Application of covariant analytic mechanics with differential forms to gravity with Dirac field

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(Dated: November 2, 2015)

We apply the covariant analytic mechanics with the differential forms to the Dirac field and the gravity with the Dirac field. The covariant analytic mechanics treats space and time on an equal footing regarding the differential forms as the basis variables. A significant feature of the covariant analytic mechanics is that the canonical equations, in addition to the Euler-Lagrange equation, are not only manifestly general coordinate covariant but also gauge covariant. Combining our study and the previous works (the scalar field, the abelian and non-abelian gauge fields and the gravity without the Dirac field), the applicability of the covariant analytic mechanics is checked for all fundamental fields. We study both the first and second order formalism of the gravitational field coupled with matters including the Dirac field. Although the first order formalism does not go well for the Hamilton formalism, the second order formalism can be successfully treated within the framework. It is suggested that the covariant analytic mechanics can not treat gravitation theories including higher order curvatures.

PACS numbers: 02.40.-k; 03.50.-z; 04.20.-q; 04.20.Fy; 11.10.Ef; 11.15.-q

I. INTRODUCTION

In the traditional analytic mechanics, the Hamilton formalism gives especial weight to time. Then, the Lorentz covariance is not trivial. Moreover, for the constrained system, for instance the gauge field, the gauge fixing or the Dirac's theory is needed. The de Donder-Weyl theory solves the former problem[1,2]. In this theory, the conjugate momentums of a field ψ are introduced as $\pi^\mu = \partial\mathcal{L}/\partial(\partial_\mu\psi)$ ($\mu = 0, 1, 2, 3$), where $\mathcal{L}$ is the Lagrangian density. π^0 is the traditional conjugate momentum. The generalization of the Hamiltonian density is given by $\mathcal{H}(\psi, \pi^\mu) = \partial_\mu\psi\pi^\mu - \mathcal{L}$. The common property of the traditional analytic mechanics and the de Donder-Weyl theory is that its basic variables of the variation are components of the tensor.

By the way, the Lagrange formalism is sometimes formulated by the differential forms[3–7], in which the basic variables are the differential forms. Because the differential form is independent of the coordinate system, the general coordinate covariance is guaranteed manifestly. And this formulation often largely reduces the cost of the calculation. Nakamura[8] generalized this formulation to the Hamilton formalism. In this method, the conjugate momentum is also a differential form, which treats space and time on an equal footing. Nakamura applied this method to the Proca field, the electromagnetic field with manifest covariance and, in the latter, with the gauge covariance. The conjugate momentum form becomes independent degree of freedoms. Kaminaga[9] formulated strictly mathematically Nakamura's idea and constructed the general theory in arbitrary dimension. We called this theory as the *covariant analytic mechanics*. Kaminaga studied that the Newtonian mechanics of a harmonic oscillator ($(1 + 0)$ dimension) and the scalar field, the abelian and non-abelian gauge fields and 4 dimension gravity without the Dirac field. The gravitational field was formulated by the second order formalism, in which the basic variable is only the frame (vielbein). And the absence of the torsion was assumed. On the other hand, Nester[10] also investigated independently the covariant analytic mechanics and constructed the general theory in 4 dimension and applied it to the Proca field, the electromagnetic field and the non-abelian gauge field. As we will explain in V A, the treatment of the gravitational field of Nester was not complete Hamilton formalism although the conjugate momentum forms of the frame and the connection were introduced. The original idea was mentioned in Ref.[11].

We investigate the Kaminaga's study. We apply the covariant analytic mechanics to the Dirac field and the gravity with the Dirac field for the first time. Combining our study and the previous works, the applicability of the covariant analytic mechanics is checked for all fundamental fields. Although the first order formalism of the gravitation does not go well for the Hamilton formalism, the second order formalism can be successfully treated. It is suggested that the covariant analytic mechanics can not treat gravitation theories including higher order curvatures.

In II, we review the covariant analytic mechanics with the application to the electromagnetic field and introduce the Poisson bracket for first time. In III, we explain the several notations and in IV we study the Dirac field. In V, we study the gravitational field coupled with matters including the Dirac field. First, we discuss the first order formalism, in which both the frame and the connection are basic variables (V A). Next, we move to the second order formalism. In V B, the Lagrange formalism is studied and we show that the Lagrange form of the pure gravity is given by subtracting the total differential term from the Einstein-Hilbert form. We discuss that if we do not drop the total differential term, it is probably impossible to derive the correct equations. In V C, we move to the Hamilton formalism and give a broad overview of the remainder discussion. In V D, we take the derivatives of the Hamilton form of the pure gravity using specialties of 4 dimension system and in V E, we discuss the canonical equations.

II. COVARIANT ANALYTIC MECHANICS

A. General theory

Let us consider D dimension space (pseudo-Riemannian or Riemannian space). Suppose a differential p-form β ($p = 0, 1, \cdots, D$) is a function of differential forms $\{\alpha^i\}_{i=1,\cdots,k}$. If there exists the differential form ω_i such that β behaves under variations $\delta\alpha^i$ as

$$\delta\beta = \sum_i \delta\alpha^i \wedge \omega_i, \tag{1}$$

we call ω_i the *derivative* of β by α^i and denote

$$\frac{\partial\beta}{\partial\alpha^i} \stackrel{\text{def}}{=} \omega_i, \tag{2}$$

namely, $\delta\beta \equiv \sum_i \delta\alpha^i \wedge \frac{\partial\beta}{\partial\alpha^i}$. If α^i is $q_i(\leq p)$-form, $\partial\beta/\partial\alpha^i$ is $(p - q_i)$-form.

As the traditional analytic mechanics starts from the Lagrangian density, the covariant analytic mechanics starts from *Lagrange D-form L*. L is a function of ψ and $d\psi$, $L = L(\psi, d\psi)$, where ψ is a set the differential forms. For simplicity, we treat ψ as single p-form. The variation of L is given by

$$\delta L = \delta\psi \wedge \frac{\partial L}{\partial\psi} + \delta d\psi \wedge \frac{\partial L}{\partial d\psi}. \tag{3}$$

Since the second term of RHS can be rewritten as

$$\delta d\psi \wedge \frac{\partial L}{\partial d\psi} = d\Big(\delta\psi \wedge \frac{\partial L}{\partial d\psi}\Big) - (-1)^p \delta\psi \wedge d\frac{\partial L}{\partial d\psi}, \tag{4}$$

we obtain

$$\delta L = \delta\psi \wedge \Big(\frac{\partial L}{\partial\psi} - (-1)^p d\frac{\partial L}{\partial d\psi}\Big) + d\Big(\delta\psi \wedge \frac{\partial L}{\partial d\psi}\Big). \tag{5}$$

Hence, the *Euler-Lagrange equation* is

$$\frac{\partial L}{\partial\psi} - (-1)^p d\frac{\partial L}{\partial d\psi} = 0. \tag{6}$$

We define the *conjugate momentum form* π as

$$\pi \stackrel{\text{def}}{=} \frac{\partial L}{\partial d\psi}. \tag{7}$$

π is $D - p - 1 = q$-form. The *Hamilton D-form* (not $(D-1)$-form) is defined by

$$H = H(\psi, \pi) \stackrel{\text{def}}{=} d\psi \wedge \pi - L, \tag{8}$$

as a function of ψ and π. The variation of H is given by

$$\delta H = (-1)^{(p+1)q} \delta\pi \wedge d\psi - \delta\psi \wedge \frac{\partial L}{\partial\psi}. \tag{9}$$

Since LHS can be written as

$$\delta H \equiv \delta\psi \wedge \frac{\partial H}{\partial\psi} + \delta\pi \wedge \frac{\partial H}{\partial\pi}, \tag{10}$$

we obtain

$$\frac{\partial H}{\partial \psi} = -\frac{\partial L}{\partial \psi}\,, \quad \frac{\partial H}{\partial \pi} = \varepsilon_{p,D} d\psi, \tag{11}$$

where $\varepsilon_{p,D} \overset{\text{def}}{=} (-1)^{(p+1)q} = 1$ if p is an odd number and $\varepsilon_{p,D} = -(-1)^D$ if p is an even number. By substituting the Euler-Lagrange equation (6), we obtain the *canonical equations*[9]

$$d\psi = \varepsilon_{p,D}\frac{\partial H}{\partial \pi}\,, \quad d\pi = -(-1)^p \frac{\partial H}{\partial \psi}. \tag{12}$$

Now we introduce the *Poisson bracket* by

$$\{A, B\} \overset{\text{def}}{=} \varepsilon_{p,D}\frac{\partial A}{\partial \psi} \wedge \frac{\partial B}{\partial \pi} - (-1)^p \frac{\partial A}{\partial \pi} \wedge \frac{\partial B}{\partial \psi}. \tag{13}$$

Then, the canonical equations can be written as

$$d\psi = \{\psi, H\}\,, \quad d\pi = \{\pi, H\}. \tag{14}$$

And we have

$$\{\psi, \pi\} = \varepsilon_{p,D}\,, \quad \{\pi, \psi\} = -(-1)^p. \tag{15}$$

The applicability of the Poisson bracket to the quantization is unclear. And the generalization of the canonical transform theory have not been studied.

Let consider D dimension space-time, which has the metric $g_{\mu\nu}$. The Hodge operator $*$ maps an arbitrary p-form $\omega = \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ $(p = 0, 1, \cdots, D)$ to $D - p = r$-form as

$$*\,\omega = \frac{1}{r!} E_{\nu_1 \cdots \nu_r}{}^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_r}. \tag{16}$$

Here, $E_{\mu_1 \cdots \mu_D}$ is the complete anti-symmetric tensor such that $E_{01 \cdots D-1} = \sqrt{-g}$ $(g = \det g_{\mu\nu})$. And $**\,\omega = -(-1)^{p(D-p)}\omega$ holds. In particular, $\Omega = *1$ is the volume form. The Lagrange form L relates to the Lagrangian density $\mathcal{L}$ as $L = \mathcal{L}\Omega$. In the following of this section, we set $D = 4$. Then, $**\,\omega = -(-1)^p\omega$ holds. The Lagrangian density $\mathcal{L}_{\mathcal{D}}$ corresponding to a Lagrange form $d\mathcal{D}$ is given by $\mathcal{L}_{\mathcal{D}}\sqrt{-g} = \partial_\mu(\sqrt{-g}d^\mu)$, where $\mathcal{D}$ is a 3-form $\mathcal{D} = d_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$ and $d^\mu = d_{\alpha\beta\gamma}E^{\alpha\beta\gamma\mu}$. If $\mathcal{D}$ dose not include $d\psi$, $d\mathcal{D}$ dose not contribute to the Euler-Lagrange equation. We will discuss an instance of $\mathcal{D}$ including $d\psi$ in V B. The Lagrange form corresponding to a Lagrangian density $a_\mu b^\mu$ is given by $*a \wedge b$ with $a = a_\mu dx^\mu$ and $b = b_\mu dx^\mu$. And the Lagrange form corresponding to a Lagrangian density $\frac{1}{2}c_{\mu\nu}d^{\mu\nu}$ is given by $*c \wedge d = c \wedge *d$ with $c = \frac{1}{2}c_{\mu\nu}dx^\mu \wedge dx^\nu$ and $d = \frac{1}{2}d_{\mu\nu}dx^\mu \wedge dx^\nu$.

B. Electromagnetic field

The Lagrange form of the electromagnetic field is given by

$$L = L(A, dA) = -\frac{1}{2}F \wedge *F + J \wedge A = \mathcal{L}\Omega, \tag{17}$$

where

$$\mathcal{L} = \mathcal{L}(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu, \tag{18}$$

$A = A_\mu dx^\mu$, $J = *(J_\mu dx^\mu)$ and $F = dA = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. A_μ is the vector potential and J^μ is the current density, which is independent of A_μ. We obtain $\partial L/\partial A = -J$ and $\partial L/\partial dA = -*F$ using $\delta * F = *\delta F$. The Euler-Lagrange equation $\partial L/\partial A + d(\partial L/\partial dA) = 0$ is

$$d * F = -J. \tag{19}$$

This equation and an identity $dF = 0$ are the Maxwell equations.

In contrast to that the basis variables of the variation of the traditional analytic mechanics are components of the tensor, A_μ, the basis variable (reality) is the differential form $A = A_\mu dx^\mu$ in the covariant analytic mechanics. In general, the Lagrange formalism is equivalent to the traditional one. However, as we show just after, the Hamilton formalism is not equivalent to the traditional one.

In the traditional analytic mechanics, the definition of the conjugate momentum, $\Pi^\mu = \partial\mathcal{L}/\partial(\partial_0 A_\mu) = -F^{0\mu}$, gives especial weight to time. Then, the Lorentz covariance is not trivial. Moreover, because $\Pi^0 = 0$, this system is a constrained system, which needs to the gauge fixing or the Dirac's theory (Dirac bracket). In contrast, the Hamilton formalism of the covariant analytic mechanics is manifestly Lorentz covariant since the differential forms are independent of the coordinate system. Moreover, the conjugate momentum form, $\pi = \partial L/\partial dA = - * F$, can represent dA as $dA = F = *\pi$. So, the gauge fixing or the Dirac's theory are not needed. This formulation is gauge free. The position variable is a 1-form, which has 4 components, and the conjugate momentum variable is a 2-from, which has 6 components (electric and magnetic fields).

The Hamilton form is given by

$$H(A,\pi) = \frac{1}{2}\pi \wedge *\pi - J \wedge A. \tag{20}$$

We have $\partial H/\partial\pi = *\pi$ and $\partial H/\partial A = J$. The canonical equations $dA = \partial H/\partial\pi$ and $d\pi = \partial H/\partial A$ are

$$dA = *\pi\,, \quad d\pi = J. \tag{21}$$

The former is equivalent to the definition of the conjugate momentum form and the latter coincides with the Euler-Lagrange equation (19).

III. NOTATION

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Let g be the metric of which signature is $(-+\cdots+)$, and let (θ^a) denote an orthonormal frame. θ^a can be expanded as $\theta^a = \theta^a{}_\mu dx^\mu$ with the vielbein $\theta^a{}_\mu$. We have $g = \eta_{ab}\theta^a \otimes \theta^b$ with $\eta_{ab} = (-+\cdots+)$. We put $e^a = *\theta^a$, $e^{ab} = *(\theta^a \wedge \theta^b)$ and $e^{abc} = *(\theta^a \wedge \theta^b \wedge \theta^c)$. Let $\omega^a{}_b$ and $w^a{}_b$ respectively be the connection and the Levi-Cività connection 1-form. All indices are lowered and raised with η_{ab} or its inverse η^{ab}. Then, $\omega_{ba} = -\omega_{ab}$ and the first structure equation

$$d\theta^a + \omega^a{}_b \wedge \theta^b = \Theta^a, \tag{22}$$

hold. Here, $\Theta^a = \frac{1}{2}C^a{}_{bc}\theta^b \wedge \theta^c$ is the torsion 2-form. From (22), we obtain

$$\begin{aligned} &\omega_{abc} = w_{abc} + \tilde{\omega}_{abc}, \\ &w_{abc} = \frac{1}{2}(\Delta_{cba} + \Delta_{abc} + \Delta_{bca})\,, \quad \tilde{\omega}_{abc} = -\frac{1}{2}(C_{cba} + C_{abc} + C_{bca}). \end{aligned} \tag{23}$$

Here, we expanded $d\theta^a$ and ω_{ab} as $d\theta^a = \frac{1}{2}\Delta^a{}_{bc}\theta^b \wedge \theta^c$ and $\omega_{ab} = \omega_{abc}\theta^c$. We have $w_{ab} = w_{abc}\theta^c$ and define $\tilde{\omega}_{ab} \overset{\text{def}}{=} \tilde{\omega}_{abc}\theta^c$, $\omega_a \overset{\text{def}}{=} \omega^b{}_{ab}$ and $C_a \overset{\text{def}}{=} C^b{}_{ab}$. In Appendix A, several identities about $\theta^a \wedge e_{a_1\cdots a_r}$, $de_{a_1\cdots a_r}$ $(r = 1,2,3)$ and $\delta e_{a_1\cdots a_r} (r = 0,1,2)$ are listed. The curvature 2-form $\Omega^a{}_b$ is given by $\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$. Expanding the curvature form as $\Omega^a{}_b = \frac{1}{2}R^a{}_{bcd}\theta^c \wedge \theta^d$, we define $R_{ab} \overset{\text{def}}{=} R^c{}_{acb}$ and $R \overset{\text{def}}{=} R^a{}_a$.

In the following, we set $D = 4$. However, up to V C, the dimension dependence appears only in the sign factor of exchanging differential forms. After V D, we use specialties of $D = 4$.

IV. DIRAC FIELD

A. Lagrange formalism

The Lagrange form of the Dirac field ψ is given by

$$L_{\mathrm{D}}(\psi,\bar{\psi},d\psi,d\bar{\psi}) = -\frac{1}{2}\bar{\psi}\gamma_c e^c \wedge (d\psi + \frac{1}{4}\gamma_{ab}\omega^{ab}\psi) + \frac{1}{2}e^c \wedge (d\bar{\psi} - \frac{1}{4}\bar{\psi}\gamma_{ab}\omega^{ab})\gamma_c\psi - m\bar{\psi}\psi\Omega, \tag{24}$$

with $\bar{\psi} = i\psi^\dagger\gamma^0$ and $\gamma_{ab} \overset{\text{def}}{=} \gamma_{[a}\gamma_{b]}$. Here, γ^a is the gamma matrix, which satisfies $\gamma^{(a}\gamma^{b)} = \eta^{ab}$. ( ) and [ ] are respectively the symmetrization and anti-symmetrization symbols. It is important that the frame (vielbein) is necessary to write down the Lagrange form even in the flat space-time. This is because that the Dirac field is a representation of the Lorentz transformation (not of the coordinate transformation) and a spinor field is defined in the tangent Minkowski space. The connection ω^{ab} is the gauge field for the local Lorentz transformations. Subtracting the total differential term $-\frac{1}{2}d(e^a\bar{\psi}\gamma_a\psi)$ from (24), and using

$$\begin{aligned}\gamma_c e^c \wedge \frac{1}{4}\gamma_{ab}\omega^{ab} &= \frac{1}{4}e^c \wedge \gamma_{ab}\omega^{ab}\gamma_c + e^c \wedge \frac{1}{4}[\gamma_c, \gamma_{ab}]\omega^{ab} \\ &= \frac{1}{4}e^c \wedge \gamma_{ab}\omega^{ab}\gamma_c + \gamma^a\omega_a\Omega, \end{aligned} \tag{25}$$

and (A11), we obtain

$$L'_{\mathrm{D}}(\psi, \bar{\psi}, d\psi, d\bar{\psi}) = -\bar{\psi}\gamma_c e^c \wedge (d + \frac{1}{4}\gamma_{ab}\omega^{ab})\psi - m\bar{\psi}\psi\Omega - \frac{1}{2}C_a\bar{\psi}\gamma^a\psi\Omega. \tag{26}$$

In the second line of (25), we used $\frac{1}{4}[\gamma_c, \gamma_{ab}] = \frac{1}{2}(-\eta_{bc}\gamma_a + \eta_{ac}\gamma_b)$ and $e_b \wedge \omega_a{}^b = \omega_a\Omega$. C_a is regarded as independent of ψ and $\bar{\psi}$. As we will show in (60), $C_a = 0$ is required. For simplicity, we treat the Dirac field as a usual number (not the Grassmann number).

From the variation by $\bar{\psi}$, we obtain

$$\frac{\partial L'_{\mathrm{D}}}{\partial \bar{\psi}} = -\gamma_c e^c \wedge (d + \frac{1}{4}\gamma_{ab}\omega^{ab})\psi - m\psi\Omega - \frac{1}{2}C_a\gamma^a\psi\Omega\,, \qquad \frac{\partial L'_{\mathrm{D}}}{\partial d\bar{\psi}} = 0. \tag{27}$$

The Euler-Lagrange equation $\partial L'_{\mathrm{D}}/\partial\bar{\psi} - d(\partial L'_{\mathrm{D}}/\partial d\bar{\psi}) = 0$ is given by

$$\gamma_c e^c \wedge (d + \frac{1}{4}\gamma_{ab}\omega^{ab})\psi + m\psi\Omega + \frac{1}{2}C_a\gamma^a\psi\Omega = 0. \tag{28}$$

This is equivalent to the Dirac equation. From the variation by ψ, we obtain

$$\frac{\partial L'_{\mathrm{D}}}{\partial \psi} = -\bar{\psi}\gamma_c e^c \wedge \frac{1}{4}\gamma_{ab}\omega^{ab} - m\bar{\psi}\Omega - \frac{1}{2}C_a\bar{\psi}\gamma^a\Omega\,, \qquad \frac{\partial L'_{\mathrm{D}}}{\partial d\psi} = \bar{\psi}\gamma_a e^a. \tag{29}$$

The Euler-Lagrange equation $\partial L'_{\mathrm{D}}/\partial\psi - d(\partial L'_{\mathrm{D}}/\partial d\psi) = 0$ is

$$\bar{\psi}\gamma_c e^c \wedge \frac{1}{4}\gamma_{ab}\omega^{ab} + m\bar{\psi}\Omega + \frac{1}{2}C_a\bar{\psi}\gamma^a\Omega + d\bar{\psi} \wedge \gamma_a e^a + \bar{\psi}\gamma_a de^a = 0. \tag{30}$$

Using (25) and (A11), (30) becomes

$$(d\bar{\psi} - \frac{1}{4}\bar{\psi}\gamma_{ab}\omega^{ab}) \wedge \gamma_c e^c + m\bar{\psi}\Omega - \frac{1}{2}C_a\bar{\psi}\gamma^a\Omega = 0. \tag{31}$$

This is the Hermitian conjugate of (28).

B. Hamilton formalism

The conjugate momentum forms of ψ and $\bar{\psi}$ are respectively $\Pi = \bar{\psi}\gamma_a e^a$ and $\bar{\Pi} = 0$. Then, the Hamilton form $H_D = d\psi \wedge \Pi + d\bar{\psi} \wedge \bar{\Pi} - L'_{\mathrm{D}}$ is given by

$$H_{\mathrm{D}} = \Pi \wedge \frac{1}{4}\gamma_{ab}\omega^{ab}\psi + m\bar{\psi}\psi\Omega + \frac{1}{2}C_a\bar{\psi}\gamma^a\psi\Omega.$$

Although the traditional Hamiltonian density includes $\partial_i\psi$ $(i = 1, \cdots, D-1)$, the Hamilton form does not include the exterior derivative of the Dirac field. Rewriting the second and third terms using Π, we obtain

$$H_{\mathrm{D}}(\psi, \Pi) = \Pi \wedge \frac{1}{4}\gamma_{ab}\omega^{ab}\psi + m\Pi \wedge \varphi\psi + \frac{1}{2}C_a\Pi \wedge \theta^a\psi, \tag{32}$$

with $\varphi \stackrel{\mathrm{def}}{=} (1/D)\gamma_a\theta^a$. We have $\gamma_a e^a \wedge \varphi = \Omega$ since (A3). The Hamilton form is regarded as function only ψ and Π. In the traditional analytic mechanics, the corresponding treatment is equivalent to the formulation using the Dirac bracket. However, in the covariant analytic mechanics, the generalization of the Dirac bracket is not known. In the covariant analytic mechanics, the similar problem does happen for the formulations of the abelian and non-abelian gauge fields and of the gravitational field in the second order formalism.

The derivatives of the Hamilton form are given by

$$\frac{\partial H_{\mathrm{D}}}{\partial \Pi} = \frac{1}{4}\gamma_{ab}\omega^{ab}\psi + m\varphi\psi + \frac{1}{2}C_a\theta^a\psi, \tag{33}$$

$$\frac{\partial H_{\mathrm{D}}}{\partial \psi} = \Pi \wedge \frac{1}{4}\gamma_{ab}\omega^{ab} + m\Pi \wedge \varphi + \frac{1}{2}C_a\Pi \wedge \theta^a. \tag{34}$$

Then, the canonical equation $d\psi = -\partial H_D/\partial \Pi$ is

$$d\psi + \frac{1}{4}\gamma_{ab}\omega^{ab}\psi + m\varphi\psi + \frac{1}{2}C_a\theta^a\psi = 0. \tag{35}$$

Applying $\gamma_a e^a$ to the above equation from the left and using $\gamma_a e^a \wedge \varphi = \Omega$, we obtain (28). The canonical equation $d\Pi = -\partial H_D/\partial\psi$ is

$$d\Pi + \Pi \wedge \frac{1}{4}\gamma_{ab}\omega^{ab} + m\Pi \wedge \varphi + \frac{1}{2}C_a\Pi \wedge \theta^a = 0. \tag{36}$$

Substituting $\Pi = \bar{\psi}\gamma_a e^a$ and using $\Pi \wedge \varphi = \bar{\psi}\Omega$, (25) and (A11), the above equation becomes (31).

V. GRAVITY WITH DIRAC FIELD

We consider the gravitational field coupled with matters including the Dirac field. We first study the first order formalism and review briefly Nester's approach[10,11] in V A. Since the first order formalism does not go well for the Hamilton formalism, next we study the second order formalism. In V B, we investigate the Lagrange formalism, and next, we investigate the Hamilton formalism from V C to V E. The formulations up to V C can be easily generalized to $D(\geq 3)$ dimension. However, after V D we use specialties of $D = 4$.

A. First order formalism

In this subsection, we consider the first order formalism, in which θ^a and $\omega^a{}_b$ are independent each other. The Lagrange form of the gravitational field coupled with the matters is given by

$$L^{(1)}(\theta, \omega, d\theta, d\omega) = L^{(1)}_{\mathrm{G}}(\theta, \omega, d\theta, d\omega) + L_{\mathrm{mat}}(\theta, \omega), \tag{37}$$

where $L^{(1)}_{\mathrm{G}}$ and L_{mat} are respectively the Lagrange forms of the pure gravity and the matters. The former is given by

$$L^{(1)}_{\mathrm{G}} = \frac{1}{2\kappa} * R = \frac{1}{2\kappa} e_{ab} \wedge (d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}), \tag{38}$$

with the Einstein constant $\kappa = 8\pi G/c^3$. In Ref.[9], L_{mat} was $L_{\mathrm{m}}(\theta)$ which does not include the connection. For instance, $L_{\mathrm{m}}(\theta)$ is the Lagrange form for the scaler field and the abelian and non-abelian gauge fields. The variation of $L^{(1)}_{\mathrm{G}}$ is given by

$$\delta L^{(1)}_{\mathrm{G}} = \frac{1}{2\kappa}\big[- \delta\theta^c \wedge e_{abc} \wedge \Omega^{ab} - \delta\omega^{ab} \wedge (\omega^c{}_a \wedge e_{cb} + \omega^c{}_b \wedge e_{ac}) + \delta d\omega^{ab} \wedge e_{ab}\big]. \tag{39}$$

Here, we used (A6). We expand the variation of L_{mat} by $\delta\theta^a$ and $\delta\omega^{ab}$ as

$$\delta L_{\mathrm{mat}}(\theta, \omega) = -\delta\theta^a \wedge *T_a + \delta\omega^{ab} \wedge \frac{\partial L_{\mathrm{mat}}}{\partial \omega^{ab}}. \tag{40}$$

If we expand T_a as $T_a = T_{ab}\theta^b$, the coefficient T_{ab} is called energy-momentum tensor. If $L_{\mathrm{mat}} = L_{\mathrm{D}}$, we obtain

$$* T_c = -\frac{1}{2} e_{dc} \wedge \gamma^d D\bar{\psi}\psi + \frac{1}{2} e_{dc} \wedge \bar{\psi}\overleftarrow{D}\gamma^d\psi - e_c m\bar{\psi}\psi, \tag{41}$$

with $D\psi \stackrel{\text{def}}{=} (d\psi + \frac{1}{4}\gamma_{ab}\omega^{ab}\psi)$ and $\bar{\psi}\overleftarrow{D} \stackrel{\text{def}}{=} (d\bar{\psi} - \frac{1}{4}\bar{\psi}\gamma_{ab}\omega^{ab})$. Applying $*$ to the above equation, we obtain

$$\begin{aligned} T_a &= -\frac{1}{2}[-\theta_b\gamma^b\bar{\psi}D_a\psi + \theta_a\gamma^b\bar{\psi}D_b\psi] + \frac{1}{2}[-\theta_b\bar{\psi}\overleftarrow{D}_a\gamma^b\psi + \theta_a\bar{\psi}\overleftarrow{D}_b\gamma^b\psi] - \theta_a m\bar{\psi}\psi \\ &= \frac{1}{2}\theta_b\gamma^b\bar{\psi}D_a\psi - \frac{1}{2}\theta_b\bar{\psi}\overleftarrow{D}_a\gamma^b\psi. \end{aligned} \tag{42}$$

Here, we used expansions $D\psi = D_a\psi\theta^a$, $\bar{\psi}\overleftarrow{D} = \bar{\psi}\overleftarrow{D}_a\theta^a$ and (A2). We have $D_a\psi = \theta_a{}^\mu(\partial_\mu + \frac{1}{4}\gamma_{bc}\omega^{bc}{}_\mu)\psi$ if we expand the connection as $\omega^{ab} = \omega^{ab}{}_\mu dx^\mu$. In the second line of the above equation, we used the Dirac equations (28) and (31). We have $T_{ab} = \frac{1}{2}\gamma_b\bar{\psi}D_a\psi - \frac{1}{2}\bar{\psi}\overleftarrow{D}_a\gamma_b\psi$, which coincides the energy-momentum tensor derived by the traditional way[12]. (39) and (40) lead

$$\frac{\partial L^{(1)}}{\partial\theta^c} = -\frac{1}{2\kappa}e_{abc}\wedge\Omega^{ab} - *T_c\,, \quad \frac{\partial L^{(1)}}{\partial d\theta^c} = 0, \tag{43}$$

$$\frac{\partial L^{(1)}}{\partial\omega^{ab}} = -\frac{1}{2\kappa}\big[\omega^c{}_a\wedge e_{cb} + \omega^c{}_b\wedge e_{ac}\big] + \frac{\partial L_{\text{mat}}}{\partial\omega^{ab}}\,, \quad \frac{\partial L^{(1)}}{\partial d\omega^{ab}} = \frac{1}{2\kappa}e_{ab}. \tag{44}$$

Then, the Euler-Lagrange equation $\partial L^{(1)}/\partial\omega^{ab} + d(\partial L^{(1)}/\partial d\omega^{ab}) = 0$ is

$$de_{ab} - \omega^c{}_a\wedge e_{cb} - \omega^c{}_b\wedge e_{ac} + 2\kappa\frac{\partial L_{\text{mat}}}{\partial\omega^{ab}} = 0. \tag{45}$$

And the Euler-Lagrange equation $\partial L^{(1)}/\partial\theta^c + d(\partial L^{(1)}/\partial d\theta^c) = 0$ is

$$-\frac{1}{2\kappa}e_{abc}\wedge\Omega^{ab} = *T_c, \tag{46}$$

which leads the Einstein equation

$$R^a_{\ b} - \frac{1}{2}R\delta^a_b = \kappa T_b{}^a.$$

We will discuss about (45) in V B.

The conjugate momentum forms of θ^a and ω^{ab} are respectively $\pi^{(1)}_a = 0$ and $p_{ab} = e_{ab}/2\kappa$. The Hamilton form is

$$H^{(1)}(\theta,\omega,\pi^{(1)},p) = d\theta^a\wedge\pi^{(1)}_a + d\omega^{ab}\wedge p_{ab} - L^{(1)} = H^{(1)}_{\text{G}} - L_{\text{mat}}, \tag{47}$$

with

$$H^{(1)}_{\text{G}}(\omega,p) = \frac{N}{2\kappa}\,, \quad N \stackrel{\text{def}}{=} e^b{}_a\wedge\omega^a{}_c\wedge\omega^c{}_b. \tag{48}$$

We have $*R = e_{ab}\wedge d\omega^{ab} - N$. We can not derive the correct equations of motions from this Hamilton form.

The start point of Nester is different from us[10,11]. For wide class of the gravitation theories, Nester started from

$$L^{(1)}_{\text{G}} = (d\theta^a + \omega^a{}_b\wedge\theta^b)\wedge\pi^{(1)}_a + (d\omega^{ab} + \omega^a{}_c\wedge\omega^{cb})\wedge p_{ab} - \Lambda(\theta,\omega,\pi^{(1)},p). \tag{49}$$

Here, Λ corresponds to the Hamilton form which is given by a hand depending on the theory. So, Nester's approach was not complete Hamilton formalism. In present theory, Λ is given by[10]

$$\Lambda = U^a\wedge\pi^{(1)}_a + V^{ab}\wedge\big(p_{ab} - \frac{e_{ab}}{2\kappa}\big). \tag{50}$$

Here, U^a and V^{ab} are the Lagrange multiplier forms.

B. Lagrange formalism

In the second order formalism, the Lagrange form is different from $L^{(1)}$:

$$L(\theta,d\theta) = L_{\text{G}}(\theta,d\theta) + L_{\text{mat}}(\theta,d\theta). \tag{51}$$

Here, $L_{\rm G}$ is the Lagrange form for the pure gravity given by

$$L_{\rm G}(\theta,d\theta)=\frac{1}{2\kappa}N',\quad N'\stackrel{\rm def}{=}*R-d(e_{ab}\wedge\omega^{ab}),\tag{52}$$

and $L_{\rm mat}(\theta,d\theta)=L_{\rm mat}(\theta,\omega(\theta,d\theta))$. $\omega_{ab}=\omega_{ab}(\theta,d\theta)$ is the connection as a function of θ^a and $d\theta^a$. The variation is given by

$$\begin{aligned}\delta L(\theta,d\theta)=&-\delta\theta^c\wedge\Big(\frac{1}{2\kappa}[e_{abc}\wedge\Omega^{ab}+d(e_{abc}\wedge\omega^{ab})]+*T_c\Big)+\delta d\theta^c\wedge\frac{1}{2\kappa}e_{abc}\wedge\omega^{ab}\\&+\delta\omega^{ab}(\theta,d\theta)\wedge\Big(\frac{1}{2\kappa}[de_{ab}-\omega^c{}_a\wedge e_{cb}-\omega^c{}_b\wedge e_{ac}]+\frac{\partial L_{\rm mat}}{\partial\omega^{ab}}\Big).\end{aligned}\tag{53}$$

We suppose that the last term vanishes:

$$\frac{1}{2\kappa}[de_{ab}-\omega^c{}_a\wedge e_{cb}-\omega^c{}_b\wedge e_{ac}]+\frac{\partial L_{\rm mat}}{\partial\omega^{ab}}=0.\tag{54}$$

It is remarkable that this condition is the same with the Euler-Lagrange equation of the connection (45) of the first order formalism. In Ref.[9], because the Levi-Cività connection ($\omega^a{}_b=w^a{}_b$) was supposed and $\partial L_{\rm mat}/\partial\omega^{ab}=0$ held since $L_{\rm mat}$ was assumed to be independent of the connection, the above requirement (54) was the identity (A10). (54) is important when $L_{\rm mat}$ includes the Dirac field. Under this supposition, (53) leads

$$\frac{\partial L}{\partial\theta^c}=-\frac{1}{2\kappa}[e_{abc}\wedge\Omega^{ab}+d(e_{abc}\wedge\omega^{ab})]-*T_c\,,\quad\frac{\partial L}{\partial d\theta^c}=\frac{1}{2\kappa}e_{abc}\wedge\omega^{ab}.\tag{55}$$

The Euler-Lagrange equation $\partial L/\partial\theta^c+d(\partial L/\partial d\theta^c)=0$ becomes the Einstein equation (46).

The Lagrange form $L_{\rm G}$ is given by subtracting the total differential term $L'_{\rm G}(\theta,d\theta)\stackrel{\rm def}{=}\frac{1}{2\kappa}d(e_{ab}\wedge\omega^{ab})$ from the Einstein-Hilbert form (38). The Lagrangian density which correspond to $L'_{\rm G}$ is $\frac{1}{2\kappa}\theta^{-1}\partial_\mu(\theta d^\mu)$ with $d^\mu=2\theta_a{}^\mu\theta_b{}^\lambda\omega^{ab}{}_\lambda$. $\omega^{ab}{}_\lambda$ and d^μ include $d\theta^a$ as (23). In the action, the total differential term becomes the surface integration of θd^μ. In the traditional derivation of the Einstein equation, the variation δd^μ is assumed to vanish on the surface even d^μ contains derivatives of the basic variables $\theta^a{}_\mu$. If we start from $L_{\rm G}$, this extra assumption is not needed. Moreover, we emphasize that it is probably impossible to derive the correct equations from the Einstein-Hilbert form including the total differential term $L'_{\rm G}(\theta,d\theta)$ by the covariant analytic mechanics. Probably, it is impossible to expand the variation of $L'_{\rm G}(\theta,d\theta)$ as

$$\delta L'_{\rm G}(\theta,d\theta)=\delta\theta^a\wedge X_a+\delta d\theta^a\wedge Y_a.$$

If it is possible, X_a and Y_a modify the Euler-Lagrange equation and the conjugate momentum form. This is very interesting because it suggests that the covariant analytic mechanics can not treat theories including higher order curvatures (R^2, $R^{ab}R_{ab}$, *etc.*).

We represent the torsion C_{abc} by the Dirac field. Using (54) and (A10), we obtain

$$\frac{1}{2\kappa}[\tilde\omega^c{}_a\wedge e_{cb}+\tilde\omega^c{}_b\wedge e_{ac}]=\frac{\partial L_{\rm mat}}{\partial\omega^{ab}}\equiv S_{c,ab}e^c.\tag{56}$$

$S_{c,ab}=0$ if $L_{\rm mat}=L_{\rm m}$ and

$$S_{c,ab}=\frac14\bar\psi\frac12(\gamma_c\gamma_{ab}+\gamma_{ab}\gamma_c)\psi=\frac14\bar\psi\gamma_{abc}\psi,$$

if $L_{\rm mat}=L_{\rm D}$[12]. Here, $\gamma_{abc}=\gamma_{[a}\gamma_b\gamma_{c]}$. If $L_{\rm mat}=L'_{\rm D}$, $S_{c,ab}=\frac14\bar\psi\gamma_c\gamma_{ab}\psi$ holds. The first term in [ ] of (56) becomes $\tilde\omega^c{}_a\wedge e_{cb}=-C_ae_b-\tilde\omega^c{}_{ab}e_c$ using (A2) and $\tilde\omega^c{}_{ac}=-C_a$. Similarly, LHS of (56) becomes $-\frac{1}{2\kappa}[C_a\eta_{cb}-C_{cab}-C_b\eta_{ca}]e^c$ using $2\tilde\omega_{c[ab]}=-C_{cab}$. Substituting this to (56), we obtain

$$\frac{1}{2\kappa}[-C_a\eta_{cb}+C_{cab}+C_b\eta_{ca}]=S_{c,ab}.\tag{57}$$

Therefore, C_{cab} is represented by the Dirac field. Contracting c and b in the above equation, we get

$$\frac{1}{2\kappa}C_a=-\frac{S_a}{D-2},\tag{58}$$

where D is the dimension and $S_a \stackrel{\text{def}}{=} S^b{}_{ab}$. Substituting this to (57), we obtain

$$\frac{1}{2\kappa}C_{cab} = S_{c,ab} - \frac{1}{D-2}[S_a\eta_{cb} - S_b\eta_{ca}]. \tag{59}$$

If $L_{\text{mat}} = L_{\text{D}}$, $S_a = 0$ holds because of the complete anti-symmetric property of γ_{abc}, and we obtain

$$C_a = 0, \tag{60}$$

from (58). Then, terms including C_a of IV vanish. S_a and C_a remain if $L_{\text{mat}} = L'_{\text{D}}$. If the Dirac field does not exist ($L_{\text{mat}} = L_{\text{m}}$), the torsion vanishes.

N' can be rewritten as

$$\begin{aligned} N' &= e_{ab} \wedge \omega^a{}_c \wedge \omega^{cb} - de_{ab} \wedge \omega^{ab} \\ &= N + \Theta^a \wedge e_{abc} \wedge \omega^{bc}. \end{aligned} \tag{61}$$

Substituting (54) to the first line, we obtain $N' = N - 2\kappa\omega^{ab} \wedge \frac{\partial L_{\text{mat}}}{\partial \omega^{ab}}$. Comparing this to the second line, we obtain

$$\omega^{ab} \wedge \frac{\partial L_{\text{mat}}}{\partial \omega^{ab}} = -\frac{1}{2\kappa}\Theta^a \wedge e_{abc} \wedge \omega^{bc}. \tag{62}$$

Kaminaga[9] used $L_{\text{G}} = \frac{1}{2\kappa}N$. Because of absence of the Dirac field, this coincides our Lagrange form. If the Dirac field exists, $\frac{1}{2\kappa}N$ is not proper. By the way, N can be rewritten as

$$N = d\theta^a \wedge \frac{1}{2}e_{abc} \wedge \omega^{bc} - \Theta^a \wedge \frac{1}{2}e_{abc} \wedge \omega^{bc}. \tag{63}$$

Here, we used $d\theta^a - \Theta^a = -\omega^a{}_b \wedge \theta^b$ and $\theta^b \wedge e_{abc} = -2e_{ca}$ derived from (A1) and the definition of N, (48).

C. Hamilton formalism

The conjugate momentum form of θ^a is given by

$$\pi_a = \frac{1}{2\kappa}e_{abc} \wedge \omega^{bc}, \tag{64}$$

and the Hamilton form is given by

$$H(\theta,\pi) = d\theta^a \wedge \pi_a - L = H_{\text{G}}(\theta,\pi) - L_{\text{mat}}(\theta,\pi), \tag{65}$$

with

$$H_{\text{G}}(\theta,\pi) = \frac{N}{2\kappa}. \tag{66}$$

Here, we used (61) and (63). Although $L_{\text{G}} \neq L_{\text{G}}^{(1)}$, $H_{\text{G}} = H_{\text{G}}^{(1)}$ holds. In Ref.[9], $H_{\text{G}} = L_{\text{G}}$ was satisfied since $\Theta^a = 0$. Although the Lagrange form of the pure gravity is different from Ref.[9], the Hamilton form is the same except for that $\omega^a{}_b$ was the Levi-Cività connection in Ref.[9]. In V D, we represent N by θ^a and π_a and takes derivatives by these. Since the torsion C_{abc} is represented by the Dirac field, it is independent of θ^a and π_a. Then, $\Theta^a = \frac{1}{2}C^a{}_{bc}\theta^b \wedge \theta^c$ is independent of π_a, however, is a function of θ^a.

The canonical equation for θ^a is $d\theta^a = \frac{1}{2\kappa}\frac{\partial N}{\partial \pi_a} - \frac{\partial L_{\text{mat}}}{\partial \pi_a}$. In RHS, the second term can be rewritten as

$$-\frac{\partial L_{\text{mat}}}{\partial \pi_a} = -\frac{\partial}{\partial \pi_a}\left[\omega^{ab} \wedge \frac{\partial L_{\text{mat}}}{\partial \omega^{ab}}\right] = \frac{\partial}{\partial \pi_a}[\Theta^a \wedge \pi_a] = \Theta^a. \tag{67}$$

Here, we used (62). Then, the canonical equation becomes

$$d\theta^a = \frac{1}{2\kappa}\frac{\partial N}{\partial \pi_a} + \Theta^a. \tag{68}$$

The canonical equation for π_a is

$$d\pi_a = \frac{\partial H}{\partial \theta^a} = \frac{1}{2\kappa}\frac{\partial N}{\partial \theta^a} - \frac{\partial L_{\mathrm{mat}}}{\partial \theta^a}. \tag{69}$$

We will show

$$\frac{\partial N}{\partial \theta^c} = e_{abc} \wedge \omega^{ad} \wedge \omega_d{}^b + (\omega^d{}_a \wedge e_{dbc} + \omega^d{}_b \wedge e_{adc} + \omega^d{}_c \wedge e_{abd}) \wedge \omega^{ab}, \tag{70}$$

in V E using the methods of Ref.[9]. The above equation is equivalent to

$$\frac{\partial N}{\partial \theta^c} = e_{abc} \wedge \Omega^{ab} + d(e_{abc} \wedge \omega^{ab}) + 2\kappa A_{c,ab} \wedge \omega^{ab},$$

because of (A9). Here, $A_{c,ab} \stackrel{\mathrm{def}}{=} \frac{1}{2\kappa}[\tilde{\omega}^d{}_a \wedge e_{dbc} + \tilde{\omega}^d{}_b \wedge e_{adc} + \tilde{\omega}^d{}_c \wedge e_{abd}]$. Introducing

$$t_c \stackrel{\mathrm{def}}{=} \frac{\partial L_{\mathrm{mat}}(\theta,\pi)}{\partial \theta^c} + *T_c = \frac{\partial L_{\mathrm{mat}}(\theta,\pi)}{\partial \theta^c} - \frac{\partial L_{\mathrm{mat}}(\theta,\omega)}{\partial \theta^c}, \tag{71}$$

we obtain

$$\frac{\partial H}{\partial \theta^c} = +\frac{1}{2\kappa}[e_{abc} \wedge \Omega^{ab} + d(e_{abc} \wedge \omega^{ab})] + *T_c + A_{c,ab} \wedge \omega^{ab} - t_c.$$

As we show in the remainder of this subsection,

$$A_{c,ab} \wedge \omega^{ab} = t_c, \tag{72}$$

holds. Then, we obtain

$$\frac{\partial H}{\partial \theta^c} = +\frac{1}{2\kappa}[e_{abc} \wedge \Omega^{ab} + d(e_{abc} \wedge \omega^{ab})] + *T_c. \tag{73}$$

Substituting this to (69), we get the Einstein equation (46). The RHS of (70), which can be rewritten as

$$\omega^d{}_b \wedge e_{adc} \wedge \omega^{ab} + \omega^d{}_c \wedge e_{abd} \wedge \omega^{ab} \equiv A_c + B_c, \tag{74}$$

is the same with Sparling's form except for a coefficient and relates to the gravitational energy-momentum pseudo-tensor[10,13,14].

We show (72). Using (A1), $\tilde{\omega}^c{}_{ac} = -C_a$ and $2\tilde{\omega}_{c[ab]} = -C_{cab}$, we obtain

$$A_{c,ab} = \frac{1}{2\kappa}[2C_{[a}e_{b]c} - C^d{}_{ab}e_{dc} + 2C^d{}_{[a|c}e_{d|b]} + C_c e_{ab}], \tag{75}$$

with $2C^d{}_{[a|c}e_{d|b]} = C^d{}_{ac}e_{db} - C^d{}_{bc}e_{da}$. By the way, using (62), (A7) and (A8), we obtain

$$\frac{\partial L_{\mathrm{D}}}{\partial \theta^c} = \frac{1}{2}\bar{\psi}\gamma_a e^a{}_c \wedge d\psi - \frac{1}{2}e^a{}_c \wedge d\bar{\psi}\gamma_a\psi + e_c m\bar{\psi}\psi - C^a{}_{cb}\theta^b \wedge \pi_a. \tag{76}$$

Then, for $L_{\mathrm{mat}} = L_{\mathrm{D}}$, we get

$$t_c = -e_{dc}S^d{}_{ab} \wedge \omega^{ab} - C^a{}_{cb}\theta^b \wedge \pi_a, \tag{77}$$

using (76) and (41). This relation also holds for $L_{\mathrm{mat}} = L'_{\mathrm{D}}$. We have $t_c = 0$ if $L_{\mathrm{mat}} = L_{\mathrm{m}}$. (77) can be rewritten as $t_c = B_{c,ab} \wedge \omega^{ab}$ with

$$B_{c,ab} = -e_{dc}S^d{}_{ab} + \frac{1}{2\kappa}[C_c e_{ab} - C^d{}_{c[a|}e_{d|b]}]. \tag{78}$$

Here, we used (64) and (A1). We can show $A_{c,ab} = B_{c,ab}$ using (75), (78) and (59). Then, (72) holds.

D. Variation of the Hamilton form

We represent ω^{ab} and N by π_a. In D dimension space-time, e_{abc} is given by

$$e_{abc} = \frac{1}{(D-3)!} E_{d_1 \cdots d_{D-3} abc} \theta^{d_1} \wedge \cdots \wedge \theta^{d_{D-3}},$$

where $E_{d_1 \cdots d_D}$ is the complete anti-symmetric tensor such that $E_{01 \cdots D-1} = 1$. In the following, we use specialties of $D = 4$. Substituting $e_{abc} = E_{dabc}\theta^d$ and $\omega^{ab} = \omega^{ab}{}_c\theta^c$ to (64), we obtain

$$\pi_c = \frac{1}{2}\pi_{c,ab}\theta^a \wedge \theta^b \,, \quad \pi_{c,ab} = \frac{1}{2\kappa}(E_{adec}\omega^{de}{}_b - E_{bdec}\omega^{de}{}_a). \tag{79}$$

Using the technique used to get (23) from (22) (this technique can not be used for $D \neq 4$), we obtain $\frac{1}{2\kappa}E_{bdea}\omega^{de}{}_c = \frac{1}{2}(\pi_{b,ca} + \pi_{c,ba} + \pi_{a,bc})$. It leads

$$\omega^{ab}{}_c = \frac{\kappa}{4}E^{abnm}\tau_{mcn} \,, \quad \tau_{abc} \stackrel{\text{def}}{=} \pi_{b,ac} + \pi_{a,bc} + \pi_{c,ab}. \tag{80}$$

We have $\tau_{abc} = -\tau_{cba}$. τ_{abc} is represented by π_c as $\tau_{abc} = -*p_{abc}$ with

$$p_{abc} \stackrel{\text{def}}{=} \pi_b \wedge e_{ac} + \pi_a \wedge e_{bc} + \pi_c \wedge e_{ab}. \tag{81}$$

Here, we used (A4). Substituting this equation to (80), we obtain

$$\omega^{ab} = -\frac{\kappa}{4}E^{abnm} * p_{mcn}\theta^c. \tag{82}$$

This equation and (22) lead

$$d\theta^a - \Theta^a = \frac{\kappa}{4}E^{abnm} * p_{mcn}\theta^c \wedge \theta_b. \tag{83}$$

Substituting $\pi_{c,ab} = -*p_{[ab]c}$ to the first equation of (79), we obtain

$$\pi_c = -\frac{1}{2} * p_{cab}\theta^a \wedge \theta^b. \tag{84}$$

We calculate the derivatives of N by θ^a and π_a. Substituting (82) to (48), we obtain[9]

$$N = n^{a_1a_2a_3a_4a_5a_6} * p_{a_1a_2a_3} * p_{a_4a_5a_6}\Omega, \tag{85}$$

by using (A4). Here,

$$\begin{aligned} n^{a_1a_2a_3a_4a_5a_6} = \frac{\kappa^2}{16}[&\eta^{a_2a_6}\eta^{a_3a_5}\eta^{a_1a_4} + \eta^{a_2a_4}\eta^{a_3a_6}\eta^{a_1a_5} + \eta^{a_5a_6}\eta^{a_3a_4}\eta^{a_1a_2} + \eta^{a_5a_4}\eta^{a_3a_2}\eta^{a_1a_6} \\ &-\eta^{a_2a_6}\eta^{a_3a_4}\eta^{a_1a_5} - \eta^{a_2a_4}\eta^{a_3a_5}\eta^{a_1a_6} - \eta^{a_5a_6}\eta^{a_3a_2}\eta^{a_1a_4} - \eta^{a_5a_4}\eta^{a_3a_6}\eta^{a_1a_2}], \end{aligned} \tag{86}$$

of which symmetries are

$$n^{a_1a_2a_3a_4a_5a_6} = -n^{a_3a_2a_1a_4a_5a_6} = -n^{a_1a_2a_3a_6a_5a_4} = n^{a_4a_5a_6a_1a_2a_3}. \tag{87}$$

Using the formula

$$\delta * p_{abc}\xi = (\delta p_{abc} + \delta\Omega * p_{abc}) * \xi,$$

for the arbitrary 4-form ξ and (87), (A6) and (A8), we obtain[9]

$$\frac{\partial N}{\partial \pi_a} = -2n^{dbcnml}(2\delta^a_d e_{bc} + \delta^a_b e_{dc}) * p_{nml}, \tag{88}$$

$$\frac{\partial N}{\partial \theta^a} = n^{dbcnml}[4\pi_d \wedge e_{bca} + 2\pi_b \wedge e_{dca} + *p_{dbc}e_a] * p_{nml}. \tag{89}$$

E. Canonical equations

We calculate RHS of (68). Substituting (86) to $n^{dbcnml}(2\delta^a_d e_{bc}+\delta^a_b e_{dc})$ of RHS of (88), and using (A1), we obtain

$$n^{dbcnml}(2\delta^a_d e_{bc}+\delta^a_b e_{dc})=\frac{\kappa^2}{4}\theta^m\wedge e^{lna}.$$

Substituting this equation to (88) and using $e^{lna}=E^{blna}\theta_b$, we obtain

$$\frac{1}{2\kappa}\frac{\partial N}{\partial \pi_a}=\frac{\kappa}{4}E^{abln}*p_{nml}\theta^m\wedge\theta_b. \tag{90}$$

Then, the canonical equation for θ^a (68) becomes

$$d\theta^a=\frac{\kappa}{4}E^{abln}*p_{nml}\theta^m\wedge\theta_b+\Theta^a, \tag{91}$$

which coincides with (83). The above equation and (22) lead (82), which is equivalent to the definition of the conjugate momentum form π_c.

We show (70). Substituting (84) to RHS of (89), and using the symmetry $p_{abc}=-p_{cba}$ and (A5), we can obtain[9]

$$\frac{\partial N}{\partial \theta^c}=-n^{abdnml}\left[2(*p_{dbc}+*p_{bdc})e_a+2(*p_{adc}-*p_{acd})e_b+*p_{abd}e_c\right]*p_{nml}.$$

Substituting (86) to this equation, we obtain

$$\begin{aligned}\frac{\partial N}{\partial \theta^c}=&\frac{\kappa^2}{2}e_n(*p^{[ab]n}*p_{acb}+*p_{ac}{}^n*p^a)\\&+\frac{\kappa^2}{2}e_n(-*p_{abc}*p^{nab}+*p^n{}_{ac}*p^a-*p_c*p^n)\\&+\frac{\kappa^2}{4}e_c(-*p_{abd}*p^{b[ad]}+*p_a*p^a),\end{aligned} \tag{92}$$

with $p_a=p^b{}_{ba}$. In Ref.[9], the anti-symmetrization symbols were missed.

Next, we show that RHS of the above equation becomes RHS of (70), namely (74). Substituting (82) to $A_c=\omega^{ab}\wedge\omega^d{}_b\wedge e_{adc}$, we obtain $A^{(1)}_c+A^{(2)}_c$ with

$$\begin{aligned}A^{(1)}_c&=\frac{\kappa^2}{16}E^{abnm}E^d{}_b{}^{kl}[(\delta^s_d\delta^t_c-\delta^s_c\delta^t_d)e_a+(\delta^s_c\delta^t_a-\delta^s_a\delta^t_c)e_d]*p_{msn}*p_{ltk}\\&=\frac{\kappa^2}{8}e_aE^{abnm}E^d{}_b{}^{kl}(\delta^s_d\delta^t_c-\delta^s_c\delta^t_d)*p_{msn}*p_{ltk},\end{aligned} \tag{93}$$

$$A^{(2)}_c=\frac{\kappa^2}{16}e_cE^{abnm}E^d{}_b{}^{kl}(\delta^s_a\delta^t_d-\delta^s_d\delta^t_a)*p_{msn}*p_{ltk}. \tag{94}$$

Here, we used (A5). We can show that

$$A^{(1)}_c=\frac{\kappa^2}{2}e_n(-*p^{[ab]n}*p_{acb}-*p_{ac}{}^n*p^a), \tag{95}$$

$$A^{(2)}_c=\frac{\kappa^2}{4}e_c(*p_{abd}*p^{b[ad]}-*p_a*p^a). \tag{96}$$

Similarly, we can write $B_c=\omega^{ab}\wedge\omega^d{}_c\wedge e_{abd}$ as $B^{(1)}_c+B^{(2)}_c$ with

$$\begin{aligned}B^{(1)}_c&=\frac{\kappa^2}{16}E^{abnm}E^d{}_c{}^{kl}[(\delta^s_b\delta^t_d-\delta^s_d\delta^t_b)e_a+(\delta^s_d\delta^t_a-\delta^s_a\delta^t_d)e_b]*p_{msn}*p_{ltk}\\&=\frac{\kappa^2}{8}e_aE^{abnm}E^d{}_c{}^{kl}(\delta^s_b\delta^t_d-\delta^s_d\delta^t_b)*p_{msn}*p_{ltk},\end{aligned} \tag{97}$$

$$\begin{aligned}B^{(2)}_c&=\frac{\kappa^2}{16}E^{abnm}E^d{}_c{}^{kl}*p_{msn}*p_{ltk}(\delta^s_a\delta^t_b-\delta^s_b\delta^t_a)e_d\\&=\frac{\kappa^2}{8}e_dE^{abnm}E^d{}_c{}^{kl}*p_{man}*p_{lbk}.\end{aligned} \tag{98}$$

And these equations lead,

$$B_c^{(1)} = \frac{\kappa^2}{2} e_n (*p^{[ab]n} * p_{acb} + *p_{ac}{}^{n} * p^a)$$
$$+\frac{\kappa^2}{2} e_n (*p_{[ab]c} * p^{anb} - *p_{abc} * p^{nab} + *p_a{}^n{}_c * p^a - *p_c * p^n)$$
$$+\frac{\kappa^2}{2} e_c (- * p_{abd} * p^{b[ad]} + *p_a * p^a), \qquad (99)$$
$$B_c^{(2)} = \frac{\kappa^2}{2} e_n (*p^{[ab]n} * p_{acb} + *p_{ac}{}^{n} * p^a)$$
$$+\frac{\kappa^2}{2} e_n (- * p_{[ab]c} * p^{anb} - *p_a{}^n{}_c * p^a + *p^n{}_{ac} * p^a). \qquad (100)$$

Using (95), (96), (99) and (100) (the anti-symmetrization symbols were missed in Ref.[9]), RHS of (92) becomes RHS of (70), namely (74). Therefore, we obtain (73), and the canonical equation for θ^a (69) becomes the Einstein equation (46).

VI. SUMMARY

We applied the covariant analytic mechanics with the differential forms to the Dirac field and the gravity with the Dirac field. In II, we reviewed the covariant analytic mechanics which treats space and time on an equal footing regarding the differential forms as the basis variables and has significant advantages that the canonical equations are gauge covariant as well as manifestly diffeomorphism covariant. Combining our study and the previous works[8–10] (the scalar field, the Proca field, the electromagnetic field, the non-abelian gauge field and the gravity without the Dirac field), the applicability of the covariant analytic mechanics was checked for all fundamental fields.

In IV, we studied the Dirac field. The frame (vielbein) is necessary to write down the Lagrange form even in the flat space-time. This fact represents a nature of the Dirac field. We regarded the basis variable of the Hamilton form of the Dirac field as only ψ and its conjugate momentum form Π. In the traditional analytic mechanics, the corresponding treatment is equivalent to the formulation using the Dirac bracket. In the covariant analytic mechanics, the similar problem does happen for the formulation of other fundamental fields. Although we introduced the Poisson bracket of the covariant analytic mechanics for the first time, the possibilities of applications to the Dirac bracket, the canonical transform theory and the quantization are unclear.

In V, we studied gravitational field coupled with matters including the Dirac field and claimed that Nester's approach[10,11] was not complete Hamilton formalism. The first order formalism did not go well for the Hamilton formalism. In this sense, the second order formalism is proper. In the second order formalism, the Lagrange form of the pure gravity is given by subtracting the total differential term $L'_{\mathrm{G}}(\theta, d\theta) = \frac{1}{2\kappa} d(e_{ab} \wedge \omega^{ab})$ from the Einstein-Hilbert form. If we do not drop the $L'_{\mathrm{G}}(\theta, d\theta)$, it is probably impossible to derive the correct equations. This is very interesting because it suggests that the covariant analytic mechanics can not treat theories including higher order curvatures (R^2, $R^{ab}R_{ab}$, *etc.*). The torsion was determined by the condition that the last term of RHS of (53) vanishes. Although the Lagrange form of the pure gravity was different from Ref.[9], the Hamilton form was the same except for that the connection was the Levi-Cività connection in Ref.[9]. We took the derivatives of the Hamilton form of the pure gravity using the specialties of 4 dimension system and corrected the errors of Ref.[9]. In other part, which can be easily generalized to arbitrary dimension, we treated the contributions due to the Dirac field.

Acknowledgments

We acknowledge helpful discussions with Y. Tokura and Y. Kaminaga.

Appendix A: Formulas

Several useful Formulas are listed. For $\theta^a \wedge e_{a_1 \cdots a_r} (r = 1, 2, 3)$,

$$\theta^a \wedge e_{bcd} = -\delta^a_b e_{cd} + \delta^a_c e_{bd} - \delta^a_d e_{bc}, \qquad (A1)$$
$$\theta^a \wedge e_{bc} = \delta^a_b e_c - \delta^a_c e_b, \qquad (A2)$$
$$\theta^a \wedge e_b = -\delta^a_b \Omega, \qquad (A3)$$

hold[14]. Using (A2) and (A3), we obtain

$$\theta^a \wedge \theta^b \wedge e_{cd} = (\delta^a_c \delta^b_d - \delta^a_d \delta^b_c)\Omega. \tag{A4}$$

Using (A1) and (A2), we have

$$\theta^a \wedge \theta^b \wedge e_{cde} = (\delta^a_d \delta^b_e - \delta^a_e \delta^b_d)e_c + (\delta^a_e \delta^b_c - \delta^a_c \delta^b_e)e_d + (\delta^a_c \delta^b_d - \delta^a_d \delta^b_c)e_e. \tag{A5}$$

For $\delta e_{a_1 \cdots a_r} (r = 0, 1, 2)$,

$$\delta e_{ab} = -\delta\theta^c \wedge e_{abc}, \tag{A6}$$
$$\delta e_a = -\delta\theta^b \wedge e_{ab}, \tag{A7}$$
$$\delta\Omega = -\delta\theta^a \wedge e_a, \tag{A8}$$

hold. For $de_{a_1 \cdots a_r} (r = 1, 2, 3)$, we have[14]

$$de_{abc} = w^d{}_a \wedge e_{dbc} + w^d{}_b \wedge e_{adc} + w^d{}_c \wedge e_{abd}, \tag{A9}$$
$$de_{ab} = w^c{}_a \wedge e_{cb} + w^c{}_b \wedge e_{ac}, \tag{A10}$$
$$de_a = w^b{}_a \wedge e_b = -(\omega_a + C_a)\Omega. \tag{A11}$$

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