Non-relativistic limit of Einstein-Cartan-Dirac equations

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Abstract

We derive the Schrödinger-Newton equation as the non-relativistic limit of the Einstein-Dirac equations. Our analysis relaxes the assumption of spherical symmetry, made in earlier work in the literature, while deriving this limit. Since the spin of the Dirac field couples naturally to torsion, we generalize our analysis to the Einstein-Cartan-Dirac (ECD) equations, again recovering the Schrödinger-Newton equation. We then consider the ECD equations with a new length scale that unifies Compton wavelength and Schwarzschild radius, and find a Poisson equation with a modified coupling constant, in the small mass limit.

Contents

1 Introduction

The Schrödinger-Newton equation has been proposed in the literature as a model for investigating the effect of self-gravity on the motion of a non-relativistic quantum particle [1, 2, 3, 4]. Assuming that the self-gravitational potential ϕ is classical, and described by the semi-classical Poisson equation

$$\nabla^2\phi = 4\pi Gm|\psi|^2 \tag{1}$$

its substitution in the Schrödinger equation

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + m\phi\ \psi(\mathbf{r},t) \tag{2}$$

gives rise to the Schrödinger-Newton [SN] equation

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) - Gm^2\int\frac{|\psi(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3r'\psi(\mathbf{r},t) \tag{3}$$

The SN equation has been studied in the context of localization of wave-packets for macroscopic objects, notably in [5], where it was shown to provide a gravitationally induced inhibition of quantum dispersion. The authors also showed how the SN equation can be obtained as the non-relativistic limit of the Einstein-Klein-Gordon equation and of the Einstein-Dirac equation [6]. In their work, they assumed the space-time to be spherically symmetric. In our present paper, we relax the assumption of spherical symmetry, and obtain the SN equation as the non-relativistic limit of the Einstein-Dirac equations. Since the spin of the Dirac field couples naturally to torsion, we derive the SN equation also as a limiting case of the Einstein-Cartan-Dirac equations. These equations are a special case of the Einstein-Cartan-Sciama-Kibble theory [7, 9, 10, 8, 14, 12, 11, 13], which we will refer to hereon as the Einstein-Cartan theory. Lastly, we consider the Einstein-Cartan-Dirac equations with a new length scale, and investigate their non-relativistic limit.

2 Preliminaries: The Einstein-Cartan-Dirac equations

Torsion is a third-rank tensor, defined as the antisymmetric part of the affine connection:

$$Q_{\alpha\beta}{}^{\mu} = \Gamma_{[\alpha\beta]}{}^{\mu} = \frac{1}{2}(\Gamma_{\alpha\beta}{}^{\mu} - \Gamma_{\beta\alpha}{}^{\mu}) \tag{4}$$

In terms of the usual Christoffel symbols, it takes the form

$$\Gamma_{\alpha\beta}{}^{\mu} = \left\{ {\mu \atop \alpha\beta} \right\} - K_{\alpha\beta}{}^{\mu} \tag{5}$$

where $K_{\alpha\beta}{}^{\mu}$ is given by $K_{\alpha\beta}{}^{\mu} = -Q_{\alpha\beta}{}^{\mu} - Q^{\mu}{}_{\alpha\beta} + Q_{\beta}{}^{\mu}{}_{\alpha}$ and is called the contorsion tensor.

When a matter field ψ is minimally coupled with gravity and torsion, its action is given as follows [8]:

$$S = \int d^4x\sqrt{-g}\Big[\mathcal{L}_m(\psi, \nabla\psi, g) - \frac{1}{2k}R(g, \partial g, Q)\Big] \tag{6}$$

Here $k = 8\pi G/c^4$ and $\mathcal{L}_m$ denotes the matter Lagrangian density. The second term represents the Lagrangian density for the gravitational field. There are three fields in this Lagrangian viz. ψ (matter field), $g_{\mu\nu}$ (metric) and $K_{\alpha\beta\mu}$ (Contorsion). Varying the action with respect to them yields three field equations as follows:

$$\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta\psi} = 0 \tag{7}$$

$$\frac{\delta(\sqrt{-g}R)}{\delta g^{\mu\nu}} = 2k\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{8}$$

$$\frac{\delta(\sqrt{-g}R)}{\delta K_{\alpha\beta\mu}} = 2k\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta K_{\alpha\beta\mu}} \tag{9}$$

Eqn. (7) yields the matter field equations on a curved space-time with torsion. The right hand side of Eqn. (8) is associated with $\sqrt{-g}kT_{\mu\nu}$ by the definition of the metric energy-momentum tensor $T_{\mu\nu}$. Similarly, the right hand side of Eqn. (9) is associated with $2\sqrt{-g}kS^{\mu\beta\alpha}$ where $S^{\mu\beta\alpha}$ is the spin density tensor. These two yield the Einstein-Cartan field equations

$$G^{\mu\nu} = k\ \ \Sigma^{\mu\nu} \tag{10}$$

$$T^{\mu\beta\alpha} = k\ \ \tau^{\mu\beta\alpha} \tag{11}$$

In Eqn. (10) the $G^{\mu\nu}$ on the left hand side is the asymmetric Einstein tensor built from the asymmetric connection. The $\Sigma^{\mu\nu}$ on the right hand side is the asymmetric canonical total energy momentum tensor, which is made from the symmetric metric energy-momentum tensor, and from the spin density tensor. In Eqn. (11), the so-called modified torsion $T^{\mu\beta\alpha}$ is the traceless part of the torsion tensor, and it is algebraically related to the $S^{\mu\beta\alpha}$ on the right. If torsion is set to zero, we recover general relativity: Eqn. (11) is no longer there, and (10) reduces to Einstein equations coupling the symmetric Einstein tensor to the metric energy-momentum tensor.

Next, we assume that the matter field is the spinorial Dirac field ψ, for which the Lagrangian is:

$$\mathcal{L}_m = \frac{i\hbar c}{2}(\overline{\psi}\gamma^{\mu}\nabla_{\mu}\psi - \nabla_{\mu}\overline{\psi}\gamma^{\mu}\psi) - mc^2\overline{\psi}\psi \tag{12}$$

We denote a Riemannian space-time by V_4 and a space-time with torsion by U_4, The theory generated by minimal coupling of Dirac field on U_4 is called Einstein-Cartan-Dirac (ECD) theory. We use the tetrad formalism to define spinors on curved space-time (both V_4 and U_4). Transformation properties of spinors are defined in a flat Minkowski space; locally tangent to the U_4 manifold. We know that, at each point, we have a coordinate basis vector field $\hat{e}^{\mu} = \partial_{\mu}$. This coordinate basis field is covariant under general coordinate transformations. However, a spinor (as defined on flat Minkowski space-time) is associated with the basis vectors which are covariant under local Lorentz transformations. To this aim, we define at each point of our manifold, a set of four orthonormal basis fields (called tetrad fields), given by $\hat{e}^i(x)$. These are four vectors (one for each μ) at every point. This tetrad field is governed by a relation $\hat{e}^i(x) = e^i_{\mu}(x)\hat{e}^{\mu}$ where the transformation matrix e^i_{μ} is such that,

$$e^{(i)}_{\mu}e^{(k)}_{\nu}\eta_{(i)(k)} = g_{\mu\nu} \tag{13}$$

The trasformation matrix $e^{(i)}_{\mu}$ allows us to convert the components of any world tensor (tensor which transforms according to general coordinate transformation) to the corresponding components in local Minkowskian space (These latter components being covariant under local Lorentz transformation). Greek indices are raised or lowered using the metric $g_{\mu\nu}$, while the Latin indices are raised or lowered using $\eta_{(i)(k)}$. Parenthesis around indices is a matter of convention.

We adopt the following conventions for the remainder of the paper:

- Greek indices, e.g. α, ζ, δ refer to world components, which transform according to *general coordinate transformations* and are raised or lowered using the metric $g_{\mu\nu}$.

- Latin indices within parenthesis e.g. (a) or (i) are the tetrad indices, which transform according to *local Lorentz transformations* in the flat tangent space, and are raised or lowered using $\eta_{(i)(k)}$.

- Latin indices (without parenthesis) e.g. i, j, b, c indicate objects in Minkowski space, which transform according to *global Lorentz transformations*).

- In general, $0, 1, 2, 3$ refer to world indices while $(0), (1), (2), (3)$ refer to tetrad indices.

- The total covariant derivative is denoted by ∇ and $\{\}$ denotes the Christoffel connection. Correspondingly, $\nabla^{\{\}}$ represents a covariant derivative with respect to the Christofell connections.

- Commas $(,)$ indicate partial derivatives while semicolons $(;)$ indicate the Riemannian covariant derivative. Thus, for tensors, $;$ and $\nabla^{\{\}}$ are same, while for spinors, $(;)$ involves both partial derivatives and the Riemannian part of the spin connection, γ, as defined in the following.

Just as we define affine connection Γ to facilitate parallel transport of geometrical objects with world (Greek) indices, we define spin connection ω for anholonomic objects (those having Latin index). Just as affine connection Γ has two parts - Riemannian ($\{\}$) part coming from Levi-Civita connection and torsional part (made up of contorsion tensor K), similarly, the spin connection ω also has two parts - Riemannian (denoted by γ) and torsional part (again made up of contorsion tensor K). γ, γ^o and K are related by following equations:

$$\gamma_\mu{}^{(i)(k)} = \gamma^o_\mu{}^{(i)(k)} - K_\mu{}^{(k)(i)} \tag{14}$$

Here, $\gamma^o_\mu{}^{(i)(k)}$ is Riemannian part and $K_\mu{}^{(k)(i)}$ is the contorsion (torsional part). The relation between spin connection and affine connection is as follows:

$$\begin{aligned}\gamma_\mu{}^{(i)(k)} &= e^{(i)}_\alpha e^{\nu(k)}\Gamma_{\mu\nu}{}^\alpha - e^{\nu(k)}\partial_\mu e^{(i)}_\nu \\ &= e^{(i)}_\alpha e^{\nu(k)}\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\} - K_\mu{}^{(k)(i)} - e^{\nu(k)}\partial_\mu e^{(i)}_\nu\end{aligned} \tag{15}$$

From the above two equations, one can obtain the following crucial equation for Riemannian part of spin connection, entirely in terms of Christoffel symbols and tetrads [13]

$$\left\{\begin{matrix}\alpha\\ \mu\nu\end{matrix}\right\} = e^\alpha_{(i)}e_{\nu(k)}\gamma^o_\mu{}^{(k)(i)} + e^\alpha_{(i)}\partial_\mu e^{(i)}_\nu \tag{16}$$

Using the above results, we define covariant derivative (CD) for Spinors on V_4 and U_4

$$\psi_{;\mu} = \partial_\mu\psi + \frac{1}{4}\gamma^o_{\mu(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi - - - - - - - - - - - - - - - - - - \text{CD on } [V_4] \tag{17}$$

$$\nabla_\mu\psi = \partial_\mu\psi + \frac{1}{4}\gamma^o_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{1}{4}K_{\mu(c)(b)}\gamma^{[(b)}\gamma^{(c)]}\psi - - - - - - - - \text{CD on } [U_4] \tag{18}$$

Substituting this into eqn (12), we obtain the explicit form of Lagrangian density; which we vary with respect to $\overline{\psi}$ as in Eqn. (7) to obtain Dirac equation on V_4 and U_4.

$$i\gamma^\mu\psi_{;\mu} - \frac{mc}{\hbar}\psi = 0 - - - - - - \text{Dirac Eqn on } [V_4] \tag{19}$$

$$i\gamma^\mu\psi_{;\mu} + \frac{i}{4}K_{(a)(b)(c)}\gamma^{[(a)}\gamma^{(b)}\gamma^{(c)]}\psi - \frac{mc}{\hbar}\psi = 0 - - - - - - \text{Dirac Eqn on } [U_4] \tag{20}$$

We next obtain gravitational field equations on both V_4 and U_4 using Eqn. (8) and Lagangian density defined in Eqn. (12)

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - - - - - - - - - - - - - - - - \text{Gravitation Eqn on } [V_4] \tag{21}$$

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} - - - - \text{Gravitation Eqn on } [U_4] \tag{22}$$

Here, $T^{\mu\nu}$ is the metric energy-momentum tensor which is symmetric and has the form:

$$T_{\mu\nu} = \Sigma_{(\mu\nu)}(\{\}) = \frac{i\hbar c}{4}\left[\bar{\psi}\gamma_\mu\psi_{;\nu} + \bar{\psi}\gamma_\nu\psi_{;\mu} - \bar{\psi}_{;\mu}\gamma_\nu\psi - \bar{\psi}_{;\nu}\gamma_\mu\psi\right] \tag{23}$$

Equations (19) and (21) together constitute the Einstein-Dirac theory.

To write the field equations of Einstein-Cartan-Dirac theory, we first define spin density tensor using Lagrangian density defined in Eqn. (12)

$$S^{\mu\nu\alpha} = \frac{-i\hbar c}{4}\bar{\psi}\gamma^{[\mu}\gamma^{\nu}\gamma^{\alpha]}\psi \tag{24}$$

Using equations (24) and (9), Eqn. (20) can be simplified to give us the Hehl-Datta equation [8], [10]. This, along with equation (22) and the equation which couples modified torsion tensor and spin density tensor together define the field equations of Einstein-Cartan-Dirac (ECD) theory; as summarized below

$$i\gamma^{\mu}\psi_{;\mu} = +\frac{3}{8}L_{Pl}^{2}\overline{\psi}\gamma^{5}\gamma_{(a)}\psi\gamma^{5}\gamma^{(a)}\psi + \frac{mc}{\hbar}\psi \tag{25}$$

$$G_{\mu\nu}(\{\}) = \frac{8\pi G}{c^4}T_{\mu\nu} - \frac{1}{2}\left(\frac{8\pi G}{c^4}\right)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{26}$$

$$T^{\mu\nu\alpha} = -K^{\mu\nu\alpha} = \frac{8\pi G}{c^4}S^{\mu\nu\alpha} \tag{27}$$

The Lorentz signature used in this paper is diag(+, -, -, -). We use Dirac basis to represent the gamma matrices. These are basically matrix representation of clifford algebra $Cl_{1,3}[\mathbb{R}]$

$$\gamma^0 = \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^5 = \frac{i}{4!}\epsilon_{ijkl}\gamma^i\gamma^j\gamma^k\gamma^l = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \alpha^i = \beta\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{28}$$

3 Non-relativistic limit of the Einstein-Dirac equations

3.1 Ansatz for the spinor and the metric

Ansatz for Dirac spinor: We need to choose an appropriate expansion ansatz for the spinor so as to obtain the non-relativistic limit. We expand $\psi(x,t)$ as $\psi(x,t) = e^{iS(x,t)\hbar}$ (which can be done for any complex function of x and t). We can either expand S as a perturbative power series in the parameter $\sqrt{\hbar}$ or $(1/c)$ and obtain the semi-classical or non-relativistic limit respectively, at various orders. The scheme for non-relativistic limit has been employed by Kiefer and Singh [15]. Giulini and Grossardt in their work [6], combine both these schemes and construct a new ansatz using the parameter $\sqrt{\hbar}/c$ as follows:

$$\psi(\mathbf{r},t) = e^{\frac{ic^2}{\hbar}S(\mathbf{r},t)}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n(\mathbf{r},t) \tag{29}$$

where $S(\mathbf{r},t)$ is a scalar function and $a_n(\mathbf{r},t)$ is a spinor field. We use this ansatz in our calculations, and by taking the limit $c \to \infty$ arrive at the non-relativistic limit.

Ansatz for metric: We first express the generic form of the metric in a power series with same parameter as that used to expand the spinor viz. $\sqrt{\hbar}/c$

$$g_{\mu\nu}(\mathbf{r},t) = \eta_{\mu\nu} + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n g_{\mu\nu}^{[n]}(\mathbf{r},t) \tag{30}$$

where $g_{\mu\nu}^{[n]}(x)$ are infinitely many metric functions indexed by n. In the non-relativistic scheme, gravitational potentials cannot produce velocities comparable to c - they are weak potentials. Therefore we assume that the leading function $g_{\mu\nu}^{[0]}(x) = \eta_{\mu\nu}$. With this, we get the following generic power series for tetrads and spin coefficients and Einstein tensor

$$e^{\mu}_{(i)} = \delta^{\mu}_{(i)} + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\mu[n]}_{(i)} \qquad \gamma_{(a)(b)(c)} = \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{[n]}_{(a)(b)(c)} \tag{31}$$

$$e^{(i)}_{\mu} = \delta^{(i)}_{\mu} + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{(i)[n]}_{\mu} \qquad G_{\mu\nu} = \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n G^{[n]}_{\mu\nu} \tag{32}$$

where $e^{\mu(n)}_{(i)}[g^{[n]}_{\mu\nu}]$, $e^{(i)[n]}_{\mu}[g^{[n]}_{\mu\nu}]$, $\gamma^{[n]}_{(a)(b)(c)}[g^{[n]}_{\mu\nu}]$ and $G^{[n]}_{\mu\nu}$ are infinitely many tetrad, spin coefficient and Einstein tensor functions indexed by n. They are functions of metric functions $g^{[n]}_{\mu\nu}$ and their various derivatives.

3.2 Analyzing Dirac equation with above ansatz

We will now expand the Dirac equation on V_4 as given in eqn (19) with the above ansatz. We also note that $\gamma^{(a)}\psi_{;(a)} = e^{(a)}_{\mu}e^{\nu}_{(a)}\gamma^{\mu}\psi_{;\nu} = \delta^{\mu}_{\nu}\gamma^{\mu}\psi_{;\nu} = \gamma^{\mu}\psi_{;\mu}$.

$$i\gamma^{\mu}\psi_{;\mu} - \frac{mc}{\hbar}\psi = 0 \tag{33}$$

$$\Rightarrow\ i\gamma^0\partial_0\psi + \frac{i}{4}\gamma^{(0)}\gamma^{o}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi + i\gamma^{\alpha}\partial_{\alpha}\psi + \frac{i}{4}\gamma^{(j)}\gamma^{o}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{mc}{\hbar}\psi = 0 \tag{34}$$

We separate spatial and temporal parts. Substituting appropriate expansions from (31), (32) into above equations and multiplying by $\gamma^{(0)}c$ on both sides yields:

$$\begin{aligned}\Rightarrow\ &\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{0[n]}_{(0)}\right]i\partial_t\psi + \frac{ic}{4}\left[\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{o[n]}_{(0)(b)(c)}\right]\gamma^{[(b)}\gamma^{(c)]}\psi + \\ &\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\alpha[n]}_{(a)}\right]ic\alpha.\nabla\psi + \frac{ic}{4}\alpha^{(j)}\left[\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n \gamma^{o[n]}_{(j)(b)(c)}\right]\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{\beta mc^2}{\hbar}\psi = 0\end{aligned} \tag{35}$$

Dividing both sides by $\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n} e_{(0)}^{0[n]}\right]$, we obtain

$$
\begin{aligned}
i\partial_t\psi = -\frac{ic}{4}\frac{\left[\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}\gamma^{o[n]}_{(0)(b)(c)}\right]}{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}e^{0[n]}_{(0)}\right]}\gamma^{[(b)}\gamma^{(c)]}\psi - \frac{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}e^{\alpha[n]}_{(a)}\right]}{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}e^{0[n]}_{(0)}\right]}ic\alpha.\nabla\psi - \\
\frac{ic}{4}\alpha^{(j)}\frac{\left[\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}\gamma^{o[n]}_{(j)(b)(c)}\right]}{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}e^{0[n]}_{(0)}\right]}\gamma^{[(b)}\gamma^{(c)]}\psi + \frac{1}{\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}e^{0[n]}_{(0)}\right]}\frac{\beta mc^2}{\hbar}\psi
\end{aligned}
\tag{36}
$$

We consider the terms of order c^2, c, 1 and neglect the terms having order of $O\left(\frac{1}{c^n}\right)$; $n \geq 1$. This is sufficient to get the behaviour of the functions in the spinor ansatz. It will turn out later that this is also sufficient to get the equation which is followed by leading order spinor term a_0. We obtain the following equations:

$$
\begin{aligned}
i\partial_t\psi + \frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi + \ ic\alpha.\nabla\psi + \frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi \\
-\beta\frac{mc^2}{\hbar}\psi + \beta\frac{mc}{\sqrt{\hbar}}e^{0[1]}_{(0)}\psi - \beta m\left[\left(e^{0[1]}_{(0)}\right)^2 - e^{0[2]}_{(0)}\right]\psi = 0
\end{aligned}
\tag{37}
$$

Substituting the spinor ansatz i.e. eqn (29) in equation (37), the various terms are evaluated as follows:

Term 1

$$
\begin{aligned}
i\partial_t\psi &= i\partial_t\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}a_n\right] \\
&= ie^{\frac{ic^2S}{\hbar}}\frac{c^2}{\hbar}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}\left[\dot{a}_{n-2}+i\dot{S}a_n\right] \\
&= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}\left[-\dot{S}a_{n-1}+i\dot{a}_{n-3}\right]
\end{aligned}
\tag{38}
$$

Term 2

$$
+\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi = +\frac{i\sqrt{\hbar}}{4}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}a_n\right] \tag{39}
$$

$$
= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}\left[i\sqrt{\hbar}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}a_{n-3}\right] \tag{40}
$$

Term 3

$$
\begin{aligned}
ic\alpha^j\partial_j\psi &= ic\overrightarrow{\alpha}\cdot\overrightarrow{\nabla}\left[e^{\frac{ic^2S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}a_n\right] \\
&= ic\overrightarrow{\alpha}\cdot\left[e^{\frac{ic^2S}{\hbar}}\frac{c^2}{\hbar}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}\left(i\overrightarrow{\nabla}Sa_n+\overrightarrow{\nabla}a_{n-2}\right)\right] \\
&= e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^{n}\left[-\sqrt{\hbar}\overrightarrow{\alpha}\cdot\overrightarrow{\nabla}Sa_n+i\sqrt{\hbar}\overrightarrow{\alpha}\cdot\overrightarrow{\nabla}a_{n-2}\right]
\end{aligned}
\tag{41}
$$

Term 4

$$+\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\psi = +\frac{i\sqrt{\hbar}}{4}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}\left[e^{\frac{ic^2 S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \tag{42}$$

$$= e^{\frac{ic^2 S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[i\sqrt{\hbar}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]}a_{n-3}\right] \tag{43}$$

Term 5

$$-\beta\frac{mc^2}{\hbar}\psi = -\beta\frac{mc^2}{\hbar}e^{\frac{ic^2 S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n$$

$$= e^{\frac{ic^2 S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n(-\beta m a_{n-1}) \tag{44}$$

Term 6

$$+\beta\frac{mc}{\sqrt{\hbar}}e^{0[1]}_{(0)}\psi = +\beta\frac{mc}{\sqrt{\hbar}}e^{0[1]}_{(0)}\left[e^{\frac{ic^2 S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \tag{45}$$

$$= e^{\frac{ic^2 S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[\beta m\ e^{0[1]}_{(0)}a_{n-2}\right] \tag{46}$$

Term 7

$$-\beta m\left[\left(e^{0[1]}_{(0)}\right)^2 - e^{0[2]}_{(0)}\right]\psi = -\beta m\left[\left(e^{0[1]}_{(0)}\right)^2 - e^{0[2]}_{(0)}\right]\left[e^{\frac{ic^2 S}{\hbar}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right] \tag{47}$$

$$= -e^{\frac{ic^2 S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left[\beta m\left(\left(e^{0[1]}_{(0)}\right)^2 - e^{0[2]}_{(0)}\right)\right]a_{n-3} \tag{48}$$

After substituting equations (38), (39), (41), (42), (44), (45) and (47) into (37) and sorting by powers of n we get,

$$e^{\frac{ic^2 S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\Bigg[\Big(-\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla}S\Big)a_n - \Big(\dot{S}+\beta m\Big)a_{n-1} + \Big(i\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla} + \beta m\ e^{0[1]}_{(0)}\Big)a_{n-2}$$
$$+i\dot{a}_{n-3} + \left(i\sqrt{\hbar}\gamma^{o[1]}_{(0)(b)(c)}\gamma^{[(b)}\gamma^{(c)]} + i\sqrt{\hbar}\alpha^{(j)}\gamma^{o[1]}_{(j)(b)(c)}\gamma^{[(b)}\gamma^{(c)]} - \beta m\Big(\big(e^{0[1]}_{(0)}\big)^2 - e^{0[2]}_{(0)}\Big)\right)a_{n-3}\Bigg] = 0 \tag{49}$$

At order $n = 0$ the equation reduces to,

$$\nabla S = 0 \tag{50}$$

which implies the scalar 'S' is a function of time only i.e., $S = S(t)$. Dirac spinor is a 4-component spinor $a_n = (a_{n,1}, a_{n,2}, a_{n,3}, a_{n,4})$. We split it into two two-component spinors $a_n^{>} = (a_{n,1}, a_{n,2})$ and $a_n^{<} = (a_{n,3}, a_{n,4})$. For order $n = 1$, the equation is $\Big(\dot{S}+\beta m\Big) = 0$; which can be written as following two equations:

$$(m+\dot{S})a_0^{>} = 0 \tag{51a}$$

$$(m-\dot{S})a_0^{<} = 0 \tag{51b}$$

This implies that either $S = -mt$ and $a_0^< = 0$ or $S = +mt$ and $a_0^> = 0$. The wave function at this order is $\psi = e^{\frac{\pm imc^2 t}{\hbar}}$. It represents the particles of positive energy (lower sign) and negative energy (upper sign) at rest. We will restrict to the former case i.e. $S = -mt$ and $a_0^< = 0$, which represents positive energy (lower sign) solutions. It has been implicitly assumed that 2 cases (of positive and negative energies) can be treated separately. We digress at this point and analyze the metric energy-momentum tensor now with the results obtained in equation (50) and the fact that $a_0^< = 0$.

3.3 Analyzing the Energy momentum tensor $T_{\mu\nu}$ with above ansatz

The dynamical Energy momentum tensor given in equation (23). Lets consider the "kT_{00}" component.

Analyzing kT_{00} (after raising the index on gamma matrices):

$$kT_{00} = \frac{4i\pi G\hbar}{c^4}\left[\bar{\psi}\gamma^0\Big(\partial_t\psi + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\psi\Big) - \Big(\partial_t\bar{\psi} + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\bar{\psi}\Big)\gamma^0\psi\right] \tag{52}$$

$$\begin{aligned}\Rightarrow kT_{00} = \frac{4i\pi G\hbar}{c^4}\Big(1 + \sum_{n=1}^{\infty}(\frac{\sqrt{\hbar}}{c})^n e^{0[n]}_{(0)}\Big)\Big[&\bar{\psi}\gamma^{(0)}\Big(\partial_t\psi + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\psi\Big) \\ &- \Big(\partial_t\bar{\psi} + \frac{c}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}]\bar{\psi}\Big)\gamma^{(0)}\psi\Big]\end{aligned} \tag{53}$$

After putting spinor anstaz eqn (29) in eqn (52), we obtain following power series for kT_{00}. We have given expression for the leading order only.

$$\begin{aligned}kT_{00} = \frac{4i\pi G}{c^2}\Bigg\{&\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n a_n^\dagger\Bigg)\Bigg(\sum_{m=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^m\Big[i\dot{S}a_m + \dot{a}_{m-2}\Big]\Bigg) \\ +&\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n\Big[i\dot{S}a_n^\dagger - \dot{a}_{n-2}^\dagger\Big]\Bigg)\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^m a_m\Bigg)\Bigg\} + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big)\end{aligned} \tag{54}$$

Explicit expression for leading order is obtained by considering $(n + m = 0)$ as follows:

$$kT_{00} = \frac{4\pi G i}{c^2}\Big\{i(-m)a_0^{>\dagger}a_0^> + i(-m)a_0^{>\dagger}a_0^>\Big\} + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big) \tag{55}$$

$$kT_{00} = \frac{8\pi Gm\,|a_0^>|^2}{c^2} + \sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big) \tag{56}$$

Analyzing $kT_{0\mu}$:

$$\begin{aligned}kT_{0\mu} = \frac{2i\pi G\hbar}{c^4}\Big[&c\bar{\psi}\gamma_0\Big(\partial_\mu\psi + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\Big) + c\bar{\psi}\gamma_\mu\Big(\partial_0\psi + \frac{1}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\Big) \\ &- c\Big(\partial_\mu\bar{\psi} + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\Big)\gamma_0\psi - c\Big(\partial_0\bar{\psi} + \frac{1}{4}[\gamma^o_{0(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\Big)\gamma_\mu\psi\Big]\end{aligned} \tag{57}$$

We will first find the coefficient of the term of order $\frac{1}{c^2}$ which is the leading order of T_{00}. Now, all the terms containing spin coefficients $\gamma_{\mu(i)(j)}$ have leading order of $\frac{1}{c^3}$. So it will not contribute at the order $\frac{1}{c^2}$. So what we get is (here, index on gamma matrices is raised):

$$kT_{0\mu} = \frac{2i\pi G\hbar}{c^4}\Bigg[c\bar{\psi}\gamma^0\partial_\mu\psi - c\bar{\psi}\gamma^\mu\partial_0\psi - c\partial_\mu\bar{\psi}\gamma^0\psi + c\partial_0\bar{\psi}\gamma^\mu\psi\Bigg] \tag{58}$$

$$= \frac{-2i\pi G\hbar}{c^3}\Big(1 + \sum_{n=1}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n e^{0[n]}_{(0)}\Big)\Bigg[\bar{\psi}\gamma^{(0)}\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^{(0)}\psi\Bigg] \tag{59}$$

$$+\frac{2i\pi G\hbar}{c^4}\Big(1 + \sum_{n=1}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n e^{\mu[n]}_{(a)}\Big)\Bigg[\partial_t\bar{\psi}\gamma^{(a)}\psi - \bar{\psi}\gamma^{(a)}\partial_t\psi\Bigg]$$

There are two types of terms in equation above. One having coefficient $\frac{2i\pi G\hbar}{c^3}$ and other with coefficient $\frac{2i\pi G\hbar}{c^4}$. We call them term 1 and 2 respectively. We analyze both of them independently. Term 1 gives

$$(term\ 1) = \frac{2i\pi G\hbar}{c^3}\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n\Big(a^\dagger_{n_1}\partial_\mu a_{n_2} - \partial_\mu a^\dagger_{n_1}a_{n_2}\Big); \qquad n = n_1 + n_2$$

$$= \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{60}$$

$$(term\ 2) = \frac{2i\pi G}{c^2}\Bigg\{\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n a^\dagger_n\Bigg)\alpha^{(a)}\Bigg(\sum_{m=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^m\Big[i\dot{S}a_m + \dot{a}_{m-2}\Big]\Bigg)$$

$$+\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n\Big[i\dot{S}a^\dagger_n - \dot{a}^\dagger_{n-2}\Big]\Bigg)\alpha^{(a)}\Bigg(\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^m a_m\Bigg)\Bigg\} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big)$$

$$= \frac{4\pi Gm}{c^2}(a^\dagger_0\alpha^{(a)}a_0) + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big)$$

$$= \frac{4\pi Gm}{c^2}\Bigg[\begin{pmatrix} a^>_0 & 0\end{pmatrix}^\dagger\begin{pmatrix} 0 & \sigma^{(a)} \\ \sigma^{(a)} & 0\end{pmatrix}\begin{pmatrix} a^>_0 \\ 0\end{pmatrix}\Bigg] + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big)$$

$$= \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{61}$$

So we find, in both term 1 and 2, terms of the $O\Big(\frac{1}{c^2}\Big)$ are zero. Hence

$$kT_{0\mu} = \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{62}$$

Analyzing $kT_{\mu\nu}$

$$\begin{aligned} kT_{\mu\nu} = \frac{2i\pi G\hbar}{c^3}\Bigg[&+ \bar{\psi}\gamma_\mu\Big(\partial_\nu\psi + \frac{1}{4}[\gamma^o_{\nu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\Big) + \bar{\psi}\gamma_\nu\Big(\partial_\mu\psi + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\psi]\Big) \\ &- \Big(\partial_\nu\bar{\psi} + \frac{1}{4}[\gamma^o_{\nu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\Big)\gamma_\mu\psi - \Big(\partial_\mu\bar{\psi} + \frac{1}{4}[\gamma^o_{\mu(i)(j)}\gamma^{[(i)}\gamma^{(j)]}\bar{\psi}]\Big)\gamma_\nu\psi\Bigg]\end{aligned} \tag{63}$$

Here also, we will first find the coefficient of the term of order $\frac{1}{c^2}$ which is the leading order of kT_{00}. All the terms containing spin coefficients $\gamma_{\mu(i)(j)}$ have leading order of $\frac{1}{c^3}$. So it will not contribute at the order $\frac{1}{c^2}$. So what we get is (here, index on gamma matrices is raised):

$$\begin{aligned} kT_{\mu\nu} &= \frac{2i\pi G\hbar}{c^3}\Bigg[-\bar{\psi}\gamma^{\mu}\partial_{\nu}\psi - \bar{\psi}\gamma^{\nu}\partial_{\mu}\psi + \partial_{\nu}\bar{\psi}\gamma^{\mu}\psi + \partial_{\mu}\bar{\psi}\gamma^{\nu}\psi\Bigg] \\ &= \frac{2i\pi G\hbar}{c^3}\left(1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\mu[n]}_{(a)}\right)\Bigg[\psi^{\dagger}\alpha^{(a)}\partial_{\nu}\psi - \partial_{\nu}\psi^{\dagger}\alpha^{(a)}\psi\Bigg] \\ &+ \frac{2i\pi G\hbar}{c^3}\left(1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e^{\nu[n]}_{(b)}\right)\Bigg[\partial_{\mu}\psi^{\dagger}\alpha^{(b)}\psi - \psi^{\dagger}\alpha^{(b)}\partial_{\mu}\psi\Bigg] \\ &= \frac{2i\pi G\hbar}{c^3}\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n\left(e^{\mu}_{(a)}a^{\dagger}_{n_1}\alpha^{(a)}\partial_{\nu}a_{n_2} - e^{\mu}_{(a)}\partial_{\nu}a^{\dagger}_{n_1}\alpha^{(a)} + e^{\nu}_{(b)}a^{\dagger}_{n_1}\alpha^{(b)}\partial_{\mu}a_{n_2} - e^{\nu}_{(b)}\partial_{\mu}a^{\dagger}_{n_1}\alpha^{(b)}a_{n_2}\right) \end{aligned} \tag{64}$$

$$kT_{\mu\nu} = \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{65}$$

From order analysis of components of the metric energy-momentum tensor, summarized in equations (56) , (62) and (65), we have proved a crucial result viz.

$$\frac{|T_{00}|}{|T_{0i}|} \ll 1, \qquad \frac{|T_{00}|}{|T_{ij}|} \ll 1, \qquad k|T_{00}| \sim O\left(\frac{1}{c^2}\right) \quad ; i,j \in (1,2,3) \tag{66}$$

Owing to Einstein's equations, the same relation then exists amongst the components of Einstein tensor as well viz.

$$\frac{|G_{00}|}{|G_{0i}|} \ll 1, \qquad \frac{|G_{00}|}{|G_{ij}|} \ll 1, \qquad |G_{00}| \sim O\left(\frac{1}{c^2}\right) \quad ; i,j \in (1,2,3) \tag{67}$$

3.4 Constraints imposed on metric as an implication of above analysis

We proved an important fact in the previous two sections viz. $|G_{00}| \sim O\left(\frac{1}{c^2}\right)$ and all other components of G are of higher order. For a generic metric ansatz, $G_{\mu\nu}$ has been explicitly calculated in appendix [6.1]. At this point, we make an important assumption – the metric field is asymptotically flat. This fact suggests the following important constraints on metric components [proved in appendix (6.2)]

1) $G^{[1]}_{\mu\nu} = 0$ $(\forall \mu,\nu)$ and non-allowance of solutions which don't respect asymptotic flatness of metric gives following result for metric and other quantities :

$$g^{[1]}_{\mu\nu} = 0, \qquad e^{\mu[1]}_{(i)} = 0, \qquad e^{(i)[1]}_{\mu} = 0, \qquad \gamma^{[1]}_{(i)(j)(k)} = 0 \quad \forall \;\; ij,k,\mu,\nu \in (0,1,2,3) \tag{68}$$

This is proved in appendix (6.2.1)

2) We also have $G^{[2]}_{\mu\nu} = 0$ (except for $\mu = 0$ and $\nu = 0$). This imposes different kind of

restrictions on $g^{[2]}_{\mu\nu}$. We see that the form which $g^{[2]}_{\mu\nu}$ can take is $g^{[2]}_{\mu\nu} = F(\mathbf{r},t)\delta_{\mu\nu}$ for some field $F(\mathbf{r},t)$. This is proved in appendix (6.2.2). The full metric is then given by:

$$g_{\mu\nu}(\mathbf{r},t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \left(\frac{\hbar}{c^2}\right) \begin{bmatrix} F & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix} (\mathbf{r},t) + \sum_{n=3}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n \begin{bmatrix} g^{[n]}_{00} & g^{[n]}_{01} & g^{[n]}_{02} & g^{[n]}_{03} \\ g^{[n]}_{10} & g^{[n]}_{11} & g^{[n]}_{12} & g^{[n]}_{13} \\ g^{[n]}_{20} & g^{[n]}_{21} & g^{[n]}_{22} & g^{[n]}_{23} \\ g^{[n]}_{30} & g^{[n]}_{31} & g^{[n]}_{32} & g^{[n]}_{33} \end{bmatrix} (\mathbf{r},t) \tag{69}$$

where $g^{[2]}_{00} = g^{[2]}_{11} = g^{[2]}_{22} = g^{[2]}_{33} = F(\mathbf{r},t)$

With this form of metric, all the other objects (tetrads, spin coefficients etc.) have been calculated in Appendix sections [6.3], [6.5], [6.4] and [6.6]. We have used these results in the next section.

3.5 Non-Relativistic (NR) limit of Einstein-Dirac equations

Dirac equation: Equation (49) becomes the following

$$e^{\frac{ic^2 S}{\hbar}} \frac{c^3}{\hbar^{3/2}} \sum_{n=0}^{\infty} \left(\frac{\sqrt{\hbar}}{c}\right)^n \left[m\, a_{n-1} + i\dot{a}_{n-3} + i\sqrt{\hbar}\vec{\alpha}\cdot\vec{\nabla} a_{n-2} - \beta m a_{n-1} - \beta \frac{mF(\mathbf{r},t)}{2} a_{n-3}\right] = 0 \tag{70}$$

We have already used the results from analysis of this equation for $n=0$ and $n=1$. We now analyze it for $n=2$ and $n=3$. At order $n=2$ the equation (49) results in

$$\begin{pmatrix} \dot{S} + \mathrm{m} & 0 \\ 0 & \dot{S} - \mathrm{m} \end{pmatrix} \begin{pmatrix} a_1^{>} \\ a_1^{<} \end{pmatrix} - i\sqrt{\hbar} \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\nabla} \\ \vec{\sigma}\cdot\vec{\nabla} & 0 \end{pmatrix} \begin{pmatrix} a_0^{>} \\ a_0^{<} \end{pmatrix} = 0 \tag{71}$$

The first of these is trivially satisfied. The second one yields an expression for $a_1^{<}$ in terms of $a_0^{>}$

$$a_1^{<} = \frac{-i\sqrt{\hbar}\vec{\sigma}\cdot\vec{\nabla}}{2m} a_0^{>} \tag{72}$$

At order $n=3$,

$$\begin{pmatrix} \dot{S} + \mathrm{m} & 0 \\ 0 & \dot{S} - \mathrm{m} \end{pmatrix} \begin{pmatrix} a_2^{>} \\ a_2^{<} \end{pmatrix} - i\sqrt{\hbar} \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{\nabla} \\ \vec{\sigma}\cdot\vec{\nabla} & 0 \end{pmatrix} \begin{pmatrix} a_1^{>} \\ a_1^{<} \end{pmatrix} - \begin{pmatrix} i\partial_t - \frac{mF(\mathbf{r},t)}{2} & 0 \\ 0 & i\partial_t + \frac{mF(\mathbf{r},t)}{2} \end{pmatrix} \begin{pmatrix} a_0^{>} \\ a_0^{<} \end{pmatrix} = 0 \tag{73}$$

Upon using equation (72), the first branch of (73) yields,

$$i\hbar \frac{\partial a_0^{>}}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^{>} + \frac{m\hbar F(\mathbf{r},t)}{2} a_0^{>} \tag{74}$$

Einstein's equations: Next, we go to Einstein's equations. G_{00} is evaluated in Appendix [6.6]. We equate it with kT_{00} and obtain:

$$\frac{\hbar\nabla^2 F(\mathbf{r},t)}{c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) = \frac{8\pi Gm\,|a_0^>|^2}{c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{75}$$

Equating the functions at order $\frac{1}{c^2}$, we obtain:

$$\nabla^2 F(\mathbf{r},t) = \frac{8\pi Gm\,|a_0^>|^2}{\hbar} \tag{76}$$

If we recognize the quantity $\frac{\hbar F(\mathbf{r},t)}{2}$ as the Newtonian potential ϕ, then we get Schrödinger-Newton system of equations with $m\phi$ as the gravitational potential energy and $m\,|a_0^>|^2$ as mass density $\rho(\mathbf{r},t)$. The physical picture, which this system of equations suggests, has been given in the introduction.

$$i\hbar\frac{\partial a_0^>}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^> + m\phi(\mathbf{r},t)a_0^> \tag{77}$$

$$\nabla^2\phi(\mathbf{r},t) = 4\pi Gm\,|a_0^>|^2 = 4\pi G\rho(\mathbf{r},t) \tag{78}$$

$$i\hbar\frac{\partial a_0^>}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 a_0^> - Gm^2\int\frac{|a_0^>(\mathbf{r}',t)|^2}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'a_0^> \tag{79}$$

This completes the derivation of the Schrödinger-Newton equation from the Einstein-Dirac equations, in the non-relativistic limit.

4 Non-relativistic limit of Einstein-Cartan-Dirac equations

Dirac equation on U_4 (which is also known as the Hehl-Datta equation) is given by equation (25)

$$i\gamma^\mu\psi_{;\mu} - \frac{3}{8}L_{Pl}^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi - \frac{mc}{\hbar}\psi = 0 \tag{80}$$

We have already evaluated first and the last term after putting ansatz for spinor (29) and metric (69). The second term (arising because of torsion) induces non-linearity into the Dirac equation. We now evaluate this term by following similar procedure as we did for the other two terms. First we multiply the mid-term by $\gamma^0 c$ as done while getting equation (35) from (34) and get the following:

$$\gamma^{(0)}\frac{3c}{8}L_{Pl}^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi = -\frac{3c}{8}l_{Pl}^2 e^{\frac{ic^2S}{\hbar}}\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n^\dagger\right)\gamma_{(a)}\left(\sum_{l=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_l\right)\gamma_5\gamma^{(a)}\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_m\right) \tag{81}$$

Next, we divide it by $\left[1+\sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e_{(0)}^{0[n]}\right]$ as done while getting equation (36) from (35). This is equivalent to dividing by $\left[1-\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]$ or equivalently multiplying by $\left[1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]$ as given in (6.4). We get the following:

The non-linear term:

$$e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\left[1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\Big(\frac{1}{c^n}\Big)\right]\frac{3G}{8}\Big(\sum_{n_1,n_2,n_3=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n a^{\dagger}_{n_1-i}\gamma_a a_{n_2-j}\gamma^5\gamma^a a_{n_3-k}\Big) \tag{82}$$

where $n=n_1+n_2+n_3$. This term modifies Equation (70) as follows

$$\begin{aligned} e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\sum_{n=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n\Bigg[& m\,a_{n-1}+i\dot{a}_{n-3}+i\sqrt{\hbar}\overrightarrow{\alpha}\cdot\overrightarrow{\nabla}a_{n-2}-\beta m a_{n-1}-\beta\frac{mF(\mathbf{r},t)}{2}a_{n-3} \\ & +\frac{3G}{8}\Big(\sum_{n_1,n_2,n_3=0}^{\infty}\Big(\frac{\sqrt{\hbar}}{c}\Big)^n a^{\dagger}_{n_1-i}\gamma_a a_{n_2-j}\gamma^5\gamma^a a_{n_3-k}\Big)\Bigg]=0 \end{aligned} \tag{83}$$

where $n=n_1+n_2+n_3, i+j+k=5$ and, whatever value of i,j,k,n_1,n_2,n_3 is chosen from (0,1,2,3,4,5) the fact that $i\leq n_1$, $j\leq n_2$ and $k\leq n_3$ is to be respected. We find from the above expression that the non-linear term with starts contributing finitely from n = 5 onwards. So, the analysis for $n=0,1,2,3$ as given in Appendix remains as it is and we obtain Schrödinger equation for $a_0^>$ viz. $i\hbar\frac{\partial a_0^>}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2 a_0^>+\frac{m\hbar F(\mathbf{r},t)}{2}a_0^>$.

Next, we go to Einstein's equations (gravitation equation of ECD theory). The equations of interest here are as given by eqn (26) as $G_{\mu\nu}(\{\})=kT_{\mu\nu}-\frac{1}{2}k^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}$. The tensors $G_{\mu\nu}$ and $T_{\mu\nu}$ are already analyzed in above section. We will analyze the second term on the right hand side, which is $(-\frac{1}{2}k^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda})$. It contains the products of spin density tensor which is given by eqn (24). We consider only first term in the expansion of metric because other terms combined with the coupling constant are already higher orders.

$$\frac{-1}{2}k^2 g_{00}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda}=-g_{00}\frac{2\pi^2G^2\hbar^2}{c^6}\sum_{N=0}^{\infty}\Big(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a^{\dagger}_k\gamma^0\gamma^{[c}\gamma^a\gamma^{b]}\Big)\Big(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a^{\dagger}_m\gamma^0\gamma_{[c}\gamma_a\gamma_{b]}n_m\Big)=\sum_{n=6}^{\infty}O\Big(\frac{1}{c^n}\Big) \tag{84}$$

We find that this addition doesn't contribute at the order $1/c^2$ on the RHS of equation (26). Hence we get back Poisson equation. Recognizing the quantity $\frac{\hbar F(\mathbf{r},t)}{2}$ as the potential ϕ, at leading order, we find that ECD theory also yields Schrödinger-Newton equation. Torsion does not contribute at leading non-relativistic order.

5 Non-relativistic limit of ECD field equations with new length scale L_{cs}

The motivation for introducing a new length scale [16] in the ECD theory is as follows. Given a relativistic particle of mass m, it could satisfy either the flat space-time Dirac equation, or the Einstein equations for a point mass, or the ECD equations which couple the Dirac field to its self-gravity and torsion. How are we to know which of these three equations does the dynamics satisfy? There is no mass scale in the equations to determine this. To resolve this problem, we introduced a new length scale L_{CS} in the ECD equations, with the following properties: for $m \gg m_{Pl}, L_{CS} = 2Gm/c^2$; for $m \ll m_{Pl} = \hbar/2mc$; for $m = m_{Pl}/2, L_{CS} = 2L_{Pl}$. In other words, for large masses the length scale in the problem is Schwarzschild radius, and for small masses the length scale is half of the Compton wavelength [17, 18, 19]. An example of a function which can satisfy these properties is

$$\frac{L_{CS}}{2L_pl} := \frac{1}{2}\left(\frac{2m}{m_Pl} + \frac{m_Pl}{2m}\right) := \cosh z \tag{85}$$

where $z = \ln 2m/m_{Pl}$. We desire that the field equations with this L_{CS} should reduce to Einstein equations for large masses, and to the Hehl-Datta equation for small masses. An action which yields such equations is

$$\frac{L_{Pl}^2}{\hbar}S = \int d^4x \sqrt{-g}\left[\frac{1}{8\pi}R - \frac{1}{2}L_{CS}\,\overline{\psi}\psi + L_{CS}^2\left\{\frac{i}{2}\overline{\psi}\gamma^\mu\nabla_\mu\psi - \frac{i}{2}(\nabla_\mu\overline{\psi})\gamma^\mu\psi\right\}\right] \tag{86}$$

The ECD field equations following from this action, with L_{cs} incorporated in them, are the following:

$$\gamma^\mu\psi_{;\mu} = +\frac{3}{8}L_{cs}^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{1}{2L_{CS}}\psi \tag{87}$$

$$G^{\mu\nu}(\{\}) = \frac{8\pi L_{CS}^2}{\hbar c}T^{\mu\nu} - \frac{1}{2}\Big(\frac{8\pi L_{CS}^2}{\hbar c}\Big)^2 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{88}$$

$$T^{\mu\nu\alpha} = -K^{\mu\nu\alpha} = \frac{8\pi L_{CS}^2}{\hbar c}S^{\mu\nu\alpha} \tag{89}$$

Here we analyze the non-relativistic limit of these equations.

5.1 Analysis for lower mass limit of L_{cs}

Lower mass limit of L_{cs} is $\frac{\lambda_C}{2} = \frac{\hbar}{2mc}$. The Dirac equation in the Riemann-Cartan spacetime with new length scale L_{CS} in its lower mass limit is given by (87):

$$i\gamma^\mu\psi_{;\mu} = \frac{3\hbar^2}{32m^2c^2}\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{1}{2L_{CS}}\psi \tag{90}$$

We have already evaluated first and the last term after putting ansatz for spinor (29) and metric (69). The second term (arising because of torsion) induces non-linearity into the Dirac equation. We now evaluate this term by following similar procedure as we did for the other two terms. First we multiply the middle term by $\gamma^0 c$ as done while getting equation (35) from (34) and get the following:

$$\gamma^{(0)}\frac{3c}{32}\lambda_C^2\overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi = \frac{3c}{32}\lambda_C^2 e^{\frac{ic^2S}{\hbar}}\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n^\dagger\right)\gamma_{(a)}\left(\sum_{l=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_l\right)\gamma_5\gamma^{(a)}\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^l a_m\right) \tag{91}$$

Next, we divide it by $\left[1 + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n e_{(0)}^{0[n]}\right]$ as done while getting equation (36) from (35). This is equivalent to dividing by $1 - \frac{\hbar F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right)$ as given in (6.4) or multiplying by

$1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)$. We get:

$$e^{\frac{ic^2S}{\hbar}}\frac{c^3}{\hbar^{3/2}}\left[1+\frac{\hbar F(\mathbf{r},t)}{2c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)\right]\frac{3\hbar^{3/2}}{32m^2}\left(\sum_{n_1,n_2,n_3=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a^{\dagger}_{n_1-i}\gamma_a a_{n_2-j}\gamma^5\gamma^a a_{n_3-k}\right) \tag{92}$$

where $n=n_1+n_2+n_3, i+j+k=4$ and, whatever value of i,j,k,n_1,n_2,n_3 is chosen from (0,1,2,3,4) the fact that $i\leq n_1$, $j\leq n_2$ and $k\leq n_3$ is to be respected. We find from the above expression that the non-linear term with L_{CS} starts contributing finitely from n $=4$ onwards. So, the analysis for $n=0,1,2,3$ as given iearlier remains as it is and we obtain Schrödinger equation for $a_0^>$ viz. $i\hbar\frac{\partial a_0^>}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2 a_0^>+\frac{m\hbar F(\mathbf{r},t)}{2}a_0^>$.

Now, the gravitational equation of ECD with L_{cs} in its lower mas limit is given by (88). We will consider terms only up till second order in (1/c). So we stick to equation for 00 component. We neglect the 2nd term om the right hand side of (88) because it is of higher order. The equation for 00 component is:

$$G_{00}=\left(\frac{2\pi\hbar}{m^2c^3}\right)\left(\frac{i\hbar c}{4}\right)\left[2\bar{\psi}\gamma_0\psi_{;0}-2\bar{\psi}_{;0}\gamma_0\psi\right] \tag{93}$$

$$G_{00}=e^0_{(0)}\left(\frac{i\pi\hbar^2}{m^2c^3}\right)\left[\psi^{\dagger}(\partial_t\psi)-(\partial_t\psi^{\dagger})\psi\right] \tag{94}$$

After substituting spinor ansatz (29), we obtain following equation for the right hand side:

$$\begin{aligned}G_{00}=\left(\frac{i\pi\hbar}{m^2c}\right)\Bigg[&\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^m a^{\dagger}_m\right)\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n[\dot{a}_{n-2}+i\dot{S}a_n]\right)\\ &-\left(\sum_{m=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^m[\dot{a}^{\dagger}_{m-2}-i\dot{S}a^{\dagger}_m]\right)\left(\sum_{n=0}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n a_n\right)\Bigg]\end{aligned} \tag{95}$$

This implies that

$$\frac{\hbar\nabla^2F}{c^2}+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right)=\frac{1}{c}\left(\frac{2\pi\hbar}{m}|a_0^>|^2\right)+\frac{1}{c^2}\left(\frac{2\pi\hbar^{3/2}}{m}\left[a_1^{>\dagger}a_0^>+a_0^{>\dagger}a_1^>\right]\right)+\sum_{n=3}^{\infty}O\left(\frac{1}{c^n}\right) \tag{96}$$

This leads us to conclude that $a_0^>=0$ and hence

$$\nabla^2F=0\implies\nabla^2\phi=0 \tag{97}$$

With this new length scale, there is no contribution to gravity in the small mass limit, at the leading order. This makes the theory different from the standard ECD theory. Another possible interpretation of the modified Poisson equation (96) might be to write it at order $1/c^2$ as

$$\nabla^2\phi=\frac{4\pi Gm}{\alpha}\left(|a_0^>|^2+\hbar^{1/2}\left[a_1^{>\dagger}a_0^>+a_0^{>\dagger}a_1^>\right]\right) \tag{98}$$

where $\alpha\equiv 4Gm^2/\hbar c$ is the dimensionless gravitational fine structure parameter. Further implications of this equation are at present under investigation.

5.2 Analysis for higher mass limit of L_{cs}

The high mass limit of L_{cs} is $2Gm/c^2$. We have shown elsewhere that in the large mass limit these equations reduce to Einstein equations for a point mass. The non-relativistic limit will then inevitably be the Poisson equation for a point mass.

This can also be seen as follows. The Einstein equation in the Riemann-Cartan spacetime with new length scale L_{CS} is given by (88). We neglect terms higher order in L_{CS} because it is easy to deduce from the fact that L_{CS}^2 in higher mass limit is already fourth order in $(1/c)$. So only first term of right hand side is significant. We consider the "00" component of the above equation

$$G_{00} = \frac{8\pi L_{CS}^2}{\hbar c} T_{00} \tag{99}$$

The stress tensor is given by (23). Its "00" component is given by [we neglect orders greater than $1/c^2$].

$$T_{00} = \frac{i\hbar c}{4}\left[2\bar{\psi}\gamma^0\psi_{;0} - 2\bar{\psi}_{;0}\gamma^0\psi\right] \tag{100}$$

The Dirac equation with L_{cs} in its higher mass limit is given by (87). Now, for large masses ($m \gg m_{Pl}$), amplitude of state ψ is negligible (except in a very narrow region where mass m gets localized). This is possible if we assume the localization process. In such a case, the kinetic energy term can be neglected and we obtain the following equations

$$\begin{aligned}\psi_{;0} &= -\frac{3}{8} i\gamma^0 L_{CS}^2 \bar{\psi}\gamma^5\gamma_a\psi\gamma^5\gamma^a\psi - \frac{i\gamma^0}{2L_{CS}}\psi \\ \psi^\dagger_{;0} &= \frac{3}{8} i L_{CS}^2 (\gamma^0\bar{\psi}\gamma^5\gamma_a\psi\gamma^5\gamma^a\psi)^\dagger + \frac{i}{2L_{CS}}\psi^\dagger\gamma^0\end{aligned} \tag{101}$$

Substituting above equation (101) in eqn (100) and neglecting higher order terms in L_{CS} we get,

$$\frac{8\pi L_{CS}^2}{\hbar c} T_{00} = 4\pi L_{CS}(\psi^\dagger\gamma^0\psi) \tag{102}$$

Substituting for L_{CS} in the large mass limit in eqn (102) ,

$$\frac{8\pi L_{CS}^2 T_{00}}{\hbar c} = 4\pi L_{CS}(\psi^\dagger\gamma^0\psi) = \frac{8\pi G m\bar{\psi}\psi}{c^2} \tag{103}$$

In the localization process we replace $\bar{\psi}\psi$ with a spatial Dirac delta function [18]. Substituting equation (103) and G_{00} from Appendix [153] in equation (99) and equating at order $\frac{1}{c^2}$, we get the Poisson equation as the non-relativistic weak field limit of the modified Einstein equation in the large mass limit,

$$\nabla^2 F(\mathbf{r}, t) = \frac{8\pi G m}{\hbar}\delta(\mathbf{r}) \tag{104}$$

As earlier, we recognize $\frac{\hbar F}{2}$ as Newtonian potential ϕ and hence, we get

$$\nabla^2\phi = 4\pi G m\delta(\mathbf{r}) \tag{105}$$

The large mass non-relativistic limit with this new length scale is not the Schrödinger-Newton equation, but the Poisson equation for a classical point mass.

5.3 Some comments on analysis for intermediate mass

For an intermediate mass, L_{cs} is given by equation (85). With this, the ECD equations become:

$$i\gamma^{\mu}\psi_{;\mu} = \frac{3}{8}\Big(\frac{2Gm}{c^2} + \frac{\hbar}{2mc}\Big)^2 \overline{\psi}\gamma^5\gamma_{(a)}\psi\gamma^5\gamma^{(a)}\psi + \frac{1}{\Big(\frac{4Gm}{c^2} + \frac{\hbar}{mc}\Big)}\psi \tag{106}$$

$$G_{\mu\nu}(\{\}) = \frac{8\pi}{\hbar c}\Big(\frac{2Gm}{c^2} + \frac{\hbar}{2mc}\Big)^2 T_{\mu\nu} - \frac{32\pi^2}{\hbar^2 c^2}\Big(\frac{2Gm}{c^2} + \frac{\hbar}{2mc}\Big)^4 g_{\mu\nu}S^{\alpha\beta\lambda}S_{\alpha\beta\lambda} \tag{107}$$

First we will analyze HD equation. The three non-linear terms appear in this equation with coefficients $\frac{3G^2m^2}{2c^4}$, $\frac{3L_{pl}^2}{4}$ and $\frac{3\hbar^2}{32m^2c^2}$. We have already done the order analysis of all these terms and shown to be higher order; not contributing to the equation at leading order. So we neglect them. What we get is:

$$i\gamma^{\mu}\psi_{;\mu} = \frac{1}{\Big(\frac{4Gm}{c^2} + \frac{\hbar}{mc}\Big)}\psi = \frac{mc\psi}{\hbar}\left(\frac{1}{1 + \frac{4m^2}{m_{pl}^2}}\right) \tag{108}$$

$$\implies \left[1 + \frac{4m^2}{m_{pl}^2}\right] i\gamma^{\mu}\psi_{;\mu} = \frac{mc\psi}{\hbar} \tag{109}$$

This is a very interesting equation. If mass m is too small compared to m_{pl}, we can neglect the second term on left hand side and this basically gives Schrödinger's equation. On the other hand, if mass is too large, we neglect the first term on the left hand side, and then the equation becomes such that we can safely assume the localization process. [basically it justifies eq. (101)]. We plan to investigate the intermediate mass case more rigorously in the future.

6 APPENDIX

6.1 Form of Einstein's tensor evaluated from the generic metric upto second order

We have used the ansatz for metric [defined in equation (30)]

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sum_{n=1}^{\infty}\left(\frac{\sqrt{\hbar}}{c}\right)^n g_{\mu\nu}^{[n]}(x)$$

The metric and its inverse, up to second order, can be written as following:

$$g_{\mu\nu} = \eta_{\mu\nu} + \Big(\frac{\sqrt{\hbar}}{c}\Big)g_{\mu\nu}^{[1]} + \Big(\frac{\hbar}{c^2}\Big)g_{\mu\nu}^{[2]} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{110}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \Big(\frac{\sqrt{\hbar}}{c}\Big)g^{\mu\nu[1]} - \Big(\frac{\hbar}{c^2}\Big)[g^{\mu[1]}_{\ \beta}g^{\beta\nu[1]} + g^{\mu\nu[2]}] + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{111}$$

We evaluate Christoffel symbols, Riemann curvature tensor, Ricci tensor and scalar curvature up to second order using above 2 equations and obtain Einstein tensor at the end. Einstein's tensor $G_{\mu\nu}$ is then given by

$$G_{\mu\nu} = \Big(\frac{\sqrt{\hbar}}{c}\Big)G^{[1]}_{\mu\nu} + \Big(\frac{\hbar}{c^2}\Big)G^{[2]}_{\mu\nu} \tag{112}$$

Where

$$G^{[1]}_{\mu\nu} = -\frac{1}{2}\Box\overline{g}^{[1]}_{\mu\nu}; \qquad \text{where } \overline{g}^{[1]}_{ij} = g^{[1]}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}g^{[1]}; \qquad g^{[1]} = (\eta^{\mu\nu}g^{[1]}_{\mu\nu}) \tag{113}$$

$$G^{[2]}_{\mu\nu} = -\frac{1}{2}\Box\overline{g}^{(2)}_{\mu\nu} + f(g^{[1]}_{\mu\nu}) \qquad \text{where } \overline{g}^{[2]}_{ij} = g^{[2]}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}g^{[2]}; \qquad g^{[2]} = (\eta^{\mu\nu}g^{[2]}_{\mu\nu}) \tag{114}$$

f is a function of $g^{[1]}_{\mu\nu}$ and is given by following equation:

$$\begin{aligned} f(g^{[1]}_{\mu\nu}) = -\frac{1}{4}\Big[& 2\partial^{\lambda}g^{[1]}\partial_{\nu}g^{[1]}_{\lambda\mu} - 2\partial^{\lambda}g^{[1]}\partial_{\lambda}g^{[1]}_{\mu\nu} - \partial_{\rho}g^{\lambda[1]}_{\nu}\partial_{\mu}g^{\rho[1]}_{\lambda} - \partial_{\rho}g^{\lambda[1]}_{\nu}\partial_{\lambda}g^{\rho[1]}_{\mu} + \\ & \partial_{\rho}g^{\lambda[1]}_{\nu}\partial^{\rho}g^{[1]}_{\lambda\mu} + \partial_{\nu}g^{\lambda[1]}_{\rho}\partial_{\mu}g^{\rho[1]}_{\lambda} + \partial_{\nu}g^{\lambda[1]}_{\rho}\partial_{\lambda}g^{\rho[1]}_{\mu} - \partial_{\nu}g^{\lambda[1]}_{\rho}\partial^{\rho}g^{[1]}_{\lambda\mu}\Big] \\ -\frac{1}{8}\Big[& 2\partial^{\lambda}g^{[1]}\partial_{\nu}g^{[1]}_{\lambda\mu} - 2\eta_{\mu\nu}\partial^{\lambda}g^{[1]}\partial_{\lambda}g^{[1]} - \partial_{\rho}g^{\lambda[1]}_{\nu}\partial_{\mu}g^{\rho[1]}_{\lambda} - \partial_{\rho}g^{\lambda[1]}_{\mu}\partial_{\lambda}g^{\rho[1]}_{\nu} \\ & +\partial_{\rho}g^{\lambda[1]}_{\mu}\partial^{\rho}g^{[1]}_{\lambda\nu} + \partial_{\mu}g^{\lambda[1]}_{\rho}\partial_{\nu}g^{\rho[1]}_{\lambda} + \partial_{\mu}g^{\lambda[1]}_{\rho}\partial_{\lambda}g^{\rho[1]}_{\nu} - \partial_{\nu}g^{\lambda[1]}_{\rho}\partial^{\rho}g_{\lambda\mu[1]}\Big] \end{aligned}$$

6.2 Constraints imposed on metric due to asymptotic flatness condition

6.2.1 Constraint on $g^{[1]}_{\mu\nu}$

First we analyze the off-diagonal form of $g^{[1]}_{\mu\nu}$. Off-diagonal components of $G^{[1]}_{\mu\nu}$ is zero. This implies (for off-diagonal components alone), from equation (113), $\Box\overline{g}^{[1]}_{\mu\nu} = \Box g^{[1]}_{\mu\nu} = 0$. Non-trivial solution to this equation (which is a gravitational wave solution) doesn't respect asymptotic flatness. So the only solution allowed is trivial solution viz. $g^{[1]}_{\mu\nu} = 0$. Now, for diagonal components, we assume the metric form to be the most generic:

$$g^{[1]}_{\mu\nu} = \begin{pmatrix} f^{[1]}_1 & 0 & 0 & 0 \\ 0 & f^{[1]}_2 & 0 & 0 \\ 0 & 0 & f^{[1]}_3 & 0 \\ 0 & 0 & 0 & f^{[1]}_4 \end{pmatrix} \tag{115}$$

$$\bar{g}^{[1]}_{00} = \frac{f^{[1]}_1 + f^{[1]}_2 + f^{[1]}_3 + f^{[1]}_4}{2} \tag{116}$$

$$\bar{g}^{[1]}_{11} = \frac{f^{[1]}_1 + f^{[1]}_2 - f^{[1]}_3 - f^{[1]}_4}{2} \tag{117}$$

$$\bar{g}^{[1]}_{22} = \frac{f^{[1]}_1 + f^{[1]}_3 - f^{[1]}_2 - f^{[1]}_4}{2} \tag{118}$$

$$\bar{g}^{[1]}_{33} = \frac{f^{[1]}_1 + f^{[1]}_4 - f^{[1]}_2 - f^{[1]}_3}{2} \tag{119}$$

And the fact that Einstein's tensor is zero for all the components implies,

$$\Box\bar{g}^{[1]}_{00} = \Box\frac{f^{[1]}_1 + f^{[1]}_2 + f^{[1]}_3 + f^{[1]}_4}{2} = 0 \implies \Box f^{[1]}_1 + \Box f^{[1]}_2 + \Box f^{[1]}_3 + \Box f^{[1]}_4 = 0 \tag{120}$$

$$\Box\bar{g}^{[1]}_{11} = \Box\frac{f^{[1]}_1 + f^{[1]}_2 - f^{[1]}_3 - f^{[1]}_4}{2} = 0 \implies \Box f^{[1]}_1 + \Box f^{[1]}_2 = \Box f^{[1]}_3 + \Box f^{[1]}_4 \tag{121}$$

$$\Box\bar{g}^{[1]}_{22} = \Box\frac{f^{[1]}_1 + f^{[1]}_3 - f^{[1]}_2 - f^{[1]}_4}{2} = 0 \implies \Box f^{[1]}_1 + \Box f^{[1]}_3 = \Box f^{[1]}_2 + \Box f^{[1]}_4 \tag{122}$$

$$\Box\bar{g}^{[1]}_{33} = \Box\frac{f^{[1]}_1 + f^{[1]}_4 - f^{[1]}_2 - f^{[1]}_3}{2} = 0 \implies \Box f^{[1]}_1 + \Box f^{[1]}_4 = \Box f^{[1]}_2 + \Box f^{[1]}_3 \tag{123}$$

One should note that individually, $\Box f^{[1]}_i = 0$ only implies $f^{[1]}_i = 0$ (no wave solution allowed) Even $f^{[1]}_i =$ constant is NOT allowed as constant solution also contradicts asymptotic flatness. Equations (121), (122) and (123) imply that

$$\Box f^{[1]}_2 = \Box f^{[1]}_1 \implies f^{[1]}_2 = f^{[1]}_1 + c_1 \tag{124}$$

$$\Box f^{[1]}_3 = \Box f^{[1]}_1 \implies f^{[1]}_3 = f^{[1]}_1 + c_2 \tag{125}$$

$$\Box f^{[1]}_4 = \Box f^{[1]}_1 \implies f^{[1]}_4 = f^{[1]}_1 + c_3 \tag{126}$$

However, all the constants c_1, c_2, c_3 should be zero [As constant + asymptotic flat function can't give overall asymptotic flat function]. Now, equation (120) implies, $4\Box f^{[1]}_1 = 0 \implies f^{[1]}_1 = 0$. Hence all the functions $f^{[1]}_i = 0 \;\forall\; i$. HENCE

$$g^{[1]}_{\mu\nu} = 0 \;\forall\; \mu, \nu \tag{127}$$

6.2.2 Constraint on $g^{[2]}_{\mu\nu}$

Here also, first we analyze the off-diagonal form of $g^{[2]}_{\mu\nu}$. Off-diagonal components of $G^{[2]}_{\mu\nu}$ is zero. This implies, from equation (114), $\Box\bar{g}^{[2]}_{\mu\nu} = \Box g^{[2]}_{\mu\nu} = 0$. Non-trivial solution to this equation (which is a gravitational wave solution) doesn't respect asymptotic flatness. So the only solution allowed is trivial solution viz. $g^{[2]}_{\mu\nu} = 0$. Now, for diagonal components, we again assume the metric form to be the most generic:

$$g^{[2]}_{\mu\nu} = \begin{pmatrix} f^{[2]}_1 & 0 & 0 & 0 \\ 0 & f^{[2]}_2 & 0 & 0 \\ 0 & 0 & f^{[2]}_3 & 0 \\ 0 & 0 & 0 & f^{[2]}_4 \end{pmatrix} \tag{128}$$

$$\bar{g}^{[2]}_{00} = \frac{f^{[2]}_1 + f^{[2]}_2 + f^{[2]}_3 + f^{[2]}_4}{2} \tag{129}$$

$$\bar{g}^{[2]}_{11} = \frac{f^{[2]}_1 + f^{[2]}_2 - f^{[2]}_3 - f^{[2]}_4}{2} \tag{130}$$

$$\bar{g}^{[2]}_{22} = \frac{f^{[2]}_1 + f^{[2]}_3 - f^{[2]}_2 - f^{[2]}_4}{2} \tag{131}$$

$$\bar{g}^{[2]}_{33} = \frac{f^{[2]}_1 + f^{[2]}_4 - f^{[2]}_2 - f^{[2]}_3}{2} \tag{132}$$

And the fact that Einstein's tensor is zero for all the components except '00' component implies,

$$\Box\bar{g}^{[2]}_{00} = \Box\frac{f^{[2]}_1 + f^{[2]}_2 + f^{[2]}_3 + f^{[2]}_4}{2} \Longrightarrow \Box f^{[2]}_1 + \Box f^{[2]}_2 + \Box f^{[2]}_3 + \Box f^{[2]}_4 \neq 0 \tag{133}$$

$$\Box\bar{g}^{[2]}_{11} = \Box\frac{f^{[2]}_1 + f^{[2]}_2 - f^{[2]}_3 - f^{[2]}_4}{2} \Longrightarrow \Box f^{[2]}_1 + \Box f^{[2]}_2 = \Box f^{[2]}_3 + \Box f^{[2]}_4 \tag{134}$$

$$\Box\bar{g}^{[2]}_{22} = \Box\frac{f^{[2]}_1 + f^{[2]}_3 - f^{[2]}_2 - f^{[2]}_4}{2} \Longrightarrow \Box f^{[2]}_1 + \Box f^{[2]}_3 = \Box f^{[2]}_2 + \Box f^{[2]}_4 \tag{135}$$

$$\Box\bar{g}^{[2]}_{33} = \Box\frac{f^{[2]}_1 + f^{[2]}_4 - f^{[2]}_2 - f^{[2]}_3}{2} \Longrightarrow \Box f^{[2]}_1 + \Box f^{[2]}_4 = \Box f^{[2]}_2 + \Box f^{[2]}_3 \tag{136}$$

Equations (134), (135) and (136) imply that

$$\Box f^{[2]}_2 = \Box f^{[2]}_1 \Longrightarrow f^{[2]}_2 = f^{[2]}_1 \tag{137}$$

$$\Box f^{[2]}_3 = \Box f^{[2]}_1 \Longrightarrow f^{[2]}_3 = f^{[2]}_1 \tag{138}$$

$$\Box f^{[2]}_4 = \Box f^{[2]}_1 \Longrightarrow f^{[2]}_4 = f^{[2]}_1 \tag{139}$$

[we have already seen why addition of constant to above solution contradicts our claim of asymptotic flatness.] With equations (137), (138) and (139), we find that $f^{[2]}_1 = f^{[2]}_2 = f^{[2]}_3 = f^{[2]}_4 = F(\mathbf{r}, t)$.

$$g^{[2]}_{\mu\nu} = \begin{pmatrix} F(\mathbf{r},t) & 0 & 0 & 0 \\ 0 & F(\mathbf{r},t) & 0 & 0 \\ 0 & 0 & F(\mathbf{r},t) & 0 \\ 0 & 0 & 0 & F(\mathbf{r},t) \end{pmatrix} \tag{140}$$

6.3 Metric and Christoffel symbol components

The form of metric defined in equation (69) is as follows:

$$g_{\mu\nu} = \begin{pmatrix} 1+\frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & -1+\frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & -1+\frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & -1+\frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{141}$$

$$g^{\mu\nu} = \begin{pmatrix} 1-\frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & -1-\frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & -1-\frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & -1-\frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{142}$$

Christoffel Connection:
The non-zero Christoffel connection components (the first term; which is second order in 1/c) corresponding to metric $g_{\mu\nu}$ defined above are as follows:

$$\begin{aligned} \Gamma^0_{0\mu} &= \frac{\hbar\partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \\ \Gamma^\mu_{00} &= \frac{\hbar\partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \\ \Gamma^\mu_{\mu\mu} &= \frac{-\hbar\partial_\mu F(\mathbf{r},t)}{2c^2} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \end{aligned} \tag{143}$$

[Here $\mu = 1, 2, 3$ i.e., it refers to the spatial coordinates.]

Other non zero Christoffel connection components have all orders of terms from order 3 viz. $\sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big)$

6.4 Tetrad components

Tetrads are introduced in the section "Preliminaries: Einstein-Cartan-Dirac equations". The metric and corresponding tetrad field on the whole manifold is defined below:

$$dS^2 = \left[1 + \frac{\hbar F(\mathbf{r},t)}{c^2}\right] c^2 dt^2 - \left[1 - \frac{\hbar F(\mathbf{r},t)}{c^2}\right] d\mathbf{r}^2 \tag{144}$$

$$\hat{e}_{(0)} = \frac{1}{c}\left(1+\frac{\hbar F}{c^2}\right)^{\frac{1}{2}}\partial_t, \quad \hat{e}_{(1)} = \left(1-\frac{\hbar F}{c^2}\right)^{\frac{1}{2}}\partial_x, \quad \hat{e}_{(2)} = \left(1-\frac{\hbar F}{c^2}\right)^{\frac{1}{2}}\partial_y, \quad \hat{e}_{(3)} = \left(1-\frac{\hbar F}{c^2}\right)^{\frac{1}{2}}\partial_z \tag{145}$$

With this, the transformation matrix which relates the world components with anholonomic components (defined in equation 13)

$$e_{\mu}^{(i)} = \begin{pmatrix} 1+\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & 1-\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & 1-\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & 1-\frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{146}$$

$$e^{\mu}_{(i)} = \begin{pmatrix} 1-\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & 1+\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & 1+\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & 1+\frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{147}$$

$$e_{\nu(k)} = \begin{pmatrix} 1+\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & -1+\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & -1+\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & -1+\frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{148}$$

$$e^{\nu(k)} = \begin{pmatrix} 1-\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 & 0 \\ 0 & -1-\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 & 0 \\ 0 & 0 & -1-\frac{\hbar F(\mathbf{r},t)}{2c^2} & 0 \\ 0 & 0 & 0 & -1-\frac{\hbar F(\mathbf{r},t)}{2c^2} \end{pmatrix} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \tag{149}$$

6.5 Components of the Riemann part of Spin Connection $\gamma^{o}_{(a)(b)(c)}$

The form of spin connections are defined in equations (14), (15). We use the relation between Christoffel connection and tetrad transformation matrix (defined in Eqn. (16)) to calculate $\gamma^{o}_{(a)(b)(c)}$ as follows:

$$\begin{aligned}
\gamma^{o}_{(0)(0)(0)} &= \frac{-\hbar\partial_0 F}{2c^2}\frac{\left(1+\frac{\hbar F}{2c^2}\right)}{\left(1-\frac{\hbar F}{2c^2}\right)} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) &\quad \gamma^{o}_{(i)(0)(0)} &= \Big(\frac{-\hbar\partial_i F}{2c^2}\Big)\frac{\hbar F/2c^2}{\left(1+\frac{\hbar F}{2c^2}\right)} + \sum_{n=5}^{\infty} O\Big(\frac{1}{c^n}\Big) \\
\gamma^{o}_{(0)(i)(0)} &= \frac{-\hbar\partial_i F}{2c^2}\frac{\left(1+\frac{\hbar F}{2c^2}\right)}{\left(1-\frac{\hbar F}{2c^2}\right)} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) &\quad \gamma^{o}_{(0)(0)(i)} &= \frac{\hbar\partial_i F}{2c^2}\frac{1}{\left(1+\frac{\hbar F}{2c^2}\right)} \\
\gamma^{o}_{(i)(i)(i)} &= \frac{\hbar\partial_i F}{2c^2}\frac{\hbar F/2c^2}{\left(1+\frac{\hbar F}{2c^2}\right)} + \sum_{n=5}^{\infty} O\Big(\frac{1}{c^n}\Big) &\quad \gamma^{o}_{(i)(i)(0)} &= \gamma^{o}_{(i)(0)(i)} = +\sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \\
\gamma^{o}_{(0)(i)(i)} &= \frac{-\hbar\partial_0 F}{2c^2} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) &\quad \gamma^{o}_{(0)(i)(j)} &= \gamma^{o}_{i0j} = \gamma^{o}_{ij0} = +\sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) \\
\gamma^{o}_{(i)(j)(j)} &= \frac{-\hbar\partial_0 F}{2c^2}\frac{\left(1-\frac{\hbar F}{2c^2}\right)}{\left(1+\frac{\hbar F}{2c^2}\right)} + \sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big) &\quad \gamma^{o}_{(i)(j)(k)} &= \gamma^{o}_{(i)(j)(i)} = \gamma^{o}_{(j)(j)(i)} = +\sum_{n=3}^{\infty} O\Big(\frac{1}{c^n}\Big)
\end{aligned} \tag{150}$$

The contorsion spin coefficients (which when gets added to Riemann spin coefficient, gives total spin connection) gets manifested as a non-linear term in Hehl-Datta equation. It is completely expressible in terms of Dirac spinor. So it can be calculated with spinor ansatz. We have done this while calculating the Non-relativistic limit of ECD system of equations.

6.6 Components for Einstein's tensor

In this appendix, we aim to calculate the components of Einstein's tensor. $G^{[1]}_{\mu\nu}$ has been proved to be zero. $G^{[2]}_{\mu\nu}$ has been defined in Eqn. (114). We found the form of $g^{[2]}_{\mu\nu}$ in appendix section (6.2.2). Since $g^{[1]}_{\mu\nu}$ is zero, f[$g^{[1]}_{\mu\nu}$] defined in Eqn. (114) is also zero. With this, we compute $G^{[2]}_{\mu\nu}$:

$$G^{[2]}_{\mu\nu} = -\frac{1}{2}\Box\overline{g}^{[2]}_{\mu\nu}; where \qquad \overline{g}^{[2]}_{\mu\nu} = g^{[2]}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}(\eta^{\alpha\beta}h_{\alpha\beta}) \tag{151}$$

$$\eta^{\mu\nu}h_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 & 0 \\ 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 & 0 \\ 0 & 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} & 0 \\ 0 & 0 & 0 & \frac{\hbar F(\mathbf{r},t)}{c^2} \end{pmatrix} = \frac{-2\hbar F(\mathbf{r},t)}{c^2} \tag{152}$$

It can easily be seen that $G_{\mu\nu}$ for $\mu \neq \nu$ is equal to 0.

We now calculate the diagonal components,

$$G_{00} = -\frac{1}{2}\Box\overline{g}^{[2]}_{00} = -\frac{\hbar}{c^2}\Box F(\mathbf{r},t) = \left[-\frac{\hbar\partial_t^2 F(\mathbf{r},t)}{c^4} + \frac{\hbar\nabla^2 F(\mathbf{r},t)}{c^2}\right] \tag{153}$$

$$G_{\alpha\alpha} = 0; \qquad because \quad \overline{g}^{[2]}_{\alpha\alpha} = 0; \qquad \alpha \in (1,2,3) \tag{154}$$

Thus,

$$G_{\mu\nu} = \frac{\hbar}{c^2}\begin{pmatrix} \nabla^2 F(\mathbf{r},t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \sum_{n=3}^{\infty} O\left(\frac{1}{c^n}\right) \tag{155}$$

6.7 Generic components of $T_{\mu\nu}$

$T_{\mu\nu}$ has been defined in equation Eqn. (23). With the spin coefficients in above sections, we get the following metric energy-momentum tensor, whose components are given on the next page.

$$
T_{\mu\nu} = \frac{i\hbar c}{4}
\begin{pmatrix}
\begin{array}{r} 2\bar{\psi}\gamma_0(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ -(\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\ +\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\bar{\psi})2\gamma_0\psi \end{array}
&
\begin{array}{r} \bar{\psi}\gamma_0\partial_1\psi + \bar{\psi}\gamma_1(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ -\partial_1\bar{\psi}\gamma_0\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\ +\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\bar{\psi})\gamma_1\psi) \end{array}
&
\begin{array}{r} \bar{\psi}\gamma_0\partial_2\psi + \bar{\psi}\gamma_2(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ -\partial_2\bar{\psi}\gamma_0\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\ +\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\bar{\psi})\gamma_2\psi) \end{array}
&
\begin{array}{r} \bar{\psi}\gamma_0\partial_3\psi + \bar{\psi}\gamma_3(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ -\partial_3\bar{\psi}\gamma_0\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha \\ +\gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\bar{\psi})\gamma_3\psi) \end{array}
\\
\begin{array}{r} \bar{\psi}\gamma_1(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ +\bar{\psi}\gamma_0\partial_1\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\ +\gamma_{0\alpha 0}\gamma^i\gamma^0]\bar{\psi})\gamma_1\psi - \partial_1\bar{\psi}\gamma_0\psi \end{array}
&
2(\bar{\psi}\gamma_1\partial_1\psi - \partial_1\psi\bar{\gamma}_1\psi)
&
\begin{array}{r} \bar{\psi}\gamma_1\partial_2\psi + \bar{\psi}\gamma_2\partial_1\psi \\ -\partial_2\bar{\psi}\gamma_1\psi - \partial_1\bar{\psi}\gamma_2\psi \end{array}
&
\begin{array}{r} \bar{\psi}\gamma_1\partial_3\psi + \bar{\psi}\gamma_3\partial_1\psi \\ -\partial_3\bar{\psi}\gamma_1\psi - \partial_1\bar{\psi}\gamma_3\psi \end{array}
\\
\begin{array}{r} \bar{\psi}\gamma_2(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ +\bar{\psi}\gamma_0\partial_2\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\ +\gamma_{0\alpha 0}\gamma^i\gamma^0]\bar{\psi})\gamma_2\psi - \partial_2\bar{\psi}\gamma_0\psi \end{array}
&
\begin{array}{r} \bar{\psi}\gamma_2\partial_1\psi + \bar{\psi}\gamma_1\partial_2\psi \\ -\partial_1\bar{\psi}\gamma_2\psi - \partial_2\bar{\psi}\gamma_1\psi \end{array}
&
2(\bar{\psi}\gamma_2\partial_2\psi - \partial_2\psi\bar{\gamma}_2\psi)
&
\begin{array}{r} \bar{\psi}\gamma_2\partial_3\psi + \bar{\psi}\gamma_3\partial_2\psi \\ -\partial_3\bar{\psi}\gamma_2\psi - \partial_2\bar{\psi}\gamma_3\psi \end{array}
\\
\begin{array}{r} \bar{\psi}\gamma_3(\partial_0\psi \\ +\frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^\alpha + \gamma_{0\alpha 0}\gamma^\alpha\gamma^0]\psi) \\ +\bar{\psi}\gamma_0\partial_3\psi - (\partial_0\bar{\psi} + \frac{1}{4}[\gamma_{00\alpha}\gamma^0\gamma^i \\ +\gamma_{0\alpha 0}\gamma^i\gamma^0]\bar{\psi})\gamma_3\psi - \partial_3\bar{\psi}\gamma_0\psi \end{array}
&
\begin{array}{r} \bar{\psi}\gamma_3\partial_1\psi + \bar{\psi}\gamma_1\partial_3\psi \\ -\partial_1\bar{\psi}\gamma_3\psi - \partial_3\bar{\psi}\gamma_1\psi \end{array}
&
\begin{array}{r} \bar{\psi}\gamma_3\partial_2\psi + \bar{\psi}\gamma_2\partial_3\psi \\ -\partial_2\bar{\psi}\gamma_3\psi - \partial_3\bar{\psi}\gamma_2\psi \end{array}
&
2(\bar{\psi}\gamma_3\partial_3\psi - \partial_3\psi\bar{\gamma}_3\psi)
\end{pmatrix}
\tag{156}
$$

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