Uncertainty principle for momentum, torsional regularization, and bare charge

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We show that in the presence of the torsion tensor $S^k{}_{ij}$, whose existence is required by the consistency of the conservation law for the total angular momentum of a Dirac particle in curved spacetime with relativistic quantum mechanics, the quantum commutation relation for the four-momentum is given by $[p_i, p_j] = 2i\hbar S^k{}_{ij} p_k$. We propose that this relation replaces the integration in the momentum space in Feynman diagrams with the summation over the discrete momentum eigenvalues. We derive a prescription for this summation that agrees with convergent integrals:

$$\int \frac{d^4 p}{(p^2 + \Delta)^s} \to 4\pi U^{s-2} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \, \frac{\sin^4 \phi \, n^{s-3}}{[\sin \phi + U\Delta n]^s},$$

where $n = \sqrt{l(l+1)}$ and $1/\sqrt{U}$ is a constant on the order of the Planck mass, determined by the Einstein-Cartan theory of gravity. We show that this prescription regularizes ultraviolet-divergent integrals in loop diagrams. We extend this prescription to tensor integrals and apply it to vacuum polarization. We derive a finite, gauge-invariant vacuum polarization tensor and a finite running coupling that agrees with the low-energy limit of the standard quantum electrodynamics. Including loops from all charged fermions, we find a finite value for the bare electric charge of an electron: $\approx -1.22\,e$. Torsional regularization, originating from the noncommutativity of the momentum and spin-torsion coupling, therefore provides a realistic, physical mechanism for eliminating infinities in quantum field theory: quantum electrodynamics with torsion is ultraviolet complete.

I. INTRODUCTION

In quantum electrodynamics (QED) [1], a calculation of the amplitude for a physical process must include perturbative corrections involving Feynman diagrams [2] with closed loops of virtual particles (radiative corrections). The integration in the resulting integrals is taken in the four-momentum space, with the magnitudes of the energy and momentum running to infinity and not restricted to the relativistic energy-momentum relation (off-shell particles). Many integrals that appear in radiative corrections are divergent, which is referred to as the ultraviolet divergence. The ultraviolet divergence results from the asymptotic, high-energy behavior of the Feynman propagators [2]. Physically, the divergence is a consequence of an incompleteness of our understanding of the physics at large energies and momenta. There are three one-loop divergent diagrams in QED: vacuum polarization (a photon creating a virtual electron-positron pair which then annihilates), self-energy (an electron emits and reabsorbs a virtual photon), and vertex (an electron emits a photon, emits a second photon, and then reabsorbs the first) [3]. More complex diagrams can be reduced to the combinations of these three diagrams. The gauge-invariant parts of all three diagrams are logarithmically divergent.

Mathematically, these divergences can be treated by a method of regularization [1]. In Pauli-Villars regularization [4], the vertices are retained but Feynman propagators are modified. From a divergent integral involving a particle of mass m one subtracts an integral of the same form but with a different mass Λ representing a fictitious particle, and then one takes a limit $\Lambda \to \infty$. The divergent term in the difference has a form $\text{const} \cdot \ln\Lambda$ and is absorbed through redefining the original (bare) mass, charge and field, leaving a finite, physical (dressed) value that is measured in experiment. Such redefinition is referred to as renormalization [1, 5, 6]. In 't Hooft-Veltman dimensional regularization [7], the number of dimensions $n = 4$ is replaced by a fictitious number $n = 4 - \epsilon$, and then one takes a limit $\epsilon \to 0$. The divergent term has a form const/ϵ and is absorbing through renormalization, leaving a finite, physical value.

Dirac was persistently critical about renormalization and expected a realistic regularization based on the established principles of physics [8]. In this article, we propose such a physical regularization, referred to as *torsional regularization*, resulting from the noncommutativity of the four-momentum in the presence of spacetime torsion coupled to spin [9]. The spin-torsion coupling is given by the Einstein-Cartan theory of gravity [10]. The existence of torsion is required by the consistency of the conservation law for the total angular momentum of a Dirac particle in curved spacetime

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with relativistic quantum mechanics [11]. We show that the four-dimensional momentum operators do not commute (uncertainty principle for momentum), which becomes significant at larger energies and momenta where regularization is needed. Consequently, the integration in the four-momentum space must be replaced with the summation over all eigenvalues of the four-momentum. We derive a prescription for this summation that gives a correct continuous limit for convergent integrals. We show that ultraviolet-divergent integrals turn into convergent sums, naturally eliminating the ultraviolet divergence in loop diagrams. Using the four-dimensional Gauss theorem, we extend this prescription to tensor integrals.

We apply our prescription to vacuum polarization and derive a finite, gauge-invariant vacuum polarization tensor. We derive a finite running coupling that agrees with the low-energy limit of the standard quantum electrodynamics. Finally, we find that the modification of the photon propagator arising from the loops involving all charged fermions give a finite value for the bare electric charge of a particle: it is approximately 1.22 times its measured, renormalized charge. The renormalization paradigm with torsion is finite and thus mathematically self-consistent. Therefore, torsion may be the source of a realistic regularization which was believed to exist by Dirac, rendering quantum electrodynamics ultraviolet complete.

II. TRANSLATION IN SPACETIME WITH TORSION

In nonrelativistic quantum mechanics, the operator $F(d\mathbf{x})$ of an infinitesimal translation by $d\mathbf{x}$ acts on a state $|\mathbf{x}\rangle$ according to [12]

$$F(d\mathbf{x})|\mathbf{x}\rangle = |\mathbf{x} + d\mathbf{x}\rangle.$$

Its unitarity requires that $F(d\mathbf{x}) = I - i\mathbf{K} \cdot d\mathbf{x}$, where I is the identity operator and $\mathbf{K}$ is a Hermitian operator. The correspondence between unitary transformations in quantum mechanics and canonical transformations in classical mechanics determines a proportionality relation between K and the momentum operator $\mathbf{p}$: $\mathbf{K} = \mathbf{p}/\hbar$. This relation introduces the Planck constant $\hbar = h/(2\pi)$. The commutator $[\mathbf{x}, F(d\mathbf{x})] = d\mathbf{x}$ gives the commutation relation between the position and momentum operators: $[x_j, p_k] = i\hbar\delta_{jk}$.

For the wave function representing a quantum state $|\alpha\rangle$ in position space, $\psi_\alpha(x) = \langle x|\alpha\rangle$, we have $(I - ip_x dx/\hbar)|\alpha\rangle = \int dx' F(dx)|x'\rangle\langle x'|\alpha\rangle = \int dx'|x' + dx\rangle\langle x'|\alpha\rangle = \int dx'|x'\rangle\langle x' - dx|\alpha\rangle = \int dx'|x'\rangle(\langle x'|\alpha\rangle - dx\frac{\partial}{\partial x'}\langle x'|\alpha\rangle)$, which gives $p_x|\alpha\rangle = \int dx'|x'\rangle(-i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle)$ [12]. Consequently, $\langle x|p_x|\alpha\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\alpha\rangle)$ and the momentum operator in position space is a partial derivative with respect to the corresponding conjugate coordinate: $p_x = -i\hbar\frac{\partial}{\partial x}$. In flat spacetime, two infinitesimal translations in two different directions commute: $[F(d\mathbf{x}), F(d\mathbf{y})] = 0$. Consequently, the momentum operator components along different directions also commute, $[p_j, p_k] = 0$, which is consistent with the commutativity of partial derivatives. This relation is also valid in curved spacetime, where partial derivatives are generalized to covariant derivatives. In general relativity, the affine connection is given by the Christoffel symbols which are symmetric in the lower indices [13]. In curved space, covariant derivatives of the wave function (which is a scalar in nonrelativistic quantum mechanics) commute.

The conservation law for the total (orbital plus spin) angular momentum of fermions in curved spacetime, consistent with the Dirac equation, requires that the torsion tensor [9] is not constrained to zero, but is determined by varying the action with respect to torsion [11]. The torsion tensor is the antisymmetric part of the affine connection $\Gamma^{i}_{j\,k}$:

$$S^i_{\ jk} = \Gamma^{\ i}_{[j\,k]}.$$

The simplest and most natural theory of gravity with torsion is the Einstein-Cartan (EC) theory [10, 14, 15], in which torsion becomes coupled to the spin of fermions and fermions are the source of torsion. The Lagrangian density for the gravitational field has the same form as that in general relativity:

$$\mathcal{L}_g = -\frac{1}{2\kappa}R\sqrt{-g}, \tag{1}$$

where R is the Ricci scalar constructed from the affine connection, g is the determinant of the metric tensor g_{ij}, and $\kappa = 8\pi G$ (we use units such that $c = 1$). The variation of the total action for the gravitational field and matter with respect to the metric tensor gives the Einstein field equations. The variation of the action with respect to the torsion tensor gives the Cartan equations that relate the spin density and torsion:

$$S^i_{\ jk} - S_j\delta^i_k + S_k\delta^i_j = -\frac{1}{2}\kappa s_{jk}^{\ \ i}, \tag{2}$$

where $S_i = S^k{}_{ik}$ is the torsion vector and $s_{ijk} = 2\delta\mathcal{L}_m/\delta C^{ijk}$ is the spin tensor of matter (the variational derivative of the Lagrangian density for matter $\mathcal{L}_m$ with respect to the contortion tensor $C_{ijk} = S_{ijk} + S_{jki} + S_{kji}$) (we use the notation of [15]).

The spin-torsion coupling generates gravitational repulsion at extremely high densities and thus avoids the formation of singularities in black holes and at the big bang [16]. The collapsing matter in a black hole bounces at a finite density and then expands into a new, finite region of space with positive curvature on the other side of the event horizon, which may be regarded as a new universe [17]. Quantum particle production caused by an extremely high curvature near a bounce (which replaces the big bang) creates enormous amounts of matter and entropy, and generates a finite period of exponential expansion (inflation) of this universe [18]. EC agrees with all solar system, binary pulsar and cosmological tests of general relativity, since even at nuclear densities, the corrections from torsion to the Einstein equations are negligible [10, 14]. EC also modifies the Dirac equation, adding a term that is cubic in spinor fields [19]. That term may solve the problem of divergent integrals in quantum field theory by providing fermions with spatial extension (about 10^{-27} m for an electron) and thus introducing an effective ultraviolet cutoff for their propagators [20].

In the presence of torsion, the parallel transports (which define the covariant derivative) do not commute, which results from the following construction [14]. The parallel transport of an infinitesimal, four-dimensional vector $\vec{PR} = dx^i$ from a point P to an infinitesimally close point Q such that $\vec{PQ} = dx'^j$ adds to dx^i a small correction:

$$\delta dx^i = -\Gamma_{jk}{}^i dx^j dx'^k.$$

After effecting the transport, the vector $dx^i + \delta dx^i$ points to a point T. The parallel transport of the vector dx'^i from a point P to an infinitesimally close point R adds to dx^i a small correction:

$$\delta dx'^i = -\Gamma_{jk}{}^i dx^j dx'^k.$$

After effecting the transport, the vector $dx'^i + \delta dx'^i$ points to a point T'. Without torsion, points T and T' would coincide and form, together with points P, Q, and R, a parallelogram because $\delta dx'^i - \delta dx^i = \Gamma_{kj}{}^i dx^j dx'^k - \Gamma_{jk}{}^i dx^j dx'^k = 0$. If the torsion tensor is not zero, however, the affine connection is asymmetric in the lower indices and

$$\delta dx'^i - \delta dx^i = -S^i{}_{jk} dx^j dx'^k.$$

Points T and T' do not coincide, the parallogram is not closed, and the combination of two displacements of point P (through dx^i and dx'^j) depends on their order. Accordingly, covariant derivatives of a scalar ψ do not commute:

$$\nabla_i \nabla_j \psi - \nabla_j \nabla_i \psi = \partial_i \nabla_j \psi - \Gamma_{ji}{}^k \nabla_k \psi - \partial_j \nabla_i \psi + \Gamma_{ij}{}^k \nabla_k \psi = \partial_i \partial_j \psi - \partial_j \partial_i \psi + 2S^k{}_{ij} \nabla_k \psi = 2S^k{}_{ij} \nabla_k \psi. \qquad (3)$$

In relativistic quantum mechanics, the wave function is a Dirac spinor [1]. The commutator of covariant derivatives of a spinor ψ is given by

$$\nabla_i \nabla_j \psi - \nabla_j \nabla_i \psi = 2S^k{}_{ij} \nabla_k \psi + \frac{1}{4} R_{klij} \gamma^k \gamma^l \psi, \qquad (4)$$

where γ^k are the Dirac matrices in curved spacetime: $\gamma^{(i}\gamma^{j)} = g^{ij} I_4$ and I_4 is the unit 4×4 matrix [14, 15]. This commutator differs from (3) by the term with the curvature tensor R_{ijkl}.

III.   COMMUTATION RELATION FOR MOMENTUM FOUR-VECTOR

Since the momentum is defined in mechanics as a generator of a translation [21] and the translation is described in terms of the covariant derivative and parallel transport, the four-dimensional momentum operator in position space is related to the covariant derivative:

$$p_k = i\hbar \nabla_k. \qquad (5)$$

This relation generalizes the standard relations $E = i\hbar\frac{\partial}{\partial t}$ and $p_x = -i\hbar\frac{\partial}{\partial x}$. Combining equations (3) and (5) gives a commutation relation for the four-dimensional momentum operators when they act on a scalar wave function:

$$[p_i, p_j] = 2i\hbar S^k{}_{ij} p_k. \qquad (6)$$

This equation indicates that the four-dimensional momentum operators do not commute. Combining equations (4) and (5) gives a commutation relation for the four-dimensional momentum operators when they act on a spinor wave function:

$$[p_i, p_j] = 2i\hbar S^k_{\ ij} p_k I_4 + \frac{1}{4} i\hbar R_{klij} \gamma^k \gamma^l. \tag{7}$$

Since torsion and curvature are significant only at extremely high densities or energies and $\hbar$ is a small constant, the right-hand side of (7) is nearly zero at the scales currently available to experiment or observation, effectively reproducing the standard commutation relation $[p_i, p_j] = 0$.

For the Dirac fields, the spin tensor is completely antisymmetric [10, 14, 15], and so is the torsion tensor. Therefore, we can define the torsion pseudovector:

$$A^i = \frac{1}{6} \epsilon^{ijkl} S_{jkl}, \quad S_{ijk} = -\epsilon_{ijkl} A^l,$$

where ϵ^{ijkl} is the Levi-Civita pseudotensor [13]. If this pseudovector is timelike, one can find a coordinate frame in which the only nonzero component is the time component A^0. Let us illustrate the implications of the noncommutativity of the four-momentum in a special case, in which we can neglect the curvature tensor and consider spacetime that is locally flat. The commutation relation (6) for Cartesian coordinates gives

$$[p_x, p_y] = -2i\hbar A^0 p_z, \ \ [p_y, p_z] = -2i\hbar A^0 p_x, \ \ [p_z, p_x] = -2i\hbar A^0 p_y. \tag{8}$$

The momentum components commute with the square of the momentum, $\mathbf{p}^2$, and the energy component p_0, and $\mathbf{p}^2$ and p_0 commute too.

Equations (8) resemble the commutation relations for the angular momentum: $[L_x, L_y] = i\hbar L_z$, $[L_y, L_z] = i\hbar L_x$, $[L_z, L_x] = i\hbar L_y$ [12]. Those relations derive the separation between adjacent eigenvalues of L_z; that separation is $\hbar$. Analogously, the separation between adjacent eigenvalues of p_z is $2\hbar|A^0|$. Since torsion increases as the length scale decreases, it increases with momentum. The Cartan equations give a power-law dependence $A^0 \sim p^3$, where $p^2 = (p^0)^2 - \mathbf{p}^2$. This dependence results from torsion being proportional to the spin density which is proportional to the inverse cube of the length scale. Therefore, the separation between the eigenvalues of the momentum increases as the magnitude of the momentum vector increases. The integration over the momentum must be replaced by the summation over the discrete momentum eigenvalues. An analogous situation occurs in a noncommutative position space [22], where the integration involves considering the eigenstates. Since the eigenvalues are more separated as they increase, the resulting summation may converge even if the corresponding integral diverges. In Section V, we will demonstrate how typical, ultraviolet divergent integrals in Feynman diagrams of QED turn into convergent sums.

IV. INTEGRATION IN NONCOMMUTATIVE MOMENTUM SPACE

To find a prescription on how to integrate in the noncommutative momentum space, we look at the classical and quantum partition functions. In classical statistical physics, the canonical-ensemble partition function is given by the integration of the exponent of the Hamiltonian $H(q_i, p_i)$ of the system over phase space:

$$Z = \frac{1}{h^s} \int dq_1 \dots dq_s \int dp_1 \dots dp_s \, e^{-\frac{H(q,p)}{kT}}, \tag{9}$$

where $i = 1, \dots, s$ counts the degrees of freedom, q_i are the generalized coordinates, p_i are the generalized momenta, k is the Boltzmann constant, and T is the temperature of the system represented by the canonical ensemble [23]. In quantum statistical physics, the partition function is the sum of the exponential of the energies of the eigenstates:

$$Z = \sum_i e^{-\frac{E_i}{kT}}. \tag{10}$$

For example, for a one-dimensional harmonic oscillator with mass m and angular frequency ω, $H = p^2/(2m) + (1/2)m\omega^2 q^2$, and the classical partition function (the integration over q and p is from $-\infty$ to ∞) is $(kT)/(\hbar\omega)$. The corresponding quantum partition function, using the energy eigenvalues of the harmonic oscillator $E_i = \hbar\omega(i + 1/2)$ with integer i running from 0 to ∞, is $(1/2)/\sinh[(\hbar\omega)/(kT)]$. In the limit $\hbar \to 0$, the quantum partition function tends to its classical value.

The correspondence between the two partition functions is

$$\frac{1}{2\pi\hbar} \int dq \int dp \leftrightarrow \sum_{\text{eigenstates}},$$

or

$$\int dq \int dp \, f(H(q,p)) \leftrightarrow 2\pi \sum_{\text{eigenstates}} f(E) \, |[q,p]|. \tag{11}$$

To account for the quantum commutation relation between the integration variables q and p, which leads to a discrete spectrum of energy eigenstates, the integration over continuous phase space in the classical partition function must be replaced with the summation over the eigenstates. The quantity under the sum is multiplied by the absolute value of the commutator of the integration variables, $|[q,p]| = \hbar$, and gives a more accurate, quantum partition function.

We propose that for three spatial components n_i of a vector quantity $\mathbf{n}$, satisfying the cyclic commutation relations:

$$[n_x, n_y] = in_z, \quad [n_y, n_z] = in_x, \quad [n_z, n_x] = in_y, \tag{12}$$

the integration over the $\mathbf{n}$ space must be replaced with a summation over eigenstates:

$$\int dn_x \int dn_y \int dn_z \, f(\mathbf{n}^2) \to 4\pi \sum_{\text{eigenstates}} f(\mathbf{n}^2) \, |n_z|, \tag{13}$$

where $f(\mathbf{n}^2)$ is any scalar function of the square of the vector, $\mathbf{n}^2$. This prescription is analogous to the partition function (11): the integration over continuous space of n_x and n_y is replaced with the summation over the eigenstates, the quantity under the sum is multiplied by the absolute value of the commutator of the integration variables, $|[n_x, n_y]| = |n_z|$, and the integration over n_z is also replaced with the summation over the eigenstates.

The summation is over the eigenvalues of $\mathbf{n}$, which are given by $|\mathbf{n}| = \sqrt{l(l+1)}$ and $n_z = m$, where l is the orbital quantum number (a nonnegative integer) and m is the magnetic quantum number (an integer $m \in [-l, l]$). These quantum numbers follow from the quantum commutation relations (12) that are analogous to the commutation relations for the angular momentum. We obtain for the prescription (13):

$$\int dn_x \int dn_y \int dn_z \, f(\mathbf{n}^2) \to 4\pi \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f(\mathbf{n}^2) \, |m| = 4\pi \sum_{l=1}^{\infty} f(\mathbf{n}^2) \, l(l+1). \tag{14}$$

The summation over l is from 1 to ∞ since for $l = 0$ we have $m = 0$ and thus $n_z = 0$, which does not contribute to (13).

If the momentum operators act on a spinor wave function, we propose that $[p_x, p_y]$ contributes to the prescription (13) through its trace divided by 4. Accordingly, in the commutator (4), we can neglect the curvature term whose trace vanishes: $R_{klij} Tr(\gamma^k \gamma^l) = R_{klij} Tr(\gamma^{[k} \gamma^{l]}) = 4R_{klij} g^{[kl]} = 0$, and use (6) and then (13). This proposition is justified by the appearance of the trace in loop integrals in Feynman diagrams involving fermions [1].

In the noncommutative momentum space, we have the commutation relations (8). Defining

$$Q = -2\hbar A^0$$

gives

$$[p_x, p_y] = iQp_z, \quad [p_y, p_z] = iQp_x, \quad [p_z, p_x] = iQp_y. \tag{15}$$

Without loss of generality, we assume $Q > 0$ (the case $Q < 0$ for the right-handed coordinate system is equivalent to the case $Q > 0$ for the left-handed one). We introduce a vector

$$\mathbf{n} = \frac{\mathbf{p}}{Q} \tag{16}$$

and denote

$$n = |\mathbf{n}|.$$

This vector satisfies the commutation relations (12). In order to use the prescription (14), we replace the integration over the momentum space with the integration over the $\mathbf{n}$ space:

$$\int dp_x \int dp_y \int dp_z \, f(\mathbf{p}^2) \to \int dn_x \int dn_y \int dn_z J \, f(Q^2 n^2) \to 4\pi \sum_{l=1}^{\infty} J \, f(Q^2 n^2) \, l(l+1), \tag{17}$$

where $J = \partial(p_x, p_y, p_z)/\partial(n_x, n_y, n_z)$ is the Jacobian of the transformation from the components of $\mathbf{p}$ to the components of $\mathbf{n}$ and $f(\mathbf{p}^2)$ is any scalar function of $\mathbf{p}^2$.

As an example illustrating this correspondence, we consider a Gaussian function $f(\mathbf{p}^2) = e^{-k\mathbf{p}^2}$, where $k > 0$ is a constant, which integrated over the three components of the momentum gives $(\pi/k)^{3/2}$. If Q is constant then $J = Q^3$ and the integral $\int e^{-k\mathbf{p}^2} d\mathbf{p}$ turns into

$$\int e^{-kQ^2 n^2} Q^3 d\mathbf{n} \to 4\pi \sum_{l=1}^{\infty} e^{-kQ^2 l(l+1)} Q^3 l(l+1),$$

where we denote $d\mathbf{p} = dp_x dp_y dp_z$ and $d\mathbf{n} = dn_x dn_y dn_z$. This sum in the limit of continuous momentum space, $Q \to 0$, tends to $(\pi/k)^{3/2}$, which can be verified numerically or using the following limit:

$$\lim_{Q \to 0} 4\pi \sum_{l=1}^{\infty} e^{-kQ^2 l(l+1)} Q^3 l(l+1) = 4\pi \int_{\sqrt{2}}^{\infty} e^{-kQ^2 \eta^2} Q^3 \eta^2 d\eta = 4\pi \int_0^{\infty} e^{-k\zeta^2} \zeta^2 d\zeta, \tag{18}$$

$\zeta = Q\eta$. In this limit, the separation between adjacent values of l does not affect significantly the function f and thus the summation over l can be replaced with the integration over η with $\eta = \sqrt{l(l+1)}$.

Physically, Q depends on the momentum according to the Cartan equations and is proportional to p^3. In the example considered, we take $Q = U|\mathbf{p}|^3$, where $U > 0$ is a constant. Accordingly, we have $Q = Un^3 Q^3$ and thus $Q = U^{-1/2} n^{-3/2}$. Using $n^2 = n_x^2 + n_y^2 + n_z^2$, we find $\partial p_x/\partial n_x = \partial(Qn_x)/\partial n_x = U^{-1/2} n^{-7/2}(n^2 - 3n_x^2/2)$, $\partial p_x/\partial n_y = \partial(Qn_x)/\partial n_y = U^{-1/2} n^{-7/2}(-3n_x n_y/2)$, and similarly for other partial derivatives. The resulting Jacobian is $J = U^{-3/2} n^{-9/2}/2$ and the integral considered turns into

$$\int e^{-kQ^2 n^2} J \, d\mathbf{n} \to 2\pi U^{-3/2} \sum_{l=1}^{\infty} e^{-k/(Un)} n^{-9/2} l(l+1) = 2\pi U^{-3/2} \sum_{l=1}^{\infty} e^{-k/[U\sqrt{l(l+1)}]} [l(l+1)]^{-5/4}.$$

This sum in the limit of continuous momentum space, $U \to 0$, also tends to $(\pi/k)^{3/2}$, which can be verified numerically or using the limit (18):

$$\lim_{U \to 0} 2\pi U^{-3/2} \sum_{l=1}^{\infty} e^{-k/[U\sqrt{l(l+1)}]} [l(l+1)]^{-5/4} = 2\pi U^{-3/2} \int_{\sqrt{2}}^{\infty} e^{-k/(U\eta)} \eta^{-5/2} d\eta = 4\pi \int_0^{\infty} e^{-k\zeta^2} \zeta^2 d\zeta,$$

where $\zeta = 1/\sqrt{U\eta}$.

V. ELIMINATION OF ULTRAVIOLET DIVERGENCES

At every point in curved spacetime, one can construct a locally flat system of coordinates with the Minkowski metric tensor $g_{\mu\nu}$ [13]. In such a system, the local four-momentum representation of a Feynman propagator has the same form as in flat spacetime. For example, the Feynman propagator for a Dirac particle with four-momentum p_μ and mass m is $\tilde{S}_{\mathrm{F}}(p) = i(\gamma^\mu p_\mu + m)/(p^2 - m^2 + i\epsilon)$ and for a photon (in the Feynman gauge) is $\tilde{D}_{\mathrm{F}}^{\mu\nu}(p) = -ig^{\mu\nu}/(p^2 + i\epsilon)$, where γ^μ are the Dirac matrices in flat spacetime: $\gamma^{(\mu}\gamma^{\nu)} = g^{\mu\nu} I_4$, $p^2 = g_{\mu\nu} p^\mu p^\nu$, $\epsilon \to 0^+$, and hereinafter we use units such that $\hbar = 1$ [1]. For a Dirac particle, using (5) gives $(i\gamma^\mu \nabla_\mu - m)\tilde{S}_{\mathrm{F}}(p) = i$, where ∇_μ is a covariant derivative (that includes the torsion tensor which turns the Dirac equation into a nonlinear equation [19]). A Fourier transform of these propagators in the four-momentum representation, with the exponential factor $e^{-ip \cdot (x-y)}$, where $A \cdot B = g_{\mu\nu} A^\mu B^\nu$ and $A^2 = A \cdot A$, gives the position representation of a Feynman propagator ($S_{\mathrm{F}}(x-y)$ or $D_{\mathrm{F}}(x-y)$) describing a particle moving from one point x in spacetime to another point y that is infinitesimally close to x. In the Fourier transform, the integration over the four-momentum must be replaced with the summation over the four-momentum eigenvalues, according to the presented prescription. The curvature and torsion tensors appear through the commutation relation for the four-momentum (7). A propagator describing the motion between two points separated by a finite distance

can be constructed as a sequence of infinitesimal propagators using the normal coordinates, which depend on the curvature and torsion tensors [24].

A typical, logarithmically divergent integral in QED has a form $(2\pi)^{-4} \int d^4p/(p^2 - \mu^2 + i\epsilon)^2$, where μ does not depend on p. The integration is taken in the four-momentum space with the Lorentzian (pseudo-Euclidean) metric. Calculations are simplified by the Wick rotation: the time component of the momentum p^0 is replaced with ip_E^0 [1]. Accordingly, the integration is taken in the Euclidean four-momentum space, with p^2 replaced with $p_E^2 = (p^0)_E^2 + \mathbf{p}_E^2$. The integral becomes $i(2\pi)^{-4} \int d^4p_E/(p_E^2 + \mu^2)^2$.

Applying the Pauli-Villars regularization to this integral (and omitting the factor i and subscript E), one obtains [4]

$$\frac{1}{(2\pi)^4}\left[\int \frac{d^4p}{(p^2+\mu^2)^2} - \int \frac{d^4p}{(p^2+\Lambda^2)^2}\right] = \frac{1}{(2\pi)^4}(2\pi^2)\left[\int_0^\infty \frac{p^3 dp}{(p^2+\mu^2)^2} - \int_0^\infty \frac{p^3 dp}{(p^2+\Lambda^2)^2}\right]$$

$$= \frac{1}{16\pi^2}\left[\int_0^\infty \frac{x\,dx}{(x+\mu^2)^2} - \int_0^\infty \frac{x\,dx}{(x+\Lambda^2)^2}\right] = \frac{1}{16\pi^2}\left[\ln\frac{x+\mu^2}{x+\Lambda^2} + \frac{\mu^2}{x+\mu^2} - \frac{\Lambda^2}{x+\Lambda^2}\right]\Big|_0^\infty = -\frac{1}{16\pi^2}\ln\frac{\mu^2}{\Lambda^2}, \quad (19)$$

since the volume of the hypersurface of a four-sphere of radius p is $2\pi^2 p^3$. Applying the dimensional regularization to this integral gives [7]

$$\frac{1}{(2\pi)^n}\int \frac{d^n p}{(p^2+\mu^2)^2} = \frac{1}{8\pi^2\epsilon} - \frac{1}{16\pi^2}\left(\ln\frac{\mu^2}{4\pi} + \gamma\right) + O(\epsilon),$$

where γ is the Euler-Mascheroni constant. This result, in the limit $\epsilon \to 0$, is consistent with that in the Pauli-Villars regularization if $1/\epsilon + [\ln(4\pi) - \gamma]/2$ is identified with $\ln\Lambda$.

We now apply the torsional regularization to this integral, using the prescription (17). Introducing a vector $\mathbf{n} = \mathbf{p}/Q$ as in (16) and the Jacobian $J = \partial(p_x, p_y, p_z)/\partial(n_x, n_y, n_z)$, we obtain

$$\int \frac{d^4p}{(p^2+\mu^2)^2} = \int \frac{dp_0\,d\mathbf{p}}{(p^2+\mu^2)^2} = \int \frac{dp_0\,J\,d\mathbf{n}}{(p^2+\mu^2)^2} \to 4\pi \int_{-\infty}^\infty dp_0 \sum_{l=1}^\infty \frac{J}{(p^2+\mu^2)^2} l(l+1), \quad (20)$$

where $p^2 = p_0^2 + Q^2 n^2 = p_0^2 + Q^2 l(l+1)$. If Q is constant then $J = Q^3$ and the sum-integral in (20) is

$$4\pi Q^3 \sum_{l=1}^\infty \int_{-\infty}^\infty dp_0 \frac{l(l+1)}{[p_0^2 + Q^2 l(l+1) + \mu^2]^2} = 2\pi^2 Q^3 \sum_{l=1}^\infty \frac{l(l+1)}{[Q^2 l(l+1) + \mu^2]^{3/2}} = 2\pi^2 \sum_{l=1}^\infty \frac{l(l+1)}{[l(l+1)+\mu^2/Q^2]^{3/2}},$$

which diverges as $\sim \sum_{l=1}^\infty l^{-1}$.

However, the Cartan equations give

$$Q = Up^3,$$

where U is a constant whose unit is the same as that of the inverse squared momentum and whose value is on the order of the inverse squared Planck mass. Without loss of generality, we assume $U > 0$. We therefore have

$$p^2 = p_0^2 + U^2 n^2 p^6. \quad (21)$$

In order to find the Jacobian J of the transformation from $\mathbf{p}$ to $\mathbf{n}$, we proceed as follows. Differentiating (21) with respect to n_x gives $2p\frac{\partial p}{\partial n_x} = 6U^2 n^2 p^5 \frac{\partial p}{\partial n_x} + 2U^2 p^6 n_x$ and thus

$$\frac{\partial p}{\partial n_x} = \frac{U^2 p^5 n_x}{1 - 3U^2 n^2 p^4}.$$

Consequently, we find

$$\frac{\partial p_x}{\partial n_x} = \frac{\partial(Qn_x)}{\partial n_x} = Q + 3U n_x p^2 \frac{\partial p}{\partial n_x} = \frac{Q}{1 - 3U^2 n^2 p^4}[1 - 3U^2 p^4(n_y^2 + n_z^2)],$$

$$\frac{\partial p_x}{\partial n_y} = \frac{\partial(Qn_x)}{\partial n_y} = 3U n_x p^2 \frac{\partial p}{\partial n_y} = \frac{Q}{1 - 3U^2 n^2 p^4}(3U^2 p^4 n_x n_y),$$

and similarly for other components. The Jacobian is

$$J = \det \begin{pmatrix} \partial p_x/\partial n_x & \partial p_x/\partial n_y & \partial p_x/\partial n_z \\ \partial p_y/\partial n_x & \partial p_y/\partial n_y & \partial p_y/\partial n_z \\ \partial p_z/\partial n_x & \partial p_z/\partial n_y & \partial p_z/\partial n_z \end{pmatrix} = \frac{Q^3}{1 - 3U^2 n^2 p^4}.$$

Substituting J into the sum-integral in (20), using $dp_0/dp = (1 - 3U^2n^2p^4)/(1 - U^2n^2p^4)^{1/2}$, which results from (21), and taking the eigenvalues $n = \sqrt{l(l+1)}$ gives

$$4\pi \int_{-\infty}^{\infty} dp_0 \sum_{l=1}^{\infty} \frac{Q^3n^2}{(1 - 3U^2n^2p^4)(p^2 + \mu^2)^2} = 4\pi \int dp \frac{dp_0}{dp} \sum_{l=1}^{\infty} \frac{Q^3n^2}{(1 - 3U^2n^2p^4)(p^2 + \mu^2)^2}$$

$$= 4\pi \int_{-1/\sqrt{Un}}^{1/\sqrt{Un}} dp \sum_{l=1}^{\infty} \frac{Q^3n^2}{(1 - U^2n^2p^4)^{1/2}(p^2 + \mu^2)^2} = 8\pi \int_0^{1/\sqrt{Un}} dp \sum_{l=1}^{\infty} \frac{U^3p^9n^2}{(1 - U^2n^2p^4)^{1/2}(p^2 + \mu^2)^2}$$

$$= 8\pi \int_0^1 d\xi \sum_{l=1}^{\infty} \frac{U^3\xi^9n^2(Un)^{-5}}{(1 - \xi^4)^{1/2}[\xi^2/(Un) + \mu^2]^2} = 8\pi \int_0^1 d\xi \sum_{l=1}^{\infty} \frac{\xi^9n^{-1}}{(1 - \xi^4)^{1/2}[\xi^2 + U\mu^2n]^2}$$

$$= 4\pi \int_0^1 d\zeta \sum_{l=1}^{\infty} \frac{\zeta^4n^{-1}}{(1 - \zeta^2)^{1/2}[\zeta + U\mu^2n]^2} = 4\pi \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, n^{-1}}{[\sin\phi + U\mu^2n]^2}$$

$$= 4\pi \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, [l(l+1)]^{-1/2}}{[\sin\phi + U\mu^2\sqrt{l(l+1)}]^2}, \tag{22}$$

where we denote $Unp^2 = \xi^2 = \zeta = \sin\phi$.

The sum-integral (22) converges, which follows from its behavior at large values of l: $\sim \sum_{l=1}^{\infty} l^{-3}$. It depends only on the nondimensional quantity $U\mu^2$. Its value, divided by $(2\pi)^4$, gives the torsional-regularized, finite value of the integral $(2\pi)^{-4} \int d^4p_{\mathrm{E}}/(p_{\mathrm{E}}^2 + \mu^2)^2$ in the noncommutative momentum space resulting from torsion. We note that regularization is possible because Q appearing in the commutation relations for the momentum (15) is not constant but increases with the magnitude of the four-momentum. Consequently, torsional regularization works for a typical, logarithmically divergent integral in QED.

Now, we consider the limit of continuous four-momentum space for the sum-integral (22) using (18). If $U = 0$ (no torsion), this sum-integral diverges as $\sim \sum_{l=1}^{\infty} l^{-1}$, as expected. In the limit $U \to 0$, this sum-integral diverges as $\ln U$:

$$4\pi \int_{\sqrt{2}}^{\infty} d\eta \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, \eta^{-1}}{[\sin\phi + U\mu^2\eta]^2} = 4\pi \int_0^{\pi/2} d\phi\, \sin^4\phi \int_{\sqrt{2}U\mu^2}^{\infty} d\zeta \frac{\zeta^{-1}}{[\sin\phi + \zeta]^2}$$

$$= 4\pi \int_0^{\pi/2} d\phi\, \sin^2\phi \Big[\ln\frac{\zeta}{\zeta + \sin\phi} + \frac{\sin\phi}{\zeta + \sin\phi}\Big]\Big|_{\zeta = \sqrt{2}U\mu^2}^{\zeta \to \infty} = -4\pi \int_0^{\pi/2} d\phi\, \sin^2\phi[\ln(U\mu^2) + \ln\sqrt{2} - \ln\sin\phi + 1]$$

$$= -\pi^2 \ln(U\mu^2) + \text{finite terms},$$

where we denote $\zeta = U\mu^2\eta$. Its divergent part, divided by $(2\pi)^4$, is equal to $-1/(16\pi^2)\ln(U\mu^2)$, which is the same as the value in (19) if $U = 1/\Lambda^2$. Accordingly, Λ^2 in Pauli-Villars regularization can be identified with $1/U$. Consequently, the fictitious mass Λ is on the order of the Planck mass. This order of magnitude originates from the spin-torsion coupling in EC. Pauli-Villars regularization can thus be regarded as a mathematical technique that is equivalent to the realistic, torsional regularization.

We now generalize the prescription (22) to an integral of form $(2\pi)^{-4} \int d^4p/(p^2 - \mu^2 + i\epsilon)^s$, where s is an arbitrary positive integer. If $s = 1$ or $s = 2$ (which is the case considered previously), such an integral is divergent. If $s \geq 3$, such an integral has a finite value. In the Euclidean four-momentum space, this integral becomes $(-1)^s i(2\pi)^{-4} \int d^4p_{\mathrm{E}}/(p_{\mathrm{E}}^2 + \mu^2)^s$. Following the steps leading to (22), we obtain

$$\int \frac{d^4p}{(p^2 + \mu^2)^s} \to 8\pi \int_0^{1/\sqrt{Un}} dp \sum_{l=1}^{\infty} \frac{U^3p^9n^2}{(1 - U^2n^2p^4)^{1/2}(p^2 + \mu^2)^s}$$

$$= 8\pi \int_0^1 d\xi \sum_{l=1}^{\infty} \frac{U^3\xi^9n^2(Un)^{-5}}{(1 - \xi^4)^{1/2}[\xi^2/(Un) + \mu^2]^s} = 8\pi \int_0^1 d\xi \sum_{l=1}^{\infty} \frac{U^{s-2}\xi^9n^{s-3}}{(1 - \xi^4)^{1/2}[\xi^2 + U\mu^2n]^s}$$

$$= 4\pi \int_0^1 d\zeta \sum_{l=1}^{\infty} \frac{U^{s-2}\zeta^4n^{s-3}}{(1 - \zeta^2)^{1/2}[\zeta + U\mu^2n]^s} = 4\pi U^{s-2} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, n^{s-3}}{[\sin\phi + U\mu^2n]^s}$$

$$= 4\pi U^{s-2} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, [l(l+1)]^{(s-3)/2}}{[\sin\phi + U\mu^2\sqrt{l(l+1)}]^s}. \tag{23}$$

The sum-integral (23) converges, which follows from its behavior at large values of l: $\sim \sum_{l=1}^{\infty} l^{-3}$. Its value, divided by $(2\pi)^4$, gives the torsional-regularized, finite value of the integral $(2\pi)^{-4} \int d^4p_{\mathrm{E}}/(p_{\mathrm{E}}^2 + \mu^2)^s$ in the noncommutative

momentum space resulting from torsion. Consequently, torsional regularization works not only for logarithmically divergent integrals ($s = 2$), but also for quadratically divergent integrals ($s = 1$) that appear in QED.

To verify that the proposed regularization procedure reproduces the values of finite integrals in the limit of continuous four-momentum space ($U \to 0$), we apply it to the integral in (23) for $s = 3$, whose value is $\pi^2/(2\mu^2)$. We obtain

$$\int \frac{d^4p}{(p^2 + \mu^2)^3} \to 4\pi U \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4 \phi}{[\sin \phi + U\mu^2 \sqrt{l(l+1)}]^3}$$

$$\to 4\pi U \int_0^{\pi/2} d\phi \int_{\sqrt{2}}^{\infty} d\eta \frac{\sin^4 \phi}{[\sin \phi + U\mu^2 \eta]^3} = \frac{4\pi}{\mu^2} \int_0^{\pi/2} d\phi \int_0^{\infty} d\zeta \frac{\sin^4 \phi}{[\sin \phi + \zeta]^3} = \frac{4\pi}{\mu^2} \int_0^{\pi/2} d\phi \frac{\sin^2 \phi}{2} = \frac{\pi^2}{2\mu^2}.$$

VI. TENSOR INTEGRALS

The integrals considered in the preceding section are scalar integrals. We now consider a tensor integral $\int d^4 p_E \, p_E^\mu p_E^\nu / (p_E^2 + \Delta)^s$, where $\Delta > 0$ does not depend on p (and is equivalent to μ^2 is the preceding section). Such an integral appears in the Feynman diagram representing vacuum polarization [1]. We omit the subscript E and use the identity

$$\int d^4 p \frac{\partial}{\partial p_\nu} \Big(\frac{p^\mu}{(p^2 + \Delta)^s} \Big) = \int d^4 p \frac{\delta^{\mu\nu}}{(p^2 + \Delta)^s} - 2s \int d^4 p \frac{p^\mu p^\nu}{(p^2 + \Delta)^{s+1}},$$

where $\delta^{\mu\nu}$ is the metric tensor in the four-dimensional Euclidean space. The left-hand side of this equation can be transformed, Using the four-dimensional Gauss theorem, into a hypersurface integral that vanishes if the integration over $d^4 p$ has no boundaries (as in Feynman diagrams) and if the integrals on the right-hand side have finite values [25]. Consequently, this hypersurface integral should also vanish for regularized integrals. We therefore have

$$\int d^4 p \frac{p^\mu p^\nu}{(p^2 + \Delta)^s} = \frac{\delta^{\mu\nu}}{2(s-1)} \int d^4 p \frac{1}{(p^2 + \Delta)^{s-1}}. \tag{24}$$

Using (23) and (24), we obtain the torsional-regularized, finite values of typical integrals appearing in Feynman diagrams, in the noncommutative momentum space resulting from torsion:

$$\int \frac{d^4 p}{(p^2 + \Delta)^s} \to 4\pi U^{s-2} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4 \phi \, n^{s-3}}{[\sin \phi + U\Delta n]^s},$$

$$\int \frac{d^4 p \, p^\mu}{(p^2 + \Delta)^s} \to 0,$$

$$\int \frac{d^4 p \, p^\mu p^\nu}{(p^2 + \Delta)^s} \to \frac{2\pi U^{s-3} \delta^{\mu\nu}}{s - 1} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4 \phi \, n^{s-4}}{[\sin \phi + U\Delta n]^{s-1}}, \tag{25}$$

with $n = \sqrt{l(l+1)}$. The second integral in (25) is zero because of symmetry. Accordingly, the following integral vanishes:

$$\int d^4 p \frac{-2p^\mu p^\nu + (p^2 + \Delta)\delta^{\mu\nu}}{(p^2 + \Delta)^2} = -2 \int \frac{d^4 p \, p^\mu p^\nu}{(p^2 + \Delta)^2} + \delta^{\mu\nu} \int \frac{d^4 p}{p^2 + \Delta}$$

$$\to -4\pi U^{-1} \delta^{\mu\nu} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4 \phi \, n^{-2}}{\sin \phi + U\Delta n} + 4\pi U^{-1} \delta^{\mu\nu} \sum_{l=1}^{\infty} \int_0^{\pi/2} d\phi \frac{\sin^4 \phi \, n^{-2}}{\sin \phi + U\Delta n} = 0, \tag{26}$$

in accordance with (24). This integral also vanishes in dimensional regularization [7]. Since the loop integrals at each order of perturbation theory in QED are of form (25) (with more vector indices), torsional regularization gives finite results at all orders.

VII. VACUUM POLARIZATION

In this section, we apply torsional regularization to vacuum polarization (photon self-energy) in QED [3]. The aim of this calculation is a finite value of the bare charge of an electron. The determination of this quantity should elucidate the nature of the fine-structure constant.

The vacuum polarization tensor at one-loop order for a photon with four-momentum q creating an electron-positron bubble is given by [1]

$$
\begin{aligned}
i\Pi^{\mu\nu}_{\text{bubble}}(q) &= -e_0^2 \int \frac{d^4k}{(2\pi)^4} \text{Tr}\left[\gamma^\mu \frac{\gamma^\rho k_\rho + m}{k^2 - m^2 + i\epsilon} \gamma^\nu \frac{\gamma^\sigma (q+k)_\sigma + m}{(q+k)^2 - m^2 + i\epsilon}\right] \\
&= -4e_0^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{k^\mu(q+k)^\nu + k^\nu(q+k)^\mu - g^{\mu\nu}k\cdot(q+k) + m^2 g^{\mu\nu}}{(k^2 + 2q\cdot kx + q^2 x - m^2 + i\epsilon)^2} \\
&= -\frac{\alpha_0}{\pi^3} \int d^4p \int_0^1 dx \frac{2p^\mu p^\nu - p^2 g^{\mu\nu} + \Delta g^{\mu\nu} + 2(q^2 g^{\mu\nu} - q^\mu q^\nu)x(1-x)}{(p^2 - \Delta + i\epsilon)^2},
\end{aligned}
$$

where e_0 is the absolute value of the (bare) electric charge of an electron, m is the (dressed) mass of an electron, $\alpha_0 = e_0^2/(4\pi)$ is the (bare) fine-structure constant, $p = k + qx$, x is the Feynman parameter, and

$$
\Delta = m^2 - q^2 x(1-x).
$$

For $q^2 < 4m^2$, below the threshold for the production of an electron-positron pair, $\Delta > 0$. Applying the Wick rotation to this integral and using the identity (26) gives

$$
\begin{aligned}
\Pi^{\mu\nu}_{\text{bubble}}(q) &= -\frac{\alpha_0}{\pi^3} \int d^4 p_{\text{E}} \int_0^1 dx \frac{-2p^\mu_{\text{E}} p^\nu_{\text{E}} + p^2_{\text{E}}\delta^{\mu\nu} + \Delta\delta^{\mu\nu} + 2(q^2 g^{\mu\nu} - q^\mu q^\nu)x(1-x)}{(p^2_{\text{E}} + \Delta)^2} \\
&= -\frac{2\alpha_0}{\pi^3} \int d^4 p_{\text{E}} \int_0^1 dx \frac{x(1-x)}{(p^2_{\text{E}} + \Delta)^2}(q^2 g^{\mu\nu} - q^\mu q^\nu) = \Pi(q^2)q^2\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right),
\end{aligned}
\tag{27}
$$

where

$$
\Pi(q^2) = -\frac{2\alpha_0}{\pi^3} \int d^4 p_{\text{E}} \int_0^1 dx \frac{x(1-x)}{(p^2_{\text{E}} + \Delta)^2}.
\tag{28}
$$

The tensor (27) is transverse: $\Pi^{\mu\nu}_{\text{bubble}}(q)q_\nu = 0$, which follows from the Ward identity (conservation of charge). This tensor is thus gauge invariant and the photon remains massless [1].

We now apply torsional regularization to (28). Using the prescription (22), or for the first integral in (25) for $s = 2$, we obtain

$$
\begin{aligned}
\Pi(q^2) &\to -\frac{8\alpha_0}{\pi^2} \sum_{l=1}^\infty \int_0^1 dx \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, n^{-1} x(1-x)}{[\sin\phi + U\Delta n]^2} \\
&= -\frac{8\alpha_0}{\pi^2} \sum_{l=1}^\infty \int_0^1 dx \int_0^{\pi/2} d\phi \frac{\sin^4\phi\,[l(l+1)]^{-1/2} x(1-x)}{[\sin\phi + U\Delta\sqrt{l(l+1)}]^2}.
\end{aligned}
\tag{29}
$$

Consequently, $\Pi(q^2)$ has a finite value (if α_0 is finite) since the sum in (29) converges as $\sim \sum_{l=1}^\infty l^{-3}$. Vacuum polarization in QED at one-loop order, with torsional regularization, therefore gives a finite, gauge-invariant correction to the transverse part of the photon propagator.

The sum-integral (29) can be rewritten as

$$
\Pi(q^2) = \Pi(0) + \delta\Pi(q^2),
$$

where

$$
\Pi(0) = -\frac{8\alpha_0}{\pi^2} \sum_{l=1}^\infty \int_0^1 dx \int_0^{\pi/2} d\phi \frac{\sin^4\phi\, n^{-1} x(1-x)}{[\sin\phi + Um^2 n]^2}.
\tag{30}
$$

Summing over the one-particle-irreducible contributions Π to the photon propagator gives $1 + \Pi + \Pi^2 + \cdots = 1/(1-\Pi)$. Since $\delta\Pi(q^2)$ is proportional to α_0, $\Pi(0)$ can be absorbed into a renormalized (dressed) value of the fine-structure constant α (on-shell renormalization) [1, 5, 6]:

$$
\alpha_{\text{run}} = \frac{\alpha_0}{1 - \Pi(q^2)} = \frac{\alpha_0}{(1 - \Pi(0))\left(1 - \frac{\delta\Pi(q^2)}{1-\Pi(0)}\right)} = \frac{\alpha}{1 - \delta\Pi_{\text{R}}(q^2)},
$$

where

$$\alpha = \frac{\alpha_0}{1 - \Pi(0)}, \quad \Pi_{\mathrm{R}}(q^2) = \frac{\alpha}{\alpha_0}\Pi(q^2), \quad \delta\Pi_{\mathrm{R}}(q^2) = \frac{\alpha}{\alpha_0}\delta\Pi(q^2). \tag{31}$$

Subtracting $\Pi(0)$ from $\Pi(q^2)$ is turned into dividing α_0 by $1 - \Pi(0)$ [26], which was proved by Dyson to all orders of perturbation theory [5]. The running coupling in QED is therefore

$$\alpha_{\mathrm{run}} = \alpha\Big[1 + \frac{8\alpha}{\pi^2}\sum_{l=1}^{\infty}\int_0^1 dx\int_0^{\pi/2}d\phi\frac{\sin^4\phi\, n^{-1}x(1-x)}{[\sin\phi + U\Delta n]^2} - \frac{8\alpha}{\pi^2}\sum_{l=1}^{\infty}\int_0^1 dx\int_0^{\pi/2}d\phi\frac{\sin^4\phi\, n^{-1}x(1-x)}{[\sin\phi + Um^2 n]^2}\Big]^{-1}, \tag{32}$$

where $\alpha = \alpha_{\mathrm{run}}(q^2 = 0) \approx 1/137.036$ [27].

For a virtual photon in scattering of two charged particles, $q^2 < 0$. In the static limit, giving the one-loop quantum correction to the Coulomb potential, $q^0 = 0$ and $q^2 = -\mathbf{q}^2$, and thus $\Delta = m^2 + \mathbf{q}^2 x(1-x)$. As $\mathbf{q}^2$ increases, so does α_{run} in (32). There is no Landau pole if α_{run} stays finite as $\mathbf{q}^2$ increases to infinity. This condition is guaranteed if

$$-\Pi_{\mathrm{R}}(0) = \frac{8\alpha}{\pi^2}\sum_{l=1}^{\infty}\int_0^1 dx\int_0^{\pi/2}d\phi\frac{\sin^4\phi\, n^{-1}x(1-x)}{[\sin\phi + Um^2 n]^2} < 1. \tag{33}$$

The relations (30) and (31) give

$$\alpha_0 = \frac{\alpha}{1 + \Pi_{\mathrm{R}}(0)} = \alpha\Big[1 - \frac{8\alpha}{\pi^2}\sum_{l=1}^{\infty}\int_0^1 dx\int_0^{\pi/2}d\phi\frac{\sin^4\phi\, n^{-1}x(1-x)}{[\sin\phi + Um^2 n]^2}\Big]^{-1}.$$

Equivalently, the absolute values of the bare charge e_0 and the renormalized (observed) charge $e = \sqrt{4\pi\alpha}$ are related by

$$e_0 = \frac{e}{(1 + \Pi_{\mathrm{R}}(0))^{1/2}} = e\Big[1 - \frac{8\alpha}{\pi^2}\sum_{l=1}^{\infty}\int_0^1 dx\int_0^{\pi/2}d\phi\frac{\sin^4\phi\, n^{-1}x(1-x)}{[\sin\phi + Um^2 n]^2}\Big]^{-1/2}. \tag{34}$$

The bare charge is finite if the condition (33) is satisfied. So is the renormalization constant for the photon wave function [1]:

$$Z_3 = \frac{e^2}{e_0^2}.$$

In the low-energy limit (relative to the Planck energy), when $|q^2| \ll U^{-2}$, we can approximate in (29) the summation over l with the integration over $\eta = \sqrt{l(l+1)}$, as in (18):

$$\Pi(q^2) \approx -\frac{8\alpha_0}{\pi^2}\int_0^1 dx\int_0^{\pi/2}d\phi\int_{\sqrt{2}}^{\infty}d\eta\frac{\sin^4\phi\, \eta^{-1}x(1-x)}{[\sin\phi + U\Delta\eta]^2} = -\frac{8\alpha_0}{\pi^2}\int_0^1 dx\int_0^{\pi/2}d\phi\int_{\sqrt{2}U\Delta}^{\infty}d\zeta\frac{\sin^4\phi\, \zeta^{-1}x(1-x)}{[\sin\phi + \zeta]^2}$$

$$\approx -\frac{8\alpha_0}{\pi^2}\int_0^1 dx\, x(1-x)\int_0^{\pi/2}d\phi\,\sin^2\phi\Big(\ln\frac{\sin\phi}{\sqrt{2}U\Delta} - 1\Big)$$

$$= -\frac{8\alpha_0}{\pi^2}\int_0^1 dx\, x(1-x)\Big(N - \frac{\pi}{4}\ln(Um^2) - \frac{\pi}{4}(\ln\sqrt{2} + 1) - \frac{\pi}{4}\ln\frac{\Delta}{m^2}\Big)$$

$$= \frac{2\alpha_0}{\pi}\int_0^1 dx\, x(1-x)\ln\Big(1 - \frac{q^2}{m^2}x(1-x)\Big) + \frac{\alpha_0}{3\pi}\Big(\ln(Um^2) + \ln\sqrt{2} + 1 - \frac{4N}{\pi}\Big), \tag{35}$$

where $N = \int_0^{\pi/2}\sin^2\phi\,\ln\sin\phi\,d\phi \approx -0.151697$. Consequently,

$$\Pi_{\mathrm{R}}(0) \approx \frac{\alpha}{3\pi}\Big(\ln(Um^2) + \ln\sqrt{2} + 1 - \frac{4N}{\pi}\Big),$$

$$\delta\Pi_{\mathrm{R}}(q^2) \approx \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\Big(1 - \frac{q^2}{m^2}x(1-x)\Big). \tag{36}$$

The first formula in (36) is valid when $m^2 \ll U^{-2}$, which holds for all known charged fermions. The running coupling (32) for $q^2 = -\mathbf{q}^2$ is therefore approximated by the known formula [1]:

$$\alpha_{\text{run}} \approx \alpha\left[1 - \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left(1 + \frac{\mathbf{q}^2}{m^2}x(1-x)\right)\right]^{-1},$$

giving the Uehling potential that modifies the Coulomb potential $1/(4\pi r)$ at the one-loop order [28]:

$$V(r) = \int \frac{d\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}\frac{1}{\mathbf{q}^2(1 - \delta\Pi_{\text{R}}(-\mathbf{q}^2))}.$$

Finally, we determine the value of the bare charge of an electron. We take $U = m_{\text{P}}^{-2}$, where m_{P} is the Planck mass. If an electron were the only charged fermion, the first formula in (36) would give $\Pi_{\text{R}}(0) \approx -0.0786$, the condition (33) for the absence of a Landau pole would be satisfied and (34) would give $e_0 \approx 1.042\,e$. Taking the reduced Planck mass $m_{\text{P}}/\sqrt{8\pi}$ instead of the Planck mass in U would give $\Pi_{\text{R}}(0) \approx -0.0761$ and $e_0 \approx 1.040\,e$. We note that the exact value of U, as long as it is on the order of the inverse squared Planck mass, is not too significant. Including all charged fermions (denoted f) contributing to vacuum polarization replaces $\Pi_{\text{R}}(0)$ in (36) with

$$\Pi_{\text{R}}(0) \approx \sum_{\text{f}}\frac{\alpha}{3\pi}\left(\ln(Um_{\text{f}}^2) + \ln\sqrt{2} + 1 - \frac{4N}{\pi}\right)k_{\text{f}},$$

where k_{f} is the squared electric charge of a fermion (in the units of the charge of an electron). Substituting the masses and charges of quarks and charged leptons [27] gives $\Pi_{\text{R}}(0) \approx -0.328$, the condition (33) for the absence of a Landau pole is satisfied, and (34) gives

$$e_0 \approx 1.220\,e.$$

The corresponding renormalization constant for the photon wave function is $Z_3 = 0.672$, which is independent of α, in agreement with the behavior found in [6]. The corresponding bare fine-structure constant is $\alpha_0 \approx 1/92.1$. These results show that the bare charge of an electron (and thus α_0) and the renormalization constant for the photon wave function are finite. Consequently, including spin-torsion coupling naturally provides a physical regularization of ultraviolet divergences and makes the renormalization procedure finite: calculations involving the bare charge have a physical meaning. Accordingly, QED with torsion is ultraviolet complete.

The factor 1.220 is universal for all fermions: the bare charge of each fermion is approximately 1.220 times its effective (renormalized) charge. The finite screening of an electric charge by virtual particle-antiparticle pairs does not depend on its value. For example, an electron has bare charge $-1.22\,e$, a positron has bare charge $1.22\,e$, and a quark u has bare charge $1.22\,(2/3)e$. The bare charge of an electron is thus a constant of Nature and its exact determination should elucidate the origin of the fine-structure constant.

VIII. SUMMARY

In this article, we showed that the noncommutativity of the four-dimensional momentum, resulting from torsion coupled to spin, regularizes a typical, ultraviolet divergent integral that appears in loop corrections to Feynman diagrams in QED. In the noncommutative momentum space, the integration over the momentum components must be replaced with the summation over the momentum eigenvalues that form a discrete spectrum. Since torsion increases with the magnitude of the four-momentum, the separation between adjacent eigenvalues also increases. Consequently, the sum over the momentum eigenvalues converges. We derived a prescription for this summation that gives a correct continuous limit for convergent integrals. We extended this prescription to tensor integrals. We showed that ultraviolet-divergent integrals turn into convergent sums, naturally eliminating the ultraviolet divergence in loop diagrams at all orders.

We applied our prescription to vacuum polarization and derived a finite, gauge-invariant vacuum polarization tensor. We derived a finite running coupling that agrees with the low-energy limit of the standard QED. Finally, we found that the modification of the photon propagator arising from the loops involving all charged fermions give a finite value for the bare electric charge of a particle: it is approximately 1.22 times its measured, renormalized charge. We estimate that including the weak bosons W and Z contributing to vacuum polarization may change this value of the bare charge by a few percent, assuming that the spin-torsion coupling for the weak bosons is on the same order as that for fermions. Torsional regularization of the W and Z loops will be considered elsewhere.

These results show that the bare charge of an electron e_0 and the renormalization constant for the photon wave function are finite. The exact determination of the bare charge should elucidate the origin of the fine-structure constant, which is one of the unsolved problems in physics. Consequently, including spin-torsion coupling naturally provides a physical regularization of ultraviolet divergences and makes renormalization paradigm finite and thus mathematically self-consistent. Torsion may therefore be the physical source of a realistic regularization in QED (and in quantum field theory in general), eliminating the necessity of using auxiliary particles or varying the number of dimensions. A more extensive analysis of torsional regularization and determination of the constant U may use the quantum commutation relations between the metric and torsion tensors, derived from the Schwinger variational principle applied to the Einstein-Cartan action for the gravitational field [29].

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