

On Einstein-Cartan Theory: I. Kinematical description

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Equations of motion for general gravitational connection and orthonormal coframe from the Einstein-Hilbert type action are derived. Additional gauge freedom is geometrically interpreted. Our formulation does not fix coframe to be tangential to spatial section hence Lorentz group is still present as a part of gauge freedom. 3+1 decomposition introduces tangent Minkowski structures hence Hamilton-Dirac approach to dynamics works with Lorentz connection over the spatial section. The second class constraints are analyzed and Dirac bracket is defined. Reduction of the phase space is performed and canonical coordinates are introduced.

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I. INTRODUCTION

Einstein theory of General Relativity is well known and understood theory of gravitation for almost a century. Gravitational interaction is described by metric tensor \mathbf{g} of a spacetime manifold \mathbf{M} . Einstein needed to assume that gravitational connection is metric-compatible and torsion-free in his derivation of equations of motion for \mathbf{g} . We will call such connection geometrical or Riemann-Levi-Civita (RLC). Important thing is that the geometrical connection ${}^{(RLC)}\hat{\nabla}$ is uniquely determined by the metric \mathbf{g} . If we want to describe gravitational system interacting with Dirac field within General Relativity then we should express metric in terms of orthonormal coframe \mathbf{e}^a . The action of such system is given by sum of Einstein-Hilbert and Dirac actions¹, where the spacetime external derivative operator \hat{d} should be replaced by ${}^{(RLC)}\hat{\nabla}$ in order to have a final theory locally Lorentz invariant.

There exist another approach to the relativistic theory of gravitation. Interaction in Standard model of elementary particles is described by gauge potentials given by the appropriate gauge group, e.g. $U(1)$ for electromagnetism or $SU(2)$ for electroweak theory, etc. Similar ideas as in Standard model can be used in construction of gravitational theory. Kibble² used Poincaré symmetry of Minkowski spacetime and he obtained the theory where gravitational potential is given by general metric-compatible connection. In contrast to General Theory of Relativity the condition of vanishing torsion is given by equation of motion and the rest of dynamics is described by Einstein equation. In other words in the case of pure gravity these two theories are physically equivalent. Since spacetime is no longer flat the Poincaré group is no longer global symmetry of solution hence only local Poincaré symmetry plays a role of gauge group of Kibble theory³. There exists another generalization⁴ of this approach called theory of affine connection, where Lorentz group is replaced by $\mathbf{GL}(4)$.

We will see in this article, that these three theories are physically equivalent at least in the case of pure gravity. Problems occur if we want to add matter fields with Lagrangian depending on connection 1-forms. In general case these three theories are no longer equivalent. We will show that general connection can be decomposed into metric-compatible connection plus something. If matter Lagrangian depends only on metric-compatible part of general connection then Kibble theory and theory of affine connection are equivalent as we will see in last paper of this series. Physically reasonable example of such matter are all Standard model fields. Bosons do not interact directly with connection while spinor part of Lagrangian depends only on metric-compatible part

of connection. Since Standard model plus Gravity is everything what we know about the Nature at the present stage of Physics we will not distinguish between these two theories and we will call them as Einstein-Cartan theory in this series except the next section and the last paper for simplicity, where we will show an example of Bičák vector field⁵, which violates this equivalence and one must consider three different descriptions of its interaction with gravity.

We will focus on Einstein-Cartan theory in this paper. The motivation for this choice is taken from loop quantum gravity, where Ashtekar connection \mathbf{A} on a spatial section Σ is defined by RLC connection of \mathbf{q} (\mathbf{q} is a metric on Σ induced from the 4-dimensional metric \mathbf{g} of the spacetime \mathbf{M}) and an external curvature of the 4-dimensional RLC connection. Ashtekar originally began with complex connection \mathbf{A} but problems with reality conditions or hermiticity of inner product of quantum Hilbert space caused that Barbero-Immirzi parameter enters the theory and \mathbf{A} becomes real. This parameter plays no role on classical level, but after quantization it causes ambiguity and must be fixed by comparison of Hawking-Bekenstein entropy with entropy computed from loop theory. Fermionic matter was successfully added to loop gravity only on kinematical level and problem of dynamics remains unresolved. And last but not least, problem is that general theory is $\mathbf{SO}(\mathbf{g})$ invariant what is still true in the case of complex Ashtekar connection but the real loop theory broke down this explicit invariance to $\mathbf{SO}(\mathbf{q})$ ^{6,7}.

If one⁸ does not fix coframe to be tangential to Σ in opposite to euclidean loop gravity then all degrees of freedom enters the theory which can then be expressed as $\mathbf{SO}(\mathbf{g})$ gauge theory. As is shown in B this leads to the theory where torsion appears as the first class constraint in the case of 2+1 dimensional gravity what is good news for 2+1 dimensional theoretical physicists, because they can work with $\mathbf{SO}(\mathbf{g})$ gauge connection instead of 2+1 analogue of Ashtekar connection and problem of vanishing torsion can be solved on quantum level as they wish. Unfortunately in the case of 3+1 dimensional Einstein-Cartan theory the condition of vanishing the torsion is split in two parts where one is the first class and other is the second class constraints. Therefore new potential problems like introduction of ghosts might be solved on quantum level.

This is the first part of a series of three papers devoted to Einstein-Cartan theory. In this paper, we will focus on the derivation of Hamiltonian-Dirac formulation of our physical system. The paper is organized as follows. In section II, Lagrangian formulation of the Einstein-Cartan theory is formulated in the the language of forms valued in the tangent tensor algebra $\Lambda\mathbf{TM}$. Equations of motion (EOM) are derived and equivalence between theory of General Relativity and Einstein-Cartan theory is also shown in this section. 3+1 decomposition is performed in section III and also

some useful formulas are evaluated there. In section IV, the Hamiltonian of the theory is written and separation of constraints into the first and second class is performed. In section V, Dirac brackets are introduced and coordinates on the reduced phase space are defined. And in the last section VI, open problems are discussed and possible solutions are suggested. Also few comments are added about possible quantization.

II. LAGRANGIAN OF EINSTEIN-CARTAN THEORY

Let $(\mathbf{M} = \mathbb{R}[t] \times \Sigma, \mathbf{g})$ be a spacetime manifold equipped with metric \mathbf{g} (signature(\mathbf{g}) = $(+, -, -, -)$). Geroch's theorem⁹ says that a spinor structure over the manifold \mathbf{M} exists iff there exists a global orthonormal frame \mathbf{e}_a over \mathbf{M} and \mathbf{M} is orientable. These two conditions restrict possible topological shapes of \mathbf{M} and Σ , e.g. if the spacetime manifold is given by product $\mathbf{M} = \mathbb{R} \times$ "3-dimensional sphere" then Geroch's conditions are not fulfilled and the spinor structure can not be defined over such manifold, in other words if one considers Friedman's models then closed model violates Geroch's conditions. We assume Geroch's conditions already now in the case of pure gravity since spinors should be added into the theory later so there is no loss of generality¹⁰.

Let \mathbf{e}_a be a global orthonormal frame and \mathbf{e}^a its dual. Then every useful geometrical or gravitational variable can be written in a global manner. Let us look at the basic quantities:

metric (η_{ab} is Minkowski matrix: $(\eta_{ab}) = \text{diag}(+1, -1, -1, -1)$)

$$\mathbf{g} = \eta_{ab} \mathbf{e}^a \otimes \mathbf{e}^b,$$

4-volume form (ε_{abcd} is Levi-Civita antisymmetric symbol, see convention in appendix A)

$$\hat{\Sigma} = \frac{1}{4!} \varepsilon_{abcd} \mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d,$$

gravitational connection 1-form $\hat{\Gamma}^b{}_a$ (\mathbf{u} is arbitrary vector)

$$\hat{\nabla}_{\mathbf{u}} \mathbf{e}_a = \hat{\Gamma}^b{}_a(\mathbf{u}) \mathbf{e}_b$$

or its curvature 2-form

$$\hat{\mathbf{F}}^a{}_b = \hat{d}\hat{\Gamma}^a{}_b + \hat{\Gamma}^a{}_c \wedge \hat{\Gamma}^c{}_b.$$

General Relativity sets connection $\hat{\nabla}$ to be geometrical and the Einstein-Hilbert action of GR is

$$S_{\text{EH}} = \int -\frac{1}{16\pi K} R_{\mathbf{g}} \omega_{\mathbf{g}}$$

where $R_{\mathbf{g}}$ is a Ricci scalar related to the RLC connection of metric tensor \mathbf{g} , $\omega_{\mathbf{g}} = \sqrt{-\det|\mathbf{g}|}d^4x$ is its volume form and κ is Newton's constant ($c=1$). The action written in this form explicitly depends on the choice of coordinates and one should overlap few coordinate's neighbourhoods and solve boundary terms if one wants to cover the whole manifold \mathbf{M} in general case. But if we use our assumption about \mathbf{e}^a then we can rewrite the Einstein-Hilbert action into the following geometrical form

$$S_{\text{EH}} = \int -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \eta^{b\bar{b}} \mathbf{R}_{\mathbf{g}}^a{}_{\bar{b}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d, \quad (1)$$

where $\mathbf{R}_{\mathbf{g}}^a{}_{\bar{b}}$ is a curvature 2-form of RLC connection. The action (1) is a functional of basic variables $\mathbf{e}^a = \mathbf{e}_{\mu}^a dx^{\mu}$ ($\mu = 0, \alpha = 0, 1, 2, 3$ are spacetime coordinate's indices) and we should make variation of the action with respect to them. The idea of the theory of affine connection is very simple, gravitational connection $\hat{\nabla}$ is no more geometrical. Variation should be made independently in both variables \mathbf{e}^a and $\hat{\Gamma}^a{}_{\bar{b}}$ in action being of Einstein-Hilbert type

$$S = \int_{\Omega} -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \eta^{b\bar{b}} \hat{\mathbf{F}}^a{}_{\bar{b}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d, \quad (2)$$

where Ω is a timelike compact set, i.e. $\Omega = \langle t_i; t_f \rangle \times \Sigma$. For simplicity we assume in this paper that Σ is compact manifold, e.g. torus; our next paper will be focused also on noncompact manifolds with boundary. Let us decompose variable $\hat{\Gamma}^a{}_{\bar{b}}$ into $\mathbf{O}(\mathbf{g})$ -irreducible parts

$$\hat{\Gamma}^{ab} = \eta^{bc} \hat{\Gamma}^a{}_c = \hat{\mathbf{A}}^{ab} + \hat{\mathbf{B}}\eta^{ab} + \hat{\mathbf{C}}^{ab} \quad (3)$$

where $\hat{\mathbf{A}}^{ab}$ is antisymmetric and $\hat{\mathbf{C}}^{ab}$ is symmetric and traceless 1-form, respectively. The curvature $\hat{\mathbf{F}}^a{}_{\bar{b}}$ can be expressed as

$$\hat{\mathbf{F}}^{ab} = \eta^{bc} \hat{\mathbf{F}}^a{}_c = \hat{\mathbf{R}}^{ab} + \hat{\mathbf{d}}\hat{\mathbf{B}}\eta^{ab} + \hat{\mathbf{D}}\hat{\mathbf{C}}^{ab} + \eta_{cd} \hat{\mathbf{C}}^{ac} \wedge \hat{\mathbf{C}}^{db}$$

where $\hat{\mathbf{D}}$ is a metric-compatible connection defined on $\Lambda\mathbf{TM}$ (see notation in appendix A) by $\hat{\mathbf{D}}\mathbf{u}^a = d\mathbf{u}^a + \eta_{bc} \hat{\mathbf{A}}^{ab} \wedge \mathbf{u}^c$ for $\forall \mathbf{u}^a \in \Lambda\mathbb{T}^1\mathbf{M}$ and $\hat{\mathbf{R}}^{ab}$ is its curvature. If $\tilde{\mathbf{e}}^a = O^a{}_{\bar{a}} \mathbf{e}^{\bar{a}}$ is a new coframe with $O^a{}_{\bar{b}}$ being Lorentz transformation, then $\hat{\mathbf{A}}^{ab}$ transforms as

$$\tilde{\hat{\mathbf{A}}}^{ab} = O^a{}_{\bar{a}} O^b{}_{\bar{b}} \hat{\mathbf{A}}^{\bar{a}\bar{b}} + O^a{}_{\bar{a}} \eta^{\bar{a}\bar{b}} dO^b{}_{\bar{b}}$$

while $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}^{ab}$ transform like tensors in their indices. The action (2) can be written in new variables $(\mathbf{e}^a, \hat{\mathbf{A}}^{ab}, \hat{\mathbf{B}}, \hat{\mathbf{C}}^{ab})$ as

$$S = \int_{\mathbf{M}} -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \hat{\mathbf{R}}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d + \int_{\mathbf{M}} -\frac{1}{32\pi\kappa} \eta_{\bar{a}\bar{b}} \varepsilon_{abcd} \hat{\mathbf{C}}^{a\bar{a}} \wedge \hat{\mathbf{C}}^{\bar{b}b} \wedge \mathbf{e}^c \wedge \mathbf{e}^d \quad (4)$$

Notice that variable $\hat{\mathbf{B}}$ does not enter the action (4). Thus variation of (4) with respect to $\hat{\mathbf{B}}$ vanishes identically and no corresponding equation of motion arises, i.e.

$$\delta_{\hat{\mathbf{B}}} S = 0, \quad (5)$$

hence $\hat{\mathbf{B}}$ is strictly gauge variable. Now if we make variation of (4) with respect to $\hat{\mathbf{C}}^{ab}$ then we get

$$\delta_{\hat{\mathbf{C}}} S = \int -\frac{1}{16\pi\kappa} \eta_{\bar{a}\bar{b}} \varepsilon_{abcd} \delta \hat{\mathbf{C}}^{a\bar{a}} \wedge \hat{\mathbf{C}}^{b\bar{b}} \wedge \mathbf{e}^c \wedge \mathbf{e}^d = 0 \quad (6)$$

for $\forall \delta \hat{\mathbf{C}}^{ab}$: $\delta \hat{\mathbf{C}}^{ab} = \delta \hat{\mathbf{C}}^{ba}$ and $\eta_{ab} \delta \hat{\mathbf{C}}^{ab} = 0$. Equation (6) is equivalent to

$$\hat{\mathbf{C}}^{ab} = 0. \quad (7)$$

If we use this fact then the action (4) can be reduced as

$$S' = \int_{\Omega} -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \hat{\mathbf{R}}^{ab} \wedge \mathbf{e}^c \wedge \mathbf{e}^d, \quad (8)$$

what is an action of Einstein-Cartan theory for metric-compatible connection $\hat{\mathbf{A}}^{ab}$, but our configuration space is little bit bigger since it is described by variables \mathbf{e}^a , $\hat{\mathbf{A}}^{ab}$, $\hat{\mathbf{B}}$ ($\hat{\mathbf{C}}^{ab} = 0$ by equation (6) or (7)) and their velocities. Hence we get Einstein-Cartan theory by gauge fixation $\hat{\mathbf{B}} = 0$. Variation of the action (8) (see, e.g., [11]) is

$$\delta_{\hat{\mathbf{A}}, \mathbf{e}} S' = \int_{\Omega} \left(\frac{1}{16\pi\kappa} \varepsilon_{abcd} \delta \hat{\mathbf{A}}^{ab} \wedge \mathbf{e}^c \wedge \hat{\mathcal{D}} \mathbf{e}^d - \frac{1}{16\pi\kappa} \varepsilon_{abcd} \delta \mathbf{e}^a \wedge \hat{\mathbf{R}}^{bc} \wedge \mathbf{e}^d \right)$$

and equations of motion are

$$0 = \frac{1}{8\pi\kappa} \varepsilon_{abcd} \mathbf{e}^c \wedge \hat{\mathcal{D}} \mathbf{e}^d = -\frac{1}{8\pi\kappa} (\hat{T}_{ab}^c + \hat{T}_{da}^d \delta_b^c - \hat{T}_{db}^d \delta_a^c) \hat{\Sigma}_c, \quad (9)$$

$$0 = -\frac{1}{16\pi\kappa} \varepsilon_{abcd} \hat{\mathbf{R}}^{bc} \wedge \mathbf{e}^d = -\frac{1}{8\pi\kappa} \hat{G}^c{}_a \hat{\Sigma}_c, \quad (10)$$

where the torsion components are given by

$$\hat{\mathcal{D}} \mathbf{e}^a = \hat{\mathbf{T}}^a = \frac{1}{2} \hat{T}_{bc}^a \mathbf{e}^b \wedge \mathbf{e}^c,$$

3-volume forms

$$\hat{\Sigma}_a = \frac{1}{3!} \varepsilon_{abcd} \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d,$$

and $\hat{G}^a{}_b$ is the Einstein tensor

$$\begin{aligned} \hat{G}^a{}_b &= \hat{R}^{ca}{}_{cb} - \frac{1}{2} \hat{R}^{cd}{}_{cd} \delta_b^a, \\ \hat{\mathbf{R}}^{ab} &= \frac{1}{2} \hat{R}^{ab}{}_{cd} \mathbf{e}^c \wedge \mathbf{e}^d. \end{aligned}$$

Equation (9) implies that connection $\hat{\mathcal{D}}$ is torsion-free and together with metricity of $\hat{\mathcal{D}}$ we have that $\hat{\mathcal{D}}$ is geometrical connection. Equations (10) are Einstein equations of General Relativity. Solution for general gravitational connection $\hat{\Gamma}^{ab}$ is

$$\hat{\Gamma}^{ab} = \hat{\mathbf{A}}^{ab} + \hat{\mathbf{B}}\eta^{ab}, \quad (11)$$

where $\hat{\mathbf{B}}$ is arbitrary 1-form and $\hat{\mathbf{A}}^{ab}$, \mathbf{e}^a are given by equations (9) and (10). Connection of type (11) will be called Cartan connection in this series. Ambiguity of $\hat{\Gamma}^{ab}$ due to $\hat{\mathbf{B}}$ represents an additional gauge freedom in $\hat{\Gamma}^{ab}$. The spacetime is given by topology of Σ which is established initially and the metric $\mathbf{g} = \eta_{ab}\mathbf{e}^a \otimes \mathbf{e}^b$ is given just by knowledge of \mathbf{e}^a , hence $\hat{\mathbf{B}}$ does not affect geometry. Thus General Relativity, Einstein-Cartan theory and theory of affine connection are physically equivalent, at least in the case of pure gravity. We will see in last paper of this series that equivalence of these three theories are no longer valid if we add matter Lagrangian depending on Γ^a_b . On the other hand if matter Lagrangian depends only on metric connection $\hat{\mathbf{A}}^{ab}$, e.g. all Standard model matter especially spinor fields, then Einstein-Cartan theory and the theory of affine connection are physically equivalent. Since this series is focused especially on pure gravitational system or gravitational system interacting with spinor fields we will not distinguish between them and we will call both of them Einstein-Cartan theory (ECT).

III. 3+1 DECOMPOSITION

We already assumed that the spacetime \mathbf{M} is given by product $\mathbb{R} \times \Sigma$. This assumption is equivalent to the existence of a global Cauchy surface and hence solution of equations (9) and (10) can be evolved from initial data on Σ uniquely upto gauge transformation¹². Our basic variables \mathbf{e}^a , $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}^{ab}$ belong to the algebra $\Lambda\mathbf{TM}$ while $\hat{\mathbf{A}}^{ab}$ are connection forms on \mathbf{M} , so it will be useful to preserve this structure even in Hamiltonian formulation. Since we assume that Geroch's conditions are valid, there exists global orthonormal frame \mathbf{e}_a . Let $x \in \Sigma$ then $\mathcal{M}_x = \text{Span}\{\mathbf{e}_a|_x\}$ together with metric $\mathbf{g}|_x$ define tangent Minkowski space at point x . Since x is arbitrary point of Σ then space $\mathcal{M} = \cup_{x \in \Sigma} \mathcal{M}_x$ plays analogue role as $\mathbb{T}^1\Sigma$ but it is little bit bigger since \mathcal{M} contains even non tangential vectors. Important thing is that \mathcal{M} can be represented as $\hat{\mathcal{M}} = F(\Sigma)^4$ and it is also equipped with Minkowski metric η_{ab} . Hat over \mathcal{M} will be omitted from now and space \mathcal{M} and its representation will be identified. \mathcal{M} is a vector space and we can define its tensor algebra $\mathbb{T}\mathcal{M}$ and algebra of forms on Σ valued in this space $\Lambda\mathbb{T}\mathcal{M}$. Let \mathbf{e}_a and $\tilde{\mathbf{e}}_a$ be two orthonormal

frames in \mathcal{M} . Then due to Geroch's conditions there exists just one $g \in \mathbf{O}(\mathfrak{g}) \times \Sigma$ such that $\tilde{\mathbf{e}}_a = g^* \mathbf{e}_a$. Thus we see that there exists trivial principal bundle $\mathbf{O}(\mathfrak{g})\Sigma = \Sigma \times \mathbf{O}(\mathfrak{g})$ over Σ which can be identified with \mathcal{M} . Now we can start detail analysis of 3+1 decomposition of our variables.

Let $\hat{\mathbf{T}} \in \Lambda \mathbb{T}\mathbf{M}$ be a p -form valued in $\mathbb{T}\mathbf{M}$, then $\hat{\mathbf{T}}$ can be uniquely decomposed into pure spatial $(p-1)$ -form $\check{\mathbf{T}}$ and p -form \mathbf{T} valued in \mathcal{M}

$$\hat{\mathbf{T}} = \check{\mathbf{T}} \wedge dt + \mathbf{T}.$$

Another important geometric object is an external derivative operator. Let us denote by \hat{d} external derivative on \mathbf{M} while we keep d for Σ . Anyway we still write dt with the hope that this will not cause any problem. Let us apply \hat{d} on $\hat{\mathbf{T}}$, we obtain

$$\hat{d}\hat{\mathbf{T}} = d\check{\mathbf{T}} \wedge dt + dt \wedge \dot{\mathbf{T}} + d\mathbf{T},$$

where dot means action of Lie derivate along ∂_t which is just simple time derivative of components, e.g. for spatial 1-form $\dot{\mathbf{T}} = \partial_t T_\alpha dx^\alpha$, etc. So we can project spacetime p -form onto the pure spatial p -form and $(p-1)$ -form on Σ and even 3 + 1 dimensional external derivative is also written in the language of spatial forms and their time and spatial derivatives.

Let us explore what happens with orthonormal coframe \mathbf{e}^a . We can write

$$\mathbf{e}^a = \lambda^a dt + \mathbf{E}^a = \lambda^a dt + E_\alpha^a dx^\alpha, \quad (12)$$

where $\alpha, \beta, \gamma, \dots = 1, 2, 3$ are spatial coordinate indices while $a, b, c, \dots = 0, 1, 2, 3$ are reserved for tensors on \mathcal{M} . It is useful for our purposes to decompose even frame \mathbf{e}_a into tangential and time parts

$$\mathbf{e}_a = \lambda_a \partial_t + \mathbf{E}_a = \lambda_a \partial_t + E_a^\alpha \partial_\alpha. \quad (13)$$

It should be noted that $\lambda_a \neq \eta_{ab} \lambda^a$. We hope that this notation is not confusing since if we need to in/de-crease indices then it will be explicitly written using metric tensor. We have $\mathbf{e}^a(\mathbf{e}_b) = \delta_b^a$ what is

$$\begin{pmatrix} \lambda^a & E_\alpha^a \end{pmatrix} \begin{pmatrix} \lambda_b \\ E_b^\alpha \end{pmatrix} = \lambda^a \lambda_b + E_\alpha^a E_b^\alpha = \delta_b^a, \quad (14)$$

thus matrices (λ^a, E_α^a) and $(\lambda_a, E_a^\alpha)^T$ are mutually inverse and since they are finite dimensional we also have

$$\begin{pmatrix} \lambda_a \\ E_a^\alpha \end{pmatrix} \begin{pmatrix} \lambda^a & E_\beta^a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta_\beta^\alpha \end{pmatrix}, \quad (15)$$

or

$$\begin{aligned}\lambda_a \lambda^a &= 1, & \lambda_a E_\alpha^a &= 0, \\ E_a^\alpha \lambda^a &= 0, & E_a^\alpha E_\beta^a &= \delta_\beta^\alpha.\end{aligned}\tag{16}$$

As we expected, variables λ^a , λ_a , \mathbf{E}^a and \mathbf{E}_a are not independent and we can express vector coefficients by using the covectors via well known formula for inverse matrix

$$e \lambda_a = \frac{\partial e}{\partial \lambda^a},\tag{17}$$

$$e E_a^\alpha = \frac{\partial e}{\partial E_\alpha^a},\tag{18}$$

where

$$e = \frac{1}{3!} \varepsilon_{abcd} \bar{\varepsilon}^{\alpha\beta\gamma} \lambda^a E_\alpha^b E_\beta^c E_\gamma^d\tag{19}$$

is determinant of matrix (λ^a, E_α^a) . Coordinate's (co)vectors can be written with the help of previous formulas as

$$\begin{aligned}dt &= \lambda_a \mathbf{e}^a & dx^\alpha &= E_\alpha^a \mathbf{e}^a \\ \partial_t &= \lambda^a \mathbf{e}_a & \partial_\alpha &= E_\alpha^a \mathbf{e}_a\end{aligned}\tag{20}$$

thus we see that vector $\partial_t \in \mathbb{T}^1 \mathbf{M}$ is represented by vector $\lambda^a \in \mathcal{M}$ and similar for $dt \in \mathbb{T}_1 \mathbf{M}$ we have $\lambda_a \in \mathbb{T}_1 \mathcal{M}$.

Since \mathcal{M} is isomorphic to $\mathbb{T}^1 \mathbf{M}$ and there exists a natural decomposition of $\mathbb{T}^1 \mathbf{M}$ into subspaces collinear with embedding of Σ and ∂_t there should also exist a similar structure on the space \mathcal{M} . We have immediately from relation $(\lambda^a \lambda_c)(\lambda^c \lambda_b) = \lambda^a \lambda_b$ that $\lambda^a \lambda_b$ is a projection on \mathcal{M} . We can rearrange the equation (14) as

$$\mathbf{E}_b^a = \mathbf{E}^a(\mathbf{E}_b) = E_\alpha^a E_b^\alpha = \delta_b^a - \lambda^a \lambda_b\tag{21}$$

and another supplemental projection \mathbf{E}_b^a on \mathcal{M} appears. It is clear from (20) that $\lambda^a \lambda_b$ maps a general vector $v^a \in \mathcal{M}$ on that part of v^a which is proportional to ∂_t and \mathbf{E}_b^a on that tangent to Σ .

We were working with general orthonormal frame until now. From this moment \mathbf{e}^a is supposed to be righthanded and future oriented. This assumption restricts our variables λ^a , \mathbf{E}^a and following conditions should be fulfilled

$$\lambda^0 > 0,\tag{22}$$

$$\eta_{ab} \lambda^a \lambda^b > 0,\tag{23}$$

$$e > 0,\tag{24}$$

$$\mathbf{q} = \eta_{ab} \mathbf{E}^a \otimes \mathbf{E}^b < 0,\tag{25}$$

where \mathbf{q} is spatial metric and $\mathbf{q} < 0$ means that this tensor on Σ is strictly negative, i.e $\forall \mathbf{v} \neq 0 \in \mathbb{T}^1\Sigma : \mathbf{q}(\mathbf{v}, \mathbf{v}) < 0$. Let $\mathbf{SO}(\mathbf{g})$ be a subgroup of $\mathbf{O}(\mathbf{g})$ preserving conditions (22)-(25). If one wants to work with the whole $\mathbf{O}(\mathbf{g})$ then configuration manifold splits into four disjoint parts given by future/past and right/left hand orientation and this discrete structure should be taken into account on quantum level, but this is far at the moment.

Decomposition of variables $\hat{\mathbf{B}}^{ab}, \hat{\mathbf{C}}^{ab}$ is given by

$$\hat{\mathbf{B}}^{ab} = \mathcal{B}^{ab} dt + \mathbf{B}^{ab}, \quad (26)$$

$$\hat{\mathbf{C}}^{ab} = \mathcal{C}^{ab} dt + \mathbf{C}^{ab} \quad (27)$$

and we can now focus on the metric connection variable $\hat{\mathbf{A}}^{ab}$. We can write

$$\hat{\mathbf{A}}^{ab} = \Lambda^{ab} dt + \mathbf{A}^{ab}. \quad (28)$$

It should be noted that Λ^{ab} transforms like tensor under $g \in \mathbf{SO}(\mathbf{g}) \times \Sigma$. Let $\tilde{\mathbf{e}}^a = g^* \hat{\mathbf{e}}^a = O^a_b \hat{\mathbf{e}}^b$ be a new coframe¹³ on $\mathbb{T}_1\mathcal{M}$ then transformation law for \mathbf{A}^{ab} is given by formula

$$\tilde{\mathbf{A}}^{ab} = O^a_{\bar{a}} O^b_{\bar{b}} \mathbf{A}^{\bar{a}\bar{b}} + O^a_{\bar{a}} \eta^{\bar{a}\bar{b}} dO^b_{\bar{b}}.$$

Let $\hat{\mathbf{v}}^a = \check{\mathbf{v}}^a \wedge dt + \mathbf{v}^a \in \Lambda\mathbb{T}\mathbf{M}$ then $\hat{\mathcal{D}}\hat{\mathbf{v}}^a$ can be written as

$$\hat{\mathcal{D}}\hat{\mathbf{v}}^a = \mathcal{D}\check{\mathbf{v}}^a \wedge dt + dt \wedge \dot{\mathcal{D}}\mathbf{v}^a + \mathcal{D}\mathbf{v}^a, \quad (29)$$

where \mathcal{D} is a spatial covariant external derivative operator on $\Lambda\mathbb{T}\mathcal{M}$ given by

$$\mathcal{D}\mathbf{v}^a = d\mathbf{v}^a + \eta_{bc} \mathbf{A}^{ab} \wedge \mathbf{v}^c \quad (30)$$

and $\dot{\mathcal{D}}$ is a covariant time derivative

$$\dot{\mathcal{D}}\mathbf{v}^a = \dot{\mathbf{v}}^a + \eta_{bc} \Lambda^{ab} \mathbf{v}^c. \quad (31)$$

Since Λ^{ab} and \mathbf{A}^{ab} are antisymmetric in their indices we have immediately that

$$\mathcal{D}\eta_{ab} = 0 \quad (32)$$

and

$$\dot{\mathcal{D}}\eta_{ab} = 0. \quad (33)$$

Thus operators \mathcal{D} and $\dot{\mathcal{D}}$ are compatible with the metric η_{ab} on \mathcal{M} .

Let us summarize our situation. We started with connection $\hat{\mathcal{D}}$ on $\Lambda\mathbb{T}\mathbf{M}$ with gauge group

$\mathbf{SO}(\mathfrak{g})$. 3 + 1 decomposition of space $\Lambda\mathbb{T}\mathbf{M}$ leads us to the pure spatial connection \mathcal{D} on Σ with the same group $\mathbf{SO}(\mathfrak{g})$ which is good news for us. Since as we wanted or expected the $\mathbf{SO}(\mathfrak{g})$ structure is preserved even in the language of spatial forms on Σ . This is in contrast with standard ADM/real Loop formulation¹⁴ where gauge group is only $\mathbf{SO}(\mathfrak{q})$. So far we are still working with real variables which is again in contrast with complex Loop theory where gauge group is $\mathbf{SO}(\mathfrak{g})$ but the prize paid for that is the loss of reality of variables.

In general theory of gauge connections a notion of curvature is well known. Vanishing of curvature expresses the condition that a horizontal subspace in fibre bundle over given manifold is integrable. In usual words this means that parallel transport along closed path of a given object (the object should be valued in nontrivial representation space of the gauge group) is given by identity (see details in ,e.g., [11]). That's why curvature plays important role even for general gauge group G (recall $\hat{\mathbf{F}} = \hat{d}\hat{\mathbf{A}}$ in Maxwell theory or more complicated objects in Standard Model). For our purposes it is sufficient to write down an explicit formula which is

$$\mathbf{R}^{ab} = \mathcal{D}\mathbf{A}^{ab} = d\mathbf{A}^{ab} + \eta_{cd}\mathbf{A}^{ac} \wedge \mathbf{A}^{db}$$

for our $\mathbf{SO}(\mathfrak{g})$ connection \mathbf{A}^{ab} on $\Lambda\mathbb{T}\mathcal{M}$. The spacetime curvature $\hat{\mathbf{R}}^{ab}$ can be decomposed as

$$\hat{\mathbf{R}}^{ab} = \mathbf{R}^{ab} + dt \wedge \dot{\mathbf{A}}^{ab} + \mathcal{D}\Lambda^{ab} \wedge dt. \quad (34)$$

Next geometrical object on \mathbf{M} which plays important role in Einstein-Cartan theory is the torsion $\hat{\mathbf{T}}^a = \hat{\mathcal{D}}\mathbf{e}^a$. How does its spatial counterpart look like? Coframe \mathbf{e}^a is not object from $\Lambda\mathbb{T}\mathcal{M}$ because it contains dt . We can project \mathbf{e}^a with \mathbf{E}_b^a and have $\mathbf{E}^a = \mathbf{E}_b^a \mathbf{e}^b$ what is already an object from $\Lambda\mathbb{T}\mathcal{M}$. Thus, let us define $\mathbf{SO}(\mathfrak{g})$ -torsion by formula

$$\mathbf{T}^a = \mathcal{D}\mathbf{E}^a. \quad (35)$$

Since we are not and will not be working with the 3-dimensional $\mathbf{SO}(\mathfrak{q})$ -connection let us call for simplicity \mathbf{T}^a as torsion on places where no confusion can arise. Another motivation for its name appears if we write spacetime torsion $\hat{\mathbf{T}}^a$ in 3 + 1 manner

$$\hat{\mathcal{D}}\mathbf{e}^a = \mathcal{D}\mathbf{E}^a + \mathcal{D}\lambda^a \wedge dt + dt \wedge \dot{\mathcal{D}}\mathbf{E}^a. \quad (36)$$

As we can see, a spatial part of the spacetime torsion $\hat{\mathbf{T}}^a$ is just $\mathbf{SO}(\mathfrak{g})$ -torsion \mathbf{T}^a .

It will be useful in a while and also in next sections to have derived few formulas. In order to do this, let us consider 2-form \mathbf{P}_{ab} which is antisymmetric in its indices ab , i.e.

$$\mathbf{P}_{ab} = \frac{1}{2} \tilde{P}_{ab}^{\alpha} \varepsilon_{\alpha\beta\gamma} dx^{\beta} \wedge dx^{\gamma}. \quad (37)$$

\mathbf{P}_{ab} can be decomposed in its tensor indices into tangential and time parallel parts as

$$\mathbf{P}_{ab} = 2\mathbf{P}_{[a}^\perp\lambda_{b]} + \hat{\mathbf{P}}_{ab}, \quad (38)$$

where

$$\mathbf{P}_a^\perp = \mathbf{P}_{ab}\lambda^b, \quad (39)$$

note that $\mathbf{P}_a^\perp\lambda^a = 0$, and

$$\hat{\mathbf{P}}_{ab} = \mathbf{E}_a^{\bar{a}}\mathbf{E}_b^{\bar{b}}\mathbf{P}_{\bar{a}\bar{b}}. \quad (40)$$

Let us focus on the tangential part $\hat{\mathbf{P}}_{ab}$. We can multiply it by \mathbf{E}^a

$$\mathbf{K}_{ab}^c = \hat{\mathbf{P}}_{ab} \wedge \mathbf{E}^c \quad (41)$$

It is easy to show that there is a one to one correspondence between $\hat{\mathbf{P}}_{ab}$ and \mathbf{K}_{ab}^c iff $\lambda_c\mathbf{K}_{ab}^c = 0$, $\mathbf{K}_{ab}^c = -\mathbf{K}_{ba}^c$ and $\lambda^a\mathbf{K}_{ab}^c = 0$. Let $\mathbf{K}_{ab}^c = \tilde{K}_{ab}^c d^3x$, then $\tilde{K}_{ab}^c = \hat{P}_{ab}^\alpha E_\alpha^c$ and due to $\lambda_c\mathbf{K}_{ab}^c = 0$ we can express $\hat{P}_{ab}^\alpha = \tilde{K}_{ab}^c E_c^\alpha$. Equation (41) can be rearranged without any loss of information by multiplying with $\bar{\varepsilon}^{ab\bar{c}\bar{d}}\lambda_{\bar{b}}$, since bottom indices are spatial and antisymmetric, into the 3-form

$$\mathbf{K}^{ab} = \frac{1}{2}\bar{\varepsilon}^{ab\bar{c}\bar{d}}\lambda_{\bar{b}}\mathbf{K}_{\bar{c}\bar{d}}^b = \frac{1}{2}\bar{\varepsilon}^{ab\bar{c}\bar{d}}\lambda_{\bar{b}}\mathbf{P}_{\bar{c}\bar{d}} \wedge \mathbf{E}^b, \quad (42)$$

which can be written as a sum of symmetric and antisymmetric parts

$$\mathbf{K}^{ab} = \mathbf{K}^{(ab)} + \mathbf{K}^{[ab]}. \quad (43)$$

Antisymmetric part can be rewritten as

$$\begin{aligned} \mathbf{P}_a^\parallel &= \varepsilon_{bcda}\lambda^b\mathbf{K}^{[cd]} = -\frac{1}{2}\varepsilon_{abcd}\bar{\varepsilon}^{c\bar{b}\bar{c}\bar{d}}\lambda^b\lambda_{\bar{b}}\hat{\mathbf{P}}_{\bar{c}\bar{d}} \wedge \mathbf{E}^d = \dots \\ \mathbf{P}_a^\parallel &= \mathbf{E}_a^b\mathbf{P}_{bc} \wedge \mathbf{E}^c. \end{aligned} \quad (44)$$

Thus whole information about \mathbf{P}_{ab} is encoded in three independent components

\mathbf{P}_a^\perp – 2-form spatial covector,

\mathbf{P}_a^\parallel – 3-form spatial covector,

σ^{ab} – spatial symmetric 3-form,

where (sign and 2 is just convention)

$$\sigma^{ab} = -2\mathbf{K}^{(ab)} = \frac{1}{2}\mathbf{P}_{\bar{a}\bar{b}}\lambda_{\bar{c}} \wedge (\bar{\varepsilon}^{\bar{a}\bar{b}\bar{c}a}\mathbf{E}^b + \bar{\varepsilon}^{\bar{a}\bar{b}\bar{c}b}\mathbf{E}^a).$$

Let us consider a linear map of \mathbf{P}_{ab} given by integral

$$\mathbf{P}(\mathbf{B}) = \int_{\Sigma} \frac{1}{2} \mathbf{P}_{ab} \wedge \mathbf{B}^{ab}, \quad (45)$$

where \mathbf{B}^{ab} is a 1-form antisymmetric in its indices. Since we can decompose \mathbf{P}_{ab} into the three parts we can expect that similar decomposition works for its dual \mathbf{B}^{ab} . We can write

$$\frac{1}{2} \mathbf{P}_{ab} \wedge \mathbf{B}^{+ab} = \mathbf{P}_a^+ \wedge \mathbf{B}^a = \frac{1}{2} \mathbf{P}_{ab} \wedge 2\mathbf{B}^{[a} \lambda^{b]}, \quad (46)$$

thus $\mathbf{B}^{+ab} = 2\mathbf{B}^{[a} \lambda^{b]}$

$$\frac{1}{2} \mathbf{P}_{ab} \wedge \mathbf{B}^{\parallel ab} = \mathbf{P}_a^{\parallel} \mathcal{B}^a = \frac{1}{2} \mathbf{P}_{ab} \wedge 2\mathcal{B}^{\bar{a}} \mathbf{E}_{\bar{a}}^{[a} \mathbf{E}^{b]}, \quad (47)$$

thus $\mathbf{B}^{\parallel ab} = 2\mathcal{B}^{\bar{a}} \mathbf{E}_{\bar{a}}^{[a} \mathbf{E}^{b]}$ and

$$\frac{1}{2} \mathbf{P}_{ab} \mathbf{B}^{Mab} = \sigma^{ab} M_{ab} = \frac{1}{2} \mathbf{P}_{ab} \lambda_{\bar{a}} \bar{\varepsilon}^{ab\bar{b}} \wedge \mathbf{E}^{\bar{c}} M_{\bar{b}\bar{c}}, \quad (48)$$

thus $\mathbf{B}^{Mab} = \bar{\varepsilon}^{ab\bar{b}} \mathbf{E}^{\bar{c}} \lambda_{\bar{a}} M_{\bar{b}\bar{c}}$. In other words we can decompose dual to \mathbf{P}_{ab} as

$$\mathbf{B}^{ab} = 2\mathbf{B}^{[a} \lambda^{b]} + 2\mathcal{B}^{\bar{a}} \mathbf{E}_{\bar{a}}^{[a} \mathbf{E}^{b]} + \bar{\varepsilon}^{ab\bar{b}} \mathbf{E}^{\bar{c}} \lambda_{\bar{a}} M_{\bar{b}\bar{c}}, \quad (49)$$

where \mathbf{B}^a is an arbitrary 1-form vector, \mathcal{B}^a is a 0-form vector and M_{ab} is a symmetric matrix.

We already derived equations of motion of the Einstein-Cartan theory from Lagrangian in section II and now it is the right time to explore them in detail. Anyway, we present here only brief description and leave the rest to the next chapters where Hamiltonian-Dirac formalism is explored in full detail. Recall that the torsion equation (9) sets the connection to be just geometrical; in other words $\hat{\mathbf{A}}^{ab}$, can be written as function(al) of the metric $g_{\mu\nu} = \eta_{ab} e_{\mu}^a e_{\nu}^b$ and initial value formulation for Einstein equations (10) written using $g_{\mu\nu}$ is well known and understood problem (see, e.g.¹⁵). If we follow ideas of Einstein-Cartan theory and work with our variables \mathbf{A}^{ab} , \mathbf{E}^a , etc. then the set of equations given by (9) and (10) is not complete. Missing equations should be derived from the condition preserving the constraints given by the equation (9) and (10). Let us look what happens here. Decomposition of (9) leads to

$$0 = \frac{1}{8\pi\kappa} \varepsilon_{abcd} \mathbf{E}^c \wedge \mathcal{D}\mathbf{E}^d, \quad (50)$$

$$0 = \frac{1}{8\pi\kappa} \varepsilon_{abcd} \left(\lambda^c \mathcal{D}\mathbf{E}^d + \mathbf{E}^c \wedge \mathcal{D}\lambda^d - \mathbf{E}^c \wedge \dot{\mathcal{D}}\mathbf{E}^d \right). \quad (51)$$

Equations (10) can be rewritten similarly as

$$0 = -\frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{R}^{bc} \wedge \mathbf{E}^d, \quad (52)$$

$$0 = -\frac{1}{16\pi\kappa} \varepsilon_{abcd} \left(\mathbf{R}^{bc} \lambda^d + \dot{\mathbf{A}}^{bc} \wedge \mathbf{E}^d - \mathcal{D}\Lambda^{bc} \wedge \mathbf{E}^d \right) \quad (53)$$

The expression on the right-hand side of (51) is a 2-form with antisymmetric indices and we can use decomposition (49). We obtain an evolution equation and a constraint

$$0 = \dot{\mathcal{D}}\mathbf{E}^a - \mathcal{D}\lambda^a, \quad (54)$$

$$0 = \mathbf{E}^{(a}\mathbf{E}_c^{b)} \wedge \mathcal{D}\mathbf{E}^c. \quad (55)$$

Here is no problem with ambiguity. The equation (53) is a 2-form with one tensor index hence it expresses $4 \times 3 = 12$ conditions for $\dot{\mathbf{A}}^{ab}$ with $6 \times 3 = 18$ degrees of freedom. We see that we are not able to determine connection velocities and some equation(s) is(are) still missing. We will see later that conditions (50) and (52) represent the first class constraints while equation (55) is the constraint of the second class. Missing equation(s) can be obtained by applying the time derivative on (55). Since (52) and (50) are the first class constraints no new conditions appear and we have closed system of equations determining \mathbf{E}^a and \mathbf{A}^{ab} . The variables λ^a and Λ^{ab} are arbitrary. The missing equation is

$$0 = \mathbf{E}^{(a}\mathbf{E}_c^{b)} \wedge (\mathbf{R}^{c\bar{a}}\eta_{\bar{a}\bar{b}}\lambda^{\bar{b}} + \mathbf{H}^{c\bar{a}}\eta_{\bar{a}\bar{b}} \wedge \mathbf{E}^{\bar{b}}), \quad (56)$$

where $\mathbf{H}^{ab} = \dot{\mathbf{A}}^{ab} - \mathcal{D}\Lambda^{ab}$. Now we can determine \mathbf{H}^{ab} as a certain function(al) of λ^a , \mathbf{E}^a , \mathbf{A}^{ab} but we will not do that because we do not need it anywhere. It is enough for our purposes to know that our set of equations determines uniquely, up to gauge transformation, evolution of our system.

IV. HAMILTONIAN

In section II we have introduced the Lagrangian of the Einstein-Cartan theory. Next step towards its quantum formulation should be done by its converting it into canonical form. Since our system contains velocities of basic variables at best linearly, standard Hamilton procedure can not be used. Therefore we must use Dirac procedure for constrained dynamic¹⁶. In the standard and even in the Dirac approach to dynamics the notion of momentum for variable q^A is introduced by $p_A = \frac{\partial L}{\partial \dot{q}^A}$, where L is Lagrangian of a system. Since action is $S = \int dt L$ we can see that action and Lagrangian for field theory can be written within 4-form \mathbf{L} called Lagrangian form as $S = \int_{\Omega} \mathbf{L}$ and $L = \int_{\Sigma} i_{\partial_t} \mathbf{L}$, where $\mathbf{L} = \mathcal{L} d^4x$ and \mathcal{L} is Lagrangian density. If we suppose that configuration space is built just by generalized n -forms $\mathbf{Q}^A = \frac{1}{n!} Q_{\alpha\dots\beta}^A dx^\alpha \wedge \dots \wedge dx^\beta$, e.g. \mathbf{E}^a , \mathbf{A}^{ab} in our system, all variables in Standard Model, etc., then we can see that their momenta $\tilde{p}_A^{\alpha\dots\beta} = \frac{\delta L}{\delta Q_{\alpha\dots\beta}^A} = \frac{\partial \mathcal{L}}{\partial Q_{\alpha\dots\beta}^A}$ transform like densities under coordinate transformation and therefore

TABLE I. Basic variables

Variables	Momenta	Velocities
λ^a	$\boldsymbol{\pi}_a = \tilde{\pi}_a d^3x$ where $\tilde{\pi}_a = \partial\mathcal{L}/\partial\dot{\lambda}^a$	$\boldsymbol{v}^a = \dot{\lambda}^a$
$\mathbf{E}^a = E^a dx^\alpha$	$\mathbf{p}_a = \frac{1}{2}\tilde{p}_a^\alpha \varepsilon_{\alpha\beta\gamma} dx^\beta \wedge dx^\gamma$ where $\tilde{p}_a^\alpha = \partial\mathcal{L}/\partial\dot{E}_\alpha^a$	$\mathbf{b}^a = \dot{\mathbf{E}}^a$
Λ^{ab}	$\boldsymbol{\Pi}_{ab} = \tilde{\Pi}_{ab} d^3x$ where $\tilde{\Pi}_{ab} = \partial\mathcal{L}/\partial\dot{\Lambda}^{ab}$	$\boldsymbol{\Gamma}^{ab} = \dot{\Lambda}^{ab}$
$\mathbf{A}^{ab} = A_\alpha^{ab} dx^\alpha$	$\mathbf{p}_{ab} = \frac{1}{2}\tilde{p}_{ab}^\alpha \varepsilon_{\alpha\beta\gamma} dx^\beta \wedge dx^\gamma$ where $\tilde{p}_{ab}^\alpha = \partial\mathcal{L}/\partial\dot{A}_\alpha^{ab}$	$\mathbf{B}^{ab} = \dot{\mathbf{A}}^{ab}$
\mathcal{B}	$\boldsymbol{\varphi} = \tilde{\varphi} d^3x$ where $\tilde{\varphi} = \partial\mathcal{L}/\partial\dot{\mathcal{B}}$	$\boldsymbol{y} = \dot{\mathcal{B}}$
$\mathbf{B} = B_\alpha dx^\alpha$	$\mathbf{u} = \frac{1}{2}\tilde{u}^\alpha \varepsilon_{\alpha\beta\gamma} dx^\beta \wedge dx^\gamma$ where $\tilde{u}^\alpha = \partial\mathcal{L}/\partial\dot{B}_\alpha$	$\mathbf{Y} = \dot{\mathbf{B}}$
\mathcal{C}^{ab}	$\boldsymbol{\Phi}_{ab} = \tilde{\Phi}_{ab} d^3x$ where $\tilde{\Phi}_{ab} = \partial\mathcal{L}/\partial\dot{\mathcal{C}}^{ab}$	$\boldsymbol{\chi}^{ab} = \dot{\mathcal{C}}^{ab}$
$\mathbf{C}^{ab} = C_\alpha^{ab} dx^\alpha$	$\mathbf{U}_{ab} = \frac{1}{2}\tilde{U}_{ab}^\alpha \varepsilon_{\alpha\beta\gamma} dx^\beta \wedge dx^\gamma$ where $\tilde{U}_{ab}^\alpha = \partial\mathcal{L}/\partial\dot{C}_\alpha^{ab}$	$\mathbf{X}^{ab} = \dot{\mathbf{C}}^{ab}$

objects $\mathbf{p}_A = \frac{1}{n!(3-n)!}\tilde{p}_A^{\alpha\dots\beta} \varepsilon_{\alpha\dots\beta\gamma\dots\delta} dx^\gamma \wedge \dots \wedge dx^\delta$ are $(3-n)$ -forms and even more $\mathbf{p}_A \wedge \dot{\mathbf{Q}}^A = \frac{1}{n!}\tilde{p}_A^{\alpha\dots\beta} \dot{Q}_{\alpha\dots\beta}^A d^3x$ what is exactly the first term in the definition of Hamiltonian $H = \int_\Sigma \mathbf{p}_A \wedge \dot{\mathbf{Q}}^A - L$.

Recall that $Q_{\alpha\dots\beta}^A$ and $\tilde{p}_A^{\alpha\dots\beta}$ are antisymmetric in their coordinate indices therefore every term in $\tilde{p}_A^{\alpha\dots\beta} \dot{Q}_{\alpha\dots\beta}^A$ is $n!$ -times repeated while every velocity should enter the Hamiltonian just once. Our configuration space is described by variables $\lambda^a, \dots, \mathbf{C}^{ab}$ and its velocities (see table I for details). Variables $\mathcal{B}, \dots, \mathbf{C}^{ab}$ enters the Lagrangian (59) in a certain special way. We can decompose it as sum of two Lagrangians $\mathbf{L} = \mathbf{L}^{(\text{EC})} + \mathbf{L}^{(\text{Rest})}$ where

$$\mathbf{L}^{(\text{Rest})} = -dt \wedge \frac{1}{16\pi\kappa} \eta_{\bar{a}\bar{b}} \varepsilon_{abcd} (C^{a\bar{a}} C^{b\bar{b}} \wedge \mathbf{E}^c \wedge \mathbf{E}^d + C^{a\bar{a}} \wedge C^{b\bar{b}} \wedge \lambda^c \mathbf{E}^d) \quad (57)$$

and $\mathbf{L}^{(\text{EC})}$ does not depend on C^{ab}, \mathbf{C}^{ab} while as we already know, the whole Lagrangian \mathbf{L} does not depend on \mathcal{B}, \mathbf{B} . Thus we can consider this subsystem independently. Hamiltonian $\mathbf{H}^{(\text{Rest})}$ is given by

$$\begin{aligned} \mathbf{H}^{(\text{Rest})} = & \boldsymbol{\varphi} \wedge \boldsymbol{y} + \mathbf{u} \wedge \mathbf{Y} + \frac{1}{2} \boldsymbol{\Phi}_{ab} \wedge \boldsymbol{\chi}^{ab} + \frac{1}{2} \mathbf{U}_{ab} \wedge \mathbf{X}^{ab} + \\ & + \frac{1}{16\pi\kappa} \eta_{\bar{a}\bar{b}} \varepsilon_{abcd} (C^{a\bar{a}} C^{b\bar{b}} \wedge \mathbf{E}^c \wedge \mathbf{E}^d + C^{a\bar{a}} \wedge C^{b\bar{b}} \wedge \lambda^c \mathbf{E}^d) \end{aligned} \quad (58)$$

with primary constraints $\boldsymbol{\varphi} = \mathbf{u} = \boldsymbol{\Phi}_{ab} = \mathbf{U}_{ab} = 0$. Secondary constraints are $C^{ab} = \mathbf{C}^{ab} = 0$. Since $\boldsymbol{\Phi}_{ab}, \mathbf{U}_{ab}$ and C^{ab}, \mathbf{C}^{ab} are canonical variables their Poisson bracket is an identity. They are thus the second class constraints and we must use the Dirac procedure. Dirac bracket for this subsystem is just Poisson bracket on canonical variables \mathcal{B}, \mathbf{B} and momenta $\boldsymbol{\varphi}, \mathbf{u}$ while reduced

Hamiltonian is $\mathbf{H}^{(\text{Rest})} = \varphi \wedge \mathcal{Y} + \mathbf{u} \wedge \mathbf{Y}$. Hence we can focus ourselves for a while just on $\mathbf{L}^{(\text{EC})}$ and its hamiltonization. Final Hamiltonian will be obtained by sum $\mathbf{H} = \mathbf{H}^{(\text{EC})} + \mathbf{H}^{(\text{Rest})}$.

Let us substitute the decomposition of the variables \mathbf{e}^a , $\hat{\mathbf{A}}^{ab}$ into the Langrangian $\mathbf{L}^{(\text{EC})}$

$$\begin{aligned} i_{\partial_t} \mathbf{L}^{(\text{EC})} = & -\frac{1}{16\pi\kappa} \varepsilon_{abcd} \lambda^a \mathbf{R}^{bc} \wedge \mathbf{E}^d + \frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathcal{D}\Lambda^{ab} \wedge \mathbf{E}^c \wedge \mathbf{E}^d \\ & -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \dot{\mathbf{A}}^{ab} \wedge \mathbf{E}^c \wedge \mathbf{E}^d. \end{aligned} \quad (59)$$

We use this in definition of Hamiltonian. Our procedure then yields the following result

$$\mathbf{H}^{(\text{EC})} = \int_{\Sigma} \mathbf{H}^{(\text{EC})} = \pi(\nu) + \Pi(\Gamma) + \mathbf{p}(\mathbf{b}) + \mathbf{P}(\mathbf{B}) + \mathbf{R}(\lambda) + \mathbf{T}(\Lambda), \quad (60)$$

where

$$\begin{aligned} \pi(\nu) &= \int_{\Sigma} \pi_a \wedge \nu^a, \\ \mathbf{p}(\mathbf{b}) &= \int_{\Sigma} \mathbf{p}_a \wedge \mathbf{b}^a, \\ \Pi(\Gamma) &= \int_{\Sigma} \frac{1}{2} \Pi_{ab} \wedge \Gamma^{ab}, \\ \mathbf{P}(\mathbf{B}) &= \int_{\Sigma} \frac{1}{2} \left(\mathbf{p}_{ab} + \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{E}^c \wedge \mathbf{E}^d \right) \wedge \mathbf{B}^{ab} = \int_{\Sigma} \frac{1}{2} \mathbf{P}_{ab} \wedge \mathbf{B}^{ab}, \\ \mathbf{R}(\lambda) &= \int_{\Sigma} \frac{1}{16\pi\kappa} \varepsilon_{abcd} \lambda^a \mathbf{R}^{bc} \wedge \mathbf{E}^d = \int_{\Sigma} \lambda^a \mathbf{R}_a, \\ \mathbf{T}(\Lambda) &= \int_{\Sigma} -\frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathcal{D}\Lambda^{ab} \wedge \mathbf{E}^c \wedge \mathbf{E}^d = \int_{\Sigma} -\frac{1}{16\pi\kappa} \varepsilon_{abcd} \Lambda^{ab} \wedge \mathbf{E}^c \wedge \mathcal{D}\mathbf{E}^d = \int_{\Sigma} \frac{1}{2} \Lambda^{ab} \mathbf{T}_{ab}, \end{aligned}$$

where $\mathbf{P}_{ab} = \mathbf{p}_{ab} + \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{E}^c \wedge \mathbf{E}^d$, $\mathbf{R}_a = \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{R}^{bc} \wedge \mathbf{E}^d$ and

$\mathbf{T}_{ab} = -\frac{1}{8\pi\kappa} \varepsilon_{abcd} \mathbf{E}^c \wedge \mathcal{D}\mathbf{E}^d$. The existence of the primary constraints represents the fact that we are working with a degenerated Lagrangian and therefore we are not able to express velocities as function(al)s of momenta (they are given by conditions $\frac{\partial \mathcal{L}}{\partial Q^A} = 0$). Our system is degenerated and primary constraints are given by

$$\begin{aligned} \pi(\nu) = 0 \quad \forall \nu^a \in \Lambda_0 \mathbb{T}\mathcal{M} &\Leftrightarrow \pi_a = 0, \\ \mathbf{p}(\mathbf{b}) = 0 \quad \forall \mathbf{b}^a \in \Lambda_1 \mathbb{T}\mathcal{M} &\Leftrightarrow \mathbf{p}_a = 0, \\ \Pi(\Gamma) = 0 \quad \forall \Gamma^{ab} \in \Lambda_0 \mathbb{T}\mathcal{M} &\Leftrightarrow \Pi_{ab} = 0, \\ \mathbf{P}(\mathbf{B}) = 0 \quad \forall \mathbf{B}^{ab} \in \Lambda_1 \mathbb{T}\mathcal{M} &\Leftrightarrow \mathbf{P}_{ab} = \mathbf{p}_{ab} + \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{E}^c \wedge \mathbf{E}^d = 0. \end{aligned}$$

Since these constraints should be valid through the whole time evolution of our physical system their time derivatives should vanish too and this implies further conditions which should be fulfilled¹⁷,

$$\frac{d\boldsymbol{\pi}(\tilde{\nu})}{dt} = \{\boldsymbol{\pi}(\tilde{\nu}); \mathbf{H}^{(\text{EC})}\} = -\mathbf{R}(\tilde{\nu}) = 0, \quad (61)$$

$$\frac{d\Pi(\tilde{\Gamma})}{dt} = \{\Pi(\tilde{\Gamma}); \mathbf{H}^{(\text{EC})}\} = -\mathbf{T}(\tilde{\Gamma}) = 0, \quad (62)$$

$$\begin{aligned} \frac{d\mathbf{p}(\tilde{\mathbf{b}})}{dt} &= \{\mathbf{p}(\tilde{\mathbf{b}}); \mathbf{H}^{(\text{EC})}\} = \\ &= \int \frac{1}{16\pi\kappa} \varepsilon_{abcd} \tilde{\mathbf{b}}^a \wedge (\mathbf{B}^{bc} \wedge \mathbf{E}^d + \lambda^b \mathbf{R}^{cd} - \mathcal{D}\Lambda^{bc} \wedge \mathbf{E}^d) = 0, \end{aligned} \quad (63)$$

$$\begin{aligned} \frac{d\mathbf{P}(\tilde{\mathbf{B}})}{dt} &= \{\mathbf{P}(\tilde{\mathbf{B}}); \mathbf{H}^{(\text{EC})}\} = \\ &= \int \frac{1}{16\pi\kappa} \varepsilon_{abcd} \tilde{\mathbf{B}}^{ab} \wedge (\mathbf{b}^c \wedge \mathbf{E}^d + \eta_{\bar{a}\bar{b}} \Lambda^{c\bar{a}} \mathbf{E}^{\bar{b}} \wedge \mathbf{E}^d - \mathcal{D}(\lambda^c \mathbf{E}^d)) = 0. \end{aligned} \quad (64)$$

The first two of them are secondary constraints. It is clear that (63) is equal to (53), while (64) is connected with (51); they determine Lagrange multipliers \mathbf{b}^a , \mathbf{B}^{ab} . As we have already promised in the previous section we will show how to do this now. Since these equations are same one can also use the same procedure there (recall that $\mathbf{b}^a = \dot{\mathbf{E}}^a$ and $\mathbf{B}^{ab} = \dot{\mathbf{A}}^{ab}$). We can express the equations (63), (64) as:

$$0 = \frac{1}{16\pi\kappa} \varepsilon_{abcd} (\mathbf{H}^{bc} \wedge \mathbf{E}^d + \mathbf{R}^{bc} \lambda^d), \quad (65)$$

$$0 = \frac{1}{8\pi\kappa} \varepsilon_{abcd} (\mathbf{h}^c \wedge \mathbf{E}^d - \lambda^c \mathcal{D}\mathbf{E}^d), \quad (66)$$

where $\mathbf{H}^{ab} = \mathbf{B}^{ab} - \mathcal{D}\Lambda^{ab}$ and $\mathbf{h}^a = \mathbf{b}^a + \Lambda^{a\bar{a}} \eta_{\bar{a}\bar{b}} \mathbf{E}^{\bar{b}} - \mathcal{D}\lambda^a$. Let us focus on the second equation (66). We can multiply it again by a general 1-form $\tilde{\mathbf{B}}^{ab}$ and since it is antisymmetric in its indices we can decompose it as (49)

$$\frac{1}{8\pi\kappa} (\tilde{\mathbf{B}}^a \lambda^b + \tilde{\mathcal{B}}^{\bar{a}} \mathbf{E}_{\bar{a}}^a \mathbf{E}^b + \frac{1}{2} \tilde{\varepsilon}^{ab\bar{a}\bar{b}} \mathbf{E}^{\bar{c}} \lambda_{\bar{a}} \tilde{M}_{\bar{b}\bar{c}}) \wedge \varepsilon_{abcd} (\mathbf{h}^c \wedge \mathbf{E}^d - \lambda^c \mathcal{D}\mathbf{E}^d) = 0. \quad (67)$$

This expression can be split into three independent equations

$$\frac{1}{8\pi\kappa} \varepsilon_{abcd} \lambda^b \mathbf{h}^c \wedge \mathbf{E}^d = 0, \quad (68)$$

$$\frac{1}{8\pi\kappa} \varepsilon_{abcd} \mathbf{E}_{\bar{a}}^a \mathbf{E}^b \wedge (\mathbf{h}^c \wedge \mathbf{E}^d - \lambda^c \mathcal{D}\mathbf{E}^d) = 0, \quad (69)$$

$$-\frac{1}{8\pi\kappa} \mathbf{E}^{(a} \mathbf{E}_c^{b)} \wedge \mathcal{D}\mathbf{E}^c = 0. \quad (70)$$

We can use constraint $\mathbf{T}_{ab} = 0$ in the second equation which together with the first one implies that $\mathbf{h}^a = 0$, while the third equation is equivalent to another secondary constraints,

$$\mathbf{S}(M) = \int_{\Sigma} \frac{1}{8\pi\kappa} M_{ab} \mathbf{E}^a \mathbf{E}_c^b \wedge \mathcal{D}\mathbf{E}^c = \int_{\Sigma} M_{ab} \mathbf{S}^{ab} = 0, \quad (71)$$

where $\mathbf{S}^{ab} = \frac{1}{8\pi\kappa} \mathbf{E}^a \mathbf{E}_c^b \wedge \mathcal{D}\mathbf{E}^c$ and M_{ab} is arbitrary function symmetric in its indices. Let us substitute the decomposition

$$\mathbf{H}^{ab} = 2\mathbf{H}^{[a} \lambda^{b]} + 2\mathcal{H}^{\bar{a}} \mathbf{E}_{\bar{a}}^{[a} \mathbf{E}^{b]} + \bar{\epsilon}^{ab\bar{a}\bar{b}} \mathbf{E}^{\bar{c}} \lambda_{\bar{a}} \gamma_{\bar{b}\bar{c}} \quad (72)$$

into the equation (65). We obtain

$$\frac{1}{16\pi\kappa} \epsilon_{abcd} (2\mathbf{H}^b \lambda^c \wedge \mathbf{E}^d + 2\mathcal{H}^{\bar{b}} \mathbf{E}_{\bar{b}}^b \mathbf{E}^c \wedge \mathbf{E}^d + \mathbf{R}^{bc} \lambda^d) = 0 \quad (73)$$

and if we multiply it with λ^a then we have immediately that $\mathbf{E}_b^a \mathcal{H}^b = 0$ while $\lambda_a \mathcal{H}^a$ is arbitrary but we do not need it since it does not enter \mathbf{H}^{ab} . Hence this equation is reduced as

$$\frac{1}{16\pi\kappa} \epsilon_{abcd} (2\mathbf{H}^b \lambda^c \wedge \mathbf{E}^d + \mathbf{R}^{bc} \lambda^d) = 0, \quad (74)$$

which can be rewritten after some algebraic manipulations as

$$2H_d^{[a} \lambda^{b]} + 2H_c^c \delta_d^{[a} \lambda^{b]} = -2R^{c[a}{}_{cd} \lambda^{b]}, \quad (75)$$

where $R^{ab}{}_{cd} = i_{\mathbf{E}_d} i_{\mathbf{E}_c} \mathbf{R}^{ab}$ and $H_b^a = i_{\mathbf{E}_b} \mathbf{H}^a$. Constraint $\mathbf{R}_a \lambda^a = 0$ is equivalent to $R^{ab}{}_{ab} = 0$ and if we sum in previous equation over $a = d$ then $H_a^a = 0$ and we finally have

$$2\mathbf{H}^{[a} \lambda^{b]} = -2i_{\mathbf{E}_c} \mathbf{R}^{c[a} \lambda^{b]} \quad (76)$$

or

$$\mathbf{H}^{ab} = -2i_{\mathbf{E}_c} \mathbf{R}^{c[a} \lambda^{b]} + \bar{\epsilon}^{ab\bar{a}\bar{b}} \mathbf{E}^{\bar{c}} \lambda_{\bar{a}} \gamma_{\bar{b}\bar{c}}, \quad (77)$$

where $\gamma_{ab} = \gamma_{ba}$ is not determined yet. But there is no need to worry since our analysis is not over. We have just finished the first level of the Dirac procedure, however conservation of the secondary constraints should be analyzed too and there will appear the missing equation for γ_{ab} . In order to

do this let us compute time derivatives of secondary constraints (61), (62) and (71)

$$\begin{aligned}\frac{d\mathbf{R}(\mu)}{dt} &= \{\mathbf{R}(\mu), \mathbf{H}^{(\text{EC})}\} = \\ &= \int_{\Sigma} \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mu^a (\mathbf{R}^{bc} \wedge \mathbf{b}^d + \mathcal{D}\mathbf{B}^{bc} \wedge \mathbf{E}^d) = 0,\end{aligned}\quad (78)$$

$$\begin{aligned}\frac{d\mathbf{T}(\Theta)}{dt} &= \{\mathbf{T}(\Theta), \mathbf{H}^{(\text{EC})}\} = \\ &= \int_{\Sigma} -\frac{1}{16\pi\kappa} \varepsilon_{abcd} \Theta^{ab} (\mathbf{E}^c \wedge \mathcal{D}\mathbf{b}^d + \mathbf{E}^c \wedge \mathbf{B}^{d\bar{a}} \eta_{\bar{a}\bar{b}} \wedge \mathbf{E}^{\bar{b}}) = 0,\end{aligned}\quad (79)$$

$$\begin{aligned}\frac{d\mathbf{S}(M)}{dt} &= \{\mathbf{S}(M), \mathbf{H}^{(\text{EC})}\} = \\ &= \int_{\Sigma} \frac{1}{8\pi\kappa} M_{ab} (\mathbf{E}^a \mathbf{E}_c^b \wedge \mathcal{D}\mathbf{b}^c + \mathbf{E}^a \mathbf{E}_c^b \wedge \mathbf{B}^{c\bar{a}} \eta_{\bar{a}\bar{b}} \wedge \mathbf{E}^{\bar{b}}) = 0,\end{aligned}\quad (80)$$

where the terms obviously proportional to the constraints are omitted. We can substitute from $\mathbf{h}^a = \mathbf{b}^a + \Lambda^{a\bar{a}} \eta_{\bar{a}\bar{b}} \mathbf{E}^{\bar{b}} = 0$ the expression for \mathbf{b}^a into (78) and thanks to generalized Bianchi $\mathcal{D}\mathbf{R}^{ab} = 0$ and Ricci $\mathcal{D}\mathcal{D}\Lambda^{ab} = \mathbf{R}^{a\bar{a}} \eta_{\bar{a}\bar{b}} \Lambda^{\bar{a}\bar{b}} + \mathbf{R}^{b\bar{a}} \eta_{\bar{a}\bar{b}} \Lambda^{a\bar{b}}$ identities we have immediately

$$\mathcal{D} \left(\frac{\varepsilon_{abcd}}{16\pi\kappa} (\mathbf{R}^{bc} \lambda^d + \mathbf{H}^{bc} \wedge \mathbf{E}^d) \right) - \frac{\varepsilon_{abcd}}{16\pi\kappa} \mathbf{R}^{bc} \Lambda^{d\bar{a}} \eta_{\bar{a}\bar{b}} \mathbf{E}^{\bar{b}} + \frac{\varepsilon_{abcd}}{8\pi\kappa} \mathbf{R}^{b\bar{a}} \eta_{\bar{a}\bar{b}} \Lambda^{\bar{b}\bar{c}} \wedge \mathbf{E}^d = 0.$$

The first term vanishes due to (65). The last term can be transformed with the help of identity $\mathbf{R}^{ab} = \frac{1}{4} \bar{\varepsilon}^{ab\bar{a}\bar{b}} \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} \mathbf{R}^{\bar{c}\bar{d}}$ into expression

$$\frac{\varepsilon_{abcd}}{8\pi\kappa} \mathbf{R}^{b\bar{a}} \eta_{\bar{a}\bar{b}} \Lambda^{\bar{b}\bar{c}} \wedge \mathbf{E}^d = \frac{\varepsilon_{abcd}}{16\pi\kappa} \mathbf{R}^{bc} \Lambda^{d\bar{a}} \eta_{\bar{a}\bar{b}} \mathbf{E}^{\bar{b}} - \frac{1}{16\pi\kappa} \eta_{ab} \Lambda^{b\bar{a}} \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} \mathbf{R}^{\bar{b}\bar{c}} \mathbf{E}^{\bar{d}}.$$

Hence no new condition appears from equation (78) since last term is proportional to $\mathbf{R}_a = 0$.

Equation (79) can be rewritten with help $\mathbf{h}^a = 0$ and due to the fact that constraints $\mathbf{T}_{ab} = \mathbf{S}^{ab} = 0$ imply $\mathcal{D}\mathbf{E}^a = 0$ as

$$\frac{1}{32\pi\kappa} \varepsilon_{abcd} \mathbf{R}^{cd} \wedge \mathbf{E}^a \eta_{\bar{a}\bar{b}} \lambda^{\bar{b}} + \frac{1}{8\pi\kappa} \eta_{\bar{a}[a} \varepsilon_{b]c\bar{b}\bar{c}} \mathbf{E}^{\bar{a}} \wedge i_{\mathbf{E}_d} \mathbf{R}^{d\bar{b}} \lambda^{\bar{c}} \wedge \mathbf{E}^c = 0,\quad (81)$$

where (77) has been already substituted. Since any 4-form on the three-dimensional manifold vanishes identically we have that $\mathbf{E}^a \wedge \mathbf{R}^{bc} \wedge \mathbf{E}^d = 0$. We can apply interior product on it $i_{\mathbf{E}_b} (\mathbf{E}^a \wedge \mathbf{R}^{bc} \wedge \mathbf{E}^d) = \mathbf{E}_b^a \mathbf{R}^{bc} \wedge \mathbf{E}^d - \mathbf{E}^a \wedge i_{\mathbf{E}_b} \mathbf{R}^{bc} \wedge \mathbf{E}^d - \mathbf{E}^a \wedge \mathbf{R}^{bc} \mathbf{E}_b^d = 0$. Now we can express from this the term proportional to $i_{\mathbf{E}_b} \mathbf{R}^{bc}$ and substitute it into previous equation. If we use again $\mathbf{R}^{ab} = \frac{1}{4} \bar{\varepsilon}^{ab\bar{a}\bar{b}} \varepsilon_{\bar{a}\bar{b}\bar{c}\bar{d}} \mathbf{R}^{\bar{c}\bar{d}}$ then we finally find out that (81) is proportional to $\mathcal{D}\mathbf{E}^a$. Hence again no new constraint appears from (79).

Equation (80) can be rewritten as

$$\frac{1}{8\pi\kappa} \mathbf{E}^a \mathbf{E}_c^b \wedge (\mathbf{R}^{c\bar{a}} \eta_{\bar{a}\bar{b}} \lambda^{\bar{b}} + \mathbf{H}^{c\bar{a}} \eta_{\bar{a}\bar{b}} \wedge \mathbf{E}^{\bar{b}}) = 0.\quad (82)$$

This is the equation which determines γ_{ab} entering (77). However, we do not need explicit expression. For our purposes it is sufficient to show that this equation determines γ_{ab} uniquely. In order to see it we should substitute the expression (77) instead of \mathbf{H}^{ab} into this equation. Since (82) is linear in \mathbf{H}^{ab} it is also linear in γ_{ab} , i.e. $c_A + Q_A^B \gamma_B = 0$, where $A, B = (ab)$, and hence it is sufficient to show that Q_B^A is invertible. The first observation is that (82) actually represents 6 equations for 6 pieces $\mathbf{E}_a^a \mathbf{E}_b^b \gamma_{ab}$ hence we can consider only the term proportional to γ_{ab} which is $\lambda_b \lambda_{\bar{b}} \eta_{c\bar{c}} \bar{\epsilon}^{dcba} \bar{\epsilon}^{\bar{a}\bar{b}\bar{c}\bar{d}} \gamma_{d\bar{d}} = \tilde{G}^{a\bar{a}b\bar{b}} \gamma_{b\bar{b}}$ and as we will see in the next section the expression $\tilde{G}^{a\bar{a}b\bar{b}}$ standing before $\gamma_{b\bar{b}}$ is invertible on spatial subspace.

Let us summarize this section. We have built the Hamiltonian formulation of Einstein-Cartan theory. The Hamiltonian is given by the sum of two Hamiltonians

$$\begin{aligned} \mathbf{H} = \mathbf{H}^{(\text{EC})} + \mathbf{H}^{(\text{Rest})} = & \pi(\nu) + \Pi(\Gamma) + \mathbf{p}(\mathbf{b}) + \mathbf{P}(\mathbf{B}) + \mathbf{R}(\mu) + \mathbf{T}(\Theta) + \mathbf{S}(M) \\ & + \varphi(\mathcal{Y}) + \mathbf{u}(\mathbf{Y}). \end{aligned} \quad (83)$$

Constraints given by $\pi(\nu)$, $\Pi(\Gamma)$, $\mathbf{R}(\mu)$, $\mathbf{T}(\Theta)$, $\varphi(\mathcal{Y})$ and $\mathbf{u}(\mathbf{Y})$ do not determine any Lagrange multipliers, therefore they are the first class constraints. The remaining constraints $\mathbf{p}(\mathbf{b})$, $\mathbf{P}(\mathbf{B})$ and $\mathbf{S}(M)$ are of the second class. Lagrange multipliers \mathbf{b}^a and \mathbf{B}^{ab} are

$$\mathbf{b}^a = \mathcal{D}\lambda^a - \Lambda^{a\bar{a}} \eta_{\bar{a}\bar{b}} \mathbf{E}^{\bar{b}}, \quad (84)$$

$$\mathbf{B}^{ab} = \mathcal{D}\Lambda^{ab} + \mathbf{H}^{ab}, \quad (85)$$

where \mathbf{H}^{ab} does not depend on Λ^{ab} and it is the solution of (65) and (82). We will continue with Dirac analysis in the next section where we will introduce Dirac bracket and consider the reduced phase space of our physical system.

V. DIRAC BRACKETS

The first level of the Hamilton-Dirac approach to the dynamics has been completed in the previous section. In the case when physical system possesses the second class constraints C_A standard Poisson bracket can not be quantized by usual rule

$$i\hbar \varrho(\{A, B\}) |\psi\rangle = [\varrho(A), \varrho(B)] |\psi\rangle,$$

where ϱ is a representation of basic variables, since in the case when A, B are the constraints C_A then there is zero vector $(\varrho(C_A)\varrho(C_B) - \varrho(C_B)\varrho(C_A)) |\psi\rangle$ on the right-hand side while the operator on the left-hand side $\varrho(\{C_A, C_B\})$ is invertible. Hence there exists only one possibility for all

physical states solving quantum analogue of classical constraints represented by quantum equation $\varrho(C_A)|\psi\rangle = 0$ given by $|\psi\rangle = 0$. Dirac solved this problem by introducing new bracket and quantization is formulated by the representation of the Dirac instead of the Poisson algebra (See details in¹⁶). Let C_A be the second class constraints and so $\{C_A, C_B\} = U_{AB}$ is invertible; then Dirac brackets are defined by

$$\{A, B\}^* = \{A, B\} - \{A, C_A\}U^{AB}\{C_B, B\}, \quad (86)$$

where $U_{AB}U^{BC} = \delta_A^C$. We divide our job in two parts. In the first part we define certain simple brackets $\{, \}'$ and then we use these partial brackets in the definition of the final Dirac brackets $\{, \}^*$.

Let us define weak equivalence before we start our analysis of constraints. We say that two variables A, A' are weakly equivalent, $A \doteq A'$, if their difference is proportional to the second class constraints. The second class constraints for our system are ($\mathbf{b}^a, \mathbf{B}^{ab}, M_{ab}$ are arbitrary)

$$\begin{aligned} \mathbf{p}(\mathbf{b}) &= \int_{\Sigma} \mathbf{p}_a \wedge \mathbf{b}^a = \int_{\Sigma} \tilde{p}_a^\alpha b_\alpha^a d^3x, \\ \mathbf{P}(\mathbf{B}) &= \int_{\Sigma} \frac{1}{2} \left(\mathbf{p}_{ab} + \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{E}^c \wedge \mathbf{E}^d \right) \wedge \mathbf{B}^{ab} = \int_{\Sigma} \frac{1}{2} \mathbf{P}_{ab} \wedge \mathbf{B}^{ab} = \int_{\Sigma} \frac{1}{2} \tilde{P}_{ab}^\alpha B_\alpha^{ab} d^3x, \\ \mathbf{S}(M) &= \int_{\Sigma} \frac{1}{8\pi\kappa} M_{ab} \mathbf{E}^a \mathbf{E}^b \wedge \mathcal{D}\mathbf{E}^c = \int_{\Sigma} M_{ab} \mathbf{S}^{ab} = \int_{\Sigma} M_{ab} \tilde{S}^{ab} d^3x. \end{aligned}$$

We start the analysis by their decompositions

$$\begin{aligned} \mathbf{p}^\perp &= \mathbf{p}_a \lambda^a && \longleftrightarrow \tilde{p}^{\perp\alpha} = \tilde{p}_a^\alpha \lambda^a, \\ \mathbf{p}_a^\parallel &= \mathbf{E}_a^\alpha \mathbf{p}_{\bar{a}} && \longleftrightarrow \tilde{p}_a^{\parallel\alpha} = \mathbf{E}_a^\alpha \tilde{p}_{\bar{a}}^\alpha, \\ \mathbf{P}_a^\perp &= \mathbf{P}_{ab} \lambda^b && \longleftrightarrow \tilde{P}_a^{\perp\alpha} = \tilde{P}_{ab}^\alpha \lambda^b, \\ \mathbf{P}_a^\parallel &= \mathbf{E}_a^\alpha \mathbf{P}_{\bar{a}b} \wedge \mathbf{E}^b && \longleftrightarrow \tilde{P}_a^{\parallel\alpha} = \mathbf{E}_a^\alpha \tilde{P}_{\bar{a}b}^\alpha E_\alpha^b, \\ \sigma^{ab} &= \mathbf{P}_{\bar{a}\bar{b}} \lambda_{\bar{c}} \wedge \tilde{\varepsilon}^{\bar{a}\bar{b}\bar{c}(a} \mathbf{E}^{b)} && \longleftrightarrow \tilde{\sigma}^{ab} = \tilde{P}_{\bar{a}\bar{b}}^\alpha \lambda_{\bar{c}} \tilde{\varepsilon}^{\bar{a}\bar{b}\bar{c}(a} E_\alpha^{b)}. \end{aligned}$$

Now we are going to eliminate constraints $\mathbf{p}^\perp, \mathbf{p}_a^\parallel$ and their "canonical friends" $\mathbf{P}_a^\perp, \mathbf{P}_a^\parallel$ by introducing "partial Dirac bracket" $\{, \}'$. This bracket plays important role even in the context of full Dirac bracket. In order to introduce it we need following expressions

$$\begin{aligned} \{\mathbf{P}(\mathbf{B}), \mathbf{p}(\mathbf{b})\} &= \int_{\Sigma} \frac{1}{16\pi\kappa} \varepsilon_{abcd} \mathbf{B}^{ab} \wedge \mathbf{b}^c \wedge \mathbf{E}^d \\ &\quad \downarrow \\ \{\tilde{P}_{ab}^\alpha(\mathbf{x}), \tilde{p}_c^\beta(\mathbf{y})\} &= \frac{1}{8\pi\kappa} \varepsilon_{abcd} \tilde{\varepsilon}^{\alpha\beta\gamma} E_\gamma^d \delta_{\mathbf{xy}}. \end{aligned}$$

Hence nontrivial Poisson brackets are

$$\begin{aligned}
\{\mathbf{P}(\mathbf{B}^\perp), \mathbf{p}(\mathbf{b}^\parallel)\} &\hat{=} \int_{\Sigma} -\frac{1}{8\pi\kappa} \varepsilon_{abcd} \mathbf{B}^a \wedge \mathbf{b}^b \lambda^c \wedge \mathbf{E}^d \\
&\Downarrow \\
\{\tilde{P}_a^{\perp\alpha}(\mathbf{x}), \tilde{p}_b^{\parallel\beta}(\mathbf{y})\} &\hat{=} -\frac{1}{8\pi\kappa} \varepsilon_{abcd} \bar{\varepsilon}^{\alpha\beta\gamma} \lambda^c E_\gamma^d \delta_{\mathbf{xy}} = U_{ab}^{\alpha\beta} \delta_{\mathbf{xy}}, \\
\\
\{\mathbf{P}(\mathbf{B}^\parallel), \mathbf{p}(\mathbf{b}^\perp)\} &\hat{=} \int_{\Sigma} \frac{1}{8\pi\kappa} \varepsilon_{abcd} \mathcal{B}^a \mathbf{b} \wedge \lambda^b \mathbf{E}^c \wedge \mathbf{E}^d \\
&\Downarrow \\
\{\tilde{P}_a^{\parallel}(\mathbf{x}), \tilde{p}^{\perp\alpha}(\mathbf{y})\} &\hat{=} \frac{1}{8\pi\kappa} \varepsilon_{abcd} \bar{\varepsilon}^{\alpha\beta\gamma} \lambda^b E_\beta^c E_\gamma^d \delta_{\mathbf{xy}} = U_a^\alpha \delta_{\mathbf{xy}}.
\end{aligned}$$

It is easy to find that matrix U_α^a inverse to U_a^α is

$$U_\alpha^a = -\frac{4\pi\kappa}{e} E_\alpha^a, \quad \text{where} \quad U_\alpha^a U_b^a = \mathbf{E}_b^a \quad \text{and} \quad U_\alpha^a U_a^\beta = \delta_\alpha^\beta. \quad (87)$$

Next step is to look for the inverse matrix to $U_{ab}^{\alpha\beta}$. We can use ansatz $U_{ab}^{\alpha\beta} = A E_\alpha^a E_\beta^b + B E_\beta^a E_\alpha^b$ and the result is given by the expression

$$U_{\alpha\beta}^{ab} = -\frac{4\pi\kappa}{e} (E_\alpha^a E_\beta^b - 2E_\beta^a E_\alpha^b), \quad \text{where} \quad U_{\alpha\beta}^{ab} U_{bc}^{\beta\gamma} = \mathbf{E}_c^a \delta_\alpha^\gamma. \quad (88)$$

Now we have prepared everything what we need in order to define the partial Dirac bracket as follows

$$\begin{aligned}
\{A, B\}' &= \{A, B\} + \int_{\Sigma} d^3x \{A, \tilde{P}_a^{\perp\alpha}(\mathbf{x})\} U_{\alpha\beta}^{ab}(\mathbf{x}) \{\tilde{p}_b^{\parallel\beta}(\mathbf{x}), B\} \\
&\quad - \int_{\Sigma} d^3x \{B, \tilde{P}_a^{\perp\alpha}(\mathbf{x})\} U_{\alpha\beta}^{ab}(\mathbf{x}) \{\tilde{p}_b^{\parallel\beta}(\mathbf{x}), A\} \\
&\quad + \int_{\Sigma} d^3x \{A, \tilde{P}_a^{\parallel}(\mathbf{x})\} U_\alpha^a(\mathbf{x}) \{\tilde{p}^{\perp\alpha}(\mathbf{x}), B\} \\
&\quad - \int_{\Sigma} d^3x \{B, \tilde{P}_a^{\parallel}(\mathbf{x})\} U_\alpha^a(\mathbf{x}) \{\tilde{p}^{\perp\alpha}(\mathbf{x}), A\}.
\end{aligned}$$

The final Dirac bracket is going to be introduced within partial brackets and remaining constraints σ^{ab} , \mathbf{S}^{ab} . First of all we should mention the following property of the partial bracket. Let A be an arbitrary variable on full phase space; then

$$\{\sigma(m), A\}' \hat{=} \{\sigma(m), A\},$$

since

$$\{\boldsymbol{\sigma}(m), \mathbf{p}(\mathbf{b})\} = \{\mathbf{P}(\mathbf{B}^m), \mathbf{p}(\mathbf{b})\} \hat{=} \int -\frac{1}{4\pi\kappa} \delta_{ab}^{\bar{a}\bar{b}} \mathbf{b}^a \lambda_{\bar{a}} m_{\bar{b}\bar{c}} \wedge \mathbf{E}^{\bar{c}} \wedge \mathbf{E}^b = 0$$

and we have also $\{\boldsymbol{\sigma}(m), \mathbf{P}(\mathbf{B})\} = \{\mathbf{P}(\mathbf{B}^m), \mathbf{P}(\mathbf{B})\} = 0$. Hence we have as a consequence

$$\{\boldsymbol{\sigma}(m), \boldsymbol{\sigma}(m')\}' \hat{=} 0 \iff \{\tilde{\sigma}^{ab}(\mathbf{x}), \tilde{\sigma}^{cd}(\mathbf{y})\} \hat{=} 0.$$

Next important classical commutator is

$$\{\boldsymbol{\sigma}(m), \mathbf{S}(M)\}' \hat{=} \{\boldsymbol{\sigma}(m), \mathbf{S}(M)\} = \int_{\Sigma} -\frac{1}{8\pi\kappa} m_{a\bar{a}} M_{b\bar{b}} \eta_{c\bar{c}} \lambda_d \lambda_{\bar{d}} \bar{\varepsilon}^{abcd} \bar{\varepsilon}^{\bar{a}\bar{b}\bar{c}\bar{d}} \boldsymbol{\omega}, \quad (89)$$

where $\boldsymbol{\omega} = \frac{1}{3!} \varepsilon_{abcd} \lambda^a \mathbf{E}^b \wedge \mathbf{E}^c \wedge \mathbf{E}^d = e d^3x$. Now it is time to pay debt from the previous section where we have stated that $\tilde{G}^{a\bar{a}b\bar{b}}$ is invertible. We are going to do even more. We are going to calculate inverse of $U^{a\bar{a}b\bar{b}} = \frac{e}{8\pi\kappa} \tilde{G}^{a\bar{a}b\bar{b}}$. We can write

$$\begin{aligned} \{\boldsymbol{\sigma}(m), \mathbf{S}(M)\} &= \int_{\Sigma} \frac{1}{8\pi\kappa} m_{a\bar{a}} M_{b\bar{b}} \tilde{G}^{a\bar{a}b\bar{b}} \boldsymbol{\omega} \\ &\quad \updownarrow \\ \{\tilde{\sigma}^{a\bar{a}}(\mathbf{x}), \tilde{S}^{b\bar{b}}(\mathbf{y})\} &= \frac{e}{8\pi\kappa} \tilde{G}^{a\bar{a}b\bar{b}} \delta_{\mathbf{xy}} = U^{a\bar{a}b\bar{b}} \delta_{\mathbf{xy}} \end{aligned}$$

and

$$\tilde{G}^{a\bar{a}b\bar{b}} = -\frac{1}{2} \eta_{c\bar{c}} \lambda_d \lambda_{\bar{d}} (\bar{\varepsilon}^{abcd} \bar{\varepsilon}^{\bar{a}\bar{b}\bar{c}\bar{d}} + \bar{\varepsilon}^{\bar{a}bcd} \bar{\varepsilon}^{\bar{a}\bar{b}\bar{c}\bar{d}}). \quad (90)$$

Let us transform $U^{a\bar{a}b\bar{b}}$ into more suitable form. In order to do so we need to use the spatial metric tensor which is due to our choice of signature strictly negative

$$\mathbf{q} = \eta_{ab} \mathbf{E}^a \otimes \mathbf{E}^b = \mathbf{q}_{ab} \mathbf{E}^a \otimes \mathbf{E}^b = q_{\alpha\beta} dx^\alpha \otimes dx^\beta, \quad (91)$$

where $\mathbf{q}_{ab} = \eta_{\bar{a}\bar{b}} \mathbf{E}_{\bar{a}}^{\bar{a}} \mathbf{E}_{\bar{b}}^{\bar{b}}$, its inverse matrix is $q^{\alpha\beta} q_{\beta\gamma} = \delta_{\beta}^{\alpha}$ or $\mathbf{q}^{ab} \mathbf{q}_{bc} = \mathbf{E}_c^a$ and determinant

$$q \varepsilon_{\alpha\beta\gamma} = q_{\alpha\bar{\alpha}} q_{\beta\bar{\beta}} q_{\gamma\bar{\gamma}} \bar{\varepsilon}^{\bar{\alpha}\bar{\beta}\bar{\gamma}}.$$

It should be noted that $\mathbf{q}^{ab} \neq \mathbf{E}_{\bar{a}}^a \mathbf{E}_{\bar{b}}^b \eta^{\bar{a}\bar{b}}$. Now we can write

$$U^{a\bar{a}b\bar{b}} = \frac{\lambda^{*2}}{16\pi\kappa} (2\mathbf{q}^{a\bar{a}} \mathbf{q}^{b\bar{b}} - \mathbf{q}^{ab} \mathbf{q}^{\bar{a}\bar{b}} - \mathbf{q}^{a\bar{b}} \mathbf{q}^{\bar{a}b}), \quad (92)$$

where we have used formula $q = -e^2 \lambda^{*2}$ and $\lambda^{*2} = \eta^{ab} \lambda_a \lambda_b$. Now we are looking for inverse matrix to $U^{a\bar{a}b\bar{b}}$ in the form $U_{a\bar{a}b\bar{b}} = A \mathbf{q}_{a\bar{a}} \mathbf{q}_{b\bar{b}} + B(\mathbf{q}_{ab} \mathbf{q}_{\bar{a}\bar{b}} + \mathbf{q}_{a\bar{b}} \mathbf{q}_{\bar{a}b})$ and the result is given by the expression

$$U_{a\bar{a}b\bar{b}} = \frac{4\pi\kappa}{\lambda^{*2} e} (\mathbf{q}_{a\bar{a}} \mathbf{q}_{b\bar{b}} - \mathbf{q}_{ab} \mathbf{q}_{\bar{a}\bar{b}} - \mathbf{q}_{a\bar{b}} \mathbf{q}_{\bar{a}b}), \quad \text{where } U^{a\bar{a}b\bar{b}} U_{b\bar{b}c\bar{c}} = \mathbf{E}_c^a \mathbf{E}_{\bar{c}}^{\bar{a}}. \quad (93)$$

Finally we can define the full Dirac bracket as

$$\begin{aligned} \{A, B\}^* &= \{A, B\}' + \int d^3x \{A, \tilde{\sigma}^{a\bar{a}}(\mathbf{x})\}' U_{a\bar{a}b\bar{b}}(\mathbf{x}) \{\tilde{S}^{b\bar{b}}(\mathbf{x}), B\}' - \\ &\quad - \int d^3x \{B, \tilde{\sigma}^{a\bar{a}}(\mathbf{x})\}' U_{a\bar{a}b\bar{b}}(\mathbf{x}) \{\tilde{S}^{b\bar{b}}(\mathbf{x}), A\}' - \\ &\quad - \int d^3x d^3y \{A, \tilde{\sigma}^{a\bar{a}}(\mathbf{x})\}' U_{a\bar{a}b\bar{b}}(\mathbf{x}) \{\tilde{S}^{b\bar{b}}(\mathbf{x}), \tilde{S}^{c\bar{c}}(\mathbf{y})\}' U_{c\bar{c}d\bar{d}}(\mathbf{y}) \{\tilde{\sigma}^{d\bar{d}}(\mathbf{y}), B\}'. \end{aligned}$$

In order to finish the phase space reduction we need to describe a reduced manifold. Let us start with full phase space $\tilde{\Gamma}$ described by canonical variables $\lambda^a, \pi_a, \dots, U_{ab}$ (see table I). As we have seen in section IV the first reduction is given by $C^{ab} = \mathbf{C}^{ab} = \Phi_{ab} = U_{ab} = 0$ while conditions $\varphi = \mathbf{u} = 0$ are the first class constraints. These constraints mean that \mathcal{B}, \mathbf{B} are arbitrary and physics does not depend on them. Hence we can write $\tilde{\Gamma}|_{\text{red}} = \hat{\Gamma} \times \Lambda\Sigma$, where $\Lambda\Sigma$ is Cartan algebra of all forms on Σ of variables $\mathcal{B}, \dots, \mathbf{u}$ and $\hat{\Gamma}$ is described by variables $\lambda^a, \dots, \mathbf{p}_{ab}$. Whole dynamics takes place in $\hat{\Gamma}$. Let us consider a set

$$\mathfrak{Conf} = \{(\lambda^a, \mathbf{E}^a); e > 0, \eta_{ab}\lambda^a\lambda^b > 0, \lambda^0 > 0, \mathbf{q} < 0\}.$$

Hence due to condition $e > 0$ we have $\mathfrak{Conf} \subset \mathbf{GL}^+(\mathcal{M})$. However \mathfrak{Conf} is not a group. Nevertheless for every sufficiently small change $(\Delta\lambda^a, \Delta\mathbf{E}^a)$ the new element is again from \mathfrak{Conf} , i.e. $(\lambda^a + \Delta\lambda^a, \mathbf{E}^a + \Delta\mathbf{E}^a) \in \mathfrak{Conf}$; in other words \mathfrak{Conf} is a manifold. Hence we can construct canonically its cotangent bundle $\mathbb{T}^*\mathfrak{Conf} = \mathbb{T}_1\mathfrak{Conf}$ with symplectic structure $\omega_{\mathfrak{Conf}}$ on it. $\mathbb{T}^*\mathfrak{Conf}$ is described by canonical coordinates $(\lambda^a, \mathbf{E}^a, \pi_a, \mathbf{p}_a)$. Another structure of $\hat{\Gamma}$ is given by space

$$\mathfrak{G} = (\Lambda_0\mathbb{A}^2\Sigma \times \Lambda_3\mathbb{A}_2\Sigma) \times (\Lambda_1\mathbb{A}^2\Sigma \times \Lambda_2\mathbb{A}_2\Sigma) \quad (94)$$

described by variables $(\Lambda^{ab}, \mathbf{\Pi}_{ab}; \mathbf{A}^{ab}, \mathbf{p}_{ab})$. Hence $\hat{\Gamma} = \mathbb{T}^*\mathfrak{Conf} \times \mathfrak{G}$.

Since \mathbf{A}^{ab} is antisymmetric matrix 1-form we can decompose it as

$$\mathbf{A}^{ab} = 2\mathbf{A}^{[a}\lambda^{b]} + 2\mathcal{A}^{\bar{a}}\mathbf{E}_{\bar{a}}^{[a}\mathbf{E}^{b]} + \bar{\epsilon}^{ab\bar{a}\bar{b}}\mathbf{E}^{\bar{c}}\lambda_{\bar{a}}\alpha_{\bar{b}\bar{c}}. \quad (95)$$

Relevant information about \mathbf{A}^a and \mathcal{A}^a is encoded in a new variable

$$\mathbf{F}_a = \frac{1}{2}\varepsilon_{abcd}\mathbf{A}^{bc} \wedge \mathbf{E}^d \quad \longleftrightarrow \quad \mathbf{F}(\mathbf{K}) = \int_{\Sigma} \frac{1}{2}\varepsilon_{abcd}\mathbf{K}^a \wedge \mathbf{A}^{bc} \wedge \mathbf{E}^d, \quad (96)$$

while α_{ab} does not enter \mathbf{F}_a . Since $\{\sigma(m), \mathbf{F}(\mathbf{K})\}' \doteq 0$ and $\{\sigma(m), \mathbf{E}(\mathbf{Q})\}' \doteq 0$, where $\mathbf{E}(\mathbf{Q}) = \int_{\Sigma} \mathbf{Q}_a \wedge \mathbf{E}^a$ we have that

$$\begin{aligned} \{\mathbf{E}(\mathbf{Q}), \mathbf{F}(\mathbf{K})\}^* &\doteq \{\mathbf{E}(\mathbf{Q}), \mathbf{F}(\mathbf{K})\}' = -8\pi\kappa \int_{\Sigma} \mathbf{Q}_a \wedge \mathbf{K}^a, \\ \{\mathbf{F}(\mathbf{K}), \mathbf{F}(\mathbf{K}')\}^* &\doteq \{\mathbf{F}(\mathbf{K}), \mathbf{F}(\mathbf{K}')\}' = 0. \end{aligned} \quad (97)$$

Analogously, we obtain the rest of Dirac brackets for our variables on $\hat{\Gamma}$. The nontrivial results are

$$\{\lambda^a, \pi(\mu)\}^* \hat{=} \mu^a, \quad (98)$$

$$\{\Lambda^{ab}, \Pi(\Gamma)\}^* \hat{=} \Gamma^{ab}. \quad (99)$$

The reduction of $\hat{\Gamma}$ is almost finished. We can express α_{ab} from the condition $\mathbf{S}^{ab} = 0$ as function(al) of λ^a , \mathbf{E}^a and \mathbf{F}_a . The remaining second class constraints are trivially soluble. Since variables \mathcal{B} , \dots , \mathbf{u} do not describe any dynamics we can cast them away by additional fixation $\mathcal{B} = 0$ and $\mathbf{B} = 0$. Similar, we can proceed with Λ^{ab} . Hence we have the final reduced phase space

$$\Gamma = \mathbb{T}^* \mathcal{C}onf \quad (100)$$

described by variables $(\lambda^a, \mathbf{E}^a, \pi_a, \mathbf{F}_a)$ with symplectic structure defined by (97) and (98).

VI. DISCUSSION AND OPEN PROBLEMS

In section II we have started with the orthonormal coframe \mathbf{e}^a and the general gravitational connection $\hat{\nabla}$ described by its forms $\hat{\Gamma}^{ab} = \eta^{b\bar{b}} \hat{\Gamma}^a_{\bar{b}}$. We have derived equations of motion which have fixed $\hat{\Gamma}^{ab} = \hat{\mathbf{A}}^{ab} + \hat{\mathbf{B}} \eta^{ab}$ where $\hat{\mathbf{A}}^{ab}$ is related to the metric connection $\hat{\mathcal{D}}$ and $\hat{\mathbf{B}}$ is an arbitrary 1-form. The torsion of $\hat{\mathcal{D}}$ vanishes as a consequence of EOM, hence $\hat{\mathbf{A}}^{ab}$ can be expressed as a functional of coframe \mathbf{e}^a which is given by the solution of Einstein equations. We have induced the geometrical structure on the spatial section Σ inherited from the spacetime \mathbf{M} and hence $\mathbf{SO}(\mathfrak{g})$ is still (part of) the gauge freedom which is in opposite to the standard euclidean loop formulation of gravity where the orthonormal coframe \mathbf{e}^a is fixed to be tangential to Σ in spatial covectors and its time covector is normal to Σ . Then we have used $\mathbf{SO}(\mathfrak{g})$ structure in the Hamilton-Dirac formulation of the Einstein-Cartan theory. Since our system is degenerated and it contains both classes of constraints the Dirac bracket has been introduced. The Dirac procedure has been finished by introducing the reduced phase space described by coordinates $(\lambda^a, \mathbf{E}^a, \pi_a, \mathbf{F}_a)$.

The first class constraints have to be analyzed. If Σ is noncompact manifold then the first class constraints $\mathbf{R}(\mu)$, $\mathbf{T}(\Lambda)$ does not belong to the set of bounded function(al)s over the phase space if meaningful fall off conditions are suggested. Thus surface terms should be added in a similar way as in ADM formulation of General Relativity and the final Hamiltonian is not identically vanishing in this case. The analysis of the first class constraints including their algebraic properties will be performed in next paper of this series.

Another problem which should be explored in detail is the existence of the second class constraints $\sigma(m)$ and $\mathbf{S}(M)$. We already know how to deal with this on the classical level. But since the goal is a quantization of the Einstein-Cartan theory then they may cause insurmountable obstacle of the whole theory. There are few ways how to solve such kind of problems. The first thing which can be done is to solve the constraints classically as we have indicated in section V and then quantize the rest of variables. Another possibility is inspired by known property in quantum mechanics. Let C and K be the second class constraints with Poisson bracket $\{C, K\} = 1$. Then one can construct its representation ϱ on an appropriate Hilbert space and proceeds to construct the creation and annihilation operator $\hat{a}^{+/-}$ associated with them. If one finds vacuum of these operators, i.e. $\hat{a}^-|\Omega\rangle = 0$ then $\langle\Omega|C|\Omega\rangle = \langle\Omega|K|\Omega\rangle = 0$ and these constraints are solved at least for mean values. The third idea how to solve these constraints lies in enlargement of the phase space in such a way, that these constraints become first class. We know from observation of 2+1 Einstein-Cartan theory that constraints \mathbf{T}_{ab} are good candidates for generators of Lorentz algebra. Constraints \mathbf{S}^{ab} look similar. Both of them look like $\mathbf{E}^a \wedge \mathcal{D}\mathbf{E}^b$. Constraints \mathbf{T}_{ab} are antisymmetric in ab while \mathbf{S}^{ab} are symmetric. Hence it looks like something which should match together and question is what happens if we enlarge phase space by metric variable g_{ab} , i.e. if we broke gauge fixing represented by orthonormality of frames. Since \mathbf{S}^{ab} lives on spatial part of $\mathbb{T}\mathcal{M}$ if this idea works then the resulting group cannot be whole $\mathbf{GL}(\mathcal{M})$. Spatial metric has six degrees of freedom plus six for its momentum is twelve which is exactly the number of degrees of freedom fixed by constraints \mathbf{S}^{ab} and σ^{ab} . Anyway if this would work then this idea can not be used in higher dimensional space time. We can still use ghosts and quite general theory behind them.

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Appendix A: Notation and conventions

Manifold structure and indices:

\mathbf{M} - spacetime, Σ - spatial section of $\mathbf{M} = \mathbb{R} \times \Sigma$

Σ - spatial section in \mathbf{M}

\mathcal{M} - tangential Minkowski space (see section III)

$a, b, \dots = 0, 1, 2, 3$ - frame indices

$\mu, \nu, \dots = 0, 1, 2, 3$ - spacetime coordinate indices

$\alpha, \beta, \dots = 1, 2, 3$ - spatial coordinate indices

η_{ab} - Minkowski metric with signature $(+, -, -, -)$

Groups:

$\mathbf{GL}(V)$ - general linear group over (real) vector space V

$\mathbf{GL}^+(V)$ - positive general linear group over (real) vector space V (elements of $\mathbf{GL}(V)$ with positive determinant)

$\mathbf{O}(\mathbf{g})$ - orthonormal group over metric vector space (V, \mathbf{g}) or manifold (\mathbf{M}, \mathbf{g})

$\overline{\mathbf{SO}}(\mathbf{g}) \subset \mathbf{O}(\mathbf{g})$ - special orthonormal group over vector space (V, \mathbf{g}) or manifold (\mathbf{M}, \mathbf{g})

$\mathbf{SO}(\mathbf{g}) \subset \overline{\mathbf{SO}}(\mathbf{g})$ - proper Lorentz group over vector space (V, \mathbf{g}) or manifold (\mathbf{M}, \mathbf{g}) preserving righthand and future time orientation

(Anti)symmetrization:

$$A^{[ab]} = \frac{1}{2}(A^{ab} - A^{ba})$$

$$S^{(ab)} = \frac{1}{2}(S^{ab} + S^{ba})$$

etc.

Antisymmetric delta and Levi-Civita symbol:

$$\delta_{c\dots d}^{a\dots b} = \delta_c^{[a} \dots \delta_d^{b]} = \delta_{[c}^a \dots \delta_{d]}^b = \delta_{[c}^{[a} \dots \delta_{d]}^{b]}$$

$$\varepsilon_{abcd} = \varepsilon_{[abcd]}, \bar{\varepsilon}^{abcd} = \bar{\varepsilon}^{[abcd]} \text{ and } \varepsilon_{0123} = \bar{\varepsilon}^{0123} = 1$$

$$\varepsilon_{\alpha\beta\gamma} = \varepsilon_{[\alpha\beta\gamma]}, \bar{\varepsilon}^{\alpha\beta\gamma} = \bar{\varepsilon}^{[\alpha\beta\gamma]} \text{ and } \varepsilon_{123} = \bar{\varepsilon}^{123} = 1$$

Tensor spaces:

$(\mathbb{T}\mathbf{M}, \otimes)$ - algebra of all tensors over the spacetime manifold \mathbf{M}

$\mathbb{T}_q^p\mathbf{M}$ - (real) vector space of all tensors of rank $_q^p$ over the spacetime manifold \mathbf{M}

$(\mathbb{T}\mathcal{M}, \otimes)$ - algebra of all tensors in \mathcal{M}

$\mathbb{T}_q^p\mathcal{M}$ - (real) vector space of all tensors of rank $_q^p$ in \mathcal{M}

Cartan algebra and exterior product:

$(\Lambda\mathbf{M}, \wedge)$ - Cartan algebra of all spacetime forms. $\Lambda_p\mathbf{M}$ - space of spacetime p-forms.

$(\Lambda\Sigma, \wedge)$ - Cartan algebra of all spatial forms. $\Lambda_p\Sigma$ - space of spatial p-forms.

$(\Lambda\mathbb{T}\mathbf{M}, \wedge)$ - algebra of forms with values in tensor space $\mathbb{T}\mathbf{M}$

$\Lambda\mathbb{T}_q^p\mathbf{M}$ - (real) vector space of forms with values in tensor space $\mathbb{T}_q^p\mathbf{M}$

$(\Lambda\mathbb{T}\mathcal{M}, \wedge)$ - algebra of forms with values in tensor space $\mathbb{T}\mathcal{M}$

$\Lambda\mathbb{T}_q^p\mathcal{M}$ - (real) vector space of forms with values in tensor space $\mathbb{T}_q^p\mathcal{M}$

If $\alpha, \beta \in \Lambda_1\mathbf{M}$ or $\Lambda_1\Sigma$ then $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$

$$d^4x = dt \wedge d^3x, d^3x = dx^1 \wedge dx^2 \wedge dx^3$$

Interior product:

$$(i_v\alpha)(u_1, \dots, u_{p-1}) = \alpha(v, u_1, \dots, u_{p-1}) \quad \forall \alpha \in \Lambda_p\mathbf{M} \text{ or } \Lambda_p\Sigma$$

Derivative operators:

\hat{d} - exterior derivative operator on spacetime $\Lambda\mathbf{M}$. Anyway we write $dt = \hat{d}t$

d - spatial exterior derivative operator on $\Lambda\Sigma$

$\hat{\nabla}$ - general covariant exterior derivative operator on $\Lambda\mathbb{T}\mathbf{M}$, or general connection associated with

$$\hat{\Gamma}^a_b$$

$\hat{\mathcal{D}}$ - $\mathbf{SO}(\mathfrak{g})$ -covariant exterior derivative operator on $\Lambda\mathbb{T}\mathbf{M}$ associated with $\hat{\mathbf{A}}^a_b = \eta_{bc}\hat{\mathbf{A}}^{ac}$

\mathcal{D} - spatial $\mathbf{SO}(\mathfrak{g})$ -covariant exterior derivative operator on $\Lambda\mathbb{T}\mathcal{M}$ associated with $\mathbf{A}^a_b = \eta_{bc}\mathbf{A}^{ac}$

Appendix B: 2+1 Dimensional Einstein-Cartan Theory

If we already start with metric (Similar analysis of general connection can be done as in 3+1 case, but for simplicity we fix connection to be compatible with metric already now.). connection

then Lagrangian for 2+1 dimensional Einstein-Cartan theory can be written as

$$\mathbf{L} = \frac{1}{2} \varepsilon_{abc} \hat{\mathbf{R}}^{ab} \wedge \mathbf{e}^c. \quad (\text{B1})$$

EOM:

$$\hat{\mathbf{R}}^{ab} = 0, \quad (\text{B2})$$

$$\hat{\mathbf{T}}^a = 0. \quad (\text{B3})$$

Using 2+1 decomposition

$$\mathbf{e}^a = \lambda^a dt + \mathbf{E}^a,$$

$$\hat{\mathbf{A}}^{ab} = \Lambda^{ab} dt + \mathbf{A}^{ab}$$

leads to Hamiltonian:

$$\mathbf{H} = \pi(\nu) + \Pi(\Gamma) + \mathbf{p}(\mathbf{b}) + \mathbf{P}(\mathbf{B}) + \mathbf{R}(\lambda) + \mathbf{T}(\Lambda), \quad (\text{B4})$$

where

$$\pi(\nu) = \int_{\Sigma} \nu^a \wedge \pi_a, \quad (\text{B5})$$

$$\Pi(\Gamma) = \int_{\Sigma} \frac{1}{2} \Gamma^{ab} \wedge \Pi_{ab}, \quad (\text{B6})$$

$$\mathbf{p}(\mathbf{b}) = \int_{\Sigma} \mathbf{b}^a \wedge \mathbf{p}_a, \quad (\text{B7})$$

$$\mathbf{P}(\mathbf{B}) = \int_{\Sigma} \frac{1}{2} \mathbf{B}^{ab} \wedge (\mathbf{p}_{ab} - \varepsilon_{abc} \mathbf{E}^c), \quad (\text{B8})$$

$$\mathbf{R}(\lambda) = \int_{\Sigma} -\frac{1}{2} \varepsilon_{abc} \lambda^a \hat{\mathbf{R}}^{bc}, \quad (\text{B9})$$

$$\mathbf{T}(\Lambda) = \int_{\Sigma} -\frac{1}{2} \varepsilon_{abc} \Lambda^{ab} \mathcal{D}\mathbf{E}^c. \quad (\text{B10})$$

Momenta and velocities variables are given by table II.

Primary constraints are

$$\pi(\nu) = 0,$$

$$\mathbf{p}(\mathbf{b}) = 0,$$

$$\Pi(\Gamma) = 0,$$

$$\mathbf{P}(\mathbf{B}) = 0.$$

TABLE II. Table of basic variables

Variables	Momentum	Velocities
λ^a	$\pi_a = \tilde{\pi}_a d^2x$ where $\tilde{\pi}_a = \partial\mathcal{L}/\partial\lambda^a$	$v^a = \dot{\lambda}^a$
$\mathbf{E}^a = E_\alpha^a dx^\alpha$	$\mathbf{p}_a = \tilde{p}_a^\alpha \varepsilon_{\alpha\beta} dx^\beta$ where $\tilde{p}_a^\alpha = \partial\mathcal{L}/\partial\dot{E}_\alpha^a$	$\mathbf{b}^a = \dot{\mathbf{E}}^a$
Λ^{ab}	$\Pi_{ab} = \tilde{\Pi}_{ab} d^2x$ where $\tilde{\Pi}_{ab} = \partial\mathcal{L}/\partial\dot{\Lambda}^{ab}$	$\Gamma^{ab} = \dot{\Lambda}^{ab}$
$\mathbf{A}^{ab} = A_\alpha^{ab} dx^\alpha$	$\mathbf{p}_{ab} = \tilde{p}_{ab}^\alpha \varepsilon_{\alpha\beta} dx^\beta$ where $\tilde{p}_{ab}^\alpha = \partial\mathcal{L}/\partial\dot{A}_\alpha^{ab}$	$\mathbf{B}^{ab} = \dot{\mathbf{A}}^{ab}$

Poisson brackets between Hamiltonian and $\mathbf{p}(\mathbf{b})$ or $\mathbf{P}(\mathbf{B})$ lead to Lagrange multipliers

$$\begin{aligned}\mathbf{B}^{ab} &= \mathcal{D}\Lambda^{ab}, \\ \mathbf{b}^a &= \mathcal{D}\lambda^a - \eta_{\bar{a}\bar{b}}\Lambda^{a\bar{a}}\bar{\mathbf{E}}^{\bar{b}},\end{aligned}$$

while $\pi(v)$ and $\Pi(\Gamma)$ give new constraints

$$\begin{aligned}\mathbf{R}(v) &= \int_{\Sigma} -\frac{1}{2}\varepsilon_{abc}v^a\hat{\mathbf{R}}^{bc}, \\ \mathbf{T}(\Gamma) &= \int_{\Sigma} -\frac{1}{2}\varepsilon_{abc}\Gamma^{ab}\mathcal{D}\mathbf{E}^c.\end{aligned}$$

No other new constraints appear and \mathbf{p}, \mathbf{P} are the second class constraints. Next step is the definition of Dirac bracket thus we need to evaluate

$$\{\mathbf{P}(\tilde{\mathbf{B}}), \mathbf{p}(\tilde{\mathbf{b}})\} = \int_{\Sigma} -\frac{1}{2}\varepsilon_{abc}\tilde{\mathbf{B}}^{ab} \wedge \tilde{\mathbf{b}}^c,$$

what is equal to

$$\{\tilde{P}_{ab}^\alpha(x), \tilde{p}_c^\beta(y)\} = -\varepsilon_{abc}\tilde{\varepsilon}^{\alpha\beta}\delta_{xy}.$$

Dirac bracket is defined as

$$\begin{aligned}\{A, B\}^* &= \{A, B\} + \int \frac{dx}{2}\{A, \tilde{P}_{ab}^\alpha\}\tilde{\varepsilon}^{abc}\varepsilon_{\alpha\beta}\{\tilde{p}_c^\beta, B\} \\ &\quad - \int \frac{dx}{2}\{B, \tilde{P}_{ab}^\alpha\}\tilde{\varepsilon}^{abc}\varepsilon_{\alpha\beta}\{\tilde{p}_c^\beta, A\}\end{aligned}\tag{B11}$$

and constraints algebra is given by commutators

$$\{\mathbf{R}(\mu), \mathbf{R}(v)\}^* = 0,\tag{B12}$$

$$\{\mathbf{R}(\mu), \mathbf{T}(\Lambda)\}^* = -\mathbf{R}(\Lambda\eta\mu),\tag{B13}$$

$$\{\mathbf{T}(\Lambda), \mathbf{T}(\Gamma)\}^* = \mathbf{T}(\tilde{\Lambda}),\tag{B14}$$

where $\tilde{\Lambda}^{ab} = 2\delta_{\tilde{a}\tilde{b}}\Lambda^{\tilde{a}\tilde{c}}\eta_{\tilde{c}\tilde{d}}\Gamma^{\tilde{d}\tilde{b}}$ and $(\Lambda\eta\mu)^a = \Lambda^{ab}\eta_{bc}\mu^c$. We see that the constraints algebra in 2+1 dimensional Einstein-Cartan theory is Poincaré algebra.

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- ¹⁰One may say that we can define the spinor structure locally and work with such structure. But there may occur some certain pathological features. We will not focus our attention to this problem. Therefore "no loss of generality".
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- ¹²One equation is still missing as we will see at the end of this section. But this equation is conservation of constraints given by (9) and (10).
- ¹³ \mathbf{e}^a is coframe on $\mathbb{T}_1\mathbf{M}$, $\hat{\mathbf{e}}^a$ is its representation on $\mathbb{T}_1\mathcal{M}$.
- ¹⁴Of course ADM formalism works with spatial metric \mathbf{q} and therefore there are no coframe variables. For example in the Loop gravity Hamiltonian formulation starts with ADM, then orthonormal coframe \mathbf{e}^i on Σ is introduced and metric is expressed by orthonormality of this coframe, i.e. $\mathbf{q}(i, j = 1, 2, 3)$.
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- ¹⁷We omitted writing of details like $\forall \tilde{\nu}^a \dots$ in constraint's expressions.