A DISCUSSION ON THE MOST GENERAL TORSION-GRAVITY WITH ELECTRODYNAMICS FOR DIRAC SPINOR MATTER FIELDS

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We consider the most general torsional completion of gravity and electrodynamics with Dirac spinorial matter fields, showing that continuity and consistency constrain torsion to be completely antisymmetric and the model to be parity-invariant and described by either the least-order derivative model or the renormalizable model.

INTRODUCTION

One of the fundamental principles that is to be considered when establishing a theory is the requirement of studying the most general situation possible.

Despite the fact that in some circumstances special cases might be tackled, these toy models have only the purpose of getting rid of unimportant details when one aims to have a quick insight into the most important properties of a system; but no theory will ever be taken seriously if arbitrary restrictions are placed upon it.

Generality was for instance the reason that moved Einstein in searching for what would be then called Einstein gravitation: as he felt uneasy with the fact that only inertial frames were used, he sought for a theory that could be invariant for the most general transformation between systems of reference and in doing so he realized that the most general theory had room enough to host the gravitational field; mathematically, the generalization of derivatives up to the covariant derivatives demanded the introduction of the connection, a geometrical quantity inside which gravitational information could be stored.

Nevertheless, despite the attribute general to this theory, the general theory of relativity is not the most general theory of relativity that can be written: this theory is the most general in the sense that in it the transformations between systems of coordinates and the metric properties of the systems of coordinates are the most general, but the connection of the spacetime is not the most general at all; to find the most general connection of the spacetime one may not employ a bottom-up approach, starting from the trivial connection and building further generalizations, but one must employ a top-down approach, starting from the most general connection to eventually decompose it in simpler parts.

So far as we can tell, the most complete discussion about the most general connection and its decomposition is in reference [1]: in this paper, it is presented the fact that the most general connection is decomposable in terms of the simplest connection, the symmetric metric connection of Einstein gravity, plus two additional tensors, one being a combination of the covariant derivatives of the metric tensor and the other being the antisymmetric part in the two lower indices of the connection itself.

The former of these two parts, the combination of the covariant derivative of the metric, vanishes if the covariant derivatives of the metric vanish themselves, and there are reasons for this to be so; essentially, the argument is that because the metric tensor is used to raise and lower tensorial indices, and that this must be possible also for tensors that are covariant derivatives of some other tensor, then the only way in which can unambiguously been done is by requiring that the covariant derivatives of the metric vanish, or as it is also said, that the connection be metric-compatible: in other words, the condition of metric-compatibility of the connection is simply the most general requirement that we can ask if we want that, whether we have tensors or derivatives of tensors, the procedure that allows us to move indices up and down leave the information content unaffected. If this were not the case, then a tensor that can be written in two different configurations of indices might happen to be constant in one but not constant in the other; as the difference between different configuration of indices, whether upper or lower, is genuinely man-made and as such all configurations must bear the same information, which is not the case if a tensor is both constant and not constant.

Conversely, for the latter of the two parts we have discussed above, that is the antisymmetric part in the two lower indices of the connection, there is no reason for it to vanish, although admittedly arguments in the past have been brought in support of this assumption; we are not going to present here the list of these arguments, which can be found for instance in [2], and it will be enough to notice that basically they are all related to the principle of equivalence: as it was discussed in [3], the vanishing of the antisymmetric part of the connection is a sufficient condition for the implementation of the Einsteinian principle of equivalence. The argument is this: the principle of equivalence requires that it is always possible to find a system of reference in which locally the gravitational field can be vanished, and because Weyl theorem states that it is always possible to find a system of coordinates in which in the neighborhood of a point the symmetric part of the connection can be set equal to zero, then the symmetric part of the connection is what encodes the gravitational information; the antisymmetric part of the connection is called Cartan torsion tensor, and if torsion vanishes then the connection is symmetric, which is a sufficient condition in order to have the principle of equivalence implemented, and consequently gravity geometrized.

Nevertheless, if for the geometrization of gravitation we want a condition that is sufficient as well as necessary then we do not need to go so far as to require that torsion be equal to zero since it is enough to require its complete antisymmetry; if torsion is non-zero but completely antisymmetric, the connection is not symmetric but it has a single symmetric part, so the gravitational field can unambiguously be represented by the symmetric part of the connection and therefore geometrized, and the principle of equivalence can still be implemented, as it has been discussed in [4, 5]. Additionally, we do not know whether the principle of equivalence is valid in the quantum domain [6, 7]; then we cannot be sure that the principle of equivalence be valid at all possible scales and therefore torsion would be unconstrained in general.

So we have that either the principle of equivalence is not valid and torsion is general, or the principle of equivalence is valid and torsion is constrained to be completely antisymmetric: both ways of the dilemma indicate that torsion is non-zero, and therefore it provides non-trivial contributions to the most general connection.

It is important to remark that when the connection in its most general form is constrained as to be metric-compatible then the system of coordinates in which locally the connection vanishes and the metric flattens are the same; on the other hand, we have just discussed how the complete antisymmetry of torsion would ensure the existence of a single symmetric part of the connection that can be vanished: thus a metric-compatible connection with a completely antisymmetric torsion ensures that there exists a unique symmetric part of the connection that can be vanished together with the flattening of the metric in the same point of the same system of coordinates. This point was the starting point for a complementary discussion, based on the analysis of light-cone structures and free-fall paths, carried on in [8, 9].

Therefore, if we base our argument on the requirement of staying in the most general case compatible with what we know about the properties of the spacetime, the torsional completion of gravity seems to be inevitable.

There are other reasons to justify the precise form of the connection we have chosen to employ.

Consider the most general theory given when gravity is taken together with electrodynamics. The geometrical background is endowed with three fundamental quantities, given by the torsion and the curvature, beside the gauge potential; on the other hand, in the Wigner classification, general matter fields are known to be characterized by three quantum numbers, that is spin and mass, beside the electric charge, and hence the quantum theory of these fields will stipulate that there be three conserved quantities for matter, namely the spin and energy, beside the gauge currents: the requirement of metric-compatibility suppresses a covariant derivative of the metric that otherwise would have been a geometrical quantity to which could couple no known conserved quantity, while torsion can couple to the spin much in the same way in which in Einstein gravity the curvature couples to the energy, and similarly in electrodynamics gauge potentials are coupled to gauge currents.

Moreover, consider the fact that of all possible matter fields with spin, the only one known in nature is the Dirac spinor field. The Dirac spinor has a spin that is completely antisymmetry, and as a consequence it would perfectly fit into a scheme in which torsion is completely antisymmetric too; conversely, the complete antisymmetry of torsion imposes the complete antisymmetry of the spin, allowing the Dirac field only [10, 11].

But if we abolish torsion from the beginning there is no way in which the spin can be coupled.

Once torsion is not neglected in gravity, it is well known that it does not couple to electrodynamics, but it couples to the the fermionic matter, rendering the matter field equations non-linear, with interesting phenomenology arising, such as for instance a dynamical version of the Pauli principle, as discussed in [12] and [13].

For quite a time however, there has been the common misconception that torsion would couple to the spin with the same strength with which curvature couple to energy, that is in terms of such small a constant that albeit torsion is present nonetheless all its effects can be neglected for practical purposes; to understand the roots of this misconception, we recall that while without torsion the field equations can be obtained by varying with respect to the metric the Lagrangian given by the torsionless Ricci scalar R in presence of torsion the field equations are obtained by varying with respect to the torsion and the metric a Lagrangian that is given by the torsionfull Ricci scalar G and here lies the problem; the Sciama-Kibble completion of Einstein theory, the one that is based on the scalar G as Lagrangian, is only the most straightforward but not the most general: in general, torsion should not only be present as included implicitly inside the curvature but also explicitly in quadratic terms, so that actually the most general Sciama-Kibble completion of Einstein theory is given by $G+Q^2$ as Lagrangian, where the notation for Q^2 formally means that one has to consider all squared-torsion terms that are allowed.

In general, there are a total of three irreducible decompositions of torsion [14]; correspondingly, there are three independent torsion-squared terms, and thus three additional coupling constants [15]. In general none of these coupling constants is forced to be the Newton constant, and the corresponding effects not necessarily negligible.

In particular, for the Dirac field, torsion is completely antisymmetric, then there is a single additional term and hence only one additional constant [16].

One of the most important features of this non-linear field equations is that negative-energy fermions are not permitted solutions: in this context negative-energy fermions do not appear in the first place [17].

The fact that only positive-energy fermions are allowed is tightly connected to the fact that the time-reversal discrete transformation is not a symmetry when non-linearities are present in the field equations [18].

Eventually, focusing on the Dirac field alone, the problem of allowing the most general parity-violating torsion terms has been considered, and for the single Dirac field solved, in reference [19]; the problem of finding the most general parity-even interaction between torsion and Dirac fields has been solved in [20]. A discussion on the intriguing situation for which the torsionally-induced non-linear interactions seems to mimic the effects due to quantum corrections has been done in [21] and references therein.

More general and comprehensive reviews have been written in past and recent times [22, 23], also with conformal symmetry and quantum effects [24] and with extensive discussions about open problems [25].

The problem of whether torsion is to be taken in its utmost generality or not, as when it is completely antisymmetric, has consequences on the type of fields allowed.

And it is the problem that we are going to investigate in some detail along the course of this paper.

FOUNDATIONS

The first thing we will specify is that we will work in the simplest space, that is the (1+3)-dimensional spacetime.

In such a spacetime, we will have both a differential structure and a metric structure, the former given in terms of the most general covariant derivative D_μ defined in terms of the most general connection, the latter given in terms of the most general metric $g_{\mu\nu}$ which is also used to raise and lower tensorial indices: as anticipated in the introduction, we will assume that the covariant derivative of the metric vanishes $D_\alpha g_{\mu\nu}=0$ so that the procedure of raising and lowering indices can be extended to indices in derivative tensors unambiguously; so the most general covariant derivative can be decomposed in terms of the simplest covariant derivative ∇_μ defined in terms of the simplest connection, entirely written in terms of the metric $g_{\mu\nu}$ alone: of course $\nabla_\alpha g_{\mu\nu}=0$ by construction.

Such a decomposition is best seen in terms of their respective connections, since they decompose as

$$\Gamma^\alpha_{\mu\nu}=\Lambda^\alpha_{\mu\nu}+K^\alpha{}_{\mu\nu} \tag{1}$$

in which we have that $\Gamma^\alpha_{\mu\nu}$ is the most general connection and $\Lambda^\alpha_{\mu\nu}=\frac{1}{2}g^{\rho\alpha}(\partial_\mu g_{\nu\rho}+\partial_\nu g_{\mu\rho}-\partial_\rho g_{\mu\nu})$ is the simplest connection written in terms of the metric while we have that $K^\alpha{}_{\mu\nu}=\frac{1}{2}(Q^\alpha{}_{\mu\nu}+Q_{\mu\nu}{}^\alpha+Q_{\nu\mu}{}^\alpha)$ is called contorsion and it is given in terms of the torsion tensor; we notice that the torsion is the antisymmetric part in the two lower indices of the connection and as such there is no torsion associated to the metric connection. The torsion tensor can be further decomposed according to the form

$$Q_{\rho\mu\nu}=\frac{1}{6}W^\alpha\varepsilon_{\alpha\rho\mu\nu}+\frac{1}{3}\left(V_\nu g_{\rho\mu}-V_\mu g_{\rho\nu}\right)+T_{\rho\mu\nu} \tag{2}$$

where $T_{\rho\mu\nu}=Q_{\rho\mu\nu}-\frac{1}{6}W^\alpha\varepsilon_{\alpha\rho\mu\nu}-\frac{1}{3}\left(V_\nu g_{\rho\mu}-V_\mu g_{\rho\nu}\right)$ is the non-completely antisymmetric irreducible tensorial part given in terms of $W^\alpha=Q_{\rho\mu\nu}\varepsilon^{\rho\mu\nu\alpha}$ as the axial vectorial part and $V_\nu=Q^\rho{}_{\rho\nu}$ as the trace vectorial part of torsion.

In terms of the most general connection alone it is possible to define the Riemann curvature tensor given by

$$G^\rho{}_{\eta\mu\nu}=\partial_\mu\Gamma^\rho_{\eta\nu}-\partial_\nu\Gamma^\rho_{\eta\mu}+\Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\eta\nu}-\Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{\eta\mu} \tag{3}$$

according to the usual definition: it is possible to write it in the form $G_{\sigma\eta\rho\nu}$ in which it is antisymmetric in both the first and second couple of indices, so with one independent contraction $G^\rho{}_{\eta\rho\nu}=G_{\eta\nu}$ which itself has a contraction given by $G_{\eta\nu}g^{\eta\nu}=G$ and they are called Ricci curvature tensor and scalar. With the simplest metric connection the Riemann metric curvature tensor is

$$R^\rho{}_{\eta\mu\nu}=\partial_\mu\Lambda^\rho_{\eta\nu}-\partial_\nu\Lambda^\rho_{\eta\mu}+\Lambda^\rho_{\sigma\mu}\Lambda^\sigma_{\eta\nu}-\Lambda^\rho_{\sigma\nu}\Lambda^\sigma_{\eta\mu} \tag{4}$$

in an analogous way: in the form $R_{\sigma\eta\rho\nu}$ it is antisymmetric in both the first and second couple of indices and it is symmetric for a switch between the first and second couple of indices, its contraction $R^\rho{}_{\eta\rho\nu}=R_{\eta\nu}$ itself has a contraction that is given by $R_{\eta\nu}g^{\eta\nu}=R$ and they are called Ricci metric curvature tensor and scalar, as usual.

Consequently, we have the decomposition given by

$$G^\rho{}_{\eta\mu\nu}=R^\rho{}_{\eta\mu\nu}+\nabla_\mu K^\rho_{\eta\nu}-\nabla_\nu K^\rho_{\eta\mu}+K^\rho_{\sigma\mu}K^\sigma_{\eta\nu}-K^\rho_{\sigma\nu}K^\sigma_{\eta\mu} \tag{5}$$

in which the most general Riemann tensor is given in terms of the Riemann metric tensor and the contorsion.

Equivalently, it is possible to pass from this coordinate formalism into the Lorentz formalism, in which the covariant derivative D_μ is defined in terms of the most general spin-connection, and the metric is written according to the expression $g_{\alpha\nu}=\xi^a_\alpha\xi^b_\nu\eta_{ab}$ in terms of the orthonormal tetrad fields ξ^σ_a and Minkowskian matrix η_{ab} used to raise and lower Lorentz indices: in the Lorentz formalism we have to assume that the covariant derivative of tetrads fields vanish $D_\alpha\xi^j_\mu=0$ in order to have the passage from the coordinate formalism to the Lorentz formalism be unambiguous and that the covariant derivative of the Minkowskian matrix vanish $D_\alpha\eta_{ij}=0$ in order to have that the raising and lowering of Lorentz indices in derivative tensors be unambiguously defined as before.

These two conditions of compatibility are expressed as

$$\Gamma^b{}_{j\mu}=\xi^\alpha_j\xi^b_\rho(\Gamma^\rho{}_{\alpha\mu}+\xi^k_\alpha\partial_\mu\xi^\rho_k) \tag{6}$$

and $\Gamma^{bj}{}_\nu=-\Gamma^{jb}{}_\nu$ showing that the spin-connection $\Gamma^{bj}{}_\nu$ is written in terms of the connection $\Gamma^\alpha_{\mu\nu}$ and the tetrad fields and that the spin-connection is antisymmetric in the two Lorentz indices compatibly with the requirement of Lorentz invariance, as we are about to see.

In the equivalent Lorentz formalism from the spin-connection we may define the Riemann curvature tensor

$$G^a{}_{b\mu\nu}=\partial_\mu\Gamma^a_{b\nu}-\partial_\nu\Gamma^a_{b\mu}+\Gamma^a_{\sigma\mu}\Gamma^\sigma_{b\nu}-\Gamma^a_{\sigma\nu}\Gamma^\sigma_{b\mu} \tag{7}$$

similarly as before: also its symmetries are as above.

As a consequence we have the relationship

$$G_{ab\mu\nu}=\xi^\rho_a\xi^\eta_b G_{\rho\eta\mu\nu} \tag{8}$$

showing that the Riemann curvature is Lorentz formalism is just the Riemann curvature in coordinate formalism after the index renaming, as it is to be expected.

The passage from coordinate formalism to Lorentz formalism is important because in this way it is possible to express the most general coordinate transformation law without any loss of generality into the special Lorentz transformation law, whose specific form makes it explicitly writable in terms of given representations, the real one but also the complex one, and when the representation is complex then fields have to be complex, and a new differential structure has to be given in terms of the gauge-covariant derivative D_μ defined in terms of the gauge-connection, as usual in gauge theories.

From the gauge-connection alone it is possible to define the tensor given by $F_{\mu\nu}$ as the Maxwell strength, as usual.

Of all Lorentz group's complex representations we will be interested in the simplest one, that is the one corresponding to the $\frac{1}{2}$-spin representation.

The differential structure is given by the most general spinorial covariant derivative $\boldsymbol{D}_\mu$ defined in terms of the most general spinorial connection and additionally we have to introduce the $\boldsymbol{\gamma}_a$ matrices belonging to the Clifford algebra $\{\boldsymbol{\gamma}_i,\boldsymbol{\gamma}_j\}=2\mathbb{I}\eta_{ij}$ from which we may define the matrices $\boldsymbol{\sigma}_{ij}=\frac{1}{4}[\boldsymbol{\gamma}_i,\boldsymbol{\gamma}_j]$ as the antisymmetric matrices belonging to the complex Lorentz algebra, called spinorial algebra, and these matrices are such that the relationship $\{\boldsymbol{\gamma}_a,\boldsymbol{\sigma}_{bc}\}=i\varepsilon_{abcd}\boldsymbol{\pi}\boldsymbol{\gamma}^d$ implicitly defines the projection matrix $\boldsymbol{\pi}$ that will be used to define the left-handed and right-handed irreducible chiral decompositions of the spinor field: here the compatibility conditions read $\boldsymbol{D}_\mu\boldsymbol{\gamma}_j=0$ and they are automatically given.

Such conditions can equivalently be expressed as

$$\boldsymbol{\Gamma}_\mu=\tfrac{1}{2}\Gamma^{ab}{}_{\mu}\boldsymbol{\sigma}_{ab}+iqA_\mu\mathbb{I} \tag{9}$$

showing that the most general spinorial connection $\boldsymbol{\Gamma}_\mu$ can be written in terms of the Lorentz-valued spin-connection $\Gamma^{ab}{}_{\mu}\boldsymbol{\sigma}_{ab}$ so that the antisymmetry of the spin-connection that is supposed to encode the Lorentz structure does so because the Lorentz complex representation is given in terms of matrices that are antisymmetric themselves plus an abelian term ieA_μ that now may be identified to the gauge field that is described in terms of the gauge-connection: it is intriguing that the most general spinorial connection contains the room to host exactly the tetrad fields as well as one abelian gauge potential, because in the interpretation that will follow quite naturally we will have that the tetrad fields are what contains the gravitational information and the abelian gauge potential is what represents the electrodynamic force.

Finally, from the spinorial connection alone it is possible to define the spinorial version of the Riemann tensor

$$\boldsymbol{G}_{\mu\nu}=\partial_\mu\boldsymbol{\Gamma}_\nu-\partial_\nu\boldsymbol{\Gamma}_\mu+\boldsymbol{\Gamma}_\mu\boldsymbol{\Gamma}_\nu-\boldsymbol{\Gamma}_\nu\boldsymbol{\Gamma}_\mu \tag{10}$$

again in the same fashion: it is of course antisymmetric.

Eventually, we have the decomposition given by

$$\boldsymbol{G}_{\mu\nu}=\tfrac{1}{2}G^{ab}{}_{\mu\nu}\boldsymbol{\sigma}_{ab}+iqF_{\mu\nu}\mathbb{I} \tag{11}$$

showing that the most general spinorial curvature is written in terms of the Lorentz-valued Riemann curvature plus the Maxwell strength: as before here too we may appreciate the fact that the most general spinorial curvature contains the contribution of the gravitational curvature as well as the electrodynamic strength, and in such a description the gravitational and electrodynamic fields fit together so well that it is tempting to regard the present circumstance as a geometrically-inspired unification.

With these definitions, the commutator of spinorial covariant derivatives of the spinor field is given by

$$[\boldsymbol{D}_\mu,\boldsymbol{D}_\nu]\psi=Q^\rho{}_{\mu\nu}\boldsymbol{D}_\rho\psi+\boldsymbol{G}_{\mu\nu}\psi \tag{12}$$

in terms of the torsion and both curvatures identically.

This introduction of the general setting that will constitute the underlying background of the paper served to settle the basic notation and conventions we will employ throughout the present article; although this has been done to render this paper somewhat self-contained, a more extensive exposition of the geometry can be found for instance in reference [21] and references therein.

A. Background Geometry

Now that the background geometry has been defined, we may proceed to study the dynamics of the geometry and the matter it will contain, which is done by assigning the dynamical action or equivalently the dynamical Lagrangian for the system we want to study.

In general, the action or the Lagrangian may contain up to an infinite number or terms, but this of course means that there will correspondingly be an infinite number of parameters to tune and in turn this diminishes the predictive power: to avoid this, one has to determine the Lagrangians by fixing them to a limited number of contributions, and this is done with some assumptions.

One such assumption is having the Lagrangian at the least-order derivative possible, that is having the contributions limited to those that have the lowest order of derivatives: from a theoretical perspective, lowest order of derivatives means fewest integration constants that will have to be chosen in looking for solutions; with this assumption, of all possible theories those that are picked are Einstein gravitation and Maxwell electrodynamics.

It would appear that this principle seems to possess a certain degree of viability, since it selects the two most successful theories ever established in physics; but on the other hand, there may be doubts cast on it, for the fact that Einstein gravity does not have some of the features modern physics would demand, such as renormalizability.

As an alternative, one may then require the Lagrangian to be renormalizable, that is the contributions are limited to those that have 4-dimension of mass in the kinetic term and 4-dimension of mass and lower for interacting terms in general: theoretically, having up to 4-dimension of mass means that when we scale the model as to reach higher energies, the kinetic terms will still be the most relevant contributions; with this assumption, of all possible theories those that are picked are various models of extended gravity together with Maxwell electrodynamics.

Despite the fact that the requirement of renormalizability demands for the replacement of Einstein gravity with one of its possible extensions, nevertheless we know of no such extension that is also free of problems: for instance, it is well known that higher-order theories of gravitation have problems of unitarity and, although in the context of second-order theories of gravitation there may be symmetries protecting these theories from being non-unitary [26, 27], nevertheless these results have never been proven a general second-order theory of gravity and let alone in higher-order theories of gravity, and so far as we can tell there is not a single problem-free extended model of gravity that is viable at present.

The assumption of least-order derivative gives rise to Einstein gravity, which is not renormalizable; the requirement of renormalizability prompts the search for an extension of gravity, which nevertheless is yet to be found.

If we wish to discriminate between these two hypotheses, a first thing we may notice is the fact that a theory at the least-order derivative is characterized by the smallest number of constants; the requirement of renormalizability does not have the same degree of theoretical value because limiting the Lagrangians to 4-dimension of mass so that high energy regimes the kinetic terms will still be the most relevant contributions is meaningful only if high energy regimes are always such that kinetic terms are the most relevant contributions, but this may not always be the case: the whole idea relies on the belief that at short distances the physical properties must be like those al large distances, which may be false; on the other hand, it is true in general the smallest number of constants makes a given model the most predictive possible.

However, we would like to introduce yet another requirement, one which might be more comprehensive than the two just discussed. We already stressed in general torsion is present and it couples to the curvature in such a way that the field equations for gravity contain torsion contributions and the field equations for torsion contain curvature contributions; and while the field equations for gravity are sourced by energy the field equations for torsion are sourced by spin. But there is no perfect symmetry between the roles of energy and spin, and whereas there all fields have an energy density not all fields have spin density: if the spin density tends to be smaller, torsion has to be smaller as well, but if in the spin-torsion field equations there are curvatures, they will not necessarily vanish; the spin-torsion field equations will remain constraints on the curvature, not identically verified.

In this sense then, a theory with torsion may always be taken in the torsionless limit, but in this limit, it may be such that the curvature will be constrained in a way that is not always verified, and therefore we will speak of non-continuity: in reference [28] we have started to discuss the non-continuity of specific gravitational models.

In the next section of the present paper, we will recall the concepts that have been first exposed in the above reference, and then proceed to deepen the investigation.

1. *Torsion-curvature crossed terms*

To begin our discussion, the first step is to consider the most general Lagrangian that can be assigned for a system describing the torsional completion of gravitation in order to study the general continuity of the dynamics.

As it is clear from (1-2), we may always separate metric and torsion and decompose the latter in three irreducible parts, and as it is clear from (5), all of these parts will have mutual interactions between one another, and as a consequence, there is no loss of generality in treating all these quantities in their split form, and accounting for all interactions as well: we have then the Riemann metric curvature $R_{\alpha\mu\rho\sigma}$ and the three irreducible parts of torsion given by $T_{\rho\mu\nu}$, W_{α} and V_{ν} which have to be taken in all possible combinations, which have to be contracted in all indices configurations in order to give rise to all possible scalar terms; furthermore, it is known that for the curvature tensor we have the validity of the condition given by $R^{\rho}{}_{\sigma\mu\nu}+R^{\rho}{}_{\nu\sigma\mu}+R^{\rho}{}_{\mu\nu\sigma}\equiv 0$ and the Bianchi identities $\nabla_{\mu}R^{\nu}{}_{\iota\sigma\rho}+\nabla_{\sigma}R^{\nu}{}_{\iota\rho\mu}+\nabla_{\rho}R^{\nu}{}_{\iota\mu\sigma}\equiv 0$ while for the non-completely antisymmetric irreducible tensorial decomposition of the torsion tensor we have the constraint that is given by $T_{\rho\mu\nu}+T_{\mu\nu\rho}+T_{\nu\rho\mu}=0$ whose contraction is given according to $T^{\mu\nu\rho}\varepsilon_{\alpha\beta\mu\nu}=-\frac{1}{2}T^{\rho\mu\nu}\varepsilon_{\alpha\beta\mu\nu}$ itself with its own contraction $T^{\rho\mu\nu}\varepsilon_{\alpha\rho\mu\nu}=0$ by construction, and these are the set of identities needed to reduce all possible scalars to the core of independent scalar terms.

The resulting Lagrangian, obtained as the sum of these scalar terms, will be distinguish in three classes: the first class includes the least-order derivative models and it is the 2-dimensional mass model; then there will be the renormalizable 4-dimensional mass model; finally there will be all the remaining n-dimensional mass models.

Now, let us start the discussion about the possibility that a given Lagrangian yield field equations that display non-continuity, whose underlying mechanics works as it follows: if in the Lagrangian there appears a term that is linear in the torsion tensor, then upon varying the Lagrangian the torsion-spin field equations will be formally written in the form of a combination of derivatives of torsion and possibly curvatures plus a spurious term without any torsion in it equals to the spin density of the system: in the limit in which both the spin density and torsion tend to vanish, the spin-torsion field equations will remain in the form of the spurious term equals to zero, which is a constraint that is in general not necessarily verified; consequently, if we want no such circumstance, we must have a spin-torsion field equation that contains no spurious term; hence in the Lagrangian there must be no linear torsion term whatsoever.

To begin our analysis, let us consider the least-order derivative model: the 2-dimensional mass model is characterized by a Lagrangian that can only contain one curvature and five square-torsion according to the form

$$\mathcal{L}=R+AT_{\rho\mu\nu}T^{\rho\mu\nu}+BW_{\nu}W^{\nu}+CV_{\mu}V^{\mu}+$$
$$+LT^{\rho}{}_{\mu\nu}T_{\rho\eta\alpha}\varepsilon^{\mu\nu\eta\alpha}+MW_{\mu}V^{\mu} \tag{13}$$

where the Newton constant is normalized to unity and with five torsional constants; terms that are linear in torsion are only divergences of vectorial parts of torsion

$$\Delta\mathcal{L}=U\nabla_{\mu}W^{\mu}+Z\nabla_{\mu}V^{\mu} \tag{14}$$

and therefore dropped as irrelevant. Consequently, we have that in the most general case continuity is ensured.

The following step consists in proceeding to the analysis of the renormalizable model: the 4-dimensional mass model is characterized by a Lagrangian that can only contain squared-curvature, quartic-torsion, products of curvatures and squared-torsion, derivatives of cubic-torsion, second-derivatives of squared-torsion, and derivatives of products between curvature and torsion; terms that are linear in torsion are the derivatives of products between curvature and torsion, and it is possible to see that the independent contractions are the derivatives of the Ricci curvature scalar times the trace and axial vectorial parts of torsion according to the combination given by

$$\Delta\mathcal{L}=K\nabla_{\alpha}RW^{\alpha}+E\nabla_{\alpha}RV^{\alpha} \tag{15}$$

and nothing else at all. In order for this contribution to disappear, because of the independence of the two terms, then each single term must vanish, and since we want to maintain continuity, then this has happen regardless the derivatives of the Ricci curvatures; we would also like to avoid arbitrary tunings of the parameters, so that we are not going to require their vanishing unless some principle justifies this assumption: although parity-evenness may be invoked to have K vanished, there is no known principle for which E should vanish too and then we have to insist that $V^{\alpha}=0$ as well. When this is done, we acknowledge that continuity is preserved in the case in which no tuning is assumed when we require parity-evenness and irreducibility of torsion: irreducibility of torsion and parity-evenness are together necessary and sufficient conditions for the most general model to have continuity.

The case of further models is easy: the n-dimensional mass models for n that is larger than 4 are characterized by Lagrangians with all possible terms; consequently, terms that are a derivative of curvature times a curvature times the non-completely antisymmetric tensorial part of torsion such as $\nabla_{\rho}R_{\alpha\nu}R^{\sigma\nu}T^{\rho\alpha}{}_{\sigma}$ and a derivative of curvature times a curvature times the axial vectorial part of torsion such as $\nabla_{\rho}R_{\alpha\nu}R_{\pi}{}^{\nu}W_{\kappa}\varepsilon^{\kappa\rho\alpha\pi}$ are parity-even and yet they do no vanish unless in addition we require the constraints $T_{\pi\sigma\eta}=0$ and $W_{\iota}=0$ as well. What this means is that for continuity to be preserved if no tuning is assumed the parity-evenness and irreducibility of torsion are no longer enough and one has to go so far as to require the vanishing of the whole torsion, which is against the possibility to provide the coupling to spin also for systems that have such coupling; hence there is no way in which the most general model may be continuous.

We may now summarize our results: 2-dimensional mass models are continuous; 4-dimensional mass models are continuous if and only if they are invariant under parity and torsion is irreducible; n-dimensional mass models are not continuous. Thus continuity allows only two models: either the one described by least-order derivative Lagrangians, or the one with irreducible torsion and parity-invariance described by renormalizable Lagrangians.

2. Abelian gauge fields

Next we shall proceed to the discussion about general Lagrangians that can be assigned for electrodynamics.

A first point that needs to be clarified is the fact that in presence of torsion there is a generalization of the covariant derivatives of tensors that might in principle create problems in electrodynamics: for example, the definition of the Maxwell tensor is given as the strength of the gauge-connection $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ and, if we see this definition as the curl of the gauge-connection, because we would like to stay in the most general case, then we should take the curl of the most general covariant derivatives $F_{\mu\nu}=D_{\mu}A_{\nu}-D_{\nu}A_{\mu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}+Q^{\rho}{}_{\nu\mu}A_{\rho}$ which is not gauge invariant precisely because of the presence of the torsion tensor; it would seem that in this case our argument of generality has failed, but to a closer look it is possible to see that instead this apparent contradiction is based on the fact that we have decided to interpret the strength as the curl of the gauge-connection, but this is not the correct interpretation that should be given, since the correct interpretation is to see the strength as the commutator of the gauge-covariant derivatives, in terms of which the most general definition of strength is exactly the one given by $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ because none of its generalizations would maintain $F_{\mu\nu}$ as to be the commutator of the gauge-covariant derivatives. Hence, Cauchy identities $\nabla_{\mu}F_{\sigma\rho}+\nabla_{\sigma}F_{\rho\mu}+\nabla_{\rho}F_{\mu\sigma}\equiv 0$ are unchanged, and similarly as above they will be needed to reduce all possible scalars to the core of independent scalar terms.

The absence of torsion within the definition of the curvature for the gauge-connection does not mean that there is no interaction between torsion-gravity and electrodynamics, and to show that in fact there can be torsional interactions in electrodynamics we study the most general model for electrodynamics: the 2-dimensional mass models cannot be defined; the least-order derivative model coincides with the renormalizable one and it is included in the class of the 4-dimensional mass model.

The construction of the Lagrangian will have to adhere to the requisite of continuity; we require the spin-torsion field equation to contain no spurious term, and hence the Lagrangian to contain no linear torsion term.

As it was anticipated, the least-order derivative is the renormalizable model: the 4-dimensional mass model is characterized by a Lagrangian that can only contain two strengths, products of one strength and two irreducible parts of torsion, and derivatives of strength times one irreducible part of torsion; terms that are linear in torsion are the derivatives of strength times one irreducible part of torsion, whose independent contractions are given by

$$\Delta\mathcal{L}=J\nabla_{\alpha}F^{\mu\nu}T^{\alpha\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}+K\nabla_{\alpha}F^{\alpha\nu}W_{\nu}+ \\ +E\nabla_{\alpha}F_{\mu\nu}T^{\alpha\mu\nu}+F\nabla_{\alpha}F^{\alpha\nu}V_{\nu} \tag{16}$$

and nothing else more. As it was before, this contribution disappears when each single term vanishes: again parity-evenness may be invoked to set J and K equal to zero, and then $T^{\alpha\mu\nu}=0$ and $V^{\alpha}=0$ too. Thus, continuity is preserved in the case in which no tuning is assumed if we require parity-evenness and the presence of only the axial vectorial part of torsion: an axial vectorial part of torsion and parity-evenness are together necessary and sufficient conditions for the most general model to be continuous.

Although in principle torsion might have affected electrodynamics we have seen that this is not the case, but conversely electrodynamics has left a mark on torsion, as the electrodynamic 4-dimensional mass model is continuous if and only if electrodynamics is invariant under parity and torsion is the completely antisymmetric dual of an axial vector. There is one model with completely antisymmetric torsion and parity-conservation described by least-order derivative renormalizable Lagrangians.

B. Material Content

So far we have extensively discussed and thoroughly investigated the theory of torsional-gravitation with electrodynamics; the next and last step is that of introducing the general Lagrangian for Dirac spinorial matter fields.

As we may always separate the left-handed and right-handed semi-spinorial chiral projections, it is easier to start with them: hence, given the left-handed and right-handed semi-spinors ψ_L and ψ_R as the two chiral projections, then $\psi_L=\frac{1}{2}(\mathbb{I}-\boldsymbol{\pi})\psi$ and $\psi_R=\frac{1}{2}(\mathbb{I}+\boldsymbol{\pi})\psi$ are the two ways in which we may get the reconstruction of the full spinor ψ which we will eventually employ in order to have a more compact notation in further developments.

The inventory of all possible terms is quick: the least-order derivative model is the renormalizable model and it has contributions up to the 4-dimensional mass terms

$$\mathcal{L}=A\tfrac{i}{2}\left[\overline{\psi}_L\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_L-\boldsymbol{\nabla}_{\mu}\overline{\psi}_L\boldsymbol{\gamma}^{\mu}\psi_L\right]+ \\ +Z\tfrac{i}{2}\left[\overline{\psi}_R\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi_R-\boldsymbol{\nabla}_{\mu}\overline{\psi}_R\boldsymbol{\gamma}^{\mu}\psi_R\right]+ \\ +C\overline{\psi}_L\boldsymbol{\gamma}^{\mu}\psi_L V_{\mu}+H\overline{\psi}_L\boldsymbol{\gamma}^{\mu}\psi_L W_{\mu}+ \\ +S\overline{\psi}\boldsymbol{\gamma}^{\mu}_{R}\psi_R V_{\mu}+X\overline{\psi}_R\boldsymbol{\gamma}^{\mu}\psi_R W_{\mu}- \\ -\beta\overline{\psi}_L\psi_R-\beta^{*}\overline{\psi}_R\psi_L \tag{17}$$

where A, Z and C, H, S, X are real while β is complex and all these parameters have still to be undetermined.

Because left-handed and right-handed semi-spinorial chiral projections must both have positive-defined energy then their kinetic terms must have the same sign, so that it is possible through a rescaling of ψ_L and ψ_R to set the parameters A and Z equal to unity and therefore we may write the Lagrangian equivalently in the compact form

$$\mathcal{L}=\tfrac{i}{2}\left(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\boldsymbol{\nabla}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi\right)+ \\ +K\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi W_{\mu}+F\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi W_{\mu}+ \\ +E\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi V_{\mu}+J\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi V_{\mu}- \\ -m\overline{\psi}\psi-ib\overline{\psi}\boldsymbol{\pi}\psi \tag{18}$$

where E, F, J, K and m and b are real parameters.

As it is clear, the non-completely antisymmetric irreducible tensorial part of torsion is absent and there is parity-invariance in the kinetic term although there is no definite parity in the potential terms, and as already said this model has the least-order derivative Lagrangian which is also the renormalizable Lagrangian.

CONSEQUENCES

In the previous sections, we have discussed what happens when a model taking into account the torsional completion of gravitation with electrodynamics and Dirac fields is investigated under the requirement of continuity in the general case in which no arbitrary tuning is imposed, and we have found that there are only two possible circumstances: one in which for torsion-gravity there are 2-dimensional mass terms and for electrodynamics there are 4-dimensional mass terms with completely antisymmetric torsion and parity-invariance, and for the Dirac matter field there are up to 4-dimensional mass terms; another in which for torsion-gravity there are 4-dimensional mass terms with irreducible torsion and parity-invariance and for electrodynamics in which there are 4-dimensional mass terms with completely antisymmetric torsion and parity-invariance, and for the Dirac matter field there are up to 4-dimensional mass terms.

Then because when the torsion is restricted to have complete antisymmetry in one sector of the theory so it is restricted in the entire theory, we will take torsion to be the completely antisymmetric dual of an axial vector, and as a consequence of this constraint all parity-odd terms in principle allowed in the 2-dimensional mass model of torsion-gravity disappear: so the two models may be condensed together into the single Lagrangian as given by

$$\mathcal{L}=NR_{\alpha\mu}R^{\alpha\mu}+PR^{2}+SR_{\alpha\nu}W^{\alpha}W^{\nu}+URW_{\nu}W^{\nu}+ \\ +X(\nabla_{\alpha}W_{\nu}-\nabla_{\nu}W_{\alpha})(\nabla^{\alpha}W^{\nu}-\nabla^{\nu}W^{\alpha})+ \\ +Y\nabla_{\alpha}W_{\nu}\nabla^{\alpha}W^{\nu}+H\left|W_{\nu}W^{\nu}\right|^{2}- \\ -kR+BW_{\nu}W^{\nu}-\tfrac{1}{4}F^{\alpha\nu}F_{\alpha\nu}+ \\ +\tfrac{i}{2}\left(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\boldsymbol{\nabla}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi\right)+ \\ +K\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi W_{\mu}+F\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi W_{\mu}-m\overline{\psi}\psi-ib\overline{\psi}\boldsymbol{\pi}\psi \tag{19}$$

where N, P, S, U, X, Y, H, B, K, F, k, m, b are real, and for $N=P=S=U=X=Y=H=0$ we have the simplest least-order derivative Lagrangian while otherwise we have the most general renormalizable Lagrangian.

This Lagrangian yields field equations whose consistency in terms of the amount of degrees of freedom and the character of propagation is to be checked with the method presented in [29] by Velo and Zwanziger.

C. Consistent Propagation

Just above we have given what, under the requirement of continuity, is the Lagrangian in its most general form

$$\begin{aligned}\mathcal{L}=&NR_{\alpha\mu}R^{\alpha\mu}+PR^2+SR_{\alpha\nu}W^{\alpha}W^{\nu}+URW_{\nu}W^{\nu}+\\&+X(\nabla_{\alpha}W_{\nu}-\nabla_{\nu}W_{\alpha})(\nabla^{\alpha}W^{\nu}-\nabla^{\nu}W^{\alpha})+\\&+Y\nabla_{\alpha}W_{\nu}\nabla^{\alpha}W^{\nu}+H\left|W_{\nu}W^{\nu}\right|^2-\\&-kR+BW_{\nu}W^{\nu}-\tfrac{1}{4}F^{\alpha\nu}F_{\alpha\nu}+\\&+\tfrac{i}{2}\left(\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\nabla}_{\mu}\psi-\boldsymbol{\nabla}_{\mu}\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi\right)+\\&+K\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi W_{\mu}+F\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi W_{\mu}-m\overline{\psi}\psi-ib\overline{\psi}\boldsymbol{\pi}\psi\end{aligned}\tag{20}$$

containing a set of 13 free parameters in total.

Now we will show how with the Velo-Zwanziger analysis the number of independent parameters is reduced.

First we notice that for the least-order derivative Lagrangian, there is no dynamical term in the torsion field equations and no such analysis that can be performed.

In the renormalizable Lagrangian varying with respect to the axial vector torsion gives the field equations

$$\begin{aligned}&2(2X+Y)\nabla^2W^{\nu}-4X\nabla^{\nu}\nabla_{\alpha}W^{\alpha}-\\&-2(S+2X)R^{\alpha\nu}W_{\alpha}-2URW^{\nu}-4HW^2W^{\nu}-\\&-2BW^{\nu}=F\overline{\psi}\boldsymbol{\gamma}^{\nu}\psi+K\overline{\psi}\boldsymbol{\gamma}^{\nu}\boldsymbol{\pi}\psi\end{aligned}\tag{21}$$

which is in fact a field equation because one may solve for the second-order time derivative of every component of the axial vector torsion; its divergence is given by

$$\begin{aligned}&2Y\nabla^2\nabla_{\alpha}W^{\alpha}-2[(S-Y)R^{\alpha\nu}+4HW^{\alpha}W^{\nu}]\nabla_{\nu}W_{\alpha}-\\&-2(UR+2HW^2+B)\nabla_{\nu}W^{\nu}-\\&-(S-Y+2U)\nabla_{\alpha}RW^{\alpha}=\nabla_{\nu}\left(F\overline{\psi}\boldsymbol{\gamma}^{\nu}\psi+K\overline{\psi}\boldsymbol{\gamma}^{\nu}\boldsymbol{\pi}\psi\right)\end{aligned}\tag{22}$$

which develops a third-order time derivative of the temporal component of the axial vector torsion, and therefore the system of field equations is not well defined, unless we require $Y=0$ hold as a constraint on the parameter.

When this is done, we find the field equation

$$\begin{aligned}&4X(\nabla^2W^{\nu}-\nabla^{\nu}\nabla_{\alpha}W^{\alpha})-\\&-2(S+2X)R^{\alpha\nu}W_{\alpha}-2URW^{\nu}-4HW^2W^{\nu}-\\&-2BW^{\nu}=F\overline{\psi}\boldsymbol{\gamma}^{\nu}\psi+K\overline{\psi}\boldsymbol{\gamma}^{\nu}\boldsymbol{\pi}\psi\end{aligned}\tag{23}$$

which is no longer a true field equation because the second-order time derivative of the temporal component of the axial vector torsion never appears; however, its divergence is now given according to the expression

$$\begin{aligned}&-2[SR^{\alpha\nu}+4HW^{\alpha}W^{\nu}]\nabla_{\nu}W_{\alpha}-\\&-2(UR+2HW^2+B)\nabla_{\nu}W^{\nu}-\\&-(S+2U)\nabla_{\alpha}RW^{\alpha}=\nabla_{\nu}\left(F\overline{\psi}\boldsymbol{\gamma}^{\nu}\psi+K\overline{\psi}\boldsymbol{\gamma}^{\nu}\boldsymbol{\pi}\psi\right)\end{aligned}\tag{24}$$

which has no second-order time derivative of the temporal component of the axial vector torsion and therefore it is a true constraint: hence it is possible to substitute the constraint (24) back into the field equation (23) getting

$$\begin{aligned}&4X\nabla^2W^{\nu}+4X[2(UR+2HW^2+B)]^{-1}\cdot\\&\cdot\nabla^{\nu}[2[SR^{\alpha\mu}+4HW^{\alpha}W^{\mu}]\nabla_{\mu}W_{\alpha}+\\&+(S+2U)\nabla_{\alpha}RW^{\alpha}\\&+\nabla_{\mu}\left(F\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi+K\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi\right)]-\\&-8X\nabla^{\nu}(UR+2HW^2)\cdot\\&\cdot[2(UR+2HW^2+B)]^{-2}\cdot\\&\cdot[2[SR^{\alpha\mu}+4HW^{\alpha}W^{\mu}]\nabla_{\mu}W_{\alpha}+\\&+(S+2U)\nabla_{\alpha}RW^{\alpha}\\&+\nabla_{\mu}\left(F\overline{\psi}\boldsymbol{\gamma}^{\mu}\psi+K\overline{\psi}\boldsymbol{\gamma}^{\mu}\boldsymbol{\pi}\psi\right)]-\\&-2(S+2X)R^{\alpha\nu}W_{\alpha}-2URW^{\nu}-4HW^2W^{\nu}-\\&-2BW^{\nu}=F\overline{\psi}\boldsymbol{\gamma}^{\nu}\psi+K\overline{\psi}\boldsymbol{\gamma}^{\nu}\boldsymbol{\pi}\psi\end{aligned}\tag{25}$$

which is again a true field equation because the second-order time derivative of every component of the axial vector torsion is present, so that the number of independent field equations corresponds to the number of physical degrees of freedom consistently. To see what happens about the propagation, we have to consider in this field equation only the highest-order derivatives, and after the substitution $i\nabla_{\alpha}\rightarrow n_{\alpha}$ the characteristic equation is

$$\begin{aligned}&(UR+2HW^2+B)n^2+SR^{\alpha\nu}n_{\alpha}n_{\nu}+\\&+4H|W^{\alpha}n_{\alpha}|^2=0\end{aligned}\tag{26}$$

and field equations (25) cease to be causal when the characteristic equation (26) allows $n^2>0$ to occur: in the case in which torsion is small, and in which also curvature is small, then the characteristic equation becomes

$$Bn^2+SR^{\alpha\nu}n_{\alpha}n_{\nu}\approx 0\tag{27}$$

and as we have no information about $R^{\alpha\nu}$ acausality may occur, unless $S=0$ holds as constraint, but even then, in the same approximation in which torsion is small, but in the complementary approximation in which curvature is large, the characteristic equations becomes

$$URn^2+4H|W^{\alpha}n_{\alpha}|^2\approx 0\tag{28}$$

and as we have no information about R acausality may occur, unless $U=0$ holds as constraint; then, in the case in which torsion is large, the characteristic equation is

$$2HW^2n^2+4H|W^{\alpha}n_{\alpha}|^2\approx 0\tag{29}$$

and because the axial vector torsion cannot have only one degree of freedom then W^2 cannot be time-like and causality may occur, unless $H=0$ holds as a constraint.

We may finally summarize: in the least-order derivative model $N=P=S=U=X=Y=H=0$ are imposed by definition of the model we are considering, with no need of further analysis; in the renormalizable model torsion is dynamical and relevant at all scales, so that there are field equations that can be studied and it is possible to assume the validity of the approximations we have considered getting $Y=S=U=H=0$ as the most stringent constraints that are possible. As a consequence, we will assume that $N=P=X=0$ for the least-order derivative Lagrangian as $Y=S=U=H=0$ will be assumed for the renormalizable Lagrangian in the most general case.

D. Effective Interaction

In this discussion, we have seen that the most general Lagrangian is constrained in 4 of its parameters as

$$\begin{aligned}\mathcal{L}=&NR_{\alpha\mu}R^{\alpha\mu}+PR^2+\\ &+X(\nabla_\alpha W_\nu-\nabla_\nu W_\alpha)(\nabla^\alpha W^\nu-\nabla^\nu W^\alpha)+\\ &-kR+BW_\nu W^\nu-\tfrac{1}{4}F^{\alpha\nu}F_{\alpha\nu}+\\ &+\tfrac{i}{2}\left(\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-\boldsymbol{\nabla}_\mu\overline{\psi}\boldsymbol{\gamma}^\mu\psi\right)+\\ &+K\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi W_\mu+F\overline{\psi}\boldsymbol{\gamma}^\mu\psi W_\mu-m\overline{\psi}\psi-ib\overline{\psi}\boldsymbol{\pi}\psi\end{aligned}\quad(30)$$

containing only 9 free parameters still undetermined.

We may proceed to study the effective interactions.

The most interesting case is the one given by the least-order derivative Lagrangian because variation respect to the axial vector torsion yields the field equation

$$-2BW^\nu=K\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi+F\overline{\psi}\boldsymbol{\gamma}^\nu\psi\quad(31)$$

which is an algebraic constraint that can be used to have torsion substituted in terms of the spin of the spinorial field so that when the spinor field is rearranged in terms of the identity $\overline{\psi}\boldsymbol{\gamma}_\nu\psi\overline{\psi}\boldsymbol{\gamma}^\nu\psi=-\overline{\psi}\boldsymbol{\gamma}_\nu\boldsymbol{\pi}\psi\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi$ together with identity $\overline{\psi}\boldsymbol{\gamma}_\nu\psi\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi=0$ then the spinorial field equations reduce to the form given by the expression

$$i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-\tfrac{F^2-K^2}{2B}\overline{\psi}\boldsymbol{\gamma}_\mu\psi\boldsymbol{\gamma}^\mu\psi-m\psi-ib\boldsymbol{\pi}\psi=0\quad(32)$$

in which the mass-like term proportional to the parameter b is the only parity-odd term that is left and the torsionally-induced non-linear potentials are identical to those we would have had if there were no torsion but effective interactions of the Nambu-Jona–Lasinio form with constant $(F^2-K^2)/B$ unknown [30]; for the renormalizable Lagrangian the torsion field equations have form

$$\nabla_\alpha\nabla^{[\alpha}W^{\nu]}-\tfrac{B}{2X}W^\nu=\tfrac{F}{4X}\overline{\psi}\boldsymbol{\gamma}^\nu\psi+\tfrac{K}{4X}\overline{\psi}\boldsymbol{\gamma}^\nu\boldsymbol{\pi}\psi\quad(33)$$

and they are differential, so that one cannot have torsion substituted and the spinorial field equations remain

$$i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi+KW_\mu\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi+FW_\mu\boldsymbol{\gamma}^\mu\psi-m\psi-ib\boldsymbol{\pi}\psi=0\quad(34)$$

as it is to be expected since interactions such as those described above are not renormalizable, and therefore there is no way they can arise within a renormalizable model.

A comparison between the two models shows that in the infra-red approximation the renormalizable torsion field equations (33) reduce to the least-order torsion field equations (31) so that the same reduction happens for the spinorial field equations (34) and (32): as expected, the least-order model can be seen as the low-energy regime of the renormalizable model, but with the fundamental difference that in growing with the energy the least-order model will always remain similar to an effective model while the renormalizable model will turn back to look like the renormalizable model and as such a torsion boson is to be expected. As a matter of fact, this can be seen precisely as the fundamental discrimination between the two instances, namely as we go up with the energy the least-order derivative Lagrangian will always look like an effective Lagrangian with no associated torsion boson while the renormalizable Lagrangian will become the Lagrangian of a massive neutral vector boson.

DISCUSSION

So far in the paper, the torsional completion of gravitation with electrodynamics and Dirac fields has been studied in the most general case although with the requirement of continuity in the torsionless limit, we could impose constraints such as the complete antisymmetry of torsion and parity-invariance in the dynamical terms of the action; we have also seen that with the further requirement of having a consistent propagation, both in terms of the amount of degrees of freedom with the correct time evolution and in view of the causality of the propagation, we could impose additional constraints on the terms allowed in the Lagrangian: we then had

$$\begin{aligned}\mathcal{L}=&NR_{\alpha\mu}R^{\alpha\mu}+PR^2+\\ &+X(\nabla_\alpha W_\nu-\nabla_\nu W_\alpha)(\nabla^\alpha W^\nu-\nabla^\nu W^\alpha)+\\ &-kR+BW_\nu W^\nu-\tfrac{1}{4}F^{\alpha\nu}F_{\alpha\nu}+\\ &+\tfrac{i}{2}\left(\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-\boldsymbol{\nabla}_\mu\overline{\psi}\boldsymbol{\gamma}^\mu\psi\right)+\\ &+K\overline{\psi}\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi W_\mu+F\overline{\psi}\boldsymbol{\gamma}^\mu\psi W_\mu-m\overline{\psi}\psi-ib\overline{\psi}\boldsymbol{\pi}\psi\end{aligned}\quad(35)$$

where $N=P=X=0$ were the constraints defining the least-order derivative Lagrangian and while otherwise we have the general renormalizable Lagrangian.

We have also discussed that the least-order model can be seen as the low-energy regime of the renormalizable model in which the spinor matter field equations are

$$i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi-\tfrac{F^2-K^2}{2B}\overline{\psi}\boldsymbol{\gamma}_\mu\psi\boldsymbol{\gamma}^\mu\psi-m\psi-ib\boldsymbol{\pi}\psi=0\quad(36)$$

while in the renormalizable model these are only the low-energy limit of the general spinor matter field equations

$$i\boldsymbol{\gamma}^\mu\boldsymbol{\nabla}_\mu\psi+KW_\mu\boldsymbol{\gamma}^\mu\boldsymbol{\pi}\psi+FW_\mu\boldsymbol{\gamma}^\mu\psi-m\psi-ib\boldsymbol{\pi}\psi=0\quad(37)$$

in which the axial vector torsion W_μ represents a massive neutral vector boson: in order to discriminate between the two instances, we may say that in the least-order derivative Lagrangian the torsion boson has to be absent or at least if it is present it cannot be fundamental while in the renormalizable Lagrangian the torsion boson must appear as fundamental at all the possible energy scales.

The reason for which at the moment the renormalizable Lagrangian tends to be favoured is that no torsion boson has ever appeared, and the limits are being pushed further and further [31–33]; it is true that in these papers the experiments that place stringent limits are always performed in non-relativistic environments, while relativistic measures must be done. Nevertheless, there is no evidence of torsion bosons at the LHC as well.

This would constitute yet another indication in the context of torsion-gravity against the assumption of having renormalizable Lagrangians; however, we do not see this situation so bad: as discussed in [34], in the case of non-minimal coupling, which is in general not renormalizable as it is well known, when taken in torsion-gravity, which is not renormalizable itself, gives rise to the peculiar circumstance for which the effective interactions are renormalizable instead. This seems to point out that the concept of renormalizability might have to be rethought in the context of torsion-gravity; for instance, the requirement that gravity must be relevant even at extremely small scales might be a prejudice coming from the fact that we think at gravity as a force like any other force while this is not. As the property for which some terms ought be relevant at all scales presupposes the knowledge of a physics that is for now precluded, renormalizability cannot not be taken as a fundamental principle.

The fundamental idea about renormalizability is based on the pillar for which we believe that at all scales and thus even for small scales, the kinetic term of a field must never be suppressed by some interaction of that field, but this assumption is arbitrary: if it were possible to find a regime in which a field were suppressed by its interactions then this would merely mean that in such regime the field would tend to vanish, which is not in contradiction with any known fact. In fact, that locally gravity should disappear is not only reasonable, but it is its very essence.

Insisting that at small scales gravity must be relevant may have the same meaning of insisting that at large scales quantum effects must be relevant.

CONCLUSION

In the present paper, we have studied torsion-gravity with electrodynamics and Dirac fields with the aim of staying in the most general case but we have seen that general arguments of continuity in the torsionless limit and consistency in time evolution and causal propagation could justify special constraints given by the fact that torsion had to be the completely antisymmetric dual of an axial vector and that parity-conservation had to be a feature of the dynamical and interacting terms in a model described by (35) with $N = P = X = 0$ defining the least-order derivative Lagrangian while otherwise being the general renormalizable Lagrangian; we have also discussed that the more general renormalizable model is however disfavoured because it predicts a torsion massive axial neutral vector boson which is undetected so far.

Although of course this does not mean that such a boson cannot be found in future researches.

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