# CLASSICAL GAUGE GRAVITATION THEORY

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Classical gravitation theory is formulated as gauge theory on natural bundles where gauge symmetries are general covariant transformations and a gravitational field is a Higgs field responsible for their spontaneous symmetry breaking.

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## 1 Introduction

Classical field theory admits a comprehensive mathematical formulation in the geometric terms of smooth fibre bundles [6, 29]. For instance, Yang–Mills gauge theory is theory of principal connections on principal bundles.

Gauge gravitation theory as particular classical field theory also is formulated in the terms of fibre bundles.

**Remark 1.** A first gauge model of gravity was suggested by Utiyama [33] in 1956 just two years after the birth of gauge theory itself. He was first who generalized the original gauge model of Yang and Mills for $SU(2)$ to an arbitrary symmetry Lie group and, in particular, to a Lorentz group in order to describe gravity. However, he met the problem of treating general covariant transformations and a pseudo-Riemannian metric which had no partner in Yang–Mills gauge theory. To eliminate this drawback, representing a gravitational field as a gauge field of a translation group was attempted (see [10, 12] for a review). Since the Poincaré group comes from the Wigner–Inönii contraction of de Sitter groups $SO(2,3)$ and $SO(1,4)$ and it is a subgroup of a conformal group, gauge theories on fibre bundles $Y \to X$ with these structure groups were also considered [13, 32]. In a different way, gravitation theory was formulated as gauge theory with a Lorentz reduced structure where a metric gravitational field was treated as the corresponding Higgs field [12, 22].

Studying gauge gravitation theory, we believe reasonable to require that it incorporates Einstein's General Relativity and, therefore, it should be based on the relativity and equivalence principles reformulated in the fibre bundle terms.

In these terms, the relativity principle states that gauge symmetries of classical gravitation theory are general covariant transformations. Let us emphasized that these gauge symmetries differ from gauge symmetries of the above mentioned Yang–Mills gauge theory which are vertical automorphisms of principal bundles. Fibre bundles possessing general covariant transformations constitute the category of so called natural bundles [6, 16].

Let $\pi : Y \to X$ be a smooth fibre bundle. Any automorphism $(\Phi, f)$ of $Y$, by definition, is projected as $\pi \circ \Phi = f \circ \pi$ onto a diffeomorphism $f$ of its base $X$. The converse is not true. A fibre bundle $Y \to X$ is called the natural bundle if there exists a monomorphism

$$\mathrm{Diff}X \ni f \to \widetilde{f} \in \mathrm{Aut}Y$$

of the group of diffeomorphisms of $X$ to the group of bundle automorphisms of $Y \to X$. Automorphisms $\widetilde{f}$ are called general covariant transformations of $Y$. The group of automorphisms of a natural bundle is a semi-direct product of a subgroup of vertical automorphisms of $Y$ (over $\mathrm{Id}\, X$) and a subgroup of general covariant transformations.

Accordingly, there exists the functorial lift of any vector field $\tau$ on $X$ to a vector field $\overline{\tau}$ on $Y$ such that $\tau \mapsto \overline{\tau}$ is a monomorphism of the Lie algebra $\mathcal{T}(X)$ of vector field on $X$ to that $\mathcal{T}(T)$ of vector fields on $Y$. This functorial lift $\overline{\tau}$ is an infinitesimal generator of a local one-parameter group of local general covariant transformations of $Y$.

The tangent bundle $TX$ of $X$ exemplifies a natural bundle. Any diffeomorphism $f$ of $X$ gives rise to the tangent automorphisms $\widetilde{f} = Tf$ of $TX$ which is a general covariant transformation of $TX$. The tangent bundle possess a structure group

$$GL_4 = GL^+(4, \mathbb{R}). \tag{1}$$

The associated principal bundle is a fibre bundle $LX$ of frames in the tangent spaces to $X$. It also is a natural bundle. Moreover, all fibre bundles associated with $LX$ are natural bundles. Principal connections on $LX$ yield linear connections on the tangent bundle $TX$ and other associated bundles. They are called the world connections.

Following the relativity principle, one thus should develop gravitation theory as a gauge theory of principal connections on a principal frame bundle $LX$ over an oriented four-dimensional smooth manifold $X$, called the world manifold $X$ [6, 27].

The equivalence principle reformulated in geometric terms requires that the structure group $GL_4$ (1) of a frame bundle $LX$ and associated bundles is reducible to a Lorentz group $SO(1, 3)$ [12, 22]. It means that these fibre bundles admit atlases with $SO(1, 3)$-valued transition functions or, equivalently, that there exist principal subbubdles of $LX$ with a Lorentz structure group. This is the case of spontaneous symmetry breaking.

Spontaneous symmetry breaking in classical gauge theory on a principal bundle $P \to X$ with a structure Lie group $G$ is characterized as a reduction of this structure group to its closed (consequently, Lie) subgroup $H$ [6, 23, 28]. By virtue of the well-known theorem [6, 15], there is one-to-one correspondence between the $H$-principal subbundles $P^h$ of $P$ and the global sections $h$ of the quotient bundle $P/H \to X$ which are treated as classical Higgs fields.

Accordingly, in gauge gravitation theory based on the equivalence principle, there is one-to-one correspondence between the Lorentz principal subbundles of a frame bundle $LX$ (called the Lorentz reduced structures) and the global sections of the quotient bundle

$$\Sigma_{\rm PR} = LX/SO(1,3), \tag{2}$$

which are pseudo-Riemannian metrics on a world manifold. In Einstein's General Relativity, they are identified with gravitational fields.

Thus, we come to gauge gravitation theory as metric-affine gravitation theory whose dynamic variables are world connections and pseudo-Riemannian metrics on a world manifold $X$ (Section 6). They are treated as gauge fields and Higgs fields, respectively.

There is the extensive literature on metric-affine gravitation theory [10, 11, 19]. However, one often formulates it as gauge theory of affine connections, that is wrong (Section 11).

The character of gravity as a Higgs field responsible for spontaneous breaking of general covariant transformations is displayed as follows. Given different gravitational fields, the representations (86) of holonomic coframes $\{dx^\mu\}$ by Dirac matrices acting on Dirac spinor fields are nonequivalent (Section 10). Consequently, Dirac operators in the presence of different gravitational fields fails to be equivalent, too. In particular, it follows that a Dirac spinor field can be considered only in a pair with a certain gravitational field. A total system of such pairs is described by sections of the composite bundle $S \to X$ (88), where $S \to \Sigma_{\rm T}$ is a spinor bundle, whereas $S \to X$ is a natural bundle.

**Remark 2.** Since the Dirac operators in the presence of different gravitational fields are nonequivalent, Dirac spinor fields fail to be considered, e.g., in the case of a superposition of different gravitational fields. Therefore, quantization of a metric gravitational field fails to satisfy the superposition principle, and one can suppose that a metric gravitational field as a Higgs field is non-quantized in principle.

Being reduced to a Lorentz group, a structure group of a frame bundle $LX$ also is reduced to a maximal compact subgroup $SO(3)$ of $SO(1,3)$. The associated Higgs field is a spatial distribution which defines a space-time structure on a world manifold $X$ (Section 5).

Since general covariant transformations are symmetries of a metric-affine gravitation Lagrangian, the corresponding conservation law holds (Section 7). It is an energy-momentum conservation law. Because general covariant transformations are gauge transformations depending on derivatives of gauge parameters, the corresponding energy-momentum current reduces to a superpotential [6, 30]. This is a generalized Komar superpotential [5, 24].

Since general covariant transformations are gauge symmetries of a gravitation Lagrangian $L_{\rm MA}$, the Euler–Lagrange operator $\delta L_{\rm MA}$ (51) of this Lagrangian obeys the complete Noether identities (74). Because they are irreducible, one obtains the BRST extension (75) of general covariant transformations and that (76) of an original metric-affine Lagrangian. This is a necessary step towards quantization of classical gauge gravitation theory [1, 6].

## 2 Natural bundles

Let $\pi : Y \to X$ be a smooth fibre bundle coordinated by $(x^\lambda, y^i)$. Given a one-parameter group $(\Phi_t, f_t)$ of automorphisms of $Y$, its infinitesimal generator is a projectable vector field

$$u = \tau^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i \tag{3}$$

on $Y$ which is projected onto a vector field $\tau = \tau^\lambda\partial_\lambda$ on $X$, whose flow is a one-parameter group $(f_t)$ of diffeomorphisms of $X$. Conversely, let $\tau = \tau^\lambda\partial_\lambda$ be a vector field on $X$. Its lift to some projectable vector field (3) on $Y$ always exists. For instance, given a connection

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda(x^\mu, y^j)\partial_i)$$

on $Y \to X$, a vector field $\tau$ on $X$ gives rise to a horizontal vector field

$$\Gamma\tau = \tau\rfloor\Gamma = \tau_\lambda(\partial_\lambda + \Gamma^i_\lambda\partial_i)$$

on $Y$. The horizontal lift $\tau \to \Gamma\tau$ yields a monomorphism of a $C^\infty(X)$-module $\mathcal{T}(X)$ of vector fields on $X$ to a $C^\infty(Y)$-module $\mathcal{T}(Y)$ of vector fields on $Y$, but this monomorphism is not a Lie algebra morphism, unless $\Gamma$ is a flat connection.

We address the category of natural bundles $Y \to X$ [6, 16] admitting the functorial lift $\widetilde\tau$ onto $Y$ of any vector field $\tau$ on $X$ such that $\tau \to \overline\tau$ is a Lie algebra monomorphism

$$\mathcal{T}(X) \to \mathcal{T}(T), \qquad [\widetilde\tau, \widetilde\tau'] = \widetilde{[\tau, \tau']}.$$

This functorial lift $\widetilde\tau$, by definition, is an infinitesimal generator of a local one-parameter group of general covariant transformations of $Y$.

Natural bundles are exemplified by tensor products

$$T = (\overset{m}{\otimes} TX) \otimes (\overset{k}{\otimes} T^*X) \tag{4}$$

of the tangent $TX$ and cotangent $T^*X$ bundles of $X$ as follows. Given a manifold atlas $\{(U_i, \phi_i)\}$ of $X$, the tangent bundle $\pi_X : TX \to X$ admits a holonomic bundle atlas

$$\{(U_i, T\phi_i : \pi_X^{-1}(U_i) \to U_i \times \mathbb{R}^4)\}, \tag{5}$$

where $T\phi_i$ is the tangent morphism to $\phi_i$. With this atlas, $TX$ is provided with holonomic bundle coordinates

$$(x^\mu, \dot x^\mu), \qquad \dot x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\dot x^\nu,$$

where $(\dot{x}^\mu)$ are fibre coordinates with respect to holonomic frames $\{\partial_\mu\}$. Accordingly, the tensor bundle (4) is endowed with holonomic bundle coordinates $(x^\mu, x^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_k})$, where

$$(x^\mu, \dot{x}_\mu), \qquad \dot{x}'_\mu = \frac{\partial x^\nu}{\partial x_\mu}\dot{x}_\nu,$$

are holonomic bundle coordinates on the cotangent bundle $T^*X$ of $X$. Then given a vector field $\tau$ on $X$, its functorial lift onto the tensor bundle (4) takes the form

$$\widetilde{\tau} = \tau^\mu\partial_\mu + [\partial_\nu\tau^{\alpha_1}\dot{x}^{\nu\alpha_2\cdots\alpha_m}_{\beta_1\cdots\beta_k} + \ldots - \partial_{\beta_1}\tau^\nu\dot{x}^{\alpha_1\cdots\alpha_m}_{\nu\beta_2\cdots\beta_k} - \ldots]\dot{\partial}^{\beta_1\cdots\beta_k}_{\alpha_1\cdots\alpha_m}, \qquad \dot{\partial}_\lambda = \frac{\partial}{\partial\dot{x}^\lambda}.$$

Tensor bundles over a world manifold $X$ have the structure group $GL_4$ (1). An associated principal bundle is the above mentioned frame bundle $LX$. Its (local) sections are called frame fields. Given the holonomic atlas (5) of the tangent bundle $TX$, every element $\{H_a\}$ of a frame bundle $LX$ takes the form $H_a = H^\mu_a\partial_\mu$, where $H^\mu_a$ is a matrix of the natural representation of a group $GL_4$ in $\mathbb{R}^4$. These matrices constitute bundle coordinates

$$(x^\lambda, H^\mu_a), \qquad H'^\mu_a = \frac{\partial x'^\mu}{\partial x^\lambda}H^\lambda_a,$$

on $LX$ associated to its holonomic atlas

$$\Psi_T = \{(U_\iota, z_\iota = \{\partial_\mu\})\}, \tag{6}$$

given by local frame fields $z_\iota = \{\partial_\mu\}$. With respect to these coordinates, the canonical right action of $GL_4$ on $LX$ reads $GL_4 \ni g : H^\mu_a \to H^\mu_b g^b{}_a$.

A frame bundle $LX$ is equipped with a canonical $\mathbb{R}^4$-valued one-form

$$\theta_{LX} = H^a_\mu dx^\mu \otimes t_a, \tag{7}$$

where $\{t_a\}$ is a fixed basis for $\mathbb{R}^4$ and $H^a_\mu$ is the inverse matrix of $H^\mu_a$.

A frame bundle $LX \to X$ is natural. Any diffeomorphism $f$ of $X$ gives rise to a principal automorphism

$$\widetilde{f} : (x^\lambda, H^\lambda_a) \to (f^\lambda(x), \partial_\mu f^\lambda H^\mu_a) \tag{8}$$

of $LX$ which is its general covariant transformation. Given a (local) one-parameter group of diffeomorphisms of $X$ and its infinitesimal generator $\tau$, the lift (8) yields a functorial lift

$$\widetilde{\tau} = \tau^\mu\partial_\mu + \partial_\nu\tau^\alpha H^\nu_a\frac{\partial}{\partial H^\alpha_a}$$

onto $LX$ of a vector field $\tau$ on $X$ which is defined by the condition $\mathbf{L}_{\widetilde{\tau}}\theta_{LX} = 0$.

Let $Y = (LX \times V)/GL_4$ be an $LX$-associated bundle with a typical fibre $V$. It admits a lift of any diffeomorphism $f$ of its base to an automorphism

$$f_Y(Y) = (\widetilde{f}(LX) \times V)/GL_4$$

of $Y$ associated with the principal automorphism $\widetilde{f}$ (8) of a frame bundle $LX$. Thus, all bundles associated with a frame bundle $LX$ are natural bundles.

## 3 World connections

Let $TX$ be the tangent bundle of a world manifold $X$. With respect to holonomic coordinates $(x^\lambda, \dot{x}^\lambda)$, a linear connection on $TX$ takes the form

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu). \tag{9}$$

Since $TX$ is associated with a frame bundle $LX$, every linear connection (9) is associated with a principal connection on $LX$. We agree to call them world connections.

A curvature of a world connection is defined as that of the connection (9). It reads

$$\begin{aligned} &R = \frac{1}{2} R_{\lambda\mu}{}^\alpha{}_\beta \dot{x}^\beta dx^\lambda \wedge dx^\mu \otimes \dot{\partial}_\alpha, \\ &R_{\lambda\mu}{}^\alpha{}_\beta = \partial_\lambda \Gamma_\mu{}^\alpha{}_\beta - \partial_\mu \Gamma_\lambda{}^\alpha{}_\beta + \Gamma_\lambda{}^\gamma{}_\beta \Gamma_\mu{}^\alpha{}_\gamma - \Gamma_\mu{}^\gamma{}_\beta \Gamma_\lambda{}^\alpha{}_\gamma. \end{aligned} \tag{10}$$

Due to the canonical splitting of the vertical tangent bundle

$$VTX = TX \times TX \tag{11}$$

of $TX$, the curvature $R$ (10) can be represented by a tangent-valued two-form

$$R = \frac{1}{2} R_{\lambda\mu}{}^\alpha{}_\beta \dot{x}^\beta dx^\lambda \wedge dx^\mu \otimes \partial_\alpha \tag{12}$$

on $TX$. Due to this representation, the Ricci tensor

$$R_c = \frac{1}{2} R_{\lambda\mu}{}^\lambda{}_\beta dx^\mu \otimes dx^\beta \tag{13}$$

of a world connection $\Gamma$ is defined.

By a torsion of a world connection is meant that of the connection $\Gamma$ (9) on the tangent bundle $TX$ with respect to the canonical soldering form

$$\theta_J = dx^\mu \otimes \dot{\partial}_\mu \tag{14}$$

on $TX$. It reads

$$T = \frac{1}{2} T_\mu{}^\nu{}_\lambda dx^\lambda \wedge dx^\mu \otimes \dot{\partial}_\nu, \qquad T_\mu{}^\nu{}_\lambda = \Gamma_\mu{}^\nu{}_\lambda - \Gamma_\lambda{}^\nu{}_\mu. \tag{15}$$

A world connection is said to be symmetric if its torsion (15) vanishes, i.e., $\Gamma_\mu{}^\nu{}_\lambda = \Gamma_\lambda{}^\nu{}_\mu$. Owing to the vertical splitting (11), the torsion form $T$ (15) of $\Gamma$ can be written as a tangent-valued two-form

$$T = \frac{1}{2} T_\mu{}^\nu{}_\lambda dx^\lambda \wedge dx^\mu \otimes \partial_\nu \tag{16}$$

on $X$. One also introduces a soldering torsion form

$$T = T_\mu{}^\nu{}_\lambda \dot{x}^\lambda dx^\mu \otimes \dot{\partial}_\nu. \tag{17}$$

Given a world connection $\Gamma$ (9) and its soldering torsion form $T$ (17), the sum $\Gamma + cT$, $c \in \mathbb{R}$, is a world connection. In particular, every world connection $\Gamma$ defines a unique symmetric world connection $\Gamma' = \Gamma - T/2$.

Being associated with a principal connection on $LX$, a world connection is represented by a section of the quotient bundle

$$C_{\rm W} = J^1LX/GL_4 \to X, \tag{18}$$

where $J^1LX$ is the first order jet manifold of sections of $LX \to X$ [6]. We agree to call $C_{\rm W}$ (18) the bundle of world connections. With respect to the holonomic atlas $\Psi_T$ (6), it is provided with the bundle coordinates

$$(x^\lambda, k_\lambda{}^\nu{}_\alpha), \qquad k'_\lambda{}^\nu{}_\alpha = \left[\frac{\partial x'^\nu}{\partial x^\gamma}\frac{\partial x^\beta}{\partial x'^\alpha}k_\mu{}^\gamma{}_\beta + \frac{\partial x^\beta}{\partial x'^\alpha}\frac{\partial^2 x'^\nu}{\partial x^\mu \partial x^\beta}\right]\frac{\partial x^\mu}{\partial x'^\lambda},$$

so that, for any section $\Gamma$ of $C_{\rm W} \to X$, its coordinates $k_\lambda{}^\nu{}_\alpha \circ \Gamma = \Gamma_\lambda{}^\nu{}_\alpha$ are components of the world connection $\Gamma$ (9).

Though the bundle of world connections $C_{\rm W} \to X$ (18) is not $LX$-associated, it is a natural bundle. It admits a functorial lift

$$\widetilde{\tau}_C = \tau^\mu\partial_\mu + [\partial_\nu\tau^\alpha k_\mu{}^\nu{}_\beta - \partial_\beta\tau^\nu k_\mu{}^\alpha{}_\nu - \partial_\mu\tau^\nu k_\nu{}^\alpha{}_\beta + \partial_{\mu\beta}\tau^\alpha]\frac{\partial}{\partial k_\mu{}^\alpha{}_\beta}$$

of any vector field $\tau$ on $X$.

The first order jet manifold $J^1C_{\rm W}$ of a bundle of world connections possesses the canonical splitting

$$\begin{aligned} k_{\lambda\mu}{}^\alpha{}_\beta = {} & \frac{1}{2}(k_{\lambda\mu}{}^\alpha{}_\beta - k_{\mu\lambda}{}^\alpha{}_\beta + k_\lambda{}^\gamma{}_\beta k_\mu{}^\alpha{}_\gamma - k_\mu{}^\gamma{}_\beta k_\lambda{}^\alpha{}_\gamma) + \\ & \frac{1}{2}(k_{\lambda\mu}{}^\alpha{}_\beta + k_{\mu\lambda}{}^\alpha{}_\beta - k_\lambda{}^\gamma{}_\beta k_\mu{}^\alpha{}_\gamma + k_\mu{}^\gamma{}_\beta k_\lambda{}^\alpha{}_\gamma) = \frac{1}{2}(\mathcal{R}_{\lambda\mu}{}^\alpha{}_\beta + \mathcal{S}_{\lambda\mu}{}^\alpha{}_\beta), \end{aligned} \tag{19}$$

so that, if $\Gamma$ is a section of $C_{\rm W} \to X$, then $\mathcal{R}_{\lambda\mu}{}^\alpha{}_\beta \circ J^1\Gamma = R_{\lambda\mu}{}^\alpha{}_\beta$ are components of the curvature (10).

A world manifold $X$ is called flat if it admits a flat world connection $\Gamma$. By virtue of the well-known theorem, there exists a bundle atlas of $TX$ with constant transition functions such that $\Gamma = dx^\lambda \otimes \partial_\lambda$ relative to this atlas [6]. However, such an atlas is not holonomic in general. Therefore, the torsion form $T$ (15) of a flat connection $\Gamma$ need not vanish.

A world manifold $X$ is called parallelizable if the tangent bundle $TX \to X$ is trivial. A parallelizable manifold is flat. A flat manifold is parallelizable if it is simply connected.

# 4 Lorentz reduced structure

As was mentioned above, gravitation theory on a world manifold $X$ is classical field theory with spontaneous symmetry breaking described by Lorentz reduced structures of a frame bundle $LX$. We deal with the following Lorentz and proper Lorentz reduced structures.

By a Lorentz reduced structure is meant a reduced principal $SO(1,3)$-subbundle $L^gX$, called the Lorentz subbundle, of a frame bundle $LX$.

Let L= $SO^0(1,3)$ be a proper Lorentz group. Recall that $SO(1,3) = \mathbb{Z}_2 \times$L. A proper Lorentz reduced structure is defined as a reduced L-subbundle $L^hX$ of $LX$.

If a world manifold $X$ is simply connected, there is one-to-ne correspondence between the Lorentz and proper Lorentz reduced structures.

One can show that different proper Lorentz subbundles $L^hX$ and $L^{h'}X$ of a frame bundle $LX$ are isomorphic as principal L-bundles. This means that there exists a vertical automorphism of a frame bundle $LX$ which sends $L^hX$ onto $L^{h'}X$ [6, 28]. If a world manifold $X$ is simply connected, the similar property of Lorentz subbundles also is true.

There is the well-known topological obstruction to the existence of a Lorentz structure on a world manifold $X$. All non-compact manifolds and compact manifolds whose Euler characteristic equals zero admit a Lorentz reduced structure [3].

By virtue of the above mentioned theorem, there is one-to-one correspondence between the principal L-subbundles $L^hX$ of a frame bundle $LX$ and the global sections $h$ of a quotient bundle

$$\Sigma_{\rm T} = LX/{\rm L}, \tag{20}$$

called the tetrad bundle. This is an $LX$-associated bundle with the typical fibre $GL_4/$L. Its global sections are called the tetrad fields. The fibre bundle (20) is a two-fold covering $\zeta : \Sigma_{\rm T} \to \Sigma_{\rm PR}$ of the metric bundle $\Sigma_{\rm PR}$ (2). In particular, every tetrad field $h$ defines a unique pseudo-Riemannian metric $g = \zeta \circ h$. For the sake of convenience, one usually identifies a metric bundle with an open subbundle of the tensor bundle $\Sigma_{\rm PR} \subset \overset{2}{\vee} TX$. Therefore, the metric bundle $\Sigma_{\rm PR}$ (2) can be equipped with bundle coordinates $(x^\lambda, \sigma^{\mu\nu})$.

Every tetrad field $h$ defines an associated Lorentz bundle atlas

$$\Psi^h = \{(U_\iota, z_\iota^h = \{h_a\})\} \tag{21}$$

of a frame bundle $LX$ such that the corresponding local sections $z_\iota^h$ of $LX$ take their values into a proper Lorentz subbundle $L^hX$ and the transition functions of $\Psi^h$ (21) between the frames $\{h_a\}$ are L-valued. The frames (21):

$$\{h_a = h_a^\mu(x)\partial_\mu\}, \qquad h_a^\mu = H_a^\mu \circ z_\iota^h, \qquad x \in U_\iota, \tag{22}$$

are called the tetrad frames.

Given a Lorentz bundle atlas $\Psi^h$, the pull-back

$$h = h^a \otimes t_a = z_\iota^{h*}\theta_{LX} = h^a_\lambda(x)dx^\lambda \otimes t_a \tag{23}$$

of the canonical form $\theta_{LX}$ (7) by a local section $z_\iota^h$ is called the (local) tetrad form. It determines tetrad coframes

$$\{h^a = h^a_\mu(x)dx^\mu\}, \qquad x \in U_\iota, \tag{24}$$

in the cotangent bundle $T^*X$. They are the dual of the tetrad frames (22). The coefficients $h_a^\mu$ and $h^a_\mu$ of the tetrad frames (22) and coframes (24) are called the tetrad functions. They are transition functions between the holonomic atlas $\Psi_T$ (6) and the Lorentz atlas $\Psi^h$ (21) of a frame bundle $LX$.

With respect to the Lorentz atlas $\Psi^h$ (21), a tetrad field $h$ can be represented by the $\mathbb{R}^4$-valued tetrad form (23). Relative to this atlas, the corresponding pseudo-Riemannian metric $g = \zeta \circ h$ takes the well-known form

$$g = \eta(h \otimes h) = \eta_{ab} h^a \otimes h^b, \qquad g_{\mu\nu} = h^a_\mu h^b_\nu \eta_{ab}, \tag{25}$$

where $\eta = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric in $\mathbb{R}^4$ written with respect to its fixed basis $\{t_a\}$. It is readily observed that the tetrad coframes $\{h^a\}$ (24) and the tetrad frames $\{h_a\}$ (22) are orthornormal relative to the pseudo-Riemannian metric (25), namely:

$$g^{\mu\nu} h^a_\mu h^b_\nu = \eta^{ab}, \qquad g_{\mu\nu} h_a^\mu h_b^\nu = \eta_{ab}.$$

Therefore, their components $h^0$, $h_0$ and $h^i$, $h_i$, $i = 1, 2, 3$, are called time-like and spatial, respectively.

Given a pseudo-Riemannian metric $g$, any world connection $\Gamma$ (9) admits a splitting

$$\Gamma_{\mu\nu\alpha} = \{_{\mu\nu\alpha}\} + S_{\mu\nu\alpha} + \frac{1}{2}C_{\mu\nu\alpha} \tag{26}$$

in the Christoffel symbols

$$\{_{\mu\nu\alpha}\} = -\frac{1}{2}(\partial_\mu g_{\nu\alpha} + \partial_\alpha g_{\nu\mu} - \partial_\nu g_{\mu\alpha}), \tag{27}$$

the non-metricity tensor

$$C_{\mu\nu\alpha} = C_{\mu\alpha\nu} = \nabla^\Gamma_\mu g_{\nu\alpha} = \partial_\mu g_{\nu\alpha} + \Gamma_{\mu\nu\alpha} + \Gamma_{\mu\alpha\nu} \tag{28}$$

and the contorsion

$$S_{\mu\nu\alpha} = -S_{\mu\alpha\nu} = \frac{1}{2}(T_{\nu\mu\alpha} + T_{\nu\alpha\mu} + T_{\mu\nu\alpha} + C_{\alpha\nu\mu} - C_{\nu\alpha\mu}), \tag{29}$$

where $T_{\mu\nu\alpha} = -T_{\alpha\nu\mu}$ are coefficients of the torsion form (16) of $\Gamma$.

A world connection $\Gamma$ is called a metric connection for a pseudo-Riemannian metric $g$ if $g$ is its integral section, i.e., the metricity condition

$$\nabla^\Gamma_\mu g_{\nu\alpha} = 0 \tag{30}$$

holds. A metric connection reads

$$\Gamma_{\mu\nu\alpha} = \{_{\mu\nu\alpha}\} + \frac{1}{2}(T_{\nu\mu\alpha} + T_{\nu\alpha\mu} + T_{\mu\nu\alpha}). \tag{31}$$

For instance, the Levi–Civita connection (27) is a torsion-free metric connection.

A principal connection on a proper Lorentz subbundle $L^hX$ of a frame bundle $LX$ is called the Lorentz connection. By virtue of the well known theorem [6, 15], this connection is extended to a principal connection $\Gamma$ on a frame bundle $LX$. It also is called the Lorentz connection. The associated linear connection (9) on the tangent bundle $TX$ with respect to a Lorentz atlas $\Psi^h$ reads

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2}A_\lambda{}^{ab}L_{ab}{}^c{}_d h^d_\mu \dot x^\mu h^\nu_c \dot\partial_\nu) \tag{32}$$

where $L_{ab}{}^c{}_d = \eta_{bd}\delta^c_a - \eta_{ad}\delta^c_b$ are generators of a right Lie algebra $\mathfrak{g}_{\mathrm{L}}$ of a proper Lorentz group L in a Minkowski space $\mathbb{R}^4$. Written relative to a holonomic atlas, the connection $\Gamma$ (32) possesses components

$$\Gamma_\lambda{}^\mu{}_\nu = h^k_\nu \partial_\lambda h^\mu_k + \eta_{ka} h^\mu_b h^k_\nu A_\lambda{}^{ab}. \tag{33}$$

**Theorem 1.** Any Lorentz connections is a metric connection for some pseudo-Riemannian metric $g$, and *vice versa*.

**Proof.** By virtue of the well-known theorem [6, 15], a metric connection $\Gamma$ for a pseudo-Riemmanian metric $g = \zeta \circ h$ is reducible to a Lorentz connection on the proper Lorentz subbundle $L^hX$, i.e., it is a Lorentz connection. Conversely, every Lorentz connection obeys the metricity condition (30) for a pseudo-Riemannian metric $g = \zeta \circ h$. □

**Theorem 2.** Any world connection $\Gamma$ defines a Lorentz connection $\Gamma_h$ on each principal L-subbundle $L^hX$ of a frame bundle.

**Proof.** The Lie algebra of $GL_4$ is a direct sum

$$\mathfrak{g}_{GL_4} = \mathfrak{g}_{\mathrm{L}} \oplus \mathfrak{m} \tag{34}$$

of the Lie algebra $\mathfrak{g}_{\mathrm{L}}$ of a Lorentz group and a subspace $\mathfrak{m}$ such that $[\mathfrak{g}_{\mathrm{L}}, \mathfrak{m}] \subset \mathfrak{m}$. Therefore, let us consider a local connection one-form of a connection $\Gamma$ with respect to a Lorentz atlas $\Psi^h$ of $LX$ given by a tetrad forms $h^a$. It reads

$$z^{h*}_\iota \overline\Gamma = -\Gamma_\lambda{}^b{}_a dx^\lambda \otimes L_b{}^a, \qquad \Gamma_\lambda{}^b{}_a = -h^b_\mu \partial_\lambda h^\mu_a + \Gamma_\lambda{}^\mu{}_\nu h^b_\mu h^\nu_a,$$

where $\{L^a_b\}$ is a basis for a Lie algebra $\mathfrak{g}_{GL_4}$. The Lorentz part of this form is precisely a local connection one-form of a connection $\Gamma_h$ on $L^hX$. We have

$$z_\zeta^{h*}\overline{\Gamma}_h = -\frac{1}{2}A_\lambda{}^{ab}dx^\lambda \otimes L_{ab}, \qquad A_\lambda{}^{ab} = \frac{1}{2}(\eta^{kb}h^a_\mu - \eta^{ka}h^b_\mu)(\partial_\lambda h^\mu_k - h^\nu_k \Gamma_\lambda{}^\mu{}_\nu). \tag{35}$$

Then combining this expression and the expression (32) gives the connection

$$\Gamma_h = dx^\lambda \otimes (\partial_\lambda + \frac{1}{4}(\eta^{kb}h^a_\mu - \eta^{ka}h^b_\mu)(\partial_\lambda h^\mu_k - h^\nu_k \Gamma_\lambda{}^\mu{}_\nu)L_{ab}{}^c{}_d h^d_\mu \dot{x}^\mu h^\nu_c \dot{\partial}_\nu) \tag{36}$$

with respect to a Lorentz atlas $\Psi^h$ and this connection

$$\Gamma_h = dx^\lambda \otimes [\partial_\lambda + \frac{1}{2}(h^k_\alpha \delta^\beta_\mu - \eta^{kc}g_{\mu\alpha}h^\beta_c)(\partial_\lambda h^\mu_k - h^\nu_k \Gamma_\lambda{}^\mu{}_\nu)\dot{x}^\alpha \partial_\beta] \tag{37}$$

relative to a holonomic atlas. If $\Gamma$ is the Lorentz connection (33) extended from $L^hX$, then obviously $\Gamma_h = \Gamma$. □

## 5 Space-time structure

If a structure group $GL_4$ (1) of a frame bundle $LX$ is reducible to a proper Lorentz group L, it is always reducible to the maximal compact subgroup $SO(3)$ of L. A structure group $GL_4$ of $LX$ also is reducible to its maximal compact subgroup $SO(4)$. Thus, there is the commutative diagram

$$\begin{array}{ccc} GL_4 & \longrightarrow & SO(4) \\ \downarrow & & \downarrow \\ \mathrm{L} & \longrightarrow & SO(3) \end{array} \tag{38}$$

of the reduction of structure groups of a frame bundle $LX$ in gravitation theory [22]. This reduction diagram results in the following.

(i) There is one-to-one correspondence between the reduced principal $SO(4)$-subbundles $L^{g^R}X$ of a frame bundle $LX$ and the global sections of the quotient bundle $LX/SO(4) \to X$. Its global sections are Riemannian metrics $g^R$ on $X$. Thus, a Riemannian metric on a world manifold always exists.

(ii) As was mentioned above, a reduction of a structure group of a frame bundle $LX$ to a proper Lorentz group means the existence of a reduced proper Lorentz subbundle $L^hX \subset LX$ associated with a tetrad field $h$ or a pseudo-Riemannian metric $g = \zeta \circ h$ on $X$.

(iii) Since a structure group $L$ of this reduced Lorentz bundle $L^hX$ is reducible to a group $SO(3)$, there exists a reduced principal $SO(3)$-subbundle

$$L_0^hX \subset L^hX \subset LX, \tag{39}$$

called the spatial structure. The corresponding global section of a quotient bundle $L^hX/SO(3) \to X$ with a typical fibre $\mathbb{R}^3$ is a one-codimensional spatial distribution $\mathbf{F} \subset TX$ on $X$. Its annihilator is a one-dimensional codistribution $\mathbf{F}^* \subset T^*X$.

Given the spatial structure $L_0^hX$ (39), let us consider the Lorentz bundle atlas $\Psi_0^h$ (21) given by local sections $z_\iota$ of $LX$ taking their values into a reduced $SO(3)$-subbundle $L_0^hX$. Its transition functions are $SO(3)$-valued.

It follows that, in gravitation theory on a world manifold $X$, one can always choose an atlas of the tangent bundle $TX$ and associated bundles with $SO(3)$-valued transition functions. This bundle atlas, called the spatial bundle atlas.

Given a spatial bundle atlas $\Psi_0^h$, its $SO(3)$-valued transition functions preserve a time-like component

$$h^0 = h_\lambda^0 dx^\lambda \tag{40}$$

of local tetrad forms (23) which, therefore, is globally defined. We agree to call it the time-like tetrad form. Accordingly, the dual time-like vector field

$$h_0 = h_0^\mu \partial_\mu \tag{41}$$

also is globally defined. In this case, a spatial distribution $\mathbf{F}$ is spanned by spatial components $h_i$, $i = 1, 2, 3$, of the tetrad frames (22), while the time-like tetrad form (40) spans the tetrad codistribution $\mathbf{F}^*$, i.e.,

$$h^0 \rfloor \mathbf{F} = 0. \tag{42}$$

Then the tangent bundle $TX$ of a world manifold $X$ admits a space-time decomposition

$$TX = \mathbf{F} \oplus T^0X, \tag{43}$$

where $T^0X$ is a one-dimensional fibre bundle spanned by the time-like vector field $h_0$ (41).

Due to the commutative diagram (38), the reduced L-subbundle $L_0^hX$ (39) of a reduced Lorentz bundle $L^hX$ is a reduced subbundle of some reduced $SO(4)$-bundle $L^{g^R}X$ too, i.e.,

$$L^hX \supset L_0^hX \subset L^{g^R}X. \tag{44}$$

Let $g = \zeta \circ h$ and $g^R$ be the corresponding pseudo-Riemannian and Riemannian metrics on $X$. Written with respect to a spatial bundle atlas $\Psi_0^h$, they read

$$g = \eta_{ab} h^a \otimes h^b, \qquad g_{\mu\nu} = h_\mu^a h_\nu^b \eta^{ab}, \tag{45}$$

$$g^R = \eta_{ab}^E h^a \otimes h^b, \qquad g_{\mu\nu}^R = h_\mu^a h_\nu^b \eta_{ab}^E, \tag{46}$$

where $\eta^E$ is an Euclidean metric in $\mathbb{R}^4$. The space-time decomposition (43) is orthonormal with respect to both the metrics (45) and (46). The following well-known theorem holds [6, 9].

**Theorem 3.** For any pseudo-Riemannian metric $g$ on a world manifold $X$, there exist a normalized time-like one-form $h^0$ and a Riemannian metric $g^R$ such that

$$g = 2h^0 \otimes h^0 - g^R. \tag{47}$$

Conversely, let a world manifold $X$ admit a nowhere vanishing one-form $\sigma$ (or, equivalently, a nowhere vanishing vector field). Then any Riemannian world metric $g^R$ on $X$ yields the pseudo-Riemannian world metric $g$ (47) where $h^0 = \sigma(g^R(\sigma,\sigma))^{-1/2}$.

**Corollary 4.** A world manifold $X$ admits a pseudo-Riemannian metric iff there exists a nowhere vanishing one-form (or a vector field) on $X$.

Note that the condition (44) gives something more.

**Theorem 5.** There is one-to-one correspondence between the reduced $SO(3)$-subbundles of a frame bundle $LX$ and the triples $(g, \mathbf{F}, g^R)$ of a pseudo-Riemannian metric $g$, a spatial distribution $\mathbf{F}$ defined by the condition (42) and a Riemannian metric $g^R$ which obey the relation (47).

A spatial distribution $\mathbf{F}$ and a Riemannian metric $g^R$ in the triple $(g, \mathbf{F}, g^R)$ in Theorem 5 are called $g$-compatible. The corresponding space-time decomposition is said to be a $g$-compatible space-time structure. A world manifold endowed with a pseudo-Riemannian metric and a compatible space-time structure is called the space-time.

**Remark 3.** A $g$-compatible Riemannian metric $g^R$ in a triple $(g, \mathbf{F}, g^R)$ defines a $g$-compatible distance function $d(x, x')$ on a world manifold $X$. Such a function brings $X$ into a metric space whose locally Euclidean topology is equivalent to a manifold topology on $X$. Given a gravitational field $g$, the $g$-compatible Riemannian metrics and the corresponding distance functions are different for different spatial distributions $\mathbf{F}$ and $\mathbf{F}'$. It follows that physical observers associated with different spatial distributions $\mathbf{F}$ and $\mathbf{F}'$ perceive a world manifold $X$ as different Riemannian spaces. The well-known relativistic changes of sizes of moving bodies exemplify this phenomenon [22].

A space-time structure is called integrable if a spatial distribution $\mathbf{F}$, given by the condition (42), is involutive. In this case, its integral manifolds constitute a spatial foliation $\mathcal{F}$ of a world manifold $X$ whose leaves are spatial three-dimensional subspaces of $X$. A spatial distribution $\mathbf{F}$ is integrable iff the one-form $h^0$ (40) obeys the condition $dh^0 \wedge h^0 = 0$. In this case, the time-like vector field $h_0$ (41) is transversal to a spatial foliation $\mathcal{F}$.

A spatial foliation $\mathcal{F}$ is called causal if no curve transversal to its leaves intersects each leave more than once. This condition is equivalent to the stable causality of Hawking [9].

**Theorem 6.** A space-time foliation $\mathcal{F}$ is causal iff it is a foliation of level surfaces of some smooth real function $f$ on $X$ whose differential nowhere vanishes. Such a foliation is simple, i.e., there is a fibred manifold $f : X \to \mathbb{R}$ whose fibres are leaves of $\mathcal{F}$.

This is not the case of a compact world manifold which can not be a fibred manifold over $\mathbb{R}$. Thus, a compact world manifold fails to satisfy the stable causality condition.

The stable causality does not provide the simplest causal structure. If a fibred manifold $X \to \mathbb{R}$ in Theorem 6 is a fibre bundle, it is trivial, i.e., a world manifold $X$ is a globally hyperbolic space $X = \mathbb{R} \times M$. Since any oriented three-dimensional manifold is parallelizable, a globally hyperbolic space also is parallelizable.

## 6 Metric-affine gravitation theory

In the absence of matter fields, dynamic variables of metric-affine gravitation theory are world connections and pseudo-Riemannian metrics on $X$. Their Lagrangian $L_{\rm MA}$ is invariant under general covariant transformations.

**Remark 4.** In view of the decomposition (26), one can choose a different collection of dynamic variables of metric-affine gravitation theory. These are a pseudo-Riemannian metric, a torsion and the non-metricity tensor (28). A problem is that, in this case, we deal with differential equations of fourth and higher order in a pseudo-Riemannian metric, unless $L_{\rm MA}$ is the Hilbert–Einstein Lagrangian $L_{\rm HE}$ (69).

World connections are represented by sections of the bundle of world connections $C_{\rm W}$ (18). Pseudo-Riemannian world metrics are described by sections of the quotient bundle (2). Therefore, let us consider the bundle product

$$Y = \Sigma_{\rm PR} \underset{X}{\times} C_{\rm W} \tag{48}$$

coordinated by $(x^\lambda, \sigma^{\mu\nu}, k_\mu{}^\alpha{}_\beta)$.

Let us restrict our consideration to first order Lagrangian theory on $Y$ (48). In this case, a configuration space of gauge gravitation theory is the jet manifold

$$J^1Y = J^1\Sigma_{\rm PR} \underset{X}{\times} J^1 C_{\rm W}, \tag{49}$$

coordinated by $(x^\lambda, \sigma^{\mu\nu}, k_\mu{}^\alpha{}_\beta, \sigma^{\mu\nu}_\lambda, k_{\lambda\mu}{}^\alpha{}_\beta)$.

A first order Lagrangian $L_{\rm MA}$ of metric-affine gravitation theory is a defined as a density

$$L_{\rm AM} = \mathcal{L}_{\rm AM}(x^\lambda, \sigma^{\mu\nu}, k_\mu{}^\alpha{}_\beta, \sigma^{\mu\nu}_\lambda, k_{\lambda\mu}{}^\alpha{}_\beta)\omega, \qquad \omega = dx^1\wedge\cdots\wedge dx^4, \tag{50}$$

on the configuration space $J^1Y$ (49) [6]. Its Euler–Lagrange operator takes the form

$$\delta L_{\rm MA} = (\mathcal{E}_{\alpha\beta} d\sigma^{\alpha\beta} + \mathcal{E}^\mu{}_\alpha{}^\beta dk_\mu{}^\alpha{}_\beta)\wedge\omega. \tag{51}$$

$$\mathcal{E}_{\alpha\beta} = \left(\frac{\partial}{\partial\sigma^{\alpha\beta}} - d_\lambda\frac{\partial}{\partial\sigma^{\alpha\beta}_\lambda}\right)\mathcal{L}_{\rm AM}, \qquad \mathcal{E}^\mu{}_\alpha{}^\beta = \left(\frac{\partial}{\partial k_\mu{}^\alpha{}_\beta} - d_\lambda\frac{\partial}{\partial k_{\lambda\mu}{}^\alpha{}_\beta}\right)\mathcal{L}_{\rm AM},$$

$$d_\lambda = \partial_\lambda + \sigma^{\alpha\beta}_\lambda\frac{\partial}{\partial\sigma^{\alpha\beta}} + k_{\lambda\mu}{}^\alpha{}_\beta\frac{\partial}{\partial k_\mu{}^\alpha{}_\beta} + \sigma^{\alpha\beta}_{\lambda\nu}\frac{\partial}{\partial\sigma^{\alpha\beta}_\nu} + k_{\lambda\nu\mu}{}^\alpha{}_\beta\frac{\partial}{\partial k_{\nu\mu}{}^\alpha{}_\beta}.$$

The corresponding Euler–Lagrange equations read

$$\mathcal{E}_{\alpha\beta} = 0, \qquad \mathcal{E}^{\mu}{}_{\alpha}{}^{\beta} = 0. \tag{52}$$

**Remark 5.** The Hilbert–Einstein Lagrangian $L_{\rm HE}$ (67) of General Relativity depends only on metric variables $\sigma^{\alpha\beta}$. It is a reduced second order Lagrangian which differs from the first order one $L'_{\rm HE}$ in a variationally trivial term and leads to second order Euler–Lagrange equations (68).

The fibre bundle (48) is a natural bundle admitting the functorial lift

$$\begin{aligned}\widetilde\tau_{\Sigma C} = \tau^\mu\partial_\mu + (\sigma^{\nu\beta}\partial_\nu\tau^\alpha + \sigma^{\alpha\nu}\partial_\nu\tau^\beta)\frac{\partial}{\partial\sigma^{\alpha\beta}} + \\ (\partial_\nu\tau^\alpha k_\mu{}^\nu{}_\beta - \partial_\beta\tau^\nu k_\mu{}^\alpha{}_\nu - \partial_\mu\tau^\nu k_\nu{}^\alpha{}_\beta + \partial_{\mu\beta}\tau^\alpha)\frac{\partial}{\partial k_\mu{}^\alpha{}_\beta}\end{aligned} \tag{53}$$

of vector fields $\tau$ on $X$ [6]. It is an infinitesimal generator of general covariant transformations. At the same time, $\widetilde\tau_{\Sigma C}$ (53) also is an infinitesimal gauge transformation whose gauge parameters are components $\tau^\lambda(x)$ of vector fields $\tau$ on $X$.

By virtue of the relativity principle, the Lagrangian $L_{\rm MA}$ (50) of metric-affine gravitation theory is assumed to be invariant under general covariant transformations. Its Lie derivative along the jet prolongation $J^1\widetilde\tau_{\Sigma C}$ of the vector field $\widetilde\tau_{\Sigma C}$ for any $\tau$ vanishes, i.e.,

$$\mathbf{L}_{J^1\widetilde\tau_{\Sigma C}} L_{\rm AM} = 0. \tag{54}$$

**Remark 6.** The total group of automorphisms of a frame bundle $LX$ also is considered in gauge gravitation theory [10]. As was mentioned above, such an automorphism is a composition of some general covariant transformation and a vertical automorphism of $LX$. A problem is that the most of gravitation Lagrangians, e.g., the Hilbert–Einstein Lagrangian are not invariant under vertical frame transformations. To overcome this difficulty, one considers frame fields (i.e., sections of a frame bundle $LX$) as dynamic variables. However, these fields fail to be global, unless a world manifold $X$ is parallelizable.

As was mentioned above a configuration space $J^1C_{\rm W}$ of world connections possesses the canonical splitting (19). The following assertion is analogous to the well-known Utiyama theorem in Yang–Mills gauge theory of principal connections [6, 29].

**Theorem 7.** If a first order Lagrangian $L_{\rm MA}$ on the configuration space (49) is invariant under general covariant transformations and it does not depend on the jet coordinates $\sigma^{\alpha\beta}_\lambda$ (i.e., derivatives of a metric), this Lagrangian factorizes through the terms $\mathcal{R}_{\lambda\mu}{}^\alpha{}_\beta$ (19).

In contrast with the well-known Lagrangian of Yang–Mills gauge theory, different contractions of a curvature tensor $\mathcal{R}_{\lambda\mu}{}^\alpha{}_\beta$ are possible. For instance, the Ricci tensor $R_c$ (13)

and a scalar curvature $\mathcal{R} = \sigma^{\mu\nu}\mathcal{R}_{\lambda\mu}{}^{\lambda}{}_{\nu}$ are defined. Moreover, a Lagrangian $L_{\rm MA}$ also can depend on a torsion

$$t_\mu{}^\nu{}_\lambda = k_\mu{}^\nu{}_\lambda - k_\lambda{}^\nu{}_\mu. \tag{55}$$

The Yang–Mills Lagrangian in metric-affine gravitation theory is given by the expression

$$\sigma^{\mu\lambda}\sigma^{\nu\gamma}\mathcal{R}_{\mu\nu}{}^\alpha{}_\beta\mathcal{R}_{\lambda\gamma}{}^\beta{}_\alpha.$$

It is invariant under the total group of automorphisms of a frame bundle $LX$. In this case, metric variables $\sigma^{\mu\lambda}$ fail to be dynamic because they are brought into a constant Minkowski metric by general frame transformations.

## 7 Energy-momentum conservation law

Since infinitesimal general covariant transformations $\widetilde\tau_{\Sigma C}$ (53) are exact symmetries of a metric-affine gravitation Lagrangian, let us study the corresponding conservation law. This is the energy-momentum conservation laws because vector fields $\widetilde\tau_{\Sigma C}$ are not vertical [6, 24]. Moreover, since infinitesimal general covariant transformations $\widetilde\tau_{\Sigma C}$ (53) are infinitesimal gauge transformations depending on derivatives of gauge parameters, the corresponding energy-momentum current reduces to a superpotential [6, 30].

In view of Theorem 7, let us assume that a metric-affine gravitation Lagrangian $L_{\rm MA}$ is independent of the derivative coordinates $\sigma_\lambda{}^{\alpha\beta}$ of a world metric and that it factorizes through the curvature terms $\mathcal{R}_{\lambda\mu}{}^\alpha{}_\beta$ (19). Then the following relations take place:

$$\pi^{\lambda\nu}{}_\alpha{}^\beta = -\pi^{\nu\lambda}{}_\alpha{}^\beta, \qquad \pi^{\lambda\nu}{}_\alpha{}^\beta = \frac{\partial\mathcal{L}_{\rm MA}}{\partial k_{\lambda\nu}{}^\alpha{}_\beta}, \tag{56}$$

$$\frac{\partial\mathcal{L}_{\rm MA}}{\partial k_\nu{}^\alpha{}_\beta} = \pi^{\lambda\nu}{}_\alpha{}^\sigma k_\lambda{}^\beta{}_\sigma - \pi^{\lambda\nu}{}_\sigma{}^\beta k_\lambda{}^\sigma{}_\alpha. \tag{57}$$

Let us follow the compact notation

$$y^A = k_\mu{}^\alpha{}_\beta, \qquad u_\mu{}^\alpha{}_\beta{}^{\varepsilon\sigma}_\gamma = \delta^\varepsilon_\mu\delta^\sigma_\beta\delta^\alpha_\gamma, \qquad u_\mu{}^\alpha{}_\beta{}^\varepsilon_\gamma = k_\mu{}^\varepsilon{}_\beta\delta^\alpha_\gamma - k_\mu{}^\alpha{}_\gamma\delta^\varepsilon_\beta - k_\gamma{}^\alpha{}_\beta\delta^\varepsilon_\mu.$$

Then the vector field (53) takes the form

$$\widetilde\tau_{\Sigma C} = \tau^\lambda\partial_\lambda + (\sigma^{\nu\beta}\partial_\nu\tau^\alpha + \sigma^{\alpha\nu}\partial_\nu\tau^\beta)\partial_{\alpha\beta} + (u^{A\beta}_{\ \ \alpha}\partial_\beta\tau^\alpha + u^{A\beta\mu}_{\ \ \ \alpha}\partial_{\beta\mu}\tau^\alpha)\partial_A.$$

We also have the equalities

$$\pi^\lambda_A u^{A\beta\mu}_{\ \ \ \alpha} = \pi^{\lambda\mu}{}_\alpha{}^\beta, \qquad \pi^\varepsilon_A u^{A\beta}_{\ \ \alpha} = -\partial^\varepsilon{}_\alpha{}^\beta\mathcal{L}_{\rm MA} - \pi^{\varepsilon\beta}{}_\sigma{}^\gamma k_\alpha{}^\sigma{}_\gamma.$$

Let $L_{\rm MA}$ be invariant under general covariant transformations, i.e., the equality (54) for any vector field $\tau$ holds. Then the first variational formula leads to an equality

$$\begin{aligned} 0 = & (\sigma^{\nu\beta}\partial_\nu\tau^\alpha + \sigma^{\alpha\nu}\partial_\nu\tau^\beta - \tau^\lambda\sigma_\lambda^{\alpha\beta})\delta_{\alpha\beta}\mathcal{L}_{\rm MA} + \\ & (u^{A\beta}{}_\alpha\partial_\beta\tau^\alpha + u^{A\beta\mu}{}_\alpha\partial_{\beta\mu}\tau^\alpha - \tau^\lambda y_\lambda^A)\delta_A\mathcal{L}_{\rm MA} - \\ & d_\lambda[\pi_A^\lambda(y_\alpha^A\tau^\alpha - u^{A\beta}{}_\alpha\partial_\beta\tau^\alpha - u^{A\varepsilon\beta}{}_\alpha\partial_{\varepsilon\beta}\tau^\alpha) - \tau^\lambda\mathcal{L}_{\rm MA}]. \end{aligned} \tag{58}$$

This equality on the shell (52) results in the weak conservation law

$$0 \approx -d_\lambda[\pi_A^\lambda(y_\alpha^A\tau^\alpha - u^{A\beta}{}_\alpha\partial_\beta\tau^\alpha - u^{A\varepsilon\beta}{}_\alpha\partial_{\varepsilon\beta}\tau^\alpha) - \tau^\lambda\mathcal{L}_{\rm MA}], \tag{59}$$

of the energy-momentum current of metric-affine gravity

$$\mathcal{J}_{\rm MA}{}^\lambda = \pi_A^\lambda(y_\alpha^A\tau^\alpha - u^{A\beta}{}_\alpha\partial_\beta\tau^\alpha - u^{A\varepsilon\beta}{}_\alpha\partial_{\varepsilon\beta}\tau^\alpha) - \tau^\lambda\mathcal{L}_{\rm MA}. \tag{60}$$

**Remark 7.** It is readily observed that, with respect to a local coordinate system where a vector field $\tau$ is constant, the energy-momentum current (60) leads to the canonical energy-momentum tensor

$$\mathcal{J}_{\rm MA}{}^\lambda{}_\alpha\tau^\alpha = (\pi^{\lambda\mu}{}_\beta{}^\nu k_{\alpha\mu}{}^\beta{}_\nu - \delta_\alpha^\lambda\mathcal{L}_{\rm MA})\tau^\alpha,$$

suggested in order to describe an energy-momentum complex in the Palatini model [2].

Due to the arbitrariness of $\tau^\lambda$, the equality (58) falls into a set of equalities

$$\pi^{(\lambda\varepsilon}{}_\gamma{}^{\sigma)} = 0, \tag{61}$$

$$(u^{A\varepsilon\sigma}{}_\gamma\partial_A + u^{A\varepsilon}{}_\gamma\partial_A^\sigma)\mathcal{L}_{\rm MA} = 0, \tag{62}$$

$$\delta_\alpha^\beta\mathcal{L}_{\rm MA} + 2\sigma^{\beta\mu}\delta_{\alpha\mu}\mathcal{L}_{\rm MA} + u^{A\beta}{}_\alpha\delta_A\mathcal{L}_{\rm MA} + d_\mu(\pi_A^\mu u^{A\beta}{}_\alpha) - y_\alpha^A\pi_A^\beta = 0, \tag{63}$$

$$\partial_\lambda\mathcal{L}_{\rm MA} = 0.$$

**Remark 8.** It is readily observed that the equalities (61) and (62) hold due to the relations (56) and (57), respectively.

Substituting the term $y_\alpha^A\pi_A^\beta$ from the expression (63) in the energy-momentum conservation law (59), one brings this conservation law into the form

$$\begin{aligned} 0 \approx & -d_\lambda[2\sigma^{\lambda\mu}\tau^\alpha\delta_{\alpha\mu}\mathcal{L}_{\rm MA} + u^{A\lambda}{}_\alpha\tau^\alpha\delta_A\mathcal{L}_{\rm MA} - \pi_A^\lambda u^{A\beta}{}_\alpha\partial_\beta\tau^\alpha + \\ & d_\mu(\pi^{\lambda\mu}{}_\alpha{}^\beta)\partial_\beta\tau^\alpha + d_\mu(\pi_A^\mu u^{A\lambda}{}_\alpha)\tau^\alpha - d_\mu(\pi^{\lambda\mu}{}_\alpha{}^\beta\partial_\beta\tau^\alpha)]. \end{aligned} \tag{64}$$

After separating the variational derivatives, the energy-momentum conservation law (64) of a metric-affine gravity takes a superpotential form

$$\begin{aligned} 0 \approx & -d_\lambda[2\sigma^{\lambda\mu}\tau^\alpha\delta_{\alpha\mu}\mathcal{L}_{\rm MA} + (k_\mu{}^\lambda{}_\gamma\delta^\mu{}_\alpha{}^\gamma\mathcal{L}_{\rm MA} - k_\mu{}^\sigma{}_\alpha\delta^\mu{}_\sigma{}^\lambda\mathcal{L}_{\rm MA} - k_\alpha{}^\sigma{}_\gamma\delta^\lambda{}_\sigma{}^\gamma\mathcal{L}_{\rm MA})\tau^\alpha + \\ & \delta^\lambda{}_\alpha{}^\mu\mathcal{L}_{\rm MA}\partial_\mu\tau^\alpha - d_\mu(\delta^\mu{}_\alpha{}^\lambda\mathcal{L}_{\rm MA})\tau^\alpha + d_\mu(\pi^{\mu\lambda}{}_\alpha{}^\nu(\partial_\nu\tau^\alpha - k_\sigma{}^\alpha{}_\nu\tau^\sigma))], \end{aligned}$$

where an energy-momentum current on-shell reduces to a generalized Komar superpotential

$$U_{\rm MA}{}^{\mu\lambda} = 2\frac{\partial\mathcal{L}_{\rm MA}}{\partial\mathcal{R}_{\mu\lambda}{}^\alpha{}_\nu}(D_\nu\tau^\alpha + t_\nu{}^\alpha{}_\sigma\tau^\sigma), \qquad (65)$$

where $D_\nu$ is a covariant derivative relative to a connection $k_\nu{}^\alpha{}_\sigma$ and $t_\nu{}^\alpha{}_\sigma$ is its torsion [5, 25].

# 8 General Relativity

In Einstein's General Relativity, dynamic variables are only a pseudo-Riemannian metric $\sigma^{\mu\nu}$, while a world connection is restricted to the Levi–Civita one

$$k_\mu{}^\beta{}_\lambda = \{_\mu{}^\beta{}_\lambda\} = -\frac{1}{2}\sigma^{\beta\nu}(d_\mu\sigma_{\nu\lambda} + d_\lambda\sigma_{\mu\nu} - d_\nu\sigma_{\mu\lambda}). \qquad (66)$$

A Lagrangian of General Relativity is the Hilbert–Einstein Lagrangian

$$L_{\rm HE} = \frac{1}{2}\mathcal{R}\sqrt{\sigma}\omega = \sigma^{\mu\beta}\mathcal{R}_{\lambda\mu}{}^\lambda{}_\beta\sqrt{\sigma}\omega, \qquad \sigma = |\det(\sigma_{\alpha\beta})|, \qquad (67)$$

$$\mathcal{R}_{\lambda\mu}{}^\alpha{}_\beta = d_\lambda\{_\mu{}^\alpha{}_\beta\} - d_\mu\{_\lambda{}^\alpha{}_\beta\} + \{_\lambda{}^\gamma{}_\beta\}\{_\mu{}^\alpha{}_\gamma\} - \{_\mu{}^\gamma{}_\beta\}\{_\lambda{}^\alpha{}_\gamma\}.$$

It is a reduced second order Lagrangian resulting in second order Euler–Lagrange equations

$$\mathcal{E}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{2}\sigma_{\alpha\beta}\mathcal{R} = 0. \qquad (68)$$

The Hilbert–Einstein Lagrangian differs from the first order one $L'_{\rm HE}$ in a variationally trivial term $d_\lambda f^\lambda\omega$. General covariant transformations are variational, but not exact symmetries of a Lagrangian $L'_{\rm HE}$.

In metric-affine gravitation theory, the Hilbert–Einstein Lagrangian takes the form

$$L = \mathcal{R}\sqrt{\sigma}\omega = \sigma^{\mu\beta}\mathcal{R}_{\lambda\mu}{}^\lambda{}_\beta\sqrt{\sigma}\omega, \qquad (69)$$

$$\mathcal{R}_{\lambda\mu}{}^\alpha{}_\beta = d_\lambda k_\mu{}^\alpha{}_\beta - d_\mu k_\lambda{}^\alpha{}_\beta + k_\lambda{}^\gamma{}_\beta k_\mu{}^\alpha{}_\gamma - k_\mu{}^\gamma{}_\beta k_\lambda{}^\alpha{}_\gamma,$$

where $\mathcal{R}$ is a scalar curvature. The corresponding Euler–Lagrange equations read

$$\mathcal{E}_{\alpha\beta} = \mathcal{R}_{\alpha\beta} - \frac{1}{2}\sigma_{\alpha\beta}\mathcal{R} = 0, \qquad (70)$$

$$\mathcal{E}^\nu{}_\alpha{}^\beta = -d_\alpha(\sigma^{\nu\beta}\sqrt{\sigma}) + d_\lambda(\sigma^{\lambda\beta}\sqrt{\sigma})\delta^\nu_\alpha + \qquad (71)$$
$$(\sigma^{\nu\gamma}k_\alpha{}^\beta{}_\gamma - \sigma^{\lambda\gamma}\delta^\nu_\alpha k_\lambda{}^\beta{}_\gamma - \sigma^{\nu\beta}k_\gamma{}^\gamma{}_\alpha + \sigma^{\lambda\beta}k_\lambda{}^\nu{}_\alpha)\sqrt{\sigma} = 0.$$

The equation (70) is an analogy of the Einstein equations, whereas the equations (71) describe the torsion $t_\mu{}^\nu{}_\lambda$ (55) and the non-metricity

$$c_{\mu\nu\alpha} = c_{\mu\alpha\nu} = d_\mu\sigma_{\nu\alpha} + k_\mu{}^\beta{}_\alpha\sigma_{\nu\beta} + k_\mu{}^\beta{}_\nu\sigma_{\beta\alpha}$$

of a world connection. They are brought into the form

$$\sqrt{\sigma^{-1}}\sigma_{\nu\varepsilon}\sigma_{\beta\mu}\mathcal{E}^\nu{}_\alpha{}^\beta = c_{\alpha\varepsilon\mu} - \frac{1}{2}\sigma_{\mu\varepsilon}\sigma^{\lambda\gamma}c_{\alpha\lambda\gamma} - \sigma_{\alpha\varepsilon}\sigma^{\lambda\beta}c_{\lambda\beta\mu} + \frac{1}{2}\sigma_{\alpha\varepsilon}\sigma^{\lambda\gamma}c_{\mu\lambda\gamma} + t_{\mu\varepsilon\alpha} + \sigma_{\mu\varepsilon}t_\alpha{}^\gamma{}_\gamma + \sigma_{\alpha\varepsilon}t_\gamma{}^\gamma{}_\mu = 0. \tag{72}$$

The Hilbert–Einstein Lagrangian (69) is invariant under general covariant transformations. The corresponding generalized Komar superpotential (65) comes to the well-known Komar superpotential if one substitutes the Levi–Civita connection $k_\nu{}^\alpha{}_\sigma = \{_\nu{}^\alpha{}_\sigma\}$ (66).

## 9 BRST extension

A BRST extension of Lagrangian field theory is a first step towards its quantization [6, 29].

Taking the vertical part of vector fields $\widetilde{\tau}_{\Sigma C}$ (53) and replacing gauge parameters $\tau^\lambda$ with ghosts $c^\lambda$, we obtain the odd vertical graded derivation

$$u = u^{\alpha\beta}\frac{\partial}{\partial\sigma^{\alpha\beta}} + u_\mu{}^\alpha{}_\beta\frac{\partial}{\partial k_\mu{}^\alpha{}_\beta} = (\sigma^{\nu\beta}c^\alpha_\nu + \sigma^{\alpha\nu}c^\beta_\nu - c^\lambda\sigma^{\alpha\beta}_\lambda)\frac{\partial}{\partial\sigma^{\alpha\beta}} + (c^\alpha_\nu k_\mu{}^\nu{}_\beta - c^\nu_\beta k_\mu{}^\alpha{}_\nu - c^\nu_\mu k_\nu{}^\alpha{}_\beta + c^\alpha_{\mu\beta} - c^\lambda k_{\lambda\mu}{}^\alpha{}_\beta)\frac{\partial}{\partial k_\mu{}^\alpha{}_\beta}. \tag{73}$$

Since the infinitesimal gauge transformations $\widetilde{\tau}_{\Sigma C}$ (53) are exact symmetries of a metric-affine Lagrangian $L_{\mathrm{MA}}$, the vertical graded derivation $u$ (73) is a variational symmetry of $L_{\mathrm{MA}}$ and, thus, is its gauge symmetry. As a consequence, the Euler–Lagrange operator $\delta L_{\mathrm{MA}}$ (51) of this Lagrangian obeys the complete Noether identities

$$-\sigma^{\alpha\beta}_\lambda\mathcal{E}_{\alpha\beta} - 2d_\mu(\sigma^{\mu\beta}\mathcal{E}_{\lambda\beta} - k_{\lambda\mu}{}^\alpha{}_\beta\mathcal{E}^\mu{}_\alpha{}^\beta - d_\mu[(k_\nu{}^\mu{}_\beta\delta^\alpha_\lambda - k_\nu{}^\alpha{}_\lambda\delta^\mu_\beta - k_\lambda{}^\alpha{}_\beta\delta^\mu_\nu)\mathcal{E}^\nu{}_\alpha{}^\beta] + d_{\mu\beta}\mathcal{E}^\mu{}_\lambda{}^\beta = 0. \tag{74}$$

These Noether identities are irreducible. Therefore, the graded derivation (73) is the gauge operator of gauge gravitation theory. It admits the nilpotent BRST extension

$$\mathbf{c} = u + c^\lambda_\mu c^\mu\frac{\partial}{\partial c^\lambda} \tag{75}$$

(which differs from that in [8]).

Accordingly, an original gravitation Lagrangian $L_{\mathrm{MA}}$ admits a BRST extension to a proper solution of the master equation which reads

$$L_E = L_{\mathrm{MA}} + u^{\alpha\beta}\overline{\sigma}_{\alpha\beta}\omega + u_\mu{}^\alpha{}_\beta\overline{k}^\mu{}_\alpha{}^\beta\omega + c^\lambda_\mu c^\mu\overline{c}_\lambda\omega, \tag{76}$$

where $\overline{\sigma}_{\alpha\beta}$, $\overline{k}^\mu{}_\alpha{}^\beta$ and $\overline{c}_\lambda$ are the corresponding antifields [1, 6].

## 10 Dirac spinor fields

Apparently, the existence of Dirac's fermion matter possessing Lorentz symmetries is an underlying physical reason of an appearance of a Lorentz reduced structure and, consequently, a (tetrad) gravitational field.

Dirac spinors are conventionally described in the framework of formalism of Clifford algebras [17].

Let $M = \mathbb{R}^4$ be a Minkowski space equipped with a Minkowski metric $\eta$. Let $\mathbb{C}_{1,3}$ be a complex Clifford algebra generated by elements of $M$. It is a complexified quotient of the tensor algebra of $M$ by a two-sided ideal spanned by elements

$$e \otimes e' + e' \otimes e - 2\eta(e, e') \in \otimes M, \qquad e, e' \in M.$$

A Dirac spinor space $V$ is defined as a minimal left ideal of $\mathbb{C}_{1,3}$ in which this algebra acts on the left. There is a representation

$$\gamma : M \otimes V \to V, \qquad \gamma(e^\alpha) = \gamma^\alpha, \tag{77}$$

of elements of a Minkowski subspace $M \subset \mathbb{C}_{1,3}$ by Dirac $\gamma$-matrices in $V$.

A Clifford group $G_{1,3} \subset \mathbb{R}_{1,3}$ is defined to consist of invertible elements $l_s$ of a real Clifford algebra $\mathbb{R}_{1,3}$ such that inner automorphisms given by these elements preserve a Minkowski space $M \subset \mathbb{R}_{1,3}$, i.e.,

$$l_s e l_s^{-1} = l(e), \qquad e \in M, \tag{78}$$

where $l$ is a Lorentz transformation of $M$. Hence, there is an epimorphism of a Clifford group $G_{1,3}$ onto a Lorentz group $O(1,3)$. However, the action (78) of a Clifford group in a Minkowski space $M$ is not effective.

A subgroup $\mathrm{Pin}(1,3)$ of $G_{1,3}$ is generated by elements $e \in M$ such that $\eta(e,e) = \pm 1$. An even part of $\mathrm{Pin}(1,3)$ is a spin group $\mathrm{Spin}(1,3)$, i.e., $\eta(e,e) = 1$, $e \in \mathrm{Spin}(1,3)$. Its component of the unity

$$\mathrm{L_s} = \mathrm{Spin}^0(1,3) \simeq SL(2,\mathbb{C})$$

is the well-known two-fold universal covering group

$$z_L : \mathrm{L_s} \to \mathrm{L} = \mathrm{L_s}/\mathbb{Z}_2 \tag{79}$$

of a proper Lorentz group L. We agree to call $\mathrm{L_s}$ (79) the spinor Lorentz group. Its Lie algebra is that of a proper Lorentz group L. We further consider an action of a spinor Lorentz group $\mathrm{L_s}$, factorizing through that of a proper Lorentz group L, in the Minkowski space $M$, but it is not effective.

A Clifford group $G_{1,3}$ acts in a Dirac spinor space $V$ by left multiplications $G_{1,3} \ni l_s : v \mapsto l_s v$, $v \in V$. This action preserves the representation (77). A spinor Lorentz group $\mathrm{L_s}$ acts in the Dirac spinor space $V$ by means of the infinitesimal generators

$$L_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]. \tag{80}$$

In classical field theory, Dirac spinor fields are described by sections of a spinor bundle on a world manifold $X$ whose typical fibre is the Dirac spinor space $V$ and whose structure group is the spinor Lorentz group $\mathrm{L_s}$. In order to construct the Dirac operator, one need a fibrewise action (77) of the whole Clifford algebra $\mathbb{C}_{1,3}$ in a spinor bundle. Therefore, a spinor bundle must be represented as a subbundle of the bundle in Clifford algebras.

Let us start with a fibre bundle in Minkowski spaces $MX \to X$ over a world manifold $X$. It is defined as a fibre bundle with a typical fibre $M$ and a structure group L. This fibre bundle is extended to a fibre bundle in Clifford algebras $CX$ whose fibres $C_xX$ are Clifford algebras generated by fibres $M_xX$ of a fibre bundle in Minkowski spaces $MX$. A fibre bundle $CX$ possesses a structure group of inner automorphisms of a complex Clifford algebra $\mathbb{C}_{1,3}$. This structure group is reducible to a proper Lorentz group L and, consequently, a bundle in Clifford algebras $CX$ contains a subbundle $MX$ of the generating Minkowski spaces. However, $CX$ need not contain a spinor subbundle because a Dirac spinor subspace $V$ of $\mathbb{C}_{1,3}$ is not stable under inner automorphisms of $\mathbb{C}_{1,3}$. A spinor subbundle $S_M$ of $CX$ exists if transition functions of $CX$ can be lifted from a Clifford group $G_{1,3}$. This condition agrees with the familiar condition of the existence of a spinor structure.

A bundle $MX$ in Minkowski spaces must be isomorphic to the cotangent bundle $T^*X$ in order that sections of a spinor bundle $S_M$ describe spinor fields on a world manifold $X$.

A Dirac spinor structure on a world manifold $X$ is defined as a pair $(P^h, z_s)$ of a principal bundle $P^h \to X$ with astructure spin group $L_s = SL(2,\mathbb{C})$ and its bundle morphism $z_s : P^h \to LX$ to a frame bundle $LX$ [17]. Any such morphism factorizes

$$P^h \to L^hX \to LX \tag{81}$$

through some reduced principal subbundle $L^hX \subset LX$ with a structure proper Lorentz group $L$. Thus, any Dirac spinor structure is associated with a Lorentz reduced structure, but the converse need not be true. There is the well-known topological obstruction to the existence of a Dirac spinor structure [4]. For instance, a Dirac spinor structure on a non-compact manifold $X$ exists iff $X$ is parallelizable.

Hereafter, we restrict our consideration to Dirac spinor structures on a non-compact (and, consequently, parallelizable) world manifold $X$. In this case, all Dirac spinor structures are isomorphic. Therefore, there is one-to-one correspondence

$$z_h : P^h_{\mathrm{s}} \to L^hX \subset LX$$

between the Lorentz reduced structures $L^hX$ and the Dirac spinor structures $(P^h_{\rm s}, z_h)$ which factorize through the corresponding $L^hX$. In particular, every Lorentz bundle atlas $\Psi^h = \{z^h_\iota\}$ (21) of $L^hX$ gives rise to an atlas

$$\overline\Psi^h = \{\overline z^h_\iota\}, \qquad z^h_\iota = z_h\circ \overline z^h_\iota, \tag{82}$$

of a principal $\mathrm{L_s}$-bundle $P^h_{\rm s}$. We agree to call $P^h_{\rm s}$ the spinor principal bundles.

Let $(P^h_{\rm s}, z_h)$ be the Dirac spinor structure associated with a tetrad field $h$. Let

$$S^h = (P^h_{\rm s}\times V)/\mathrm{L_s}\to X \tag{83}$$

be a $P^h_{\rm s}$-associated spinor bundle whose typical fibre $V$ carriers the spinor representation (80) of a spinor Lorentz group $\mathrm{L_s}$. One can think of sections of $S^h$ (83) as describing Dirac spinor fields in the presence of a tetrad field $h$.

Let us consider an $L^hX$-associated bundle in Minkowski spaces

$$M^hX = (L^hX\times M)/\mathrm{L} = (P^h_{\rm s}\times M)/\mathrm{L_s}$$

and a $P^h_{\rm s}$-associated spinor bundle $S^h$ (83). It is isomorphic to the cotangent bundle

$$T^*X = (L^hX\times M)/\mathrm{L}. \tag{84}$$

Then, using the morphism (77), one can define a representation

$$\gamma_h : T^*X\otimes S^h = (P^h_{\rm s}\times (M\otimes V))/\mathrm{L_s}\to (P^h_{\rm s}\times \gamma(M\otimes V))/\mathrm{L_s} = S^h \tag{85}$$

of covectors to $X$ by Dirac $\gamma$-matrices on elements of a spinor bundle $S^h$. Relative to a Lorentz bundle atlas $\{z^h_\iota\}$ of $LX$ and the corresponding atlas $\{\overline z_\iota\}$ (82) of a spinor principal bundle $P^h_{\rm s}$, the representation (85) reads

$$y^A(\gamma_h(h^a(x)\otimes v)) = \gamma^{aA}{}_B y^B(v), \qquad v\in S^h_x,$$

where $y^A$ are associated bundle coordinates on $S^h$, and $h^a$ are tetrad coframes. For brevity, we write

$$\widehat h^a = \gamma_h(h^a) = \gamma^a, \qquad \widehat{dx}^\lambda = \gamma_h(dx^\lambda) = h^\lambda_a(x)\gamma^a. \tag{86}$$

Given the representation (86), one can introduce a Dirac operator on $S^h$ with respect to a principal connection on $P^h$. Then, sections of a spinor bundle $S^h$ describe Dirac spinor fields in the presence of a tetrad field $h$. Note that there is one-to-one correspondence between the principal connections on $P^h$ and those on a proper Lorentz bundle $L^hX = z_s(P^h)$. Then it follows from Theorem 2 that, since Lie algebras of $GL_4$ and L obey the decomposition (34), any world connection $\Gamma$ yields a spinor connection $\Gamma_s$ on $P^h$ and $S^h$ [6, 26].

This fact enables one to describe Dirac spinor fields in the framework of metric-affine gravitation theory with general world connections.

Dirac spinor fields in the presence of different tetrad field $h$ and $h'$ are described by sections of different spinor bundles $S^h$ and $S^{h'}$. A problem is that, though reduced Lorentz bundles $L^hX$ and $L^{h'}X$ are isomorphic, the associated structures of bundles in Minkowski spaces $M^hX$ and $M^{h'}X$ (84) on the cotangent bundle $T^*X$ are non-equivalent because of non-equivalent actions of a Lorentz group on a typical fibre of $T^*X$ seen both as a typical fibre of $M^hX$ and that of $M^{h'}X$. As a consequence, the representations $\gamma_h$ and $\gamma_{h'}$ (85) for different tetrad fields $h$ and $h'$ are non-equivalent [22, 26]. Indeed, let

$$t^* = t_\mu dx^\mu = t_a h^a = t'_a h'^a$$

be an element of $T^*X$. Its representations $\gamma_h$ and $\gamma_{h'}$ (85) read

$$\gamma_h(t^*) = t_a\gamma^a = t_\mu h^\mu_a\gamma^a, \qquad \gamma_{h'}(t^*) = t'_a\gamma^a = t_\mu h'^\mu_a\gamma^a.$$

They are non-equivalent because no isomorphism $\Phi_s$ of $S^h$ onto $S^{h'}$ can obey the condition

$$\gamma_{h'}(t^*) = \Phi_s\gamma_h(t^*)\Phi_s^{-1}, \qquad t^* \in T^*X.$$

Since the representations (86) for different tetrad fields fail to be equivalent, one meets a problem of describing Dirac spinor fields in the presence of different tetrad fields and under general covariant transformations.

In order to solve this problem, let us consider a universal two-fold covering $\widetilde{GL}_4$ of a group $GL_4$ and a $\widetilde{GL}_4$-principal bundle $\widetilde{LX} \to X$ which is a two-fold covering bundle of a frame bundle $LX$ [17]. Then we have a commutative diagram

$$\begin{array}{ccc} \widetilde{LX} & \xrightarrow{\zeta} & LX \\ \uparrow & & \uparrow \\ P^h & \longrightarrow & L^hX \end{array}$$

for any Dirac spinor structure (81) [26]. As a consequence, $\widetilde{LX}/\mathrm{L}_s = LX/L = \Sigma_\mathrm{T}$. Since $\widetilde{LX} \to \Sigma_T$ is an $\mathrm{L}_s$-principal bundle, one can consider an associated spinor bundle $S \to \Sigma_T$ whose typical fibre is a Dirac spinor space $V$. We agree to call it the universal spinor bundle because, given a tetrad field $h$, the pull-back $S^h = h^*S \to X$ of $S$ onto $X$ is a spinor bundle $S^h$ on $X$ which is associated with an $\mathrm{L}_s$-principal bundle $P^h$. A universal spinor bundle $S$ is endowed with bundle coordinates $(x^\lambda, \sigma^\mu_a, y^A)$, where $(x^\lambda, \sigma^\mu_a)$ are bundle coordinates on $\Sigma_T$ and $y^A$ are coordinates on a spinor space $V$. A universal spinor bundle $S \to \Sigma_T$ is a subbundle of a bundle in Clifford algebras which is generated by a bundle of Minkowski spaces associated with an L-principal bundle $LX \to \Sigma_T$. As a consequence, there is a representation

$$\gamma_\Sigma : T^*X \underset{\Sigma_T}{\otimes} S \to S, \qquad \gamma_\Sigma(dx^\lambda) = \sigma^\lambda_a \gamma^a, \tag{87}$$

whose restriction to a subbundle $S^h \subset S$ restarts the representation (**??**).

Sections of a composite bundle

$$S \to \Sigma_{\rm T} \to X \tag{88}$$

describe Dirac spinor fields in the presence of different tetrad fields as follows [6, 26]. Due to the splitting (34), any world connection $\Gamma$ on $X$ yields a connection

$$\begin{aligned} A_\Sigma = dx^\lambda \otimes (\partial_\lambda - \frac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)\sigma^\nu_k \Gamma_\lambda{}^\mu{}_\nu L_{ab}{}^A{}_B y^B \partial_A) + \\ d\sigma^\mu_k \otimes (\partial^k_\mu + \frac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu) L_{ab}{}^A{}_B y^B \partial_A) \end{aligned} \tag{89}$$

on the universal spinor bundle $S \to \Sigma_T$. Its restriction to $S^h$ is the above mentioned spinor connection

$$\Gamma_s = dx^\lambda \otimes [\partial_\lambda + \frac{1}{4}(\eta^{kb}h^a_\mu - \eta^{ka}h^b_\mu)(\partial_\lambda h^\mu_k - h^\nu_k \Gamma_\lambda{}^\mu{}_\nu) L_{ab}{}^A{}_B y^B \partial_A] \tag{90}$$

defined by $\Gamma$. The connection (89) yields the so called vertical covariant differential

$$\widetilde{D} = dx^\lambda \otimes [y^A_\lambda - \frac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k \Gamma_\lambda{}^\mu{}_\nu) L_{ab}{}^A{}_B y^B]\partial_A, \tag{91}$$

on a fibre bundle $S \to X$ (88). Its restriction to $J^1S^h \subset J^1S$ recovers the familiar covariant differential on the spinor bundle $S^h \to X$ relative to the spin connection (90). Combining (87) and (91) gives the first order differential operator

$$\mathcal{D} = \sigma^\lambda_a \gamma^{aB}{}_A [y^A_\lambda - \frac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_k \Gamma_\lambda{}^\mu{}_\nu) L_{ab}{}^A{}_B y^B],$$

on the fibre bundle $S \to X$ (88). Its restriction to $J^1S^h \subset J^1S$ is the familiar Dirac operator on a spinor bundle $S^h$ in the presence of a background tetrad field $h$ and a general linear connection $\Gamma$.

Due to the decomposition (34), there is a canonical lift of any vector field on $X$ onto bundles $P^h$ and $S^h$ though they are not natural bundles [7, 26]. However, this lift, called the Kosmann's Lie derivative, fails to be an infinitesimal generator of general covariant transformations.

Since a world manifold $X$ is assumed to be parallelizable, a universal spinor structure is unique, and a principal $\widetilde{GL}_4$-bundle $\widetilde{LX} \to X$ as well as a linear frame bundle $LX$ admits a functorial lift of any diffeomorphism $f$ of a base $X$. This lift is defined by a commutative diagram

$$\begin{array}{ccc}
\widetilde{LX} & \xrightarrow{\widetilde{f}_{\rm s}} & \widetilde{LX} \\
{\scriptstyle \widetilde{z}}\downarrow & & \downarrow{\scriptstyle \widetilde{z}} \\
LX & \xrightarrow{\widetilde{f}} & LX \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & X
\end{array}$$

where $\widetilde{f}$ is the holonomic bundle automorphism of $LX$ (8) induced by $f$. Consequently, a universal spinor bundle $\widetilde{LX}$ is a natural bundle, and it admits the functorial lift $\widetilde{\tau}_{\rm s}$ of vector fields $\tau$ on its base $X$. These lifts $\widetilde{\tau}_{\rm s}$ are infinitesimal general covariant transformations of $\widetilde{LX}$. Consequently, the composite bundle $S \to X$ (88) also is a natural bundle, and it possesses infinitesimal general covariant transformations [6, 26].

## 11 Appendix. Affine world connections

The tangent bundle $TX$ of a world manifold $X$ as like as any vector bundle possesses a natural structure of an affine bundle. It is associated with a principal bundle $AX$ of oriented affine frames in $TX$ whose structure group is a general affine group $GA(4,\mathbb{R})$. This structure group is always reducible to a linear subgroup $GL_4$ since the quotient $GA(4,\mathbb{R})/GL_4$ is a vector space $\mathbb{R}^4$. The corresponding quotient bundle $AX/GL_4$ is isomorphic to the tangent bundle $TX$. There is the canonical injection of a frame bundle $LX \to AX$ onto a reduced $GL_4$-principal subbundle of $AX$ which corresponds to the zero section $\widehat{0}$ of $TX$.

Treating as an affine bundle, the tangent bundle $TX$ admits affine connections

$$A = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^\alpha{}_\mu(x)\dot x^\mu \dot\partial_\alpha + \sigma^\alpha_\lambda(x)\dot\partial_\alpha), \tag{92}$$

called affine world connections. They are associated with principal connections on an affine frame bundle $AX$. Every affine connection $\Gamma$ (92) on $TX$ yields a unique linear connection

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^\alpha{}_\mu(x)\dot x^\mu \dot\partial_\alpha), \tag{93}$$

on $TX$. It is associated with a principal connection on a frame bundle $LX \subset AX$. Conversely, by virtue of the well-known theorem, any principal connection on a frame bundle $LX \subset AX$ gives rise to a principal connection on an affine frame bundle $AX$, i.e., every linear connection on $TX$ can be seen as the affine one. It follows that any affine connection $A$ (92) on the tangent bundle $TX$ is represented by a sum of the associated linear connection $\overline{\Gamma}$ (93) and a soldering form

$$\sigma = \sigma^\alpha_\lambda(x) dx^\lambda \otimes \dot\partial_\alpha \tag{94}$$

on $TX$, which is the $(1,1)$-tensor field

$$\sigma = \sigma_\lambda^\alpha(x) dx^\lambda \otimes \partial_\alpha \tag{95}$$

on $X$ due to the canonical splitting (11).

In particular, let us consider the canonical soldering form $\theta_J$ (14) on $TX$. Given an arbitrary world connection $\Gamma$ (9) on $TX$, the corresponding affine connection

$$A = \Gamma + \theta_X, \qquad A_\lambda^\mu = \Gamma_\lambda{}^\mu{}_\nu \dot{x}^\nu + \delta_\lambda^\mu, \tag{96}$$

on $TX$ is a Cartan connection. Its torsion coincides with the torsion $T$ (15) of the world connection $\Gamma$, while its curvature is the sum $R+T$ of the curvature and the torsion of $\Gamma$.

There is a problem of a physical meaning of the tensor field $\sigma$ (95).

In the framework of so-called Poincaré gauge theory [10, 19], it is treated as a non-holonomic frame, which is a dynamic variable describing a gravitational field. This treatment of $\sigma$ is wrong because a soldering form and a frame are different mathematical objects. A frame is a (local) section of a principal frame bundle $LX$, while a soldering form is a global section of the $LX$-associated bundle

$$\overline{LX} = (LX \times GL_4)/GL_4$$

whose typical fibre is a group $GL_4$, acting on itself by the adjoint representation, but not left multiplications.

At the same time, a translation part of an affine connection on $\mathbb{R}^3$ characterise an elastic distortion in gauge theory of dislocations in continuous media [14, 18]. By analogy with this gauge theory, a gauge model of hypothetic deformations of a world manifold has been developed [20, 22]. They are described by the translation part $\sigma$ (95) of affine world connections on $X$ and, in particular, they are responsible for the so called "fifth force" [21, 22].

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