Poincaré gauge gravity: selected topics

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Abstract

In the gauge theory of gravity based on the Poincaré group (the semidirect product of the Lorentz group and the spacetime translations) the mass (energy-momentum) and the spin are treated on an equal footing as the sources of the gravitational field. The corresponding spacetime manifold carries the Riemann-Cartan geometric structure with the nontrivial curvature and torsion. We describe some aspects of the classical Poincaré gauge theory of gravity. Namely, the Lagrange-Noether formalism is presented in full generality, and the family of quadratic (in the curvature and the torsion) models is analyzed in detail. We discuss the special case of the spinless matter and demonstrate that Einstein's theory arises as a degenerate model in the class of the quadratic Poincaré theories. Another central point is the overview of the so-called double duality method for constructing of the exact solutions of the classical field equations.

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I. INTRODUCTION

In this paper, we do not aim to give an exhaustive review of the Poincaré gauge theory of gravity. Correspondingly, our list of references is far from being complete. The inter-

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ested reader can find more details and more literature in the review papers and the books [9,17–19,22,23,31,34,42,45,48].

Within the framework of the gauge approach to gravity, the *kinematic* scheme of the theory is well understood at present, see, for example, [9,17–19,23,50–52,42,45]. The latter is based on the fiber bundle formalism and the connection theory combined with a certain spontaneous symmetry breaking mechanism, see also [54,53,41]. These aspects are described in the references given above, and we will not discuss them here. Instead, we will start from the final point of the kinematic scheme which introduces the Riemann-Cartan geometry on the spacetime manifold with the coframe and linear connection as the fundamental variables which describe the gravitational field.

However, the *dynamic* aspects of Poincaré gauge gravity have been rather poorly studied up to now. The choice of the basic Lagrangian of the theory still remains an open problem, and this, in turn, prevents a detailed analysis of possible physical effects. As a first step, one can use a correspondence principle. It is well known that Einstein's general relativity theory is satisfactorily supported by experimental tests on the macroscopic level. Thus, whereas the gravitational gauge models provide an alternative description of the gravitational physics in the microworld, it is natural to require their correspondence with the general relativity at large distances. In other words, of particular interest is the limit of spinless matter and the possibility of a reduction of the field equations to the (effective) Einstein theory. In this paper we will address these questions.

The structure of the paper is as follows. In Sec. II we fix the notation and set the general framework for the Lagrangian theory of the interacting gravitational and material fields. Sec. III is devoted to the analysis of the symmetries of the action of matter. The energy-momentum and the spin currents are introduced. They satisfy the Noether identities derived in Sec. IV. The gravitational field momenta are introduced in Sec. V and the corresponding Noether identities are obtained for the diffeomorphism and the local Lorentz invariance of the action of the Poincaré gravitational field. The general system of the gravitational field equations is established in Sec. VI. Then, in Sec. VII we discuss the limit of spinless matter.

After these preparations, we turn attention to the general quadratic Poincaré gauge models and derive the corresponding field equations in Sec. VIII. This class of models is important because of its similarity with the Yang-Mills theories of the internal symmetries. In Sec. IX we demonstrate that the Einstein theory can be interpreted as a special degenerate case of the quadratic Poincaré gauge theory. The notion of double duality for the tensor-valued 2-forms is introduced in Sec. X and the double duality properties of the irreducible parts of the Riemann-Cartan curvature are derived. These properties underlie the *double duality ansatz* (DDA) which proved to be a very powerful tool for solving the classical gravitational field equations in vacuum and with the nontrivial matter sources. The latter, under the DDA assumption, reduce to the effective Einstein theory. As an example, in Sec. XII we describe the exact *kinky-torsion* solution of the equations for the coupled gravitational and the Higgs-type scalar fields. Finally, in Sec. XIII we demonstrate that the Einstein spaces are the generic *torsion-free* solutions of the quadratic Poincaré gauge models. We collect the technical details of the computations in the three appendices.

Our basic notation and conventions are those of the ref. [18]. In particular, the Greek indices $\alpha, \beta, \ldots = 0, \ldots, 3$, denote the anholonomic components (for example, of a coframe ϑ^α), while the Latin indices $i, j, \ldots = 0, \ldots, 3$, label the holonomic components (dx^i, e.g.). The volume 4-form is denoted η, and the η-basis in the space of exterior forms is constructed with the help of the interior products as $\eta_{\alpha_1 \ldots \alpha_p} := e_{\alpha_p} \rfloor \ldots e_{\alpha_1} \rfloor \eta$, $p = 1, \ldots, 4$. They are related to the θ-basis via the Hodge dual operator *, for example, $\eta_{\alpha\beta} = \frac{1}{2} {}^\star (\vartheta_\alpha \wedge \vartheta_\beta)$.

II. LAGRANGE FORMALISM FOR THE GAUGE GRAVITY THEORY

The Poincaré gauge gravitational potentials are the local coframe 1-form ϑ^α and the 1-form $\Gamma_\alpha{}^\beta$ of the metric-compatible connection. Moreover, the metric structure $\mathbf{g}$ is defined on the spacetime manifold M. We can describe the latter explicitly by using the metric components $g_{\alpha\beta} := \mathbf{g}(e_\alpha, e_\beta)$ calculated on the frame e_α dual to ϑ^α. The compatibility of the metric and the connection is formulated in terms of the vanishing nonmetricity:

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma_\alpha{}^\gamma g_{\gamma\beta} + \Gamma_\beta{}^\gamma g_{\alpha\gamma} = 0. \tag{2.1}$$

Similarly to the freedom of the choice of the local coordinates, we notice that there is no a priori any reason to confine oneself to the case of the orthonormal frames for which $g_{\alpha\beta}$ is reduced to $o_{\alpha\beta} = \mathrm{diag}(+1, -1, -1, -1)$. Mathematically, the vanishing of the nonmetricity (2.1) means that the symmetric part of the connection is completely determined by the metric,

$$\Gamma_{(\alpha\beta)} = \frac{1}{2} dg_{\alpha\beta}, \tag{2.2}$$

and thus only the antisymmetric piece

$$\Gamma_{[\alpha\beta]} \tag{2.3}$$

is the true independent (of the coframe and the metric) gravitational potential.

With the help of the local general linear transformation

$$e'_\alpha = \Lambda(x)_\alpha{}^\beta e_\beta, \qquad \left(\text{hence} \quad \vartheta'^\alpha = \Lambda^{-1}(x)_\beta{}^\alpha \vartheta^\beta\right), \tag{2.4}$$

one may go to the *gauge* in which $g_{\alpha\beta}$ is *constant* (but not necessarily equal to $o_{\alpha\beta}$), thus eliminating the symmetric part of connection, $\Gamma_{(\alpha\beta)} = 0$. The well known examples of such a gauge are provided by the null (or Newman-Penrose) and semi-null frames which are not orthonormal.

It is worthwhile to note that (2.1) guarantees the skew symmetry of the curvature 2-form,

$$R_{\alpha\beta} = R_{[\alpha\beta]} \tag{2.5}$$

which is true for all choices of the frame. Indeed, taking the covariant derivative of (2.1) and using the Ricci identity, we find

$$2R_{(\alpha\beta)} = -DDg_{\alpha\beta} = DQ_{\alpha\beta} = 0. \tag{2.6}$$

The material fields may be scalar-, tensor- or spinor-valued forms of any rank, and we will denote them collectively as Ψ^A, where the superscript A indicates the appropriate index (tensor and/or spinor) structure. In many cases we will suppress the generalized index A. We will assume that the matter fields Ψ^A belong to the space of some (reducible, in general) representation of the Lorentz subgroup $SO(1,3)$ of (2.4) which is, as usually, defined by the condition of invariance of the metric,

$$\Lambda(x)_\alpha{}^\beta \in SO(1,3) \Longleftrightarrow \Lambda(x)_\alpha{}^\mu \Lambda(x)_\beta{}^\nu g_{\mu\nu} = g_{\alpha\beta}. \tag{2.7}$$

For infinitesimal transformations,

$$\Lambda(x)_\alpha{}^\beta = \delta_\alpha^\beta + \omega_\alpha{}^\beta, \tag{2.8}$$

we find the standard skew symmetry condition

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha}, \tag{2.9}$$

and hence the corresponding transformation of the material fields is given by

$$\Psi'^A = \Psi^A + \delta\Psi^A, \qquad \delta\Psi^A = -\omega_\alpha{}^\beta \rho(L^\alpha{}_\beta)^A{}_B \Psi^B, \tag{2.10}$$

where $L^{\alpha\beta} = L^{[\alpha\beta]}$ are the Lorentz group generators, and ρ describes the matrix representation of the Lorentz group. For brevity, we will use more compact notation, $\rho^{\alpha A}_{\beta B} := \rho(L^\alpha{}_\beta)^A{}_B$.

The dynamics of the theory is determined by choosing a scalar-valued Lagrangian 4-form

$$L_{\rm tot} = L(g_{\alpha\beta}, \Psi^A, d\Psi^A, \vartheta^\alpha, \Gamma_\alpha{}^\beta) + V(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta, d\vartheta^\alpha, d\Gamma_\alpha{}^\beta)\,, \tag{2.11}$$

(plus possible surface term $V_{\rm surface}$ which is often necessary for ensuring the correct boundary conditions), where L is the material and V the gravitational Lagrangian. Note that the Lagrangian does not contain derivatives of the metric in view of (2.2). The field equations are then found by requiring that the action integral

$$W = \int\limits_U L_{\rm tot}\,, \tag{2.12}$$

should have a stationary value for arbitrary variations $\delta\Psi^A$, $\delta\vartheta^\alpha$, $\delta\Gamma_\alpha{}^\beta$ of Ψ^A, ϑ^α and $\Gamma_\alpha{}^\beta$, which vanish on the boundary ∂U of an arbitrary 4-dimensional region $U \subset M$ of spacetime. In other words, we require

$$\delta W = \int_U \delta L_{\rm tot} = 0\,, \tag{2.13}$$

under the stated conditions. Noether identities will follow from the requirement that L is scalar-valued 4-form with respect to frame transformations (2.4) and, since there is no explicit coordinate dependence, it is invariant under arbitrary diffeomorphisms.

A separate remark is necessary about the status of the metric $g_{\alpha\beta}$ which can enter explicitly in (2.11). At the first sight it may seem to be an additional dynamical variable. However, it is not. The variation with respect to the metric vanishes as a result of the Noether identities (see below). As a matter of fact, this result is fairly clear if one recalls that, using the invariance of (2.11) under the change of the frame and local coordinates, we may completely eliminate the metric by choosing the convenient gauge, e.g., $g_{\alpha\beta} = o_{\alpha\beta}$.

Having thus outlined the programme that we intend to pursue, it remains to carry out this programme step by step. We do this initially without specifying the precise form of L or V. General relativity, the Einstein-Cartan theory and the quadratic gauge theory enter as special cases of V. As a historic remark, let us mention that the general aspects of the Lagrangian approach for the Poincaré gauge gravity were analyzed, for example, in [13,24,49] and in the review papers [17,18].

III. MATERIAL SOURCES

We assume that the material Lagrangian 4–form L depends most generally on Ψ, $d\Psi$, the gravitational potentials ϑ^α, $\Gamma_\alpha{}^\beta$, and the metric $g_{\alpha\beta}$ which may have nontrivial components in an arbitrary frame. According to the *minimal coupling* prescription, derivatives of the gravitational potentials are not permitted. We usually adhere to this principle. However, the *Pauli type terms* and the Jordan–Brans–Dicke type terms may occur in the phenomenological models or in the context of a symmetry breaking mechanism. Also the Gordon decomposition of the matter currents and the discussion of the gravitational moments necessarily requires the inclusion of the Pauli type terms, see [21,39]. Therefore, we develop our Lagrangian formalism in a sufficient generality in order to cope with such models by including in the Lagrangian also the derivatives $d\vartheta^\alpha$, and $d\Gamma_\alpha{}^\beta$ of the gravitational potentials:

$$L = L(g_{\alpha\beta}\,, \vartheta^\alpha\,, d\vartheta^\alpha\,, \Gamma_\alpha{}^\beta\,, d\Gamma_\alpha{}^\beta\,, \Psi^A, d\Psi^A)\,. \tag{3.1}$$

As a further bonus, we can then also read off the Noether identities for the gravitational gauge fields by considering the subcase $\Psi^A = 0$.

One consequence of the invariance of L under frame transformations (2.4) is that it can be recast in the form

$$L = L(g_{\alpha\beta}, \Psi^A, D\Psi^A, \vartheta^\alpha, T^\alpha, R_\alpha{}^\beta)\,. \tag{3.2}$$

That is, the exterior derivative d and the connection form $\Gamma_\alpha{}^\beta$ can only occur in the combination which gives the *covariant* exterior derivative

$$D = d + \rho^{\nu A}_{\mu B}\,\Gamma_\mu{}^\nu\wedge, \tag{3.3}$$

whereas the derivatives of the coframe and connection can only appear via the torsion and the curvature 2-forms. In order to see this, we use the fact that at any given event x there exists a frame such that $\Gamma_\alpha{}^\beta = 0$ at x. Then, in that frame,

$$L(g_{\alpha\beta}, \Psi^A, d\Psi^A, \vartheta^\alpha, d\vartheta^\alpha\,, \Gamma_\alpha{}^\beta\,, d\Gamma_\alpha{}^\beta) \stackrel{*}{=} \hat{L}(g_{\alpha\beta}, \Psi^A, D\Psi^A, \vartheta^\alpha, T^\alpha, R_\alpha{}^\beta)\,, \tag{3.4}$$

at x. Now, $\hat{L}$ is a scalar-valued 4-form constructed from tensorial and spinorial quantities and it is therefore invariant under frame transformations, whereas L is likewise invariant, by hypothesis. Hence (3.4) holds for all frames at x. The same argument holds at every event x and consequently the result (3.2) is proved.

We are now in a position to study the consequences of the various symmetries of the action. Independent variations of the arguments yield for the matter Lagrangian

$$\begin{aligned}\delta L = \delta g_{\alpha\beta}\,\frac{\partial L}{\partial g_{\alpha\beta}} + \delta\vartheta^\alpha\wedge\frac{\partial L}{\partial\vartheta^\alpha} + \delta T^\alpha\wedge\frac{\partial L}{\partial T^\alpha} + \delta R_\alpha{}^\beta\wedge\frac{\partial L}{\partial R_\alpha{}^\beta} \\ + \delta\Psi^A\wedge\frac{\partial L}{\partial\Psi^A} + \delta(D\Psi^A)\wedge\frac{\partial L}{\partial(D\Psi^A)}\,,\end{aligned} \tag{3.5}$$

where the partial derivatives are implicitly *defined* by (3.5). Note that in order to avoid counting the nondiagonal components twice in the variation procedure, a *strict ordering* of the indices is assumed in the first term of (3.5). The variation δ and the exterior derivative d commute, i.e. $[\delta\,,\,d]=0$, since from the very definition of the variation of a p–form $\delta\Psi := \Psi' - \Psi$ it follows $d\delta\Psi := d\Psi' - d\Psi = \delta d\Psi$. Using this fact, we can transform the variations with respect to the torsion and curvature T^α and $R_\alpha{}^\beta$ into the variations with respect to the gravitational potentials ϑ^α, and $\Gamma_\alpha{}^\beta$. We find

$$\delta T^\alpha = D\delta\vartheta^\alpha + \delta\Gamma_\beta{}^\alpha \wedge \vartheta^\beta, \qquad \delta R_\alpha{}^\beta = D\delta\Gamma_\alpha{}^\beta, \tag{3.6}$$

and thus

$$\begin{aligned}\delta L = {} & \frac{1}{2}\delta g_{\alpha\beta}\,\sigma^{\alpha\beta} + \delta\vartheta^\alpha \wedge \Sigma_\alpha + \delta\Gamma_\alpha{}^\beta \wedge \tau^\alpha{}_\beta + \delta\Psi^A \wedge \frac{\delta L}{\delta\Psi^A} \\ & + d\left[\frac{1}{2}\delta g_{\alpha\beta}\;\vartheta^\alpha \wedge \frac{\partial L}{\partial T_\beta} + \delta\vartheta^\alpha \wedge \frac{\partial L}{\partial T^\alpha} + \delta\Gamma_\alpha{}^\beta \wedge \frac{\partial L}{\partial R_\alpha{}^\beta} + \delta\Psi^A \wedge \frac{\partial L}{\partial D\Psi^A}\right].\end{aligned} \tag{3.7}$$

Here, for a gauge–invariant Lagrangian L, the expression

$$\frac{\delta L}{\delta\Psi^A} = \frac{\partial L}{\partial\Psi^A} - (-1)^p D\,\frac{\partial L}{\partial(D\Psi^A)} \tag{3.8}$$

is the covariant *variational derivative* of L with respect to the matter p–form Ψ^A. The *matter currents* in (3.7) are given by

$$\sigma^{\alpha\beta} := 2\,\frac{\delta L}{\delta g_{\alpha\beta}} = 2\,\frac{\partial L}{\partial g_{\alpha\beta}} - D\left(\vartheta^{(\alpha} \wedge \frac{\partial L}{\partial T_{\beta)}}\right), \tag{3.9}$$

$$\Sigma_\alpha := \frac{\delta L}{\delta\vartheta^\alpha} = \frac{\partial L}{\partial\vartheta^\alpha} + D\,\frac{\partial L}{\partial T^\alpha}\,, \tag{3.10}$$

$$\tau_{\alpha\beta} := \frac{\delta L}{\delta\Gamma_{[\alpha\beta]}} = \rho_{\alpha\beta}{}^A{}_B\,\Psi^B \wedge \frac{\partial L}{\partial(D\Psi^A)} + \vartheta_{[\alpha} \wedge \frac{\partial L}{\partial T^{\beta]}} + D\,\frac{\partial L}{\partial R_{\alpha\beta}}\,. \tag{3.11}$$

The equations (3.7)-(3.11) were derived with an account of the vanishing nonmetricity (2.1), from which the variation of the symmetric part of connection is expressed in terms of the variation of the metric,

$$\delta\Gamma_{(\alpha\beta)} = \frac{1}{2}D\delta g_{\alpha\beta}. \tag{3.12}$$

This ultimately leaves only the antisymmetric part of the connection $\Gamma_{[\alpha\beta]}$ as an independent variable.

The last term on the right hand side of (3.7) is an exact form which does not contribute to the action integral because of the usual assumption that $\delta g_{\alpha\beta} = 0$, $\delta\vartheta^\alpha = 0$, $\delta\Gamma_\alpha{}^\beta = 0$, and $\delta\Psi = 0$ on the boundary ∂U of the spacetime domain U of integration.

A. Energy–momentum

The 4–form $\sigma^{\alpha\beta}$ and the 3–form Σ_α are the *metrical* (Hilbert) and the *canonical* (Noether) *energy–momentum currents*, respectively.

Since the metric $g_{\alpha\beta}$ can be completely gauged away by the the frame transformations, it is clear that $\sigma^{\alpha\beta}$ is a secondary object, the very existence of which is due to the arbitrariness of the choice of the frames. That conclusion will be clarified later: after we have the Noether theorems at our disposal, we will demonstrate that the metrical energy-momentum is related to the symmetric part of the canonical energy-momentum.

On the contrary, the canonical energy–momentum 3–form has a clear physical meaning as the Noether current corresponding to the local translational (general coordinate) invariance of the field theory. It is an important dynamical object in the structure of the gravity theory.

From the canonical energy–momentum current we can extract its *trace*

$$\vartheta^\alpha \wedge \Sigma_\alpha, \tag{3.13}$$

with one independent component, and find

$$\not\Sigma_\alpha := \Sigma_\alpha - \frac{1}{4}\, e_\alpha \rfloor (\vartheta^\gamma \wedge \Sigma_\gamma)\,, \tag{3.14}$$

that is traceless:

$$\vartheta^\alpha \wedge \not\Sigma_\alpha = 0\,. \tag{3.15}$$

The *antisymmetric* piece $\vartheta_{[\alpha} \wedge \Sigma_{\beta]}$ is a 4-form which has 6 independent components, exactly as the scalar–valued 2–form

$$\Sigma := g^{\alpha\beta} e_\alpha \rfloor \Sigma_\beta = e_\alpha \rfloor \Sigma^\alpha \,. \tag{3.16}$$

With the help of some contractions, we find

$$\vartheta_{[\alpha} \wedge \Sigma_{\beta]} = \frac{1}{2} \vartheta_\alpha \wedge \vartheta_\beta \wedge \Sigma \,. \tag{3.17}$$

Consequently, the irreducible decomposition of the canonical energy–momentum 3–form Σ_α into a symmetric tracefree, trace, and antisymmetric piece reads

$$\Sigma_\alpha = \widehat{\not\Sigma}_\alpha + \frac{1}{4}\, e_\alpha \rfloor (\vartheta^\gamma \wedge \Sigma_\gamma) + \frac{1}{2}\, \vartheta_\alpha \wedge \Sigma \,. \tag{3.18}$$

This equation can be understood as defining the symmetric tracefree piece $\widehat{\not\Sigma}_\alpha$ with its 9 components. For the symmetric piece

$$\widehat{\Sigma}_\alpha = \Sigma_\alpha - \frac{1}{2}\, \vartheta_\alpha \wedge \Sigma \,, \tag{3.19}$$

we find

$$e_\alpha \rfloor \widehat{\Sigma}^\alpha = 0 \,, \qquad \vartheta^\alpha \wedge \widehat{\not\Sigma}_\alpha = 0 \,, \qquad \text{and} \qquad e_\alpha \rfloor \widehat{\not\Sigma}^\alpha = 0 \,. \tag{3.20}$$

Moreover, in analogy to (3.17), we have

$$\vartheta_{(\alpha} \wedge \Sigma_{\beta)} = \vartheta_{(\alpha} \wedge \widehat{\not\Sigma}_{\beta)} + \frac{1}{4}\, g_{\alpha\beta}\, (\vartheta^\gamma \wedge \Sigma_\gamma) \,. \tag{3.21}$$

B. Spin current

The (dynamical) *spin current* 3–form

$$\tau_{\alpha\beta} := \vartheta_{[\alpha} \wedge \mu_{\beta]} \tag{3.22}$$

can be equivalently expressed in terms of a vector–valued 2–form μ_α. As a first step to prove this, observe that the antisymmetric 3-form $\tau_{\alpha\beta} = \tau_{[\alpha\beta]}$ has the same number of independent components (namely, 24), as μ_α. Now, let us find the explicit form of *the spin energy potential* 2–form μ_α. Contracting (3.22) with e^β, we obtain

$$e^\beta\rfloor\tau_{\alpha\beta} = \frac{1}{2}\Big(\mu_\alpha - \vartheta_\alpha\wedge(e^\beta\rfloor\mu_\beta) - 4\mu_\alpha + \vartheta_\beta\wedge e^\beta\rfloor\mu_\alpha\Big)$$
$$= -\frac{1}{2}\Big(\mu_\alpha + \vartheta_\alpha\wedge(e^\beta\rfloor\mu_\beta)\Big)\,. \tag{3.23}$$

The second contraction with e^α yields

$$e^\alpha\rfloor e^\beta\rfloor\tau_{\alpha\beta} = -2e^\beta\rfloor\mu_\beta, \tag{3.24}$$

and substituting this into (3.23), we find finally:

$$\mu_\alpha = -2e^\beta\rfloor\tau_{\alpha\beta} + \frac{1}{2}\vartheta_\alpha\wedge(e^\beta\rfloor e^\gamma\rfloor\tau_{\beta\gamma})\;. \tag{3.25}$$

The 3-form spin current can be decomposed with respect to the η_μ basis of the space of 3-forms,

$$\tau_{\alpha\beta} = \tau_{\mu\alpha\beta}\eta^\mu. \tag{3.26}$$

The components $\tau_{\mu\alpha\beta}$ comprise the *spin density tensor*. It is easy to see that

$$e^\beta\rfloor\tau_{\alpha\beta} = -\tau_{\mu\nu\alpha}\eta^{\mu\nu}, \qquad e^\alpha\rfloor e^\beta\rfloor\tau_{\alpha\beta} = -\tau_{\mu\nu\alpha}\eta^{\mu\nu\alpha}, \tag{3.27}$$

and hence the spin energy potential reads:

$$\mu_\alpha = \frac{1}{2}\left(\tau_{\mu\nu\alpha} + \tau_{\nu\alpha\mu} - \tau_{\alpha\mu\nu}\right)\eta^{\mu\nu}, \tag{3.28}$$

which realises its expansion with respect to the η-basis of 2-forms.

The dynamical spin $\tau_{\alpha\beta}$ is an additional source term which has the equal importance as the energy–momentum current Σ_α in the Einstein–Cartan theory, and in a broader context, in the Poincaré gauge theory of gravity.

The field equation for the matter fields Ψ^A is given by the familiar Euler–Lagrange equation

$$\frac{\delta L}{\delta\Psi^A} = 0. \tag{3.29}$$

If (3.29) is assumed to be fulfilled in the course of the derivation of identities, we call the latter the *weak* identities in the following ("on shell" in the parlance of the particle physicists).

IV. NOETHER IDENTITIES FOR ENERGY-MOMENTUM AND SPIN CURRENTS

According to the Noether theorem, the conservation identities of the matter system result from the postulated invariance of L under a local symmetry group. Actually, this is only true "weakly", i.e., provided the Euler–Lagrange equation (3.29) for the matter fields is satisfied.

Here we consider the consequences of the invariance of L under the group of diffeomorphisms on the spacetime manifold M, and under the linear transformations of the frame field according to (2.4).

A. Diffeomorphisms

Let ξ be a vector field generating an arbitrary one–parameter group $\mathcal{T}_t$ of diffeomorphisms. In order to obtain the *covariant* Noether identity from the invariance of L under the one–parameter group of *local* translations $\mathcal{T}_t \subset \mathcal{T} \approx \mathit{Diff}\,(4, R)$, we need the conventional Lie derivative $\ell_\xi := \xi\rfloor d{+}d\xi\rfloor$ on M with respect to ξ. Since our Lagrangian L is also assumed to be a *scalar* under the linear transformations of the frames, we can equivalently replace ℓ_ξ by the covariant Lie derivative $\text{Ł}_\xi := \xi\rfloor D + D\xi\rfloor$. Then we find directly the *covariant* Noether identity by substituting Ł_ξ into (3.5):

$$\text{Ł}_\xi L = (\text{Ł}_\xi\, g_{\alpha\beta})\,\frac{\partial L}{\partial g_{\alpha\beta}} + (\text{Ł}_\xi\vartheta^\alpha)\wedge\frac{\partial L}{\partial\vartheta^\alpha} + (\text{Ł}_\xi T^\alpha)\wedge\frac{\partial L}{\partial T^\alpha} + (\text{Ł}_\xi R_\alpha{}^\beta)\wedge\frac{\partial L}{\partial R_\alpha{}^\beta}$$
$$+(\text{Ł}_\xi\Psi^A)\wedge\frac{\partial L}{\partial\Psi^A} + (\text{Ł}_\xi D\Psi^A)\wedge\frac{\partial L}{\partial D\Psi^A}\,. \tag{4.1}$$

Recall that the interior product $\xi\rfloor$, which formally acts analogously to a derivative of degree -1, obeys the Leibniz rule. Since the Lagrangian L is the 4–form on a four-dimensional manifold, its Lie derivative reduces to $\text{Ł}_\xi L = D(\xi\rfloor L)$. Similarly, since the metric $g_{\alpha\beta}$ is a 0-form with the vanishing covariant derivative, we have $\text{Ł}_\xi g_{\alpha\beta} = 0$. After expanding the Lie derivatives and performing some rearrangements, we get

$$
\begin{aligned}
D(\xi\rfloor L) = D\Big[&(\xi\rfloor\vartheta^\alpha)\,\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\partial L}{\partial T^\alpha} + (\xi\rfloor R_\alpha{}^\beta)\wedge\frac{\partial L}{\partial R_\alpha{}^\beta}\\
&+(\xi\rfloor\Psi^A)\wedge\frac{\partial L}{\partial\Psi^A} + (\xi\rfloor D\Psi^A)\wedge\frac{\partial L}{\partial D\Psi^A}\Big]\\
&-(\xi\rfloor\vartheta^\alpha)D\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor\vartheta^\alpha)\,R_\alpha{}^\beta\wedge\frac{\partial L}{\partial T^\beta}\\
&+(\xi\rfloor T^\alpha)\wedge D\frac{\partial L}{\partial T^\alpha} + (\xi\rfloor R_\beta{}^\gamma)\wedge\vartheta^\beta\wedge\frac{\partial L}{\partial T^\gamma} + (\xi\rfloor R_\beta{}^\gamma)\wedge D\frac{\partial L}{\partial R_\beta{}^\gamma}\\
&+(\xi\rfloor R_\beta{}^\gamma)\wedge\,\rho^{\beta A}_{\gamma B}\Psi^B\wedge\frac{\partial L}{\partial D\Psi^A}\\
&+(\xi\rfloor D\Psi^A)\,\wedge\frac{\delta L}{\delta\Psi^A} + (-1)^p(\xi\rfloor\Psi^A)\wedge D\frac{\delta L}{\delta\Psi^A}.
\end{aligned}
\tag{4.2}
$$

Collecting together the terms which form the variational derivatives, we obtain

$$
A + dB = 0\,, \tag{4.3}
$$

where

$$
\begin{aligned}
A :=& -(\xi\rfloor\vartheta^\alpha)D\frac{\delta L}{\delta\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\delta L}{\delta\vartheta^\alpha} + (\xi\rfloor R_\beta{}^\gamma)\wedge\frac{\delta L}{\delta\Gamma_\beta{}^\gamma}\\
&+(\xi\rfloor D\Psi^A)\,\wedge\frac{\delta L}{\delta\Psi^A} + (-1)^p(\xi\rfloor\Psi^A)\wedge D\frac{\delta L}{\delta\Psi^A}\,,
\end{aligned}
\tag{4.4}
$$

$$
\begin{aligned}
B :=& \,\xi\rfloor L - \Big[(\xi\rfloor\vartheta^\alpha)\,\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\partial L}{\partial T^\alpha} + (\xi\rfloor R_\alpha{}^\beta)\wedge\frac{\partial L}{\partial R_\alpha{}^\beta}\\
&+(\xi\rfloor\Psi^A)\wedge\frac{\partial L}{\partial\Psi^A} + (\xi\rfloor D\Psi^A)\wedge\frac{\partial L}{\partial D\Psi^A}\Big]\,.
\end{aligned}
\tag{4.5}
$$

The functions A and B have the form

$$
A = \xi^\alpha\,A_\alpha, \qquad\qquad B = \xi^\alpha\,B_\alpha\,. \tag{4.6}
$$

Thus (4.3) yields

$$
\xi^\alpha(A_\alpha + dB_\alpha) + d\xi^\alpha\wedge B_\alpha = 0, \tag{4.7}
$$

where *both* ξ^α *and* $d\xi^\alpha$ are *pointwise arbitrary.* Hence we conclude that both B_α and A_α vanish

$$
A = 0, \quad\text{and}\quad B = 0\,. \tag{4.8}
$$

From $B = 0$ we can read off the identity

$$
\begin{aligned}
\xi\rfloor L = (\xi\rfloor\vartheta^\alpha)\,\frac{\partial L}{\partial\vartheta^\alpha} + (\xi\rfloor T^\alpha)\wedge\frac{\partial L}{\partial T^\alpha} + (\xi\rfloor R_\alpha{}^\beta)\wedge\frac{\partial L}{\partial R_\alpha{}^\beta} \\
+(\xi\rfloor\Psi^A)\wedge\frac{\partial L}{\partial\Psi^A} + (\xi\rfloor D\Psi^A)\wedge\frac{\partial L}{\partial D\Psi^A}\,.
\end{aligned}
\tag{4.9}
$$

After replacing the vector field by the vector basis, $\xi \to e_\alpha$, Eq.(4.9) yields directly the explicit form of the *canonical energy–momentum current*

$$
\begin{aligned}
\Sigma_\alpha = e_\alpha\rfloor L - (e_\alpha\rfloor D\Psi^A)\wedge\frac{\partial L}{\partial D\Psi^A} - (e_\alpha\rfloor\Psi^A)\wedge\frac{\partial L}{\partial\Psi^A} \\
+D\frac{\partial L}{\partial T^\alpha} - (e_\alpha\rfloor T^\beta)\wedge\frac{\partial L}{\partial T^\beta} - (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\frac{\partial L}{\partial R_\beta{}^\gamma}.
\end{aligned}
\tag{4.10}
$$

The first line in (4.10) represents the result known in the context of the *special* relativistic classical field theory. For the case of the Maxwell electrodynamics, for example, Ψ stands for the electromagnetic potential one–form $A = A_i\, dx^i$, with the field strength two–form $F = DA = dA$. Then (4.10) describes Minkowski's $U(1)$–gauge invariant canonical energy–momentum current of the Maxwell field. The second line in (4.10) accounts for the possible Pauli terms as well as for the Lagrange multiplier terms in the variations with the constraints and it is absent for the case of the minimal coupling.

From $A = 0$, we read off the *first Noether identity*

$$
\begin{aligned}
D\Sigma_\alpha &\equiv (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\tau^\beta{}_\gamma + W_\alpha \\
&\cong (e_\alpha\rfloor T^\beta)\wedge\Sigma_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\tau^\beta{}_\gamma, \qquad \text{(1st)}
\end{aligned}
\tag{4.11}
$$

where

$$
W_\alpha := (e_\alpha\rfloor D\Psi^A)\,\frac{\delta L}{\delta\Psi^A} + (-1)^p(e_\alpha\rfloor\Psi^A)\wedge D\frac{\delta L}{\delta\Psi^A}\,. \tag{4.12}
$$

The first line in (4.11) is given in the *strong* form, without using the field equations. Note that the metrical energy-momentum $\sigma^{\alpha\beta}$ does not show up in the conservation law at all which proves its non-dynamical character.

In the right hand side of the differential identity (4.11) for the canonical energy–momentum current we find the typical Lorentz–type force terms. They have the general structure *field strength* $\times$ *current*.

B. Lorentz invariance and general frame transformation

The invariance of L with respect to the local Lorentz transformations (2.7) of the frames gives rise to a further identity. Under the infinitesimal transformations (2.8)-(2.9), the variations of the geometrical objects and of the matter fields read as follows

$$\delta g_{\alpha\beta} = 0, \quad \delta\vartheta^{\alpha} = -\omega_{\beta}{}^{\alpha}\,\vartheta^{\beta}, \quad \delta\Gamma_{\alpha}{}^{\beta} = D\omega_{\alpha}{}^{\beta}, \quad \delta\Psi^{A} = -\omega^{\alpha\beta}\,\rho_{\alpha\beta}{}^{A}_{B}\,\Psi^{B}. \tag{4.13}$$

If we insert (4.13) into (3.7), we obtain

$$\begin{aligned}\delta L = & -\omega^{\alpha\beta}\left(\vartheta_{[\alpha}\wedge\Sigma_{\beta]} + D\tau_{\alpha\beta} + \rho_{\alpha\beta}{}^{A}_{B}\Psi^{B}\wedge\frac{\delta L}{\delta\Psi^{A}}\right) \\ & +d\left[\omega^{\alpha\beta}\left(\tau_{\alpha\beta} - \rho_{\alpha\beta}{}^{A}_{B}\Psi^{B}\wedge\frac{\partial L}{\partial D\Psi^{A}} - \vartheta_{[\alpha}\wedge\frac{\partial L}{\partial T^{\beta]}} - D\,\frac{\partial L}{\partial R^{\alpha\beta}}\right)\right].\end{aligned} \tag{4.14}$$

The boundary term vanishes identically in view of the definition (3.11) of the spin current $\tau_{\alpha\beta}$. Then, from the arbitrariness of $\omega_{\alpha}{}^{\beta}$, we find the *second Noether identity*

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha}\wedge\Sigma_{\beta]} \equiv -\rho_{\alpha\beta}{}^{A}_{B}\Psi^{B}\wedge\frac{\delta L}{\delta\Psi^{A}} \cong 0\,. \qquad \text{(2nd)} \tag{4.15}$$

Again, the weak Noether identity holds provided the matter field equation (3.29) is satisfied.

Now, let us consider the linear transformations of the frame (2.4) which are necessarily non-Lorentz. In the infinitesimal form,

$$\Lambda(x)_{\alpha}{}^{\beta} = \delta_{\alpha}^{\beta} + \omega_{\alpha}{}^{\beta}, \tag{4.16}$$

but this time, unlike (2.9), the transformation parameters are symmetric

$$\omega_{\alpha\beta} = \omega_{\beta\alpha}. \tag{4.17}$$

Under (4.16)-(4.17), the geometrical objects and the matter fields transform as

$$\delta g_{\alpha\beta} = 2\omega_{\alpha\beta}, \quad \delta\vartheta^{\alpha} = -\omega_{\beta}{}^{\alpha}\,\vartheta^{\beta}, \quad \delta\Gamma_{\alpha}{}^{\beta} = D\omega_{\alpha}{}^{\beta}, \quad \delta\Psi^{A} = 0. \tag{4.18}$$

Substituting (4.18) into (3.7), we have

$$\delta L = \omega^{\alpha\beta}\left(\sigma_{\alpha\beta} - \vartheta_{(\alpha}\wedge\Sigma_{\beta)}\right). \tag{4.19}$$

From this we find that the metrical energy-momentum is equal to the symmetric part of the canonical energy-momentum,

$$\sigma_{\alpha\beta} \equiv \vartheta_{(\alpha} \wedge \Sigma_{\beta)}. \tag{4.20}$$

Note that this relation is a *strong identity* which holds true in any Lorentz-covariant field theory when the conditions (4.18) are satisfied. In particular, this is true for the Dirac field (described by the spinor-valued 0-form) and for the Rarita-Schwinger field (the spinor-valued 1-form). [The condition (4.18) is, however, not true in the Proca theory where the spin 1 particle is described by the covector-valued 1-form.] Thus we see that indeed the metrical energy-momentum current is a secondary object which arises as a symmetric part of the canonical energy-momentum current. In addition, no conservation law can be established for $\sigma_{\alpha\beta}$ directly from the invariance of the Lagrangian under the general coordinate or the frame transformation.

In order to get some more insight into the relation between the metrical and canonical energy-momentum currents, let us introduce instead of the 4-form $\sigma_{\alpha\beta}$ an equivalent vector-valued 3-form

$$\sigma_\alpha := e_\beta \rfloor \sigma_\alpha{}^\beta. \tag{4.21}$$

Evidently,

$$e_\beta \rfloor \sigma^\beta = 0. \tag{4.22}$$

The identity (4.15) yields for the antisymmetric part of the canonical energy-momentum

$$\Sigma = e_\alpha \rfloor \Sigma^\alpha = e_\alpha \rfloor e_\beta \rfloor D\tau^{\alpha\beta}, \tag{4.23}$$

and we then straightforwardly see that the Noether identity (4.20) can be rewritten as

$$\begin{aligned} \sigma_\alpha = \widehat{\Sigma}_\alpha &= \Sigma_\alpha - \frac{1}{2}\vartheta_\alpha \wedge \Sigma \\ &= \Sigma_\alpha - e^\beta \rfloor D\tau_{\alpha\beta}. \end{aligned} \tag{4.24}$$

As it is well known, the relation between the metrical ("Hilbert") and the canonical ("Noether") energy–momentum currents is established in the so-called Belinfante-Rosenfeld symmetrization procedure. The last formula does not substitute the Belinfante-Rosenfeld result, it rather can be understood as a specific symmetrization of an otherwise asymmetric energy–momentum current for the models which satisfy the condition (4.18).

V. GRAVITATIONAL FIELD MOMENTA AND NOETHER IDENTITIES FOR THE GRAVITATIONAL LAGRANGIAN

The total Lagrangian L_{tot} (2.11) includes the pure gravitational Lagrangian V. We assume that the 4–form V depends on the metric $g_{\alpha\beta}$, and the gravitational potentials ϑ^α, ${\Gamma_\alpha}^\beta$ and their first derivatives, $d\vartheta^\alpha$ and $d{\Gamma_\alpha}^\beta$. By an argument similar to the one used in Sec. III, we can verify that invariance of V under the tetrad deformations requires V to be of the form

$$V = V(g_{\alpha\beta}, \vartheta^\alpha, T^\alpha, {R_\alpha}^\beta)\,. \tag{5.1}$$

Consequently, we can use the results of Sec. IV and apply them to the gravitational Lagrangian simply by replacing L by V and by dropping all Ψ–dependent terms in the end. For convenience, we condense our notation and introduce, according to the conventional canonical prescription, the following *gauge field momenta* 2-forms:

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha}\,, \qquad {H^\alpha}_\beta := -\frac{\partial V}{\partial {R_\alpha}^\beta}\,. \tag{5.2}$$

Moreover, we define the *metrical* energy–momentum 4–form

$$m^{\alpha\beta} := 2\,\frac{\partial V}{\partial g_{\alpha\beta}}, \tag{5.3}$$

the *canonical* energy–momentum 3–form

$$E_\alpha := \frac{\partial V}{\partial \vartheta^\alpha} \tag{5.4}$$

and the *spin* 3–form

$$E^{\alpha\beta} := \frac{\partial V}{\partial \Gamma_{[\alpha\beta]}} = -\vartheta^{[\alpha} \wedge H^{\beta]} \tag{5.5}$$

for the gravitational gauge fields themselves. If we apply the variational principle (3.7) with respect to the *independent variables* $g_{\alpha\beta}$, ϑ^α, and $\Gamma_\alpha{}^\beta$ and compare it with (3.9)-(3.11), we find

$$2\,\frac{\delta V}{\delta g_{\alpha\beta}} = -DM^{\alpha\beta} + m^{\alpha\beta}, \tag{5.6}$$

$$\frac{\delta V}{\delta \vartheta^\alpha} = -DH_\alpha + E_\alpha, \tag{5.7}$$

$$\frac{\delta V}{\delta \Gamma_{[\alpha\beta]}} = -DH^{\alpha\beta} + E^{\alpha\beta}\,. \tag{5.8}$$

Here

$$M^{\alpha\beta} := -\vartheta^{(\alpha} \wedge H^{\beta)} \tag{5.9}$$

plays the role of the metric field momentum.

The Noether machinery can be applied to the gravitational Lagrangian (5.1) in a precisely the same way as it was done for the material Lagrangian in Sec. IV. As a result, we find:

(i) The diffeomorphism invariance yields the explicit structure of the canonical energy–momentum 3–form

$$E_\alpha = e_\alpha\rfloor V + (e_\alpha\rfloor T^\beta)\wedge H_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge H^\beta{}_\gamma \tag{5.10}$$

of the gauge fields, cf. (4.10) for the material case. This implies for its trace

$$\vartheta^\alpha \wedge E_\alpha = 4V + 2T^\beta\wedge H_\beta + 2R_\beta{}^\gamma \wedge H^\beta{}_\gamma. \tag{5.11}$$

Furthermore we find the *first Noether identity*

$$D\,\frac{\delta V}{\delta\vartheta^\alpha} \equiv (e_\alpha\rfloor T^\beta)\wedge\frac{\delta V}{\delta\vartheta^\beta} + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge\frac{\delta V}{\delta\Gamma_\beta{}^\gamma}, \qquad \text{(1st)} \tag{5.12}$$

as a gravitational counterpart of the identity (4.11) for the matter Lagrangian.

(ii) The invariance with respect to the (infinitesimal) local Lorentz transformations yields the *second Noether identity*

$$D\,\frac{\delta V}{\delta\Gamma^{[\alpha\beta]}} + \vartheta_{[\alpha}\wedge\frac{\delta V}{\delta\vartheta^{\beta]}} \equiv 0. \qquad \text{(2nd)} \tag{5.13}$$

(iii) The invariance with respect to the (infinitesimal) local non-Lorentz transformations of the frames yields an additional identity

$$2\,\frac{\delta V}{\delta g_{\alpha\beta}} - \vartheta_{(\alpha}\wedge\frac{\delta V}{\delta\vartheta^{\beta)}} \equiv 0. \tag{5.14}$$

Observe that the Noether identities for the gravitational gauge fields are all *strong*, since no field equation is involved in their derivation.

By inserting (5.6)-(5.7) into the Noether identity (5.14), we obtain the following explicit form of the metrical gravitational energy–momentum current

$$m^{\alpha\beta} = \vartheta^{(\alpha}\wedge E^{\beta)} - T^{(\alpha}\wedge H^{\beta)}. \tag{5.15}$$

Consequently, its trace is given by

$$m^{\alpha}{}_{\alpha} = \vartheta^{\alpha}\wedge E_{\alpha} - T^{\alpha}\wedge H_{\alpha} = 4V + T^{\beta}\wedge H_{\beta} + 2R_{\beta}{}^{\gamma}\wedge H^{\beta}{}_{\gamma}\,. \tag{5.16}$$

Analogously, after using (5.6)-(5.7) and (5.5) in (5.13), we find

$$\vartheta_{[\alpha}\wedge E_{\beta]} - T_{[\alpha}\wedge H_{\beta]} + R_{\alpha}{}^{\gamma}\wedge H_{\gamma\beta} + R_{\beta}{}^{\gamma}\wedge H_{\alpha\gamma} = 0, \tag{5.17}$$

where we used the definition of the curvature as a commutator of covariant derivatives, $DDH_{\alpha\beta} \equiv R_{\alpha}{}^{\gamma}\wedge H_{\gamma\beta} + R_{\beta}{}^{\gamma}\wedge H_{\alpha\gamma}$. Alternatively, we may collect the symmetric and skew-symmetric identities (5.15) and (5.17) into a single equation,

$$m_{\alpha\beta} = \vartheta_{\alpha}\wedge E_{\beta} - T_{\alpha}\wedge H_{\beta} + R_{\alpha}{}^{\gamma}\wedge H_{\gamma\beta} + R_{\beta}{}^{\gamma}\wedge H_{\alpha\gamma}. \tag{5.18}$$

VI. GRAVITATIONAL FIELD EQUATIONS

Now we are in the position to formulate the action principle in full generality: The total action of the gravitational gauge fields and of the minimally coupled matter fields reads

$$W = \int [V(g_{\alpha\beta}, \vartheta^\alpha, T^\alpha, R_\alpha{}^\beta) + L(g_{\alpha\beta}, \vartheta^\alpha, \Psi, D\Psi)]. \tag{6.1}$$

The *independent variables* are Ψ, $g_{\alpha\beta}$, ϑ^α, and $\Gamma_\alpha{}^\beta$. Their independent variation then yields, by means of (5.6)-(5.8) and the definitions (5.2)-(5.5),(5.9) and (3.8)-(3.11), the *Yang–Mills type gauge field equations* of gravity:

$$\frac{\delta L}{\delta \Psi} = 0\,, \qquad \text{(MATTER)} \tag{6.2}$$

$$DM^{\alpha\beta} - m^{\alpha\beta} = \sigma^{\alpha\beta}\,, \qquad \text{(ZEROTH)} \tag{6.3}$$

$$DH_\alpha - E_\alpha = \Sigma_\alpha\,, \qquad \text{(FIRST)} \tag{6.4}$$

$$DH_{\alpha\beta} - E_{\alpha\beta} = \tau_{\alpha\beta}\,. \qquad \text{(SECOND)} \tag{6.5}$$

The covariant exterior derivatives D of the gauge field momenta describe the terms of the Yang–Mills type. In addition, due to the universality of the gravitational interaction, we find the *self–coupling* terms which involve the metrical energy–momentum $m^{\alpha\beta}$, the canonical energy–momentum E_α, or the spin $E_{\alpha\beta}$ of the gravitational fields, respectively. They, together with the corresponding material currents $\sigma^{\alpha\beta}$, Σ_α, and $\tau_{\alpha\beta}$, act as sources of the gauge field potentials.

This dynamical framework is very general. As special cases, it contains the field equations of the general relativity (GR) theory and those of the Einstein–Cartan theory. Both are the dynamically degenerate cases of the Poincaré gauge theory.

As soon as an explicit gauge Lagrangian V is specified, all we have to do is *to partially differentiate* this Lagrangian with respect to the field strengths, the torsion T^α and the curvature $R_\alpha{}^\beta$, respectively. Thereby we find the gauge field momenta in (5.2). If we substitute the latter into (5.5), (5.10), and (5.15) and, subsequently, into the field equations (6.3)-(6.5), we obtain the field equations explicitly. Our framework allows to investigate the different gauge Lagrangians in a straightforward way.

Note, in particular, that we do not need to vary the Hodge star, a computation which would complicate things appreciably. The idea to use the gauge field momenta as the operationally meaningful quantities in their own right — together with the temporary suspension of the relations between the momenta and the field strengths — is taken from the Kottler–Cartan–van Dantzig representation of the electrodynamics (see the book [20]).

Compared to the earlier work on this subject, in which only the two field equations occur, we have obtained a system of the *three* gauge field equations for the gravitational potentials. This can be traced back to the assumption that the coframe field ϑ^α is *not* necessarily *orthonormal*. This allows for a more flexibility in the process of the eventual solving of the field equations. However, it is clear that the price for this flexibility is a certain duplication of the dynamical equations. It is straightforward to verify that not all of the gravitational field equations (6.3)-(6.5) are independent.

This fact is obvious already from the direct counting of the number of the field variables. For the description of the gravitational field, we have $10 + 16 + 24 = 50$ components of the metric $g_{\alpha\beta}$, coframe ϑ^α and connection $\Gamma_{[\alpha\beta]}$. We count only the antisymmetric part, since the symmetric piece of connection is expressed in terms of the metric via (2.2). Formally, we have exactly 50 field equations (6.3)-(6.5). However, there is a wide symmetry group of the action which includes the local Lorentz rotations plus the general linear transformations of the frame (2.4). Together, they amount to $6+10 = 16$ arbitrary functions which parametrize these transformations. In principle, we always have an option to "gauge away" completely either the metric or the coframe everywhere on the spacetime manifold M, and to work in one of the following gauges:

- The *constant-metric-gauge* is obtained by choosing the frames in such a way that the metric

$$g_{\alpha\beta} = \text{constant} \tag{6.6}$$

  everywhere. By imposing this gauge, we fix the freedom of the non-Lorentz linear transformations (2.4), and reduce the number of gravity field variables to $16+24 = 40$

(for the coframe ϑ^α and connection $\Gamma_{[\alpha\beta]}$). Note, however, that the local Lorentz rotations which, by definition (2.7), preserve (6.6), are still available. This remaining freedom (involving 6 arbitrary functions) can be used for the further reduction of the number of variables. In particular, one can eliminate any 6 of the 16 components of the coframe, so that finally we end up with $10 + 24 = 34$ independent variables. For example, a convenient choice is to make a 4×4 matrix of the coframe coefficients, e_i^α, *symmetric*.

- *Constant-coframe-gauge* is achieved after the combined use of the local Lorentz and general linear transformations, reducing the coframe to

$$\vartheta^\alpha = \delta_i^\alpha\, dx^i = dx^\alpha \tag{6.7}$$

everywhere. This eliminates the coframe components completely, and one is left again with $10 + 24 = 34$ independent variables (this time they are the metric $g_{\alpha\beta}$ and the connection $\Gamma_{[\alpha\beta]}$). One can call this a holonomic gauge since effectively the holonomic components of the metric g_{ij} and the metric-compatible connection $\Gamma_i{}^j$ describe the gravitational field configurations now.

The possibility of "gauging away" either the metric or the coframe (6.6), (6.7), clearly suggests that exactly 16 of the 50 field equations (6.3)-(6.5) are redundant. Indeed, let us rewrite the first field equation (6.4) in the equivalent form,

$$\vartheta_\alpha \wedge DH_\beta - \vartheta_\alpha \wedge E_\beta = \vartheta_\alpha \wedge \Sigma_\beta\,. \tag{6.8}$$

Substituting the second Noether identity (5.17), one can transform the left-hand side to

$$\begin{aligned}\vartheta_\alpha \wedge DH_\beta - T_\alpha \wedge H_\beta + R_\alpha{}^\gamma \wedge H_{\gamma\beta} + R_\beta{}^\gamma \wedge H_{\alpha\gamma} - m_{\alpha\beta}\\ = -D(\vartheta_\alpha \wedge H_\beta) - DDH_{\alpha\beta} - m_{\alpha\beta}.\end{aligned} \tag{6.9}$$

Thus the first equation is equivalent to

$$-D(\vartheta_\alpha \wedge H_\beta + DH_{\alpha\beta}) - m_{\alpha\beta} = \vartheta_\alpha \wedge \Sigma_\beta\,. \tag{6.10}$$

We can decompose this equation into the symmetric and antisymmetric parts. Then we immediately see that the *antisymmetric part is identically vanishing* due to the Noether identity (4.15) and the second field equation (6.5). On the other hand, the *symmetric part is identically reproducing* the zeroth equation (6.3), because the metric energy-momentum is equal to the symmetric part of the canonical energy-momentum, (4.20). Since the matter field equation (6.2) is a prerequisite for the validity of the differential Noether identity, we obtain the important result that one of the first two gravitational field equations is "weakly" *redundant*, and the number of truly independent field equations is indeed $50 - 16 = 34$.

Not unexpectedly, the second field equation does *not* follow from the other field equations. While working in the constant-metric-gauge (6.6), it is convenient to solve the coupled system of the second equation and the symmetric part of the first field equation. Analogously, in the constant-frame gauge (6.7), it may be more natural to consider the equivalent set of the coupled system of the zeroth and the second field equations.

VII. LIMITING CASE: SPINLESS MATTER

The Noether identities (4.11) and (4.15) contain a plenty of information about the interaction of the spin of the classical matter with the post-Riemannian geometry of a spacetime. They also allow for a transparent limit when the spin vanishes $\tau_{\alpha\beta} = 0$. Let us study this in some detail.

Recall that the metric-compatible connection can be decomposed into the Riemannian and post-Riemannian parts as

$$\Gamma_\alpha{}^\beta = \widetilde{\Gamma}_\alpha{}^\beta - K_\alpha{}^\beta. \tag{7.1}$$

Here the tilde denotes the purely Riemannian connection and $K_\alpha{}^\beta$ is the contortion which is related to the torsion via the identity

$$T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta. \tag{7.2}$$

Accordingly, the curvature is decomposed as

$$R_\beta{}^\gamma = \widetilde{R}_\beta{}^\gamma - \widetilde{D}K_\beta{}^\gamma - K_\beta{}^\lambda \wedge K_\lambda{}^\gamma. \tag{7.3}$$

Hence, the last term in the 1st Noether identity (4.11) reads

$$\begin{aligned}(e_\alpha \rfloor R_\beta{}^\gamma) \wedge \tau^\beta{}_\gamma = &\ (e_\alpha \rfloor \widetilde{R}_{\beta\gamma}) \wedge \tau^{\beta\gamma} \\ &- (e_\alpha \rfloor \widetilde{D}K_{\beta\gamma}) \wedge \tau^{\beta\gamma} - e_\alpha \rfloor (K_\beta{}^\lambda \wedge K_{\lambda\gamma}) \wedge \tau^{\beta\gamma}.\end{aligned} \tag{7.4}$$

Using (7.2), we find

$$e_\alpha \rfloor T^\beta = (e_\alpha \rfloor K^\beta{}_\gamma)\vartheta^\gamma + K_\alpha{}^\beta, \tag{7.5}$$

and hence

$$\begin{aligned}(e_\alpha \rfloor T^\beta) \wedge \Sigma_\beta &= (e_\alpha \rfloor K^{\beta\gamma})\vartheta_{[\gamma} \wedge \Sigma_{\beta]} + K_\alpha{}^\beta \wedge \Sigma_\beta \\ &= K_\alpha{}^\beta \wedge \Sigma_\beta + (e_\alpha \rfloor K^{\beta\gamma}) \wedge D\tau_{\beta\gamma} \\ &= K_\alpha{}^\beta \wedge \Sigma_\beta + (e_\alpha \rfloor K^{\beta\gamma}) \wedge \widetilde{D}\tau_{\beta\gamma} + e_\alpha \rfloor (K_\beta{}^\lambda \wedge K_{\lambda\gamma}) \wedge \tau^{\beta\gamma},\end{aligned} \tag{7.6}$$

where we used the second Noether identity (4.15), and then decomposed the covariant derivative into the Riemannian and post-Riemannian parts according to (7.1). Finally, again with the help of (7.1), one finds

$$D\Sigma_\alpha = \widetilde{D}\Sigma_\alpha + K_\alpha{}^\beta \wedge \Sigma_\beta. \tag{7.7}$$

Substituting (7.4), (7.6), (7.7) into the 1st Noether identity, we finally obtain the conservation law

$$\widetilde{D}\Sigma_\alpha = (e_\alpha \rfloor K_{\beta\gamma}) \wedge \widetilde{D}\tau^{\beta\gamma} - (e_\alpha \rfloor \widetilde{D}K_{\beta\gamma}) \wedge \tau^{\beta\gamma} + (e_\alpha \rfloor \widetilde{R}_{\beta\gamma}) \wedge \tau^{\beta\gamma}. \tag{7.8}$$

Making use of the definition of the covariant Lie derivative, $\widetilde{\text{Ł}}_{e_\alpha} = e_\alpha \rfloor \widetilde{D} + \widetilde{D} e_\alpha \rfloor$, the last equation can be rewritten in the equivalent form,

$$\widetilde{D}\left(\Sigma_\alpha - \tau^{\beta\gamma} e_\alpha \rfloor K_{\beta\gamma}\right) - \tau^{\beta\gamma} \wedge \widetilde{\text{Ł}}_{e_\alpha} K_{\beta\gamma} = (e_\alpha \rfloor \widetilde{R}_{\beta\gamma}) \wedge \tau^{\beta\gamma}. \tag{7.9}$$

The "decomposed" form of the first Noether identity, (7.9), shows that the post-Riemannian geometrical variables (contortion) are coupled directly to the spin of matter.

In particular, when the matter is spinless, $\tau^{\beta\gamma} = 0$, we are left with the purely Riemannian conservation law (although the geometry is still non-Riemannian!):

$$\widetilde{D}\Sigma_\alpha = \widetilde{D}\sigma_\alpha = 0. \tag{7.10}$$

The first equality arises from (4.24) which shows that the metric and canonical energy-momenta are coinciding for spinless case.

An important physical conclusion is thus that test particles without spin are always moving along the *Riemannian geodesics*, in complete agreement with the equivalence principle. To put it differently, the spacetime torsion can only be detected with the help of test matter with spin.

On the other hand, it is worthwhile to note that in the absence of torsion, the equation (7.9) displays the standard Mathisson-Papapetrou force of GR which gives rise to a non-geodesic motion of a test particle with spin.

VIII. GENERAL QUADRATIC MODELS

The general Lagrangian which is at most quadratic in the Poincaré gauge field strengths – in the torsion and the curvature – reads

$$\begin{aligned} V_{\rm Q} = \frac{1}{2\kappa} & \left[a_0\, R^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda\,\eta - T^\alpha \wedge {}^*\!\left(\sum_{I=1}^{3} a_I\, {}^{(I)}T_\alpha\right)\right] \\ & - \frac{1}{2}\, R^{\alpha\beta} \wedge {}^*\!\left(\sum_{I=1}^{6} b_I\, {}^{(I)}R_{\alpha\beta}\right). \end{aligned} \tag{8.1}$$

We use the unit system in which the dimension of the gravitational constant is $[\kappa] = \ell^2$ with the unit length ℓ. The coupling constants a_0, a_1, a_2, a_3 and $b_1, ..., b_6$ are *dimensionless*, whereas $[\lambda] = \ell^{-2}$. These coupling constants determine the particle contents of the qudratic Poincaré gauge models, the corresponding analysis can be found in [26,38,46,47,40].

The Lagrangian (8.1) has the general structure similar to that of the Yang–Mills Lagrangian for the gauge theory of internal symmetry group.

In order to be able to compare (8.1) to the Lagrangians studied in the literature, let us rewrite $V_{\rm Q}$ using the tensor language:

$$V_{\rm Q} = -\,\frac{1}{2}\,\eta\left[\frac{1}{\kappa}\,(a_0 R + 2\lambda + \alpha_1\,T_{\mu\nu}{}^{\alpha}\,T^{\mu\nu}{}_{\alpha} + \alpha_2\,T_{\mu\alpha}{}^{\mu}\,T^{\nu\alpha}{}_{\nu} + \alpha_3\,T_{\mu\nu}{}^{\alpha}\,T_{\alpha}{}^{\mu\nu})\right.$$
$$+\beta_1\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \beta_2\,R_{\mu\nu\alpha\beta}R^{\mu\alpha\nu\beta} + \beta_3\,R_{\mu\nu\alpha\beta}R^{\alpha\beta\mu\nu}$$
$$\left.+\beta_4\,{\rm Ric}_{\mu\nu}{\rm Ric}^{\mu\nu} + \beta_5\,{\rm Ric}_{\mu\nu}{\rm Ric}^{\nu\mu} + \beta_6\,R^2\right]. \tag{8.2}$$

Using the definitions of the irreducible torsion and curvature parts (see Appendix 1), we find the relation between the coupling constants:

$$a_1 = 2\alpha_1 - \alpha_3, \quad a_2 = 2\alpha_1 + 3\alpha_2 - \alpha_3, \quad a_3 = 2\alpha_1 + 2\alpha_3, \tag{8.3}$$

$$b_1 = 2\beta_1 + 2\beta_3, \tag{8.4}$$

$$b_2 = 2\beta_1 + \beta_2 - 2\beta_3, \tag{8.5}$$

$$b_3 = 2\beta_1 - 2\beta_2 + 2\beta_3, \tag{8.6}$$

$$b_4 = 2\beta_1 + \beta_2 + 2\beta_3 + \beta_4 + \beta_5, \tag{8.7}$$

$$b_5 = 2\beta_1 - 2\beta_3 + \beta_4 - \beta_5, \tag{8.8}$$

$$b_6 = 2\beta_1 + \beta_2 + 2\beta_3 + 3\beta_4 + 3\beta_5 + 12\beta_6. \tag{8.9}$$

The inverse of (8.3) reads

$$\alpha_1 = \frac{2a_1 + a_3}{6}, \quad \alpha_2 = \frac{a_2 - a_1}{3}, \quad \alpha_3 = \frac{a_3 - a_1}{3}. \tag{8.10}$$

The Poincaré gauge field equations are derived from the total Lagrangian $V_{\rm Q} + L_{\rm mat}$ within the general framework described in Sec. VI. The resulting system of the *first* (6.4) and *second* (6.5) field equations reads

$$DH_\alpha - E_\alpha = \Sigma_\alpha\,, \tag{8.11}$$

$$DH^\alpha{}_\beta - E^\alpha{}_\beta = \tau^\alpha{}_\beta\,. \tag{8.12}$$

The right hand sides describe the material sources of the Poincaré gauge gravity: the canonical energy–momentum (3.10) and the spin (3.11) three–forms.

The explicit form for the gauge field momenta (5.2) which enter the left hand sides of (8.11)–(8.12) is given by

$$H_\alpha := -\frac{\partial V_{\rm Q}}{\partial T^\alpha} = \frac{1}{\kappa}\,{}^*\!\left(\sum_{I=1}^{3} a_I\,{}^{(I)}T_\alpha\right), \tag{8.13}$$

$$H^\alpha{}_\beta := -\frac{\partial V_{\rm Q}}{\partial R_\alpha{}^\beta} = -\frac{a_0}{2\kappa}\,\eta^\alpha{}_\beta + {}^*\!\left(\sum_{I=1}^{6} b_I\,{}^{(I)}R^\alpha{}_\beta\right). \tag{8.14}$$

The three-forms E_α and $E^\alpha{}_\beta$ describe, respectively, the *canonical energy-momentum* and *spin* densities determined by the Poincaré gauge gravitational field via (5.4) and (5.5).

IX. EINSTEIN'S THEORY – A DEGENERATE CASE OF QUADRATIC POINCARÉ GRAVITY

In Einstein's general relativity theory, the metric is the only fundamental field variable. The linear connection is constructed from the first derivatives of the metric components, it is a unique metric compatible and torsion-free connection which we denote hereafter $\widetilde{\Gamma}_\alpha{}^\beta$. In the local Lorentz invariant formulation, the metric is effectively replaced by the (orthonormal) coframe, $\vartheta^\alpha = e_i^\alpha\,dx^i$, which is a kind of a "square root" of the spacetime metric: $g_{ij} = e_i^\alpha\,e_j^\beta\,o_{\alpha\beta}$. In terms of the metric/coframe components, the purely Riemannian connection reads

$$\widetilde{\Gamma}_{\alpha\beta} = e_{[\alpha}\rfloor C_{\beta]} - \frac{1}{2}\,(e_\alpha\rfloor e_\beta\rfloor C_\gamma)\,\vartheta^\gamma, \tag{9.1}$$

where the anholonomity two–form is defined as usual,

$$C^\alpha := d\vartheta^\alpha. \tag{9.2}$$

It is straightforward to check that the *Christoffel symbol* (9.1) indeed has zero torsion and nonmetricity:

$$\widetilde{D}\vartheta^\alpha \equiv 0, \qquad \widetilde{D}g_{\alpha\beta} \equiv 0. \tag{9.3}$$

Hereafter the differential operators defined by the Riemannian connection and geometrical objects constructed from (9.1) (e.g., the exterior covariant differential $\widetilde{D}$ and the curvature

$\widetilde{R}_\alpha{}^\beta$ two–form) will be denoted by the tilde. In view of (9.3), the Riemannian covariant derivatives of the dual η-forms are also vanishing,

$$\widetilde{D}\eta_{\mu_1\dots\mu_p} \equiv 0, \qquad p = 0, 1, ..., 4. \tag{9.4}$$

In this section we will demonstrate that the standard general relativity (Einstein's theory of gravity) is the special case of the Poincaré gauge gravity. It is described by the particular quadratic model (8.1) when the coupling constants are chosen as follows:

$$a_0 = 1, \quad a_1 = -1, \quad a_2 = 2, \quad a_3 = \frac{1}{2}, \qquad b_I = 0. \tag{9.5}$$

This choice is *degenerate* in the sense we explain below. Let us write down explicitly the Lagrangian of this model,

$$V^{(0)} = \frac{1}{2\kappa}\left(R^{\mu\nu}\wedge\eta_{\mu\nu} + {}^{(1)}T^\alpha\wedge{}^{*(1)}T_\alpha - 2{}^{(2)}T^\alpha\wedge{}^{*(2)}T_\alpha - \frac{1}{2}{}^{(3)}T^\alpha\wedge{}^{*(3)}T_\alpha\right). \tag{9.6}$$

In accordance with the general scheme we find the gauge momenta for (9.6):

$$H^{(0)}_\alpha = -\frac{\partial V^{(0)}}{\partial T^\alpha} = \frac{1}{\kappa}{}^*\left(-{}^{(1)}T_\alpha + 2\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha\right), \tag{9.7}$$

$$H^{(0)\alpha}{}_\beta = -\frac{\partial V^{(0)}}{\partial R_\alpha{}^\beta} = -\frac{1}{2\kappa}\,\eta^\alpha{}_\beta. \tag{9.8}$$

The degeneracy of the model under consideration is manifested in the fact that the left hand side of the *second field equation is identically zero*:

$$DH^{(0)\alpha}{}_\beta - E^{(0)\alpha}{}_\beta \equiv 0. \tag{9.9}$$

Here, see (5.5), $E_{\alpha\beta} = -\vartheta_{[\alpha}\wedge H^{(0)}_{\beta]}$. The proof relies on two geometrical identities, (17.1) and (17.2), which we prove in Appendix 3 at the end of the paper. In particular, of fundamental importance is to observe that

$$H^{(0)}_\alpha \equiv \frac{1}{2\kappa}\,K^{\mu\nu}\wedge\eta_{\alpha\mu\nu} \tag{9.10}$$

which follows from the identity (17.1).

Let us now turn to the *first* field equation. With the help of (9.7), one can rewrite the Lagrangian (9.6) as

$$V^{(0)} = \frac{1}{2\kappa}\, R^{\mu\nu} \wedge \eta_{\mu\nu} - \frac{1}{2}\, T^{\beta} \wedge H^{(0)}_{\beta}. \tag{9.11}$$

Hence, see (5.10), the gravitational energy–momentum three-form reads

$$\begin{aligned} E^{(0)}_{\alpha} &= e_{\alpha}\rfloor V^{(0)} + (e_{\alpha}\rfloor T^{\beta}) \wedge H^{(0)}_{\beta} + (e_{\alpha}\rfloor R_{\beta}{}^{\gamma}) \wedge H^{(0)\beta}{}_{\gamma} \\ &= \frac{1}{2\kappa}\, R^{\mu\nu} \wedge \eta_{\alpha\mu\nu} + \frac{1}{2}\,(e_{\alpha}\rfloor T^{\beta}) \wedge H^{(0)}_{\beta} - \frac{1}{2}\, T^{\beta} \wedge (e_{\alpha}\rfloor H^{(0)}_{\beta}), \end{aligned} \tag{9.12}$$

where we substituted (9.8).

Some efforts are required to calculate the term

$$DH^{(0)}_{\alpha} = \widetilde{D}H^{(0)}_{\alpha} - K_{\alpha}{}^{\beta} \wedge H^{(0)}_{\beta}. \tag{9.13}$$

At first, directly from the identity (9.10) we find:

$$T^{\beta} \wedge (e_{\alpha}\rfloor H^{(0)}_{\beta}) = \frac{1}{2\kappa}\left((e_{\alpha}\rfloor K^{\mu\nu})\, T^{\beta} \wedge \eta_{\beta\mu\nu} + T^{\beta} \wedge K^{\mu\nu}\, \eta_{\alpha\beta\mu\nu}\right). \tag{9.14}$$

Recalling that $T^{\beta} \wedge \eta_{\beta\mu\nu} = D\eta_{\mu\nu}$ and using the fundamental identity (17.2), we get

$$\begin{aligned} \frac{1}{2\kappa}\,(e_{\alpha}\rfloor K^{\mu\nu})\, T^{\beta} \wedge \eta_{\beta\mu\nu} &= \frac{1}{2\kappa}\,(e_{\alpha}\rfloor K^{\mu\nu})\, \vartheta_{\mu} \wedge K^{\rho\sigma} \wedge \eta_{\nu\rho\sigma} \\ &= (e_{\alpha}\rfloor T^{\beta}) \wedge H^{(0)}_{\beta} + K_{\alpha}{}^{\beta} \wedge H^{(0)}_{\beta}, \end{aligned} \tag{9.15}$$

where we used the Leibniz rule for the interior product, $(e_{\alpha}\rfloor K^{\mu\nu})\, \vartheta_{\mu} = e_{\alpha}\rfloor(K^{\mu\nu} \wedge \vartheta_{\mu}) + K_{\alpha}{}^{\nu}$, and identities (7.2) and (9.10). Hence, (9.14) and (9.15) yield

$$\frac{1}{2\kappa}\, T^{\beta} \wedge K^{\mu\nu}\, \eta_{\alpha\beta\mu\nu} \equiv -K_{\alpha}{}^{\beta} \wedge H^{(0)}_{\beta} + T^{\beta} \wedge (e_{\alpha}\rfloor H^{(0)}_{\beta}) - (e_{\alpha}\rfloor T^{\beta}) \wedge H^{(0)}_{\beta}. \tag{9.16}$$

On the other hand,

$$T^{\beta}\, \eta_{\alpha\beta\mu\nu} = D\eta_{\alpha\mu\nu} = -K_{\alpha}{}^{\beta} \wedge \eta_{\beta\mu\nu} - K_{\mu}{}^{\beta} \wedge \eta_{\alpha\beta\nu} - K_{\nu}{}^{\beta} \wedge \eta_{\alpha\mu\beta}. \tag{9.17}$$

Exterior product of this with $\frac{1}{2\kappa}\, K^{\mu\nu}$ yields, taking into account (9.10),

$$\frac{1}{2\kappa}\, T^{\beta} \wedge K^{\mu\nu}\, \eta_{\alpha\beta\mu\nu} \equiv K_{\alpha}{}^{\beta} \wedge H^{(0)}_{\beta} - \frac{1}{\kappa}\, K^{\mu\nu} \wedge K_{\mu}{}^{\beta} \wedge \eta_{\alpha\beta\nu}. \tag{9.18}$$

Comparing (9.16) and (9.18), we obtain:

$$\begin{aligned} K_\alpha{}^\beta \wedge H^{(0)}_\beta \equiv\, & \frac{1}{2\kappa}\, K^{\mu\nu} \wedge K_\mu{}^\beta \wedge \eta_{\alpha\beta\nu} \\ & + \frac{1}{2}\, T^\beta \wedge (e_\alpha \rfloor H^{(0)}_\beta) - \frac{1}{2}\, (e_\alpha \rfloor T^\beta) \wedge H^{(0)}_\beta . \end{aligned} \tag{9.19}$$

Substituting (9.10) and (9.19) into (9.13), we finally get

$$\begin{aligned} DH^{(0)}_\alpha \equiv\, & \frac{1}{2\kappa}\left(\widetilde{D}K^{\mu\nu} - K_\gamma{}^\mu \wedge K^{\nu\gamma}\right) \wedge \eta_{\alpha\mu\nu} \\ & - \frac{1}{2}\, T^\beta \wedge (e_\alpha \rfloor H^{(0)}_\beta) + \frac{1}{2}\, (e_\alpha \rfloor T^\beta) \wedge H^{(0)}_\beta . \end{aligned} \tag{9.20}$$

We are now in a position to compute the left hand side of the *first* field equation. Collecting together (9.20), (9.12), and using the decomposition of the curvature (7.3), one obtains

$$DH^{(0)}_\alpha - E^{(0)}_\alpha \equiv -\frac{1}{2\kappa}\, \widetilde{R}^{\mu\nu} \wedge \eta_{\alpha\mu\nu}. \tag{9.21}$$

For the Lagrangian (9.6), the complete system of the *first* and *second* gravitational field equations thus reads:

$$-\frac{1}{2\kappa}\, \widetilde{R}^{\mu\nu} \wedge \eta_{\alpha\mu\nu} = \Sigma_\alpha, \tag{9.22}$$

$$0 = \tau^\alpha{}_\beta. \tag{9.23}$$

This is the true Einstein's theory in which only the matter with the vanishing spin $\tau^\alpha{}_\beta = 0$ is allowed, and the energy-momentum of matter Σ_α determines the *purely Riemannian geometry* via the Einstein's equations (the components of the three-form on the left hand side of (9.22) comprise the standard Einstein tensor).

The degenerate features of the model (9.6) are explained by the existence of the *auxiliary* (or occasional) symmetry of the Lagrangian. The symmetry in question arises from the transformation of the linear connection *alone*:

$$\delta_\varepsilon \Gamma_\beta{}^\alpha = \varepsilon_\beta{}^\alpha, \qquad \delta_\varepsilon \vartheta^\alpha = 0. \tag{9.24}$$

Here $\varepsilon_\beta{}^\alpha$ is an arbitrary tensor-valued one form, antisymmetric in its indices $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$. Taking into account that $\delta_\varepsilon R_\beta{}^\alpha = D\varepsilon_\beta{}^\alpha$ and $\delta_\varepsilon T^\alpha = \varepsilon_\beta{}^\alpha \wedge \vartheta^\beta$, we can straightforwardly calculate the variation of the Lagrangian (9.6):

$$\delta_\varepsilon V^{(0)} = \frac{1}{2\kappa}\,\delta_\varepsilon R_\beta{}^\alpha\wedge\eta^\beta{}_\alpha - \delta_\varepsilon T^\alpha\wedge H^{(0)}_\alpha = \frac{1}{2\kappa}\left((D\varepsilon_\beta{}^\alpha)\wedge\eta^\beta{}_\alpha - \varepsilon_\beta{}^\alpha\wedge D\eta^\beta{}_\alpha\right)$$
$$= \frac{1}{2\kappa}\,d\left(\varepsilon_\beta{}^\alpha\wedge\eta^\beta{}_\alpha\right). \tag{9.25}$$

The identities (9.10) and (17.2) were used above. Thus the *action* is unchanged $\delta_\varepsilon(\int V^{(0)}) = 0$, and we have demonstrated that the model (9.6) is *invariant* under the transformation (9.24). This "auxiliary" symmetry has nothing to do with the linear or Poincaré gauge group underlying the gravity theory. The number of free parameters involved is 24 ($\varepsilon^{\alpha\beta}$ is a one-form with skew symmetry). This is exactly equal to the number of components of the torsion (or contortion). In principle, it is possible to use the symmetry (9.24) and "gauge away" the torsion completely, transforming from the total connection to the *purely Riemannian* one:

$$\Gamma_\beta{}^\alpha,\quad K_\beta{}^\alpha\neq 0\quad\longrightarrow\quad \Gamma_\beta{}^\alpha=\widetilde{\Gamma}_\beta{}^\alpha,\quad K_\beta{}^\alpha=0. \tag{9.26}$$

The problem of the "auxiliary" symmetry in the teleparallel and the Poincaré gauge gravity was discussed in [16,25,37,27,28,57].

X. DOUBLE DUALITY PROPERTIES OF THE IRREDUCIBLE PARTS OF THE RIEMANN-CARTAN CURVATURE

In this section we will demonstrate that the irreducible parts of the Riemann-Cartan curvature are all characterized by the *double-duality* property

$$^{\star(I)}R_{\alpha\beta} = K_I\,\frac{1}{2}\,\eta_{\alpha\beta\mu\nu}\,{}^{(I)}R^{\mu\nu},\qquad I=1,\dots,6, \tag{10.1}$$

where the double duality index K_I is either $+1$ or -1. We will prove (10.1) and compute K_I for each irreducible curvature part.

The terminology is explained as follows. In addition to the Hodge ("left") duality operator, for the Lorentz algebra-valued objects $\psi_{\alpha\beta}=-\psi_{\beta\alpha}$ we can define the "right" duality operator by

$$\psi^{\star}_{\alpha\beta} := \frac{1}{2}\,\eta_{\alpha\beta\mu\nu}\,\psi^{\mu\nu}. \tag{10.2}$$

Then, from (10.1) we find for the irreducible curvature parts

$${}^{\star(I)}R^{\star}_{\alpha\beta} = -K_I\,{}^{(I)}R^{\mu\nu}, \qquad I = 1,\dots,6. \tag{10.3}$$

For $K_I = -1$ we have the *double dual* objects, whereas for $K_I = 1$ we find *anti-double dual* objects.

A. Mathematical preliminaries

In our proof of (10.1) we will use the possibility of generating of the 2-forms with the double duality properties from the different irreducible pieces of the vector-valued 1-forms. This fact can be formulated in terms of the following three lemmas.

Lemma A: Let A_α be a vector-valued 1-form such that

$$A_\alpha \wedge \eta_\beta = -A_\beta \wedge \eta_\alpha. \tag{10.4}$$

Then this 1-form satisfies the identity

$$\frac{1}{2}\eta^{\alpha\beta\mu\nu}\vartheta_\alpha \wedge A_\beta - {}^\star\Big(\vartheta^{[\mu} \wedge A^{\nu]}\Big) \equiv 0. \tag{10.5}$$

Lemma B: Let B_α be a vector-valued 1-form such that

$$B_\alpha \wedge \eta_\beta = B_\beta \wedge \eta_\alpha, \qquad B_\alpha \wedge \eta^\alpha = 0. \tag{10.6}$$

Then this 1-form satisfies the identity

$$\frac{1}{2}\eta^{\alpha\beta\mu\nu}\vartheta_\alpha \wedge B_\beta + {}^\star\Big(\vartheta^{[\mu} \wedge B^{\nu]}\Big) \equiv 0. \tag{10.7}$$

The proofs of these lemmas are given in the Appendix 2. The identities (10.5) and (10.7) mean that any 1-form A_α (resp., any 1-form B_α) satisfying the conditions (10.4) (resp., (10.6)) defines an *anti-double-dual* 2-form $\vartheta_{[\alpha} \wedge A_{\beta]}$ (resp., a *double-dual* 2-form $\vartheta_{[\alpha} \wedge B_{\beta]}$).

Now we will show that the different irreducible pieces of an arbitrary vector-valued 1-form generate the double dual 2-forms. At first, we recall that any 1-form ρ_α is decomposed into the sum

$$\rho_\alpha = {}^{(1)}\rho_\alpha + {}^{(2)}\rho_\alpha + {}^{(3)}\rho_\alpha, \tag{10.8}$$

where the irreducible parts are defined by

$$\begin{align}
{}^{(1)}\rho_\alpha &:= \rho_\alpha - \frac{1}{4}\,\vartheta_\alpha\,(e^\beta\rfloor\rho_\beta) - \frac{1}{2}\,e_\alpha\rfloor(\vartheta^\beta\wedge\rho_\beta), \tag{10.9}\\
{}^{(2)}\rho_\alpha &:= \frac{1}{2}\,e_\alpha\rfloor(\vartheta^\beta\wedge X_\beta), \tag{10.10}\\
{}^{(3)}\rho_\alpha &:= \frac{1}{4}\,\vartheta_\alpha\,(e^\beta\rfloor\rho_\beta). \tag{10.11}
\end{align}$$

In tensor language, ρ_α describes the second rank tensor, and the decomposition (10.8) splits this tensor into the traceless symmetric piece (10.9), antisymmetric piece (10.10), and the trace (10.11).

Lemma C: For an arbitrary 1-form ρ_α, the 2-forms $\vartheta_{[\alpha}\wedge{}^{(2)}\rho_{\beta]}$ and $\vartheta_{[\alpha}\wedge{}^{(3)}\rho_{\beta]}$ are anti-double-dual, whereas the 2-form $\vartheta_{[\alpha}\wedge{}^{(1)}\rho_{\beta]}$ is double dual.

Using the definition (10.9), one immediately proves:

$$\begin{align}
{}^{(1)}\rho_\alpha\wedge\eta_\beta &= \rho_\alpha\wedge\eta_\beta - \frac{1}{4}\vartheta_\alpha\wedge\eta_\beta\,(e^\gamma\rfloor\rho_\gamma) - \frac{1}{2}(\rho_\alpha - \vartheta^\lambda e_\alpha\rfloor\rho_\lambda)\wedge\eta_\beta\\
&= \frac{1}{2}\,\rho_\alpha\wedge\eta_\beta - \frac{1}{4}g_{\alpha\beta}\,\eta\,(e^\gamma\rfloor\rho_\gamma) + \frac{1}{2}\eta\,e_\alpha\rfloor\rho_\beta\\
&= \frac{1}{2}(\rho_\alpha\wedge\eta_\beta + \rho_\beta\wedge\eta_\alpha) - \frac{1}{4}g_{\alpha\beta}\,\eta\,(e^\gamma\rfloor\rho_\gamma). \tag{10.12}
\end{align}$$

Here we repeatedly used the basic identity $\vartheta_\alpha\wedge\eta_\beta = g_{\alpha\beta}\,\eta$. Hence, ${}^{(1)}\rho_\alpha$ satisfies the conditions (10.6):

$${}^{(1)}\rho_\alpha\wedge\eta_\beta = {}^{(1)}\rho_\beta\wedge\eta_\alpha, \qquad {}^{(1)}\rho_\alpha\wedge\eta^\alpha = 0. \tag{10.13}$$

Accordingly, by the Lemma B, the 2-form $\vartheta_{[\alpha}\wedge{}^{(1)}\rho_{\beta]}$ is double dual.

As to the second irreducible part (10.10), we find

$${}^{(2)}\rho_\alpha\wedge\eta_\beta = \frac{1}{2}\,(\rho_\alpha\wedge\eta_\beta - \eta\,e_\alpha\rfloor\rho_\beta) = \frac{1}{2}\,(\rho_\alpha\wedge\eta_\beta - \rho_\beta\wedge\eta_\alpha)\,, \tag{10.14}$$

and thus the condition (10.4) of the Lemma A is explicitly fulfilled for ${}^{(2)}\rho_\alpha$.

The third irreducible part (10.11) is proportional to the coframe 1-form ϑ^α. The latter does not satisfy either (10.4) or (10.6). However, the coframe is directly involved in the construction of Hodge duals, and, by definition,

$$ {}^\star(\vartheta_\alpha \wedge \vartheta_\beta) = \eta_{\alpha\beta} = \frac{1}{2}\eta_{\mu\nu\alpha\beta}\vartheta^\mu \wedge \vartheta^\nu. \tag{10.15} $$

Thus, the 2–form $\vartheta_\alpha \wedge \vartheta_\beta$ is *anti-double dual*, and the same is true for $\vartheta_{[\alpha} \wedge {}^{(3)}\rho_{\beta]} = \frac{1}{4}(e^\gamma\rfloor\rho_\gamma)\,\vartheta_\alpha \wedge \vartheta_\beta$. As a by-product of our analysis, we note that

$$ {}^\star\eta_{\alpha\beta} = {}^{\star\star}(\vartheta_\alpha \wedge \vartheta_\beta) = -\vartheta_\alpha \wedge \vartheta_\beta = \frac{1}{2}\eta_{\mu\nu\alpha\beta}\eta^{\mu\nu}, \tag{10.16} $$

hence $\eta_{\alpha\beta}$ is *anti-double dual* too.

B. Dual properties of the curvature

Now we are in a position to prove the double duality properties (10.1). As a first step, we notice that the definitions of the irreducible parts of the Riemann-Cartan curvature (15.4)-(15.9) involve the pair of the vector-valued 1-forms W_α and X_α defined in (15.10). As a result, we can straightforwardly apply the lemmas A-C. Since $\Psi_\alpha = {}^{(1)}X_\alpha$ and $\Phi_\alpha = {}^{(1)}W_\alpha$, see eqs. (15.11) and (15.12), we can immediately apply the Lemma B to the 2-nd and the 4-th irreducible parts of the Riemann-Cartan curvature to demonstrate that these two pieces are both double dual. All the rest irreducible parts are anti-double dual. Indeed, for the 5-th part this is proved via the lemma A, using the fact that it involves ${}^{(2)}W_\alpha$, whereas for the 3-rd and the 6-th pieces this is obvious from (10.15) and (10.16). The proof for the Weyl type 1-st curvature piece is somewhat more nontrivial and will be given below. The complete list of the double duality properties for the Riemann-Cartan curvature reads:

$$ {}^{\star(1)}R_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\,{}^{(1)}R^{\mu\nu}, \tag{10.17} $$

$$ {}^{\star(2)}R_{\alpha\beta} = -\frac{1}{2}\eta_{\alpha\beta\mu\nu}\,{}^{(2)}R^{\mu\nu}, \tag{10.18} $$

$$ {}^{\star(3)}R_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\,{}^{(3)}R^{\mu\nu}, \tag{10.19} $$

$$
{}^{\star(4)}R_{\alpha\beta} = -\frac{1}{2}\eta_{\alpha\beta\mu\nu}\,{}^{(4)}R^{\mu\nu}, \tag{10.20}
$$

$$
{}^{\star(5)}R_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\,{}^{(5)}R^{\mu\nu}, \tag{10.21}
$$

$$
{}^{\star(6)}R_{\alpha\beta} = \frac{1}{2}\eta_{\alpha\beta\mu\nu}\,{}^{(6)}R^{\mu\nu}. \tag{10.22}
$$

As a comment to the proof of (10.17), we first notice that the definition (15.9) yields

$$
\begin{aligned}
\eta^{\mu\nu}\wedge{}^{(1)}R_{\alpha\beta} = -\frac{1}{6}\eta^{\mu\nu}{}_{\alpha\beta}\,\eta\,X - \frac{1}{3}\delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta}\,\eta\,W + 2\,\eta^{[\mu}{}_{[\alpha}\wedge W^{\nu]}{}_{\beta]}\\
-\delta^{[\mu}_{[\alpha}\left(\eta_{\beta]}\wedge W^{\nu]} + \eta^{\nu]}\wedge W_{\beta]}\right).
\end{aligned} \tag{10.23}
$$

From this we find $\eta^{\alpha\gamma}\wedge{}^{(1)}R_{\beta\gamma} = 0$ (use the obvious identities $\eta_\alpha\wedge W^\beta = \eta_{\alpha\gamma}\wedge W^{\beta\gamma}$ and $\eta_\alpha\wedge W^\alpha = -W\,\eta$). Since $(e_\mu\rfloor e_\nu\rfloor W_{\alpha\beta})\,\eta = -\eta_{\mu\nu}\wedge W_{\alpha\beta}$ (use the Leibniz rule twice for the interior product), the above two equations imply

$$
e_\mu\rfloor e_\nu\rfloor{}^{(1)}R_{\alpha\beta} = e_\alpha\rfloor e_\beta\rfloor{}^{(1)}R_{\mu\nu}, \tag{10.24}
$$

and

$$
e_\mu\rfloor e_\nu\rfloor{}^{(1)}R^{\alpha\nu} = 0. \tag{10.25}
$$

Now, using the evident identity

$$
{}^{(1)}R^{\alpha\beta} = \frac{1}{2}\vartheta^\nu\wedge\vartheta^\mu\,(e_\mu\rfloor e_\nu\rfloor{}^{(1)}R^{\alpha\beta}), \tag{10.26}
$$

one can straightforwardly compute

$$
\begin{aligned}
\frac{1}{2}\eta_{\mu\nu\alpha\beta}\,{}^{\star(1)}R^{\alpha\beta} &= \frac{1}{8}\eta_{\mu\nu\alpha\beta}\eta^{\rho\sigma\delta\gamma}\,\vartheta_\gamma\wedge\vartheta_\delta\,(e_\rho\rfloor e_\sigma\rfloor{}^{(1)}R^{\alpha\beta})\\
&= \frac{1}{2}\vartheta_\gamma\wedge\vartheta_\delta\,(e_\mu\rfloor e_\nu\rfloor{}^{(1)}R^{\gamma\delta})\\
&= \frac{1}{2}\vartheta_\gamma\wedge\vartheta_\delta\,(e^\gamma\rfloor e^\delta\rfloor{}^{(1)}R_{\mu\nu}) = -{}^{(1)}R_{\mu\nu},
\end{aligned} \tag{10.27}
$$

where one have to expand the product $\eta_{\mu\nu\alpha\beta}\,\eta^{\rho\sigma\delta\gamma}$ in terms of four Kronecker deltas, then use (10.25) to arrive at the second line, and subsequently use (10.24) to get the final result (10.17).

A *simple corollary* of the double duality properties (10.17)-(10.22) is the identity valid for any $I = 1, ..., 6$:

$$(e_\alpha\rfloor{}^{(I)}R^{\mu\nu})\wedge{}^{*(I)}R_{\mu\nu}\equiv\frac{1}{2}\,e_\alpha\rfloor\left({}^{(I)}R^{\mu\nu}\wedge{}^{*(I)}W_{\mu\nu}\right), \tag{10.28}$$

where for the sign coefficient K_I see (10.17)-(10.22), although its value is in fact not important. Now we compute straightforwardly:

$$\begin{aligned}(e_\alpha\rfloor{}^{(I)}R^{\mu\nu})\wedge{}^{*(I)}R_{\mu\nu} &= (e_\alpha\rfloor{}^{(I)}R^{\mu\nu})\wedge[K_I\,\frac{1}{2}\,\eta_{\mu\nu\rho\sigma}\,{}^{(I)}R^{\rho\sigma}]\\ &= (e_\alpha\rfloor[K_I\,\frac{1}{2}\,\eta_{\mu\nu\rho\sigma}\,{}^{(I)}R^{\mu\nu}])\wedge{}^{(I)}R^{\rho\sigma}\\ &= (e_\alpha\rfloor{}^{*(I)}R_{\rho\sigma})\wedge{}^{(I)}R^{\rho\sigma}\\ &= e_\alpha\rfloor\left({}^{*(I)}R_{\rho\sigma}\wedge{}^{(I)}W^{\rho\sigma}\right)-{}^{*(I)}R_{\rho\sigma}\wedge(e_\alpha\rfloor{}^{(I)}W^{\rho\sigma}).\end{aligned} \tag{10.29}$$

This concludes the proof of (10.28).

Completely analogously, one can demonstrate that the following identities hold true among the elements of the two subsets of irreducible parts of curvature with the *same* duality coefficient K_I (namely, within the subset $I = 1, 3, 5, 6$ with $K_I = 1$ and within the subset $J = 2, 4$ with $K_J = -1$):

$$(e_\alpha\rfloor{}^{(1)}R^{\mu\nu})\wedge{}^{*(3)}R_{\mu\nu}=(e_\alpha\rfloor{}^{(1)}R^{\mu\nu})\wedge{}^{*(6)}R_{\mu\nu}=0, \tag{10.30}$$

$$(e_\alpha\rfloor{}^{(1)}R^{\mu\nu})\wedge{}^{*(5)}R_{\mu\nu}=(e_\alpha\rfloor{}^{(3)}R^{\mu\nu})\wedge{}^{*(5)}R_{\mu\nu}=0, \tag{10.31}$$

$$(e_\alpha\rfloor{}^{(3)}R^{\mu\nu})\wedge{}^{*(6)}R_{\mu\nu}=(e_\alpha\rfloor{}^{(5)}R^{\mu\nu})\wedge{}^{*(6)}R_{\mu\nu}=0, \tag{10.32}$$

$$(e_\alpha\rfloor{}^{(2)}R^{\mu\nu})\wedge{}^{*(4)}R_{\mu\nu}=0. \tag{10.33}$$

Among the elements belonging to the *different* subsets:

$$(e_\alpha\rfloor{}^{(I)}R^{\mu\nu})\wedge{}^{*(J)}R_{\mu\nu}\;=\;(e_\alpha\rfloor{}^{(J)}R^{\mu\nu})\wedge{}^{*(I)}R_{\mu\nu},\qquad\text{for } I=1,3,5,6,\quad\text{and } J=2,4. \tag{10.34}$$

The identities (10.28) and (10.30)-(10.34) are extremely helpful in the computations for the gravitational energy-momentum (5.10) in the general quadratic models with the Lagrangians containing the curvature square terms.

XI. DOUBLE DUALITY SOLUTIONS

Let us consider now the general quadratic model (8.1). The Double Duality Ansatz (DDA) technique provides an effective method of finding exact solutions of the field equations of the Poincaré gauge theory (8.11) and (8.12). This method was developed in the numerous papers [1–8,32–35,55,57].

The general DDA represents the Lorentz gauge momentum in the form:

$$H^{\alpha}{}_{\beta} = \zeta\,\frac{1}{2}\,\eta^{\alpha}{}_{\beta\mu\nu}\,R^{\mu\nu} - \frac{1}{2\kappa}\left(\xi\,\eta^{\alpha}{}_{\beta} + \chi\,\vartheta^{\alpha}\wedge\vartheta_{\beta}\right), \tag{11.1}$$

where ζ, ξ, χ are three constant parameters.

Let us consider in detail how the DDA works, separating the whole scheme into the simple steps, listed below in the following subsections.

A. Second equation: solution for the translational momentum

The exterior covariant derivative for (11.1) is calculated straightforwardly:

$$\begin{aligned} DH_{\alpha\beta} &= \zeta\,\frac{1}{2}\,\eta_{\alpha\beta\mu\nu}\,DR^{\mu\nu} - \frac{1}{2\kappa}\Big(\xi\,D\eta_{\alpha\beta} + 2\chi\,T_{[\alpha}\wedge\vartheta_{\beta]}\Big) \\ &= -\,\xi\,\vartheta_{[\alpha}\wedge H^{(0)}_{\beta]} + \frac{\chi}{\kappa}\,\vartheta_{[\alpha}\wedge T_{\beta]}, \end{aligned} \tag{11.2}$$

where we used the Bianchi identity $DR^{\mu\nu}\equiv 0$, and the fundamental identities (17.2) and (9.10). Substituting the gravitational spin density (5.5) into (8.12), we obtain the *second* field equation in the form

$$-\,\xi\,\vartheta_{[\alpha}\wedge H^{(0)}_{\beta]} + \frac{\chi}{\kappa}\,\vartheta_{[\alpha}\wedge T_{\beta]} + \vartheta_{[\alpha}\wedge H_{\beta]} = \vartheta_{[\alpha}\wedge\mu_{\beta]}. \tag{11.3}$$

Here we rewrote the matter source in terms of the *spin energy potential* two-form μ_α introduced in (3.22) and (3.25). Equation (11.3) is formally solved with respect to the translational momentum:

$$H_\alpha = \xi\,H^{(0)}_\alpha - \frac{\chi}{\kappa}\,T_\alpha + \mu_\alpha. \tag{11.4}$$

The analysis of this formal solution will be given in the subsequent Sec. XI C.

B. First equation: reduction to the effective Einstein equation

In terms of the gauge field momenta (8.13) and (8.14), the Lagrangian (8.1) reads:

$$V_{\rm Q} = \frac{a_0}{4\kappa}\, R^{\alpha\beta}\wedge\eta_{\alpha\beta} - \frac{\lambda}{\kappa}\,\eta - \frac{1}{2}\,T^\alpha\wedge H_\alpha - \frac{1}{2}\,R^{\alpha\beta}\wedge H_{\alpha\beta}. \tag{11.5}$$

Inserting the DDA (11.1) and the solution (11.4), we find

$$\begin{aligned} V_{\rm Q} = &\frac{a_0}{4\kappa}\, R^{\alpha\beta}\wedge\eta_{\alpha\beta} - \frac{\lambda}{\kappa}\,\eta - \frac{\zeta}{4}\,\eta_{\alpha\beta\mu\nu}\,R^{\alpha\beta}\wedge R^{\mu\nu} - \frac{1}{2}\,T^\alpha\wedge\mu_\alpha \\ &+ \frac{\xi}{2\kappa}\left(\frac{1}{2}\,R^{\alpha\beta}\wedge\eta_{\alpha\beta} - \kappa\,T^\alpha\wedge H^{(0)}_\alpha\right) \\ &+ \frac{\chi}{2\kappa}\left(\frac{1}{2}\,R^{\alpha\beta}\wedge\vartheta_\alpha\wedge\vartheta_\beta + T^\alpha\wedge T_\alpha\right). \end{aligned} \tag{11.6}$$

In the similar way, we obtain

$$(e_\alpha\rfloor T^\beta)\wedge H_\beta = \xi\,(e_\alpha\rfloor T^\beta)\wedge H^{(0)}_\beta - \frac{\chi}{2\kappa}\,e_\alpha\rfloor(T^\beta\wedge T_\beta) + (e_\alpha\rfloor T^\beta)\wedge\mu_\beta, \tag{11.7}$$

$$\begin{aligned} (e_\alpha\rfloor R^{\mu\nu})\wedge H_{\mu\nu} = &\frac{\zeta}{4}\,\eta_{\rho\sigma\mu\nu}\,e_\alpha\rfloor(R^{\rho\sigma}\wedge R^{\mu\nu}) \\ &-\frac{1}{2\kappa}\left[\xi\,(e_\alpha\rfloor R^{\mu\nu})\wedge\eta_{\mu\nu} + \chi\,(e_\alpha\rfloor R^{\mu\nu})\wedge\vartheta_\mu\wedge\vartheta_\nu\right]. \end{aligned} \tag{11.8}$$

Substituting (11.6) and (11.7)-(11.8) into (5.10), after some simple algebra we get the gravitational energy-momentum density:

$$\begin{aligned} E_\alpha &= \frac{1}{4\kappa}\,e_\alpha\rfloor\left(a_0\,R^{\mu\nu}\wedge\eta_{\mu\nu} - 4\lambda\,\eta - \xi\,R^{\mu\nu}\wedge\eta_{\mu\nu} - \chi\,R^{\mu\nu}\wedge\vartheta_\mu\wedge\vartheta_\nu\right) \\ &\quad + \xi\,E^{(0)}_\alpha + \frac{\chi}{\kappa}\,R_{\alpha\beta}\wedge\vartheta^\beta + \overset{(s)}{E}_\alpha \\ &= -\frac{1}{4\kappa}\,\eta_\alpha\left[4\lambda + (a_0-\xi)\,W - \chi\,X\right] + \xi\,E^{(0)}_\alpha + \frac{\chi}{\kappa}\,R_{\alpha\beta}\wedge\vartheta^\beta + \overset{(s)}{E}_\alpha. \end{aligned} \tag{11.9}$$

Here the *effective spin energy* three-form is introduced by

$$\overset{(s)}{E}_\alpha := \frac{1}{2}\left[(e_\alpha\rfloor T^\beta)\wedge\mu_\beta - T^\beta\wedge(e_\alpha\rfloor\mu_\beta)\right]. \tag{11.10}$$

We used (9.12) at the intermediate stage, and inserted the contractions $R^{\mu\nu}\wedge\eta_{\mu\nu} = {}^{(6)}R^{\mu\nu}\wedge\eta_{\mu\nu} = -W\,\eta$ and $R^{\mu\nu}\wedge\vartheta_\mu\wedge\vartheta_\nu = {}^{(3)}R^{\mu\nu}\wedge\vartheta_\mu\wedge\vartheta_\nu = -X\,\eta$. The scalar functions W and X are the Riemann-Cartan curvature scalar and pseudoscalar, respectively. The covariant exterior differential of (11.4) is

$$DH_\alpha = \xi\, DH^{(0)}_\alpha + \frac{\chi}{\kappa}\, R_{\alpha\beta}\wedge\vartheta^\beta, \tag{11.11}$$

where the first Bianchi identity $DT_\alpha \equiv R_{\alpha\beta}\wedge\vartheta^\beta$ was inserted.

Finally, making use of the identity (9.21), we find the *first* gauge field equation (8.11) in the form of the *effective Einstein* equation:

$$-\frac{\xi}{2\kappa}\,\widetilde{R}^{\mu\nu}\wedge\eta_{\alpha\mu\nu} + \frac{\Lambda_{\rm eff}}{\kappa}\,\eta_\alpha = \Sigma^{\rm eff}_\alpha. \tag{11.12}$$

Here we denote the effective energy-momentum of matter and the effective cosmological term:

$$\Sigma^{\rm eff}_\alpha := \Sigma_\alpha + \overset{(s)}{E}_\alpha, \tag{11.13}$$

$$\Lambda_{\rm eff} := \lambda + \frac{1}{4}\,[(a_0-\xi)\,W - \chi\,X]. \tag{11.14}$$

In general, the Riemann-Cartan curvature scalar W and pseudoscalar X are not constant, but the algebraic conditions on the curvature force them to be constant, see below. The specific combination (11.14) *must* be constant *in vacuum* ($\Sigma_\alpha = \tau^\alpha{}_\beta = 0$) since taking covariant Riemannian exterior derivative of (11.12), we find $\widetilde{D}(\Lambda_{\rm eff}\,\eta_\alpha) = d\Lambda_{\rm eff}\wedge\eta_\alpha = 0$, hence $\Lambda_{\rm eff}$=const.

C. Algebraic conditions on torsion

The formal solution (11.4) for the translational gauge momentum represents an algebraic system on the components of the torsion. We can write it down explicitly substituting (8.13), (9.7) into (11.4), and using the irreducible decomposition of the dual torsion,

$${}^{(1)}({}^*T^\alpha) = {}^*({}^{(1)}T^\alpha),\quad {}^{(2)}({}^*T^\alpha) = {}^*({}^{(3)}T^\alpha),\quad {}^{(3)}({}^*T^\alpha) = {}^*({}^{(2)}T^\alpha). \tag{11.15}$$

The resulting system of the irreducible parts of the equation (11.4) reads

$$(a_1+\xi)\,{}^{(1)}T^\alpha - \chi\,{}^*({}^{(1)}T^\alpha) = -\,{}^{(1)}({}^*\mu^\alpha), \tag{11.16}$$

$$(a_2-2\xi)\,{}^{(2)}T^\alpha - \chi\,{}^*({}^{(3)}T^\alpha) = -\,{}^{(2)}({}^*\mu^\alpha), \tag{11.17}$$

$$\left(a_3-\frac{\xi}{2}\right)\,{}^{(3)}T^\alpha - \chi\,{}^*({}^{(2)}T^\alpha) = -\,{}^{(3)}({}^*\mu^\alpha). \tag{11.18}$$

Here the irreducible parts of the spin potential two-form ${}^{(I)}({}^*\mu^\alpha)$ are defined in the same way as for the torsion.

In generic *non-vacuum* case, when all irreducible parts of spin are nontrivial, the choice of the constants on the left hand sides must allow for the unique torsion solution. In particular, (11.16) yields the traceless irreducible part of torsion:

$$
{}^{(1)}T^\alpha = \frac{1}{(a_1+\xi)^2+\chi^2}\left[\chi\,{}^{(1)}\mu^\alpha - (a_1+\xi)\,{}^{*(1)}\mu^\alpha\right]. \tag{11.19}
$$

Certainly, the denominator should be nonzero. Recall that until now we have not fixed the constant parameters ξ and χ which enter the original DDA representation (11.1). At this stage, it is enough just to assume that $\chi \neq 0$ and this guarantees the nonzero denominator in (11.19) for any choice of ξ and any value of the coupling constant a_1. Analogously, we obtain from (11.17)-(11.18) the trace and axial trace irreducible parts of torsion:

$$
{}^{(2)}T^\alpha = \frac{1}{(a_2-2\xi)(a_3-\frac{\xi}{2})+\chi^2}\left[\chi\,{}^{(2)}\mu^\alpha - \left(a_3-\frac{\xi}{2}\right){}^{*(3)}\mu^\alpha\right], \tag{11.20}
$$

$$
{}^{(3)}T^\alpha = \frac{1}{(a_2-2\xi)(a_3-\frac{\xi}{2})+\chi^2}\left[\chi\,{}^{(3)}\mu^\alpha - (a_2-2\xi)\,{}^{*(2)}\mu^\alpha\right]. \tag{11.21}
$$

Here again the denominator must be nonzero which can always be guaranteed by the proper choice of the constant parameters ξ and χ.

A special word is necessary about the *vacuum DDA* solutions. When $\Sigma_\alpha = 0, \tau_{\alpha\beta} = 0$, hence $\mu_\alpha = 0$, the generic solution of the system (11.16)-(11.18) is $T^\alpha = 0$, which is clearly described by the formulas (11.19)-(11.21). Hence, the geometry becomes purely Riemannian with the metric determined from the vacuum Einstein equation (11.12).

The *nontrivial vacuum torsion* is only possible when

$$
(a_1+\xi)^2+\chi^2 = 0, \tag{11.22}
$$

or/and

$$
(a_2-2\xi)(a_3-\frac{\xi}{2})+\chi^2 = 0. \tag{11.23}
$$

With such a special choice of coefficients many DDA solutions were obtained in the literature. However, it was immediately noticed that most of these solutions involve *free* functions which means that the torsion configurations are *not* determined unambiguously by the physical sources. This observation had stirred a confusion among the gravitational community [33,16,25,37]: indeed, how can (at least part of) the torsion be nondynamical and hence arbitrary when one "apparently" can measure the torsion with the help of the particles with spin? So (citing the title of the paper [16]), "can Poincaré gauge theory be saved?"

Quite fortunately, the theory cures itself due to the self-consistency of its general scheme. Indeed, one simply has to recall what is a measurement in a physical theory. For example, if we want to measure the torsion, what do we need for this? Clearly, we need a "measuring device" which feels the torsion. In physical terms, "to feel" means "to interact with". Thus we are again returning from the vacuum case to the theory with sources. Note that, evidently, one cannot choose one set of the coupling constants a_I, b_J for vacuum and a different set for nontrivial sources. One must keep the coupling constants a_I, b_J fixed in both cases, working within one and the same particular model.

To be specific, let us consider the vacuum DDA solutions which allow for a nontrivial tracefree torsion ${}^{(1)}T^\alpha$. In vacuum, this is only possible when (11.22) is fulfilled. Hence one must put $a_1 + \xi = \chi = 0$. But in turn, such a choice (see (11.16)) means that the tracefree part of spin ${}^{(1)}\mu_\alpha = 0$ always! To put it in a different way, no "measuring device" which feels ${}^{(1)}T^\alpha$ is allowed in this model. Hence, ${}^{(1)}T^\alpha$ is *unobservable*, and there is no reason to worry about free functions which may occur in the solutions: one cannot measure these configurations anyway.

The Dirac particles with spin $\frac{1}{2}$ represent the matter source which appears to be the most suitable for the measurement of torsion. As we know, the Dirac spin is totally antisymmetric, which in terms of the spin energy potential means that only the second irreducible part ${}^{(2)}\mu_\alpha$ is nontrivial and ${}^{(1)}\mu_\alpha = {}^{(3)}\mu_\alpha = 0$. [There is no misprint: axial torsion is described by ${}^{(3)}T^\alpha$ whereas axial spin is ${}^{(2)}\mu_\alpha$]. One can demonstrate that the vacuum DDA solutions with the vanishing axial torsion (which is consistent with the equation (11.21)) involve free functions in the trace and trace-free torsion parts. In order to allow for such solutions one should restrict the choice of the constants to (11.22)-(11.23). However such a choice then demands

${}^{(1)}\mu_\alpha = {}^{(3)}\mu_\alpha = 0$, and hence again the nondynamical torsion parts are truly unobservable: spin $\frac{1}{2}$ particles cannot detect them.

D. Algebraic conditions on Riemann-Cartan curvature

Similarly, the double duality ansatz itself (11.1) represents an algebraic system on the Riemann-Cartan curvature. Here we analyze this system. At first, we substitute the explicit Lorentz gauge momentum (8.14) into (11.1), and use the double duality properties (10.17)-(10.22) for the Riemann-Cartan curvature. Then the irreducible parts of (11.1) read as follows:

$$(b_1 - \zeta)\, {}^{(1)}R_{\alpha\beta} = 0, \tag{11.24}$$

$$(b_2 + \zeta)\, {}^{(2)}R_{\alpha\beta} = 0, \tag{11.25}$$

$$(b_4 + \zeta)\, {}^{(4)}R_{\alpha\beta} = 0, \tag{11.26}$$

$$(b_5 - \zeta)\, {}^{(5)}R_{\alpha\beta} = 0, \tag{11.27}$$

for the traceless (1st, 2nd, 4th and 5th) parts of the curvature, and we find it convenient to write the trace and pseudotrace parts (6th and 3rd) separately,

$$\frac{(b_3 - \zeta)}{6}\, X - \frac{\chi}{\kappa} = 0, \tag{11.28}$$

$$\frac{(b_6 - \zeta)}{6}\, W + \frac{(a_0 - \xi)}{\kappa} = 0. \tag{11.29}$$

Since we still have one free parameter of DDA, namely ζ, one can choose it in such a way that one of the coefficients in (11.24)-(11.29) vanishes. Usual choice is $\zeta = -b_4$ which eliminates the contribution of the fourth irreducible curvature part. In the generic case, when no other coefficients vanish, one have to use the remaining equations as the constraints on the components of the nontrivial torsion.

XII. A TORSION KINK

The DDA technique works also for non-vacuum solutions. As a particular example [7], let us consider the gauge gravity coupled to the Higgs-type massless scalar field φ. The latter is described by the Lagrangian

$$L_{\rm mat} = \frac{1}{2}\,d\varphi\wedge{}^*d\varphi. \tag{12.1}$$

The total Lagrangian $V_{\rm Q}+L_{\rm mat}$ yields the non-vacuum field equations (8.11) and (8.12) with the sources:

$$\Sigma_\alpha = \frac{\delta L_{\rm mat}}{\delta\vartheta^\alpha} = -\frac{1}{2}\left[(e_\alpha\rfloor d\varphi)^*d\varphi + d\varphi\wedge(e_\alpha\rfloor{}^*d\varphi)\right], \tag{12.2}$$

$$\tau^\alpha{}_\beta = \frac{\delta L_{\rm mat}}{\delta\Gamma_\alpha{}^\beta} = 0. \tag{12.3}$$

The vanishing spin (12.3) evidently leads to $\overset{(s)}{E}_\alpha = 0$ in the effective Einstein equation.

Besides, the matter (Klein-Gordon) field equation arises from the variation of (12.1) with respect to the scalar field:

$$d\,{}^*d\varphi = 0. \tag{12.4}$$

Let us look for the spherically symmetric solution within the DDA approach. We introduce the standard coordinate system (t,r,θ,ϕ), and assume the spherically symmetric ansatz for the coframe

$$\vartheta^{\hat 0} = e^{\mu(r)}\,dt,\quad \vartheta^{\hat 1} = e^{\nu(r)}\,dr,\quad \vartheta^{\hat 2} = r\,d\theta,\quad \vartheta^{\hat 3} = r\,\sin\theta\,d\phi. \tag{12.5}$$

The functions $\mu=\mu(r)$ and $\nu=\nu(r)$ depend only on the radial coordinate r, as well as the scalar field $\varphi=\varphi(r)$. Substituting this into (12.2) and subsequently into the effective Einstein equation (11.12), we find the following system:

$$\frac{\kappa}{2}\,(\varphi')^2 - 2\frac{\nu'}{r} + \frac{1-e^{2\nu}}{r^2} + \Lambda_{\rm eff}\,e^{2\nu} = 0, \tag{12.6}$$

$$\frac{\kappa}{2}\,(\varphi')^2 - 2\frac{\mu'}{r} - \frac{1-e^{2\nu}}{r^2} - \Lambda_{\rm eff}\,e^{2\nu} = 0, \tag{12.7}$$

$$\frac{\kappa}{2}\,(\varphi')^2 + \mu'' + (\mu'-\nu')\left(\mu'+\frac{1}{r}\right) + \Lambda_{\rm eff}\,e^{2\nu} = 0. \tag{12.8}$$

Primes denote differentiation with respect to r. The Klein-Gordon equation (12.4) yields

$$\varphi'' + \left(\mu' - \nu' + \frac{2}{r}\right)\varphi' = 0. \tag{12.9}$$

From (12.6) and (12.7) we obtain

$$\frac{\kappa}{2}\,(\varphi')^2 = \frac{\mu' + \nu'}{r}, \tag{12.10}$$

and the scalar field equation (12.9) gives

$$\varphi' = C\,\frac{e^{\nu-\mu}}{r^2}, \tag{12.11}$$

where C is an integration constant.

We will not analyze the general solutions of the couple Einstein-Klein-Gordon system. Let us confine ourselves to the particular case with

$$\mu = 0. \tag{12.12}$$

Then (12.8) and (12.10) demand $\Lambda_{\text{eff}} = 0$, whereas (12.6)-(12.7) yield the solution

$$e^{2\nu} = \left(1 + \frac{C^2}{r^2}\right)^{-1}. \tag{12.13}$$

Here C is the same integration constant as in (12.11). Finally, equation (12.11) is solved for the scalar field:

$$\varphi = \sqrt{\frac{2}{\kappa}}\,\left(\pm\text{arcsinh}\left[\frac{C}{r}\right] + C_1\right). \tag{12.14}$$

Spherically symmetric static torsion configurations are described by the general ansatz:

$$T^{\hat{0}} = f\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{1}}, \tag{12.15}$$

$$T^{\hat{1}} = h\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{1}}, \tag{12.16}$$

$$T^{\hat{2}} = \frac{1}{r}\left(k\,\vartheta^{\hat{0}} + g\,\vartheta^{\hat{1}}\right)\wedge\vartheta^{\hat{2}}, \tag{12.17}$$

$$T^{\hat{3}} = \frac{1}{r}\left(k\,\vartheta^{\hat{0}} + g\,\vartheta^{\hat{1}}\right)\wedge\vartheta^{\hat{3}}, \tag{12.18}$$

where the functions $f = f(r), h = h(r), k = k(r), g = g(r)$ depend only on the radial coordinate. Given the coframe (12.5) and the torsion (12.15)-(12.18), it is straightforward to calculate the Riemann-Cartan curvature. Let us introduce, for convenience, the functions

$$F := f + e^{-\nu}\,\mu', \qquad G := g - e^{-\nu}. \tag{12.19}$$

The direct calculation gives the 1st (Weyl) irreducible curvature part:

$$^{(1)}R_{\hat{0}\hat{1}} = \frac{1}{3}\,U\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{1}}, \qquad ^{(1)}R_{\hat{2}\hat{3}} = -\,\frac{1}{3}\,U\,\vartheta^{\hat{2}}\wedge\vartheta^{\hat{3}}, \tag{12.20}$$

$$^{(1)}R_{\hat{0}\hat{2}} = -\,\frac{1}{6}\,U\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{2}}, \quad ^{(1)}R_{\hat{3}\hat{1}} = \frac{1}{6}\,U\,\vartheta^{\hat{3}}\wedge\vartheta^{\hat{1}}, \tag{12.21}$$

$$^{(1)}R_{\hat{0}\hat{3}} = -\,\frac{1}{6}\,U\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{3}}, \quad ^{(1)}R_{\hat{1}\hat{2}} = \frac{1}{6}\,U\,\vartheta^{\hat{1}}\wedge\vartheta^{\hat{2}}, \tag{12.22}$$

where

$$U = e^{-\nu}\left(F' + F\mu' + \frac{G'}{r}\right) + \frac{1}{r^2}\Big((FG + hk)\,r + G^2 - k^2 - 1\Big). \tag{12.23}$$

The 2nd and the 3rd parts are trivial,

$$^{(2)}R_{\alpha\beta} = {}^{(3)}R_{\alpha\beta} = 0\,, \tag{12.24}$$

whereas the 5th part reads

$$^{(5)}R_{\alpha\beta} = -\,\vartheta_{[\alpha}\wedge e_{\beta]}\rfloor\Phi\,, \tag{12.25}$$

where the 2-form

$$\Phi := \frac{1}{r}\Big(e^{-\nu}\,k' + Fk + Gh\Big)\,\vartheta^{\hat{0}}\wedge\vartheta^{\hat{1}}. \tag{12.26}$$

The 6th irreducible part:

$$^{(6)}R_{\alpha\beta} = -\,\frac{1}{12}\,W\,\vartheta_{\alpha}\wedge\vartheta_{\beta}, \tag{12.27}$$

where the curvature scalar is

$$W = 2U - \frac{6}{r}\Big(e^{-\nu}\,G' + FG + kh\Big). \tag{12.28}$$

For completeness, let us write down the 4th irreducible part of curvature (15.6) which describes the traceless symmetric Ricci tensor (15.13). The corresponding one-form Φ_α has the following components

$$\Phi_{\hat{0}} = (A+P)\,\vartheta^{\hat{0}} + B\,\vartheta^{\hat{1}}, \tag{12.29}$$

$$\Phi_{\hat{1}} = (A-P)\,\vartheta^{\hat{1}} + B\,\vartheta^{\hat{0}}, \tag{12.30}$$

$$\Phi_{\hat{2}} = P\,\vartheta^{\hat{2}}, \qquad \Phi_{\hat{3}} = P\,\vartheta^{\hat{3}}, \tag{12.31}$$

where we denoted

$$A := \frac{1}{r}\,(e^{-\nu}G' - FG + kh), \tag{12.32}$$

$$B := \frac{1}{r}\,(e^{-\nu}k' - Fk + Gh), \tag{12.33}$$

$$P := \frac{1}{2}\,\left[e^{-\nu}(F' + F\mu') + \frac{1}{r^2}(-G^2 + k^2 + 1)\right]. \tag{12.34}$$

Now we are in a position to solve the algebraic curvature equations (11.24)-(11.29). We choose the DDA parameter $\zeta = -b_4$, which makes (11.26) automatically satisfied. The remaining equations yield vanishing of the 1st and 5th irreducible parts (recall that 2nd and 3rd are already zero, (12.24), hence $\chi = 0$), and the constancy of the curvature scalar. Denote the constant

$$A_0 := \frac{a_0 - \xi}{\kappa(b_4 + b_6)}. \tag{12.35}$$

From (11.29) we have $W = -6A_0$, and (12.23), (12.26), (12.28) yield the final differential system:

$$e^{-\nu}\,(F' + F\mu') + \frac{1}{r^2}\,(G^2 - k^2 - 1) = -A_0, \tag{12.36}$$

$$e^{-\nu}\,k' + Fk + Gh = 0, \tag{12.37}$$

$$e^{-\nu}\,G' + FG + kh = A_0\,r. \tag{12.38}$$

We will not analyze the complete solution of this system. Instead, consider a particular solution:

$$f = g = 0, \qquad h = k', \qquad k = \pm\sqrt{A_0\, r^2 + \frac{C^2}{r^2}}. \tag{12.39}$$

We can verify for the effective (zero) cosmological constant that

$$\Lambda_{\rm eff} = \lambda - \frac{3}{2}\frac{(a_0 - \xi)^2}{\kappa(b_4 + b_6)} = 0. \tag{12.40}$$

XIII. TORSION-FREE SOLUTIONS

The numerous classical exact and approximate solutions (including the demonstration of the generalized Birkhoff theorem for the spherical symmetry) were derived in [14,15,29,30,36,43,44,56–58], to mention but a few papers. The detailed overview can be found, for example, in [40].

Let us consider the vacuum solutions with *vanishing torsion* in the general quadratic models (8.1). For $T^\alpha = 0$, from (8.13) and (5.5) we immediately find (to recall, the tilde denotes the torsion-free Riemannian objects)

$$\widetilde{H}_\alpha = 0, \qquad \widetilde{E}^\alpha{}_\beta = 0. \tag{13.1}$$

The only nontrivial irreducible parts of the curvature are the Weyl form ${}^{(1)}\widetilde{W}^{\mu\nu}$, the traceless Ricci form ${}^{(4)}\widetilde{W}^{\mu\nu}$ and the curvature scalar ${}^{(6)}\widetilde{W}^{\mu\nu} = -\frac{1}{12}\widetilde{W}\vartheta^\mu \wedge \vartheta^\nu$. The Lorentz gauge momentum (8.14) reads

$$\widetilde{H}_{\alpha\beta} = -\frac{a_0}{2\kappa} + b_1\, {}^{*(1)}\widetilde{W}_{\alpha\beta} + b_4\, {}^{*(4)}\widetilde{W}_{\alpha\beta} + b_6\, {}^{*(6)}\widetilde{W}_{\alpha\beta}, \tag{13.2}$$

and hence the second vacuum equation (8.12) reduces to

$$b_1\, \widetilde{D}({}^{*(1)}\widetilde{W}_{\alpha\beta}) + b_4\, \widetilde{D}({}^{*(4)}\widetilde{W}_{\alpha\beta}) + b_6\, \widetilde{D}({}^{*(6)}\widetilde{W}_{\alpha\beta}) = 0. \tag{13.3}$$

This can be simplified with the help of the Bianchi identity

$$\widetilde{D}\widetilde{R}^{\mu\nu} \equiv \widetilde{D}\, {}^{(1)}\widetilde{W}^{\mu\nu} + \widetilde{D}\, {}^{(4)}\widetilde{W}^{\mu\nu} + \widetilde{D}\, {}^{(6)}\widetilde{W}^{\mu\nu} \equiv 0. \tag{13.4}$$

Contracting the last identity with $\frac{1}{2}\eta_{\alpha\beta\mu\nu}$ and using the double duality properties (10.17)-(10.22), we obtain

$$\widetilde{D}({}^{*(1)}\widetilde{W}_{\alpha\beta}) - \widetilde{D}({}^{*(4)}\widetilde{W}_{\alpha\beta}) + \widetilde{D}({}^{*(6)}\widetilde{W}_{\alpha\beta}) \equiv 0. \tag{13.5}$$

We can eliminate the derivative of the Weyl form in (13.3), and write the *second* field equation as

$$(b_1 + b_4)\,\widetilde{D}({}^{*(4)}\widetilde{W}_{\alpha\beta}) + \frac{1}{12}(b_1 - b_6)\, d\widetilde{W} \wedge \eta_{\alpha\beta} = 0, \tag{13.6}$$

where we substituted the 6th irreducible curvature explicitly in terms of the curvature scalar $\widetilde{W}$.

The gravitational energy (5.10) is calculated straightforwardly with the help of the identities (10.28) and (10.30)-(10.34), and the *first* vacuum field equation is written in the form:

$$\begin{aligned} -\widetilde{E}_\alpha = & -\frac{a_0}{2\kappa}\,\widetilde{R}^{\mu\nu} \wedge \eta_{\alpha\mu\nu} + \frac{\lambda}{\kappa}\,\eta_\alpha \\ & + (b_1 + b_4)\,\widetilde{\Phi}^\beta \wedge {}^{*(1)}\widetilde{W}_{\alpha\beta} - \frac{1}{6}(b_4 + b_6)\,\widetilde{W}\,{}^*\widetilde{\Phi}_\alpha\; = 0. \end{aligned} \tag{13.7}$$

Here we used the explicit representation of the 4th irreducible curvature part ${}^{(4)}\widetilde{W}_{\alpha\beta} = -\vartheta_{[\alpha} \wedge \Phi_{\beta]}$ in terms of the one-form (15.12). Note that $\widetilde{R}^{\mu\nu} \wedge \eta_{\alpha\mu\nu} \equiv 2{}^*\widetilde{\Phi}_\alpha - \frac{1}{2}\,\widetilde{W}\,\eta_\alpha$.

One can somewhat simplify the resulting field equations. Transvecting (13.7) with ϑ^α, we find that the curvature scalar is constant

$$\widetilde{W} = -\,\frac{4\lambda}{a_0}. \tag{13.8}$$

(For the purely quadratic models with $a_0 = 0$ the cosmological term should also be zero $\lambda = 0$). Hence the last term in (13.6) vanishes. Substituting (13.8) back into (13.6)-(13.7) we obtain the final system of algebraic-differential equations:

$$(b_1 + b_4)\,\widetilde{\Phi}^\beta \wedge {}^{*(1)}\widetilde{W}_{\alpha\beta} = \left(\frac{a_0}{\kappa} - \frac{2\lambda}{3a_0}(b_4 + b_6)\right){}^*\widetilde{\Phi}_\alpha, \tag{13.9}$$

$$(b_1 + b_4)\,\widetilde{D}(e_{[\alpha}\rfloor{}^*\widetilde{\Phi}_{\beta]}) = 0. \tag{13.10}$$

All the solutions of the vacuum Einstein equations with a cosmological term

$$\widetilde{\Phi}_\alpha = 0, \tag{13.11}$$

see (15.13), are evidently also the torsion-free solutions of the general quadratic Poincaré gauge models. One can prove [10–12] that the Einstein spaces (13.11) are *the only* torsion-free vacuum solutions of (13.9)-(13.10) except for the three very specific degenerate choices of the coupling constants [40]:

$$b_6 - \frac{3a_0^2}{2\lambda\kappa} = \begin{cases} b_1, \\ -b_4, \\ -2b_1 - 3b_4. \end{cases} \tag{13.12}$$

XIV. CONCLUSION

In this paper, an overview of the selected aspects of the Poincaré gauge gravity is given. The Lagrange-Noether approach is formulated in a general way and the conservation laws and the field equations are derived. As a particular application, we analyze the family of quadratic (in the curvature and the torsion) models. The new results obtained include the discussion of the special case of the spinless matter and the demonstration that Einstein's theory arises as a degenerate model in the class of the quadratic Poincaré theories. Finally, we outlined the main features of the so-called double duality method for constructing of the exact solutions of the quadratic Poincaré gauge theories.

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XV. APPENDIX 1: IRREDUCIBLE DECOMPOSITIONS

At first, we recall that the torsion 2-form can be decomposed into the three irreducible pieces, $T^\alpha = {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha$, where

$$^{(2)}T^\alpha = \frac{1}{3}\vartheta^\alpha \wedge (e_\nu \rfloor T^\nu), \tag{15.1}$$

$$^{(3)}T^\alpha = -\frac{1}{3}{}^*(\vartheta^\alpha \wedge {}^*(T^\nu \wedge \vartheta_\nu)) = \frac{1}{3}e^\alpha \rfloor (T^\nu \wedge \vartheta_\nu), \tag{15.2}$$

$$^{(1)}T^\alpha = T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha. \tag{15.3}$$

The Riemann-Cartan curvature 2-form is decomposed $R^{\alpha\beta} = \sum_{I=1}^{6} {}^{(I)}R^{\alpha\beta}$ into the 6 irreducible parts

$$^{(2)}R^{\alpha\beta} = -{}^*(\vartheta^{[\alpha} \wedge \Psi^{\beta]}), \tag{15.4}$$

$$^{(3)}R^{\alpha\beta} = -\frac{1}{12}{}^*(X\,\vartheta^\alpha \wedge \vartheta^\beta), \tag{15.5}$$

$$^{(4)}R^{\alpha\beta} = -\vartheta^{[\alpha} \wedge \Phi^{\beta]}, \tag{15.6}$$

$$^{(5)}R^{\alpha\beta} = -\frac{1}{2}\vartheta^{[\alpha} \wedge e^{\beta]} \rfloor (\vartheta^\alpha \wedge W_\alpha), \tag{15.7}$$

$$^{(6)}R^{\alpha\beta} = -\frac{1}{12} W\,\vartheta^\alpha \wedge \vartheta^\beta, \tag{15.8}$$

$$^{(1)}R^{\alpha\beta} = R^{\alpha\beta} - \sum_{I=2}^{6} {}^{(I)}R^{\alpha\beta}, \tag{15.9}$$

where

$$W^\alpha := e_\beta \rfloor R^{\alpha\beta}, \quad W := e_\alpha \rfloor W^\alpha, \quad X^\alpha := {}^*(R^{\beta\alpha} \wedge \vartheta_\beta), \quad X := e_\alpha \rfloor X^\alpha, \tag{15.10}$$

and

$$\Psi_\alpha := X_\alpha - \frac{1}{4}\,\vartheta_\alpha\, X - \frac{1}{2}\, e_\alpha \rfloor (\vartheta^\beta \wedge X_\beta), \tag{15.11}$$

$$\Phi_\alpha := W_\alpha - \frac{1}{4}\,\vartheta_\alpha\, W - \frac{1}{2}\, e_\alpha \rfloor (\vartheta^\beta \wedge W_\beta). \tag{15.12}$$

The curvature tensor $R_{\mu\nu\alpha}{}^\beta$ is constructed from the components of the 2-form $R_\alpha{}^\beta = \frac{1}{2}R_{\mu\nu\alpha}{}^\beta\,\vartheta^\mu \wedge \vartheta^\nu$. The Ricci tensor is defined as $\mathrm{Ric}_{\alpha\beta} := R_{\gamma\alpha\beta}{}^\gamma$. The curvature scalar $R = g^{\alpha\beta}\mathrm{Ric}_{\alpha\beta}$ determines the 6-th irreducible part (15.8) since $W \equiv R$. The first irreducible part (15.9) introduces the generalized Weyl tensor $C_{\mu\nu\alpha}{}^\beta$ via the expansion of the 2-form $^{(1)}R_\alpha{}^\beta = \frac{1}{2}C_{\mu\nu\alpha}{}^\beta\,\vartheta^\mu \wedge \vartheta^\nu$. From (15.12) we learn that the 4-th part of the curvature is given by the symmetric traceless Ricci tensor,

$$\Phi_\alpha = \left(R_{(\alpha\beta)} - \frac{1}{4}\,R\,g_{\alpha\beta}\right)\vartheta^\beta. \tag{15.13}$$

Accordingly, the 1-st, 4-th and 6-th curvature parts generalize the well-known irreducible decomposition of the Riemannian curvature tensor. The 2-nd, 3-rd and 5-th curvature parts are purely non-Riemannian since they all arise from the nontrivial right-hand side of the first Bianchi identity $R_\alpha{}^\beta \wedge \vartheta^\alpha = DT^\beta$, see (15.10) and (15.11).

XVI. APPENDIX 2: PROOF OF LEMMAS

Proof of Lemma A: Consider a chain of identical transformations for the 4-form

$$\begin{aligned}
\left(\frac{1}{2}\eta^{\alpha\beta\mu\nu}\vartheta_\alpha \wedge A_\beta\right) \wedge \vartheta_\rho \wedge \vartheta_\sigma &= \frac{1}{2}\eta^{\alpha\beta\mu\nu}\eta_{\alpha\rho\sigma\lambda}\eta^\lambda \wedge A_\beta \\
&= \frac{1}{2}(-\delta^\mu_\sigma\eta^\nu \wedge A_\rho + \delta^\nu_\sigma\eta^\mu \wedge A_\rho \\
&\qquad +\delta^\mu_\rho\eta^\nu \wedge A_\sigma - \delta^\nu_\rho\eta^\mu \wedge A_\sigma) - \delta^{[\mu}_\rho\delta^{\nu]}_\sigma\,\eta^\beta \wedge A_\beta \\
&= \frac{1}{2}(\delta^\mu_\sigma\eta_\rho \wedge A^\nu - \delta^\nu_\sigma\eta_\rho \wedge A^\mu - \delta^\mu_\rho\eta_\sigma \wedge A^\nu + \delta^\nu_\rho\eta_\sigma \wedge A^\mu) \\
&= \vartheta^{[\mu} \wedge \eta_{\rho\sigma} \wedge A^{\nu]} = \vartheta^{[\mu} \wedge A^{\nu]} \wedge {}^\star(\vartheta_\rho \wedge \vartheta_\sigma) \\
&= {}^\star\left(\vartheta^{[\mu} \wedge A^{\nu]}\right) \wedge \vartheta_\rho \wedge \vartheta_\sigma. \qquad (16.1)
\end{aligned}$$

Comparing the beginning and the end, by Cartan's lemma, one finds the identity (10.5). In this calculation we used (10.4), the identity for transvection of two Levi-Civita tensors, and the identity $\vartheta^\mu \wedge \eta_{\rho\sigma} = \delta^\mu_\sigma\eta_\rho - \delta^\nu_\rho\eta_\sigma$. Notice that (10.4) implies $A_\alpha \wedge \eta^\alpha = 0$.

Proof of Lemma B: Completely analogously to Lemma A, we have

$$\begin{aligned}
\left(\frac{1}{2}\eta^{\alpha\beta\mu\nu}\vartheta_\alpha \wedge B_\beta\right) \wedge \vartheta_\rho \wedge \vartheta_\sigma &= \frac{1}{2}\eta^{\alpha\beta\mu\nu}\eta_{\alpha\rho\sigma\lambda}\eta^\lambda \wedge B_\beta \\
&= \frac{1}{2}(-\delta^\mu_\sigma\eta^\nu \wedge B_\rho + \delta^\nu_\sigma\eta^\mu \wedge B_\rho \\
&\qquad +\delta^\mu_\rho\eta^\nu \wedge B_\sigma - \delta^\nu_\rho\eta^\mu \wedge B_\sigma) - \delta^{[\mu}_\rho\delta^{\nu]}_\sigma\,\eta^\beta \wedge B_\beta \\
&= \frac{1}{2}(-\delta^\mu_\sigma\eta_\rho \wedge B^\nu + \delta^\nu_\sigma\eta_\rho \wedge B^\mu + \delta^\mu_\rho\eta_\sigma \wedge B^\nu - \delta^\nu_\rho\eta_\sigma \wedge B^\mu) \\
&= -\,\vartheta^{[\mu} \wedge \eta_{\rho\sigma} \wedge B^{\nu]} = -\,\vartheta^{[\mu} \wedge B^{\nu]} \wedge {}^\star(\vartheta_\rho \wedge \vartheta_\sigma) \\
&= -\,{}^\star\left(\vartheta^{[\mu} \wedge B^{\nu]}\right) \wedge \vartheta_\rho \wedge \vartheta_\sigma. \qquad (16.2)
\end{aligned}$$

We thus find the identity (10.7).

XVII. APPENDIX 3: PROOF OF GEOMETRICAL IDENTITIES

Here we prove two important geometrical identities for contortion and torsion:

$$\frac{1}{2}\,K^{\mu\nu}\wedge\eta_{\alpha\mu\nu}\equiv{}^*(-{}^{(1)}T_\alpha+2{}^{(2)}T_\alpha+\frac{1}{2}{}^{(3)}T_\alpha), \tag{17.1}$$

$$D\eta_{\alpha\beta}\equiv\vartheta_{[\alpha}\wedge K^{\mu\nu}\wedge\eta_{\beta]\mu\nu}. \tag{17.2}$$

Hereafter we use the identity

$${}^*(\Phi\wedge\vartheta^\alpha)\equiv e^\alpha\rfloor{}^*\Phi, \tag{17.3}$$

valid for any form Φ. Next, we need the product

$$\vartheta^\beta\wedge\eta_{\alpha\mu\nu}=\delta^\beta_\alpha\,\eta_{\mu\nu}+\delta^\beta_\mu\,\eta_{\nu\alpha}+\delta^\beta_\nu\,\eta_{\alpha\mu}. \tag{17.4}$$

Let us now compute $K^{\mu\nu}\wedge\eta_{\alpha\mu\nu}$. Using (7.2), we find:

$$K^{\mu\nu}\wedge\eta_{\alpha\mu\nu}=-(e^\mu\rfloor T^\nu)\wedge\eta_{\alpha\mu\nu}+\frac{1}{2}(e^\mu\rfloor e^\nu\rfloor T_\beta)\vartheta^\beta\wedge\eta_{\alpha\mu\nu}. \tag{17.5}$$

In order to calculate the first term, we start with

$$\begin{aligned}(e^\mu\rfloor T^\nu)\wedge\eta_{\mu\nu}&=e^\mu\rfloor(T^\nu\wedge\eta_{\mu\nu})=e^\mu\rfloor(\eta_\mu\wedge T)=-\eta_\mu\,e^\mu\rfloor T\\&=-{}^*(\vartheta_\mu\,e^\mu\rfloor T)=-{}^*T,\end{aligned} \tag{17.6}$$

where $T:=e_\nu\rfloor T^\nu$, and we used the identity $0\equiv e_\nu\rfloor(T^\nu\wedge\eta_\mu)=T\wedge\eta_\mu+T^\nu\wedge\eta_{\mu\nu}$. Applying the interior product $e_\alpha\rfloor$, we find

$$(e_\alpha\rfloor e^\mu\rfloor T^\nu)\,\eta_{\mu\nu}-(e^\mu\rfloor T^\nu)\wedge\eta_{\alpha\mu\nu}=-e_\alpha\rfloor{}^*T, \tag{17.7}$$

and thus the first term on the right hand side of (17.5) reads

$$\begin{aligned}-(e^\mu\rfloor T^\nu)\wedge\eta_{\alpha\mu\nu}&\equiv-e_\alpha\rfloor{}^*T-(e_\alpha\rfloor e^\mu\rfloor T^\nu)\,\eta_{\mu\nu}\\&=-{}^*(T\wedge\vartheta_\alpha)-{}^*(\vartheta_\mu\wedge\vartheta_\nu\,e_\alpha\rfloor e^\mu\rfloor T^\nu)\\&={}^*(\vartheta_\alpha\wedge T-T_\alpha+e_\alpha\rfloor(\vartheta^\nu\wedge T_\nu)).\end{aligned} \tag{17.8}$$

The second term on the right hand side of (17.5) is easily computed with the help of (17.4):

$$\frac{1}{2}(e^\mu\rfloor e^\nu\rfloor T_\beta)\vartheta^\beta\wedge\eta_{\alpha\mu\nu}\equiv\frac{1}{2}{}^*(\vartheta_\mu\wedge\vartheta_\nu\, e^\mu\rfloor e^\nu\rfloor T_\alpha$$
$$-\vartheta_\nu\wedge\vartheta_\alpha\, e^\nu\rfloor T+\vartheta_\alpha\wedge\vartheta_\mu\, e^\mu\rfloor T)$$
$$={}^*(-T_\alpha+\vartheta_\alpha\wedge T). \tag{17.9}$$

Collecting (17.8) and (17.9) together, we find:

$$K^{\mu\nu}\wedge\eta_{\alpha\mu\nu}\equiv{}^*(-2T_\alpha+2\vartheta_\alpha\wedge T+e_\alpha\rfloor(\vartheta^\nu\wedge T_\nu)). \tag{17.10}$$

Substituting the definitions (15.1)-(15.3), one proves the identity (17.1).

The proof of the second identity (17.2) is more simple. Using the decomposition (7.1) and (9.4), we find for the left hand side:

$$D\eta_{\alpha\beta}=-K_\alpha{}^\gamma\wedge\eta_{\gamma\beta}-K_\beta{}^\gamma\wedge\eta_{\alpha\gamma}. \tag{17.11}$$

However, from (17.4) we derive

$$\vartheta_{[\alpha}\wedge\eta_{\beta]\mu\nu}=g_{\alpha[\mu}\,\eta_{\nu]\beta}-g_{\beta[\mu}\,\eta_{\nu]\alpha}, \tag{17.12}$$

and hence for the right hand side one finds

$$\vartheta_{[\alpha}\wedge K^{\mu\nu}\wedge\eta_{\beta]\mu\nu}=-K^{\mu\nu}\wedge\vartheta_{[\alpha}\wedge\eta_{\beta]\mu\nu}=-2\,K_{[\alpha}{}^\nu\wedge\eta_{\nu|\beta]}, \tag{17.13}$$

which proves (17.2).

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