# Lecture on Gauge Gravitation Theory. Gravity as a Higgs Field

G. Sardanashvily

Moscow State University, Russia Lepage Research Institute, Czech Republic

20th International Summer School on Global Analysis and its Applications "General Relativity: 100 years after Hilbert" (Stará Lesná, Slovakia, 2015)

**Abstract.** Gravitation theory is formulated as gauge theory on natural bundles with spontaneous symmetry breaking where gauge symmetries are general covariant transformations, gauge fields are general linear connections, and Higgs fields are pseudo-Riemannian metrics.

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### 1 Introduction

Theory of classical fields admits a comprehensive mathematical formulation in the geometric terms of smooth fibre bundles over X [11, 52, 55]. For instance, Yang–Mills gauge theory is theory of principal connections on principal bundles.

Gravitation theory on a world manifold X is formulated as gauge theory on natural bundles over X which admit general covariant transformations as the canonical functorial lift of diffeomorphisms of their base X [11, 54]. This is metric-affine gravitation theory where gauge fields are general linear connections (Section 5), and a metric gravitational field is treated as a classical Higgs field responsible for reducing a structure group of natural bundles to a Lorentz group (Section 6). The underlying physical reason of this reduction is both the geometric Equivalence principle and the existence of Dirac spinor fields. Herewith, a structure Lorentz group always is reducible to its maximal compact subgroup of spatial rotations that provides a world manifold X with an associated space-time structure and metric space topology (Section 7).

Spontaneous symmetry breaking is a quantum phenomenon when automorphism of a quantum algebra need not preserve its vacuum state [51, 58]. In this case, we have inequivalent vacuum states of a quantum system which are classical objects. The physical nature of gravity as a Higgs field is characterized by the fact that, given different gravitational fields, the representations (11.30) of holonomic coframes  $\{dx^{\mu}\}$  on a world manifold X by  $\gamma$ -matrices acting on spinor fields are non-equivalent and, consequently, the Dirac operators in the presence of different gravitational field fails to be equivalent, too (Section 10). This fact motivates us to think that a metric gravitational field is not quantized in principle.

## 2 History

A first model of gauge gravitation theory was suggested by R.Utiyama [69] in 1956 just two years after the birth of gauge theory itself. He was first who generalized the original gauge model of Yang and Mills for SU(2) to an arbitrary symmetry Lie group and, in particular, to a Lorentz group in order to describe gravity. However, he met the problem of treating general covariant transformations and a pseudo-Riemannian metric which had no partner in Yang-Mills gauge theory.

To eliminate this drawback, representing a tetrad gravitational field as a gauge field of a translation subgroup of a Poincaré group was attempted because, by analogy with gauge potentials in Yang–Mills gauge theory, the indices a of a tetrad field  $h^a_{\mu}$  were treated as those of a translation group (see [1, 2, 6, 18, 21, 39, 49] and references therein). Since the Poincaré group comes from the Wigner–Inönii contraction of de Sitter groups SO(2,3) and SO(1,4) and it is a subgroup of a conformal group, gauge theories on fibre bundles  $Y \to X$  with these structure groups were also considered [13, 17, 22, 29, 63, 68]. Because these fibre bundles fail to be natural, the lift of the group Diff(X) of diffeomorphisms of X onto Y should be defined [30, 31]. In a general setting, one can study a gauge theory on a fibre bundle with the typical fibre  $\mathbb{R}^n$ and the topological structure group  $Diff(\mathbb{R}^n)$  or its subgroup of analytical diffeomorphisms [3, 25]. The Poincaré gauge theory also is generalized to the higher s-spin gauge theory of tensor coframes and generalized Lorentz connections, which satisfy certain symmetry, skew symmetry and traceless conditions [70].

A problem however is that that a non-linear (translation) summand of an affine connection (Section 12) is a soldering form, but neither frame (vierbein) field nor tetrad field. The latter thus has no the status of a gauge field [21, 41, 54]. At the same time, a translation part of an affine connection on  $\mathbb{R}^3$  characterizes an elastic distortion in gauge theory of dislocations in continuous media [23, 32]. A similar gauge model of hypothetic deformations of a world manifold has been developed and, in particular, they may be responsible for the so called "fifth force" [42, 43, 44].

At the same time, gauge theory in a case of spontaneous symmetry breaking also contains classical Higgs fields, besides the gauge and matter ones [11, 21, 24, 36, 45, 50, 51, 56, 67]. Therefore, basing on the mathematical definition of a pseudo-Riemnnian metric, we have formulated gravitation theory as gauge theory with a Lorentz reduced structure where a metric gravitational field is treated as a Higgs field [11, 21, 40, 44, 49, 54].

### 3 Main Theses

Gauge gravitation theory in comparison with the Yang–Mills one possesses the following peculiarities [11, 54].

• Gauge symmetries of gravitation theory are general covariant transformations which are not vertical automorphisms of principal bundles in Yang–Mills gauge theory.

• Gauge gravitation theory necessarily is theory with spontaneous symmetry breaking in the presence of the corresponding Higgs fields. Since gauge symmetries of gravitation theory are general covariant transformations, but not vertical automorphisms of fibre bundles, these Higgs fields, unlike Higgs fields in Yang–Mills gauge theory, are dynamic variables.

• In comparison with Yang–Mills gauge theory, e.g., the Standard Model of particle physics [37, 61], matter fields in gauge gravitation theory admits only exact symmetries. These are Dirac spinor fields with Lorentz spin symmetries, and there is a problem of describing their general covariant transformations.

• The gauge invariance gauge gravitation theory under general covariant transformation leads to a conservation law of an energy-momentum symmetry current, but not the Noether one in Yang–Mills gauge theory.

Studying gauge gravitation theory, we believe reasonable to require that it incorporates Einstein's General Relativity and, therefore, it should be based on Relativity and Equivalence Principles reformulated in the fibre bundle terms [20, 21].

In these terms, Relativity Principle states that gauge symmetries of classical gravitation theory are general covariant transformations [11, 54].

Let  $\pi: Y \to X$  be a smooth fibre bundle. Any automorphism  $(\Phi, f)$  of Y, by definition, is projected as  $\pi \circ \Phi = f \circ \pi$  onto a diffeomorphism f of its base X. The converse is not true.

A fibre bundle  $Y \to X$  is called the natural bundle if there exists a monomorphism

$$\operatorname{Diff} X \ni f \to f \in \operatorname{Aut} Y$$

of the group of diffeomorphisms of X to the group of bundle automorphisms of  $Y \to X$ . Automorphisms  $\tilde{f}$  are called general covariant transformations of Y.

Accordingly, there is the functorial lift of any vector field  $\tau$  on X to a vector field  $\overline{\tau}$  on Y such that  $\tau \mapsto \overline{\tau}$  is a monomorphism of the Lie algebra  $\mathcal{T}(X)$  of vector field on X to that  $\mathcal{T}(T)$ of vector fields on Y. This functorial lift  $\overline{\tau}$  is an infinitesimal generator of a local one-parameter group of local general covariant transformations of Y.

As was mentioned above, general covariant transformations differ from gauge symmetries of Yang–Mills gauge theory which are vertical automorphisms of principal bundles. Fibre bundles possessing general covariant transformations constitute the category of so called natural bundles [27, 65].

The tangent bundle TX of X exemplifies a natural bundle. Any diffeomorphism f of X gives rise to the tangent automorphisms  $\tilde{f} = Tf$  of TX which is a general covariant transformation of TX. The associated principal bundle is a fibre bundle LX of linear frames in the tangent spaces to X. It also is a natural bundle. Moreover, all fibre bundles associated to LX are natural bundles, but not they are only. Principal connections on LX yield linear connections on the tangent bundle TX and other associated bundles over a world manifold. They are called the world connections.

Following Relativity Principle, one thus should develop gravitation theory as a gauge theory of principal connections on a principal frame bundle LX over an oriented four-dimensional connected smooth manifold X, called the world manifold.

**Remark 3.1:** Smooth manifolds throughout are assumed to be Hausdorff second-countable (consequently, locally compact and paracompact) topological spaces.  $\Box$ 

Equivalence Principle reformulated in geometric terms requires that the structure group

$$GL_4 = GL^+(4,\mathbb{R}) \tag{3.1}$$

of a frame bundle LX and associated bundles over a world manifold X is reducible to a Lorentz group SO(1,3) [21, 44, 54]. It means that these fibre bundles admit atlases with SO(1,3)valued transition functions or, equivalently, that there exist principal subbubdles of LX with a Lorentz structure group. This is the case of spontaneous symmetry breaking in classical gauge theory.

As was mentioned above, spontaneous symmetry breaking is a quantum phenomenon when automorphism of a quantum algebra need not preserve its vacuum state [51, 58]. In this case, we have inequivalent vacuum states of a quantum system which are classical objects. For instance, spontaneous symmetry breaking in Standard Model of particle physics is ensured by the existence of a constant vacuum Higgs field which takes a value into the quotient G/H of a broken symmetry group G by the exact one H [37, 61].

Therefore, classical gauge theory on principal bundles with spontaneous symmetry breaking also is considered. This phenomenon is characterized as a reduction of a structure Lie group Gof a principal bundle  $P \to X$  to its closed Lie subgroup H [11, 50, 51, 56]. One refers to the following reduction theorem [26].

THEOREM 3.1: There exists one-to-one correspondence between the principal *H*-subbundles  $P^h$  of *P* and the global sections *h* of the quotient bundle  $P/H \to X$  with a typical fibre G/H.

These global sections are treated as classical Higgs fields [11, 50, 56].

Accordingly, in gauge gravitation theory based on Equivalence Principle, there is one-toone correspondence between the Lorentz principal subbundles of a frame bundle LX (called the Lorentz reduced structures) and the global sections of the quotient bundle

$$\Sigma_{\rm PR} = LX/SO(1,3),\tag{3.2}$$

which are pseudo-Riemannian metrics on a world manifold. In Einstein's General Relativity, they are identified with gravitational fields.

Thus, gauge gravitation theory leads us to metric-affine gravitation theory whose dynamic variables are linear world connections and pseudo-Riemannian metrics on a world manifold X (Section 8). They are treated as gauge fields and Higgs fields, respectively [11, 54].

There is the extensive literature on metric-affine gravitation theory [1, 18, 19, 39]. However, one often formulates it as gauge theory of affine connections, that is incorrect (Section 12). Let us also emphasize that gauge gravitation theory deals with general linear connections which need not be the Lorentz connections.

The character of gravity as a Higgs field responsible for spontaneous breaking of general covariant transformations is displayed as follows. Given different gravitational fields, the representations (11.30) of holonomic coframes  $\{dx^{\mu}\}$  by  $\gamma$ -matrices acting on spinor fields are inequivalent (Remark 10.6). In particular, it follows that a Dirac spinor field can be considered

only in a pair with a certain gravitational field. A total system of such pairs is described by sections of the composite bundle  $S \to \Sigma_T \to X$  (11.25), where  $S \to \Sigma_T$  is a spinor bundle.

Being reduced to a Lorentz group, a structure group of a frame bundle LX also is reduced to a maximal compact subgroup SO(3) of SO(1,3). The associated Higgs field is a spatial distribution which defines a space-time structure on a world manifold X (Section 7).

Since general covariant transformations are symmetries of a metric-affine gravitation Lagrangian, the corresponding conservation law holds (Section 9). It is an energy-momentum conservation law. Because general covariant transformations are gauge transformations depending on derivatives of gauge parameters, the corresponding energy-momentum current reduces to a superpotential [11, 53, 54]. This is the generalized Komar superpotential (9.5).

### 4 Natural bundles

Let  $\pi: Y \to X$  be a smooth fibre bundle coordinated by  $(x^{\lambda}, y^{i})$ . Given a one-parameter group  $(\Phi_t, f_t)$  of automorphisms of Y, its infinitesimal generator is a projectable vector field

$$u = \tau^{\lambda}(x^{\mu})\partial_{\lambda} + u^{i}(x^{\mu}, y^{j})\partial_{i}$$

on Y which is projected onto a vector field  $\tau = \tau^{\lambda} \partial_{\lambda}$  on X, whose flow is a one-parameter group  $(f_t)$  of diffeomorphisms of X. Conversely, let  $\tau = \tau^{\lambda} \partial_{\lambda}$  be a vector field on X. Its lift to some projectable vector field on Y always exists. For instance, given a connection

$$\Gamma = dx^{\lambda} \otimes (\partial_{\lambda} + \Gamma^{i}_{\lambda}(x^{\mu}, y^{j})\partial_{i})$$

on  $Y \to X$ , a vector field  $\tau$  on X gives rise to a horizontal vector field

$$\Gamma \tau = \tau \rfloor \Gamma = \tau_{\lambda} (\partial_{\lambda} + \Gamma^{i}_{\lambda} \partial_{i})$$

on Y. The horizontal lift  $\tau \to \Gamma \tau$  yields a monomorphism of a  $C^{\infty}(X)$ -module  $\mathcal{T}(X)$  of vector fields on X to a  $C^{\infty}(Y)$ -module  $\mathcal{T}(Y)$  of vector fields on Y, but this monomorphism is not a Lie algebra morphism, unless  $\Gamma$  is flat.

We address the category of **natural bundles**  $Y \to X$  admitting the functorial lift  $\tilde{\tau}$  onto Y of any vector field  $\tau$  on X such that  $\tau \to \overline{\tau}$  is a Lie algebra monomorphism  $\mathcal{T}(X) \to \mathcal{T}(T)$ ,  $[\tilde{\tau}, \tilde{\tau}'] = [\tilde{\tau}, \tau']$  [11, 27, 65]. This functorial lift  $\tilde{\tau}$ , by definition, is an infinitesimal generator of a local one-parameter group of **general covariant transformations** of Y.

Natural bundles are exemplified by tensor products

$$T = (\overset{m}{\otimes} TX) \otimes (\overset{\kappa}{\otimes} T^*X) \tag{4.1}$$

of the tangent TX and cotangent  $T^*X$  bundles of X. Given a coordinate atlas  $(x^{\mu})$  of X, the tangent bundle  $\pi_X : TX \to X$  is provided with holonomic bundle coordinates

$$(x^{\mu}, \dot{x}^{\mu}), \qquad \dot{x}'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \dot{x}^{\nu},$$

where  $(\dot{x}^{\mu})$  are fibre coordinates with respect to holonomic frames  $\{\partial_{\mu}\}$ . Accordingly, the tensor bundle (4.1) is endowed with holonomic bundle coordinates  $(x^{\mu}, x^{\alpha_1 \cdots \alpha_m}_{\beta_1 \cdots \beta_k})$ , where

$$(x^{\mu}, \dot{x}_{\mu}), \qquad \dot{x}'_{\mu} = \frac{\partial x^{\nu}}{\partial x_{\mu}} \dot{x}_{\nu},$$

are those on the cotangent bundle  $T^*X$  of X. Then given a vector field  $\tau$  on X, its functorial lift onto the tensor bundle (4.1) takes a form

$$\widetilde{\tau} = \tau^{\mu} \partial_{\mu} + [\partial_{\nu} \tau^{\alpha_1} \dot{x}^{\nu \alpha_2 \cdots \alpha_m}_{\beta_1 \cdots \beta_k} + \dots - \partial_{\beta_1} \tau^{\nu} \dot{x}^{\alpha_1 \cdots \alpha_m}_{\nu \beta_2 \cdots \beta_k} - \dots] \dot{\partial}^{\beta_1 \cdots \beta_k}_{\alpha_1 \cdots \alpha_m}, \qquad \dot{\partial}_{\lambda} = \frac{\partial}{\partial \dot{x}^{\lambda}}$$

Tensor bundles over a world manifold X have the structure group  $GL_4$  (3.1). An associated principal bundle is the above mentioned frame bundle LX. Its (local) sections are called **frame** (vierbein) fields. Given a holonomic atlas of the tangent bundle TX, every element  $\{H_a\}$  of a frame bundle LX takes a form  $H_a = H_a^{\mu}\partial_{\mu}$ , where  $H_a^{\mu}$  is a matrix of the natural representation of a group  $GL_4$  in  $\mathbb{R}^4$ . These matrices constitute bundle coordinates

$$(x^{\lambda}, H^{\mu}_{a}), \qquad H^{\prime \mu}_{a} = rac{\partial x^{\prime \mu}}{\partial x^{\lambda}} H^{\lambda}_{a},$$

on LX associated to its holonomic atlas

$$\Psi_T = \{ (U_\iota, z_\iota = \{\partial_\mu\}) \}, \tag{4.2}$$

given by local frame fields  $z_{\iota} = \{\partial_{\mu}\}.$ 

A frame bundle LX is equipped with a canonical  $\mathbb{R}^4$ -valued one-form

$$\theta_{LX} = H^a_\mu dx^\mu \otimes t_a, \tag{4.3}$$

where  $\{t_a\}$  is a fixed basis for  $\mathbb{R}^4$  and  $H^a_\mu$  is the inverse matrix of  $H^\mu_a$ .

A frame bundle  $LX \to X$  is natural. Indeed, any diffeomorphism f of X gives rise to an automorphism

$$\widetilde{f}: (x^{\lambda}, H_a^{\lambda}) \to (f^{\lambda}(x), \partial_{\mu} f^{\lambda} H_a^{\mu})$$

$$(4.4)$$

of LX which is its general covariant transformation. Given a (local) one-parameter group of diffeomorphisms of X and its infinitesimal generator  $\tau$ , the lift (4.4) yields a functorial lift

$$\widetilde{\tau} = \tau^{\mu} \partial_{\mu} + \partial_{\nu} \tau^{\alpha} H^{\nu}_{a} \frac{\partial}{\partial H^{\alpha}_{a}}$$

onto LX of a vector field  $\tau$  on X which is defined by the condition  $\mathbf{L}_{\tilde{\tau}}\theta_{LX} = 0$ .

Let  $Y = (LX \times V)/GL_4$  be an LX-associated bundle with a typical fibre V. It admits a lift of any diffeomorphism f of its base to an automorphism

$$f_Y(Y) = (\widetilde{f}(LX) \times V)/GL_4$$

of Y associated to the principal automorphism  $\tilde{f}$  (4.4) of a frame bundle LX. Thus, all bundles associated to a frame bundle LX are natural bundles.

**Remark 4.1:** In a general setting, one also considers the total group  $\operatorname{Aut}(LX)$  of automorphisms of a frame bundle LX [18]. Such an automorphism is the composition of some general covariant transformation and a vertical automorphism of LX, which is a non-holonomic frame transformation. Subject to vertical automorphisms, the tangent bundle TX is provided with non-holonomic frames  $\{\vartheta_a\}$  and the corresponding bundle coordinates  $(x^{\mu}, y^{a})$ . A problem is that Lagrangians of gravitation theory which factorize through the Ricci tensor (5.5), e.g. the Hilbert–Einstein Lagrangian (8.7) are not invariant under non-holonomic frame transformations (see Remark 5.1 and Example 8.3). To overcome this difficulty, one can additionally introduce frame  $\vartheta_a = \vartheta_a^{\mu} \partial_{\mu}$  (or coframe  $\vartheta^a = \vartheta_{\mu}^a dx^{\mu}$ ) fields, which are sections of a frame bundle LX. These sections are necessarily local, unless LX is a trivial bundle, i.e., X is a parallelizable manifold (Remark 5.2). In particular, this is the case of theory of teleparallel gravity [5, 38].  $\Box$ 

### 5 World connections

Let TX be the tangent bundle of a world manifold X. With respect to holonomic coordinates  $(x^{\lambda}, \dot{x}^{\lambda})$ , a linear connection on TX takes a form

$$\Gamma = dx^{\lambda} \otimes (\partial_{\lambda} + \Gamma_{\lambda}{}^{\mu}{}_{\nu} \dot{x}^{\nu} \dot{\partial}_{\mu}).$$
(5.1)

It is called a **linear world connection** on X. Since TX is associated to a frame bundle LX, every linear connection (5.1) is associated to a principal connection on LX.

A curvature of a linear world connection is defined as that of the connection (5.1). It reads

$$R = \frac{1}{2} R_{\lambda\mu}{}^{\alpha}{}_{\beta} \dot{x}^{\beta} dx^{\lambda} \wedge dx^{\mu} \otimes \dot{\partial}_{\alpha},$$

$$R_{\lambda\mu}{}^{\alpha}{}_{\beta} = \partial_{\lambda} \Gamma_{\mu}{}^{\alpha}{}_{\beta} - \partial_{\mu} \Gamma_{\lambda}{}^{\alpha}{}_{\beta} + \Gamma_{\lambda}{}^{\gamma}{}_{\beta} \Gamma_{\mu}{}^{\alpha}{}_{\gamma} - \Gamma_{\mu}{}^{\gamma}{}_{\beta} \Gamma_{\lambda}{}^{\alpha}{}_{\gamma}.$$
(5.2)

Due to the canonical splitting of the vertical tangent bundle

$$VTX = TX \times TX \tag{5.3}$$

of TX, the curvature R (5.2) can be represented by a tangent-valued two-form

$$R = \frac{1}{2} R_{\lambda\mu}{}^{\alpha}{}_{\beta} \dot{x}^{\beta} dx^{\lambda} \wedge dx^{\mu} \otimes \partial_{\alpha}$$
(5.4)

on TX. Due to this representation, the **Ricci tensor** 

$$R_c = \frac{1}{2} R_{\lambda\mu}{}^{\lambda}{}_{\beta} dx^{\mu} \otimes dx^{\beta}$$
(5.5)

of a linear world connection  $\Gamma$  is defined.

**Remark 5.1:** The vertical splitting (5.3) with respect to the holonomic atlases (4.2) of TX takes place only. Accordingly, the Ricci tensor (5.5) with respect to holonomic atlases is ill defined.  $\Box$ 

By a **torsion** of a linear world connection is meant that of the connection  $\Gamma$  (5.1) on the tangent bundle TX with respect to the canonical soldering form

$$\theta_J = dx^\mu \otimes \dot{\partial}_\mu \tag{5.6}$$

on TX. It reads

$$T = \frac{1}{2} T_{\mu}{}^{\nu}{}_{\lambda} dx^{\lambda} \wedge dx^{\mu} \otimes \dot{\partial}_{\nu}, \qquad T_{\mu}{}^{\nu}{}_{\lambda} = \Gamma_{\mu}{}^{\nu}{}_{\lambda} - \Gamma_{\lambda}{}^{\nu}{}_{\mu}.$$
(5.7)

A world connection is said to be symmetric if its torsion (5.7) vanishes, i.e.,  $\Gamma_{\mu}{}^{\nu}{}_{\lambda} = \Gamma_{\lambda}{}^{\nu}{}_{\mu}$ . Owing to the vertical splitting of VTX, the torsion form T (5.7) of  $\Gamma$  can be written as a tangent-valued two-form

$$T = \frac{1}{2} T_{\mu}^{\ \nu}{}_{\lambda} dx^{\lambda} \wedge dx^{\mu} \otimes \partial_{\nu}$$
(5.8)

on X.

Being associated to a principal connection on LX, a world connection is represented by a section of the quotient bundle

$$C_{\rm W} = J^1 L X / G L_4 \to X, \tag{5.9}$$

where  $J^1LX$  is the first order jet manifold of sections of  $LX \to X$ . We agree to call  $C_W$  (5.9) the bundle of world connections [11, 33, 55]. With respect to the holonomic atlas  $\Psi_T$  (4.2), it is provided with the bundle coordinates  $(x^{\lambda}, k_{\lambda}{}^{\nu}{}_{\alpha})$  so that, for any section  $\Gamma$  of  $C_W \to X$ , its coordinates  $k_{\lambda}{}^{\nu}{}_{\alpha} \circ \Gamma = \Gamma_{\lambda}{}^{\nu}{}_{\alpha}$  are components of the world connection  $\Gamma$  (5.1).

Though the bundle of world connections  $C_W \to X$  (5.9) is not LX-associated, it is a natural bundle. It admits a functorial lift

$$\widetilde{\tau}_C = \tau^{\mu} \partial_{\mu} + \left[ \partial_{\nu} \tau^{\alpha} k_{\mu}{}^{\nu}{}_{\beta} - \partial_{\beta} \tau^{\nu} k_{\mu}{}^{\alpha}{}_{\nu} - \partial_{\mu} \tau^{\nu} k_{\nu}{}^{\alpha}{}_{\beta} + \partial_{\mu\beta} \tau^{\alpha} \right] \frac{\partial}{\partial k_{\mu}{}^{\alpha}{}_{\beta}}$$

of any vector field  $\tau$  on X.

The first order jet manifold  $J^1C_W$  of a bundle of world connections possesses the canonical splitting

$$k_{\lambda\mu}{}^{\alpha}{}_{\beta} = \frac{1}{2} (k_{\lambda\mu}{}^{\alpha}{}_{\beta} - k_{\mu\lambda}{}^{\alpha}{}_{\beta} + k_{\lambda}{}^{\gamma}{}_{\beta}k_{\mu}{}^{\alpha}{}_{\gamma} - k_{\mu}{}^{\gamma}{}_{\beta}k_{\lambda}{}^{\alpha}{}_{\gamma}) +$$

$$\frac{1}{2} (k_{\lambda\mu}{}^{\alpha}{}_{\beta} + k_{\mu\lambda}{}^{\alpha}{}_{\beta} - k_{\lambda}{}^{\gamma}{}_{\beta}k_{\mu}{}^{\alpha}{}_{\gamma} + k_{\mu}{}^{\gamma}{}_{\beta}k_{\lambda}{}^{\alpha}{}_{\gamma}) = \frac{1}{2} (\mathcal{R}_{\lambda\mu}{}^{\alpha}{}_{\beta} + \mathcal{S}_{\lambda\mu}{}^{\alpha}{}_{\beta})$$
(5.10)

so that, if  $\Gamma$  is a section of  $C_{W} \to X$ , then  $\mathcal{R}_{\lambda\mu}{}^{\alpha}{}_{\beta} \circ J^{1}\Gamma = R_{\lambda\mu}{}^{\alpha}{}_{\beta}$  are components of the curvature (5.2) [11, 33].

**Remark 5.2:** A world manifold X is called flat if it admits a flat world connection  $\Gamma$ , called the Weitzenböck connection. By virtue of the well-known theorem, there exists a bundle atlas of TX with constant transition functions such that  $\Gamma = dx^{\lambda} \otimes \partial_{\lambda}$  relative to this atlas. However, such an atlas is not holonomic in general. Therefore, the torsion form T (5.7) of a flat connection  $\Gamma$  need not vanish. A world manifold X is called parallelizable if the tangent bundle  $TX \to X$  is trivial. A parallelizable manifold is flat. A flat manifold is parallelizable if it is simply connected. Flat connections together with global frame fields (Remark 4.1) on a parallelizable world manifold are attributes of theory of teleparallel gravity [5, 38].  $\Box$ 

#### 6 Lorentz reduced structure

As was mentioned above, gravitation theory on a world manifold X is classical field theory with spontaneous symmetry breaking described by Lorentz reduced structures of a frame bundle LX[11, 21, 44, 54]. We deal with the following Lorentz and proper Lorentz reduced structures.

By a Lorentz reduced structure is meant a reduced principal SO(1,3)-subbundle  $L^{g}X$ , called the Lorentz subbundle, of a frame bundle LX. By virtue of the Theorem 3.1, there is one-to-one correspondence between the principal Lorentz subbundles  $L^{g}X$  of a frame bundle LX and the global sections of g the quotient bundle  $\Sigma_{PR}$  (3.2) which are **pseudo-Riemannian metrics** of signature (+, ---) on a world manifold X. For the sake of convenience, one usually identifies the quotient bundle  $\Sigma_{PR}$  (3.2), called the **metric bundle**, with an open subbundle of the tensor bundle  $\Sigma_{PR} \subset \sqrt[2]{TX}$ . Therefore, a metric bundle  $\Sigma_{PR}$  can be equipped with bundle coordinates  $(x^{\lambda}, \sigma^{\mu\nu})$ .

Let  $L = SO^0(1,3)$  be a **proper Lorentz group**, i.e., a connected component of the unit of SO(1,3). Recall that  $SO(1,3) = \mathbb{Z}_2 \times L$ , where  $\mathbb{Z}_2$  is the total reflection group. A **proper Lorentz reduced structure** is defined as a reduced L-subbundle  $L^hX$  of LX. One needs the proper Lorentz reduced structure when Dirac spinor fields in gravitation theory are considered (Section 11).

If a world manifold X is simply connected, there is one-to-ne correspondence between the Lorentz and proper Lorentz reduced structures.

One can show that different proper Lorentz subbundles  $L^hX$  and  $L^{h'}X$  of a frame bundle LX are isomorphic as principal L-bundles. This means that there exists a vertical automorphism of a frame bundle LX which sends  $L^hX$  onto  $L^{h'}X$ . If a world manifold X is simply connected, the similar property of Lorentz subbundles also is true.

There is the well-known topological obstruction to the existence of a Lorentz structure on a world manifold X. All non-compact manifolds and compact manifolds whose Euler characteristic equals zero admit a Lorentz reduced structure [9, 44].

By virtue of Theorem 3.1, there is one-to-one correspondence between the principal Lsubbundles  $L^h X$  of a frame bundle LX and the global sections h of the quotient bundle

$$\Sigma_{\rm T} = LX/{\rm L} \to X,\tag{6.1}$$

called the **tetrad bundle**. This is an *LX*-associated bundle with a typical fibre  $GL_4/L$ . Its global sections are named the **tetrad fields**. The fibre bundle (6.1) is a two-fold covering  $\zeta : \Sigma_{\rm T} \to \Sigma_{\rm PR}$  of the metric bundle  $\Sigma_{\rm PR}$  (3.2). In particular, every tetrad field *h* defines a unique pseudo-Riemannian metric  $g = \zeta \circ h$ .

Every tetrad field h defines an associated Lorentz bundle atlas

$$\Psi^{h} = \{ (U_{\iota}, z_{\iota}^{h} = \{h_{a}\}) \}$$
(6.2)

of a frame bundle LX such that the corresponding local sections  $z_{\iota}^{h}$  of LX take their values into a proper Lorentz subbundle  $L^{h}X$  and the transition functions of  $\Psi^{h}$  (6.2) between the frames  $\{h_{a}\}$  are L-valued. The frames (6.2):

$$\{h_a = h_a^{\mu}(x)\partial_{\mu}\}, \qquad h_a^{\mu} = H_a^{\mu} \circ z_{\iota}^h, \qquad x \in U_{\iota}, \tag{6.3}$$

are called the tetrad frames.

Given a Lorentz bundle atlas  $\Psi^h$ , the pull-back

$$h = h^a \otimes t_a = z_\iota^{h*} \theta_{LX} = h^a_\lambda(x) dx^\lambda \otimes t_a \tag{6.4}$$

of the canonical form  $\theta_{LX}$  (4.3) by a local section  $z_{\iota}^{h}$  is called the (local) tetrad form. It determines tetrad coframes

$$\{h^{a} = h^{a}_{\mu}(x)dx^{\mu}\}, \qquad x \in U_{\iota}, \tag{6.5}$$

in the cotangent bundle  $T^*X$ . They are the dual of the tetrad frames (6.3). The coefficients  $h^{\mu}_a$  and  $h^{\mu}_{\mu}$  of the tetrad frames (6.3) and coframes (6.5) are called the tetrad functions. They are transition functions between the holonomic atlas  $\Psi_T$  (4.2) and the Lorentz atlas  $\Psi^h$  (6.2) of a frame bundle LX.

With respect to the Lorentz atlas  $\Psi^h$  (6.2), a tetrad field h can be represented by the  $\mathbb{R}^4$ -valued tetrad form (6.4). Relative to this atlas, the corresponding pseudo-Riemannian metric  $g = \zeta \circ h$  takes the well-known form

$$g = \eta(h \otimes h) = \eta_{ab}h^a \otimes h^b, \qquad g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}, \tag{6.6}$$

where  $\eta = \text{diag}(1, -1, -1, -1)$  is the Minkowski metric in  $\mathbb{R}^4$  written with respect to its fixed basis  $\{t_a\}$ . It is readily observed that the tetrad coframes  $\{h^a\}$  (6.5) and the tetrad frames  $\{h_a\}$  (6.3) are orthornormal relative to the pseudo-Riemannian metric (6.6), namely:

$$g^{\mu\nu}h^{a}_{\mu}h^{b}_{\nu} = \eta^{ab}, \qquad g_{\mu\nu}h^{\mu}_{a}h^{\nu}_{b} = \eta_{ab}.$$

Therefore, their components  $h^0$ ,  $h_0$  and  $h^i$ ,  $h_i$ , i = 1, 2, 3, are called time-like and spatial, respectively.

**Remark 6.1:** It should be emphasized the difference between tetrad and frame fields. Tetrad fields are global sections of the quotient bundle  $\Sigma_{\rm T} = LX/{\rm L}$  (6.1), whereas frame fields are local sections of a frame bundle LX. Since there is one-to-one correspondence between these sections h and principal L-subbundles  $L^hX$  of a frame bundle LX, a tetrad field h locally is represented by a family of particular frame fields  $z_i^h$  (6.2) taking values into the corresponding Lorentz subbundle  $L^hX \subset LX$ , but modulo L-valued transition functions.  $\Box$ 

Given a pseudo-Riemannian metric g, any linear world connection  $\Gamma$  (5.1) admits a splitting

$$\Gamma_{\mu\nu\alpha} = \{_{\mu\nu\alpha}\} + S_{\mu\nu\alpha} + \frac{1}{2}C_{\mu\nu\alpha}$$
(6.7)

in Christoffel symbols

$$\{_{\mu\nu\alpha}\} = -\frac{1}{2}(\partial_{\mu}g_{\nu\alpha} + \partial_{\alpha}g_{\nu\mu} - \partial_{\nu}g_{\mu\alpha}), \qquad (6.8)$$

a non-metricity tensor

$$C_{\mu\nu\alpha} = C_{\mu\alpha\nu} = \nabla^{\Gamma}_{\mu}g_{\nu\alpha} = \partial_{\mu}g_{\nu\alpha} + \Gamma_{\mu\nu\alpha} + \Gamma_{\mu\alpha\nu}, \qquad (6.9)$$

and a contorsion

$$S_{\mu\nu\alpha} = -S_{\mu\alpha\nu} = \frac{1}{2} (T_{\nu\mu\alpha} + T_{\nu\alpha\mu} + T_{\mu\nu\alpha} + C_{\alpha\nu\mu} - C_{\nu\alpha\mu}), \qquad (6.10)$$

where  $T_{\mu\nu\alpha} = -T_{\alpha\nu\mu}$  are coefficients of the torsion form (5.8) of  $\Gamma$ . The tensor fields T and C, in turn, are decomposed into three and four irreducible summands, respectively [18, 34].

A linear world connection  $\Gamma$  is called the **metric connection** for a pseudo-Riemannian metric g if g is its integral section, i.e., the metricity condition

$$\nabla^{\Gamma}_{\mu}g_{\nu\alpha} = 0 \tag{6.11}$$

holds. A metric connection reads

$$\Gamma_{\mu\nu\alpha} = \{_{\mu\nu\alpha}\} + \frac{1}{2}(T_{\nu\mu\alpha} + T_{\nu\alpha\mu} + T_{\mu\nu\alpha}).$$
(6.12)

The Levi–Civita connection, by definition, is a torsion-free metric connection  $\Gamma_{\mu\nu\alpha} = \{\mu\nu\alpha\}$ .

A principal connection on a proper Lorentz subbundle  $L^h X$  of a frame bundle LX is called the **Lorentz connection**. Since connections on a principal bundle are equivariant, this Lorentz connection is extended to a principal connection  $\Gamma$  on a frame bundle LX. The associated linear connection (5.1) on the tangent bundle TX with respect to the Lorentz atlas  $\Psi^h$  (6.2) reads

$$\Gamma = dx^{\lambda} \otimes \left(\partial_{\lambda} + \frac{1}{2} A_{\lambda}{}^{ab} L_{ab}{}^{c}{}_{d} h^{d}_{\mu} \dot{x}^{\mu} h^{\nu}_{c} \dot{\partial}_{\nu}\right)$$
(6.13)

where

$$L_{ab}{}^{c}{}_{d} = \eta_{bd}\delta^{c}_{a} - \eta_{ad}\delta^{c}_{b}$$

are generators of a right Lie algebra  $\mathfrak{g}_{L}$  of a proper Lorentz group L in a Minkowski space  $\mathbb{R}^{4}$ . Written relative to the holonomic atlas  $\Psi_{T}$  (4.2), the connection  $\Gamma$  (6.13) possesses components

$$\Gamma_{\lambda}{}^{\mu}{}_{\nu} = h^k_{\nu}\partial_{\lambda}h^{\mu}_k + \eta_{ka}h^{\mu}_b h^k_{\nu}A_{\lambda}{}^{ab}.$$
(6.14)

This also is called the Lorentz connection. Its holonomy group is a subgroup of the proper Lorentz group L. Conversely, let  $\Gamma$  be a world connection with the holonomy group L. By virtue of the well known theorem [26, 33], it defines a Lorentz subbundle of a frame bundle LX, and is a Lorentz connection on this subbundle (see also [60]).

One can show that any Lorentz connections is a metric world connection for some pseudo-Riemannian metric g (which is not necessarily unique [66]), and vice versa [26, 33],.

At the same time, any linear world connection  $\Gamma$  (5.1) yields a Lorentz connection  $\Gamma_h$  on each principal L-subbundle  $L^h X$  of a frame bundle [11, 33, 54]. It follows from the fact that the Lie algebra of  $GL_4$  is a direct sum

$$\mathfrak{g}_{GL_4} = \mathfrak{g}_{\mathcal{L}} \oplus \mathfrak{m} \tag{6.15}$$

of the Lie algebra  $\mathfrak{g}_{\mathrm{L}}$  of a Lorentz group and a subspace  $\mathfrak{m}$  such that  $[\mathfrak{g}_{\mathrm{L}}, \mathfrak{m}] \subset \mathfrak{m}$ . Therefore, let us consider a local connection one-form of a connection  $\Gamma$  with respect to the Lorentz atlas  $\Psi^{h}$  (6.2) of LX given by tetrad coframes  $h^{a}$  (6.5). It reads

$$z_{\iota}^{h*}\overline{\Gamma} = -\Gamma_{\lambda}{}^{b}{}_{a}dx^{\lambda} \otimes L_{b}{}^{a}, \qquad \Gamma_{\lambda}{}^{b}{}_{a} = -h^{b}_{\mu}\partial_{\lambda}h^{\mu}_{a} + \Gamma_{\lambda}{}^{\mu}{}_{\nu}h^{b}_{\mu}h^{\nu}_{a}.$$

where  $\{L_b^a\}$  is a basis for a Lie algebra  $\mathfrak{g}_{GL_4}$ . The Lorentz part of this form is precisely a local connection one-form of a connection  $\Gamma_h$  on  $L^h X$ . We have

$$z_{\zeta}^{h*}\overline{\Gamma}_{h} = -\frac{1}{2}A_{\lambda}^{ab}dx^{\lambda} \otimes L_{ab}, \qquad A_{\lambda}^{ab} = \frac{1}{2}(\eta^{kb}h_{\mu}^{a} - \eta^{ka}h_{\mu}^{b})(\partial_{\lambda}h_{k}^{\mu} - h_{k}^{\nu}\Gamma_{\lambda}{}^{\mu}{}_{\nu}). \tag{6.16}$$

Then combining this expression and the expression (6.13) gives a connection

$$\Gamma_h = dx^{\lambda} \otimes (\partial_{\lambda} + \frac{1}{4} (\eta^{kb} h^a_{\mu} - \eta^{ka} h^b_{\mu}) (\partial_{\lambda} h^{\mu}_k - h^{\nu}_k \Gamma_{\lambda}{}^{\mu}{}_{\nu}) L_{ab}{}^c{}_d h^d_{\mu} \dot{x}^{\mu} h^{\nu}_c \dot{\partial}_{\nu})$$
(6.17)

with respect to a Lorentz atlas  $\Psi^h$  and this connection

$$\Gamma_h = dx^{\lambda} \otimes \left[\partial_{\lambda} + \frac{1}{2} (h^k_{\alpha} \delta^{\beta}_{\mu} - \eta^{kc} g_{\mu\alpha} h^{\beta}_c) (\partial_{\lambda} h^{\mu}_k - h^{\nu}_k \Gamma_{\lambda}{}^{\mu}{}_{\nu}) \dot{x}^{\alpha} \partial_{\beta}\right]$$
(6.18)

relative to a holonomic atlas. If  $\Gamma$  is the Lorentz connection (6.14) extended from  $L^h X$ , then obviously  $\Gamma_h = \Gamma$ .

#### 7 Space-time structure

There is the well-known theorem [11, 50, 56].

THEOREM 7.1: A structure Lie group G of a principal bundle over a paracompact manifold always is reducible to its maximal compact subgroup H.  $\Box$ 

This follows from Theorem 3.1 and the facts that the quotient G/H of a Lie group G by its maximal compact subgroup H is diffeomorphic to an Euclidean space  $\mathbb{R}^m$  and that a fibre bundle over a paracompact manifold admits a global section if its typical fibre is an Euclidean space  $\mathbb{R}^m$  [62].

A corollary of Theorem 7.1) is that a structure group  $GL_4$  of a frame bundle LX is reducible to its maximal compact subgroup SO(4). In gravitation theory, if a structure group  $GL_4$  of LX is reducible to a proper Lorentz group L, it is always reducible to the maximal compact subgroup SO(3) of L. Thus, there is a commutative diagram

of the reduction of structure groups of a frame bundle LX in gravitation theory. This reduction diagram results in the following.

• There is one-to-one correspondence between the reduced principal SO(4)-subbundles  $L^{g^R}X$  of a frame bundle LX and the global sections of the quotient bundle  $LX/SO(4) \to X$ . Its global sections are Riemannian metrics  $g^R$  on X. Thus, a Riemannian metric on a world manifold always exists.

• As was mentioned above, a reduction of a structure group of a frame bundle LX to a proper Lorentz group implies the existence of a reduced proper Lorentz subbundle  $L^h X \subset LX$  associated to a tetrad field h or a pseudo-Riemannian metric  $g = \zeta \circ h$  on X.

• Since a structure group L of this reduced Lorentz bundle  $L^hX$  is reducible to a group SO(3), there exists a reduced principal SO(3)-subbundle

$$L_0^h X \subset L^h X \subset LX, \tag{7.2}$$

called the spatial structure. The corresponding global section of the quotient fibre bundle  $L^h X/SO(3) \to X$  with a typical fibre  $\mathbb{R}^3$  is a one-codimensional spatial distribution  $\mathbf{F} \subset TX$  on X. Its annihilator is a one-dimensional codistribution  $\mathbf{F}^* \subset T^*X$ .

Given the spatial structure  $L_0^h X$  (7.2), let us consider the Lorentz bundle atlas  $\Psi_0^h$  (6.2) given by local sections  $z_i$  of LX taking their values into a reduced SO(3)-subbundle  $L_0^h X$ . Its transition functions are SO(3)-valued.

It follows that, in gravitation theory on a world manifold X, one can always choose an atlas of the tangent bundle TX and associated bundles with SO(3)-valued transition functions. It is called the spatial bundle atlas.

Given a spatial bundle atlas  $\Psi_0^h$ , its SO(3)-valued transition functions preserve a time-like component

$$h^0 = h^0_\lambda dx^\lambda \tag{7.3}$$

of local tetrad forms (6.4) which, therefore, is globally defined. We agree to call it the time-like tetrad form. Accordingly, the dual time-like vector field

$$h_0 = h_0^\mu \partial_\mu \tag{7.4}$$

also is globally defined. In this case, a spatial distribution  $\mathbf{F}$  is spanned by spatial components  $h_i$ , i = 1, 2, 3, of the tetrad frames (6.3), while the time-like tetrad form (7.3) spans the tetrad codistribution  $\mathbf{F}^*$ , i.e.,

$$h^0 | \mathbf{F} = 0. \tag{7.5}$$

Then the tangent bundle TX of a world manifold X admits a space-time decomposition

$$TX = \mathbf{F} \oplus T^0 X, \tag{7.6}$$

where  $T^0X$  is a one-dimensional fibre bundle spanned by the time-like vector field  $h_0$  (7.4).

Due to the commutative diagram (7.1), the reduced L-subbundle  $L_0^h X$  (7.2) of a reduced Lorentz bundle  $L^h X$  is a reduced subbundle of some reduced SO(4)-bundle  $L^{g^R} X$  too, i.e.,

$$L^h X \supset L^h_0 X \subset L^{g^R} X. \tag{7.7}$$

Let  $g = \zeta \circ h$  and  $g^R$  be the corresponding pseudo-Riemannian and Riemannian metrics on X. Written with respect to a spatial bundle atlas  $\Psi_0^h$ , they read

$$g = \eta_{ab}h^a \otimes h^b, \qquad g_{\mu\nu} = h^a_\mu h^b_\nu \eta^{ab}, \tag{7.8}$$

$$g^R = \eta^E_{ab} h^a \otimes h^b, \qquad g^R_{\mu\nu} = h^a_\mu h^b_\nu \eta^E_{ab}, \tag{7.9}$$

where  $\eta^E$  is an Euclidean metric in  $\mathbb{R}^4$ . The space-time decomposition (7.6) is orthonormal with respect to both the metrics (7.8) and (7.9). Thus, we come to the following well-known results [11, 16, 44].

• For any pseudo-Riemannian metric g on a world manifold X, there exist a normalized time-like one-form  $h^0$  and a Riemannian metric  $g^R$  such that

$$g = 2h^0 \otimes h^0 - g^R. (7.10)$$

Conversely, let a world manifold X admit a nowhere vanishing one-form  $\sigma$  (or, equivalently, a nowhere vanishing vector field). Then any Riemannian world metric  $g^R$  on X yields the pseudo-Riemannian world metric g (7.10) where  $h^0 = \sigma (g^R(\sigma, \sigma))^{-1/2}$ .

• A world manifold X admits a pseudo-Riemannian metric iff there exists a nowhere vanishing one-form (or a vector field) on X.

Note that the condition (7.7) gives something more. Namely, there is one-to-one correspondence between the reduced SO(3)-subbundles of a frame bundle LX and the triples  $(g, \mathbf{F}, g^R)$ of a pseudo-Riemannian metric g, a spatial distribution  $\mathbf{F}$  defined by the condition (7.5) and a Riemannian metric  $g^R$  which obey the relation (7.10). A spatial distribution  $\mathbf{F}$  and a Riemannian metric  $g^R$  in the triple  $(g, \mathbf{F}, g^R)$  are called g-compatible. The corresponding space-time decomposition is said to be a g-compatible **space-time structure**. A world manifold endowed with a pseudo-Riemannian metric and a compatible space-time structure is called the **space-time**.

**Remark 7.1:** A g-compatible Riemannian metric  $g^R$  in a triple  $(g, \mathbf{F}, g^R)$  defines a gcompatible distance function d(x, x') on a world manifold X. Such a function brings X into a metric space whose locally Euclidean topology is equivalent to a manifold topology on X. Given a gravitational field g, the g-compatible Riemannian metrics and the corresponding distance functions are different for different spatial distributions  $\mathbf{F}$  and  $\mathbf{F'}$ . It follows that physical observers associated to different spatial distributions  $\mathbf{F}$  and  $\mathbf{F'}$  perceive a world manifold X as different Riemannian spaces. The well-known relativistic changes of sizes of moving bodies exemplify this phenomenon. Note that there were attempts of deriving a world topology directly from its pseudo-Riemannian structure (e.g., path topology,  $C^0$ -topology, etc.) [12, 16]. However, these topologies are rather extraordinary, e.g., they are the non-Hausdorff ones.  $\Box$ 

### 8 Metric-affine gauge gravitation theory

In the absence of matter fields, dynamic variables of gauge gravitation theory are linear world connections and pseudo-Riemannian metrics on X [11, 33, 57]. Their Lagrangian  $L_{\rm MA}$  is invariant under general covariant transformations.

This is the case of metric-affine gravitation theory [1, 6, 18, 19, 34, 39]. Let us however emphasize that we consider general linear connections which need not be metric (Lorentz) connections.

**Remark 8.1:** In view of the decomposition (6.7), one can choose a different collection of dynamic variables of metric-affine gauge gravitation theory. These are a pseudo-Riemannian metric, the torsion (5.7) and the non-metricity tensor (6.9).  $\Box$ 

World connections are represented by sections of the bundle of world connections  $C_{\rm W}$  (5.9). World metrics are described by sections of the quotient bundle (3.2). Therefore, let us consider the bundle product

$$Y_{\mathrm{M}A} = \Sigma_{\mathrm{PR}} \underset{X}{\times} C_{\mathrm{W}},\tag{8.1}$$

coordinated by  $(x^{\lambda}, \sigma^{\mu\nu}, k_{\mu}{}^{\alpha}{}_{\beta})$ .

Let us restrict our consideration to first order Lagrangian theory on  $Y_{MA}$ . Then a configuration space of gauge gravitation theory is the first order jet manifold

$$J^{1}Y_{\mathrm{M}A} = J^{1}\Sigma_{\mathrm{PR}} \underset{X}{\times} J^{1}C_{\mathrm{W}}, \qquad (8.2)$$

coordinated by  $(x^{\lambda}, \sigma^{\mu\nu}, k_{\mu}{}^{\alpha}{}_{\beta}, \sigma^{\mu\nu}_{\lambda}, k_{\lambda\mu}{}^{\alpha}{}_{\beta})$  [11, 54, 55]. A first order Lagrangian  $L_{\rm MA}$  of metricaffine gauge gravitation theory is a defined as a density

$$L_{\rm MA} = \mathcal{L}_{\rm AM}(x^{\lambda}, \sigma^{\mu\nu}, k_{\mu}{}^{\alpha}{}_{\beta}, \sigma^{\mu\nu}_{\lambda}, k_{\lambda\mu}{}^{\alpha}{}_{\beta})\omega, \qquad \omega = dx^1 \wedge \dots \wedge dx^4, \tag{8.3}$$

on the configuration space  $J^{1}Y$  (8.2). Its Euler-Lagrange operator is

$$\begin{split} \delta L_{\mathrm{MA}} &= \left(\mathcal{E}_{\alpha\beta}d\sigma^{\alpha\beta} + \mathcal{E}^{\mu}{}_{\alpha}{}^{\beta}dk_{\mu}{}^{\alpha}{}_{\beta}\right) \wedge \omega. \\ \mathcal{E}_{\alpha\beta} &= \left(\frac{\partial}{\partial\sigma^{\alpha\beta}} - d_{\lambda}\frac{\partial}{\partial\sigma^{\alpha\beta}_{\lambda}}\right)\mathcal{L}_{\mathrm{AM}}, \qquad \mathcal{E}^{\mu}{}_{\alpha}{}^{\beta} &= \left(\frac{\partial}{\partial k_{\mu}{}^{\alpha}{}_{\beta}} - d_{\lambda}\frac{\partial}{\partial k_{\lambda\mu}{}^{\alpha}{}_{\beta}}\right)\mathcal{L}_{\mathrm{AM}}, \\ d_{\lambda} &= \partial_{\lambda} + \sigma^{\alpha\beta}_{\lambda}\frac{\partial}{\partial\sigma^{\alpha\beta}} + k_{\lambda\mu}{}^{\alpha}{}_{\beta}\frac{\partial}{\partial k_{\mu}{}^{\alpha}{}_{\beta}} + \sigma^{\alpha\beta}_{\lambda\nu}\frac{\partial}{\partial\sigma^{\alpha\beta}_{\nu}} + k_{\lambda\nu\mu}{}^{\alpha}{}_{\beta}\frac{\partial}{\partial k_{\nu\mu}{}^{\alpha}{}_{\beta}}. \end{split}$$

The corresponding Euler–Lagrange equations read

$$\mathcal{E}_{\alpha\beta} = 0, \qquad \mathcal{E}^{\mu}{}_{\alpha}{}^{\beta} = 0.$$

The fibre bundle  $Y_{MA}$  (8.1) is a natural bundle admitting the functorial lift

$$\widetilde{\tau}_{\Sigma C} = \tau^{\mu} \partial_{\mu} + (\sigma^{\nu\beta} \partial_{\nu} \tau^{\alpha} + \sigma^{\alpha\nu} \partial_{\nu} \tau^{\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (\partial_{\nu} \tau^{\alpha} k_{\mu}{}^{\nu}{}_{\beta} - \partial_{\beta} \tau^{\nu} k_{\mu}{}^{\alpha}{}_{\nu} - \partial_{\mu} \tau^{\nu} k_{\nu}{}^{\alpha}{}_{\beta} + \partial_{\mu\beta} \tau^{\alpha}) \frac{\partial}{\partial k_{\mu}{}^{\alpha}{}_{\beta}}$$

$$(8.4)$$

of vector fields  $\tau$  on X. It is an infinitesimal generator of general covariant transformations. At the same time,  $\tilde{\tau}_{\Sigma C}$  (8.4) also is a gauge transformation whose gauge parameters are components  $\tau^{\lambda}(x)$  of vector fields  $\tau$  on X.

By virtue of Relativity Principle, the Lagrangian  $L_{\rm MA}$  (8.3) of metric-affine gauge gravitation theory is assumed to be invariant under general covariant transformations. Its Lie derivative along the jet prolongation  $J^1 \tilde{\tau}_{\Sigma C}$  of the vector field  $\tilde{\tau}_{\Sigma C}$  (8.4) for any  $\tau$  vanishes, i.e.,

$$\mathbf{L}_{J^1 \widetilde{\tau}_{\Sigma C}} L_{\mathrm{MA}} = 0. \tag{8.5}$$

Since a configuration space  $J^1C_W$  of world connections possesses the canonical splitting (5.10), the following analogy to the well-known Utiyama theorem in Yang–Mills gauge theory is true.

THEOREM 8.1: If the first order Lagrangian  $L_{MA}$  (8.3) on the configuration space (8.2) is invariant under general covariant transformations and it does not depend on the jet coordinates  $\sigma_{\lambda}^{\alpha\beta}$  (i.e., derivatives of a metric), this Lagrangian factorizes through the terms  $\mathcal{R}_{\lambda\mu}{}^{\alpha}{}_{\beta}$  (5.10).

In contrast with the well-known Lagrangian of Yang–Mills gauge theory, different contractions of a curvature tensor  $\mathcal{R}_{\lambda\mu}{}^{\alpha}{}_{\beta}$  are possible. For instance, the Ricci tensor  $R_c$  (5.5) and a scalar curvature  $\mathcal{R}$  are defined. Moreover, a Lagrangian  $L_{\rm MA}$  also can depend separately on a torsion

$$t_{\mu}{}^{\nu}{}_{\lambda} = k_{\mu}{}^{\nu}{}_{\lambda} - k_{\lambda}{}^{\nu}{}_{\mu}. \tag{8.6}$$

**Example 8.2:** In metric-affine gravitation theory, the Hilbert–Einstein Lagrangian of General Relativity takes a form

$$L_{\rm GR} = \mathcal{R}\sqrt{\sigma\omega} = \sigma^{\mu\beta} \mathcal{R}_{\lambda\mu}{}^{\lambda}{}_{\beta}\sqrt{\sigma\omega}.$$
(8.7)

The corresponding Euler–Lagrange equations read

$$\mathcal{E}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{2}\sigma_{\alpha\beta}\mathcal{R} = 0, \qquad (8.8)$$

$$\mathcal{E}^{\nu}{}_{\alpha}{}^{\beta} = -d_{\alpha}(\sigma^{\nu\beta}\sqrt{\sigma}) + d_{\lambda}(\sigma^{\lambda\beta}\sqrt{\sigma})\delta^{\nu}_{\alpha} +$$

$$(8.9)$$

$$(\sigma^{\nu\gamma}k_{\alpha}{}^{\beta}{}_{\gamma} - \sigma^{\lambda\gamma}\delta^{\nu}_{\alpha}k_{\lambda}{}^{\beta}{}_{\gamma} - \sigma^{\nu\beta}k_{\gamma}{}^{\gamma}{}_{\alpha} + \sigma^{\lambda\beta}k_{\lambda}{}^{\nu}{}_{\alpha})\sqrt{\sigma} = 0.$$

The equation (8.8) is an analogy of the Einstein equations, whereas the equation (8.9) describes the torsion (8.6) and the non-metricity

$$c_{\mu\nu\alpha} = c_{\mu\alpha\nu} = d_{\mu}\sigma_{\nu\alpha} + k_{\mu}{}^{\beta}{}_{\alpha}\sigma_{\nu\beta} + k_{\mu}{}^{\beta}{}_{\nu}\sigma_{\beta\alpha}$$

of a linear world connection. It is brought into a form

$$\sqrt{\sigma^{-1}}\sigma_{\nu\varepsilon}\sigma_{\beta\mu}\mathcal{E}^{\nu}{}_{\alpha}{}^{\beta} = c_{\alpha\varepsilon\mu} - \frac{1}{2}\sigma_{\mu\varepsilon}\sigma^{\lambda\gamma}c_{\alpha\lambda\gamma} - \sigma_{\alpha\varepsilon}\sigma^{\lambda\beta}c_{\lambda\beta\mu} + \frac{1}{2}\sigma_{\alpha\varepsilon}\sigma^{\lambda\gamma}c_{\mu\lambda\gamma} + t_{\mu\varepsilon\alpha} + \sigma_{\mu\varepsilon}t_{\alpha}{}^{\gamma}{}_{\gamma} + \sigma_{\alpha\varepsilon}t_{\gamma}{}^{\gamma}{}_{\mu} = 0.$$

**Example 8.3:** The Yang–Mills Lagrangian

$$L_{\rm YM} = \sigma^{\mu\lambda} \sigma^{\nu\gamma} \mathcal{R}_{\mu\nu}{}^{\alpha}{}_{\beta} \mathcal{R}_{\lambda\gamma}{}^{\beta}{}_{\alpha} \sqrt{\sigma} \omega$$

in metric-affine gauge gravitation theory also is considered. It is invariant under a total group  $\operatorname{Aut}(LX)$  of automorphisms of a frame bundle LX (Remark 4.1). In this case, metric variables  $\sigma^{\mu\lambda}$  fail to be dynamic because they are brought into a constant Minkowski metric by general frame transformations.  $\Box$ 

### 9 Energy-momentum conservation law

Since infinitesimal general covariant transformations  $\tilde{\tau}_{\Sigma C}$  (8.4) are exact symmetries of a metricaffine gravitation Lagrangian, let us study the corresponding conservation law. This is the energy-momentum conservation laws because vector fields  $\tilde{\tau}_{\Sigma C}$  are not vertical [11, 46]. Moreover, since infinitesimal general covariant transformations  $\tilde{\tau}_{\Sigma C}$  (8.4) are gauge transformations depending on derivatives of gauge parameters, the corresponding energy-momentum current reduces to a superpotential [11, 53].

In view of Theorem 8.1, let us assume that the metric-affine gravitation Lagrangian  $L_{\rm MA}$ (8.3) is independent of the derivative coordinates  $\sigma_{\lambda}{}^{\alpha\beta}$  of a world metric and that it factorizes through the curvature terms  $\mathcal{R}_{\lambda\mu}{}^{\alpha}{}_{\beta}$  (5.10). Then the following relations hold:

$$\pi^{\lambda\nu}{}_{\alpha}{}^{\beta} = -\pi^{\nu\lambda}{}_{\alpha}{}^{\beta}, \qquad \pi^{\lambda\nu}{}_{\alpha}{}^{\beta} = \frac{\partial \mathcal{L}_{\mathrm{MA}}}{\partial k_{\lambda\nu}{}^{\alpha}{}_{\beta}},$$
$$\frac{\partial \mathcal{L}_{\mathrm{MA}}}{\partial k_{\nu}{}^{\alpha}{}_{\beta}} = \pi^{\lambda\nu}{}_{\alpha}{}^{\sigma}k_{\lambda}{}^{\beta}{}_{\sigma} - \pi^{\lambda\nu}{}_{\sigma}{}^{\beta}k_{\lambda}{}^{\sigma}{}_{\alpha}.$$

Let us use the compact notation

$$y^{A} = k_{\mu}{}^{\alpha}{}_{\beta}, \qquad u_{\mu}{}^{\alpha}{}_{\beta}{}_{\gamma}{}^{\varepsilon\sigma} = \delta^{\varepsilon}_{\mu}\delta^{\sigma}_{\beta}\delta^{\alpha}_{\gamma}, \qquad u_{\mu}{}^{\alpha}{}_{\beta}{}_{\gamma}{}^{\varepsilon} = k_{\mu}{}^{\varepsilon}{}_{\beta}\delta^{\alpha}_{\gamma} - k_{\mu}{}^{\alpha}{}_{\gamma}\delta^{\varepsilon}_{\beta} - k_{\gamma}{}^{\alpha}{}_{\beta}\delta^{\varepsilon}_{\mu}.$$

Then the vector field (8.4) takes a form

$$\widetilde{\tau}_{\Sigma C} = \tau^{\lambda} \partial_{\lambda} + (\sigma^{\nu\beta} \partial_{\nu} \tau^{\alpha} + \sigma^{\alpha\nu} \partial_{\nu} \tau^{\beta}) \partial_{\alpha\beta} + (u^{A\beta}{}_{\alpha} \partial_{\beta} \tau^{\alpha} + u^{A\beta\mu}{}_{\alpha} \partial_{\beta\mu} \tau^{\alpha}) \partial_{A}.$$

Let  $L_{\rm MA}$  be invariant under general covariant transformations, i.e., the equality (8.5) for any vector field  $\tau$  is satisfied. On-shell, we then have a weak conservation law

$$0 \approx -d_{\lambda} [\pi^{\lambda}_{A} (y^{A}_{\alpha} \tau^{\alpha} - u^{A\beta}_{\ \alpha} \partial_{\beta} \tau^{\alpha} - u^{A\varepsilon\beta}_{\ \alpha} \partial_{\varepsilon\beta} \tau^{\alpha}) - \tau^{\lambda} \mathcal{L}_{\mathrm{MA}}]$$
(9.1)

of the **energy-momentum current** of metric-affine gravity

$$\mathcal{J}_{\mathrm{MA}}{}^{\lambda} = \pi_{A}^{\lambda} (y_{\alpha}^{A} \tau^{\alpha} - u_{\ \alpha}^{A\beta} \partial_{\beta} \tau^{\alpha} - u_{\ \alpha}^{A\varepsilon\beta} \partial_{\varepsilon\beta} \tau^{\alpha}) - \tau^{\lambda} \mathcal{L}_{\mathrm{MA}}.$$
(9.2)

**Remark 9.1:** It is readily observed that, with respect to a local coordinate system where a vector field  $\tau$  is constant, the energy-momentum current (9.2) leads to a canonical energy-momentum tensor

$$\mathcal{J}_{\mathrm{MA}}{}^{\lambda}{}_{\alpha}\tau^{\alpha} = (\pi^{\lambda\mu}{}_{\beta}{}^{\nu}k_{\alpha\mu}{}^{\beta}{}_{\nu} - \delta^{\lambda}_{\alpha}\mathcal{L}_{\mathrm{MA}})\tau^{\alpha},$$

suggested in order to describe an energy-momentum complex in the Palatini model [8].  $\Box$ 

Due to the arbitrariness of  $\tau^{\lambda}$ , we have a set of equalities

$$\pi^{(\lambda\varepsilon_{\gamma}\sigma)} = 0,$$

$$(u^{A\varepsilon\sigma}_{\gamma}\partial_{A} + u^{A\varepsilon}_{\gamma}\partial_{A}^{\sigma})\mathcal{L}_{MA} = 0,$$

$$\delta^{\beta}_{\alpha}\mathcal{L}_{MA} + 2\sigma^{\beta\mu}\delta_{\alpha\mu}\mathcal{L}_{MA} + u^{A\beta}_{\alpha}\delta_{A}\mathcal{L}_{MA} + d_{\mu}(\pi^{\mu}_{A}u^{A\beta}_{\ \alpha}) - y^{A}_{\alpha}\pi^{\beta}_{A} = 0,$$

$$\partial_{\lambda}\mathcal{L}_{MA} = 0.$$
(9.3)

Substituting the term  $y^A_{\alpha} \pi^{\beta}_A$  from the expression (9.3) in the energy-momentum conservation law (9.1), one brings this conservation law into a form

$$0 \approx -d_{\lambda} [2\sigma^{\lambda\mu}\tau^{\alpha}\delta_{\alpha\mu}\mathcal{L}_{\mathrm{MA}} + u^{A\lambda}_{\ \alpha}\tau^{\alpha}\delta_{A}\mathcal{L}_{\mathrm{MA}} - \pi^{\lambda}_{A}u^{A\beta}_{\ \alpha}\partial_{\beta}\tau^{\alpha} + d_{\mu}(\pi^{\lambda\mu}_{A}u^{A\lambda}_{\ \alpha})\tau^{\alpha} - d_{\mu}(\pi^{\lambda\mu}_{\ \alpha}{}^{\beta}\partial_{\beta}\tau^{\alpha})].$$

$$(9.4)$$

After separating the variational derivatives, the energy-momentum conservation law (9.4) of a metric-affine gravity takes a superpotential form

$$0 \approx -d_{\lambda} [2\sigma^{\lambda\mu}\tau^{\alpha}\delta_{\alpha\mu}\mathcal{L}_{\mathrm{MA}} + (k_{\mu}{}^{\lambda}{}_{\gamma}\delta^{\mu}{}_{\alpha}{}^{\gamma}\mathcal{L}_{\mathrm{MA}} - k_{\mu}{}^{\sigma}{}_{\alpha}\delta^{\mu}{}_{\sigma}{}^{\lambda}\mathcal{L}_{\mathrm{MA}} - k_{\alpha}{}^{\sigma}{}_{\gamma}\delta^{\lambda}{}_{\sigma}{}^{\gamma}\mathcal{L}_{\mathrm{MA}})\tau^{\alpha} + \delta^{\lambda}{}_{\alpha}{}^{\mu}\mathcal{L}_{\mathrm{MA}}\partial_{\mu}\tau^{\alpha} - d_{\mu}(\delta^{\mu}{}_{\alpha}{}^{\lambda}\mathcal{L}_{\mathrm{MA}})\tau^{\alpha} + d_{\mu}(\pi^{\mu\lambda}{}_{\alpha}{}^{\nu}(\partial_{\nu}\tau^{\alpha} - k_{\sigma}{}^{\alpha}{}_{\nu}\tau^{\sigma}))],$$

where an energy-momentum current on-shell reduces to a **generalized Komar superpoten**tial

$$U_{\rm MA}{}^{\mu\lambda} = 2 \frac{\partial \mathcal{L}_{\rm MA}}{\partial \mathcal{R}_{\mu\lambda}{}^{\alpha}{}_{\nu}} (D_{\nu}\tau^{\alpha} + t_{\nu}{}^{\alpha}{}_{\sigma}\tau^{\sigma}), \qquad (9.5)$$

where  $D_{\nu}$  is a covariant derivative relative to a connection  $k_{\nu}^{\alpha}{}_{\sigma}$  and  $t_{\nu}^{\alpha}{}_{\sigma}$  is its torsion [10, 47, 54].

In particular, the Hilbert–Einstein Lagrangian (8.7) is invariant under general covariant transformations. The corresponding generalized Komar superpotential (9.5) comes to the well-known Komar superpotential if one substitutes the Levi–Civita connection  $k_{\nu}{}^{\alpha}{}_{\sigma} = \{\nu^{\alpha}{}_{\sigma}\}$ .

#### **10** Spinor structure

In classical field theory, Dirac spinor fields usually are represented by sections of a spinor bundle on a world manifold X whose typical fibre is a Dirac spinor space  $\Psi(1,3)$  and whose structure group is a Lorentz spin group Spin(1,3). In order to introduce the Dirac operator, one however must assume that Dirac spinors carry out a representation of a Clifford algebra. Moreover, we describe spinor spaces as subspaces of Clifford algebras and define spinor bundles as subbundles of fibre bundles in Clifford algebras [11, 54, 59].

Note that spinor representations of Lie algebras so(m, n - m) of pseudo-orthogonal Lie groups SO(m, n - m),  $n \ge 1$ , m = 0, 1, ..., n, were discovered by E. Cartan in 1913, when he classified finite-dimensional representations of simple Lie algebras [7]. Though, there is a problem of spinor representations of pseudo-orthogonal Lie groups SO(m, n - m) themselves. Spinor representations are attributes of Spin groups Spin(m, n-m). Spin groups Spin(m, n-m) are two-fold coverings (10.18) of pseudo-orthogonal groups SO(m, n - m).

Spin groups  $\operatorname{Spin}(m, n - m)$  are defined as certain subgroups of real Clifford algebras  $\mathcal{C}\ell(m, n-m)$  (10.16). Moreover, spinor representations of Spin groups in fact are the restriction of spinor representation of real Clifford algebras to its Spin subgroups. As was mentioned above, one needs an action of a whole real Clifford algebra in a spinor space in order to construct a Dirac operator. In 1935, R. Brauer and H. Weyl described spinor representations in terms of Clifford algebras [4, 28]. This description is based on the following.

• Real Clifford algebras  $\mathcal{C}\ell(m, n-m)$  and complex Clifford algebras  $\mathbb{C}\mathcal{C}\ell(n)$  of even dimension n are isomorphic to matrix algebras (Theorems 10.2 and 10.4, respectively). Therefore, they are simple, and all their automorphisms are inner (Theorems 10.7 and 10.9). Their invertible elements constitute general linear matrix groups. They act on Clifford algebras by a left-regular representation, and their adjoint representation exhaust all automorphisms of Clifford algebras.

• Given a real Clifford algebra  $\mathcal{C}\ell(m, n-m)$ , the corresponding spinor space  $\Psi(m, n-m)$  is defined as a carrier space of its exact irreducible representation. This representation of a real Clifford algebra  $\mathcal{C}\ell(m, n-m)$  of even dimension n is unique up to an equivalence (Theorem 10.3).

However, spinor spaces  $\Psi(m, n-m)$  and  $\Psi(m', n-m')$  need not be isomorphic vector spaces for  $m' \neq m$ . For instance, a Dirac spinor space is defined to be a spinor space  $\Psi(1,3)$  of a real Clifford algebra  $\mathcal{C}\ell(1,3)$ . It differs from a Majorana spinor space  $\Psi(3,1)$  of a real Clifford algebra  $\mathcal{C}\ell(3,1)$ . In contrast with the four-dimensional real matrix representation (11.3) of  $\mathcal{C}\ell(3,1)$ , the representation (11.5) of a real Clifford algebra  $\mathcal{C}\ell(3,1)$  by complex Dirac's matrices is not a representation of a real Clifford algebra. By this reason and because, from the physical viewpoint, Dirac spinor fields describing charged fermions are complex fields, we focus our consideration on complex Clifford algebras and complex spinors.

• A complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  of even dimension n is proved to be isomorphic to a ring  $\operatorname{Mat}(2^{n/2},\mathbb{C})$  of complex  $(2^{n/2} \times 2^{n/2})$ -matrices (Theorem 10.4). The corresponding complex spinor space  $\Psi(n)$  is defined as a carrier space of its exact irreducible representation. Due to the canonical monomorphism  $\mathcal{C}\ell(m, n-m) \to \mathbb{C}\mathcal{C}\ell(n)$  (10.10) of real Clifford algebras to the complex ones, a complex spinor space  $\Psi(n)$  admits a representation of a real Clifford algebra  $\mathcal{C}\ell(m, n-m)$ , though it need not be irreducible.

• Similarly to a case of real Clifford algebras, an exact irreducible representation of a complex Clifford algebra  $\mathbb{CC}\ell(n)$  of even dimension n is unique up to an equivalence (Theorem 10.6). Therefore, we define a complex spinor space  $\Psi(n)$  in a case of even n as a minimal left ideal of a complex Clifford algebra  $\mathbb{CC}\ell(n)$ . Thus, a spinor representation

$$\gamma: \mathbb{C}\mathcal{C}\ell(n) \times \Psi(n) \to \Psi(n) \tag{10.1}$$

of a Clifford algebra  $\mathbb{CC}\ell(n)$  is equivalent to the canonical representation of  $\operatorname{Mat}(2^{n/2}, \mathbb{C})$  by matrices in a complex vector space  $\Psi(n) = \mathbb{C}^{2^{n/2}}$ .

Treating a complex spinor space  $\Psi(n)$  as a subspace of a complex Clifford algebra  $\mathbb{CC}\ell(n)$ which carries out its left-regular representation (10.1), we believe reasonable to consider a fibre bundle in spinor spaces  $\Psi(n)$  as a subbundle of a fibre bundle in Clifford algebras. However, one usually considers fibre bundles in Clifford algebras whose structure group is a group of automorphisms of these algebras [11, 28]. A problem is that this group fails to preserve spinor subspaces  $\Psi(n)$  of a complex Clifford algebra  $\mathbb{CC}\ell(n)$  (Remark 10.3) and, thus, it can not be a structure group of spinor bundles.

Therefore, we define fibre bundles  $\mathcal{C}$  (10.24) in Clifford algebras  $\mathbb{CC}\ell(n)$  whose structure group is a general linear group  $GL(2^{n/2}, \mathbb{C})$  of invertible elements of  $\mathbb{CC}\ell(n)$  which acts on this algebra by left multiplications [59]. Certainly, it preserves minimal left ideals of this algebra and, consequently, is a structure group of spinor subbundles S of a Clifford algebra bundle  $\mathcal{C}$ .

It should be emphasized that, though there is the ring monomorphism  $\mathcal{C}\ell(m, n - m) \to \mathbb{C}\mathcal{C}\ell(n)$  (10.10), the Clifford algebra bundle  $\mathcal{C}$  (10.24) need not contain a subbundle in real Clifford algebras  $\mathcal{C}\ell(m, n - m)$  unless a structure group  $GL(2^{n/2}, \mathbb{C})$  of  $\mathcal{C}$  is reducible to a group  $\mathcal{G}\mathcal{C}\ell(m, n - m)$  of invertible elements of  $\mathcal{C}\ell(m, n - m)$ . Let X be an n-dimensional smooth manifold and LX a principal frame bundle over X. In accordance with Theorem 3.1, any global section h of the quotient bundle  $\Sigma(m, n - m) = LX/O(m, n - m) \to X$  (10.36) is associated to the fibre bundle  $\mathcal{C}^h \to X$  (10.30) in complex Clifford algebras  $\mathcal{C}\ell(m, n - m)$  and a spinor subbundle  $S^h \to X$ .

A key point is that, given different sections h and h' of the quotient bundle  $\Sigma(m, n-m) \to X$  (10.36), the Clifford algebra bundles  $\mathcal{C}^h$  and  $\mathcal{C}^{h'}$  need not be isomorphic.

In order to describe all these non-isomorphic Clifford algebra bundles  $\mathcal{C}^h$ , we follow a construction of composite bundles. We consider composite Clifford algebra bundles  $\mathcal{C}_{\Sigma}$  (10.43) and  $\mathcal{C}(m, n - m)_{\Sigma}$  (10.44), and the spinor bundle  $S_{\Sigma}$  (10.45) over a base  $\Sigma(m, n - m)$  (10.36). Then given a global section h of the quotient bundle  $\Sigma(m, n - m) \to X$  (10.36), the pullback bundles  $h^*\mathcal{C}_{\Sigma}$ ,  $h^*\mathcal{C}(m, n - m)_{\Sigma}$  and  $h^*S_{\Sigma}$  are the above mentioned fibre bundles  $\mathcal{C}^h \to X$ ,  $\mathcal{C}^h(m, n - m) \to X$  and  $S^h \to X$ , respectively.

#### 10.1 Clifford algebras

A real Clifford algebra is defined as a ring (i.e., a unital associative algebra) possessing a certain vector subspace of generating elements. However, such a ring can possess different generating spaces. Therefore, we also consider a real Clifford algebra without specifying its generating space.

Let  $V = \mathbb{R}^n$  be an *n*-dimensional real vector space provided with a non-degenerate bilinear

form (a pseudo-Euclidean metric)  $\eta$ . Let us consider a tensor algebra

$$\otimes V = \mathbb{R} \oplus V \oplus \overset{2}{\otimes} V \oplus \cdots \oplus \overset{k}{\otimes} V \oplus \cdots$$

of V and its two-sided ideal  $I_\eta$  generated by the elements

$$v \otimes v' + v' \otimes v - 2\eta(v, v')e, \qquad v, v' \in V,$$

where e denotes the unit element of  $\otimes V$ . The quotient  $\otimes V/I_{\eta}$  is a real non-commutative ring. A real ring  $\otimes V/I_{\eta}$  together with a fixed generating subspace  $(V, \eta)$  is called the **real Clifford algebra**  $\mathcal{C}\ell(V, \eta)$  modelled over a pseudo-Euclidean space  $(V, \eta)$ .

There is the canonical monomorphism of a real vector space V to the quotient  $\otimes V/I_{\eta}$ . It is a generating subspace of a real ring  $\otimes V/I_{\eta}$ . Its elements obey the relations

$$vv' + v'v - 2\eta(v, v')e = 0, \quad v, v' \in V.$$

Given real Clifford algebras  $\mathcal{C}\ell(V,\eta)$  and  $\mathcal{C}\ell(V',\eta')$ , by their isomorphism is meant an isomorphism of them as rings:

$$\phi: \mathcal{C}\ell(V,\eta) \to \mathcal{C}\ell(V',\eta'), \qquad \phi(qq') = \phi(q)\phi(q'), \tag{10.2}$$

which also is an isometric isomorphism of their generating pseudo-Euclidean spaces:

$$\phi : \mathcal{C}\ell(V,\eta) \supset (V,\eta) \rightarrow (V',\eta') \subset \mathcal{C}\ell(V',\eta'),$$

$$2\eta'(\phi(v),\phi(v')) = \phi(v)\phi(v') + \phi(v')\phi(v) = \phi(vv'+v'v) = 2\eta(v,v').$$

$$(10.3)$$

It follows from the isomorphism (10.3) that two real Clifford algebras  $\mathcal{C}\ell(V,\eta)$  and  $\mathcal{C}\ell(V',\eta')$ are isomorphic iff they are modelled over pseudo-Euclidean spaces  $(V,\eta)$  and  $(V',\eta')$  of the same signature. Let a pseudo-Euclidean metric  $\eta$  be of signature (m; n - m) = (1, ..., 1; -1, ..., -1). Let  $\{v^1, ..., v^n\}$  be a basis for V such that  $\eta$  takes a diagonal form

$$\eta^{ab} = \eta(v^a, v^b) = \pm \delta^{ab}.$$

Then a ring  $\mathcal{C}\ell(V,\eta)$  is generated by elements  $v^1, ..., v^n$  which obey relations

$$v^a v^b + v^b v^a = 2\eta^{ab} e.$$

We agree to call  $\{v^1, ..., v^n\}$  the basis for a real Clifford algebra  $\mathcal{C}\ell(\mathbb{R}^n, \eta)$ . Given this basis, let us denote  $\mathcal{C}\ell(\mathbb{R}^n, \eta) = \mathcal{C}\ell(m, n - m)$ .

Certainly, any isomorphism (10.2) - (10.3) of real Clifford algebras is their ring isomorphism (10.2). However, the converse is not true, because their ring isomorphism (10.2) need not be

the isometric isomorphism (10.3) of their generating spaces. Therefore, we also consider real Clifford algebras, without specifying their generating spaces.

LEMMA 10.1: Any isometric isomorphism (10.3) of generating vector spaces  $\phi : V \to V'$  of real Clifford algebras  $\mathcal{C}\ell(V,\eta)$  and  $\mathcal{C}\ell(V',\eta')$  is prolonged to their ring isomorphism (10.2):

$$\phi: \mathcal{C}\ell(V,\eta) \to \mathcal{C}\ell(V',\eta') \qquad \phi(v_1 \cdots v_k) = \phi(v_1) \cdots \phi(v_k), \tag{10.4}$$

which also is an isomorphism of real Clifford algebras.  $\Box$ 

**Remark 10.1:** It may happen that a ring  $\mathcal{C}\ell(V,\eta)$  admits a generating pseudo-Euclidean space  $(V',\eta')$  whose signature differs from that of  $(V,\eta)$ . In this case,  $\mathcal{C}\ell(V,\eta)$  possesses the structure of a real Clifford algebra  $\mathcal{C}\ell(V',\eta')$  which is not isomorphic to a real Clifford algebra  $\mathcal{C}\ell(V,\eta)$ .  $\Box$ 

There is the following classification of real Clifford algebras [28, 59].

THEOREM 10.2: Real Clifford algebras  $\mathcal{C}\ell(p,q)$  as rings are isomorphic to the following matrix algebras.

$$\mathcal{C}\ell(p,q) = \begin{cases}
\operatorname{Mat}(2^{(p+q)/2}, \mathbb{R}) = \bigotimes_{\mathbb{R}}^{(p+q)/2} \operatorname{Mat}(2, \mathbb{R}) & p-q = 0, 2 \mod 8 \\
\operatorname{Mat}(2^{(p+q-1)/2}, \mathbb{R}) \oplus \operatorname{Mat}(2^{(p+q-1)/2}, \mathbb{R}) & p-q = 1 \mod 8 \\
\operatorname{Mat}(2^{(p+q-1)/2}, \mathbb{C}) & p-q = 3, 7 \mod 8 \\
\operatorname{Mat}(2^{(p+q-2)/2}, \mathbb{H}) & p-q = 4, 6 \mod 8 \\
\operatorname{Mat}(2^{(p+q-3)/2}, \mathbb{H}) \oplus \operatorname{Mat}(2^{(p+q-3)/2}, \mathbb{H}) & p-q = 5 \mod 8
\end{cases}$$
(10.5)

Since matrix algebras  $Mat(r, \mathcal{K}), \mathcal{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , are simple, a glance at Table 10.5 shows that real Clifford algebras  $\mathcal{C}\ell(V, \eta)$  modelled over even dimensional vector spaces V (i.e., p - q is even) are simple.

By a representation of a real Clifford algebra  $\mathcal{C}\ell(V,\eta)$  is meant its ring homomorphism  $\rho$  to a real ring of linear endomorphisms of a finite-dimensional real vector space  $\Xi$ , whose dimension is called the dimension of a representation. A representation is said to be exact if  $\rho$  is an isomorphism. A representation is called irreducible if there is no proper subspace of  $\Xi$  which is a carrier space of a representation of  $\mathcal{C}\ell(V,\eta)$ .

Two representations  $\rho$  and  $\rho'$  of a Clifford algebra  $\mathcal{C}\ell(V,\eta)$  in vector spaces  $\Xi$  and  $\Xi'$  are said to be equivalent if there is an isomorphism  $\xi : \Xi \to \Xi'$  of these vector spaces such that  $\rho' = \xi \circ \rho \circ \xi^{-1}$  is a real ring isomorphism of  $\rho(\mathcal{C}\ell(V,\eta))$  and  $\rho'(\mathcal{C}\ell(V,\eta))$ . The following is a corollary of Theorem 10.2 [28].

THEOREM 10.3: If  $n = \dim V$  is even, an exact irreducible representation of a real ring  $\mathcal{C}\ell(m, n-m)$  is unique up to an equivalence. If n is odd there exist two inequivalent exact irreducible representations of a real Clifford algebra  $\mathcal{C}\ell(m, n-m)$ .  $\Box$ 

Now, let us consider the complexification

$$\mathbb{CC}\ell(m,n-m) = \mathbb{C} \bigotimes_{\mathbb{R}} \mathcal{C}\ell(m,n-m)$$
(10.6)

of a real ring  $\mathcal{C}\ell(m, n-m)$ . It is readily observed that all complexifications  $\mathbb{C}\mathcal{C}\ell(m, n-m)$ ,  $m = 0, \ldots, n$ , are isomorphic:

$$\mathbb{CC}\ell(m, n-m) = \mathbb{CC}\ell(m', n-m'), \tag{10.7}$$

both as real and complex rings. Though the isomorphisms (10.7) are not unique, one can speak about an abstract complex ring  $\mathbb{CC}\ell(n)$  (10.7) so that, given a real Clifford algebra  $\mathcal{C}\ell(m, n-m)$ and its complexification  $\mathbb{CC}\ell(m, n-m)$  (10.6), there exists the complex ring isomorphism of  $\mathbb{CC}\ell(m, n-m)$  to  $\mathbb{CC}\ell(n)$ . We call  $\mathbb{CC}\ell(n)$  (10.7) the **complex Clifford algebra**, and define it as a complex ring

$$\mathbb{C}\mathcal{C}\ell(n) = \mathbb{C} \bigotimes_{\mathbb{R}} \mathcal{C}\ell(n,0), \tag{10.8}$$

generated by n elements  $(e^i)$  such that

$$e^{i}e^{j} + e^{j}e^{i} = 2\kappa(e^{i}, e^{j})e = 2\delta^{ij}e.$$
 (10.9)

Let us call  $\{e^i\}$  (10.9) the Euclidean basis for a complex Clifford algebra  $\mathbb{CC}\ell(n)$ . A complex vector space  $\mathcal{V}$ , spanned by an Euclidean basis  $\{e^i\}$  and provided with the bilinear form  $\kappa$  (10.9), is termed the Euclidean generating space of a complex Clifford algebra  $\mathbb{CC}\ell(n)$ . With this basis, the complex ring  $\mathbb{CC}\ell(n)$  (10.8) possesses a canonical real subring

$$\mathcal{C}\ell(m,n-m) \to \mathbb{C}\mathcal{C}\ell(n)$$
 (10.10)

with a basis  $\{e^1, ..., e^m, ie^{m+1}, ..., ie^n\}$ .

Theorem 10.2 provides the following classification of complex Clifford algebras  $\mathbb{CC}\ell(n)$  (10.8) [28, 59].

THEOREM 10.4: Complex Clifford algebras are isomorphic to the following matrix ones

$$\mathbb{C}\mathcal{C}\ell(n) = \begin{cases} \operatorname{Mat}(2^{n/2}, \mathbb{C}) = \bigotimes_{\mathbb{C}}^{n/2} \operatorname{Mat}(2, \mathbb{C}) = \bigotimes_{\mathbb{C}}^{n/2} \mathbb{C}\mathcal{C}\ell(2) \ n = 0 \mod 2\\ \operatorname{Mat}(2^{(n-1)/2}, \mathbb{C}) \oplus \operatorname{Mat}(2^{(n-1)/2}, \mathbb{C}) & n = 1 \mod 2 \end{cases}$$
(10.11)

COROLLARY 10.5: Since matrix algebras  $Mat(n, \mathbb{C})$  are simple and central (i.e., their center is proportional to the unit matrix), complex Clifford algebras  $\mathbb{CC}\ell(n)$  of even n are central simple algebras.  $\Box$ 

By a representation of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  is meant its morphism  $\rho$  to a complex algebra of linear endomorphisms of a finite-dimensional complex vector space. The following is a corollary of Theorem 10.4 [28].

THEOREM 10.6: If n is even, an exact irreducible representation of a complex Clifford algebra  $\mathbb{CC}\ell(n)$  is unique up to an equivalence. If n is odd there exist two inequivalent exact irreducible representations of a complex Clifford algebra  $\mathbb{CC}\ell(n)$ .  $\Box$ 

In view of Corollary 10.5 and Theorem 10.6, we hereafter focus our consideration on real and complex Clifford algebras modelled over even vector spaces.

### **10.2** Automorphisms of Clifford algebras

We consider both generic ring automorphisms of a Clifford algebra and its automorphisms which preserve a specified generating space [59].

Let  $\mathcal{C}\ell(V,\eta)$  be a real Clifford algebra modelled over an even-dimensional pseudo-Euclidean space  $(V,\eta)$ . By Aut $[\mathcal{C}\ell(V,\eta)]$  is denoted the group of automorphisms of a real ring  $\mathcal{C}\ell(V,\eta)$ . A key point is the following.

THEOREM 10.7: Any automorphism of a real ring  $\mathcal{C}\ell(V,\eta)$  is inner.  $\Box$ 

Indeed, Theorem 10.2 states that any real Clifford algebra  $\mathcal{C}\ell(p,q), p-q=0 \mod 2$  as a ring is isomorphic to some matrix algebra  $\operatorname{Mat}(m,\mathcal{K}), \mathcal{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ . Such an algebra is simple. Algebras  $\operatorname{Mat}(m,\mathcal{K}), \mathcal{K} = \mathbb{R}, \mathbb{H}$ , are central simple real algebras with the center  $\mathcal{Z} = \mathbb{R}$ . Algebras  $\operatorname{Mat}(m,\mathbb{C})$  are central simple complex algebras with the center  $\mathcal{Z} = \mathbb{C}$ . In accordance with the well-known Skolem–Noether theorem automorphisms of these algebras are inner.

Invertible elements of a real Clifford algebra  $\mathcal{C}\ell(V,\eta) = \operatorname{Mat}(m,\mathcal{K})$  constitute a general linear matrix group  $\mathcal{G}\mathcal{C}\ell(V,\eta) = Gl(m,\mathcal{K})$ . In particular, this group contains all elements  $v \in V \subset \mathcal{C}\ell(V,\eta)$  such that  $\eta(v,v) \neq 0$ . Acting in  $\mathcal{C}\ell(V,\eta)$  by left and right multiplications, the group  $\mathcal{G}\mathcal{C}\ell(V,\eta)$  also acts in a real Clifford algebra by the adjoint representation

$$\widehat{g}: q \to gqg^{-1}, \qquad g \in \mathcal{GC}\ell(V,\eta), \qquad q \in \mathcal{C}\ell(V,\eta).$$
 (10.12)

By virtue of Theorem 10.7, this representation provides an epimorphism

$$\zeta: \mathcal{GC}\ell(V,\eta) = Gl(m,\mathcal{K}) \to Gl(m,\mathcal{K})/\mathcal{Z} = \operatorname{Aut}[\mathcal{C}\ell(V,\eta)].$$
(10.13)

Any ring automorphism g of  $\mathcal{C}\ell(V,\eta)$  sends a generating pseudo-Euclidean space  $(V,\eta)$  of  $\mathcal{C}\ell(V,\eta)$  onto an isometrically isomorphic pseudo-Euclidean space  $(V',\eta')$  such that

$$2\eta'(g(v), g(v'))e = g(v)g(v') + g(v')g(v) = 2\eta(v, v')e, \qquad v, v' \in V.$$

It also is a generating space of a ring  $\mathcal{C}\ell(V,\eta)$ . Conversely, let  $(V,\eta)$  and  $(V',\eta')$  be two different pseudo-Euclidean generating spaces of the same signature of a ring  $\mathcal{C}\ell(V,\eta)$ . In accordance with

Lemma 10.1, their isometric isomorphism  $(V, \eta) \to (V', \eta')$  gives rise to an automorphism of a ring  $\mathcal{C}\ell(V, \eta)$  which also is an isomorphism of Clifford algebras  $\mathcal{C}\ell(V, \eta) \to \mathcal{C}\ell(V', \eta')$ .

In particular, any (isometric) automorphism

$$g: V \ni v \to g(v) \in V, \qquad \eta(g(v), g(v')) = \eta(v, v'), \qquad g \in O(V, \eta),$$

of a pseudo-Euclidean generating space  $(V, \eta)$  is prolonged to an automorphism of a ring  $\mathcal{C}\ell(V, \eta)$ which also is an automorphism of a real Clifford algebra  $\mathcal{C}\ell(V, \eta)$ . Then we have a monomorphism

$$O(V,\eta) \to \operatorname{Aut}[\mathcal{C}\ell(V,\eta)]$$
 (10.14)

of a group  $O(V,\eta)$  of automorphisms of a pseudo-Euclidean space  $(V,\eta)$  to a group of ring automorphisms of  $\mathcal{C}\ell(V,\eta)$ . Herewith, an automorphism  $g \in O(V,\eta)$  of a ring  $\mathcal{C}\ell(V,\eta)$  is the identity one iff its restriction to V is an identity map of V. Consequently, the following is true.

THEOREM 10.8: A subgroup  $O(V, \eta) \subset \operatorname{Aut}[\mathcal{C}\ell(V, \eta)]$  (10.14) exhausts all automorphisms of a ring  $\mathcal{C}\ell(V, \eta)$  which are automorphisms of a Clifford algebra  $\mathcal{C}\ell(V, \eta)$ .  $\Box$ 

Let us consider a subgroup  $\operatorname{Cliff}(V,\eta) \subset \mathcal{GC}\ell(V,\eta)$  generated by all invertible elements of  $V \subset \mathcal{C}\ell(V,\eta)$ . It is called the Clifford group. One can show that the homomorphism  $\zeta$  (10.13) of a Clifford group  $\operatorname{Cliff}(V,\eta)$  to  $\operatorname{Aut}[\mathcal{C}\ell(V,\eta)]$  is its epimorphism

$$\zeta : \mathcal{GC}\ell(V,\eta) \supset \operatorname{Cliff}(V,\eta) \to O(V,\eta) \subset \operatorname{Aut}[\mathcal{C}\ell(V,\eta)]$$
(10.15)

onto  $O(V, \eta)$ . Due to the factorization (10.15), any ring automorphism  $\hat{v}, v \in \text{Cliff}(V, \eta)$ , of  $\mathcal{C}\ell(V, \eta)$  also is an automorphism of a real Clifford algebra  $\mathcal{C}\ell(V, \eta)$ .

The epimorphism (10.15) yields an action of a Clifford group  $\text{Cliff}(V, \eta)$  in a pseudo-Euclidean space  $(V, \eta)$  by the adjoint representation (10.12). However, this action is not effective. Therefore, one consider subgroups  $\text{Pin}(V, \eta)$  and  $\text{Spin}(V, \eta)$  of  $\text{Cliff}(V, \eta)$ . The first one is generated by elements  $v \in V$  such that  $\eta(v, v) = \pm 1$ . A group  $\text{Spin}(V, \eta)$  is defined as an intersection

$$\operatorname{Spin}(V,\eta) = \operatorname{Pin}(V,\eta) \cap \mathcal{C}\ell^0(V,\eta)$$
(10.16)

of a group  $\operatorname{Pin}(V,\eta)$  and the even subring  $\mathcal{C}\ell^0(V,\eta)$  of a real Clifford algebra  $\mathcal{C}\ell(V,\eta)$ . In particular, generating elements  $v \in V$  of  $\operatorname{Pin}(V,\eta)$  do not belong to its subgroup  $\operatorname{Spin}(V,\eta)$ . The epimorphism (10.15) restricted to the Pin and Spin groups leads to short exact sequences of groups

$$e \to \mathbb{Z}_2 \longrightarrow \operatorname{Pin}(V, \eta) \xrightarrow{\zeta} O(V, \eta) \to e.$$
 (10.17)

$$e \to \mathbb{Z}_2 \longrightarrow \operatorname{Spin}(V, \eta) \xrightarrow{\zeta} SO(V, \eta) \to e,$$
 (10.18)

where  $\mathbb{Z}_2 \to (e, -e) \subset \operatorname{Spin}(V, \eta)$ .

**Remark 10.2:** It should be emphasized that an epimorphism  $\zeta$  in (10.17) and (10.18) is not a trivial bundle unless  $\eta$  is of signature (1, 1). It is a universal coverings over each component of  $O(V, \eta)$ .  $\Box$ 

Let  $\mathbb{CC}\ell(n)$  be the complex Clifford algebra (10.8) of even n.

THEOREM 10.9: All automorphisms of a complex Clifford algebra  $\mathbb{CC}\ell(n)$  are inner.

Indeed, by virtue of Theorem 10.4, there is the ring isomorphism (10.11):

$$\mathbb{CC}\ell(n) = \operatorname{Mat}(2^{n/2}, \mathbb{C}).$$
(10.19)

In accordance with Corollary 10.5, this algebra is a central simple complex algebra with the center  $\mathcal{Z} = \mathbb{C}$ . In accordance with the above-mentioned Skolem–Noether theorem automorphisms of these algebras are inner. Invertible elements of the Clifford algebra (10.19) constitute a general linear group

$$\mathcal{GCC}\ell(n) = GL(2^{n/2}, \mathbb{C}).$$
(10.20)

Acting in  $\mathbb{CC}\ell(n)$  by left and right multiplications, this group also acts in a Clifford algebra by the adjoint representation, and we obtain its epimorphism

$$GL(2^{n/2}, \mathbb{C}) \to \operatorname{Aut}[\mathcal{C}\ell(n)] = PGL(2^{n/2}, \mathbb{C}) =$$

$$GL(2^{n/2}, \mathbb{C})/\mathbb{C} = SL(2^{n/2}, \mathbb{C})/\mathbb{Z}_{2^{n/2}}$$
(10.21)

onto a projective linear group  $PGL(2^{n/2}, \mathbb{C})$ .

Any automorphism g of a complex Clifford algebra  $\mathbb{CCl}(n)$  sends its Euclidean generating space  $(\mathcal{V}, \kappa)$  onto some generating space

$$(\mathcal{V}', \kappa'), \qquad \kappa'(g(v), g(v')) = \kappa(v, v'), \qquad v, v' \in \mathcal{V},$$

which is the Euclidean one with respect to the basis  $\{g(e^i)\}$ . Conversely, any automorphism of an Euclidean generating space  $(\mathcal{V}, \kappa)$  is prolonged to an automorphism of a ring  $\mathbb{CC}\ell(n)$ . Then we have a monomorphism

$$O(n, \mathbb{C}) \to \operatorname{Aut}[\mathbb{C}\mathcal{C}\ell(n)]$$
 (10.22)

of a group  $O(n, \mathbb{C})$  of automorphisms of an Euclidean generating space  $(\mathcal{V}, \kappa)$  to a group of ring automorphisms of  $\mathbb{C}\mathcal{C}\ell(n)$ . Herewith, an automorphism  $g \in O(n, \mathbb{C})$  of a complex ring  $\mathbb{C}\mathcal{C}\ell(n)$  is the identity one iff its restriction to  $\mathcal{V}$  is an identity map of  $\mathcal{V}$ . Consequently, all ring automorphisms of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$  preserving its Euclidean generating space form a group  $O(n, \mathbb{C})$ . Given a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(n)$ , let  $\mathcal{C}\ell(m, n-m)$  be a real Clifford algebra. Due to the canonical ring monomorphism  $\mathcal{C}\ell(m, n-m) \to \mathbb{C}\mathcal{C}\ell(n)$  (10.10), there is the canonical group monomorphism

$$\mathcal{GC}\ell(m, n-m) \to \mathcal{GCC}\ell(n) = GL(2^{n/2}, \mathbb{C}).$$
 (10.23)

Since all ring automorphisms of a real Clifford algebra are inner (Theorem 10.7), they are extended to inner automorphisms of a complex Clifford algebra  $\mathbb{CC}\ell(n)$ .

#### **10.3** Spinor spaces

As was mentioned above, we define spinor spaces in terms of Clifford algebras [11, 59].

A real spinor space  $\Psi(m, n - m)$  is defined as a carrier space of an irreducible representation of a real Clifford algebra  $\mathcal{C}\ell(m, n - m)$ . It also carries out a representation of the corresponding group  $\operatorname{Spin}(m, n - m) \subset \mathcal{C}\ell(m, n - m)$  [28].

If n is even, such a real spinor space is unique up to an equivalence in accordance with Theorem 10.3. However, spinor spaces  $\Psi(m, n - m)$  and  $\Psi(m', n - m')$  need not be isomorphic vector spaces for  $m' \neq m$ .

A complex spinor space  $\Psi(n)$  is defined as a carrier space of an irreducible representation of a complex Clifford algebra  $\mathbb{CC}\ell(n)$ .

Since n is even, a representation  $\Psi(n)$  is unique up to an equivalence in accordance with Theorem 10.6. Therefore, it is sufficient to describe a complex spinor space  $\Psi(n)$  as a subspace of a complex Clifford algebra  $\mathbb{CC}\ell(n)$  which acts on  $\Psi(n)$  by left multiplications.

Given a complex Clifford algebra  $\mathbb{CC}\ell(n)$ , let us consider its non-zero minimal left ideal which  $\mathcal{C}\ell(n)$  acts on by left multiplications. It is a finite-dimensional complex vector space. Therefore, an action of a complex Clifford algebra  $\mathbb{CC}\ell(n)$  in a minimal left ideal by left multiplications defines a linear representation of  $\mathbb{CC}\ell(n)$ . It obviously is irreducible. In this case, a minimal left ideal of  $\mathbb{CC}\ell(n)$  is a complex spinor space  $\Psi(n)$ . Thus, we come to an equivalent definition of a **complex spinor space** as a minimal left ideal of a complex  $\mathbb{CC}\ell(n)$  which carry out its irreducible representation (10.1) [59].

By virtue of Theorem 10.4, there is a ring isomorphism  $\mathbb{CC}\ell(n) = \operatorname{Mat}(2^{n/2}, \mathbb{C})$  (10.19). Consequently, a spinor representation of a complex Clifford algebra  $\mathbb{CC}\ell(n)$  is equivalent to the canonical representation of  $\operatorname{Mat}(2^{n/2}, \mathbb{C})$  by matrices in a complex vector space  $\mathbb{C}^{2^{n/2}}$ , i.e.,  $\Psi(n) = \mathbb{C}^{2^{n/2}}$ . A spinor space  $\Psi(n) \subset \mathbb{CC}\ell(n)$  also carries out the left-regular irreducible representation of the group  $\mathcal{GCC}\ell(n) = GL(2^{n/2}, \mathbb{C})$  (10.20) which is equivalent to the natural matrix representation of  $GL(2^{n/2}, \mathbb{C})$  in  $\mathbb{C}^{2^{n/2}}$ .

Owing to the monomorphism  $\mathcal{C}\ell(m, n-m) \to \mathbb{C}\mathcal{C}\ell(n)$  (10.10), a spinor space  $\Psi(n)$  also carries out a representation of real Clifford algebras  $\mathcal{C}\ell(m, n-m)$ , their Pin and Spin groups, though these representation need not be reducible.

**Remark 10.3:** Certainly, an automorphism of a Clifford algebra  $\mathbb{CC}\ell(n)$  sends a spinor space

onto a spinor space, but not the same one. An action of a group  $PGL(2^{n/2}, \mathbb{C})$  of automorphisms of  $\mathbb{CC}\ell(n)$  in a set  $S\Psi(n)$  of spinor spaces is transitive.  $\Box$ 

#### 10.4 Clifford algebra bundles and spinor bundles

Treating spinor spaces as subspaces of Clifford algebras, we can describe spinor bundles as subbundles of a fibre bundle in complex Clifford algebras [59].

One usually consider fibre bundles in Clifford algebras whose structure group is a group of automorphisms of these algebras [11, 28]. A problem is that, as was mentioned above, this group fails to preserve spinor subspaces of a complex Clifford algebra (Remark 10.3) and, thus, it can not be a structure group of spinor bundles. Therefore, we define fibre bundles in Clifford algebras whose structure group is a group of invertible elements of a complex Clifford algebra which acts on this algebra by left multiplications. Certainly, it preserves minimal left ideals of this algebra and, consequently, it is a structure group of spinor bundles.

Let  $\mathbb{CC}\ell(n)$  be a complex Clifford algebra modelled over an even dimensional complex space  $\mathbb{C}^n$ . It is isomorphic to a ring  $\operatorname{Mat}(2^{n/2}, \mathbb{C})$  of complex  $(2^{n/2} \times 2^{n/2})$ -matrices (Theorem 10.4). Its invertible elements constitute the general linear group  $\mathcal{GCC}\ell(n) = GL(2^{n/2}, \mathbb{C})$  (10.20) whose adjoint representation in  $\mathbb{CC}\ell(n)$  yields the projective linear group  $PGL(2^{n/2}, \mathbb{C})$  (10.21) of automorphisms of  $\mathbb{CC}\ell(n)$  (Theorem 10.9).

Given a smooth manifold X, let us consider a principal bundle  $P \to X$  with a structure group  $GL(2^{n/2}, \mathbb{C})$ . A fibre bundle in complex Clifford algebras  $\mathbb{CCl}(n)$  is defined to be the *P*-associated bundle:

$$\mathcal{C} = (P \times \operatorname{Mat}(2^{n/2}, \mathbb{C}))/GL(2^{n/2}, \mathbb{C}) \to X$$
(10.24)

with a typical fibre  $\mathbb{CC}\ell(n) = \operatorname{Mat}(2^{n/2}, \mathbb{C})$  which carries out the left-regular representation of a group  $GL(2^{n/2}, \mathbb{C})$ .

Owing to the canonical inclusion  $GL(2^{n/2}, \mathbb{C}) \to \operatorname{Mat}(2^{n/2}, \mathbb{C})$ , a principal  $GL(2^{n/2}, \mathbb{C})$ bundle P is a subbundle  $P \subset \mathcal{C}$  of the Clifford algebra bundle  $\mathcal{C}$  (10.24). Herewith, the canonical right action of a structure group  $GL(2^{n/2}, \mathbb{C})$  on a principal bundle P is extended to the fibrewise action of  $GL(2^{n/2}, \mathbb{C})$  on the Clifford algebra bundle  $\mathcal{C}$  (10.24) by right multiplications. This action is globally defined because it is commutative with transition functions of  $\mathcal{C}$  acting on its typical fibre  $\operatorname{Mat}(2^{n/2}, \mathbb{C})$  on the left.

**Remark 10.4:** As was mentioned above, one usually considers a fibre bundle in Clifford algebras  $\mathbb{CC}\ell(n) = \operatorname{Mat}(2^{n/2})$  (10.19) whose structure group is the group  $PGL(2^{n/2}, \mathbb{C})$  (10.21) of automorphisms of  $\mathbb{CC}\ell(n)$ . This also is a *P*-associated bundle

$$\mathcal{AC} = (P \times \mathbb{CC}\ell(n))/GL(2^{n/2}, \mathbb{C}) \to X$$
(10.25)

where  $GL(2^{n/2}, \mathbb{C})$  acts on  $\mathbb{CC}\ell(n)$  by the adjoint representation.

Let  $\Psi(n)$  be a spinor space of a complex Clifford algebra  $\mathbb{CC}\ell(n)$ . Being a minimal left ideal of  $\mathbb{CC}\ell(n)$ , it is a subspace  $\Psi(n)$  of  $\mathbb{CC}\ell(n)$  which inherits the left-regular representation of a group  $GL(2^{n/2}, \mathbb{C})$  in  $\mathbb{CC}\ell(n)$ . Given a principal  $GL(2^{n/2}, \mathbb{C})$ -bundle P, a **spinor bundle** then is defined as a P-associated bundle

$$S = (P \times \Psi(n))/GL(2^{n/2}, \mathbb{C}) \to X$$
(10.26)

with a typical fibre  $\Psi(n) = \mathbb{C}^{2^{n/2}}$  and a structure group  $GL(2^{n/2}, \mathbb{C})$  which acts on  $\Psi(n)$  by left multiplications.

Obviously, the spinor bundle S (10.26) is a subbundle of the Clifford algebra bundle C (10.24). However, S (10.26) need not be a subbundle of the fibre bundle  $\mathcal{AC}$  (10.25) in Clifford algebras because a spinor space  $\Psi(n)$  is not stable under automorphisms of a complex Clifford algebra  $\mathbb{CCl}(n)$ .

At the same time, given the spinor representation (10.1) of a complex Clifford algebra, there is a fibrewise representation morphism

$$\gamma : \mathcal{AC} \underset{X}{\times} S \underset{X}{\longrightarrow} S,$$

$$\gamma : (P \times (\mathbb{CC}\ell(n) \times \Psi(n))) / \mathcal{GCC}\ell(n) \to (P \times \gamma(\mathbb{CC}\ell(n) \times \Psi(n))) / \mathcal{GCC}\ell(n),$$
(10.27)

of the *P*-associated fibre bundles  $\mathcal{AC}$  (10.25) and *S* (10.26) with a structure group  $\mathcal{GCC}\ell(n)$ .

It should be emphasized that, though there is the ring monomorphism  $\mathcal{C}\ell(m, n-m) \to \mathbb{C}\mathcal{C}\ell(n)$  (10.10), the Clifford algebra bundle  $\mathcal{C}$  (10.24) need not contains a subbundle in real Clifford algebras  $\mathcal{C}\ell(m, n-m)$ , unless a structure group  $GL(2^{n/2}, \mathbb{C})$  of  $\mathcal{C}$  is reducible to a subgroup  $\mathcal{G}\mathcal{C}\ell(m, n-m)$ . This problem can be solved as follows.

Let X be a smooth real manifold of even dimension n. Let  $T^*X$  be the cotangent bundle over X and LX the associated principal frame bundle. Let us assume that their structure group is  $GL(n, \mathbb{R})$  is reducible to a pseudo-ortohogonal subgroup O(m, n-m). In particular, a structure group  $GL(n, \mathbb{R})$  always is reducible to a maximal compact subgroup  $O(n, \mathbb{R})$  (Theorem 7.1). There is the exact sequence of groups (10.17):

$$e \to \mathbb{Z}_2 \longrightarrow \operatorname{Pin}(m, n-m) \xrightarrow{\zeta} O(m, n-m) \to e.$$
 (10.28)

A problem is that this exact sequence need not be split, i.e., there is no monomorphism  $\kappa$ :  $O(m, n - m) \rightarrow Pin(m, n - m)$  so that  $\zeta \circ \kappa = Id$ , in general.

In this case, we say that a principal Pin(m, n - m)-bundle  $\widetilde{Y} \to X$  is an extension of a principal O(m, n - m)-bundle  $Y \to X$  if there is an epimorphism of principal bundles

$$\widetilde{Y} \xrightarrow{X} Y.$$
(10.29)

Such an extension need not exist.

**Remark 10.5:** The topological obstruction to that a principal O(m, n - m)-bundle  $Y \to X$ lifts to a principal Pin(m, n - m)-bundle  $\tilde{Y} \to X$  is given by the Čech cohomology group  $H^2(X;\mathbb{Z}_2)$  of X [11, 15, 28]. Namely, a principal bundle Y defines an element of  $H^2(X;\mathbb{Z}_2)$ which must be zero so that  $Y \to X$  can give rise to  $\tilde{Y} \to X$ . Inequivalent lifts of  $Y \to X$ to principal  $\operatorname{Pin}(m, n - m)$ -bundles are classified by elements of the Čech cohomology group  $H^1(X;\mathbb{Z}_2)$ .  $\Box$ 

Let  $L^hX$  be a reduced principal O(m, n - m)-subbundle of a frame bundle LX. In this case, the above mentioned topological obstruction to that this bundle  $L^hX$  is extended to a principal  $\operatorname{Pin}(m, n-m)$ -bundle  $\tilde{L}^hX$  (Remark 10.5) is the second Stiefel–Whitney class  $w_2(X) \in$  $H^2(X;\mathbb{Z}_2)$  of X [28]. Let us assume that a manifold X is orientable, i.e., the Čech cohomology group  $H^1(X;\mathbb{Z}_2)$  is trivial, and that the second Stiefel–Whitney class  $w_2(X) \in H^2(X;\mathbb{Z}_2)$  of Xalso is trivial. Let  $\tilde{L}^hX$  be the desired  $\operatorname{Pin}(m, n - m)$ -lift (10.29) of a principal O(m, n - m)bundle  $L^hX$ . Owing to the canonical monomorphism (10.10) of Clifford algebras, there is the group monomorphism  $\operatorname{Pin}(m, n - m) \to \mathcal{GCCl}(n)$  (10.23). Due to this monomorphism, there exists a principal  $\mathcal{GCCl}(n)$ -bundle  $P^h$  whose reduced  $\operatorname{Pin}(m, n - m)$ -subbundle is  $\tilde{L}^hX$ , and whose structure group  $\mathcal{GCCl}(n) = GL(2^{n/2}, \mathbb{C})$  (10.20) thus is reducible to  $\operatorname{Pin}(m, n - m)$ . Let

$$\mathcal{C}^{h} = (P^{h} \times \operatorname{Mat}(2^{n/2}, \mathbb{C}))/GL(2^{n/2}, \mathbb{C}) \to X$$
(10.30)

be the  $P^h$ -associated bundle (10.24) in complex Clifford algebras  $\mathbb{CC}\ell(n)$ . Then it contains a subbundle

$$\mathcal{C}^{h}(m, n-m) = (\widetilde{L}^{h}X \times \mathcal{C}\ell(m, n-m)) / \operatorname{Pin}(m, n-m) \to X$$
(10.31)

in real Clifford algebras  $\mathcal{C}\ell(m, n - m)$ . The Clifford algebra bundle  $\mathcal{C}^h$  (10.30) also contains spinor subbundles (10.26):

$$S^{h} = (P^{h} \times \Psi(4))/GL(2^{n/2}, \mathbb{C}) \to X.$$
 (10.32)

Let us consider a  $P^h$ -associated fibre bundle  $\mathcal{AC}^h$  (10.25) in complex Clifford algebras  $\mathbb{CC}\ell(n)$  whose structure group acts on  $\mathbb{CC}\ell(n)$  by the adjoint representation and, thus, it is the group  $\operatorname{Aut}[\mathcal{C}\ell(n)]$  (10.21) of its automorphisms. Since a structure group of  $P^h$  is reducible to  $\operatorname{Pin}(m, n - m)$ , a fibre bundle  $\mathcal{AC}^h$  contains a  $\widetilde{L}^h X$ -associated subbundle

$$\mathcal{AC}^{h}(m, n-m) = (\tilde{L}^{h}X \times \mathcal{C}\ell(m, n-m)) / \operatorname{Pin}(m, n-m) \to X$$
(10.33)

in real Clifford algebras  $\mathcal{C}\ell(m, n-m)$  where the group  $\operatorname{Pin}(m, n-m)$  acts on  $\mathcal{C}\ell(m, n-m)$  by the adjoint representation. Then there is the fibrewise representation (10.27):

$$\gamma : \mathcal{AC}^{h}(m, n-m) \underset{X}{\times} S^{h} \underset{X}{\longrightarrow} S^{h}.$$
(10.34)

Due to the epimorphism  $\zeta$  (10.28), the Clifford algebra bundle  $\mathcal{AC}^{h}(m, n - m)$  contains a subbundle  $M^{h}X$  in pseudo-Euclidean generating spaces of fibres of  $\mathcal{AC}^{h}(m, n - m)$  with a structure group O(m, n - m). It is associated to an original reduced principal subbundle  $L^{h}X$  of a frame bundle LX and, thus, is isomorphic to the cotangent bundle  $T^*X$  of X. Accordingly, the fibrewise representation  $\gamma$  (10.34) leads to a fibrewise Clifford algebra representation

$$\gamma: M^h X \underset{X}{\times} S^h \underset{X}{\longrightarrow} S^h \tag{10.35}$$

of elements of the cotangent bundle  $TX = M^h X$ .

Of course, with a different reduced principal O(m, n - m)-subbundle  $L^{h'}X$  of LX, we come to a different Clifford algebra bundle  $\mathcal{C}^{h'}$  (10.30). By virtue of Theorem 3.1, there is one-to-one correspondence between the reduced principal O(m, n - m)-subbundle  $L^hX$  of LX and the global sections h of the quotient bundle

$$\Sigma(m, n-m) = LX/O(m, n-m) \to X, \tag{10.36}$$

which are pseudo-Riemannian metrics of signature (m, n - m) on X.

**Remark 10.6:** A key point is that, given different global sections h and h' of the quotient bundle  $\Sigma(m, n - m)$  (10.36), neither complex Clifford algebra bundles  $\mathcal{C}^h$  and  $\mathcal{C}^{h'}$  (10.30) nor real Clifford algebra bundles  $\mathcal{AC}^h(m, n - m)$  and  $\mathcal{AC}^{h'}(m, n - m)$  are not isomorphic. These fibre bundles are associated to principal  $\operatorname{Pin}(m, n - m)$ -bundles  $\tilde{L}^h X$  and  $\tilde{L}^{h'} X$  which are the two-fold covers (10.29) of the reduced principal O(m, n - m)-subbundles  $L^h X$  and  $L^{h'} X$  of a frame bundle LX, respectively. These subbundles need not be isomorphic, and then the principal bundles  $\tilde{L}^h X$  and  $\tilde{L}^{h'} X$  are well. Moreover, let principal bundles  $L^h X$  and  $L^{h'} X$ be isomorphic. For instance, this is the case of an orthogonal group  $O(n, \mathbb{R})$ . However, their covers  $\tilde{L}^h X$  and  $\tilde{L}^{h'} X$  need not be isomorphic. An isomorphism of  $L^h X$  and  $L^{h'} X$  yields an isomorphism of fibre bundles  $M^h X$  and  $M^{h'} X$  in generating pseudo-Euclidean spaces, but it is not isometric, and, therefore, fails to provide an isomorphism of real Clifford algebra bundles  $\mathcal{AC}^h(m, n-m)$  and  $\mathcal{AC}^{h'}(m, n-m)$ . Consequently, a Clifford algebra bundle must be considered only in a pair with a certain pseudo-Riemannian metric h.  $\Box$ 

In order to describe a whole family of non-isomorphic Clifford algebra bundles  $\mathcal{C}^h$ , let us call into play a composite bundle

$$LX \xrightarrow{X} \Sigma(m, n-m) \longrightarrow X$$
 (10.37)

where

$$LX \xrightarrow{X} \Sigma(m, n-m)$$
 (10.38)

is a principal bundle with a structure group O(m, n - m) [59]. Let us consider its principal Pin(m, n - m)-lift (10.29):

$$\widetilde{L}X \xrightarrow{X} \Sigma(m, n-m),$$
 (10.39)

if this exists. It is a composite bundle

$$\widetilde{L}X \xrightarrow{X} \Sigma(m, n-m) \longrightarrow X.$$
 (10.40)

Then, given a global section h of  $\Sigma(m, n-m) \to X$  (10.36), the pull-back  $h^*LX$  of  $LX \to \Sigma(m, n-m)$  (10.38) is a reduced principal O(m, n-m)-subbundle  $L^hX$  of the frame bundle  $LX \to X$  (10.37). Accordingly, the pull-back  $h^*\widetilde{L}X$  of  $\widetilde{L}X \to \Sigma(m, n-m)$  (10.39) is a principal  $\operatorname{Pin}(m, n-m)$ -subbundle of the composite bundle  $\widetilde{L}X \to X$  (10.40), and it is a  $\operatorname{Pin}(m, n-m)$ -lift

$$h^* \widetilde{L} X = \widetilde{L}^h X \xrightarrow{X} L^h X \tag{10.41}$$

of  $L^h X = h^* L X$ .

Owing to the group monomorphism  $\operatorname{Pin}(m, n - m) \to \mathcal{GCCl}(n)$  (10.23), there exists a principal  $\mathcal{GCCl}(n)$ -bundle

$$P_{\Sigma} \xrightarrow{X} \Sigma(m, n-m), \tag{10.42}$$

whose reduced principal Pin(m, n - m)-subbundle is the fibre bundle (10.39). Let

$$\mathcal{C}_{\Sigma} \xrightarrow{X} \Sigma(m, n-m) \tag{10.43}$$

be the  $P_{\Sigma}$ -associated bundle (10.24) in complex Clifford algebras  $\mathbb{CCl}(n)$ . It contains a  $\widetilde{LX}$ -associated subbundle

$$\mathcal{C}_{\Sigma}(m, n-m) \xrightarrow{X} \Sigma(m, n-m)$$
 (10.44)

in real Clifford algebras  $\mathcal{C}\ell(m, n-m)$ . The Clifford algebra bundle  $\mathcal{C}_{\Sigma}$  (10.43) also has  $P_{\Sigma}$ -associated spinor subbundles

$$S_{\Sigma} \xrightarrow{X} \Sigma(m, n-m).$$
 (10.45)

Given a global section h of  $\Sigma(m, n - m) \to X$  (10.36), the pull-back bundles  $h^* \mathcal{C}_{\Sigma} \to X$ ,  $h^* \mathcal{C}_{\Sigma}(m, n - m) \to X$  and  $h^* S_{\Sigma} \to X$  are subbundles of the composite bundles  $\mathcal{C}_{\Sigma} \to X$ ,  $\mathcal{C}_{\Sigma}(m, n - m) \to X$  and  $S_{\Sigma} \to X$  and are the bundles  $\mathcal{C}^h \to X$  (10.30),  $\mathcal{C}^h(m, n - m) \to X$ (10.31) and  $S^h \to X$  (10.32), respectively.

Similarly, we define an LX-associated bundle

$$\mathcal{AC}_{\Sigma}(m,n-m) \xrightarrow{X} \Sigma(m,n-m)$$
 (10.46)

in real Clifford algebras  $\mathcal{C}\ell(m, n-m)$  where the group  $\operatorname{Pin}(m, n-m)$  acts on  $\mathcal{C}\ell(m, n-m)$  by the adjoint representation. Then there is a fibrewise Clifford algebra representation

$$\gamma : \mathcal{AC}_{\Sigma}(m, n-m) \underset{\Sigma}{\times} S_{\Sigma} \xrightarrow{}{\longrightarrow} S_{\Sigma}.$$
(10.47)

Given a global section h of  $\Sigma(m, n-m) \to X$  (10.36), the pull-back bundle  $h^* \mathcal{AC}_{\Sigma}(m, n-m) \to X$  restarts the Clifford algebra bundle  $\mathcal{AC}^h(m, n-m)$  (10.33) and the fibrewise representation  $\gamma$  (10.34).

Due to the epimorphism  $\zeta$  (10.28), the Clifford algebra bundle  $\mathcal{AC}_{\Sigma}(m, n - m)$  (10.46) contains a subbundle  $M_{\Sigma}$  in pseudo-Euclidean generating spaces of fibres of  $\mathcal{AC}_{\Sigma}(m, n - m)$ 

with a structure group O(m, n-m). This fibre bundle is associated to the principal O(m, n-m)bundle (10.38), and it inherits the fibrewise representation (10.47):

$$\gamma: M_{\Sigma} \underset{\Sigma}{\times} S_{\Sigma} \xrightarrow{\Sigma} S_{\Sigma}.$$
(10.48)

Given a global section h of  $\Sigma(m, n-m) \to X$  (10.36), its pull-back  $h^*M_{\Sigma} \to X$  coincides with a fibre bundle  $M^hX$  in pseudo-Euclidean generating spaces, and it is isomorphic to the cotangent bundle  $T^*X$  of X. Accordingly, the fibrewise representation  $\gamma$  (10.48) reproduces that (10.35).

### 11 Dirac spinor fields in gauge gravitation theory

A Dirac spinor space is defined to be a spinor space  $\Psi(1,3)$  of an irreducible representation of real Clifford algebra  $\mathcal{C}\ell(1,3)$ .

There are ring isomorphisms of real Clifford algebras

$$\mathcal{C}\ell(1,3) = \mathcal{C}\ell(4,0) = \mathcal{C}\ell(0,4) = \operatorname{Mat}(2,\mathbb{H}),$$
(11.1)

which as rings fail to be isomorphic to real Clifford algebras

$$\mathcal{C}\ell(3,1) = \mathcal{C}\ell(2,2) = \operatorname{Mat}(4,\mathbb{R}).$$
(11.2)

Due to the isomorphism (11.2), a real Clifford algebra  $\mathcal{C}\ell(3,1)$  possesses an irreducible fourdimensional representation by real matrices

$$\begin{pmatrix} 0 \mathbf{1} \\ \mathbf{1} \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} 0 - \mathbf{1} \\ \mathbf{1} \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \quad \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}, \quad (11.3)$$

where **1** is the unit  $(2 \times 2)$ -matrix and  $\sigma^k$ , k = 1, 2, 3, are Pauli matrices

$$\sigma^{1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \sigma^{2} = \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}, \qquad \sigma^{3} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$
(11.4)

By virtue of Theorem 10.3, the representation (11.3) is unique up to an equivalence. Its carrier space is  $\Psi(3, 1)$  of real **Majorana spinors**.

In contrast with the representation (11.3) of  $\mathcal{C}\ell(3,1)$ , a representation of a real Clifford algebra  $\mathcal{C}\ell(3,1)$ , the matrix representation  $\mathcal{C}\ell(1,3) = \text{Mat}(2,\mathbb{H})$  (11.1) by Dirac's  $\gamma$ -matrices

$$\gamma^{0} = \begin{pmatrix} 0 \ \mathbf{1} \\ \mathbf{1} \ 0 \end{pmatrix}, \qquad \gamma^{j} = \begin{pmatrix} 0 \ -\sigma^{j} \\ \sigma^{j} \ 0 \end{pmatrix}$$
(11.5)

is not real. As was mentioned above, we therefore consider complex spinors which form a carrier space  $\Psi(4)$  of an irreducible representation of a complex Clifford algebra  $\mathbb{CC}\ell(4)$ . This representation is unique up to an equivalence in accordance with Theorem 10.6.

Let  $\{e^0, e^i\}$  be the Euclidean basis (10.9) for a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(4)$ . With this basis, the complex ring  $\mathbb{C}\mathcal{C}\ell(4)$  possesses a canonical real subring  $\mathcal{C}\ell(1,3)$  (10.10) with a basis  $\{e^0, ie^i\}$ . Then  $\Psi(4)$  admits a representation of a complex Clifford algebra  $\mathbb{C}\mathcal{C}\ell(4)$  by the matrices  $e^0 = \gamma^0$ ,  $e^i = -i\gamma^i$  whose restriction to a real Clifford algebra  $\mathcal{C}\ell(1,3)$  restarts its representation (11.5) and provides a representation of a group Spin(1,3).

A group Spin(1,3) contains two connected components  $\text{Spin}^+(1,3)$  and  $\text{Spin}^-(1,3)$ . Being a connected component of the unity, the first one is a group  $SL(2,\mathbb{C})$ . We have the exact sequence (10.18):

$$e \to \mathbb{Z}_2 \longrightarrow \operatorname{Spin}(1,3) \xrightarrow{\zeta} SO(1,3) \to e.$$

It is restricted to the exact sequence

$$e \to \mathbb{Z}_2 \longrightarrow \operatorname{Spin}^+(1,3) \xrightarrow{\zeta} \mathcal{L} \to e,$$
 (11.6)

where a proper Lorentz group L is a connected component of the unit of SO(1,3). Let us call

$$L_s = Spin^+(1,3) = SL(2,\mathbb{C})$$
 (11.7)

#### the Lorentz spin group.

Group spaces of  $L_s$  and L are topological spaces  $S^3 \times \mathbb{R}^3$  and  $\mathbb{R}P^3 \times \mathbb{R}^3$ , respectively. Their Lie algebras coincide with each other. It can be provided with a basis  $\{I_{ab} = -I_{ba}\}, a, b = 0, 1, 2, 3$  whose elements obey the commutation relations

$$[I_{ab}, I_{cd}] = \eta_{ad}I_{bc} + \eta_{bc}I_{ad} - \eta_{ac}I_{bd} - \eta_{bd}I_{ac},$$

where  $\eta$  is the Minkowski metric. Its representation (11.5) in  $\Psi(4)$  reads

$$I_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]. \tag{11.8}$$

Let X be a world manifold. Let us assume that the second Stiefel–Whitney class  $w_2(X) \in H^2(X; \mathbb{Z}_2)$  of X is trivial (Remark 10.5). We follow the procedure in Section 10 in order to describe a Dirac spinor structure on X [11, 48, 54].

For this purpose, let us assume that the structure group  $GL_4$  (3.1) of a linear frame bundle LX is reducible to a proper Lorentz group L. By virtue of Theorem 3.1, there is one-to-one correspondence between the principal L-subbundles  $L^hX$  of a frame bundle LX and the global sections h of the quotient bundle  $\Sigma_T \to X$  (6.1) called the tetrad fields. Let us consider the composite bundle (10.37):

$$LX \xrightarrow{X} \Sigma_{\mathrm{T}} \longrightarrow X,$$
 (11.9)

where

$$LX \xrightarrow{X} \Sigma_{\mathrm{T}}$$
 (11.10)

is a principal bundle with a structure group L. Given a tetrad field h, the pull-back  $h^*LX$  of  $LX \to \Sigma_T$  (11.10) is a reduced principal L-subbundle  $L^hX$  of a frame bundle  $LX \to X$ .

Let us note that a structure group  $GL_4$  of a frame bundle LX is not simply-connected. Its first homotopy group is

$$\pi_1(GL_4) = \pi_1(\mathrm{SO}(4)) = \mathbb{Z}_2$$

[14]. Therefore, a group  $GL_4$  also admits the universal two-fold covering group  $\widetilde{GL}_4$  such that the diagram

$$\begin{array}{cccc} \widetilde{GL}_4 \longrightarrow GL_4 \\ \uparrow & \uparrow \\ L_s \xrightarrow{\zeta} & L \end{array}$$
(11.11)

is commutative [18, 28, 64].

**Remark 11.1:** Though a group  $\widetilde{GL}_4$  admits finite-dimensional representations, its fundamental spinor representation is infinite-dimensional [18, 35]. Elements of this representation are called **world spinors**. Their field model has been developed (see [18] and references therein).

Given a group  $\widetilde{GL}_4$ , there exists a unique principal  $\widetilde{GL}_4$ -bundle  $\widetilde{LX} \to X$  which is a two-fold cover

$$\widetilde{L}X \xrightarrow[X]{z} LX \tag{11.12}$$

of a frame bundle LX. Due to the commutative diagram (11.11), there is a commutative diagram of principal bundles



where

$$\widetilde{L}X \xrightarrow{X} \widetilde{L}X/L_{s} = \Sigma_{T}$$
 (11.13)

is a principal L<sub>s</sub>-bundle. It is just the L<sub>s</sub>-lift (10.39) of the principal L-bundle  $LX \to \Sigma_T$  (11.10).

Let us consider the composite bundle (10.40):

$$\widetilde{L}X \xrightarrow{X} \Sigma_{\mathrm{T}} \longrightarrow X.$$
 (11.14)

Given a tetrad field h, the pull-back  $h^* \widetilde{L} X$  of  $\widetilde{L} X \to \Sigma_T$  (11.13) is a reduced principal L<sub>s</sub>subbundle  $\widetilde{L}^h X$  of the composite bundle  $\widetilde{L} X \to X$  (11.14). Due to the commutative diagram (11.11), there is a commutative diagram of principal bundles

Owing to the group monomorphism (11.6):

$$L_{s} \to \mathcal{GCC}\ell(4) = GL(4,\mathbb{C}),$$

there exists a principal  $GL(4, \mathbb{C})$ -bundle

$$P_{\Sigma} \xrightarrow{X} \Sigma_{\mathrm{T}},$$
 (11.16)

whose reduced principal  $L_s$ -subbundle is the fibre bundle (11.13). Let

$$\mathcal{C}_{\Sigma} \xrightarrow{X} \Sigma_{\mathrm{T}} \tag{11.17}$$

be the  $P_{\Sigma}$ -associated bundle (10.24) in complex Clifford algebras  $\mathbb{CC}\ell(4)$ . It contains a  $\mathbb{L}_{\sim}$ -associated subbundle

$$\mathcal{C}_{\Sigma}(1,3) \xrightarrow{X} \Sigma_{\mathrm{T}}$$
 (11.18)

in real Clifford algebras  $\mathcal{C}\ell(1,3)$ . The Clifford algebra bundle  $\mathcal{C}_{\Sigma}$  (11.17) also has  $P_{\Sigma}$ -associated spinor subbundles

$$S_{\Sigma} \xrightarrow{X} \Sigma_{\mathrm{T}}$$
 (11.19)

with a typical fibre  $\Psi(4)$ .

Similarly, we define a  $\widetilde{L}X$ -associated bundle

$$\mathcal{AC}_{\Sigma}(1,3) \xrightarrow{X} \Sigma_{\mathrm{T}}$$
 (11.20)

in real Clifford algebras  $\mathcal{C}\ell(1,3)$  where a group  $L_s$  acts on  $\mathcal{C}\ell(1,3)$  by the adjoint representation. Then there is a fibrewise representation morphism

$$\gamma : \mathcal{AC}_{\Sigma}(1,3) \underset{\Sigma_{\mathrm{T}}}{\times} S_{\Sigma} \xrightarrow{\Sigma_{\mathrm{T}}} S_{\Sigma}.$$
(11.21)

Due to the epimorphism  $\zeta$  (11.6), the Clifford algebra bundle  $\mathcal{AC}_{\Sigma}(1,3)$  (11.20) contains a subbundle  $M_{\Sigma}$  x in Minkowski generating spaces  $\mathbb{R}^4 \subset \mathcal{Cl}(1,3)$  with a structure group L. This fibre bundle

$$M_{\Sigma} = (\widetilde{L}X \times \mathbb{R}^4) / \mathcal{L}_{\mathrm{s}} = (LX \times \mathbb{R}^4) / \mathcal{L}$$
(11.22)

is associated to the principal L-bundle (11.10), and it inherits the fibrewise representation (11.21):

$$\gamma: M_{\Sigma} \underset{\Sigma_{\mathrm{T}}}{\times} S_{\Sigma} \xrightarrow{}_{\Sigma_{\mathrm{T}}} S_{\Sigma}.$$
(11.23)

Given a tetrad field h,

$$S^h = h^* S_{\Sigma} \to X \tag{11.24}$$

of the spinor bundle  $S_{\Sigma}$  (11.19) is a subbundle of a composite bundle

$$S = S_{\Sigma} \xrightarrow{X} \Sigma_{\mathrm{T}} \to X \tag{11.25}$$

and, in view of the commutative diagram (11.15), it is a  $\tilde{L}^h X$ -associated bundle with the structure Lorentz spin group  $L_s$  (11.7).

With a tetrad field h, let us consider the pull-back

$$\mathcal{AC}^{h}(1,3) = h^* \mathcal{AC}^{(1,3)} \to X \tag{11.26}$$

of the Clifford algebra bundle  $\mathcal{AC}_{\Sigma}(1,3)$  (11.21). It contains the pull-back bundle

$$M^{h}X = h^{*}M_{\Sigma} = (L^{h}X \times \mathbb{R}^{4})/L$$
(11.27)

of generating Minkowski spaces. It is isomorphic to the cotangent bundle

$$T^*X = (L^hX \times \mathbb{R}^4)/\mathcal{L}$$

of X if it is endowed the Lorentz atlas  $\Psi^h$  (6.2). The fibre bundle  $\mathcal{AC}^h(1,3)$  (11.26) inherits the fibrewise representation (11.21):

$$\gamma_h : \mathcal{AC}^h(1,3) \underset{X}{\times} S^h \xrightarrow{}_X S^h, \tag{11.28}$$

and  $M^h X$  (11.27) does fibrewise representation (11.23):

$$\gamma_h: M^h X \underset{X}{\times} S^h \underset{X}{\longrightarrow} S^h.$$
(11.29)

**Remark 11.2:** Given a tetrad field, let the Lorentz bundle atlas  $\Psi^h = \{z_{\iota}^h\}$  (6.2) of a reduced Lorentz bundle  $L^h X$  gives rise to an atlas  $\overline{\Psi}^h = \{\overline{z}_{\iota}^h\}$ ,  $z_{\iota}^h = z_h \circ \overline{z}_{\iota}^h$ , of the principal L<sub>s</sub>-bundle  $\widetilde{L}^h X$  in the diagram (11.15). With respect to these and associated atlases the representations (11.28) – (11.29) takes a form

$$\hat{h}^a = \gamma_h(h^a) = \gamma^a, \qquad \hat{d}x^\lambda = \gamma_h(dx^\lambda) = h_a^\lambda(x)\gamma^a,$$
(11.30)

where  $\gamma^a$  are Dirac's  $\gamma$ -matrices (11.5) and  $h^a$  are the tetrad coframes (6.5).  $\Box$ 

In view of the representations (11.28) - (11.29), one can treat sections of the fibre bundle  $S^h$  (11.24) as Dirac spinor fields in the presence of a tetrad field h.

However, the representations  $\gamma_h$  and  $\gamma_{h'}$  (11.30) for different tetrad fields h and h' are inequivalent. Indeed, given elements  $t = t_{\mu}dx^{\mu} = t_ah^a = t'_ah'^a$  of  $T^*X$ , their representations  $\gamma_h$  and  $\gamma_{h'}$  (11.30) read

$$\gamma_h(t) = t_a \gamma^a = t_\mu h_a^\mu \gamma^a, \qquad \gamma_{h'}(t) = t_a' \gamma^a = t_\mu h_a'^\mu \gamma^a,$$

and lead to non-isomorphic Clifford algebras because  $\gamma_h(t)\gamma_h(t') \neq \gamma_{h'}(t)\gamma_{h'}(tt')$ .

Treating sections of spinor bundles  $S^h$  as Dirac spinor fields in the presence of tetrad fields h, one can consider the composite spinor bundle S (11.25) in order to describe the totality of Dirac spinor fields in the presence of gravitational field in gauge gravitation theory [11, 48, 54]. We agree to call it the **universal spinor bundle** because, given a tetrad field h, the pullback  $S^h = h^*S \to X$  (11.24) of  $S_{\Sigma}$  (11.19) is a spinor bundle on X which is associated to an L<sub>s</sub>-principal bundle  $\tilde{L}^h X$ . A universal spinor bundle S is endowed with bundle coordinates  $(x^{\lambda}, \sigma_a^{\mu}, y^A)$ , where  $(x^{\lambda}, \sigma_a^{\mu})$  are bundle coordinates on  $\Sigma_{\mathrm{T}}$  and  $y^A$  are coordinates on a spinor space  $\Psi(4)$ . A universal spinor bundle  $M_{\Sigma}$  (11.22) in Minkowski spaces associated to an L-principal bundle  $LX \to \Sigma_{\mathrm{T}}$  (11.10). As a consequence, the fibrewise representation (11.23) is defined. It reads

$$\gamma(dx^{\lambda}) = \sigma_a^{\lambda} \gamma^a. \tag{11.31}$$

Given the fibrewise Clifford algebra representation (11.31), one can introduce a Dirac operator on a spinor bundle  $S^h$  for each tetrad field h as the pull-back of the total Dirac operator  $\mathcal{D}$  (11.35) on the universal spinor bundle S as follows [11, 48, 54].

One can show that, due to the splitting (6.15), any world connection  $\Gamma$  (5.1) on X yields a connection

$$A_{\Sigma} = dx^{\lambda} \otimes (\partial_{\lambda} - \frac{1}{4} (\eta^{kb} \sigma^{a}_{\mu} - \eta^{ka} \sigma^{b}_{\mu}) \sigma^{\nu}_{k} \Gamma_{\lambda}{}^{\mu}{}_{\nu} I_{ab}{}^{A}{}_{B} y^{B} \partial_{A}) +$$

$$d\sigma^{\mu}_{k} \otimes (\partial^{k}_{\mu} + \frac{1}{4} (\eta^{kb} \sigma^{a}_{\mu} - \eta^{ka} \sigma^{b}_{\mu}) I_{ab}{}^{A}{}_{B} y^{B} \partial_{A})$$

$$(11.32)$$

on the spinor bundle  $S_{\Sigma} \to \Sigma_{T}$  (11.19), where  $I_{ab}$  are the generators (11.8). Its pall-back to  $S^{h}$  is the spin Lorentz connection

$$\Gamma_s = dx^{\lambda} \otimes \left[\partial_{\lambda} + \frac{1}{4} (\eta^{kb} h^a_{\mu} - \eta^{ka} h^b_{\mu}) (\partial_{\lambda} h^{\mu}_k - h^{\nu}_k \Gamma_{\lambda}{}^{\mu}{}_{\nu}) I_{ab}{}^A{}_B y^B \partial_A\right]$$
(11.33)

associated to the Lorentz connection  $\Gamma_h$  (6.17) defined by  $\Gamma$  on a reduced Lorentz bundle  $L^h X$ . . The connection (11.32) yields the vertical covariant differential

$$\widetilde{D} = dx^{\lambda} \otimes [y^{A}_{\lambda} - \frac{1}{4}(\eta^{kb}\sigma^{a}_{\mu} - \eta^{ka}\sigma^{b}_{\mu})(\sigma^{\mu}_{\lambda k} - \sigma^{\nu}_{k}\Gamma_{\lambda}{}^{\mu}{}_{\nu})L_{ab}{}^{A}{}_{B}y^{B}]\partial_{A}, \qquad (11.34)$$

on the universal spinor bundle  $S \to X$  (11.25). Its restriction to  $S^h \subset S$  recovers the familiar covariant differential on the spinor bundle  $S^h \to X$  relative to the spin connection (11.33). Combining (11.31) and (11.34) gives the first order differential operator

$$\mathcal{D} = \sigma_a^{\lambda} \gamma^{aB}{}_A [y_{\lambda}^A - \frac{1}{4} (\eta^{kb} \sigma_{\mu}^a - \eta^{ka} \sigma_{\mu}^b) (\sigma_{\lambda k}^{\mu} - \sigma_k^{\nu} \Gamma_{\lambda}{}^{\mu}{}_{\nu}) L_{ab}{}^A{}_B y^B], \qquad (11.35)$$

on the universal spinor bundle  $S \to X$  (11.25). Its restriction to  $S^h \subset S$  is the familiar Dirac operator on a spinor bundle  $S^h$  in the presence of a background tetrad field h and a general world connection  $\Gamma$ .

#### 12 Affine world connections

The tangent bundle TX of a world manifold X as like as any vector bundle possesses a natural structure of an affine bundle. It is associated to a principal bundle AX of oriented affine frames in TX whose structure group is a general affine group  $GA(4, \mathbb{R})$ . This structure group is always reducible to a linear subgroup  $GL_4$  since the quotient  $GA(4, \mathbb{R})/GL_4$  is a vector space  $\mathbb{R}^4$ . Treating as an affine bundle, the tangent bundle TX admits affine connections

$$A = dx^{\lambda} \otimes (\partial_{\lambda} + \Gamma_{\lambda}{}^{\alpha}{}_{\mu}(x)\dot{x}^{\mu}\dot{\partial}_{\alpha} + \sigma^{\alpha}_{\lambda}(x)\dot{\partial}_{\alpha}), \qquad (12.1)$$

called the **affine world connections**. They are associated to principal connections on an affine frame bundle AX. Every affine connection  $\Gamma$  (12.1) on TX yields a unique linear connection

$$\Gamma = dx^{\lambda} \otimes (\partial_{\lambda} + \Gamma_{\lambda}{}^{\alpha}{}_{\mu}(x)\dot{x}^{\mu}\dot{\partial}_{\alpha})$$
(12.2)

on TX. It is associated to a principal connection on a frame bundle  $LX \subset AX$ . Conversely, being equivariant, any principal connection on a frame bundle  $LX \subset AX$  gives rise to a principal connection on an affine frame bundle AX, i.e., every linear connection on TX can be seen as the affine one. It follows that any affine connection A (12.1) on the tangent bundle TX is represented by a sum of the associated linear connection  $\overline{\Gamma}$  (12.2) and a **soldering form**  $\sigma = \sigma_{\lambda}^{\alpha}(x)dx^{\lambda} \otimes \dot{\partial}_{\alpha}$  on TX, which is a (1, 1)-tensor field

$$\sigma = \sigma_{\lambda}^{\alpha}(x)dx^{\lambda} \otimes \partial_{\alpha} \tag{12.3}$$

on X due to the canonical splitting  $VTX = TX \times TX$ .

In particular, let us consider the canonical soldering form  $\theta_J$  (5.6) on TX. Given an arbitrary world connection  $\Gamma$  (5.1) on TX, the corresponding affine connection on TX is a **Cartan connection** 

$$A = \Gamma + \theta_X, \qquad A^{\mu}_{\lambda} = \Gamma_{\lambda}{}^{\mu}{}_{\nu}\dot{x}^{\nu} + \delta^{\mu}_{\lambda}.$$

There is a problem of a physical meaning of the tensor field  $\sigma$  (12.3).

In the framework of above mentioned Poincaré gauge theory, it is treated as a non-holonomic frame field or a tetrad field (Remark 6.1). This treatment of  $\sigma$  is wrong because a soldering form and a frame field are different mathematical objects. A frame field is a (local) section of a principal frame bundle LX, while a soldering form is a global section of the LX-associated tensor bundle

$$TX \otimes T^*X = (LX \times \operatorname{Mat}(4, \mathbb{R}))/GL_4$$

whose typical fibre is an algebra  $Mat(4, \mathbb{R})$  of four-dimensional real matrices. It contains a group  $GL_4$  which acts on  $Mat(4, \mathbb{R})$  by the adjoint representation, but not left multiplications.

At the same time, a translation part of an affine connection on  $\mathbb{R}^3$  characterizes an elastic distortion in gauge theory of dislocations in continuous media [23, 32]. By analogy with this gauge theory, a gauge model of hypothetic deformations of a world manifold has been developed. They are described by the translation part  $\sigma$  (12.3) of affine world connections on X and, in particular, they are responsible for the so called "fifth force" [42, 43, 44].

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