma_{C}$ coefficients have the opposite sign from ours. This is because the $\Gamma_{C}$ coefficients are defined with the opposite sign in the software. However, this is compensated in CARTAN by inserting another minus sign so that the Dirac equation is now

$$
\begin{equation*}
i \gamma^{C}\left(e_{C}-\Gamma_{C}\right) \psi-m \psi=0, \quad(\mathrm{CARTAN}) \tag{C.7}
\end{equation*}
$$

thus identical to Eq. (C.6).

## C. 2 Signature (+2)

We begin by again letting $\varepsilon=+1$ in Eqs. (C.2), (C.4). In order to satisfy Eq. (C.4), we multiply the $\gamma$ matrices by $(+i)$ so that now, instead of Eqs. (C.5), we have

$$
\begin{equation*}
\left(\gamma^{0}\right)^{2}=-I, \quad\left(\gamma^{K}\right)^{2}=I \tag{C.8}
\end{equation*}
$$

It is easy to see from Eq. (C.2) that the change of signature will also change the sign of the coefficients $\omega_{A B \mu}$. Thus the latter sign change along with the product of the two $(+i)$ factors from the $\gamma$ matrices in Eq. (C.2), will give us the same $\Gamma_{C}$ coefficients as before (Sec. C.1). The Dirac equation is now written as

$$
\begin{equation*}
\gamma^{C}\left(e_{C}+\Gamma_{C}\right) \psi-m \psi=0 \tag{C.9}
\end{equation*}
$$

since the $(+i)$ factor has been absorbed in the $\gamma^{C}$.
Remark 16. The software package CARTAN, also uses $(+i)$ for this signature, but recall that the $\Gamma_{C}$ coefficients have the opposite sign from ours. So that Cartan's Dirac equation is now

$$
\begin{equation*}
\gamma^{C}\left(e_{C}-\Gamma_{C}\right) \psi-m \psi=0, \quad(\text { CARTAN }) \tag{C.10}
\end{equation*}
$$

A number of authors prefer to multiply their $\gamma$ matrices with a factor $(-i)$, e.g., [3], [55]. With the definitions in our paper or the ones in ref. [55] the Dirac equation would be

$$
\begin{equation*}
\gamma^{C}\left(e_{C}+\Gamma_{C}\right) \psi+m \psi=0 \tag{C.11}
\end{equation*}
$$

Parker in [3], using the Dirac $\beta$ and $\alpha^{K}$ matrices, Eq. (B.10), has $\gamma^{0}=\eta^{00} \gamma_{0}=$ $-i \beta$, and $\gamma^{K}=\gamma^{0} \alpha^{K}$. In addition Parker defines his $\Gamma_{\mu}$ with the opposite sign from the one adopted here and compensates with the usual ( - ) sign change in the Dirac equation. Thus his Dirac equation is

$$
\begin{equation*}
\gamma^{C}\left(e_{C}-\Gamma_{C}\right) \psi+m \psi=0 \tag{C.12}
\end{equation*}
$$

As another example we consider Ryder in ref. [11]. Ryder uses $\varepsilon=-1$ in Eqs. in Eqs. (C.2), (C.4), so he can use the usual $\gamma$ matrix representations with

$$
\begin{equation*}
\left(\gamma^{0}\right)^{2}=I, \quad\left(\gamma^{K}\right)^{2}=-I \tag{C.13}
\end{equation*}
$$

Note, however, that the $\varepsilon=-1$ along with the sign change due to the signature, ultimately gives the same $\Gamma_{\mu}$ and $\Gamma_{C}$ coefficients as ours obtained in Sec. C.1. Clearly, using $\varepsilon=-1$ is just completely equivalent to multiplying the $\gamma$ matrices with $( \pm i)$, except that now we don't have to hide the $(i)$ in the Dirac equation, which retains its standard form (see [11], Eq. (11.129)).

$$
\begin{equation*}
i \gamma^{C}\left(e_{C}+\Gamma_{C}\right) \psi-m \psi=0 \tag{C.14}
\end{equation*}
$$

Finally one may use the Jauch-Rohrlich representation (see Secs. 2.3 and B.3).

## D Dirac plane wave solutions in SR

## D. 1 Notation

In the calculations below we use the metric sign convention $(+,-,-,-)$. As usual $c=\hbar=1$. Thus we write the Minkowski metric as

$$
\begin{equation*}
d s^{2}=\eta_{\mu \nu} d x^{\mu} d x^{\nu}=d t^{2}-d x^{2}-d y^{2}-d z^{2} \tag{D.1}
\end{equation*}
$$

where $\mu$ and $\nu$ run over $(0,1,2,3)$ or $(t, x, y, z)$, Thus there is no distinction between the upper case latin indices used in Sec. 1 and the greek indices in the present section D. We write $p=\left(p^{0}, \boldsymbol{p}\right)$, where,

$$
\begin{align*}
& p^{0}=p_{0}  \tag{D.2}\\
& p^{j}=-p_{j}, \quad j=(1,2,3)=(x, y, z) \tag{D.3}
\end{align*}
$$

and

$$
\begin{align*}
p^{0} & =p_{0}=i \partial_{t}  \tag{D.4}\\
p^{j} & =-p_{j}=-i \partial_{j} . \tag{D.5}
\end{align*}
$$

## D. 2 The Dirac equation

The Dirac equation for a free, spin $1 / 2$, particle of mass $m$ in Minkowski spacetime is usually written as

$$
\begin{equation*}
i \gamma^{\mu} \partial_{\mu} \psi-m \psi=0 \tag{D.6}
\end{equation*}
$$

and introducing the Feynman "slash" notation,

$$
\begin{equation*}
\not p=\gamma^{\mu} p_{\mu} \tag{D.7}
\end{equation*}
$$

we may rewrite the Dirac Eq. (D.6) in the shorthand version

$$
\begin{equation*}
(\not p-m I) \psi=0 \tag{D.8}
\end{equation*}
$$

or,

$$
\begin{equation*}
i \gamma^{0} \partial_{t} \psi+i \gamma^{1} \partial_{x} \psi+i \gamma^{2} \partial_{y} \psi+i \gamma^{3} \partial_{z} \psi-m I \psi=0 \tag{D.9}
\end{equation*}
$$

where $\psi$ is a 4 -component spinor,

$$
\psi=\left(\begin{array}{l}
\psi_{1}  \tag{D.10}\\
\psi_{2} \\
\psi_{3} \\
\psi_{4}
\end{array}\right)
$$

The $\gamma$ matrices are reviewed in Appendix B.

## D. 3 Plane wave solutions in the standard representation

In this section we will write the plane wave solutions of the Dirac equation, first using the standard representation of the $\gamma$ matrices (see Appendix B), then we will show how this set of solutions can be transformed into the corresponding set in the chiral representation of the $\gamma$ matrices. The most useful references for some of the material below are [16], [57], [58].

In general, to obtain solutions of the Dirac equation, one would have to solve a set of coupled partial differential equations. For example, in the standard representation of the $\gamma$ matrices, the Dirac equation (D.9), for the $\psi_{i}$ of Eq. (D.10) becomes the set of coupled partial differential equations below

$$
\begin{array}{r}
i \partial_{t} \psi_{1}+i \partial_{z} \psi_{3}+i \partial_{x} \psi_{4}+\partial_{y} \psi_{4}-m \psi_{1}=0 \\
i \partial_{t} \psi_{2}+i \partial_{x} \psi_{3}-\partial_{y} \psi_{3}-i \partial_{z} \psi_{4}-m \psi_{2}=0 \\
-i \partial_{z} \psi_{1}-i \partial_{x} \psi_{2}-\partial_{y} \psi_{2}-i \partial_{t} \psi_{3}-m \psi_{3}=0 \\
-i \partial_{x} \psi_{1}+\partial_{y} \psi_{1}+i \partial_{z} \psi_{2}-i \partial_{t} \psi_{4}-m \psi_{4}=0 \tag{D.14}
\end{array}
$$

However in the present case we seek solutions for plane waves of the form

$$
\begin{align*}
\psi^{(+)} & =u(p) e^{-i p_{\mu} x^{\mu}}  \tag{D.15}\\
\psi^{(-)} & =v(p) e^{i p_{\mu} x^{\mu}} \tag{D.16}
\end{align*}
$$

where $\psi^{(+)}$will be the positive energy solutions and $\psi^{(-)}$the negative energy solutions. Substituting Eqs. (D.15) and (D.16) in Eq. (D.8), we find the set of algebraic equations below,

$$
\begin{align*}
& (\not p-m I) u(p)=0  \tag{D.17}\\
& (\not p+m I) v(p)=0 \tag{D.18}
\end{align*}
$$

We note that Eqs. (D.17) and (D.18) are systems of homogeneous equations for the components of $u(p)$ and $v(p)$. These systems will have a non-trivial solutions only if

$$
\begin{equation*}
\operatorname{det}(\not p \pm m I)=0 \tag{D.19}
\end{equation*}
$$

Equation (D.19) ${ }^{6}$ gives us the (representation independent) condition

$$
\begin{equation*}
\left(p^{2}-m^{2}\right)^{2}=0 \tag{D.20}
\end{equation*}
$$

where $p^{2}=\left(p^{0}\right)^{2}-\boldsymbol{p}^{2} \equiv E^{2}-\boldsymbol{p}^{2}$, and therefore Eq. (D.20) may be rewritten as

$$
\begin{equation*}
\left[\left(E-\sqrt{\boldsymbol{p}^{2}+m^{2}}\right)\left(E+\sqrt{\boldsymbol{p}^{2}+m^{2}}\right)\right]^{2}=0 \tag{D.21}
\end{equation*}
$$

We see that condition (D.19) is satisfied for both $E= \pm \sqrt{\boldsymbol{p}^{2}+m^{2}}$.

[^3]Our set of solutions consists of four linearly independent 4-component spinors. We will use $\psi$ for the spinors in the standard representation and $\phi$ for the spinors in the chiral representation (Sec. B.3). To avoid confusion, we define $\epsilon$ by

$$
\begin{equation*}
p_{0}=p^{0} \equiv \epsilon=+\sqrt{\left(p^{x}\right)^{2}+\left(p^{y}\right)^{2}+\left(p^{z}\right)^{2}+m^{2}} \tag{D.22}
\end{equation*}
$$

For the case of $m \neq 0$ we adopt the normalization of ref. [16] (see Section D.5 below),

$$
\begin{equation*}
N=\sqrt{\frac{\epsilon+m}{2 m}} . \tag{D.23}
\end{equation*}
$$

It is now easy to verify, using Eq. (D.17) with $p_{0}=\epsilon$, that we obtain the two positive energy spinors $u^{(1)}(p)$ and $u^{(2)}(p),\left(S_{z}=+1 / 2\right.$ and $S_{z}=-1 / 2$, respectively) below,

$$
u^{(1)}(p)=N\left(\begin{array}{c}
1  \tag{D.24}\\
0 \\
\frac{p^{z}}{\epsilon+m} \\
\frac{p^{x}+i p^{y}}{\epsilon+m}
\end{array}\right), \quad u^{(2)}(p)=N\left(\begin{array}{c}
0 \\
1 \\
\frac{p^{x}-i p^{y}}{\epsilon+m} \\
\frac{-p^{z}}{\epsilon+m}
\end{array}\right) .
$$

We could repeat the calculations, with $p_{0}=-\epsilon$, and obtain two negative energy spinors. However it is preferable to use the convention adopted in most textbooks (following the Feynman-Stückelberg interpretation). So from Eq. (D.18), we obtain the two negative energy spinors $v^{(1)}(p)$ and $v^{(2)}(p)$ below,

$$
v^{(1)}(p)=N\left(\begin{array}{c}
\frac{p^{z}}{\epsilon+m}  \tag{D.25}\\
\frac{p^{x}+i p^{y}}{\epsilon+m} \\
1 \\
0
\end{array}\right), \quad v^{(2)}(p)=N\left(\begin{array}{c}
\frac{p^{x}-i p^{y}}{\epsilon+m} \\
\frac{-p^{z}}{\epsilon+m} \\
0 \\
1
\end{array}\right)
$$

We summarize here some notation and results. To avoid notational ambiguities
we will write when necessary,

$$
\psi^{(+)(\alpha)}(x)=\left(\begin{array}{c}
u_{1}^{(\alpha)}(p)  \tag{D.26}\\
u_{2}^{(\alpha)}(p) \\
u_{3}^{(\alpha)}(p) \\
u_{4}^{(\alpha)}(p)
\end{array}\right) e^{-i p_{\mu} x^{\mu}},
$$

and so on, thus,

$$
\begin{align*}
& \psi^{(+)(\alpha)}(x)=u^{(\alpha)}(p) e^{-i p_{\mu} x^{\mu}},  \tag{D.27}\\
& \psi^{(-)(\alpha)}(x)=v^{(\alpha)}(p) e^{i p_{\mu} x^{\mu}}, \quad \alpha=(1,2) . \tag{D.28}
\end{align*}
$$

Each member of the set $\psi$ (or $\phi$, Sec. B.3) is an eigenstate of the energy, and momentum. We adopt the notation below so that Eqs. (D.15) and (D.16) are:

$$
\left.\begin{array}{ll}
\psi^{(+)(1)}, & S_{z}=+\frac{1}{2} \\
\psi^{(+)(2)}, & S_{z}=-\frac{1}{2} \tag{D.30}
\end{array}\right\} \quad \text { positive energy }
$$

We note that since $i \partial_{t} \psi=H \psi=E \psi$, we have

$$
\begin{align*}
& i \partial_{t} \psi^{(+)(\alpha)}(x)=p_{0} \psi^{(+)(\alpha)}(x)  \tag{D.31}\\
& i \partial_{t} \psi^{(-)(\alpha)}(x)=-p_{0} \psi^{(-)(\alpha)}(x) \tag{D.32}
\end{align*}
$$

Likewise,

$$
\begin{align*}
& -i \partial_{z} \psi^{(+)(\alpha)}(x)=p^{z} \psi^{(+)(\alpha)}(x)  \tag{D.33}\\
& -i \partial_{z} \psi^{(-)(\alpha)}(x)=-p^{z} \psi^{(-)(\alpha)}(x) \tag{D.34}
\end{align*}
$$

Remark 17. Note that although $p_{0}>0$ in both Eqs. (D.27) and (D.28), some of the 3-momenta have opposite directions.

## D. 4 Chiral representation set

There is a fundamental theorem by Pauli which states that for any two fourdimensional representations of the Dirac $\gamma$ matrices there exists a nonsingular $4 \times 4$ matrix $U$, such that $\gamma^{A}=U \gamma^{A} U^{-1}$. Moreover if $\gamma^{0 \dagger}=\gamma^{0}, \gamma^{K \dagger}=$ $-\gamma^{K}$, for $K=1,2,3$, the matrix $U$ is unitary (e.g., see [58]). The unitary
matrix $U$ below relates the standard representation to our version of the chiral representation, Eqs. (B.11).

$$
U=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}
I_{2} & I_{2}  \tag{D.35}\\
-I_{2} & I_{2}
\end{array}\right)
$$

We have that

$$
\begin{equation*}
\gamma_{(\text {chiral })}^{\mu}=U \gamma_{(\text {stand })}^{\mu} U^{-1} \tag{D.36}
\end{equation*}
$$

compare Eqs. (B.8) and (B.11). Using Eq. (D.6) we write

$$
\begin{equation*}
i U \gamma^{\mu} U^{-1} U \partial_{\mu} \psi-m U \psi=0 \tag{D.37}
\end{equation*}
$$

so instead of Eqs. (D.11) - (D.14), we now have,

$$
\begin{array}{r}
-i \partial_{t} \phi_{3}+i \partial_{z} \phi_{3}+i \partial_{x} \phi_{4}+\partial_{y} \phi_{4}-m \phi_{1}=0 \\
i \partial_{x} \phi_{3}-\partial_{y} \phi_{3}-i \partial_{t} \phi_{4}-i \partial_{z} \phi_{4}-m \phi_{2}=0 \\
-i \partial_{t} \phi_{1}-i \partial_{z} \phi_{1}-i \partial_{x} \phi_{2}-\partial_{y} \phi_{2}-m \phi_{3}=0 \\
-i \partial_{x} \phi_{1}+\partial_{y} \phi_{1}-i \partial_{t} \phi_{2}+i \partial_{z} \phi_{2}-m \phi_{4}=0 \tag{D.41}
\end{array}
$$

where

$$
\begin{equation*}
\phi=U \psi \tag{D.42}
\end{equation*}
$$

We use Eq. (D.42) and Eqs. (D.27), (D.28), and obtain the chiral positive and negative energy solutions below,

$$
\left.\begin{array}{l}
\phi^{(+)(1)}=\frac{N}{\sqrt{2}}\left(\begin{array}{c}
1+\frac{p^{z}}{\epsilon+m} \\
\frac{p^{x}+i p^{y}}{\epsilon+m} \\
-1+\frac{p^{z}}{\epsilon+m} \\
\frac{p^{x}+i p^{y}}{\epsilon+m}
\end{array}\right) e^{-i p_{\mu} x^{\mu}}, \quad \phi^{(+)(2)}=\frac{N}{\sqrt{2}}\left(\begin{array}{c}
\frac{p^{x}-i p^{y}}{\epsilon+m} \\
1-\frac{p^{z}}{\epsilon+m} \\
\frac{p^{x}-i p^{y}}{\epsilon+m} \\
\phi^{(-)(1)}=\frac{N}{\sqrt{2}}\binom{1+\frac{p^{z}}{\epsilon+m}}{-1-\frac{p^{z}}{\epsilon+m}} e^{-i p_{\mu} x^{\mu}}, \\
1-\frac{p^{x}+i p^{y}}{\epsilon+m} \\
\epsilon+m \\
-\frac{p^{x}+i p^{y}}{\epsilon+m}
\end{array}\right) e^{\frac{p^{x} p_{\mu} x^{\mu}}{\epsilon+m},} \quad \phi^{(-)(2)}=\frac{N}{\sqrt{2}}\left(\begin{array}{c}
1-\frac{p^{z}}{\epsilon+m} \\
\frac{-p^{x}+i p^{y}}{\epsilon+m} \\
1+\frac{p^{z}}{\epsilon+m}
\end{array}\right) e^{i p_{\mu} x^{\mu}} \\
\hline
\end{array}\right)
$$

Remark 18. It is worth pointing out that if we use the notation of Eq. (D.26) for the above (chiral) wavefunctions, and substitute these functions in the Dirac equations Eqs. (D.38) - (D.41), and let $p^{x}=p^{y}=p^{z}=0$, we will find that for the $\phi^{(+)(i)}, i=1,2$, the equations are satisfied with $p^{0}=\epsilon=m$, while for the $\phi^{(-)(i)}$, the equations are satisfied with $p^{0}=-m$.

The chiral operator $\gamma^{5}$, Eq. (B.13), in the chiral representation is

$$
\gamma^{5}=\left(\begin{array}{cc}
I_{2} & 0  \tag{D.44}\\
0 & -I_{2}
\end{array}\right)
$$

We introduce the bispinor $\chi$, [59] below to represent the spinors in Eqs. (D.43),

$$
\begin{equation*}
\chi=\binom{\chi_{R}}{\chi_{L}} \tag{D.45}
\end{equation*}
$$

where each entry is a 2 -component spinor. We then have

$$
\begin{equation*}
\gamma^{5} \chi=\binom{+\chi_{R}}{-\chi_{L}} \tag{D.46}
\end{equation*}
$$

$\chi_{R}$ being right-handed and $\chi_{L}$ left-handed.
Remark 19. As mentioned above, the results given in Eqs. (D.31) - (D.32) for the $\psi$ 's also hold for the $\phi$ 's of Eq. (D.43).

## D. 5 Normalization of $\psi$

We follow Itzykson and Zuber [16] and adopt the Lorentz invariant normalizations

$$
\begin{array}{ll}
\bar{u}^{(\alpha)}(p) u^{(\beta)}(p)=\delta^{\alpha \beta}, & \bar{u}^{(\alpha)}(p) v^{(\beta)}(p)=0 \\
\bar{v}^{(\alpha)}(p) v^{(\beta)}(p)=-\delta^{\alpha \beta}, &  \tag{D.48}\\
\bar{v}^{(\alpha)}(p) u^{(\beta)}(p)=0
\end{array}
$$

Remark 20. We remark that different authors adopt different normalizations for the Lorentz invariant product $\bar{\psi} \psi$ (the invariance itself is a little tedious to show, see, e.g., the last page of [57]).

The normalization factor $N$ for the plane waves, given by Eq. (D.23), follows from Eqs. (D.47), (D.48), using the solutions (D.24) and (D.25).

For easy reference for the proofs to follow, we write again some of the formulas derived above:

$$
\begin{align*}
(\not p-m I) u^{(\alpha)}(p)=0, & \bar{u}^{(\alpha)}(p)(\not p-m I)=0  \tag{D.49}\\
(\not p+m I) v^{(\alpha)}(p)=0, & \bar{v}^{(\alpha)}(p)(\not p+m I)=0 \tag{D.50}
\end{align*}
$$

We shall derive an expression for $j^{\mu}=\bar{\psi} \gamma^{\mu} \psi$. In the derivation we make use of Eqs. (D.47) - (D.50). For the positive energy solutions we have,

$$
\begin{align*}
\bar{\psi}^{(+)(\alpha)} \gamma^{\mu} \psi^{(+)(\beta)} & =\bar{u}^{(\alpha)}(p) \gamma^{\mu} u^{(\beta)}(p) \\
& =\frac{1}{2}\left[\left(\bar{u}^{(\alpha)} \gamma^{\mu}\right) u^{(\beta)}+\bar{u}^{(\alpha)}\left(\gamma^{\mu} u^{(\beta)}\right)\right] \\
& =\frac{1}{2 m}\left[\left(\bar{u}^{(\alpha)} m \gamma^{\mu}\right) u^{(\beta)}+\bar{u}^{(\alpha)}\left(\gamma^{\mu} m u^{(\beta)}\right)\right] \\
& =\frac{1}{2 m}\left[\left(\bar{u}^{(\alpha)} \not p \gamma^{\mu}\right) u^{(\beta)}+\bar{u}^{(\alpha)}\left(\gamma^{\mu} \not p u^{(\beta)}\right)\right] \\
& =\frac{1}{2 m}\left[\bar{u}^{(\alpha)}\left\{\not p, \gamma^{\mu}\right\} u^{(\beta)}\right] \\
& =\frac{1}{2 m}\left[\bar{u}^{(\alpha)} p_{\nu}\left\{\gamma^{\nu}, \gamma^{\mu}\right\} u^{(\beta)}\right] \\
& =\frac{1}{2 m}\left[\bar{u}^{(\alpha)} p_{\nu} 2 \eta^{\nu \mu} I u^{(\beta)}\right]=\frac{p^{\mu}}{m} \delta^{\alpha \beta} \tag{D.51}
\end{align*}
$$

Repeating this derivation for the negative energy solutions we get

$$
\begin{align*}
\bar{\psi}^{(-)(\alpha)} \gamma^{\mu} \psi^{(-)(\beta)} & =\bar{v}^{(\alpha)}(p) \gamma^{\mu} v^{(\beta)}(p) \\
& =-\frac{1}{2 m}\left[\bar{v}^{(\alpha)}\left\{\not p, \gamma^{\mu}\right\} v^{(\beta)}\right] \\
& =-\frac{1}{2 m}\left[\bar{v}^{(\alpha)} p_{\nu} 2 \eta^{\nu \mu} I v^{(\beta)}\right]=\frac{p^{\mu}}{m} \delta^{\alpha \beta} \tag{D.52}
\end{align*}
$$

It is important to show that positive and negative energy states are mutually orthogonal if we consider states with opposite energies but the same 3-momentum. We recall Eqs. (D.31), (D.32) and write explicitly

$$
\begin{align*}
& \psi^{(+)(\alpha)}(x)=u^{(\alpha)}(p) e^{-i\left(p^{0} x^{0}-p^{i} x^{i}\right)},  \tag{D.53}\\
& \psi^{(-)(\beta)}(x)=v^{(\beta)}(q) e^{i\left(p^{0} x^{0}+p^{i} x^{i}\right)}, \tag{D.54}
\end{align*}
$$

where the vector (not the covector $p_{\mu}$ ) momenta are $p=\left(p^{0}, \boldsymbol{p}\right), q=\left(p^{0},-\boldsymbol{p}\right)$, respectively, see Remark 17. Therefore, using again Eqs. (D.47) - (D.50), we have

$$
\begin{align*}
\bar{\psi}^{(-)(\beta)} \psi^{(+)(\alpha)} & =e^{-2 i p^{0} x^{0}} \bar{v}^{(\beta)}(q) \gamma^{0} u^{(\alpha)}(p), \\
& =\frac{1}{2 m} e^{-2 i p^{0} x^{0}}\left[\left(\bar{v}^{(\beta)}(q) m\right) \gamma^{0} u^{(\alpha)}(p)+\bar{v}^{(\beta)}(q) \gamma^{0}\left(m u^{(\alpha)}(p)\right)\right], \\
& =\frac{1}{2 m} e^{-2 i p^{0} x^{0}} \bar{v}^{(\beta)}(q)\left(-q \gamma^{0}+\gamma^{0} \not p\right) u^{(\alpha)}(p)=0 . \tag{D.55}
\end{align*}
$$

Showing the last step above requires care!
We now define the scalar product with the standard delta function normalization for free particles,

$$
\begin{equation*}
\left(\psi_{\boldsymbol{p}^{\prime}}^{(\alpha)} \mid \psi_{\boldsymbol{p}}^{(\beta)}\right)=\int \bar{\psi}_{\boldsymbol{p}^{\prime}}^{(\alpha)} \gamma^{0} \psi_{\boldsymbol{p}}^{(\beta)} d^{3} x=\delta^{\alpha \beta} \delta^{3}\left(\boldsymbol{p}^{\prime}-\boldsymbol{p}\right), \tag{D.56}
\end{equation*}
$$

where the delta function is given by

$$
\begin{equation*}
\delta^{3}\left(\boldsymbol{p}^{\prime}-\boldsymbol{p}\right)=\frac{1}{(2 \pi)^{3}} \int e^{i\left(\boldsymbol{p}^{\prime}-\boldsymbol{p}\right) \boldsymbol{r}} d^{3} x, \quad \boldsymbol{r}=\left(x^{1}, x^{2}, x^{3}\right) \tag{D.57}
\end{equation*}
$$

As a consequence of result (D.55) we see that if, in the integrand of (D.56), the $\psi$ 's have opposite energies, the result of the integration is zero (either from Eq. (D.55) or from the $\delta^{3}\left(\boldsymbol{p}^{\prime}-\boldsymbol{p}\right)$ ).

Remark 21. Since $\epsilon=+\sqrt{\boldsymbol{p}^{2}+m^{2}}$, if the absolute value of the $\boldsymbol{p}$ 's is the same, then the absolute value $\epsilon$ of the energy is the same.

At this point we must introduce another normalization factor, $\tilde{N}$, required by the integration over space (we could have introduced $\tilde{N}$ in Eqs. (D.27) and (D.28) but it would only have complicated the writing). Effectively this amounts to re-defining the $\psi$ 's as follows,

$$
\begin{align*}
& \psi^{(+)(\alpha)}(x)=\tilde{N} u^{(\alpha)}(p) e^{-i p_{\mu} x^{\mu}}  \tag{D.58}\\
& \psi^{(-)(\alpha)}(x)=\tilde{N} v^{(\alpha)}(p) e^{i p_{\mu} x^{\mu}} \tag{D.59}
\end{align*}
$$

The calculation of $\tilde{N}$ is simpler if we consider "box normalization" with periodic boundary conditions in a volume $V$. Then, instead of Eq. (D.56) we have

$$
\begin{equation*}
\left(\psi_{\boldsymbol{p}^{\prime}}^{(\alpha)} \mid \psi_{\boldsymbol{p}}^{(\beta)}\right)=\int \bar{\psi}_{\boldsymbol{p}^{\prime}}^{(\alpha)} \gamma^{0} \psi_{\boldsymbol{p}}^{(\beta)} d^{3} x=\delta^{\alpha \beta} \delta_{\boldsymbol{p}^{\prime} \boldsymbol{p}} \tag{D.60}
\end{equation*}
$$

thus

$$
\begin{equation*}
\int \bar{\psi}_{\boldsymbol{p}}^{(\alpha)} \gamma^{0} \psi_{\boldsymbol{p}}^{(\beta)} d^{3} x=\int j^{0} d^{3} x=\frac{p^{0}}{m} \delta^{\alpha \beta} \tilde{N}^{2} V \tag{D.61}
\end{equation*}
$$

and so

$$
\begin{align*}
\psi^{(+)(\alpha)}(x) & =\frac{1}{\sqrt{V}} \sqrt{\frac{m}{p^{0}}} u^{(\alpha)}(p) e^{-i p_{\mu} x^{\mu}}  \tag{D.62}\\
\psi^{(-)(\alpha)}(x) & =\frac{1}{\sqrt{V}} \sqrt{\frac{m}{p^{0}}} v^{(\alpha)}(p) e^{i p_{\mu} x^{\mu}} \tag{D.63}
\end{align*}
$$

where we recall that $p^{0}=\epsilon$. If we had used Eq. (D.56) instead of Eq. (D.60), we would have to do the replacement (cf. ref. [60]),

$$
\begin{equation*}
\frac{1}{\sqrt{V}} \rightarrow \frac{1}{(2 \pi)^{3 / 2}} \tag{D.64}
\end{equation*}
$$

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Unprotect[RiemannR]
Remove[RiemannR]
Unprotect[Symmetrize]
Remove[Symmetrize]
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[^1]:    ${ }^{3}$ We shall use the term vielbein whenever the dimensionality is not necessarily $(3+1)$. We reserve the term tetrad for the $(3+1)$ case.
    ${ }^{4}$ A shorthand for Eq. (A.28) is $e=\Lambda^{-1} h$, while Eq. (A.37) is $L^{-1} \gamma L=\Lambda \gamma$, so that our index positions agree with refs. [52], [53] and [54].

[^2]:    ${ }^{5}$ So more precisely Eq. (B.2) is $i\left(\gamma^{A}\right)^{i}{ }_{k} \partial_{A} \psi^{k}-m \psi^{i}=0, i, k=(1,2,3,4)$.

[^3]:    ${ }^{6}$ Best evaluated with Mathematica for a couple of representations.

