Uncertainty relation for momentum with torsion

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We show that in the presence of the torsion tensor $S^k{}_{ij}$, the quantum commutation relation for the momentum is given by $[p_i, p_j] = 2i\hbar S^k{}_{ij} p_k$.

In nonrelativistic quantum mechanics, the operator $F(d\mathbf{x})$ of an infinitesimal translation by $d\mathbf{x}$ acts on a state $|\mathbf{x}\rangle$ according to [1]

$$F(d\mathbf{x})|\mathbf{x}\rangle = |\mathbf{x} + d\mathbf{x}\rangle.$$

Its unitarity requires that $F(d\mathbf{x}) = I - i\mathbf{K}\cdot d\mathbf{x}$, where I is the identity operator and $\mathbf{K}$ is a Hermitian operator. The correspondence between unitary transformations in quantum mechanics and canonical transformations in classical mechanics determines a proportionality relation between K and the momentum operator $\mathbf{p}$: $\mathbf{K} = \mathbf{p}/\hbar$. This relation introduces the Planck constant $\hbar = h/(2\pi)$. The commutator $[\mathbf{x}, F(d\mathbf{x})] = d\mathbf{x}$ gives the commutation relation between the position and momentum operators: $[x_j, p_k] = i\hbar\delta_{jk}$.

For the wave function representing a quantum state $|\alpha\rangle$ in position space, $\psi_\alpha(x) = \langle x|\alpha\rangle$, we have $(I - ip_x dx/\hbar)|\alpha\rangle = \int dx' F(dx)|x'\rangle\langle x'|\alpha\rangle = \int dx'|x' + dx\rangle\langle x'|\alpha\rangle = \int dx'|x'\rangle\langle x' - dx|\alpha\rangle = \int dx'|x'\rangle(\langle x'|\alpha\rangle - dx\frac{\partial}{\partial x'}\langle x'|\alpha\rangle)$, which gives $p_x|\alpha\rangle = \int dx'|x'\rangle(-i\hbar\frac{\partial}{\partial x'}\langle x'|\alpha\rangle)$ [1]. Consequently, $\langle x|p_x|\alpha\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\alpha\rangle)$ and the momentum operator in position space is a partial derivative with respect to the corresponding conjugate coordinate: $p_x = -i\hbar\frac{\partial}{\partial x}$. In flat spacetime, two infinitesimal translations in two different directions commute: $[F(d\mathbf{x}), F(d\mathbf{y})] = 0$. Consequently, the momentum operator components along different directions also commute, $[p_j, p_k] = 0$, which is consistent with the commutativity of partial derivatives. This relation is also valid in curved spacetime, where partial derivatives are generalized to covariant derivatives. In general relativity, the affine connection is given by the Christoffel symbols which are symmetric in the lower indices [2]. In curved space, covariant derivatives of the wave function (which is a scalar in nonrelativistic quantum mechanics) commute.

The conservation law for the total (orbital plus spin) angular momentum of fermions in curved spacetime, consistent with the Dirac equation, requires that the torsion tensor [3] is not constrained to zero, but is determined by varying the action with respect to torsion [4]. The torsion tensor is the antisymmetric part of the affine connection:

$$S^i{}_{jk} = \Gamma^{\ \,i}_{[jk]}.$$

The simplest and most natural theory of gravity with torsion is the Einstein-Cartan (EC) theory [5, 6], in which torsion becomes coupled to the spin of fermions and fermions are the source of torsion. The Cartan equation relating spin and torsion is given by

$$S^i{}_{jk} - S_j\delta^i_k + S_k\delta^i_j = -\frac{1}{2}\kappa s_{jk}{}^i,$$

where S_i is the torsion vector, $s_{jk}{}^i$ is the spin tensor of matter, and $\kappa = 8\pi G/c^4$ (we use the notation of [6]). This coupling generates gravitational repulsion at extremely high densities and thus avoids the formation of singularities in black holes and at the big bang [7]. The collapsing matter in a black hole bounces at a finite density and then expands into a new, finite region of space with positive curvature on the other side of the event horizon, which may be regarded as a new universe [8]. Quantum particle production caused by an extremely high curvature near a bounce (which replaces the big bang) creates enormous amounts of matter and entropy, and generates a finite period of exponential expansion (inflation) of this universe [9]. EC agrees with all solar system, binary pulsar and cosmological tests of general relativity, since even at nuclear densities, the corrections from torsion to the Einstein equations are negligible [5]. EC also modifies the Dirac equation, adding a term that is cubic in spinor fields [10]. That term may solve the problem of divergent integrals in quantum field theory by providing fermions with spatial extension (about 10^{-27} m for an electron) and thus introducing an effective ultraviolet cutoff for their propagators [11].

In the presence of torsion, the parallel transports (which define the covariant derivative) do not commute, which results from the following construction. The parallel transport of an infinitesimal, four-dimensional vector $\vec{PR} = dx^i$

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from a point P to an infinitesimally close point Q such that $\vec{PQ} = dx'^j$ adds to dx^i a small correction:

$$\delta dx^i = -\Gamma^{\,i}_{jk} dx^j dx'^k.$$

After effecting the transport, the vector $dx^i + \delta dx^i$ points to a point T. The parallel transport of the vector dx'^i from a point P to an infinitesimally close point R adds to dx^i a small correction:

$$\delta dx'^i = -\Gamma^{\,i}_{jk} dx^j dx'^k.$$

After effecting the transport, the vector $dx'^i + \delta dx'^i$ points to a point T'. Without torsion, points T and T' would coincide and form, together with points P, Q, and R, a parallelogram because $\delta dx'^i - \delta dx^i = \Gamma^{\,i}_{kj} dx^j dx'^k - \Gamma^{\,i}_{jk} dx^j dx'^k = 0$. If the torsion tensor is not zero, however, the affine connection is asymmetric in the lower indices and

$$\delta dx'^i - \delta dx^i = -S^i_{\ jk} dx^j dx'^k.$$

Points T and T' do not coincide, the parallogram is not closed, and the combination of two displacements of point P (through dx^i and dx'^j) depends on their order. Accordingly, covariant derivatives of a scalar ψ do not commute:

$$\nabla_i \nabla_j \psi - \nabla_j \nabla_i \psi = \partial_i \nabla_j \psi - \Gamma^{\,k}_{ji} \nabla_k \psi - \partial_j \nabla_i \psi + \Gamma^{\,k}_{ij} \nabla_k \psi = \partial_i \partial_j \psi - \partial_j \partial_i \psi + 2S^k_{\ ij} \nabla_k \psi = 2S^k_{\ ij} \nabla_k \psi. \tag{1}$$

Since the momentum is defined in mechanics as a generator of a translation [12] and the translation is described in terms of the covariant derivative and parallel transport, the four-dimensional momentum operator in position space is related to the covariant derivative:

$$p_k = i\hbar \nabla_k. \tag{2}$$

This relation generalizes the standard relations $E = i\hbar\frac{\partial}{\partial t}$ and $p_x = -i\hbar\frac{\partial}{\partial x}$. Combining equations (1) and (2) gives

$$[p_i, p_j] = 2i\hbar S^k_{\ ij} p_k. \tag{3}$$

This equation indicates that the four-dimensional momentum operators do not commute.

Since torsion is significant only at extremely high densities or energies (probing the distances on the order of 10^{-27} m and smaller), the right-hand side of (3) is nearly zero at the scales currently available, effectively reproducing the standard commutation relation. However, the integration in momentum space to calculate radiative corrections to the photon and electron propagators in Feynman diagrams in quantum field theory [13] may be affected. Currently, such integration (to infinity) involves divergent integrals that are treated by regularization. We argue below that at larger energies and momenta, the noncommutativity of the four-dimensional momentum (3) should affect the integration, as it does for position space [14], and eliminate the divergence of radiative corrections and the necessity of regularization.

For the Dirac fields, the spin tensor is completely antisymmetric, and so is the torsion tensor. Therefore, we can define the torsion pseudovector:

$$A^i = \epsilon^{ijkl} S_{jkl},$$

where ϵ^{ijkl} is the Levi-Civita pseudotensor (in spacetime where the metric tensor is locally flat). If this pseudovector is timelike, one can find a coordinate frame in which the only nonzero component is the time component A^0. The commutation relation (3) becomes

$$[p_x, p_y] = -2i\hbar A^0 p_z, \ \ [p_y, p_z] = -2i\hbar A^0 p_x, \ \ [p_z, p_x] = -2i\hbar A^0 p_y,$$

resembling the commutation relations for the angular momentum: $[L_x, L_y] = i\hbar L_z]$, etc. [1]. Those relations derive the separation between adjacent eigenvalues of L_z; this separation is $\hbar$. The momentum components commute with $\mathbf{p}^2$ and the energy component p_0, and $\mathbf{p}^2$ and p_0 commute too. It is reasonable to assume that A^0 is an increasing function of $p = \sqrt{\mathbf{p}^2}$. Defining $C = -2\hbar A^0 = kp^\alpha$, where C, k, and $\alpha > 0$ are constants, $n_x = p_x/C$ (similarly for y and z), and $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$ gives

$$\int \frac{d^4p}{p^4} \sim \int \frac{dp}{p} \sim \sum \frac{1}{p} \sim \sum_{n=1}^{\infty} \frac{1}{Cn} = \sum_{n=1}^{\infty} (nk)^{\frac{1}{\alpha-1}}.$$

The last sum is convergent for $0 < \alpha < 1$, showing that the logarithmically divergent integrals of form $\int dp^4/p^4$ that appear in radiative corrections in quantum electrodynamics should become convergent (in this range of α) if the noncommutativity of the momentum is taken into account. The noncommutativity of the momentum, resulting from torsion, regularizes those integrals. The value of α could be determined from the commutation relations for the torsion tensor, which we started calculating in [15].

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