

LECTURES ON GRAVITATION

Pietro Menotti

Dipartimento di Fisica, Università di Pisa, Italy

March 2017

INDEX

Foreword		7
Acknowledgments		7
Chapter 1:	The Lorentz group	9
	1. The group of inertial transformations: the three solutions	
	2. Lorentz transformations	
	3. $SL(2, C)$ is connected	
	4. $SL(2, C)$ is the double covering of the restricted Lorentz group	
	5. The restricted Lorentz group is simple	
	6. $SL(2, C)$ is simply connected	
	7. Little groups	
	8. There are no non-trivial unitary finite dimensional representation of the Lorentz group	
	9. The conjugate representation	
	10. Relativistically invariant field equations	
	11. The Klein-Gordon equation	
	12. The Weyl equation	
	13. The Majorana mass	
	14. The Dirac equation	
	15. The Rarita- Schwinger equation	
	16. Transformation properties of quantum fields	
Chapter 2:	Equivalence principle and the path to general relativity	35
	1. Introduction	
	2. Motion of a particle in a gravitational field	
	3. Hamiltonian of a particle in a gravitational field	
Chapter 3:	Manifolds	41
	1. Introduction	
	2. Mappings, pull-back and push-forward	
	3. Vector fields and diffeomorphisms	
	4. The Lie derivative	
	5. Components of the Lie derivative	

6. Commuting vector fields
7. Stokes's theorem

Chapter 4:	The covariant derivative	49
	<ol style="list-style-type: none"> 1. Introduction 2. Transformation properties of the connection 3. Parallel transport 4. Geodesics 5. Curvature 6. Torsion 7. The structure equations 8. Non-abelian gauge fields on differential manifolds 9. Non compact electrodynamics 10. Meaning of torsion 11. Vanishing of the torsion 12. Metric structure 13. Isometries 14. Metric compatibility 15. Contracted Bianchi identities 16. Orthonormal reference frames 17. Compact electrodynamics 18. Symmetries of the Riemann tensor 19. Sectional curvature and Schur theorem 20. Spaces of constant curvature 21. Geometry in diverse dimensions 22. Flat and conformally flat spaces 23. The non abelian Stokes theorem 24. Vanishing of torsion and of the Riemann tensor 25. The Weyl-Schouten theorem 26. Hodge * operation 27. The current 28. Action for gauge fields 	
Chapter 5:	The action for the gravitational field	79
	<ol style="list-style-type: none"> 1. The Hilbert action 2. Einstein $\Gamma - \Gamma$ action 	

	3. Palatini first order action	
	4. Einstein-Cartan formulation	
	5. The energy momentum tensor	
	6. Coupling of matter to the gravitational field	
	7. Dimensions of the Hilbert action	
	8. Coupling of the Dirac field to the gravitational field	
	9. The field equations	
	10. The generalized energy- momentum tensor	
	11. Einstein-Eddington affine theory	
Chapter 6:	Submanifolds	101
	1. Introduction	
	2. Extrinsic curvature	
	3. The trace-K action	
	4. The boundary terms in the Palatini formulation	
	5. The boundary terms in the Einstein-Cartan formulation	
	6. The Gauss and Codazzi relations	
Chapter 7:	Hamiltonian formulation of gravity	111
	1. Introduction	
	2. Constraints and dynamical equations of motion	
	3. The Cauchy problem for the Maxwell and Einstein equations in vacuum	
	4. The action in canonical form	
	5. The Poisson algebra of the constraints	
	6. Fluids	
	7. The energy conditions	
	8. The cosmological constant	
Chapter 8:	The energy of a gravitational system	125
	1. The non abelian charge conservation	
	2. Background Bianchi identities	
	3. The superpotential	
	4. The positive energy theorem	
Chapter 9:	The linearization of gravity	135

1. Introduction
2. The Fierz-Pauli equation
3. The massless case
4. The VDVZ discontinuity
5. Classical general relativity from field theory in Lorentz space

Chapter 10:	Quantization in external gravitational fields <ol style="list-style-type: none"> 1. Introduction 2. Scalar product of two solutions 3. The scalar product for general stationary metric 4. Schwarzschild solution and its maximal extension 5. Proper acceleration of stationary observers in Schwarzschild metric and surface gravity 6. The accelerated detector 7. Elements of causal structure of space-time 8. Rindler metric 9. Eigenfunctions of K in the Rindler metric 10. The Bogoliubov transformation 11. Quantum field theory in the Rindler 12. Rindler space with a reflecting wall 13. Resolution in discrete wave packets 14. Hawking radiation in 4-dimensional Schwarzschild space 	149
Chapter 11:	$N = 1$ Supergravity <ol style="list-style-type: none"> 1. The Wess-Zumino model 2. Noether currents 3. $N = 1$ supergravity 	189
Appendix:	Derivation of the hamiltonian equations of motion and of the Poisson algebra of the constraints	201

FOREWORD

The suggested textbook for this course are, in the order

R.M. Wald, General Relativity, The University of Chicago Press, Chicago and London, 1984 referred to as [Wald]

S.W.Hawking and G.F.R. Ellis, The large scale structure of space-time, Cambridge University Press, 1974 referred to as [HawkingEllis]

S. Weinberg, Gravitation and Cosmology, John Wiley and Sons, New York, 1972, referred to as [WeinbergGC]

S. Weinberg, The quantum theory of fields, Cambridge University Press, 1995, referred to as [WeinbergQFT]

Reference to other books and papers are given at the end of each section.

ACKNOWLEDGMENTS

I am grateful to Damiano Anselmi, Giovanni Morchio, Carlo Rovelli and Domenico Seminara for useful discussions.

Chapter 1

The Lorentz group

1.1 The group of inertial transformations: the three solutions

In this section we examine a group theoretical argument which shows that the Lorentz transformations are a solution of a symmetry problem in which the independence of the velocity of light from the inertial frame does not play a direct role (see [1],[2] and reference therein). Actually there might be no signal with such a property and still Lorentz transformation would maintain their validity. As we shall see the solutions of the symmetry problem are three: One is Galilei transformations, the other is Lorentz transformations while the third solution has to be discarded of physical grounds.

We work for simplicity in $1 + 1$ dimensions (x, t) . Extension to $1 + n$ dimensions is trivial. We accept Galilean invariance as a symmetry group in the following sense: in addition to space-time translations we assume that the equivalent frames form a continuous family which depends on a single continuous parameter α .

In setting up the rods and synchronizing the clocks we must respect homogeneity; e.g. for synchronizing the clocks we can use any signal (elastic waves, particles etc.) provided we divide by 2 the time of come back.

The most general transformation for the coordinates of the events is

$$x' = f(x, t, \alpha)$$

$$t' = g(x, t, \alpha).$$

We require:

1) Homogeneity of space time under space time translations

i.e. in

$$\begin{aligned} dx' &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial t} dt \\ dt' &= \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial t} dt \end{aligned}$$

the partial derivatives do not depend on x and t . Thus we have a linear transformation. Choosing the origin of space and time properly we have

$$\begin{aligned} x' &= a(\alpha)x + b(\alpha)t \\ t' &= c(\alpha)x + d(\alpha)t. \end{aligned}$$

If v is the speed of O' in the frame L we have

$$0 = a(\alpha)x + b(\alpha)t = a(\alpha)vt + b(\alpha)t; \quad \text{i.e.} \quad b(\alpha) = -va(\alpha).$$

From now on we shall use as parameter v instead of α i.e. write

$$\begin{aligned} x' &= a(v)x + b(v)t \\ t' &= c(v)x + d(v)t. \end{aligned}$$

and the equivalent frames move one with respect to the other with constant speed.

2) Isotropy of space

Inverting $x \rightarrow y = -x$ and $x' \rightarrow y' = -x'$ we admit to have also a transformation of the group with some speed u

$$\begin{aligned} y' &= a(u)y + b(u)t \\ t' &= c(u)y + d(u)t. \end{aligned}$$

Substituting the previous into the above we get

$$a(v) = a(u); \quad b(v) = -b(u); \quad c(v) = -c(u); \quad d(v) = d(u)$$

and thus

$$u = -v; \quad a = a(|v|); \quad d = d(|v|); \quad c(-v) = -c(v).$$

3) Group law

The equivalence of all inertial frames implies that the transformation form a group. Thus there must exist the inverse (unique)

$$\begin{pmatrix} a(w) & -wa(w) \\ c(w) & d(w) \end{pmatrix} \begin{pmatrix} a(v) & -va(v) \\ c(v) & d(v) \end{pmatrix} = I$$

which provides

$$w = -v \frac{a(v)}{d(v)} = -v\rho(v)$$

where

$$\rho(v) \equiv \frac{a(v)}{d(v)}.$$

We obtain also

$$\frac{a(v)}{c(v)} = -\frac{d(w)}{c(w)}.$$

But as $A^{-1}A = I = AA^{-1}$ we must also have

$$\frac{a(w)}{c(w)} = -\frac{d(v)}{c(v)}$$

and dividing

$$\frac{a(v)}{d(v)} = \frac{d(w)}{a(w)}; \quad \text{i.e.} \quad \rho(w) = \frac{1}{\rho(v)}$$

that is we have reached the functional equation

$$\rho(-v\rho(v)) = \frac{1}{\rho(v)} = \rho(v\rho(v))$$

where in the last passage we took into account that a and d and thus ρ are even functions.

To solve the equation set

$$x\rho(x) = \zeta(x); \quad \rho(x) = \frac{\zeta(x)}{x}$$

and we have

$$\rho(\zeta(v)) = \frac{1}{\rho(v)}$$

i.e.

$$\rho(\zeta(v)) = \frac{v}{\zeta(v)}$$

or

$$\zeta(\zeta(v)) = v. \tag{1.1}$$

As $a(0) = d(0) = 1$ we have the additional information $\zeta(0) = 0$ and $\zeta'(0) = 1$. Expand in power series

$$\zeta(x) = x + a_2x^2 + a_3x^3 + \dots$$

Eq.(1.1) becomes

$$x + a_2x^2 + a_3x^3 + \dots + a_2(x + a_2x^2 + a_3x^3 + \dots)^2 + a_3(x + a_2x^2 + a_3x^3 + \dots)^3 + \dots = x$$

which gives

$$2a_2 = 0.$$

Then the equation becomes

$$a_3x^3 + \dots + a_3(x + a_3x^3 + \dots)^3 + a_4(x + a_3x^3 + \dots)^4 + \dots = 0$$

which gives

$$2a_3 = 0$$

and so on. Finally we have $\zeta(v) = v$, with the consequences $a(v) = d(v)$ and $w = -v$.

For a proof of $\zeta(v) = v$ without recourse to a power series expansion see [1].

We now exploit the relation

$$\begin{pmatrix} a(v) & va(v) \\ -c(v) & a(v) \end{pmatrix} \begin{pmatrix} a(v) & -va(v) \\ c(v) & a(v) \end{pmatrix} = I$$

obtaining

$$a^2(v) + va(v)c(v) = 1.$$

We notice that

$$\begin{pmatrix} a(v) & -va(v) \\ \frac{1-a^2(v)}{va(v)} & a(v) \end{pmatrix} \in SL(2, R).$$

Composing two transformations of the above type, recalling that $d = a$, we must obtain a transformation of the same type

$$\begin{pmatrix} a' & b' \\ c' & a' \end{pmatrix} \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'a \\ c'a + a'c & c'b + a'a \end{pmatrix}$$

i.e. $b'c = c'b$ or

$$\frac{c'}{b'} = \frac{c}{b} = k \quad \text{independent of } v.$$

From the expression of $c(v)$ and $b(v)$ we have

$$a(v) = \frac{1}{\sqrt{1 - kv^2}}$$

where the + determination of the square root has to be chosen, as we know that $a(0) = 1$.

Substituting we have

$$\begin{pmatrix} \frac{1}{\sqrt{1-kv^2}} & -v\frac{1}{\sqrt{1-kv^2}} \\ -\frac{kv}{\sqrt{1-kv^2}} & \frac{1}{\sqrt{1-kv^2}} \end{pmatrix}$$

1. $k = 0$

In this case we have the Galileo transformations

$$\begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}$$

2. $0 < k \equiv 1/c^2$

In this case we have the Lorentz transformations; setting $x^0 = ct$

$$\begin{pmatrix} \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} & -\frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} \\ -\frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{pmatrix} = \begin{pmatrix} \cosh \alpha & -\sinh \alpha \\ -\sinh \alpha & \cosh \alpha \end{pmatrix} \in SO(1,1)$$

with $v/c = \tanh \alpha$. The product of two transformation characterized by α and β gives the transformation characterized by $\alpha + \beta$.

Composition of velocities

$$\frac{v}{c} \oplus \frac{w}{c} = \frac{\frac{v}{c} + \frac{w}{c}}{1 + \frac{vw}{c^2}} \quad (1.2)$$

shows that c is a limit velocity. Eq.(1.2) can also be written as

$$\frac{v}{c} = \tanh \alpha, \quad \frac{w}{c} = \tanh \beta, \quad \frac{v}{c} \oplus \frac{w}{c} = \tanh(\alpha + \beta).$$

With respect to the event $(0,0)$, events can be invariantly classified as past present and future.

3. $0 > k \equiv -1/c^2$

In this case setting $x^0 = ct$ we have the transformation

$$\begin{pmatrix} \frac{1}{\sqrt{1+\frac{v^2}{c^2}}} & -\frac{\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}} \\ \frac{\frac{v}{c}}{\sqrt{1+\frac{v^2}{c^2}}} & \frac{1}{\sqrt{1+\frac{v^2}{c^2}}} \end{pmatrix} \in SO(2).$$

Combining more of these transformations we can turn time into space, and there is no more distinction between present past and future. There are obvious problems with causality.

References

- [1] J-M. Levy-Leblond, "One more derivation of the Lorentz transformation" American Journal of Physics, 44 (1976) 271.
- [2] B. Preziosi, "Inertia principle and transformation laws" Nuovo Cim. 109 B (1994) 1331.

1.2 Lorentz transformations

Combinations of rotations, boosts and inversions gives rise to a transformation

$$x' = \Lambda x \quad (1.3)$$

with the property $\langle x, \eta x \rangle = \langle x', \eta x' \rangle$, where $x^0 = ct$ and $\eta = \text{Diag}(-1, 1, 1, 1)$ i.e.

$$\eta = \Lambda^T \eta \Lambda. \quad (1.4)$$

The topology is defined by the metric $\text{tr}((\Lambda' - \Lambda)^T(\Lambda' - \Lambda))$. From (1.4) we have also $\Lambda^{-1}\eta^{-1} = \eta^{-1}\Lambda^T$, i.e.

$$\eta^{-1} = \Lambda \eta^{-1} \Lambda^T. \quad (1.5)$$

Obviously $\det \Lambda = \pm 1$ and from Eq.(1.4), $-1 = -(\Lambda^0_0)^2 + (\Lambda^j_0)^2$ and thus $|\Lambda^0_0| \geq 1$. Real transformations satisfying (1.4) and with $\det \Lambda = 1$ and $\Lambda^0_0 \geq 1$ are the restricted Lorentz group. They form a group because the determinant is a representation and we have that $\Lambda^0_0 \geq 1$ is conserved in the composition. In fact

$$(\Lambda' \Lambda)^0_0 = \Lambda'^0_0 \Lambda^0_0 + \Lambda'^j_0 \Lambda^j_0$$

and from Eq.(1.4)

$$(\Lambda'^0_0)^2 > \Lambda'^j_0 \Lambda'^j_0$$

while from Eq.(1.5)

$$(\Lambda^0_0)^2 > \Lambda^j_0 \Lambda^j_0$$

and using Schwarz inequality we have

$$|\Lambda'^j_0 \Lambda^j_0| < \Lambda'^0_0 \Lambda^0_0$$

which provides $(\Lambda' \Lambda)^0_0 > 0$. The elements of the Lorentz group can be classified by the sign of the determinant $(+, -)$ and by the sign of Λ^0_0 , (\uparrow, \downarrow) , $L^{\uparrow}_+, L^{\downarrow}_+, L^{\uparrow}_-, L^{\downarrow}_-$. Due to the discontinuity in the determinant and in Λ^0_0 they form disjoint sets. L^{\uparrow}_+ is called the restricted Lorentz group. The other sets are not groups and can be reached from the elements of L^{\uparrow}_+ by applying I_s (parity) I_t (time inversion), $I_{st} = I_s I_t$ (strong reflections). The sets $L^{\uparrow}_+ \cup L^{\downarrow}_+$, $L^{\uparrow}_+ \cup L^{\downarrow}_+$, $L^{\uparrow}_+ \cup L^{\downarrow}_-$, are subgroups, respectively the orthochronous, the proper and the orthocorous subgroups.

We have 16 elements with 10 conditions thus 6 real degrees of freedom. Given a Λ belonging to the restricted Lorentz group it can be always written as RZS where Z is a boost along z and R, S rotations. This can be done by deconstruction. Such decomposition is not unique as seen by counting the degrees of freedom.

Lorentz transformations can be defined also as the diffeomorphisms preserving the distance

$$ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu. \quad (1.6)$$

and such that $x'^\mu(0) = 0$. If Eq.(1.6) is left invariant under $x'^\mu = x'^\mu(x)$ then the transformation is linear. In fact from

$$0 = \frac{\partial^2 x'^\lambda}{\partial x^\sigma \partial x^\mu} \eta_{\lambda\rho} \frac{\partial x'^\rho}{\partial x^\nu} + \frac{\partial x'^\lambda}{\partial x^\mu} \eta_{\lambda\rho} \frac{\partial^2 x'^\rho}{\partial x^\nu \partial x^\sigma} \quad (I)$$

we have

$$0 = \frac{\partial^2 x'^\lambda}{\partial x^\mu \partial x^\nu} \eta_{\lambda\rho} \frac{\partial x'^\rho}{\partial x^\sigma} + \frac{\partial x'^\lambda}{\partial x^\nu} \eta_{\lambda\rho} \frac{\partial^2 x'^\rho}{\partial x^\sigma \partial x^\mu} \quad (II)$$

$$0 = \frac{\partial^2 x'^\lambda}{\partial x^\nu \partial x^\sigma} \eta_{\lambda\rho} \frac{\partial x'^\rho}{\partial x^\mu} + \frac{\partial x'^\lambda}{\partial x^\sigma} \eta_{\lambda\rho} \frac{\partial^2 x'^\rho}{\partial x^\mu \partial x^\nu} \quad (III)$$

$$0 = I - II + III = 2 \frac{\partial x'^\lambda}{\partial x^\mu} \eta_{\lambda\rho} \frac{\partial^2 x'^\rho}{\partial x^\nu \partial x^\sigma}$$

and as $x \rightarrow x'$ is a diffeomorphisms and thus invertible

$$\frac{\partial^2 x'^\rho}{\partial x^\nu \partial x^\sigma} = 0$$

which implies x' linear functions of the x .

References

- [1] R.F. Streater and A.S. Wightman, “PCT, spin and statistics and all that” W.A.Benjamin, New York, 1964

1.3 $SL(2, C)$ is connected

$A \in SL(2, C)$ can be written as $A = a\sigma_0 + \mathbf{b} \cdot \boldsymbol{\sigma}$ with a complex number and \mathbf{b} complex vector related by $\det A \equiv a^2 - \mathbf{b}^2 = 1$. We have $A^{-1} = a\sigma_0 - \mathbf{b} \cdot \boldsymbol{\sigma}$. To see the connectedness of $SL(2, C)$ bring \mathbf{b} to 0 avoiding $\mathbf{b}^2 = -1$. a is constrained to move by continuity. At the end we have $a = \pm 1$. If $a = -1$ bring the element to the identity by

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \in SL(2, C).$$

1.4 $SL(2, C)$ is the double covering of the restricted Lorentz group

Let $B = \sigma_\mu x^\mu$ with σ_0 the identity and σ_k the Pauli matrices. B is hermitean iff x^μ is real. $\det B = -x^T \eta x$. If $A \in SL(2, C)$

$$ABA^+ = B' = \sigma_\mu x'^\mu = \sigma_\mu \Lambda^\mu{}_\nu x^\nu \quad (1.7)$$

defines a Lorentz transformation; in fact B' is hermitean and $\det B' = \det B$. If A_1 induces Λ_1 and A_2 induces Λ_2 , $A_2 A_1$ induces $\Lambda_2 \Lambda_1$.

If A is unitary i.e. an element of $SU(2)$, Λ is a rotation. If A is hermitean Λ is a boost e.g.

$$\cosh \frac{\phi}{2} + \sigma_3 \sinh \frac{\phi}{2}$$

induces the Lorentz boost

$$x'^0 = \cosh \phi x^0 + \sinh \phi x^3, \quad x'^3 = \sinh \phi x^0 + \cosh \phi x^3, \quad x'^1 = x^1, \quad x'^2 = x^2$$

Thus Λ covers all the restricted Lorentz group. It is a double covering. In fact if for all hermitean B

$$ABA^+ = CBC^+$$

then

$$C^{-1}AB = BC^+(A^+)^{-1}$$

Putting $B = 1$ we have

$$C^{-1}A = C^+(A^+)^{-1}$$

and then $C^{-1}A$ has to commute with all hermitean matrices and thus with *all* 2×2 matrices. By Schur lemma $C^{-1}A = \alpha$. But $SL(2, C)$ tells us $\alpha^2 = 1$ thus $\alpha = \pm 1$.

Thus due to the discontinuity in the determinant and in $\Lambda^0{}_0$, A cannot describe elements of L_-^\uparrow , L_+^\downarrow , L_-^\downarrow and thus $SL(2, C)$ is exactly the double covering of L_+^\uparrow .

1.5 The restricted Lorentz group is simple

Here we follow Wigner [1] writing explicitly the transformations that in Wigner paper are described by words.

To start: the rotation group is simple.

In fact suppose \mathcal{G} to be an invariant subgroup of $SO(3)$ and $g \in \mathcal{G}$, $g \neq I$ described by $A \in SU(2)$

$$A = \cos \alpha/2 - i\mathbf{n} \cdot \boldsymbol{\sigma} \sin \alpha/2; \quad \alpha \neq 0, \quad \alpha \neq \pm\pi.$$

Then with $U \in SU(2)$ representing a rotation by an angle β around the axis \mathbf{k}

$$UAU^+ = UAU^{-1}$$

induces a new transformation belonging to \mathcal{G} i.e. the rotations induced by

$$B = \cos \alpha/2 - i\mathbf{m} \cdot \boldsymbol{\sigma} \sin \alpha/2$$

with \mathbf{m} the rotation of the unit vector \mathbf{n} by the angle β around the axis \mathbf{k} . Thus to \mathcal{G} belong all rotations by α around any axis. BA also induces a rotation belonging to \mathcal{G}

$$BA = \cos^2 \alpha/2 - \mathbf{m} \cdot \mathbf{n} \sin^2 \alpha/2$$

$$-i(\mathbf{m} + \mathbf{n}) \cdot \boldsymbol{\sigma} \sin \alpha/2 \cos \alpha/2 - i\mathbf{m} \wedge \mathbf{n} \cdot \boldsymbol{\sigma} \sin^2 \alpha/2$$

which is a rotation by an angle γ with $\cos \gamma/2 = \cos^2 \alpha/2 - \mathbf{m} \cdot \mathbf{n} \sin^2 \alpha/2$ and thus we have rotations of any angle γ with $0 < \gamma < 2\alpha$ and from the above results rotations by γ around any axis. Composing such rotations we have rotations by any angle around any axis and thus the full rotation group. On the other hand $SU(2)$ is not simple due to the presence of the invariant subgroup $I, -I$.

We come now to the Lorentz group. Given an element $\Lambda \neq I$ of the restricted Lorentz group belonging to the invariant subgroup \mathcal{G} let A be a representative of Λ in $SL(2, C)$ (the other is $-A$). By means of a similitude transformation SAS^{-1} it is possible to reduce it (Jordan reduction) to the one of the two forms

$$\begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \quad \text{with} \quad \rho^2 \neq 1 \quad \text{or} \quad \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$$

with $S \in SL(2, C)$ and thus as $A \neq \pm I$, \mathcal{G} must contain the transformations induced by one of the two above elements and also their inverses. In the first case by taking the conjugation with

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

we obtain

$$\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \rho & b(\rho - \rho^{-1}) \\ 0 & \rho^{-1} \end{pmatrix}$$

and thus

$$\begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix} \begin{pmatrix} \rho & x \\ 0 & \rho^{-1} \end{pmatrix} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

for any y ; taking the conjugation with $i\sigma_1$ we also obtain

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}.$$

In the second case by conjugation with elements

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \in SL(2, C)$$

we obtain the same matrices where 1 has to be substituted with $k = \pm 1$. Then keeping in mind that \mathcal{G} is a group

$$\begin{pmatrix} k & i \\ 0 & k \end{pmatrix} \begin{pmatrix} k & 0 \\ i & k \end{pmatrix} \begin{pmatrix} k & i \\ 0 & k \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \Rightarrow X_\pi \in \mathcal{G}.$$

X_π is the rotation by π around x . Thus due to the previous theorem all rotations belong to \mathcal{G} . Then denoting by Z a boost along z and keeping in mind that $X_\pi Z^{-1} X_\pi = Z$ we have

$$\mathcal{G} \ni ZX_\pi Z^{-1} \cdot X_\pi = Z \cdot X_\pi Z^{-1} X_\pi = Z^2$$

and as any boost along z can be written as Z^2 , all boosts along z belong to \mathcal{G} and combining with rotations all elements of the Lorentz group belong to \mathcal{G} . We conclude that the restricted Lorentz group is simple. On the other hand $SL(2, C)$ is not simple due to the presence of the invariant subgroup $I, -I$.

References

- [1] E.P. Wigner, “On unitary representations of the inhomogeneous Lorentz group” *Ann. Mathematics*, **40** (1939) 149, pag. 167.

1.6 $SL(2, C)$ is simply connected

Consider a closed contour of $\mathbf{b}(\xi)$ in C^3 and a corresponding contour $a(\xi)$ with the restriction a continuous and $a^2 - \mathbf{b}^2 = 1$, $0 \leq \xi \leq 1$ and $a(1) = a(0)$ (see Figure 1.1). Look at the closed contour in the complex \mathbf{b}^2 plane. If such a contour does not enclose the point -1 we can deform it to a point e.g. by $\mathbf{b}(\xi, \lambda) = \lambda \mathbf{b}(\xi)$, $1 \geq \lambda \geq 0$. $a(\xi, \lambda) = \sqrt{1 + \mathbf{b}(\xi, \lambda)^2}$ is determined by continuity. At the end for $\lambda = 0$ we have $\mathbf{b} \equiv 0$ and $a \equiv \pm 1$ where

the $+$ or $-$ depends on the original determination of the square root for $\lambda = 1$ and we succeeded in shrinking the contour to a point (see Figure 1.2).

If the contour in the complex \mathbf{b}^2 plane encloses the point -1 it has to do it an even number of times (see Figure 1.3); otherwise $a(\xi)$ does not follow a closed contour i.e. $a(1) = -a(0)$. In this case perform the same shrinking as in the previous case and stop when the contour touches the point -1 i.e. when $\lambda_0^2 \mathbf{b}^2(\xi_0) = -1$ (see Figure 1.4). Then continue the contraction as follows

$$\mathbf{b}(\xi, \rho) = \lambda_0 \mathbf{b}(\xi_0) + [\lambda_0 \mathbf{b}(\xi) - \lambda_0 \mathbf{b}(\xi_0)]\rho$$

with ρ varying from 1 to 0. We have for all ρ , $\mathbf{b}(\xi_0, \rho) \equiv \lambda_0 \mathbf{b}(\xi_0)$ and thus $a(\xi_0, \rho) \equiv 0$. For $\rho = 0$ we have that the contour has shrunk to the point $\lambda_0 \mathbf{b}(\xi_0)$ and $a = 0$.

On the other hand L_+^\uparrow is not simply connected. In fact consider a path $A(\xi)$ in $SL(2, C)$ for $0 \leq \xi \leq 1$ such that $A(0) = I$ and $A(1) = -I$. To such a path there corresponds a closed path in L_+^\uparrow . However such a path in L_+^\uparrow cannot be shrunk to a point because the corresponding path in $SL(2, C)$ is constrained to have $A(0) = -A(1)$. $SL(2, C)$ is the universal covering of L_+^\uparrow .

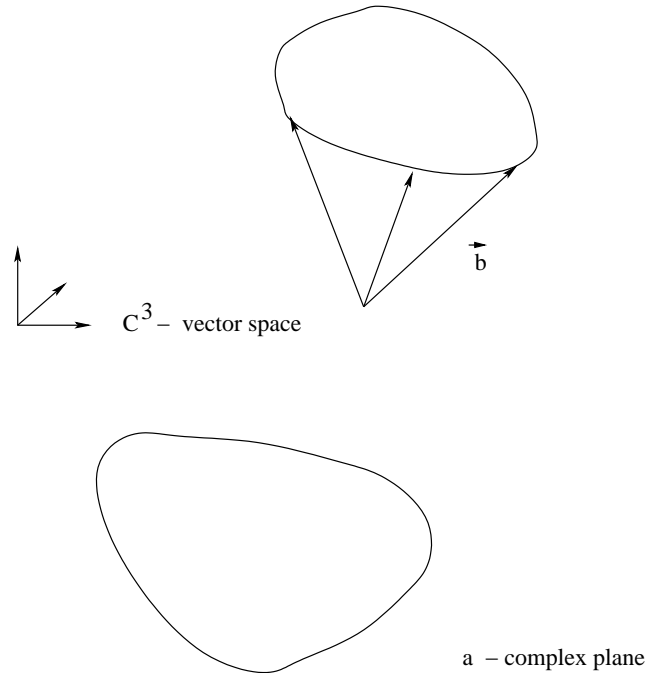


Figure 1.1: Contour in C^3 and in the a complex plane

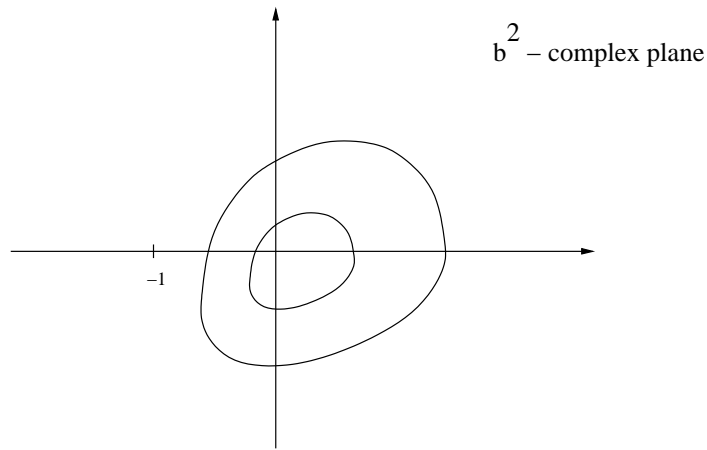
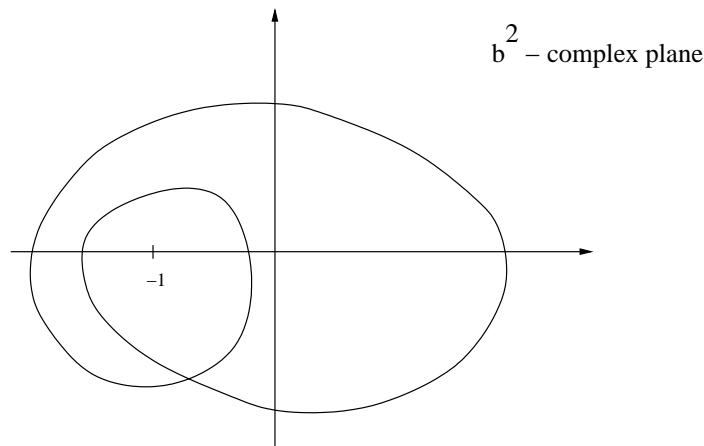
Figure 1.2: Contour of b^2 

Figure 1.3: First deformation

1.7 Little groups

These are the subgroups of the restricted Lorentz group which leave unchanged a given four vector.

1. Time like four vector: we can always choose $(1, 0, 0, 0)$ gives $SU(2)$ the group of rotations and vice versa. (3 parameter group). In fact if

$$A\sigma_0A^+ = \sigma_0 \equiv I$$

we have that A is unitary. It classifies the particle of non zero mass.

2. Light-like four vector: we can always choose $(1, 0, 0, 1)$. It gives the little group of the

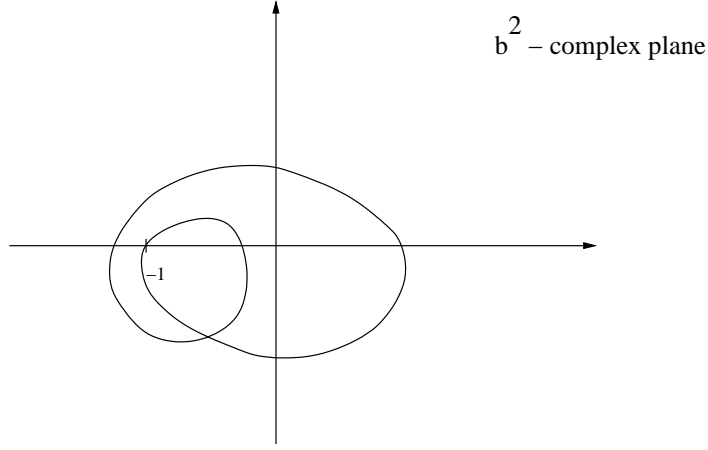


Figure 1.4: Second deformation

photon. In fact

$$A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A^+ = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

gives

$$A = \begin{pmatrix} e^{i\alpha/2} & b \\ 0 & e^{-i\alpha/2} \end{pmatrix} \in E(2)$$

$E(2)$ being the inhomogeneous two dimensional euclidean group. In fact defined

$$T_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$$

we have

$$T_x T_y = T_{x+y}, \quad R_\alpha R_\beta = R_{\alpha+\beta}, \quad T_x R_\alpha = R_\alpha T_{e^{-i\alpha}x}$$

which are the composition relations of $E(2)$. It classifies the particles of zero mass. Barring the existence of a continuous degree of freedom [1], the states of the particle are classified by the eigenvalues of the operator of rotation around the z axis which is the direction of the space part of the momentum i.e. the helicity. The irreducible representation of the little group is then given by a single state; if we introduce parity to this the state of opposite helicity state has to be added.

3. Space-like four vector: we can always choose $(0, 0, 1, 0)$. It gives the little group of the tachyon, given by $SL(2, R)$ as seen by

$$\sigma_2 \det A = A \sigma_2 A^T = \sigma_2 = A \sigma_2 A^+$$

from which $A^+ = A^T$. As one can expand the scattering amplitude at *fixed energy* (little group (1)) in representations of $SU(2)$ (spherical harmonics), similarly one can expand the scattering amplitude at *fixed momentum transfer* (little group (3)) in representations of $SL(2, R)$. Regge poles are the singularities of the Fourier transform of the scattering amplitude, in the unitary irreducible representations of $SL(2, R)$.

References

[1] [WeinbergQFT] I Chap. 2

1.8 There are no non-trivial unitary finite dimensional representations of the restricted Lorentz group

Consider

$$HPH^{-1}$$

with

$$H = \begin{pmatrix} c+s & 0 \\ 0 & c-s \end{pmatrix}$$

with $c = \cosh \frac{\phi}{2}$ and $s = \sinh \frac{\phi}{2}$ and

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{1.8}$$

we have

$$HPH^{-1} = \begin{pmatrix} 1 & e^\phi \\ 0 & 1 \end{pmatrix} \in SL(2, C).$$

For $e^\phi = n$ positive integer we have

$$\begin{pmatrix} c+s & 0 \\ 0 & c-s \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c-s & 0 \\ 0 & c+s \end{pmatrix} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n$$

thus we have matrices H_n and P such that

$$H_n P H_n^{-1} = P^n.$$

The induced transformations of L_+^\uparrow satisfy

$$L_n L_P L_n^{-1} = L_P^n.$$

For a unitary representation of L_+^\uparrow we have

$$V_n U V_n^+ = U^n$$

with V_n and U unitary operators. If the representation is finite dimensional (D dimensional) we have that U and U^n must have the same eigenvalues, for any integer n . Thus if λ_1 is an eigenvalue of U also $\lambda_1^2, \lambda_1^3, \dots, \lambda_1^{D+1}$ must be eigenvalues of U and as there are at most D different eigenvalues we have that $\lambda_1^{k_1} = 1$ for some $k_1 \leq D$ and thus $\lambda_1 = e^{2\pi i n_1 / k_1}$ and the same holds for all eigenvalues. As a result U^N for $N = k_1 k_2 \dots k_D$ has all eigenvalues equal to 1 and U^N being unitary we have $U^N = I$. Thus

$$U = V_N^+ U^N V_N = V_N^+ I V_N = I.$$

The kernel of the representation, i.e. the elements which are represented by I form an invariant subgroup of L_+^\uparrow . But the restricted Lorentz group is simple and as to P there corresponds an element of L_+^\uparrow different from the identity, the kernel of the representation is the whole L_+^\uparrow and thus the representation is trivial.

The result holds also for the finite dimensional unitary representations of the universal covering group of L_+^\uparrow i.e. $SL(2, C)$. In fact the only invariant subgroup of $SL(2, C)$ is $(I, -I)$ and P given by (1.8) does not belong to this subgroup.

References

[1] E.P. Wigner, Ann. Mathematics, "On unitary representations of the inhomogeneous Lorentz group" **40** (1939) 149, pag. 167.

1.9 The conjugate representation

A^+, A^{-1}, A^T are not representations. $(A^+)^T \equiv A^*$ is a representation, called the conjugate representation. $(A^+)^{-1}$ is also a representation.

In $SU(2)$ the conjugate representation is equivalent to the fundamental representation A as seen from

$$U = a - i\boldsymbol{\sigma} \cdot \mathbf{b}; \quad a, \mathbf{b} \text{ real and } a^2 + \mathbf{b}^2 = 1.$$

$$U^* = \sigma_2 U \sigma_2.$$

This is a peculiarity of $SU(2)$. Moreover for unitary representations $(U^+)^{-1} = U$. For $SL(2, C)$, A^* is inequivalent to A as seen by taking the trace e.g. of $\begin{pmatrix} i & -2 \\ 1 & i \end{pmatrix}$. On the other hand $(A^+)^{-1}$ is equivalent to A^* as seen by the $\det A = 1$ relation written as

$$\sigma_2 = A^T \sigma_2 A$$

which gives

$$\sigma_2 = A^+ \sigma_2 A^*; \quad (A^+)^{-1} = \sigma_2 A^* \sigma_2.$$

Let p^μ be a four vector (in the following $p_\mu = -i\hbar \frac{\partial}{\partial x^\mu}$, $p^\mu = \eta^{\mu\nu} p_\nu$). We have

$$A \sigma_\mu p^\mu A^+ = \sigma_\mu p'^\mu = \sigma_\mu \Lambda^\mu{}_\nu p^\nu$$

and also

$$A^* \sigma_\mu^* p^\mu A^T = \sigma_\mu^* p'^\mu = \sigma_\mu^* \Lambda^\mu{}_\nu p^\nu$$

equivalent to

$$\tilde{A} \tilde{\sigma}_\mu p^\mu \tilde{A}^+ = \tilde{\sigma}_\mu \Lambda^\mu{}_\nu p^\nu$$

with $\tilde{A} \equiv \sigma_2 A^* \sigma_2 = (A^+)^{-1}$ and $\tilde{\sigma}_\mu = \sigma_2 \sigma_\mu^* \sigma_2$, i.e. $\tilde{\sigma}_\mu = (\sigma_0, -\boldsymbol{\sigma})$. Being the representation \tilde{A} more elegant we shall use it instead of A^* .

It is not too difficult to prove that A and \tilde{A} are the only inequivalent two-dimensional continuous representations of $SL(2, C)$.

1.10 Relativistically invariant field equations

By this we mean linear differential equations invariant under space time translations i.e. whose coefficients do not depend on x and invariant under the local linear transformation

$$\psi'(x') = S(\Lambda_1) \psi(x), \quad x' = \Lambda_1 x$$

$S(\Lambda_1)$ being a linear transformation. They can always be reduced to first order differential equations. As

$$\psi''(x'') = S(\Lambda_2) \psi'(x') = S(\Lambda_2) S(\Lambda_1) \psi(x) = S(\Lambda_2 \Lambda_1) \psi(x)$$

$S(\Lambda)$ has to be a representation of the Lorentz group (or better of its universal covering $SL(2, C)$).

Simplest cases which will be relevant in the future.

References

[1] G.Ya. Lyubarskii, "The applications of group theory in physics" Pergamon Press, New York 1960, Chap. XVI.

1.11 The Klein-Gordon equation

$$\phi'(x') = \phi(x)$$

In the following we shall use the notation $p_\mu = -i\hbar \frac{\partial}{\partial x^\mu}$ and $p^\mu = \eta^{\mu\nu} p_\nu$. The invariant Klein-Gordon equation is

$$-p_\mu p^\mu \phi(x) = m^2 c^2 \phi(x)$$

i.e.

$$\partial_\mu \partial^\mu \phi(x) = \frac{m^2 c^2}{\hbar^2} \phi(x).$$

The Klein-Gordon equation can be derived from the invariant action

$$S = \int d^4x (\eta^{\mu\nu} \partial_\mu \phi^+ \partial_\nu \phi + \frac{m^2 c^2}{\hbar^2} \phi^+ \phi)$$

1.12 The Weyl equation

We look for an equation whose field is represented by a two component vector which transforms accordingly to the $(0, 1/2)$ representation

$$\psi'(x') = \tilde{A} \psi(x)$$

where $A \sigma_\mu x^\mu A^+ = \sigma_\mu x'^\mu$ equivalent to $\tilde{A} \tilde{\sigma}_\mu x^\mu \tilde{A}^+ = \tilde{\sigma}_\mu x'^\mu$.

Such an equation is given by

$$\sigma_\mu p^\mu \psi(x) = 0. \tag{1.9}$$

In fact if (1.9) holds we have

$$\sigma_\mu p'^\mu \tilde{A} \psi(x) = \sigma_\mu p'^\mu \psi'(x') = 0 \tag{1.10}$$

as seen by rewriting (1.9) as

$$0 = A \sigma_\mu p^\mu A^+ \tilde{A} \psi = \sigma_\mu p'^\mu \tilde{A} \psi.$$

Multiplying on the left by $\tilde{\sigma}_\mu p^\mu$ we have

$$-p_\mu p^\mu \psi(x) = 0$$

i.e. Weyl equation describes particles of zero mass.

It is not possible to add to it a mass term

$$\sigma_\mu p^\mu \psi(x) = M \psi(x) \tag{1.11}$$

M being a 2×2 matrix, without violating Lorentz invariance. In fact if $\psi(x) \in (0, 1/2)$ we have $\sigma_\mu p^\mu \psi(x) \in (1/2, 0)$ because

$$\sigma_\mu p'^\mu \psi'(x') = A \sigma_\mu p^\mu A^+ \tilde{\psi}(x) = A \sigma_\mu p^\mu \psi(x)$$

from which

$$M\psi(x) = A^{-1} M \tilde{A} \psi(x); \quad \forall A \in SL(2, C). \quad (1.12)$$

or

$$M\psi(x) = A^+ M A \psi(x); \quad \forall A \in SL(2, C). \quad (1.13)$$

Choose a point x such that $\psi(x) \neq 0$ and replacing A with AU , $A \in SL(2, C)$, $U \in SU(2)$ we have

$$M\psi(x) = U^+ A^+ M A U \psi(x) .$$

Taking the scalar product

$$(U\psi(x), A^+ M A U \psi(x)) = (\psi(x), M\psi(x)) \equiv c(\psi(x), \psi(x)) = c(U\psi(x), U\psi(x))$$

for all $U \in SU(2)$ and as $U\psi(x)$ covers the whole two dimensional space we have $A^+ M A = cI$, which has to hold for all $A \in SL(2, C)$. For $A = I$ we have $M = cI$ so that $cI = cA^+ A$ for all $A \in SL(2, C)$, which implies it implies $c = 0$.

Going over to the Fourier transform we have

$$\sigma_\mu p^\mu \psi_f(p^0, \mathbf{p}) = 0, \quad \text{with} \quad (p^0)^2 - \mathbf{p}^2 = 0. \quad (1.14)$$

Equation (1.9) is not invariant under parity. In fact let us look for an invertible matrix P such that

$$\psi'(x') = P\psi(x) \quad \text{with} \quad x' = (x^0, -\mathbf{x})$$

and such that

$$\sigma_\mu p'^\mu \psi'(x') = 0$$

i.e.

$$(\sigma_0 p^0 - \boldsymbol{\sigma} \cdot \mathbf{p}) P \psi(x) = 0.$$

Going over to Fourier transform we have

$$(\sigma_0 p^0 + \boldsymbol{\sigma} \cdot \mathbf{p}) \psi_f(p^0, \mathbf{p}) = 0 \quad \text{and} \quad (\sigma_0 p^0 - \boldsymbol{\sigma} \cdot \mathbf{p}) P \psi_f(p^0, \mathbf{p}) = 0. \quad (1.15)$$

i.e. we must have

$$-\boldsymbol{\sigma} \cdot \mathbf{p} \psi_f(p^0, \mathbf{p}) = P^{-1} \boldsymbol{\sigma} \cdot \mathbf{p} P \psi_f(p^0, \mathbf{p}) .$$

Taking $\mathbf{p} \rightarrow -\mathbf{p}$ we have also

$$-\boldsymbol{\sigma} \cdot \mathbf{p} \psi_f(p^0, -\mathbf{p}) = P^{-1} \boldsymbol{\sigma} \cdot \mathbf{p} P \psi_f(p^0, -\mathbf{p}).$$

$\psi_f(p^0, \mathbf{p})$ and $\psi_f(p^0, -\mathbf{p})$ span the two dimensional space otherwise from $\psi_f(p^0, -\mathbf{p}) = \alpha \psi_f(p^0, \mathbf{p})$ it follows

$$\sigma_0 p^0 \psi_f(p^0, \mathbf{p}) = 0.$$

Thus

$$-\boldsymbol{\sigma} \cdot \mathbf{p} = P^{-1} \boldsymbol{\sigma} \cdot \mathbf{p} P, \quad \forall \mathbf{p}.$$

But such a matrix P does not exist as seen from

$$\{\boldsymbol{\sigma} \cdot \mathbf{p}, a\sigma_0 + \boldsymbol{\sigma} \cdot \mathbf{b}\} = 2a \boldsymbol{\sigma} \cdot \mathbf{p} + 2\mathbf{b} \cdot \mathbf{p}.$$

Similar results hold for a $\psi(x)$ which transforms as $\psi'(x') = A\psi(x)$ and obeys $\tilde{\sigma}_\mu p^\mu \psi(x) = 0$. Weyl equation can be derived from the invariant action

$$S = \int d^4x \psi^\dagger(x) \sigma_\mu (-i\partial^\mu) \psi(x)$$

Such an action is hermitean as seen by taking the hermitean conjugate and integrating by parts.

1.13 The Majorana mass

It is possible to give a mass to a two component field transforming according to the $(1/2, 0)$ representation of the restricted Lorentz group as follows (similar considerations can be done for the $(0, 1/2)$ representation). Let us consider the equation

$$\tilde{\sigma}_\mu p^\mu \phi(x) = -imc \sigma_2 \phi^*(x). \quad (1.16)$$

We saw already that if $\phi'(x') = A\phi(x)$ then $\tilde{\sigma}_\mu p'^\mu \phi'(x') = \tilde{A} \tilde{\sigma}_\mu p^\mu \phi(x) \in (0, 1/2)$. Thus we have simply to prove that $\sigma_2 \phi^*(x) \in (0, 1/2)$. In fact

$$\sigma_2 (A\phi(x))^* = \sigma_2 A^* \sigma_2 \sigma_2 \phi^*(x) = \tilde{A} \sigma_2 \phi^*(x). \quad (1.17)$$

Multiplying Eq.(1.16) on the left by $\sigma_\mu p^\mu$ and using again Eq.(1.16) we obtain, taking into account that $(p^\mu)^* = -p^\mu$,

$$-p^2 \phi(x) = m^2 c^2 \phi(x)$$

proving that both component of the spinor $\phi(x)$ satisfy the massive Klein-Gordon equation.

Equation (1.16) can be derived by varying the action

$$S = \int d^4x [\phi^+(x) i\tilde{\sigma}_\mu \partial^\mu \phi(x) + \frac{imc}{2} (\phi^T \sigma_2 \phi(x) - \phi^+ \sigma_2 \phi^*)] \quad (1.18)$$

Again integrating by parts it is proved that the action S is hermitean.

The action (1.18) is invariant under the parity transformation

$$\phi'(x') = P\phi(x) = \sigma_2 \phi^*(x), \quad \text{with} \quad x' = (x^0, -\mathbf{x}) \quad (1.19)$$

and thus also the equations of motion are invariant under such a transformation.

1.14 The Dirac equation

Consider again a $\psi(x)$ which transforms according to

$$\psi'(x') = \tilde{A}\psi(x).$$

Posing

$$\sigma_\mu p^\mu \psi(x) = \phi(x) \quad (1.20)$$

we have that $\phi(x)$ transforms like

$$\phi'(x') = A\phi(x).$$

In fact

$$A\sigma_\mu p^\mu A^+ \tilde{A}\psi(x) = A\phi(x)$$

i.e.

$$\sigma_\mu p'^\mu \psi'(x') = \phi'(x')$$

We have now four components and we must supply other two equations, invariant under Lorentz transformations. These are given by

$$\tilde{\sigma}_\mu p^\mu \phi(x) = \kappa\psi(x)$$

which is invariant as

$$\tilde{A}\tilde{\sigma}_\mu p^\mu \tilde{A}^+ A\phi(x) = \kappa\tilde{A}\psi(x)$$

i.e.

$$\sigma_\mu p'^\mu \phi'(x') = \kappa\psi'(x').$$

Multiplying now (1.20) on the left by $\tilde{\sigma}_\mu p^\mu$ we have

$$-p_\mu p^\mu \psi(x) = \kappa\psi(x)$$

i.e. the two equations

$$\begin{aligned}\sigma_\mu p^\mu \psi(x) &= \phi(x) \\ \tilde{\sigma}_\mu p^\mu \phi(x) &= \kappa \psi(x)\end{aligned}$$

describe particles of mass m such that $m^2 c^2 = \kappa$. It is better to distribute in more symmetrical way the κ

$$\begin{aligned}\sigma_\mu p^\mu \psi(x) &= mc \phi(x) \\ \tilde{\sigma}_\mu p^\mu \phi(x) &= mc \psi(x).\end{aligned}$$

Thus the

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \phi(x) \end{pmatrix}$$

transforms according to the $(1/2, 0) \oplus (0, 1/2)$ representation. From $\tilde{\sigma} = -\sigma$ it is immediately seen that the parity transformation is

$$\Psi'(x') = P\Psi(x), \quad \text{with } x' = (x^0, -\mathbf{x})$$

and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can also write

$$\begin{aligned}\sigma^\mu p_\mu \psi(x) &= mc \phi(x) \\ \tilde{\sigma}^\mu p_\mu \phi(x) &= mc \psi(x)\end{aligned}$$

with $\sigma^\mu = \eta^{\mu\nu} \sigma_\nu$ or

$$\left[- \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} p_0 + \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} p_k \right] \Psi(x) = mc \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi(x)$$

which can also be written as

$$(\gamma^\mu \partial_\mu + \frac{mc}{\hbar}) \Psi(x) = 0$$

with

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \quad \text{antihermitean} \quad (1.21)$$

$$\gamma^j = \begin{pmatrix} 0 & -i\sigma^j \\ i\sigma^j & 0 \end{pmatrix} \quad \text{hermitean.} \quad (1.22)$$

These are the same γ matrices as adopted in [1]. Notice that defined

$$\Sigma_\mu = \begin{pmatrix} \tilde{\sigma}_\mu & 0 \\ 0 & \sigma_\mu \end{pmatrix} \quad (1.23)$$

and

$$\mathcal{A} = \begin{pmatrix} \tilde{A} & 0 \\ 0 & A \end{pmatrix} \quad (1.24)$$

we have obviously

$$\mathcal{A}\Sigma_\mu\mathcal{A}^+ = \Sigma_\nu\Lambda^\nu{}_\mu.$$

Taking into account that $\gamma_\mu = i\Sigma_\mu P$ we have

$$\mathcal{A}\Sigma_\mu(iP)(-iP)\mathcal{A}^+(iP) = \Sigma_\nu(iP)\Lambda^\nu{}_\mu$$

i.e.

$$\mathcal{A}\gamma_\mu\mathcal{A}^{-1} = \gamma_\nu\Lambda^\nu{}_\mu. \quad (1.25)$$

In addition it will be useful to define $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ which in the same representation assumes the form

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{hermitean.}$$

In the following also the symbol $\not{\partial} = \gamma^\mu\partial_\mu$ will be used.

References

[1] [WeinbergQFT] I pag. 216.

1.15 The Rarita- Schwinger equation

In developing supergravity we shall need the field theoretical description of spin 3/2 particles. These are described by a vector- spinor ψ^μ . We recall that a 4-vector V^μ has spin content $0 \oplus 1$ because $\partial_\mu V^\mu$ is a scalar. On the other hand a spinor describes a particle of spin 1/2. As a consequence the spin content of ψ^μ is $(0 \oplus 1) \otimes 1/2 = 1/2 \oplus 1/2 \oplus 3/2$ and thus we have two unwanted spin 1/2 particles. These can be eliminated by imposing the following supplementary conditions $\gamma^\lambda\psi_\lambda = 0$ and $\partial_\lambda\psi^\lambda = 0$, which set to zero two spinors. For massive particles both conditions are contained in the Rarita- Schwinger equation, which in addition contains the dynamical equations for the ψ_λ . Such equation is

$$\varepsilon^{\mu\nu\rho\sigma}\gamma_\nu\gamma_5(\partial_\rho + \frac{m}{2}\gamma_\rho)\psi_\sigma = 0 \quad (1.26)$$

where

$$\varepsilon_{0123} = 1, \quad \varepsilon^{0123} = -1.$$

The only properties of the γ^μ we shall use are the Clifford algebra

$$\{\gamma^\mu, \gamma^\lambda\} = 2\eta^{\mu\lambda}$$

and the ensuing relation $\{\gamma^\lambda, \gamma_5\} = 0$.

By taking the divergence of Eq.(1.26) we have, for $m \neq 0$

$$\varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_5 \gamma_\rho \partial_\mu \psi_\sigma = 0. \quad (1.27)$$

But

$$\varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_5 \gamma_\rho = -i(\gamma^\mu \gamma^\sigma - \gamma^\sigma \gamma^\mu)$$

and thus

$$(\gamma^\mu \gamma^\sigma - \gamma^\sigma \gamma^\mu) \partial_\mu \psi_\sigma = 0.$$

Rewriting the above as

$$(2\gamma^\mu \gamma^\sigma - 2\eta^{\mu\sigma}) \partial_\mu \psi_\sigma = 0$$

we obtain

$$\not{\partial}(\gamma^\sigma \psi_\sigma) - \partial_\mu \psi^\mu = 0 \quad (1.28)$$

with $\not{\partial} = \gamma^\mu \partial_\mu$. We contract now Eq.(1.26) with γ_μ to obtain

$$\varepsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_5 \partial_\rho \psi_\sigma + \frac{m}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_5 \gamma_\rho \psi_\sigma = 0.$$

The first term vanishes due to Eq.(1.27) and thus we have

$$0 = (\gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho) \gamma_\rho \psi_\sigma = -2(\gamma^\sigma \gamma^\rho - \eta^{\sigma\rho}) \gamma_\rho \psi_\sigma = -6\gamma^\sigma \psi_\sigma = 0.$$

Due to Eq.(1.28) it implies also $\partial_\mu \psi^\mu = 0$. Now we prove that each component of the Rarita- Schwinger field satisfies the Dirac equation. The term of the Rarita- Schwinger equation proportional to m can be written as

$$i\frac{m}{2}(\gamma^\mu \gamma^\sigma - \gamma^\sigma \gamma^\mu) \psi_\sigma = im(\gamma^\mu \gamma^\sigma - \eta^{\mu\sigma}) \psi_\sigma = -im\psi^\mu.$$

With regard to the derivative term, it can be rewritten as

$$\begin{aligned} \varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_5 \partial_\rho \psi_\sigma &= \varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_5 \delta_\sigma^\lambda \partial_\rho \psi_\lambda = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_5 (\gamma^\lambda \gamma_\sigma + \gamma_\sigma \gamma^\lambda) \partial_\rho \psi_\lambda = \\ &= \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_5 \gamma^\lambda \gamma_\sigma \partial_\rho \psi_\lambda. \end{aligned}$$

But the following equality holds

$$\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_5 \gamma^\lambda \gamma_\sigma = i(\eta^{\rho\lambda} \gamma^\mu - \eta^{\mu\lambda} \gamma^\rho)$$

which substituted in the previous expression gives

$$-i\not{\partial} \psi^\mu.$$

Summing the two terms we have

$$\not{\partial}\psi^\mu + m\psi^\mu = 0.$$

Obtaining the constraints from the Rarita- Schwinger equation is similar to what happens in the massive vector theory in which the equations of motion are

$$\partial_\mu F^{\mu\nu} - m^2 A^\nu = 0,$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Taking the divergence of the above equation we have $\partial_\nu A^\nu = 0$. We come now to the massless Rarita- Schwinger equation

$$\varepsilon^{\mu\nu\rho\sigma}\gamma_\nu\gamma_5\partial_\rho\psi_\sigma = 0. \quad (1.29)$$

Such equation is invariant under the gauge transformation

$$\psi_\sigma \rightarrow \psi_\sigma + \partial_\sigma\eta$$

with η an arbitrary spinor. Contracting Eq.(1.29) with γ_μ gives

$$0 = (\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)\partial_\rho\psi_\sigma = 2(\gamma^\rho\gamma^\sigma - \eta^{\sigma\rho})\partial_\rho\psi_\sigma = 2\not{\partial}(\gamma^\sigma\psi_\sigma) - 2\partial_\sigma\psi^\sigma.$$

A useful gauge choice is $\gamma^\sigma\psi_\sigma = 0$, which induces $\partial_\sigma\psi^\sigma = 0$. Such a gauge is at least locally attainable because the equation

$$\gamma^\sigma\psi_\sigma + \not{\partial}\eta = 0$$

is solvable in η ; the inverse of $\not{\partial}$ is given by $\not{\partial}\partial^{-2}$.

1.16 Transformation properties of quantum fields

We saw that classically the transformation properties of (local) fields are

$$\psi'(x') = S(\Lambda)\psi(x), \quad \text{with} \quad x' = \Lambda x. \quad (1.30)$$

Eq.(1.30) can be rewritten as

$$\psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x).$$

This is the notation universally adopted. It would be more proper to write instead of $S(\Lambda)$, $S(A)$ where A is an element of $SL(2, C)$.

In the quantum treatment we shall adopt the Heisenberg picture where the state vector is given *sub specie aeternitatis* i.e. constant in time and unchanged under Lorentz transformations. The transformation of field is induced by a unitary transformation which is a representation of the Lorentz group, more properly of $SL(2, C)$ i.e. the one valued representation of the universal covering of the Lorentz group.

$$\psi'(x) = U^+ \psi(x) U = S(\Lambda) \psi(\Lambda^{-1}x). \quad (1.31)$$

In order to be a representation U has to stay on the right. In fact if

$$U_1^+ \psi(x) U_1 = S(\Lambda_1) \psi(\Lambda_1^{-1}x)$$

and

$$U_2^+ \psi(x) U_2 = S(\Lambda_2) \psi(\Lambda_2^{-1}x)$$

we have

$$\begin{aligned} U_2^+ U_1^+ \psi(x) U_1 U_2 &= S(\Lambda_1) U_2^+ \psi(\Lambda_1^{-1}x) U_2 = S(\Lambda_1) S(\Lambda_2) \psi(\Lambda_2^{-1} \Lambda_1^{-1}x) = \\ &= S(\Lambda_1) S(\Lambda_2) \psi(\Lambda_2^{-1} \Lambda_1^{-1}x) = S(\Lambda_1 \Lambda_2) \psi((\Lambda_1 \Lambda_2)^{-1}x). \end{aligned}$$

Transformation (1.31) can also be rewritten as

$$U^+ \psi(\Lambda x) U = S(\Lambda) \psi(x)$$

or

$$U \psi(x) U^{-1} = S(\Lambda)^{-1} \psi(\Lambda x)$$

which is the form given in [1]

$$\begin{aligned} U_2 U_1 \psi(x) U_1^+ U_2^+ &= S(\Lambda_1)^{-1} U_2 \psi(\Lambda_1 x) U_2^+ = S(\Lambda_1)^{-1} S(\Lambda_2)^{-1} \psi(\Lambda_2 \Lambda_1 x) = \\ &= S((\Lambda_2 \Lambda_1)^{-1}) \psi(\Lambda_2 \Lambda_1 x). \end{aligned}$$

One could work also in the Schroedinger picture in which the fields in different reference frames are represented by the same operator $\psi(x)$ and the state vector changes according to $\Omega \rightarrow U\Omega$, U being a unitary representation of the Lorentz group. Then

$$(U\Omega, \psi(x)U\Omega) = S(\Lambda)(\Omega, \psi(\Lambda^{-1}x)\Omega) = (\Omega, U^+ \psi(x)U\Omega).$$

References

- [1] [WeinbergQFT] I p.192.

Chapter 2

Equivalence principle and the path to general relativity

2.1 Introduction

We saw that special relativity does not differentiate from classical mechanics with regard to the invariance under inertial transformations, except for the existence of a limit velocity. If a field A_μ exists whose equation of motion are

$$\partial_\mu F^{\mu\nu} = 0 \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

we have that such a limit velocity is actually reached by the signals described by the field $F^{\mu\nu}$.

One may attempt the inclusion of gravitational phenomena within the framework of special relativity. The simplest idea is to describe gravitational phenomena by means of a scalar field e.g. by changing the metric from $\eta_{\mu\nu}$ to $\eta_{\mu\nu}e^\phi(x)$ (Nordström theory). A scalar field provides an attractive force and if such a field is of zero mass one has the Newtonian behavior $1/r^2$ of the force at large distances. On the other hand such a theory does not forecast the correct value for the advance of the perihelion of Mercury and most important does not satisfy the equivalence principle as seen most clearly in the absence of the deflection of light [1]. Equivalence principle on material bodies is verified with great accuracy (Eötvös 10^{-9} ; Dicke 10^{-11} ; Shapiro et al. (free fall) 10^{-13} and more accurate experiments are under way).

The equivalence principle shows that the avenue of extending the invariance of the laws of physics to more general coordinate transformations than Lorentz transformations, is a promising path toward a theory of gravitation.

Beware that the principle of equivalence can hold only “ultralocally”, i.e. in the abstract case of point bodies; there are phenomena related to the gravitational field which do not vanish even when the size of the laboratory is shrunk to zero. For example a quadrupole formed by two masses m joined by a rod of length $2l$ posed in the gravitational field described by the acceleration $g > 0$ for which $\frac{\partial g}{\partial z} = k^2 \neq 0$ when aligned near the z axis is subject to oscillations of pulsation $\omega = k$, independent of l , i.e. a frequency which does not vanish for $l \rightarrow 0$.

Given a general metric $g_{\mu\nu}$ describing a gravitational field the principle of equivalence requires that given one event x , it must be possible to reduce by means of a congruence the metric in x to the Lorentzian form, i.e. a non singular real matrix A must exist such that

$$A^T g A = \eta \quad \text{where} \quad A^\mu{}_\nu = \frac{\partial x^\mu}{\partial y^\nu} \quad (2.1)$$

Thus the signature of g has to be $(3, 1)$ and as a consequence $\det g < 0$. Obviously given x there are infinite diffeomorphisms $x^\mu = x^\mu(y)$ which realize (2.1).

References

- [1] [HawkingEllis] Chap. 3.
- [2] A. Einstein, “Die Grundlage der allgemeinen Relativitätstheorie” Ann.Phys. 49 (1916). Translated in “The principle of relativity” Dover Pub. Inc.
- [3] [WeinbergGC] Chap. 3 par. 1, 2.

2.2 Motion of a particle in a gravitational field

In absence of gravitational fields we know that the action of a particle is given by

$$S = -mc \int_{x_1}^{x_2} \sqrt{-ds^2} = \int_{q_1 t_1}^{q_2 t_2} L(q, \dot{q}) dt$$

with

$$L(q, \dot{q}) = -mc \sqrt{c^2 - \dot{q}^i \dot{q}^j \eta_{ij}} .$$

From the variational principle of analytical mechanics we know that the correct action in presence of gravitational fields generated by a change of coordinates is

$$S = -mc \int_{x_1}^{x_2} \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \quad (2.2)$$

and this gives a coordinate independent formulation of the law of motion.

If we accept the idea that a gravitational field is described by a generic metric $g_{\mu\nu}$ the laws of motion have to be postulated, but the form Eq.(2.2) appears the most natural [1]. Thus we shall assume that also in presence of gravitational fields which cannot be removed over a finite region of space time by means of a coordinate transformation, the law of motion for a material body is given by Eq.(2.2).

The principle is to render maximum the length of the trajectory when keeping the initial and final events fixed. The principle of equivalence is then satisfied as we have reduced the problem of motion to a geometrical problem, where the mass, the shape, the substance of the body do not intervene. Thus one has a geometrization of the gravitational field and of the motion of particles under the influence of a gravitational field.

References

[1] A. Einstein, "Die Grundlage der allgemeinen Relativitätstheorie" Ann.Phys. 49 (1916) paragraphs 1, 2, 3, 4. Translated in "The principle of relativity" Dover Pub. Inc.

2.3 Hamiltonian of a particle in a gravitational field

The Lagrangian is

$$L = -mc\sqrt{-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = -mc\sqrt{-c^2g_{00} - g_{ij}\dot{q}^i\dot{q}^j - 2cg_{0i}\dot{q}^i}.$$

It is instrumental to write the metric in the ADM form (see section 7.1 for more details)

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$

We shall give a thorough interpretation of N and N^i in the future. The hypersurfaces $t = \text{const.}$ are space like i.e. the vector $(0, dx^i)$ has positive norm i.e. $h_{ij}dx^i dx^j$ is definite positive. h_{ij} is usually called the space metric even if it coincides with the space metric obtained by the coincidence method only for $N^i = 0$. In terms of N, h_{ij}, N^i the $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \begin{pmatrix} \frac{N^i h_{ij} N^j - N^2}{c^2} & \frac{h_{nl} N^l}{c} \\ \frac{h_{ml} N^l}{c} & h_{mn} \end{pmatrix} \quad \text{with inverse} \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^n}{N^2} \\ \frac{N^m}{N^2} & h^{mn} - \frac{N^m N^n}{N^2} \end{pmatrix}.$$

The Lagrangian becomes

$$L = -mc\sqrt{Z}$$

with

$$Z = N^2 - N^i h_{ij} N^j - h_{ij} \dot{q}^i \dot{q}^j - 2N^i h_{ij} \dot{q}^j.$$

We have

$$p_i = \frac{mc(\dot{q}_i + N_i)}{\sqrt{Z}} \quad (2.3)$$

where we have lowered the indices of \dot{q}^i and N^i with the metric h_{ij} . Keep in mind that the conjugate momenta are the p_i (lower indices).

We have the identity

$$p_i p^i = \frac{m^2 c^2 (-Z + N^2)}{Z}$$

or

$$Z = \frac{m^2 c^2 N^2}{p_i p^i + m^2 c^2}$$

which allows, using Eq.(2.3) to solve in the \dot{q}^i

$$\dot{q}_i + N_i = \frac{\sqrt{Z}}{mc} p_i$$

from which we have

$$\begin{aligned} H = p_i \dot{q}^i - L &= \frac{\sqrt{Z}}{mc} p_i p^i - p_i N^i + mc\sqrt{Z} = \\ &= \frac{\sqrt{Z}}{mc} (p_i p^i + m^2 c^2) - p_i N^i = N \sqrt{p_i h^{ij} p_j + m^2 c^2} - p_i N^i \end{aligned}$$

h^{ij} being the inverse of h_{ij} . N, N^i, h^{ij} are functions of t and q^i .

The relevant Hamilton equations are

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

We can now write the Hamilton-Jacobi equation for $S(q, q_0, t, t_0)$

$$N \sqrt{\partial_i S h^{ij} \partial_j S + m^2 c^2} - N^j \partial_j S = -\partial_t S.$$

Removing the square root by squaring we have

$$g^{\mu\nu} \partial_\mu S \partial_\nu S + m^2 c^2 = 0$$

which has a covariant form being S a scalar. If we denote by q_0^i the coordinates of the initial event, the relevant equation of motion are

$$\frac{\partial S}{\partial q_0^j} = \text{const}$$

which give implicitly q^j as a function of t . We shall see an application of such equation in computing the geodesic motion in a time independent metric.

For a “static” metric i.e. $N^i \equiv 0$ and small velocities we have

$$\dot{p}_i = -mc \frac{\partial N}{\partial q^i}.$$

Let us write for the static metric $-g_{00} = N^2/c^2 = 1 + 2\phi/c^2$ which for $N^2/c^2 \approx 1$ (weak gravitational field) gives $N = c + \phi/c$ and thus

$$\dot{p}_i = -m \frac{\partial \phi}{\partial q^i}$$

i.e. ϕ is the Newtonian gravitational potential.

Chapter 3

Manifolds

3.1 Introduction

The most important characteristics of a manifold are its dimension n and its degree of smoothness, given by the degree of smoothness of the transition functions. The three most important categories are TOP_n where the transition functions are simply continuous and they define the topological manifolds; PL_n where the transition functions are piece-wise linear and the $DIFF_n$ in which the transition functions are C^∞ . For the relations between them and the possibility and number of inequivalent ways of *smoothing* a manifold i.e. of extracting a compatible C^∞ atlas see [1,2,3].

In two dimensions one can consider holomorphic transition function giving rise to the concept of Riemann surface. Sometime real-analytic transition function in n dimensions, i.e. transition function given locally by convergent power expansions, are considered [4].

References

- [1] C.P. Rourke and B.J. Sanderson, "Introduction to Piecewise-linear topology", Springer Verlag, Berlin
- [2] M. Freedman and F. Quinn, "Topology of 4-manifolds", Princeton University Press, (1990)
- [3] C. Nash, "Differential topology and quantum field theory", Academic -Press, London (1991); Chap. I
- [4] S. Klainerman, "Recent results in mathematical GR" The 13th Marcel Grossmann meeting, vol I pag.93-104, World Scientific

3.2 Mappings, pull-back and push-forward

For any regular mapping ϕ between manifolds

$$\begin{array}{c} \phi \\ M \longrightarrow M' \end{array}$$

M and M' not necessarily of the same dimensions, it is possible to define the pull-back of a function f defined on M' , given by $f(\phi(p))$ and denoted by $\phi^*f(p)$. It is also possible to define the push-forward of a vector \mathbf{V} denoted by $\phi_*\mathbf{V}(p')$ as follows: Given a vector $\mathbf{V} \in T(p)$, $p \in M$, defined by the motion $\lambda(t)$ with $\lambda(0) = p$, one considers the vector in $T(p')$, $p' = \phi(p) \in M'$ defined by the motion on M' , $\phi(\lambda(t))$ again at $t = 0$. Such a vector which belongs to $T(p')$ i.e. to the tangent space of M' at p' will be called the push-forward of the vector $\mathbf{V}(p)$ defined at $T(p)$ and written as $\phi_*\mathbf{V}(p')$.

Using the push-forward of vectors, forms are naturally pulled back e.g. for the one-form w , ϕ^*w is defined by

$$\langle \mathbf{V}(p), \phi^*w(p) \rangle \equiv \langle \phi_*\mathbf{V}(p'), w(p') \rangle.$$

In the case of diffeomorphisms between manifolds the inverse of ϕ exists and it is possible to define the push-forward of forms and the pull-back of vectors. For vectors the pull-back $\phi^*\mathbf{V}(p)$ is defined by $(\phi^{-1})_*\mathbf{V}(p)$, where $p \in M$, while for forms the push-forward $\phi_*\omega(p')$ is defined by $(\phi^{-1})^*\omega(p')$, where $p' \in M'$. One verifies that $(\phi^{-1})_* = \phi^*$ and $(\phi^{-1})^* = \phi_*$. Beware that in the pull-back process, the argument of the result is the starting point of the mapping, while in the push-forward process it is the end point of the mapping.

This notation, which is the most useful, can be source of confusion in the case of the push-forward. For this reason it useful to define also the linear operator $\hat{\phi}$ from the space $T(p)$ to $T(p')$

$$\phi_*\mathbf{V}(p') = \hat{\phi}\mathbf{V}(p) . \quad (3.1)$$

For a product of two mappings

$$\begin{array}{ccccc} & \alpha & & \beta & \\ M & \longrightarrow & M' & \longrightarrow & M'' \end{array}$$

one has with $p' = \alpha(p)$, $p'' = \beta(p') = \beta \circ \alpha(p)$

$$(\beta \circ \alpha)_*\mathbf{V}(p'') = \hat{\beta}\alpha_*\mathbf{V}(p') = \hat{\beta}\hat{\alpha}\mathbf{V}(p).$$

If α and β are diffeomorphisms and thus invertible transformations we can write

$$(\beta \circ \alpha)_*\mathbf{V}(p'') = \hat{\beta}\alpha_*\mathbf{V}(\beta^{-1}p'') = \hat{\beta}\hat{\alpha}\mathbf{V}(\alpha^{-1}\beta^{-1}p'') .$$

References

- [1] [HawkingEllis] Chap. 2 [2.1,2.2,2.3,2.4,2.5,2.6,2.7].
 [2] [Wald] Chap. 2, Chap. 3, Appendix A, Appendix B.
 [3] H. Flanders, “Differential forms” Academic Press, New York (1963) par. Chap.III 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8

3.3 Vector fields and diffeomorphisms

We state now an important theorem [1]:

If \mathbf{X} is a C^∞ vector field with compact support (in particular if the manifold is compact) then there is a unique family of diffeomorphisms ϕ_t which for all $t \in \mathcal{R}$ satisfy the properties

- (1) ϕ_t is C^∞
- (2) $\phi_{s+t}(q) = \phi_s \circ \phi_t(q)$
- (3) $\mathbf{X}(q)$ is the tangent vector at $t = 0$ of the motion $\phi_t(q)$.
- (4) Every collection of diffeomorphisms ψ_t satisfying (3) must coincide with ϕ_t .

The outcome is that for compact manifolds there is a one to one correspondence between one parameter abelian groups of diffeomorphisms and C^∞ vector fields. Such a result will allow the definition of the Lie derivative.

If the vector field has a non compact support we can still obtain a local result by considering, given a point p , an open neighborhood of it U_p and working with the vector field $\rho(x)\mathbf{X}$ where $\rho(x)$ is a C^∞ function which is equal to 1 in a neighborhood of p and vanishes outside a compact support.

References

- [1] M. Spivak, “A comprehensive introduction to differential geometry I”, Publish or Perish, Inc. Boston (1970)

3.4 The Lie derivative

This is a concept related to a vector field i.e. a vector $\mathbf{X}(p)$ defined at each point of the manifold; it operates on functions, forms, vector and tensor fields, giving as a result functions, forms, vector and tensor fields of the same order.

In the following we shall denote by ϕ_t the abelian group of diffeomorphisms generated by the vector field \mathbf{X} .

Lie derivative of a function

$$L_{\mathbf{X}}f(p) = \lim_{t \rightarrow 0} \frac{1}{t} [\phi_t^* f(p) - f(p)] = X^i \partial_i f(p) \equiv X(f)(p)$$

Lie derivative of a form field

Forms ω are naturally pulled-back; the definition of the Lie derivative is

$$L_{\mathbf{X}}\omega(p) = \lim_{t \rightarrow 0} \frac{1}{t} [\phi_t^* \omega(p) - \omega(p)]$$

Lie derivative of a vector field

ϕ_t being a diffeomorphisms it has inverse $\phi_t^{-1} = \phi_{-t}$ and so the pull-back of a vector is well defined.

$$\begin{aligned} L_{\mathbf{X}}\mathbf{Y}(p) &= \lim_{t \rightarrow 0} \frac{1}{t} [\phi_t^* \mathbf{Y}(p) - \mathbf{Y}(p)] = \lim_{t \rightarrow 0} \frac{1}{t} [(\phi_t^{-1})_* \mathbf{Y}(p) - \mathbf{Y}(p)] = \\ &= \lim_{t \rightarrow 0} \frac{1}{t} [\phi_{-t}^* \mathbf{Y}(p) - \mathbf{Y}(p)] = \lim_{t \rightarrow 0} \frac{1}{t} [\mathbf{Y}(p) - \phi_{t*} \mathbf{Y}(p)]. \end{aligned} \quad (3.2)$$

3.5 Components of the Lie derivative

We write down now the components of the Lie derivatives of a form $\omega = \omega_k dx^k$. Denoting on a given chart the diffeomorphism ϕ_t by

$$x' = \phi(x, t), \quad \phi(x, 0) = x$$

the pull-back of ω is

$$\omega(\phi(x, t))_m = \frac{\partial \phi^m(x, t)}{\partial x^k} dx^k.$$

Taking the derivative with respect to t at $t = 0$ and recalling that

$$\left. \frac{\partial \phi^m(x, t)}{\partial t} \right|_{t=0} = X^m(x)$$

we obtain

$$L_X(\omega)_k = \frac{\partial \omega_k}{\partial x^m} X^m + \omega_m \frac{\partial X^m}{\partial x^k} \quad (3.3)$$

which easily generalizes to a Λ^n form

$$L_X(\omega)_{k_1\dots k_n} = \frac{\partial \omega_{k_1\dots k_n}}{\partial x^m} X^m + \omega_{mk_2\dots k_n} \frac{\partial X^m}{\partial x^{k_1}} + \dots + \omega_{k_1 k_2\dots m} \frac{\partial X^m}{\partial x^{k_n}}.$$

In computing the components of the Lie derivative of a vector one has to keep in mind that the argument of the push-forward is the end point. We shall use a chart with coordinates x and for clearness sake we shall write for the diffeomorphism

$$(\phi_t(p))^k = \phi^k(x, t)$$

and

$$\partial_m \phi^k(x, t) = \frac{\partial \phi^k(x, t)}{\partial x^m}.$$

Notice that $\phi^k(x, 0) = x^k$ and

$$\partial_m \phi^k(x, 0) = \delta_m^k.$$

We have

$$[\phi_{t*} \mathbf{Y}(x)]^k = Y^m(\phi(x, -t)) \partial_m \phi^k(\phi(x, -t), t)$$

and taking the derivative with respect to t , we have

$$\begin{aligned} \frac{d}{dt} [\phi_{t*} \mathbf{Y}(x)]^k &= \frac{d}{dt} [Y^m(\phi(x, -t)) \partial_m \phi^k(\phi(x, -t), t)] |_{t=0} = \\ &= -\partial_l Y^m(x) X^l(x) \delta_m^k + Y^m(x) \left(\frac{\partial}{\partial t} \delta_m^k + \partial_m X^k(x) \right) = \\ &= Y^m(x) \partial_m X^k(x) - X^m(x) \partial_m Y^k(x). \end{aligned}$$

Taking into account Eq.(3.2) we have

$$(L_{\mathbf{X}} \mathbf{Y})^k = X^m \partial_m Y^k - Y^m \partial_m X^k \quad (3.4)$$

Thus $L_{\mathbf{X}} \mathbf{Y} = -L_{\mathbf{Y}} \mathbf{X} = [\mathbf{X}, \mathbf{Y}]$. The last is not simply a symbol but a real commutator of the operations.

$$\mathbf{X}(\mathbf{Y}(f)) = X^m \frac{\partial}{\partial x^m} (Y^n \frac{\partial}{\partial x^n} f)$$

and antisymmetrizing

$$\mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)) = (X^m \frac{\partial Y^k}{\partial x^m} - Y^m \frac{\partial X^k}{\partial x^m}) \frac{\partial f}{\partial x^k}$$

which by the way, proves independently that the commutator of two vector fields is a vector field.

A simpler method to obtain the components of the Lie derivative of a vector field is to notice that given any form $\omega = \omega_m dx^m$ we have that $s = \omega_m Y^m$ is a scalar and the Lie derivative of a scalar is the simple derivative. Then we have

$$L_{\mathbf{X}}s = \partial_m \omega_k X^m Y^k + \omega_k \partial_m Y^k X^m = (L_{\mathbf{X}}\omega)_k Y^k + \omega_k (L_{\mathbf{X}}\mathbf{Y})^k$$

and from the expression of the components of the Lie derivative of 1-forms (3.3) we have again (3.4).

3.6 Commuting vector fields

We recall that there is a bijection between C^∞ vector fields with compact support and one-parameter abelian groups of diffeomorphisms. Given the vector field \mathbf{X} we shall call ϕ_t the associated group of diffeomorphisms. Let now be α an other diffeomorphism which commutes with ϕ_t

$$\alpha \circ \phi_t \circ \alpha^{-1} = \phi_t.$$

By taking the derivative with respect to t at $t = 0$ we have

$$\alpha_* \mathbf{X}(p) = \mathbf{X}(p)$$

(always recall that the argument of the push-forward is the end point of the mapping). Vice-versa given a whatever diffeomorphism α also $\alpha \circ \phi_t \circ \alpha^{-1}$ is a one-parameter group of diffeomorphisms and according to the previous results it is generated by $\alpha_* \mathbf{X}$. If it turns out that $\alpha_* \mathbf{X} = \mathbf{X}$ then due to the bijection we have

$$\alpha \circ \phi_t \circ \alpha^{-1} = \phi_t.$$

Thus we have at present the following result: necessary and sufficient condition for α to commute with ϕ_t is that $\alpha_* \mathbf{X} = \mathbf{X}$.

Given now the field \mathbf{Y} and the diffeomorphisms ψ_s generated by it, if ψ_s commutes with ϕ_t we have

$$\phi_t \circ \psi_s \circ \phi_t^{-1} = \psi_s$$

and thus

$$\phi_{t*} \mathbf{Y} = \mathbf{Y}$$

which through the definition of Lie derivative implies

$$L_{\mathbf{X}}\mathbf{Y} = 0.$$

We prove now the reverse, that is if

$$L_{\mathbf{X}}\mathbf{Y} \equiv [\mathbf{X}, \mathbf{Y}] = 0$$

then the diffeomorphism ϕ_t generated by \mathbf{X} commutes with the diffeomorphism ψ_s generated by \mathbf{Y} .

Let us consider, given a point p , the vector \mathbf{C} defined by

$$\mathbf{C}(t) = \phi_{t*}\mathbf{Y}(p).$$

We shall see in a moment that $\mathbf{C}'(t) = 0$ and thus

$$\phi_{t*}\mathbf{Y}(p) = \mathbf{Y}(p)$$

which implies from the above reasonings that ϕ_t and ψ_s commute.

To prove that $\mathbf{C}'(t) = 0$ let us write

$$\mathbf{C}'(t) = \lim_{h \rightarrow 0} \frac{1}{h} (\phi_{t+h*}\mathbf{Y}(p) - \phi_{t*}\mathbf{Y}(p)) = \lim_{h \rightarrow 0} \frac{1}{h} \left[\hat{\phi}_t \hat{\phi}_h \mathbf{Y}(\phi_h^{-1} \circ \phi_t^{-1}(p)) - \hat{\phi}_t \mathbf{Y}(\phi_t^{-1}(p)) \right]$$

where we used the notation of Eq.(3.1). Then

$$\begin{aligned} \mathbf{C}'(t) &= \hat{\phi}_t \lim_{h \rightarrow 0} \frac{1}{h} \left[\hat{\phi}_h \mathbf{Y}(\phi_h^{-1} \circ \phi_t^{-1}(p)) - \mathbf{Y}(\phi_t^{-1}(p)) \right] = \\ &= \hat{\phi}_t \lim_{h \rightarrow 0} \frac{1}{h} \left[\phi_{h*} \mathbf{Y}(\phi_t^{-1}(p)) - \mathbf{Y}(\phi_t^{-1}(p)) \right] = \\ &= -\hat{\phi}_t L_{\mathbf{X}} \mathbf{Y}(\phi_t^{-1}(p)) = -\phi_{t*} L_{\mathbf{X}} \mathbf{Y}(p) = 0 \end{aligned}$$

due to the vanishing of $L_{\mathbf{X}}\mathbf{Y}$. Summing up: Necessary and sufficient condition for the two abelian groups of diffeomorphisms ϕ_t and ψ_s to commute is that their associated vector fields $\mathbf{X}(p)$, $\mathbf{Y}(p)$ commute.

Given two commuting one-parameter abelian groups of diffeomorphisms ϕ_t and ψ_s , we can construct a two dimensional surface by moving a given point p as follows

$$\psi_s \circ \phi_t(p)$$

and on this surface, set the coordinates system t, s . The coordinate basis of the tangent space then contain

$$\frac{\partial}{\partial t} = \mathbf{X}(p); \quad \frac{\partial}{\partial s} = \mathbf{Y}(p)$$

The above results can be extended to an arbitrary number of commuting vector fields.

References

- [1] M. Spivak, "A comprehensive introduction to differential geometry I", Publish or Perish, Inc. Boston (1970); from pag.5-30 to pag.5-39

3.7 Stokes' theorem

Given an n -dimensional manifold M , a local system of coordinates $x^1 \dots x^n$ defines, if the manifold is orientable, an orientation. This means that we associate to the symbol

$$\int_V f dx^1 \wedge \dots dx^n$$

the value

$$\int_V f dx^1 \dots dx^n$$

where $dx^1 \dots dx^n$ is the usual measure.

The orientation on V induces an orientation on the boundary ∂V as follows: Choose around a point of ∂V a local system of coordinates $u^1 \dots u^n$ which is equioriented with $x^1 \dots x^n$ i.e.

$$\frac{\partial(x^1 \dots x^n)}{\partial(u^1 \dots u^n)} > 0$$

and such that ∂V is described by $u^1 = 0$ and u^1 is negative inside and positive outside V . Then the orientation on ∂V is given by

$$\int_{\partial V} g du^2 \wedge \dots du^n = \int_{\partial V} g du^2 \dots du^n .$$

This convention is equivalent to the one adopted in [1]. Once defined the orientation on ∂V we have Stokes' theorem for an $n - 1$ form on M

$$\int_V d\omega = \int_{\partial V} \omega .$$

Thus if

$$\omega = \omega_{\nu_2 \dots \nu_n} \frac{dx^{\nu_2} \wedge \dots dx^{\nu_n}}{(n-1)!}$$

we have

$$\int_V \partial_\lambda \omega_{\nu_2 \dots \nu_n} \frac{dx^\lambda \wedge dx^{\nu_2} \wedge \dots dx^{\nu_n}}{(n-1)!} = \int_{\partial V} \omega_{\nu_2 \dots \nu_n} \frac{\partial x^{\nu_2}}{\partial u^{\sigma_2}} \dots \frac{\partial x^{\nu_n}}{\partial u^{\sigma_n}} \frac{du^{\sigma_2} \wedge \dots \wedge du^{\sigma_n}}{(n-1)!}$$

References

[1] S.S. Chern, W.H. Chen, K.S. Lam, "Lectures in differential geometry" World Scientific, 1999

Chapter 4

The covariant derivative

4.1 Introduction

We give the defining properties of the covariant derivative. Given a vector field \mathbf{Y} , $\mathbf{d}_\mathbf{X}\mathbf{Y}$ is a vector field with the properties

$$\mathbf{d}_{f\mathbf{X}+g\mathbf{Z}}\mathbf{Y} = f\mathbf{d}_\mathbf{X}\mathbf{Y} + g\mathbf{d}_\mathbf{Z}\mathbf{Y} \quad (I)$$

$$\mathbf{d}_\mathbf{X}(\alpha\mathbf{Y} + \beta\mathbf{Z}) = \alpha\mathbf{d}_\mathbf{X}\mathbf{Y} + \beta\mathbf{d}_\mathbf{X}\mathbf{Z} \quad \alpha, \beta = \text{const.} \quad (II)$$

$$\mathbf{d}_\mathbf{X}(f\mathbf{Y}) = df(\mathbf{X})\mathbf{Y} + f\mathbf{d}_\mathbf{X}\mathbf{Y}. \quad (III)$$

From the property (I) we see that such covariant derivative can be written as the result of a vector valued 1-form $\mathbf{d}\mathbf{Y}$ applied to \mathbf{X} i.e.

$$\mathbf{d}_\mathbf{X}\mathbf{Y} = \langle \mathbf{d}\mathbf{Y}, \mathbf{X} \rangle \equiv \mathbf{d}\mathbf{Y}(\mathbf{X}).$$

$\mathbf{d}\mathbf{Y}$ is called the covariant differential. From the property (III) we have that

$$\mathbf{d}(f\mathbf{Y}) = df\mathbf{Y} + f\mathbf{d}\mathbf{Y}.$$

Writing $\mathbf{Y} = \mathbf{e}_a Y^a$ we have

$$\mathbf{d}\mathbf{Y} = \mathbf{d}\mathbf{e}_a Y^a + \mathbf{e}_a dY^a.$$

Thus the covariant derivative is completely defined by the 1-forms Γ^b_a

$$\mathbf{d}\mathbf{e}_a = \mathbf{e}_b \Gamma^b_a; \quad \Gamma^b_a \in \Lambda^1.$$

By calling e^b the basis in the dual space dual to \mathbf{e}_a i.e.

$$e^b(\mathbf{e}_a) = \delta^b_a$$

we can write

$$\Gamma^b{}_a = \Gamma^b{}_{ac} e^c$$

or equivalently

$$\Gamma^b{}_{ac} = \langle \Gamma^b{}_a, \mathbf{e}_c \rangle \equiv \Gamma^b{}_a(\mathbf{e}_c).$$

We have

$$d\mathbf{Y} = d\mathbf{e}_a Y^a + \mathbf{e}_a dY^a = \mathbf{e}_a(dY^a + \Gamma^a{}_b Y^b) \equiv \mathbf{e}_a \nabla Y^a$$

where ∇Y^a is defined by

$$\nabla Y^a \equiv dY^a + \Gamma^a{}_b Y^b \in \Lambda^1_{(1,0)}.$$

References

[1] [HawkingEllis] Chap. 2

4.2 Transformation properties of the connection

We have imposed $d\mathbf{x}\mathbf{Y}$ to be a vector field, or equivalently $d\mathbf{Y}$ to be a vector valued 1-form. This imposes the transformation properties of the connection components under local changes of the frame \mathbf{e}_a . Under $\mathbf{e}_b = \mathbf{e}'_a \Omega^a{}_b$ with $\Omega \in GL(n, R)$ we have

$$d\mathbf{e}_b \equiv \mathbf{e}_a \Gamma^a{}_b = \mathbf{e}'_a \Gamma'^a{}_c \Omega^c{}_b + \mathbf{e}'_a d\Omega^a{}_b = \mathbf{e}'_a \Omega^a{}_c \Gamma^c{}_b$$

i.e.

$$\Gamma = \Omega^{-1} d\Omega + \Omega^{-1} \Gamma' \Omega. \quad (4.1)$$

or

$$\Gamma' = \Omega d\Omega^{-1} + \Omega \Gamma \Omega^{-1}. \quad (4.2)$$

Writing

$$\Gamma^a{}_b = \Gamma^a{}_{bc} e^c$$

we have

$$\Gamma^a{}_{bc} = (\Omega^{-1})^a{}_d d\Omega^d{}_b(\mathbf{e}_c) + (\Omega^{-1})^a{}_d \Gamma'^d{}_{fk} \Omega^f{}_b \Omega^k{}_c. \quad (4.3)$$

In coordinate base under a change of coordinates we have

$$\mathbf{u}_\mu = \mathbf{u}'_\nu \Omega^\nu{}_\mu = \mathbf{u}'_\nu \frac{\partial x'^\nu}{\partial x^\mu} \quad \text{and} \quad \Omega^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}.$$

Substituting in Eq.(4.3) we obtain the familiar transformation of the connection in the coordinate frame where the argument of $\Gamma^a{}_{bc}$ on the l.h.s. is x and the argument of $\Gamma'^d{}_{fk}$ on the r.h.s. is x' and they represent the same event.

Notice that for the Lorentz transformation of Chapter 1, $\Lambda = \Omega = \text{const.}$

4.3 Parallel transport

Given a path $\gamma(t)$ and a vector $\mathbf{Y}(\gamma(t))$ defined along γ we say that the vector \mathbf{Y} is parallel transported along $\gamma(t)$ if, being $\mathbf{X}(t)$ the tangent vector to $\gamma(t)$, we have $d\mathbf{Y}(\mathbf{X}(t)) = 0$ for all t i.e.

$$X^\nu \partial_\nu Y^\mu + \Gamma^\mu_{\beta\nu} Y^\beta X^\nu = 0$$

It appears that the above expression involves values of Y^μ outside the curve $\gamma(t)$. Thus one extends the vector \mathbf{Y} to a smooth field in a neighborhood of $\gamma(t)$. We have

$$X^\nu \partial_\nu Y^\mu = \frac{dY^\mu(\gamma(t))}{dt}$$

showing that such term does not depend on the particular extension.

Giving the connection at a point p is equivalent to giving the parallel transport of a frame of n independent vectors $\mathbf{Y}_{(k)}$ in n independent directions starting from p

$$dY_{(k)}^a(\mathbf{e}_b) + \Gamma^a_c(\mathbf{e}_b) Y_{(k)}^c = 0$$

which can be solved as

$$\Gamma^a_c(\mathbf{e}_b) = -dY_{(k)}^a(\mathbf{e}_b) R_c^{(k)}$$

being $R_c^{(k)}$ the inverse matrix of $Y_{(k)}^c$ which exists being the vectors $\mathbf{Y}_{(k)}$ independent. These result are immediately extended to a general gauge theory by replacing the tangent space T_p with a general N dimensional vector space V .

4.4 Geodesics

A curve $\gamma(t)$ is called a geodesic curve if its tangent vector $\mathbf{X}(\gamma(t))$ has along $\gamma(t)$ a covariant derivative proportional to $\mathbf{X}(\gamma(t))$ itself i.e.

$$d\mathbf{X}(\mathbf{X}) = \alpha(\gamma(t))\mathbf{X}.$$

In component form

$$\nabla X^a(\mathbf{X}) = dX^a(\mathbf{X}) + \Gamma^a_b(\mathbf{X})X^b = \alpha(\gamma(t))X^a$$

without any commitment up to now about the reference frame in the tangent space. Using the coordinate basis \mathbf{u}_μ we have

$$X^\nu \partial_\nu X^\mu + \Gamma^\mu_{\beta\nu} X^\beta X^\nu = \alpha(\gamma(t))X^\mu$$

If we write $\gamma(t)$ in the form $x(t)$ and $X^\mu = \frac{dx^\mu}{dt}$ we have

$$\frac{d^2x^\mu}{dt^2} + \Gamma_{\beta\nu}^\mu \frac{dx^\beta}{dt} \frac{dx^\nu}{dt} = \alpha(t) \frac{dx^\mu}{dt}.$$

The r.h.s can be put to zero by a change in the parametrization $\gamma_s(s) = \gamma(t(s))$. In fact we obtain

$$\frac{d^2x^\mu}{ds^2} + \Gamma_{\beta\nu}^\mu \frac{dx^\beta}{ds} \frac{dx^\nu}{ds} = \left(\alpha(t(s)) \left(\frac{dt}{ds} \right)^2 + \frac{d^2t}{ds^2} \right) \frac{dx^\mu}{dt}.$$

Going over to the inverse function, using $s = s(t(s))$, $1 = s'(t(s))t'(s)$, $0 = s''(t(s))(t'(s))^2 + s'(t(s))t''(s)$ we have to solve

$$\alpha(t) = \frac{s''(t)}{s'(t)}$$

whose general solution is

$$s(t) = c_0 + c_1 \int_0^t dt' \exp \left(\int_0^{t'} \alpha(t'') dt'' \right).$$

Thus s is defined up to an affine transformation and as such called the affine parameter. A geodesic curve parametrized by an affine parameter is called simply a geodesic.

4.5 Curvature

Take the exterior covariant differential of $\mathbf{d}\mathbf{e}_a$

$$\mathbf{d}\mathbf{d}\mathbf{e}_a = \mathbf{d}(\mathbf{e}_b \Gamma^b{}_a) = \mathbf{e}_b d\Gamma^b{}_a + \mathbf{e}_c \Gamma^c{}_b \wedge \Gamma^b{}_a \equiv \mathbf{e}_b R^b{}_a$$

$R^b{}_a \in \Lambda^2$ is the curvature 2-form. In matrix form

$$R = d\Gamma + \Gamma \wedge \Gamma$$

(Notice: being Γ a matrix $\Gamma \wedge \Gamma$ is not necessarily zero; similarly being \mathbf{d} the covariant differential $\mathbf{d}\mathbf{d}$ is not necessarily zero). Given a vector $\mathbf{Y} = \mathbf{e}_a Y^a$ we have

$$\begin{aligned} \mathbf{d}\mathbf{d}(\mathbf{e}_a Y^a) &= \mathbf{d}(\mathbf{e}_b \Gamma^b{}_a Y^a + \mathbf{e}_a dY^a) = \\ &= \mathbf{e}_a \Gamma^a{}_b \wedge \Gamma^b{}_c Y^c + \mathbf{e}_b d\Gamma^b{}_a Y^a - \mathbf{e}_b \Gamma^b{}_a \wedge dY^a + \mathbf{e}_a \Gamma^a{}_b \wedge dY^b = \mathbf{e}_b R^b{}_a Y^a \end{aligned}$$

thus the curvature $R^a{}_b$ is a linear mapping of T_p into itself which does not depend on the field \mathbf{Y} .

We derive now the transformation properties of the curvature. Given $\mathbf{e}_a = \mathbf{e}'_b \Omega^b{}_a$ with $\Omega \in GL(n)$ using the above formula we have

$$\mathbf{d}\mathbf{d}\mathbf{e}_a \equiv \mathbf{e}_b R^b{}_a = \mathbf{e}'_c \Omega^c{}_b R^b{}_a = \mathbf{d}\mathbf{d}(\mathbf{e}'_b \Omega^b{}_a) = \mathbf{d}(\mathbf{d}\mathbf{e}'_b \Omega^b{}_a + \mathbf{e}'_b d\Omega^b{}_a) \quad (4.4)$$

$$= \mathbf{e}'_c R^c{}_b \Omega^b{}_a - \mathbf{d}\mathbf{e}'_b \wedge \Omega^b{}_a + \mathbf{d}\mathbf{e}'_b \wedge d\Omega^b{}_a + \mathbf{e}'_b dd\Omega^b{}_a = \mathbf{e}'_c R^c{}_b \Omega^b{}_a \quad (4.5)$$

i.e.

$$R = \Omega^{-1} R' \Omega \quad \text{or} \quad R' = \Omega R \Omega^{-1}$$

and thus $R^a{}_b$ are the components of a (1,1) tensor 2-form.

4.6 Torsion

We shall introduce torsion formally and then go over to its physical meaning. The torsion S^a is a (1,0) valued 2-form defined by

$$S^a = \nabla e^a.$$

We recall that being $e^a(\mathbf{e}_b) = \delta_b^a$, e^a transforms under local changes of frames according to Ω i.e. like the components of a vector: $e'^a = \Omega^a{}_b e^b$. Thus the explicit expression of S^a is

$$S^a = \nabla e^a = de^a + \Gamma^a{}_b \wedge e^b \in \Lambda^2_{(1,0)}.$$

We have

$$S'^a \equiv \nabla e'^a = de'^a + \Gamma'^a{}_b \wedge e'^b = d[\Omega^a{}_b e^b] + \Gamma'^a{}_b \wedge e^b \quad (4.6)$$

$$= d\Omega^a{}_b \wedge e^b + \Omega^a{}_b de^b + (\Omega d\Omega^{-1} \Omega)^a{}_b \wedge e^b + \Omega^a{}_b \Gamma^b{}_c \wedge e^c \quad (4.7)$$

$$= \Omega^a{}_b S^b \quad (4.8)$$

or $S' = \Omega S$ i.e S^a transforms like the components of a vector.

Given an n component field W_a which transforms like $W_a = W'_b \Omega^b{}_a$ we shall call it of type (0, 1) while those V^a which transform like $V'^a = \Omega^a{}_b V^b$ we shall call of type (1, 0). We shall define ∇ acting on W_a , by imposing the validity of Leibniz rule $\nabla(W_a V^b) = \nabla(W_a) V^b + W_a \nabla(V^b)$, linearity and that on the invariant function $W_a V^a$ we have $\nabla(W_a V^a) = d(W_a V^a)$. As a result

$$\nabla W_a = dW_a - W_b \Gamma^b{}_a.$$

We can extend such definition to the case in which W_a and V^a are differential form by imposing

$$\nabla(W_a \wedge V^a) = d(W_a \wedge V^a)$$

We cannot write for the l.h.s.

$$\nabla(W_a \wedge V^a) = \nabla W_a \wedge V^a + W_a \wedge \nabla V^a$$

because this is inconsistent with

$$d(W_a \wedge V^a) = dW_a \wedge V^a + (-)^w W_a \wedge dV^a$$

being w the order of W_a , when e.g. the connection is identically zero. Thus we shall write

$$\nabla(W_a \wedge V^a) = \nabla W_a \wedge V^a + (-)^w W_a \wedge \nabla V^a$$

from which we obtain

$$\nabla W_a = dW_a - (-)^w W_b \wedge \Gamma_a^b = dW_a - \Gamma_a^b \wedge W_b$$

being Γ_a^b a one form.

By taking linear combinations of fields $W_b V^a$ one extends the above rules to tensors T_b^a and actually to any tensorial form of any order $T_{b_1 \dots b_n}^{a_1 \dots a_m} \in \Lambda_{(m,n)}^t$ e.g.

$$\nabla T_b^a = dT_b^a + \Gamma_c^a \wedge T_b^c - (-)^t T_c^a \wedge \Gamma_b^c$$

which can be written in matrix form as

$$\nabla T = dT + \Gamma \wedge T - (-)^t T \wedge \Gamma.$$

4.7 The structure equations

First Bianchi identity: compute the Λ^3 form

$$\begin{aligned} \nabla R_b^a &= \nabla(d\Gamma + \Gamma \wedge \Gamma)_b^a \\ \nabla R &= dd\Gamma + d\Gamma \wedge \Gamma - \Gamma \wedge d\Gamma + \Gamma \wedge R - R \wedge \Gamma = \\ d\Gamma \wedge \Gamma - \Gamma \wedge d\Gamma + \Gamma \wedge (d\Gamma + \Gamma \wedge \Gamma) - (d\Gamma + \Gamma \wedge \Gamma) \wedge \Gamma &= 0. \end{aligned}$$

These are identities i.e. satisfied by any connection Γ_b^a .

Second Bianchi identity: take the covariant differential of S^a

$$\nabla S^a = dde^a + d\Gamma_b^a \wedge e^b - \Gamma_b^a \wedge de^b + \Gamma_b^a \wedge (de^b + \Gamma_c^b \wedge e^c) = R_b^a \wedge e^b \in \Lambda_{(1,0)}^3.$$

These are identities i.e. satisfied by any connection Γ_b^a and any forms e^a .

Summarizing in matrix form

$$\nabla R = 0; \quad \nabla S = R \wedge e.$$

4.8 Non abelian gauge fields on differential manifolds

One replaces the tangent space T_p with an abstract N dimensional vector space V_p . The covariant differential is defined similarly; to avoid confusion we shall use the symbol \mathbf{D} . If \mathbf{v}_m is a base of the space V_p

$$\mathbf{D}\mathbf{v}_m = \mathbf{v}_n A_n^m$$

where A_n^m is the gauge field one-form. The field strength is defined as the curvature induced by $A_n^m \in \Lambda^1$

$$\mathbf{D}\mathbf{D}\mathbf{v}_n = \mathbf{v}_m F_n^m$$

$$F = dA + A \wedge A.$$

The expanded version is the following: putting $A = A_\mu dx^\mu$ and $F = \frac{1}{2}F_{\mu\nu} dx^\mu \wedge dx^\nu$ we have

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

The first Bianchi identity is proven in exactly the same way; denoting with D the covariant differential acting on the components, we have

$$\Lambda^3 \ni DF = 0.$$

The expanded form of the above equation is

$$D_\lambda F_{\mu\nu} + \text{cyclic} = 0$$

with

$$D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}].$$

Under $\mathbf{v}_n = \mathbf{v}'_m U_n^m$ one has

$$A' = U dU^{-1} + U A U^{-1}; \quad F' = U F U^{-1}. \quad (4.9)$$

The above described is the gauge theory of the group $GL(N)$. We can restrict however the connection to the Lie algebra of a less general group G and the gauge transformations to the elements of G .

It is very easy to see that if $U \in G$ both $U dU^{-1}$ and $U A U^{-1}$ are elements of the Lie algebra \mathcal{L} of G . Consider in fact a trajectory $\gamma(t) \in G$ with $\gamma(0) = I$

$$d\gamma = A \in \mathcal{L}.$$

Then also the trajectory $U\gamma(t)U^{-1} \in G$ goes through I for $t = 0$ and thus we have

$$U d\gamma U^{-1} = U A U^{-1} \in \mathcal{L}.$$

Similarly given a $U(t) \in G$ with $U(0)$ not necessarily I we have $U(0)U^{-1}(t) \in G$ and $U(0)U^{-1}(0) = I$ and thus $U(0)dU^{-1}(0) \in \mathcal{L}$.

In the following we shall often make use of the non degeneracy of the trace form of a faithful representation of a given Lie algebra.

If the group G is compact we know that, as all its representations are equivalent to unitary representations (i.e. antihermitean generators), any faithful representation of its Lie algebra has non degenerate trace form i.e. if $\text{tr}AB = 0$, for any B we have $A = 0$.

Such non degeneracy holds also for faithful representations of non compact groups which have a semi-simple Lie algebra. E.g. the non compact group $SL(2, C)$ is not semi-simple due to the presence of the invariant subgroup $I, -I$ but a simple calculation shows that the Lie algebra of $SL(2, C)$ is semi-simple and thus the trace form of a faithful representation of the Lie algebra of $SL(2, C)$ is non degenerate.

References

[1] N. Jacobson, "Lie algebras" Dover Publication Chap.II

4.9 Non compact electrodynamics

Consider the case in which the vector space is one dimensional on the reals. Then the general vector is $\mathbf{V} = \mathbf{v}V$; the connection is

$$\mathbf{D}\mathbf{v} = \mathbf{v}A$$

with A real 1-form. The field strength is given by

$$\mathbf{D}\mathbf{D}\mathbf{v} = \mathbf{v}(dA + A \wedge A) = \mathbf{v}dA = \mathbf{v}F.$$

Explicitly

$$\begin{aligned} F &= dA = d(A_\nu dx^\nu) = \partial_\mu A_\nu dx^\mu \wedge dx^\nu = (\partial_\mu A_\nu - \partial_\nu A_\mu) \frac{dx^\mu \wedge dx^\nu}{2} = \\ &= F_{\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2}. \end{aligned}$$

Consider the general linear transformation on \mathbf{v} , $\mathbf{v} = \mathbf{v}'e^{-\Lambda(x)}$. We have

$$A' = e^{-\Lambda(x)} de^{\Lambda(x)} + e^{-\Lambda(x)} A e^{\Lambda(x)} = d\Lambda + A$$

and

$$F' = e^{-\Lambda(x)} F e^{\Lambda(x)} = F.$$

Λ is the usual gauge transformation.

4.10 Meaning of torsion

As already stated torsion is a space related concept. Consider two infinitesimal vectors \mathbf{v} and $\mathbf{w} \in T_p$. Let \mathbf{v}_1 be the vector \mathbf{v} parallel transported along \mathbf{w} . In the coordinate basis $\mathbf{v} = \mathbf{u}_\mu v^\mu$ and $\mathbf{w} = \mathbf{u}_\mu w^\mu$ we have

$$v_1^\mu = v^\mu - \Gamma_{\nu\lambda}^\mu v^\nu w^\lambda.$$

Moving now by \mathbf{v}_1 we reach the point of coordinates $x_0^\mu + w^\mu + v_1^\mu$. Reversing the process the two final points agree iff

$$w^\mu + v_1^\mu = w_1^\mu + v^\mu.$$

If this has to happen for all \mathbf{v} and \mathbf{w} we have

$$\Gamma_{\nu\lambda}^\mu = \Gamma_{\lambda\nu}^\mu.$$

Thus a connection symmetric in the lower indices in the coordinate basis is equivalent to the commutativity of infinitesimal displacements. Writing e^a in terms of the vierbeins e_μ^a , $e^a \equiv e_\mu^a dx^\mu$ we have

$$S^a = \partial_\lambda e_\mu^a dx^\lambda \wedge dx^\mu + \Gamma_{b\lambda}^a dx^\lambda \wedge e_\mu^b dx^\mu.$$

Vanishing of the torsion means

$$\partial_\lambda e_\mu^a + \Gamma_{b\lambda}^a e_\mu^b = \partial_\mu e_\lambda^a + \Gamma_{b\mu}^a e_\lambda^b.$$

Multiplying both members by e_a^ν and recalling that $\mathbf{u}_\mu = \mathbf{e}_a e_\mu^a$ as seen from $e^a(\mathbf{u}_\mu) = e_\mu^a$ we have, using Eq.(4.3)

$$\Gamma_{\mu\lambda}^\nu = \Gamma_{\lambda\mu}^\nu.$$

Thus the vanishing of the torsion implies symmetry of the connection in the lower indices in any *coordinate basis*. This can be more simply achieved by

$$S^\mu = 0 \quad \text{implies} \quad 0 = \nabla dx^\mu = ddx^\mu + \Gamma_{\nu}^\mu \wedge dx^\nu = \Gamma_{\nu\lambda}^\mu dx^\lambda \wedge dx^\nu.$$

4.11 Vanishing of the torsion

If the torsion S^a vanishes identically we have from the second Bianchi identity

$$0 = R^a_b \wedge e^b = \frac{1}{2} R^a_{bcd} e^c \wedge e^d \wedge e^b$$

i.e.

$$R^a_{[bcd]} = 0.$$

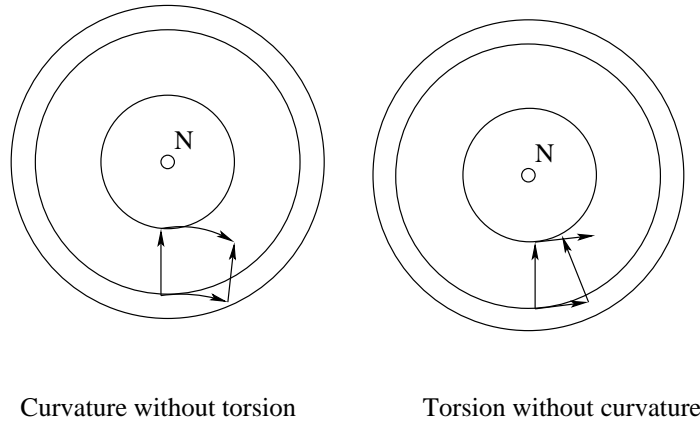


Figure 4.1: Torsion vs. Curvature

4.12 Metric structure

The metric is a bilinear real symmetric functional on $T_p \times T_p$

$$g(\mathbf{W}, \mathbf{V}) \in R.$$

Writing $\mathbf{V} = \mathbf{e}_a V^a = \mathbf{u}_\mu V^\mu$ and similarly for \mathbf{W} we have

$$g(\mathbf{W}, \mathbf{V}) = W^a g_{ab} V^b = W^\mu g_{\mu\nu} V^\nu.$$

Under $\mathbf{e}_a = \mathbf{e}'_b \Omega^b{}_a$ we have

$$g = \Omega^T g' \Omega \quad \text{or} \quad g' = \Omega^{-1T} g \Omega^{-1}.$$

All manifolds (Hausdorff, paracompact) admit a positive definite metric. Necessary and sufficient condition for a manifold to support a Lorentzian metric is that it supports a smooth never vanishing line element. All non compact manifolds support a Lorentzian metric.

References

[1] [HawkingEllis] Chap. 2

4.13 Isometries

Given a mapping α between two manifolds M and M' i.e. $M \xrightarrow{\alpha} M'$ or a diffeomorphism mapping the manifold in itself, and given a metric on M' it is possible to pull back the metric from M' to M as we did with the one-forms. We shall again denote the pulled

back metric by α^*g . If $M = M'$ we shall call α an isometry if $\alpha^*g(p) = g(p)$. If we have an abelian group of diffeomorphisms ϕ_t generated by the vector field \mathbf{X} we can compute the Lie derivative of g

$$L_{\mathbf{X}} g(p) = \lim_{t \rightarrow 0} \frac{1}{t} (\phi_t^* g(p) - g(p)).$$

If ϕ_t is a group of isometries we have $L_{\mathbf{X}} g(p) = 0$ and \mathbf{X} is called a Killing vector field. The explicit form of $\alpha^*g(p)$ in the coordinate basis is, with $x' = \alpha(x)$

$$\alpha^* g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} g_{\rho\sigma}(x') \frac{\partial x'^{\sigma}}{\partial x^{\nu}} .$$

4.14 Metric compatibility

We impose that parallel transported vectors maintain the value of their scalar product. The result is

$$0 = dg - \Gamma^T g - g\Gamma. \quad (4.10)$$

The above equation can also be read as

$$\nabla g_{ab} = dg_{ab} - \Gamma^c_a g_{cb} - g_{ac} \Gamma^c_b = 0$$

i.e. the vanishing of the covariant differential.

By taking the differential of Eq.(4.10) and using again Eq.(4.10) we have

$$\begin{aligned} 0 &= -d\Gamma^T g + \Gamma^T \wedge dg - dg \wedge \Gamma - gd\Gamma = -d\Gamma^T g + \Gamma^T \wedge (\Gamma^T g + g\Gamma) - (\Gamma^T g + g\Gamma) \wedge \Gamma - gd\Gamma = \\ &= -R^T g - gR . \end{aligned} \quad (4.11)$$

Written explicitly

$$g_{ac} R^c_b + R^c_a g_{cb} = R_{ab} + R_{ba} = 0 .$$

4.15 Contracted Bianchi identities

The Bianchi identities for the curvature 2-form $R^a_b \in \Lambda^2_{(1,1)}$

$$\nabla R^a_b = 0$$

can be expanded as

$$0 = \nabla(R^a_{bcd} e^c \wedge e^d) = \nabla(R^a_{bcd}) e^c \wedge e^d + R^a_{bcd} \nabla e^c \wedge e^d - R^a_{bcd} e^c \wedge \nabla e^d .$$

In absence of torsion i.e $S^a = \nabla e^a = 0$ we have

$$\nabla(R^a_{bcd}) e^c \wedge e^d = \nabla_f (R^a_{bcd}) e^f \wedge e^c \wedge e^d = 0$$

i.e.

$$\nabla_f R^a{}_{bcd} + \nabla_c R^a{}_{bdf} + \nabla_d R^a{}_{bfc} = 0 .$$

The Ricci tensor is defined by

$$R_{bd} = R^a{}_{bad} .$$

In presence of metric and metric compatibility we can contract indices to have

$$0 = \nabla_f R^a{}_{bad} + \nabla_a R^a{}_{bdf} + \nabla_d R^a{}_{bfa} = \nabla_f R_{bd} + \nabla_a R^a{}_{bdf} - \nabla_d R_{bf}$$

and contracting again

$$0 = \nabla_f R - \nabla_a R^a{}_f - \nabla_b R^b{}_f = \nabla_f R - 2\nabla_a R^a{}_f$$

with $R = R_{bd} g^{bd}$ the Ricci scalar. These are the contracted Bianchi identities.

4.16 Orthonormal reference frames

In presence of a metric we can choose as basis vectors on the tangent space, an orthonormal frame for each point, i.e. $\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab}$. Metric compatibility now becomes

$$\Gamma^T \eta + \eta \Gamma = d\eta = 0$$

or

$$\Gamma = -\eta^{-1} \Gamma^T \eta \quad \text{i.e.} \quad \Gamma \in so(3, 1).$$

Also from Eq.(4.11)

$$R = -\eta^{-1} R^T \eta \quad \text{i.e.} \quad R \in so(3, 1).$$

Thus Γ and R belong to the algebra of $SO(3, 1)$. Now the changes of frame are elements of the Lorentz group, $\mathbf{e}_a = \mathbf{e}'_b \Omega^b{}_a$, with $\Omega^T \eta \Omega = \eta$ and we have

$$\Gamma' = \Omega d\Omega^{-1} + \Omega \Gamma \Omega^{-1}, \quad R' = \Omega R \Omega^{-1} .$$

4.17 Compact electrodynamics

We consider the case of gauge theory in which space V_p is 1-dimensional on the complex. Given the basic vector \mathbf{v} , all vectors are given by $\mathbf{v}V$ with $V \in C$. Choosing \mathbf{v} normalized we have

$$(\mathbf{W}, \mathbf{V}) = W^* V.$$

The connection is given by

$$\mathbf{D}\mathbf{v} = \mathbf{v}A$$

with A complex valued 1-form. Imposition of “metric compatibility” gives

$$W'^*V' = (W - \varepsilon A(\mathbf{X})W)^*(V - \varepsilon A(\mathbf{X})V) + O(\varepsilon^2) = W^*V$$

for any $\mathbf{X} \in T_p$. As a consequence

$$A^* + A = 0$$

i.e. A is a pure imaginary 1-form. Thus it is useful to change the notation for the connection to

$$\mathbf{D}\mathbf{v} = \mathbf{v}ikA$$

with k real constant and A real 1-form. Keep in mind that now the connection is ikA . The gauge transformation now are $\mathbf{v} = \mathbf{v}' e^{-ik\Lambda} \equiv \mathbf{v}' U$

$$ikA' = e^{-ik\Lambda} de^{ik\Lambda} + e^{-ik\Lambda} ikAe^{ik\Lambda} = ikd\Lambda + ikA$$

i.e. $A' = A + d\Lambda$. Define now F as

$$\mathbf{D}\mathbf{D}\mathbf{v} = \mathbf{v}ikF = ik(dA + A \wedge A) \quad \text{i.e.} \quad F = dA .$$

The gauge group is now compact i.e. for $\Lambda = 2\pi/k$ we have the identity transformation on the vector space V_p . We recall now the transformation properties of the Schrödinger wave function describing a particle of charge e

$$\psi(x) = e^{-\frac{ie}{\hbar c}\Lambda(x)}\psi'(x).$$

For $\Lambda = 2\pi/k$ we must have the identity transformation i.e.

$$\frac{e}{\hbar c} = n$$

with integer n which implies that all charges are integer multiples of a fundamental charge $k\hbar c$.

References

- [1] T.T. Wu, C.N. Yang, “Concept of nonintegrable phase factors and global formulation of gauge fields”, Phys. Rev. D12 (1975) 3845

4.18 Symmetries of the Riemann tensor

$$R_{bcd}^a = -R_{bdc}^a \quad (4.12)$$

is simply due to the definition. We have $n^3(n-1)/2$ independent components. In the torsionless case we have

$$R_{[bcd]}^a = 0 \quad (4.13)$$

and we have to subtract to the above number $n^2(n-1)(n-2)/3!$ which is the number of independent identities given by Eq.(4.13), to obtain $n^2(n^2-1)/3$ independent components. Metric compatibility gives

$$R_{abcd} = -R_{bacd} . \quad (4.14)$$

If we combine Eq.(4.12), Eq.(4.13) with Eq.(4.14) we have the further symmetry

$$R_{abcd} = R_{cdab} . \quad (4.15)$$

In fact let us define

$$S_{abcd} = R_{abcd} + R_{acdb} + R_{adbc}$$

which are identically zero as a consequence of the second Bianchi identities in absence of torsion. We have identically

$$0 = S_{abcd} - S_{bcda} - S_{cdab} + S_{dabc} = 2R_{abcd} - 2R_{cdab} .$$

Eq.(4.15) has been derived without exploiting the explicit form of the connection. We can now compute the number of independent components of the Riemann tensor in presence of metric compatibility and absence of torsion. The components with only 2 different indices are $n(n-1)/2$; with three different indices we have n choices for the double index and $(n-1)(n-2)/2$ choices for the other two thus giving $n(n-1)(n-2)/2$; with 4 different indices we have $n(n-1)(n-2)(n-3)/4!$ choices which should be multiplied by 3 due to the three different pairings. However due to Eq.(4.13) only 2 of the three pairings are independent. Notice that the second Bianchi identity does not play any role when only two or three indices are different as they are identically satisfied. Summing the contributions we have $n^2(n^2-1)/12$.

4.19 Sectional curvature and Schur theorem

We refer to the *torsionless and metric compatible* case in which the Riemann tensor has the symmetries

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab} .$$

By simple algebra [1] from the knowledge of

$$R_{abcd}v_1^av_2^bv_1^cv_2^d$$

one determines completely R_{abcd} . The quantity

$$\frac{R_{abcd}v_1^av_2^bv_1^cv_2^d}{\mathbf{v}_1 \cdot \mathbf{v}_1 \ \mathbf{v}_2 \cdot \mathbf{v}_2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2} \equiv K(p, \text{plane}) \quad (4.16)$$

with $\mathbf{v}_i \cdot \mathbf{v}_j = v_i^ag_{ab}v_j^b$ is invariant under linear invertible substitutions of \mathbf{v}_1 and \mathbf{v}_2 . In fact given the transformation $\mathbf{w}_j = \mathbf{v}_l a_j^l$, due to the antisymmetry in the first two indices we have

$$R_{abcd}w_1^aw_2^bw_1^cw_2^d = \det(a)R_{abcd}v_1^av_2^bv_1^cv_2^d$$

and similarly for the second pair of indices. The denominator can be written as

$$\varepsilon_{ij}(\mathbf{v}_i \cdot \mathbf{v}_l)(\mathbf{v}_j \cdot \mathbf{v}_k) = \varepsilon_{lk}[(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_2 \cdot \mathbf{v}_1)]$$

and thus

$$\begin{aligned} \varepsilon_{ij}(\mathbf{w}_i \cdot \mathbf{w}_l)(\mathbf{w}_j \cdot \mathbf{w}_k) &= \det(a)\varepsilon_{ij}(\mathbf{v}_i \cdot \mathbf{w}_l)(\mathbf{v}_j \cdot \mathbf{w}_k) = \\ &= (\det(a))^2[(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2)(\mathbf{v}_2 \cdot \mathbf{v}_1)]\varepsilon_{lk}. \end{aligned}$$

$K(p, \text{plane})$ is called the sectional curvature; it is a functional of a plane. In the case of positive definite metric, the denominator in Eq.(4.16) (Gram determinant) is non vanishing if the two vectors $\mathbf{v}_1, \mathbf{v}_2$ are independent. This is not necessarily true for Lorentz metric. In this case we define the sectional curvature only for those pairs of vectors which have non vanishing Gram determinant. However any couple of vectors which have a vanishing Gram determinant can be reached continuously by pairs of vector with non vanishing Gram determinant.

If at a point p

$$\frac{R_{abcd}v_1^av_2^bv_1^cv_2^d}{\mathbf{v}_1 \cdot \mathbf{v}_1 \ \mathbf{v}_2 \cdot \mathbf{v}_2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2} = c(p)$$

i.e. K is independent of the plane in the case of positive definite metric, and independent of the plane with non vanishing Gram determinant in the case of Lorentz metric, we have due to the previous result

$$R_{abcd} = c(p)(g_{ac}g_{bd} - g_{ad}g_{bc})$$

which can be rewritten also as

$$R_{abcd} = \frac{R(p)}{n(n-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \quad (4.17)$$

and the Ricci tensor is

$$R_{bd} = \frac{R(p)}{n} g_{bd}.$$

Computing the contracted Bianchi identity of Eq.(4.17) we have

$$0 = \nabla_a \left(\frac{R}{n} \delta_b^a - \frac{1}{2} R \delta_b^a \right) = \left(\frac{1}{n} - \frac{1}{2} \right) \nabla_b R.$$

Thus we have Schur theorem: For $n > 2$, if K is independent of the plane K does not depend on the point.

Such manifolds are said to have *constant curvature*.

Notice that the proof of Schur theorem fails for $n = 2$. In two dimensions for each point there is only one plane and the curvature scalar can well depend on the point.

References

[1] S. Kobayashi, K. Nomitzu “Foundations of differential geometry”, J. Wiley& Sons, New York, vol. I p. 198

4.20 Spaces of constant curvature

The spaces of constant curvature i.e. those for which

$$R_{abcd} = k(g_{ac}g_{bd} - g_{ad}g_{bc})$$

coincide with the maximally symmetric spaces, which in 4 dimensions with signature $(-+++)$ are Minkowski, de Sitter and anti-de Sitter. They admit the maximum number of Killing vectors i.e. $n(n+1)/2$ independent Killing vectors.

4.21 Geometry in diverse dimensions

$n = 2$

The Riemann tensor has only one independent component; set in a well defined frame and at the point p

$$R_{1212} = c(p)(g_{11}g_{22} - g_{12}g_{21})$$

$c(p)$ is well defined as $\det g \neq 0$. The tensor

$$T_{abcd} = c(p)(g_{ac}g_{bd} - g_{ad}g_{bc})$$

has all the component equal to those of R_{abcd} at the point p in the given frame and thus in all frames. We have

$$R_{abcd} = \frac{R(p)}{2}(g_{ac}g_{bd} - g_{ad}g_{bc}).$$

The Ricci tensor is

$$R_{bd} = \frac{R(p)}{2} g_{bd}.$$

$n = 3$

Here the Riemann tensor has 6 independent components and the Ricci tensor also 6 independent components. We can write

$$R_{abcd} = \epsilon_{abk} \epsilon_{cdl} F^{kl}$$

where F^{kl} is a symmetric tensor. We have

$$g^{ac} \epsilon_{abk} \epsilon_{cdl} = -g_{bd} g_{kl} + g_{dk} g_{bl}$$

with $\epsilon_{cdl} = \sqrt{-g} \varepsilon_{cdl}$ and ε_{cdl} the usual antisymmetric symbol (see Section 4.26). Thus we have

$$R_{bd} = F_{bd} - g_{bd} F; \quad R = -2F$$

from which

$$R_{abcd} = \epsilon_{abk} \epsilon_{cdl} G^{kl}$$

being $G^{kl} = R^{kl} - \frac{g^{kl}}{2} R$ the Einstein tensor. The Weyl tensor (see below) vanishes identically in $n = 3$.

$n \geq 4$

We can write

$$R_{ijkl} = W_{ijkl} - \frac{2}{n-2} (g_{i[l} R_{k]j} + g_{j[k} R_{l]i}) - \frac{2}{(n-1)(n-2)} R g_{i[k} g_{l]j}$$

being W_{ijkl} the Weyl tensor. W_{ijkl} is traceless and W^i_{jkl} is invariant under Weyl transformations $g_{ij} \rightarrow e^{2f} g_{ij}$ (see Section 4.25).

4.22 Flat and conformally flat spaces

A manifold with metric g_{ij} , is said to be flat if there exists a coordinate system such that in such a system the metric takes the form \hat{g}_{ij} with \hat{g}_{ij} a constant metric.

In Section 4.23 it is proven that, with a torsionless and metric compatible connection, necessary and sufficient condition for a space to be locally flat is the vanishing of the Riemann tensor.

A manifold with metric g_j , is said to be conformally flat if there exists a coordinate system such that in such a system the metric takes the form

$$g_{ij} = e^{2f} \hat{g}_{ij}$$

with \hat{g}_{ij} a constant metric.

In Section 4.24 it is proven the Weyl-Schouten theorem:

For $n \geq 4$ necessary and sufficient condition for a metric to be locally conformally flat is the vanishing of the Weyl tensor.

The necessity of the condition is simple to prove. Suppose that by means of a diffeomorphism we are able to bring the metric in the form

$$g_{ij}(x) = e^{2f} \hat{g}_{ij}$$

with \hat{g}_{ij} constant. We have $\hat{R}_{ijkl} = 0$, $\hat{R}_{jl} = 0$, $\hat{R} = 0$ and thus $\hat{W}^i_{jkl} = 0$. But W^i_{jkl} is invariant under Weyl transformations (see Section 4.25) and thus $W^i_{jkl} = 0$ and being W a tensor it implies the vanishing of the original Weyl tensor. For the sufficient condition see Section 4.25.

For $n = 3$ the Weyl tensor vanishes identically and necessary and sufficient condition for a geometry to be locally conformally flat is the vanishing of the Cotton tensor as proven in Section 4.25.

In dimension 2 the curvature transform under a Weyl transformation as (see Section 4.24)

$$\tilde{R} = e^{-2f}(R - 2\Delta f) .$$

Thus by solving locally the equation

$$\Delta f = \frac{R}{2} \tag{4.18}$$

we obtain the vanishing of the Riemann tensor and thus we have local conformal flatness. Eq.(4.18) can always be locally solved. Given a point p we consider a C^∞ function $\rho(x)$ with compact support which around p is identically 1 and the solution is given by

$$f(x) = \frac{1}{8\pi} \int \rho(x') R(x') \log[(x_1 - x'_1)^2 + (x_2 - x'_2)^2] d^2 x' .$$

At the global level in $n = 2$ for genus $g = 0$ i.e. the topology of the sphere we can reduce the metric to $g_{ij} = \delta_{ij} e^{2f}$; for $g = 1$ i.e. the torus topology, we can reduce the metric to $g_{ij} = \hat{g}_{ij} e^{2f}$ where \hat{g} is a constant metric which depends on two parameters; for $g > 1$ we can reduce the metric to $g_{ij} = \hat{g}_{ij} e^{2f}$ where \hat{g} is a metric of constant negative curvature taken equal to -1 and \hat{g} depends on $6g - 6$ parameters.

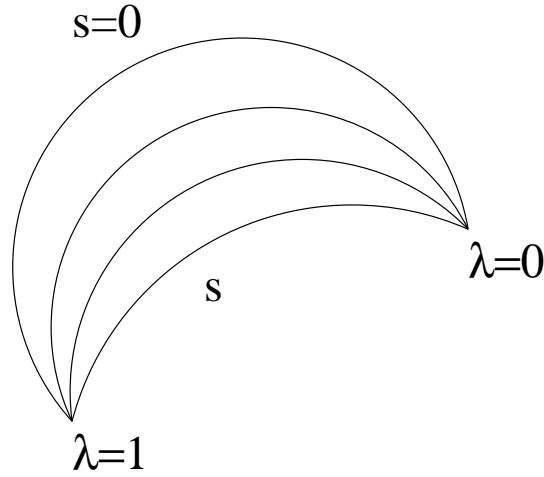


Figure 4.2: The non abelian Stokes theorem-1

4.23 The non abelian Stokes theorem

Given a family of paths $x(s, \lambda)$ (see fig. 4.2) we want to write the variation of the line integral $W(s, 1)$, called the Wilson line,

$$W(s, \lambda) = \text{Pexp} \left(- \int_0^\lambda A_\lambda(s, \lambda') d\lambda' \right) = \text{Pexp} \left(- \int_0^\lambda A_\mu(x(s, \lambda')) \frac{dx^\mu}{d\lambda'} d\lambda' \right)$$

when the parameter s changes from 0 to s and the end points are fixed, i.e. $x(s, 0) = x_0$ $x(s, 1) = x_1$, as a surface integral containing the field strength $F_{s\lambda}$. It is possible to consider also the case in which the end points are moving in s .

We recall that the Wilson line $W(s, \lambda)$ provides the parallel transport of a vector along the line $s = \text{const}$.

$$v(s, \lambda) = W(s, \lambda)v(s, 0)$$

as $W(s, \lambda)$ satisfies the equation

$$\frac{\partial W(s, \lambda)}{\partial \lambda} = -A_\lambda(s, \lambda)W(s, \lambda) . \quad (4.19)$$

The derivative of $W(s, \lambda)$ with respect to s is given by

$$X(s, \lambda) = -W(s, \lambda) \int_0^\lambda W(s, \lambda')^{-1} \frac{\partial A_\lambda(s, \lambda')}{\partial s} W(s, \lambda') d\lambda' . \quad (4.20)$$

In fact from Eq.(4.19) we have

$$\frac{\partial}{\partial \lambda} \frac{\partial W}{\partial s} = - \frac{\partial A_\lambda(s, \lambda)}{\partial s} W(s, \lambda) - A_\lambda(s, \lambda) \frac{\partial W(s, \lambda)}{\partial s}$$

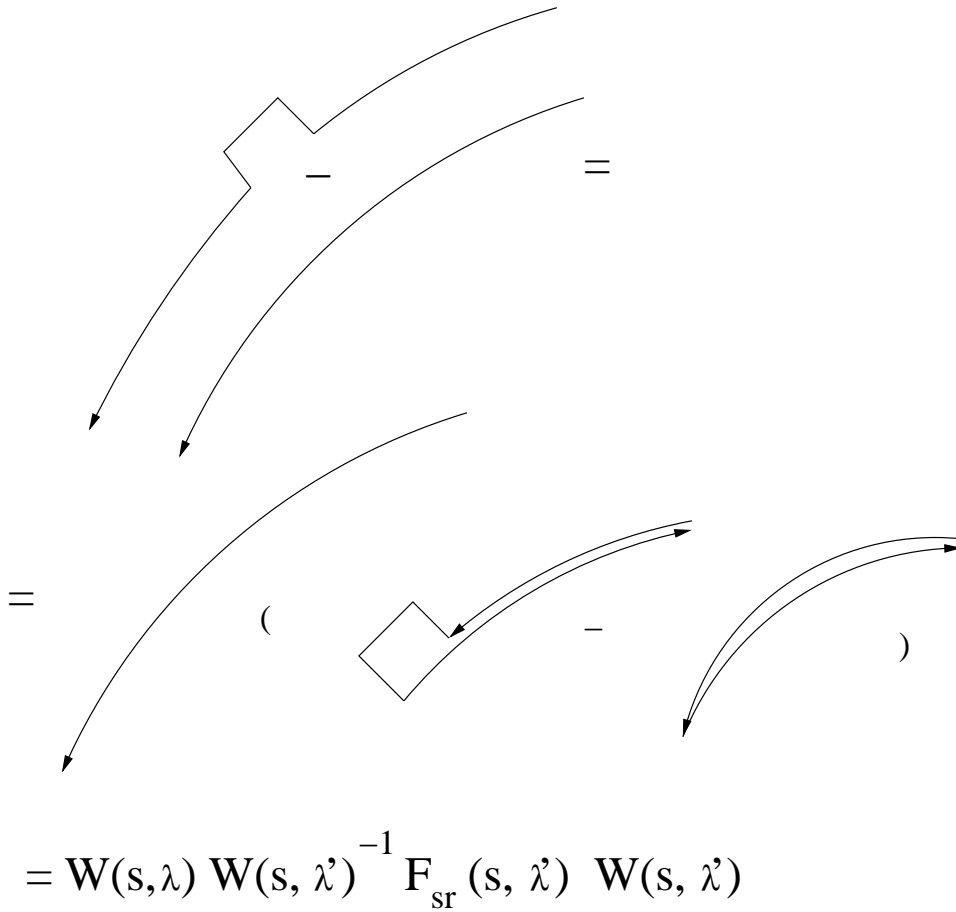


Figure 4.3: The non abelian Stokes theorem-2

and trivially from Eq.(4.20)

$$\frac{\partial}{\partial \lambda} X(s, \lambda) = -\frac{\partial A_\lambda(s, \lambda)}{\partial s} W(s, \lambda) - A_\lambda(s, \lambda) X(s, \lambda) .$$

Thus $\frac{\partial W}{\partial s}$ and X satisfy the same first order equation with the same initial conditions

$$\frac{\partial W(s, 0)}{\partial s} = 0; \quad X(s, 0) = 0 .$$

We can now add to Eq.(4.20)

$$0 = W(s, 1) \int_0^1 \frac{\partial}{\partial \lambda} (W(s, \lambda)^{-1} A_s(s, \lambda) W(s, \lambda)) d\lambda$$

due to

$$A_s(s, 1) = A_\mu(x(s, 1)) \frac{dx^\mu(s, 1)}{ds} = 0$$

to obtain

$$\frac{\partial W(s, 1)}{\partial s} = -W(s, 1) \int_0^1 W(s, \lambda)^{-1} F_{s\lambda}(s, \lambda) W(s, \lambda) d\lambda \equiv -W(s, 1) G(s) . \quad (4.21)$$

An intuitive picture of such equation is given in fig.4.3. Eq.(4.21) is already an interesting result. It can be further integrated to obtain

$$W^{-1}(s, 1)W(0, 1) = \text{Pexp}\left(\int_0^s G(s')ds'\right)$$

References

[1] P. Menotti, D. Seminara, “Energy theorem for (2+1)-dimensional gravity” Ann. Phys. 240 (1995) 203, Appendix A

[2] P. M. Fishbane, S. Gasiorowicz, P. Kraus, “Stokes’s theorems for non-Abelian fields” Phys. Rev. D 24 (1981)2324

4.24 Vanishing of torsion and of the Riemann tensor

If $R_{abcd} = 0$ due to the non abelian Stokes theorem we have that the space is teleparallel i.e. the parallel transport of a vector does non depend on the path. If in addition we have vanishing torsion and metric compatibility the space is locally flat.

In fact teleparallelism means that we can define n independent vector fields $\mathbf{v}^{(a)}$, $a = 1 \dots n$ with the property, using the covariant components $v_\mu^{(a)} = \mathbf{v}^{(a)} \cdot \mathbf{u}_\mu$

$$0 = \nabla_\nu v_\mu^{(a)} = \partial_\nu v_\mu^{(a)} - v_\rho^{(a)} \Gamma_{\mu\nu}^\rho.$$

If we suppose also $S^a = 0$ (symmetric $\Gamma_{\lambda\mu}^\nu$) we have

$$0 = \nabla_\nu v_\mu^{(a)} - \nabla_\mu v_\nu^{(a)} = \partial_\nu v_\mu^{(a)} - \partial_\mu v_\nu^{(a)}.$$

The above equation tells us that the vector field $v_\mu^{(a)}$ is locally integrable i.e. there exists functions y^a of space such that

$$v_\mu^{(a)} = \frac{\partial y^a}{\partial x^\mu}.$$

Let us now go over to the coordinates y^a . The components of the vectors $\mathbf{v}^{(a)}$ in these new coordinates are

$$v_b^{(a)} = v_\mu^{(a)} \frac{\partial x^\mu}{\partial y^b} = \frac{\partial y^a}{\partial x^\mu} \frac{\partial x^\mu}{\partial y^b} = \delta_b^a$$

and then $\mathbf{d}\mathbf{v}^{(a)} = 0$ becomes

$$0 = \nabla_b v_c^{(a)} = \partial_b \delta_c^a - \Gamma_{cb}^f \delta_f^a = -\Gamma_{cb}^a.$$

But $\Gamma_{bc}^a = 0$ combined with metric compatibility $0 = dg - g\Gamma - \Gamma^T g$ implies $\partial_c g_{ab} = 0$ i.e. $g_{ab} = \text{const.}$

4.25 The Weyl-Schouten theorem

We give below a proof of the Weyl-Schouten theorem which gives the necessary and sufficient conditions for a geometry to be locally conformally flat. The setting is that of the torsionless metric compatible i.e. Levi-Civita connection.

From the previous result we see that necessary and sufficient condition for a space to be conformally flat is the existence of a Weyl transformation $\tilde{g}_{ij} = e^{2f} g_{ij}$ such that $\tilde{R}^i_{jkl} = 0$. It is useful to introduce the Weyl tensor via the Schouten tensor

$$S_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{1}{2(n-1)} R g_{ij} \right)$$

and define the Weyl tensor W through

$$R_{ijkl} = W_{ijkl} + (S_{ik}g_{jl} + S_{jl}g_{ik} - S_{il}g_{jk} - S_{jk}g_{il}) \equiv W_{ijkl} + (S \odot g)_{ijkl}$$

where for conciseness we introduced the Kulkarni notation \odot . One immediately verifies that

$$R^i_{jil} = R_{jl} = W^i_{jil} + R_{ij}$$

and thus the Weyl tensor W is completely traceless and has the same symmetry properties in the indices as the Riemann tensor.

Under $\tilde{g}_{ij} = e^{2f} g_{ij}$ we have

$$\tilde{\Gamma}^i_{jl} = \Gamma^i_{jl} + g^{im} (\partial_j f g_{lm} + \partial_l f g_{jm} - \partial_m f g_{jl}) = \Gamma^i_{jl} + \delta \Gamma^i_{jl} .$$

For the Riemann 2-form we have

$$\tilde{R} = R + d \delta \Gamma + \Gamma \wedge \delta \Gamma + \delta \Gamma \wedge \Gamma + \delta \Gamma \wedge \delta \Gamma = R + \nabla \delta \Gamma + \delta \Gamma \wedge \delta \Gamma \quad (4.22)$$

where ∇ is the covariant derivative with the connection Γ . Using Eq.(4.22) one finds

$$\tilde{R}^i_{jkl} = R^i_{jkl} - g^{im} (a \odot g)_{mjkl}$$

and

$$\tilde{S}_{ij} = S_{ij} - a_{ij}$$

where

$$a_{ij} = \nabla_i \nabla_j f - \nabla_i f \nabla_j f + \frac{1}{2} |\nabla f|^2 g_{ij} .$$

From this it follows that the Weyl tensor W^i_{jkl} is invariant because

$$\begin{aligned} \tilde{R}^i_{jkl} &= \tilde{W}^i_{jkl} + \tilde{g}^{im} (\tilde{S} \odot \tilde{g})_{mjkl} = \tilde{W}^i_{jkl} + g^{im} (S \odot g)_{mjkl} - g^{im} (a \odot g)_{mjkl} \\ &= R^i_{jkl} - g^{im} (a \odot g)_{mjkl} = W^i_{jkl} + g^{im} (S \odot g)_{mjkl} - g^{im} (a \odot g)_{mjkl} . \end{aligned} \quad (4.23)$$

Using

$$\nabla_i R^i{}_{jkl} - \nabla_k R_{jl} + \nabla_l R_{jk} = 0$$

and the contracted Bianchi identities one finds

$$\nabla_i W^i{}_{jkl} = (n-3)(\nabla_j S_{kl} - \nabla_k S_{jl}) \equiv 2(n-3)C_{jkl} . \quad (4.24)$$

C_{jkl} is the Cotton tensor. This means that in $n > 3$ the identical vanishing of the Weyl tensor implies the vanishing of the Cotton tensor.

If $W^i{}_{jkl} = 0$ the condition for having $\tilde{R}^i{}_{jkl} = 0$ is from (4.23)

$$\tilde{S} \odot g = 0$$

i.e.

$$a_{ij} = S_{ij} \quad (4.25)$$

because for $n > 2$

$$F \odot g = 0$$

implies $F = 0$ as it is easily verified by taking traces. Solubility of (4.25) in terms of f implies the existence of a vector field $v_i = \partial_i f$ which solves

$$\nabla_i v_j = S_{ij} + v_i v_j - \frac{1}{2} v^l v_l g_{ij} . \quad (4.26)$$

We are interested in the inverse problem, i.e. given S_{ij} find an f such that $v_i = \partial_i f$ solves (4.26). On the other hand in we find a vector field v_i which solves (4.26) due to the symmetry in the indices of the r.h.s. of Eq.(4.26) and the absence of torsion we have $\partial_i v_j - \partial_j v_i = 0$, which locally assures the existence of a generating function f . Thus we have simply to find the necessary and sufficient condition for the existence of a vector field which satisfies Eq.(4.26).

It is useful to rewrite the previous equation as

$$\nabla v^k = S^k + v^k V - \frac{1}{2} v^l v_l dx^k \equiv A^k = S^k + r^k \quad (4.27)$$

where we introduced the 1-forms

$$S^k = S_i^k dx^i, \quad V = v_i dx^i .$$

After rewriting Eq.(4.27) as

$$dv^k = A^k - \Gamma_m^k v^m$$

we have the integrability condition for Eq.(4.27)

$$\nabla A^k - R^k{}_l v^l = ddv^k = 0$$

where R^k_l is the Riemann two form. As the Weyl tensor is assumed zero, the above equation becomes

$$0 = \nabla A^k - (S \odot g)^k_{lij} v^l \frac{dx^i \wedge dx^j}{2} = \nabla S^k + \nabla r^k - (S \odot g)^k_{lij} v^l \frac{dx^i \wedge dx^j}{2}. \quad (4.28)$$

But we have

$$\nabla r^k = S^k \wedge V + v^k dV - v_l S^l \wedge dx^k = (S^k_i v_j + v^k \partial_i v_j - v_l S^l_i \delta_j^k) dx^i \wedge dx^j \quad (4.29)$$

$$= (S^k_i v_j - v_l S^l_i \delta_j^k) dx^i \wedge dx^j = (S \odot g)^k_{lij} v^l \frac{dx^i \wedge dx^j}{2} \quad (4.30)$$

which cancels the last term in Eq.(4.28). Thus we are left with the necessary condition for integrability

$$\nabla S^k = 0 = dS^k + \Gamma_l^k \wedge S^l. \quad (4.31)$$

Due to zero torsion and metric compatibility it is also equivalent to

$$\nabla_i S_{kj} - \nabla_k S_{ij} = 0 \quad (4.32)$$

i.e. the vanishing of the Cotton tensor.

Due to Frobenius integrability theorem Eq.(4.31) is not only necessary but also sufficient for the integrability of Eq.(4.27) [2].

Notice that the Weyl tensor vanishes identically in dimension 3 but this does not mean that in dimension 3 the space is conformally flat as Eq.(4.24) does not imply the vanishing of the Cotton tensor. Thus in dimension 3 necessary and sufficient condition for being a geometry conformally flat is the vanishing of the Cotton tensor.

In dimension equal or higher than 4 being the Weyl tensor an invariant if it is different from zero the geometry cannot be conformally flat. If it is zero then Eq.(4.24) implies that the Cotton tensor is zero and the space is conformally flat. We conclude that in dimension equal or higher than 4 necessary and sufficient condition for being geometry conformally flat is the vanishing of the Weyl tensor.

References

- [1] [Wald] Appendix D
- [2] J.C. Gerretsen, "Lectures on tensor calculus and differential geometry", P. Noordhoff N.V. Groningen (1962)

4.26 Hodge * operation

This is a bijection between Λ^p and $\Lambda^{(n-p)}$ which can be established when we have a metric on the manifold. There are several level of abstraction with which such a concept can be introduced. Given an orientable manifold i.e. such that there exists an atlas with all transition functions with positive Jacobian, the expression (for definiteness we assume a metric with signature $(-, + + \dots)$).

$$\epsilon_{\mu_1 \dots \mu_n} = \sqrt{-g} \varepsilon_{\mu_1 \dots \mu_n}, \quad \text{with} \quad \varepsilon_{0,1,\dots,n-1} = 1$$

is an invariant tensor under diffeomorphisms with positive Jacobian. One can raise indices with the $g^{\mu\nu}$ and we have also

$$\epsilon^{\mu_1 \dots \mu_n} = \frac{1}{\sqrt{-g}} \varepsilon^{\mu_1 \dots \mu_n}, \quad \text{with} \quad \varepsilon^{0,1,\dots,n-1} = -1.$$

Taking the covariant differential of

$$\epsilon_{\mu_1, \dots, \mu_n} \epsilon^{\mu_1, \dots, \mu_n} = -n!$$

we have that $\epsilon_{\mu_1, \dots, \mu_n}$ is covariantly constant. Accordingly one can define an invariant form belonging to Λ^n given by

$$\epsilon = \frac{1}{n!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}.$$

More generally we have

$$\epsilon = \frac{1}{n!} \epsilon_{a_1 \dots a_n} e^{a_1} \wedge \dots \wedge e^{a_n}$$

where

$$\epsilon_{a_1 \dots a_n} = \sqrt{-\det g_{ab}} \varepsilon_{a_1 \dots a_n}.$$

Given a p -form

$$f = \frac{1}{p!} f_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

we define

$$*f = \frac{1}{p!(n-p)!} f^{\mu_1 \dots \mu_p} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_n} \in \Lambda^{(n-p)}.$$

The following relation holds

$$**f = -(-)^{p(n-p)} f.$$

The form $*f$ is called the form dual to f .

This allows to define product of two p -forms f and h which is a n form as

$$f \wedge *h = (f, h)\epsilon.$$

Due to the invariant nature of ϵ , (f, h) is an invariant function which is linear in f, h . It is immediately shown that

$$f \wedge *h = h \wedge *f = (-)^{p(n-p)} *h \wedge f$$

and thus $(f, h) = (h, f)$.

4.27 The current

Given the gauge connection A and $F = dA + A \wedge A$, we recall that $DF = 0$ (Bianchi identity). For any form $G \in \Lambda_{(1,1)}^g$ we have

$$DDG = F \wedge G - G \wedge F \in \Lambda_{(1,1)}^{g+2}. \quad (4.33)$$

In fact

$$\begin{aligned} DG &= dG + A \wedge G - (-)^g G \wedge A \in \Lambda_{(1,1)}^{g+1}; \\ DDG &= ddG + dA \wedge G - A \wedge dG - (-)^g dG \wedge A - (-)^{2g} G \wedge dA + \\ &+ A \wedge (dG + A \wedge G - (-)^g G \wedge A) - (-)^{g+1} (dG + A \wedge G - (-)^g G \wedge A) \wedge A = \\ &= (dA + A \wedge A) \wedge G - G \wedge (dA + A \wedge A) \end{aligned}$$

which is Eq.(4.33). We apply now Eq.(4.33) to $G = *F \in \Lambda_{(1,1)}^{n-2}$. Defined J by

$$D * F = *J$$

we have

$$DD * F = F \wedge *F - *F \wedge F = 0 \quad \text{as} \quad F \in \Lambda^2$$

i.e.

$$D * J = 0 .$$

This is the covariant conservation of the Yang-Mills current valid on a differential manifold in any dimension n and for arbitrary metric.

In the case of the Maxwell field in n -dimensions the above formulas simplify to

$$F = dA, \quad dF = 0, \quad d * F = *J, \quad 0 = dd * F = d * J .$$

The last is equivalent to

$$\partial_\lambda(\sqrt{-g}J^\lambda) = 0 . \quad (4.34)$$

Integrating Eq.(4.34) on a cylinder V of base Σ_1 with $x^0 = c_1 = \text{const.}$ and of cover Σ_2 with $x^0 = c_2 = \text{const.}$ and a mantle at large distances, provided the field F and thus J decreases sufficiently rapidly at large distances we obtain the conservation law

$$Q = \int_{\Sigma_2} \sqrt{-g} J^0 dx^1 \dots dx^{n-1} = \int_{\Sigma_1} \sqrt{-g} J^0 dx^1 \dots dx^{n-1} .$$

Using the ADM metric with $c = 1$ i.e. using as time coordinate $x^0 = ct$ we have

$$g \equiv \det g_{\mu\nu} = -N^2 \det h_{mn} \equiv -N^2 h$$

Moreover the normal to the space like surface $x^0 = \text{const.}$ is given by the time-like vector n whose covariant components are defined by $0 = n_\mu dx^\mu$ with $dx^\mu = (0, dx^1, dx^2, \dots)$ i.e. $n_\mu = (n_0, 0, 0, \dots)$. The normalization is provided by $-1 = n_\mu g^{\mu\nu} n_\nu = n_0 g^{00} n_0 = -n_0^2 N^{-2}$ and thus $n_\mu = (N, 0, 0, \dots)$. We can now rewrite the expression of the charge as

$$Q = \int_{\Sigma} \sqrt{h} n_\lambda J^\lambda dx^1 \dots dx^{n-1}$$

which in covariant form can be written

$$Q = \int_{\Sigma} n_\lambda J^\lambda d\Sigma$$

being $d\Sigma$ the invariant area element of the surface $\Sigma = \partial V$. In case of torsionless metric compatible connection as $\Gamma^\rho_{\lambda\rho} = \frac{1}{\sqrt{-g}} \partial_\lambda \sqrt{-g}$ we have

$$\frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g} J^\lambda) = \nabla_\lambda J^\lambda$$

and thus the current conservation can be written as

$$\nabla_\lambda J^\lambda = 0 .$$

In the non abelian case $D * F = *J$ is explicitly written as

$$\partial_\rho (\sqrt{-g} F^{\nu\rho}) + \sqrt{-g} (A_\rho F^{\nu\rho} - F^{\nu\rho} A_\rho) = \sqrt{-g} J^\nu$$

and $D * J = 0$ as

$$\partial_\rho (\sqrt{-g} J^\rho) + \sqrt{-g} (A_\rho J^\rho - J^\rho A_\rho) = 0.$$

This is the covariant conservation of the covariant current; due to the additional term containing the A_ρ from the previous equation we cannot derive a conservation law.

4.28 Action for gauge fields

In four dimensions the expression

$$\int \text{Tr}(F \wedge F)$$

is invariant under gauge transformations and under proper diffeomorphisms. It is however a pseudoscalar.

Under a variation of A we have

$$\delta(dA + A \wedge A) = d\delta A + \delta A \wedge A + A \wedge \delta A = D\delta A$$

and thus

$$\delta \int \text{Tr}(F \wedge F) = 2 \int \text{Tr}(D\delta A \wedge F) = 2 \int \text{Tr}(\delta A \wedge DF)$$

where we have integrated by parts, neglecting the surface term, provided δA vanishes sufficiently rapidly at infinity. Due to the Bianchi identity $DF = 0$ such a variation is zero independently of any equation of motion. A quantity with such a property is called a topological invariant and in our case is called the second Chern class.

It is useful even if not strictly necessary at the classical level, to have an action from which to derive the equations of motion. If we deal with pure gauge theory we have that

$$S_{YM} = \int \text{Tr}(F \wedge *F) \quad (4.35)$$

is invariant under diffs and under local gauge transformations. Notice that Eq.(4.35) is meaningful in n dimensions as $F \in \Lambda_{(1,1)}^2$ and $*F \in \Lambda_{(1,1)}^{n-2}$. Moreover due to the appearance of two antisymmetric structures it is a scalar. It provides also the equation of motion as under a variation of the gauge field we have

$$\delta S_{YM} = 2 \int \text{Tr}(D\delta A \wedge *F) .$$

Then using

$$d\text{Tr}(\delta A \wedge *F) = D\text{Tr}(\delta A \wedge *F) = \text{Tr}(D\delta A \wedge *F) - \text{Tr}(\delta A \wedge D *F) ,$$

for a variation of A which vanishes on the boundary we have

$$\delta S_{YM} = 2 \int \text{Tr}(\delta A \wedge D *F) .$$

For a faithful representation of the Lie algebra of a compact group or for a faithful representation of a semi-simple Lie algebra the vanishing of $\text{Tr}(\lambda\rho)$ for any λ , λ and ρ

belonging to the Lie algebra, gives $\rho = 0$ and thus we have $D * F = 0$. In presence of an external current we add

$$-2 \int \text{Tr}(A \wedge *J) .$$

and we reach the equation of motion $D * F = *J$. $DD * F = 0$ imposes $D * J = 0$. The current J as a rule arises from the coupling of the Yang-Mills field A to the matter fields ψ and the total action is given by

$$S = S_{YM} + S_{\text{matter}} .$$

S is constructed invariant under gauge transformations. The current is defined as

$$\delta S_{\text{matter}} = -2 \int \text{Tr}(\delta A \wedge *J) .$$

Let us consider an infinitesimal gauge transformation

$$\delta A = D\lambda .$$

S_{YM} is invariant while

$$\delta S_{\text{matter}} = -2 \int \text{Tr}(\delta A \wedge *J) + \delta_{\psi} S_{\text{matter}}$$

where $\delta_{\psi} S_{\text{matter}}$ is the variation of S_{matter} due to the change of the matter fields ψ under the gauge transformation. But on the equation of motion $\delta_{\psi} S_{\text{matter}} = 0$ as δS_{matter} on the equation of motion is zero for any variation of ψ . Thus on the equations of motion we have reached

$$0 = - \int \text{Tr}(D\lambda \wedge *J) = \int \text{Tr}(\lambda \wedge D * J)$$

which implies $D * J = 0$. This result is in agreement with the kinematic derivation of the same relation obtained from the identity $DD * F = 0$.

Finally notice that in four dimensions, if we have a configuration such that $F = \pm * F$ (self-dual (antiself)- dual field), F satisfies the equation of motion

$$D * F = DF \equiv 0 .$$

Chapter 5

The action for the gravitational field

5.1 The Hilbert action

Hilbert action is given by

$$S_H = \int_V \sqrt{-g} R d^n x = \int_V \sqrt{-g} g^{\mu\nu} R_{\mu\nu} d^n x. \quad (5.1)$$

From the variational viewpoint this action is not correct as it contains second derivatives of $g_{\mu\nu}$. It is like writing the action of a particle in one dimension as

$$S = \int_{t_0 q_0}^{t_1 q_1} \left[-\frac{1}{2} q \ddot{q} - V(q) \right] dt. \quad (5.2)$$

We have

$$\delta S = \int_{t_0 q_0}^{t_1 q_1} \delta q \left(-\ddot{q} - \frac{\partial V}{\partial q} \right) dt - \left[\frac{1}{2} q \delta \dot{q} \right]_{t_0}^{t_1}$$

i.e. in order to obtain the correct equations of motion it is not sufficient to impose $\delta q = 0$ at t_0 and t_1 but it is necessary to impose also $\delta \dot{q} = 0$ at t_0 and t_1 . That would give 4 conditions at t_0 and t_1 which is as a rule inconsistent with the equations of motion. I.e. action (5.2) cannot be used in a variational treatment.

Such action can be mended by adding a proper boundary term

$$S_1 = S + \frac{1}{2} (q\dot{q})(t_1) - \frac{1}{2} (q\dot{q})(t_0).$$

A similar cure has to be applied to the Hilbert action by adding a proper boundary term giving rise to the so called $\text{Tr}K$ -action. This will be done in Section 6.3.

5.2 Einstein's $\Gamma\Gamma$ action

In performing the variation with respect to $g_{\mu\nu}$ we shall use the convention to treat $\delta g_{\mu\nu}$ as a tensor i.e.

$$\delta g^{\mu\nu} = g^{\mu\alpha} \delta g_{\alpha\beta} g^{\beta\nu}$$

which means that $\delta(g^{\mu\nu}) = -\delta g^{\mu\nu}$. From

$$S_H = \int_V d^n x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} = \int_V d^n x \sqrt{-g} g^{\mu\nu} [\partial_\lambda \Gamma_\nu - \partial_\nu \Gamma_\lambda + \Gamma_\lambda \Gamma_\nu - \Gamma_\nu \Gamma_\lambda]^\lambda_\mu$$

we have

$$\begin{aligned} \delta S_H &= \int_V d^n x \delta(\sqrt{-g} g^{\mu\nu}) [\partial_\lambda \Gamma_\nu - \partial_\nu \Gamma_\lambda + \Gamma_\lambda \Gamma_\nu - \Gamma_\nu \Gamma_\lambda]^\lambda_\mu + \int_V d^n x \sqrt{-g} g^{\mu\nu} [D_\lambda \delta \Gamma_\nu - D_\nu \delta \Gamma_\lambda]^\lambda_\mu \\ &= - \int_V d^n x \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} + \int_V d^n x \sqrt{-g} g^{\mu\nu} [D_\lambda \delta \Gamma_\nu - D_\nu \delta \Gamma_\lambda]^\lambda_\mu \end{aligned} \quad (5.3)$$

being $G_{\mu\nu}$ the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R .$$

D is the space analogue of the gauge gauge covariant differential of Section 4.8 where we had $\delta F = D\delta A$. Here we have

$$D_\lambda \delta \Gamma_{\mu\nu}^\rho = \partial_\lambda \delta \Gamma_{\mu\nu}^\rho + \Gamma_{\sigma\lambda}^\rho \delta \Gamma_{\mu\nu}^\sigma - \delta \Gamma_{\sigma\nu}^\rho \Gamma_{\mu\lambda}^\sigma .$$

Using the 0-torsion property of the Levi-Civita connection i.e. $\Gamma_{\beta\gamma}^\alpha = \Gamma_{\gamma\beta}^\alpha$, the above equation can be rewritten as

$$\delta S_H = - \int_V d^n x \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} + \int_V d^n x \sqrt{-g} g^{\mu\nu} [\nabla_\lambda \delta \Gamma_\nu - \nabla_\nu \delta \Gamma_\lambda]^\lambda_\mu .$$

where

$$\nabla_\lambda \delta \Gamma_{\beta\gamma}^\alpha$$

is the full covariant derivative of the (1,2) tensor $\delta \Gamma_{\beta\gamma}^\alpha$ taking into account also the covariant vector index γ

$$\nabla_\lambda \delta \Gamma_{\beta\gamma}^\alpha = D_\lambda \delta \Gamma_{\beta\gamma}^\alpha - \delta \Gamma_{\beta\sigma}^\alpha \Gamma_{\gamma\lambda}^\sigma .$$

We recall that $\delta \Gamma$ is a tensor and thus

$$v^\lambda \equiv \delta \Gamma^\lambda_{\mu\nu} g^{\mu\nu} - \delta \Gamma^\nu_{\mu\nu} g^{\mu\lambda} \equiv 2\delta \Gamma^{[\lambda\nu]}_{\nu} \quad (5.4)$$

is a vector. Thus we found the identity

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\lambda v^\lambda$$

which is known as Palatini identity and we have reached the result

$$\delta S_H = - \int_V \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} d^n x + \int_V \partial_\lambda (\sqrt{-g} v^\lambda) d^n x = - \int_V \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} d^n x + \int_{\partial V} v^\lambda \Sigma_\lambda. \quad (5.5)$$

where

$$\Sigma_\lambda = \epsilon_{\lambda\sigma_2\dots\sigma_n} \frac{\partial x^{\sigma_2}}{du^2} \dots \frac{\partial x^{\sigma_n}}{du^n} du^2 \dots du^n$$

and ∂V is given by $x^\mu(0, u^2 \dots u^{n-1})$ being the u 's the local coordinates with $u_1 = 0$ on the boundary, $u^1 < 0$ inside V and $u^1 > 0$ outside V and $\frac{\partial x^\mu}{\partial u^\nu} > 0$ (see Section 3.7).

We have *algebraically*, treating Γ as a tensor

$$\begin{aligned} \partial_\lambda \Gamma_{\mu\nu}^\alpha - \partial_\nu \Gamma_{\mu\lambda}^\alpha + [\Gamma_\lambda, \Gamma_\nu]^\alpha{}_\mu &= D_\lambda \Gamma_{\mu\nu}^\alpha - D_\nu \Gamma_{\mu\lambda}^\alpha - [\Gamma_\lambda, \Gamma_\nu]^\alpha{}_\mu = \\ &= \nabla_\lambda \Gamma_{\mu\nu}^\alpha - \nabla_\nu \Gamma_{\mu\lambda}^\alpha - [\Gamma_\lambda, \Gamma_\nu]^\alpha{}_\mu \end{aligned}$$

and thus

$$\begin{aligned} g^{\mu\nu} R_{\mu\nu} &= g^{\mu\nu} (\partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + [\Gamma_\lambda, \Gamma_\nu]^\lambda{}_\mu) = -g^{\mu\nu} [\Gamma_\lambda, \Gamma_\nu]^\lambda{}_\mu + 2 \nabla_\lambda \Gamma^{[\lambda\nu]}{}_\nu = \\ &= -g^{\mu\nu} [\Gamma_\lambda, \Gamma_\nu]^\lambda{}_\mu + \frac{2}{\sqrt{-g}} \partial_\lambda (\sqrt{-g} \Gamma^{[\lambda\nu]}{}_\nu) \equiv -g^{\mu\nu} [\Gamma_\lambda, \Gamma_\nu]^\lambda{}_\mu + \frac{1}{\sqrt{-g}} \partial_\lambda (\sqrt{-g} W^\lambda) \quad (5.6) \end{aligned}$$

with the obvious definition of W^λ . We stress that Eq.(5.6) is an algebraic identity, where the transformation properties of the symbol $\Gamma_{\mu\nu}^\lambda$ play no role. Then we define Einstein's $\Gamma\Gamma$ -action

$$S_{\Gamma\Gamma} = - \int_V \sqrt{-g} g^{\mu\nu} [\Gamma_\lambda, \Gamma_\nu]^\lambda{}_\mu d^n x = S_H - \int_V \partial_\lambda (\sqrt{-g} W^\lambda) d^n x = S_H - \oint_{\partial V} W^\lambda \Sigma_\lambda.$$

$S_{\Gamma\Gamma}$ has only first order derivatives and from Eq.(5.5) its variation keeping $g_{\mu\nu}$ fixed on ∂V is given simply by

$$\delta S_{\Gamma\Gamma} = - \int_V \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} d^n x.$$

In fact on ∂V we have

$$\begin{aligned} \delta W^\lambda &= 2 \delta (\sqrt{-g} \Gamma^{[\lambda\nu]}{}_\nu) = \sqrt{-g} \delta (\Gamma^\lambda{}_{\nu'\nu} g^{\nu'\nu} - \Gamma^\nu{}_{\nu'\nu} g^{\nu'\lambda}) = \\ &= \sqrt{-g} (\delta \Gamma^\lambda{}_{\nu'\nu} g^{\nu'\nu} - \delta \Gamma^\nu{}_{\nu'\nu} g^{\nu'\lambda}) = 2\sqrt{-g} \delta \Gamma^{[\lambda\nu]}{}_\nu = \sqrt{-g} v^\lambda. \end{aligned}$$

The algebraic manipulations are not covariant and the resulting action $S_{\Gamma\Gamma}$ is not invariant. Despite that, it is a good action. It is an example of a non invariant action which gives rise to covariant equations.

5.3 Palatini first order action

The previous approach is called the second order approach as the fundamental variables $g_{\mu\nu}$ appear with second order derivatives.

In the Palatini approach we are given a metric $g_{\mu\nu}$ and a torsionless connection Γ , independent of the metric. In the following all covariant derivative will refer the connection Γ . No metric compatibility of Γ is assumed.

$R^\alpha_{\mu\beta\nu}$ is well defined independently of $g_{\mu\nu}$, and also the Ricci tensor $R_{\mu\nu}$. An invariant action is

$$\int_V \sqrt{-g} g^{\mu\nu} R_{\mu\nu} d^n x + \text{boundary terms.}$$

where the covariant boundary terms are given in Section 6.4.

We remark that

$$\sqrt{-g} \nabla_\mu v^\mu = \sqrt{-g} (\partial_\mu v^\mu + \Gamma^\rho_{\mu\rho} v^\mu) = \partial_\mu (\sqrt{-g} v^\mu) + \sqrt{-g} (\Gamma^\rho_{\mu\rho} - \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g}) v^\mu.$$

The variation with respect to $g_{\mu\nu}$ gives

$$0 = - \int \sqrt{-g} \delta g^{\mu\nu} G_{\mu\nu} d^n x = 0, \quad \text{i.e.} \quad G_{(\mu\nu)} = 0,$$

while the variation with respect to $\Gamma^\lambda_{\mu\nu}$ gives

$$\begin{aligned} 0 &= \int \sqrt{-g} g^{\mu\nu} [D_\lambda \delta \Gamma_\nu - D_\nu \delta \Gamma_\lambda]^\lambda_\mu d^n x = \\ &= \int \sqrt{-g} g^{\mu\nu} [\nabla_\lambda \delta \Gamma_\nu - \nabla_\nu \delta \Gamma_\lambda]^\lambda_\mu d^n x \end{aligned}$$

due to the absence of torsion.

Consider the first term

$$\begin{aligned} \sqrt{-g} g^{\mu\nu} \nabla_\lambda \delta \Gamma^\lambda_{\mu\nu} &= \sqrt{-g} \nabla_\lambda (g^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu}) - \sqrt{-g} \delta \Gamma^\lambda_{\mu\nu} \nabla_\lambda g^{\mu\nu} \\ &= \partial_\lambda (\sqrt{-g} g^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu}) + \sqrt{-g} (\Gamma^\rho_{\lambda\rho} - \partial_\lambda \log \sqrt{-g}) g^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu} - \sqrt{-g} \delta \Gamma^\lambda_{\mu\nu} \nabla_\lambda g^{\mu\nu} \\ &\equiv \partial_\lambda (\sqrt{-g} g^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu}) + \sqrt{-g} C^\mu_{\lambda\rho} \delta \Gamma^\lambda_{\mu\nu} \end{aligned} \quad (5.7)$$

with

$$C^\mu_{\lambda\rho} = g^{\mu\nu} (\Gamma^\rho_{\lambda\rho} - \partial_\lambda \log \sqrt{-g}) - \nabla_\lambda g^{\mu\nu}. \quad (5.8)$$

The divergence term in (5.7) is canceled by the variation of the boundary term. Adding the second term gives

$$(C^\mu_{\lambda\rho} - C^{\mu\rho}_{\lambda\rho}) \delta \Gamma^\lambda_{\mu\nu}$$

and due to the symmetry of $\Gamma_{\mu\nu}^\lambda$ in the lower indices

$$C_\lambda^{\mu\nu} - \frac{1}{2}(C_\rho^{\mu\rho}\delta_\lambda^\nu + C_\rho^{\nu\rho}\delta_\lambda^\mu) = 0. \quad (5.9)$$

Contracting ν with λ we have

$$\frac{1-n}{2}C_\nu^{\mu\nu} = 0$$

and substituting in Eq.(5.9)

$$C_\lambda^{\mu\nu} = 0. \quad (5.10)$$

In Eqs.(5.8,5.10) write Γ as $\hat{\Gamma} + \Delta$, where $\hat{\Gamma}$ is the metric compatible connection; then we have

$$\Delta_{\mu'\lambda}^\mu g^{\mu'\nu} + \Delta_{\nu'\lambda}^\nu g^{\mu\nu'} - g^{\mu\nu} \Delta_{\lambda\rho}^\rho = 0.$$

Taking the trace

$$(2-n)\Delta_{\lambda\rho}^\rho = 0$$

which combined with the previous equation gives

$$\Delta_{\mu\nu\lambda} + \Delta_{\nu\mu\lambda} = 0.$$

Rotating twice and subtracting one from the other two, recalling that $\Delta_{\mu\nu\lambda} = \Delta_{\mu\lambda\nu}$ we have $\Delta_{\mu\nu\lambda} = 0$ and thus $\Gamma_{\mu\nu}^\lambda$ is the metric compatible torsionless connection i.e. the Levi-Civita connection.

The Palatini result renders possible the first order formulation of gravity. If the matter action depends only on $g_{\mu\nu}$ the variation with respect to $\Gamma_{\mu\nu}^\lambda$ is still zero and the same procedure holds.

5.4 Einstein-Cartan formulation

Here again we shall use a first order formalism, with the difference that now metric compatibility is assumed from the start and the vanishing of torsion is obtained as the consequence of the equations of motion in absence of matter; the formalism is apt to describe the coupling with fermions and here a non vanishing torsion will appear.

We shall consider a connection on the tangent space in which we have chosen an orthonormal system of vectors \mathbf{e}_a , $\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab}$ and the connection 1-form due to metric compatibility must belong to the algebra of $SO(n-1, 1)$, $\Gamma_b^a \in so(n-1, 1)$. From these we compute the curvature 2-form R_b^a . The action is given by

$$S_{EC} = - \int \text{Tr}(R\eta^{-1} \wedge H) + \text{boundary terms}$$

where the covariant boundary terms are given in Section 6.5. H is the $n - 2$ form

$$H_{ba} = \frac{1}{(n-2)!} \varepsilon_{baa_3 \dots a_n} e^{a_3} \wedge \dots \wedge e^{a_n} \in \Lambda_{(0,2)}^{n-2}$$

being $\varepsilon_{a_1 \dots a_n}$ the standard antisymmetric symbol, with $\varepsilon_{01 \dots n-1} = 1$. Such action is obviously invariant under diffeomorphisms; it is invariant also under local $SO(n-1, 1)$ transformations. In fact under

$$\mathbf{e}_a = \mathbf{e}'_b \Omega^b{}_a, \quad \text{from which } e'^c = \Omega^c{}_d e^d$$

we already know that

$$R' = \Omega R \Omega^{-1}.$$

From

$$\varepsilon_{a_1 \dots a_n} \Omega^{a_1}{}_{b_1} \dots \Omega^{a_n}{}_{b_n} = \det(\Omega) \varepsilon_{b_1 \dots b_n}$$

one obtains

$$\varepsilon_{baa_3 \dots a_n} \Omega^{a_3}{}_{b_3} \dots \Omega^{a_n}{}_{b_n} = \det(\Omega) (\Omega^{-1})^{b_1}{}_b (\Omega^{-1})^{b_2}{}_a \varepsilon_{b_1 \dots b_n}$$

from which one gets

$$H' = \det(\Omega) (\Omega^{-1})^T H \Omega^{-1}.$$

Then

$$\text{Tr}(R' \eta^{-1} \wedge H') = \det(\Omega) \text{Tr}(\Omega R \Omega^{-1} \eta^{-1} (\Omega^{-1})^T \wedge H \Omega^{-1}) = \det(\Omega) \text{Tr}(R \eta^{-1} \wedge H)$$

as $\Omega^{-1} \eta^{-1} (\Omega^{-1})^T = \eta^{-1}$. Thus strictly speaking the action is invariant only under proper local Lorentz transformations.

In the following indices are raised and lowered with the tensors $\eta^{-1} = \eta = \text{diag}(-1, 1 \dots 1)$. S_{EC} can also be written as

$$S_{EC} = \frac{1}{(n-2)!} \int R^{ab} \wedge e^{a_3} \wedge \dots \wedge e^{a_n} \varepsilon_{aba_3 \dots a_n}. \quad (5.11)$$

Before extracting the equations of motion we want to relate S_{EC} with the Hilbert action.

Writing

$$R^{ab} = \frac{1}{2!} R^{ab}{}_{fg} e^f \wedge e^g$$

we have

$$\begin{aligned} S_{EC} &= \frac{1}{2(n-2)!} \int R^{ab}{}_{fg} \varepsilon_{abcd \dots} (-\varepsilon^{fgcd \dots}) \det(e^a_\mu) dx^0 \wedge dx^1 \wedge \dots \wedge dx^{n-1} = \\ &= \frac{1}{2} \int R^{ab}{}_{fg} \delta^{fg}{}_{ab} \epsilon = \int R \epsilon. \end{aligned}$$

$\epsilon = \sqrt{-\det g_{ab}} e^0 \wedge \cdots \wedge e^{n-1}$ is the volume form as

$$g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b, \quad e^2 = \det(e_\mu^a)^2 = -g.$$

It is however much better to work directly with action (5.11). For simplicity we work from now on in four dimensions, but the extension to n dimensions is trivial. Variation of Γ gives

$$\delta\Gamma^{ab} \wedge (De^c \wedge e^d - e^c \wedge De^d) \varepsilon_{abcd} = 0$$

where to keep in touch with gauge theories we denoted the ∇ appearing in Section 4.8 as D , but they are the same operation. Thus

$$De^c \wedge e^d \varepsilon_{abcd} = 0; \quad \text{i.e.} \quad S^c \wedge e^d \varepsilon_{bacd} = 0 \quad (5.12)$$

where S^c is the torsion. But this implies $S^c = 0$ i.e. vanishing torsion in absence of matter. In fact write

$$S^c = \frac{1}{2} S_{uv}^c e^u \wedge e^v$$

and multiply Eq.(5.12) by $\wedge e^f$ to obtain

$$S_{uv}^c \delta_{abc}^{uvf} = 0$$

i.e.

$$S_{ab}^f + S_{ca}^c \delta_b^f - S_{cb}^c \delta_a^f = 0.$$

Taking the trace

$$(n-2)S_{cb}^c = 0 \quad \text{from which} \quad S_{ab}^c = 0.$$

Thus in absence of matter Γ in addition of being metric compatible is also torsionless, i.e. it is the Levi-Civita connection.

Variation of e^d implies

$$R^{ab} \wedge e^c \varepsilon_{abcd} = 0 \quad (5.13)$$

i.e. Einstein's equations. In fact multiplying by $\wedge e^h$ we have

$$\frac{1}{2} R^{ab}{}_{fg} \varepsilon^{fgh} \varepsilon_{abcd} \epsilon = 0$$

or

$$0 = R^{ab}{}_{fg} \delta_{abd}^{fgh} = 2(R\delta_d^h - 2R_d^h) = -4G_d^h$$

being G_d^h the Einstein tensor.

References

- [1] A. Trautman, “On the Einstein-Cartan equation” Bull. Acad. Pol. Sci. Ser. Math. Astron. Phys. 20 (1972) 185, 503, 895
- [2] Y. Ne’eman, T. Regge, “Gauge theory of gravity and supergravity on a group manifold” Riv. Nuovo Cim. I (1978) 1

5.5 The energy momentum tensor

If the Lagrangian L does not contain explicitly the coordinates we have

$$\partial_\mu L = \frac{\partial L}{\partial \phi} \partial_\mu \phi + \frac{\partial L}{\partial \partial_\lambda \phi} \partial_\mu \partial_\lambda \phi = (\text{via eq. of motion}) = \partial_\lambda \left(\frac{\partial L}{\partial \partial_\lambda \phi} \partial_\mu \phi \right)$$

and thus

$$\partial_\lambda \left(-\frac{\partial L}{\partial \partial_\lambda \phi} \partial_\mu \phi + \delta_\mu^\lambda L \right) = 0$$

i.e.

$$-\frac{\partial L}{\partial \partial_\lambda \phi} \partial_\mu \phi + \delta_\mu^\lambda L = T_\mu^{c\lambda}$$

is conserved.

Example:

$$L = \frac{1}{2} [-\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 - V(\phi)]$$

$$T_{\lambda\mu}^c = \partial_\lambda \phi \partial_\mu \phi + \eta_{\lambda\mu} \frac{1}{2} [-\eta^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi - m^2 \phi^2 - V(\phi)].$$

The conserved energy-momentum vector (covariant components) is given by

$$P^\mu = \int n_\lambda T^{c\lambda\mu} d^{(n-1)}x$$

with $n_\mu = (1, 0, 0, 0)$.

$$T_{00}^c = \frac{1}{2} [\partial_0 \phi \partial_0 \phi + \partial_j \phi \partial_j \phi + m^2 \phi^2 + V(\phi)]$$

is the energy density.

5.6 Coupling of matter to the gravitational field

We give an alternative definition of the energy momentum tensor by means of the coupling to a gravitational field.

It is not always possible to render the matter Lagrangian invariant under diffeomorphisms by introducing the metric tensor $g_{\mu\nu}$. If it is possible the lagrangian density is given by $\sqrt{-g} L_s$ where L_s is a *scalar* and the action is

$$S_m = \int_V \sqrt{-g} L_s d^n x = \int_V L d^n x.$$

Under diffeomorphisms

$$x^\mu = x'^\mu + \varepsilon \xi^\mu(x')$$

(which obviously include the translations) we have (pull-back from x to x')

$$\begin{aligned} g'_{\alpha\beta}(x') &= g_{\alpha\beta}(x') + \varepsilon \xi^\lambda(x') \partial_\lambda g_{\alpha\beta}(x') + \varepsilon \partial_\alpha \xi^\lambda(x') g_{\lambda\beta}(x') + \varepsilon \partial_\beta \xi^\lambda(x') g_{\alpha\lambda}(x') = \\ &= g_{\alpha\beta}(x') + \varepsilon L_\xi g_{\alpha\beta}(x') \end{aligned}$$

being L_ξ the Lie derivative. In absence of torsion we can compute the Lie derivative replacing usual derivatives with covariant derivatives and assuming metric compatibility we obtain

$$g'_{\alpha\beta} = g_{\alpha\beta} + \varepsilon \nabla_\beta \xi_\alpha + \varepsilon \nabla_\alpha \xi_\beta .$$

Putting to zero the change in the action we have

$$0 = \int (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) \frac{\partial \sqrt{-g} L_s}{\partial g_{\mu\nu}} d^n x + \int \frac{\partial \sqrt{-g} L_s}{\partial \partial_\lambda g_{\mu\nu}} \partial_\lambda (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu) d^n x + \delta_\phi S_m$$

where the last term is the contribution due to the variation of the matter fields under the diffeomorphisms. Such contribution vanishes on the Lagrange equations of motion for the matter fields. Integrating by parts we have

$$0 = 2 \int \nabla_\mu \xi_\nu \left(\frac{\partial \sqrt{-g} L_s}{\partial g_{\mu\nu}} - \partial_\lambda \frac{\partial \sqrt{-g} L_s}{\partial \partial_\lambda g_{\mu\nu}} \right) d^n x$$

and again integrating by parts

$$0 = - \int \sqrt{-g} \xi_\nu \nabla_\mu T^{\mu\nu} d^n x$$

from which

$$\nabla_\mu T^{\mu\nu} = 0$$

having defined

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \left[\frac{\partial \sqrt{-g} L_s}{\partial g_{\mu\nu}} - \partial_\lambda \frac{\partial \sqrt{-g} L_s}{\partial \partial_\lambda g_{\mu\nu}} \right] .$$

We can summarize the result as follows

$$\int \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} d^n x = - \int \sqrt{-g} T_{\mu\nu} \delta (g^{\mu\nu}) d^n x = \int \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} d^n x = 2\delta S_m \quad (5.14)$$

where we used the notation of Section 5.2 $\delta g^{\mu\nu} = g^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu}$.

Example: The invariant Lagrangian of the previous case is

$$L_s = \frac{1}{2}[-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - V(\phi)]$$

and taking into account that L_s does not depend on $\partial_\lambda g_{\mu\nu}$ we have

$$T^{\mu\nu} = 2\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial g_{\mu\nu}}L_s + 2\frac{\partial L_s}{\partial g_{\mu\nu}} = \nabla^\mu\phi\nabla^\nu\phi + g^{\mu\nu}L_s$$

with $\nabla^\mu\phi = g^{\mu\nu}\partial_\nu\phi$, which agrees with the previous one for $g_{\mu\nu} = \eta_{\mu\nu}$.

5.7 Dimensions of the Hilbert action

Giving $g_{\mu\nu}$ the dimensions l^2 and x^μ dimensions l^0 we have in n dimensions $\Gamma_{\beta\gamma}^\alpha \sim l^0$, $R_{\beta\gamma\delta}^\alpha \sim l^0$, $R_{\beta\delta} \sim l^0$, $R \sim l^{-2}$, $\sqrt{-g} \sim l^n$ and thus $S_H \sim l^{n-2}$.

If we give the dimensions $g_{\mu\nu} \sim l^0$, and $x^\mu \sim l^1$ we have in n dimensions $\Gamma_{\beta\gamma}^\alpha \sim l^{-1}$, $R_{\beta\gamma\delta}^\alpha \sim l^{-2}$, $R_{\beta\delta} \sim l^{-2}$, $R \sim l^{-2}$, $\sqrt{-g} \sim l^0$ and thus again $S_H \sim l^{n-2}$.

From (Poisson equation in $n - 1$ dimensions)

$$G\frac{m^2}{l^{n-3}} \sim mc^2$$

we have

$$mc^2 \sim \frac{c^4 l^{n-3}}{G}$$

and

$$\text{action} \sim mc^2 t \sim mc^2 \frac{l}{c} \sim \frac{c^3 l^{n-2}}{G}$$

and thus we have the action, in all dimensions

$$\frac{c^3}{16\pi G}S_H + S_{\text{matter}}$$

where the factor 16π is obtained by fitting Newton's law. From

$$\hbar = \frac{c^3 l_P^{n-2}}{G}$$

we derive the expression of the Planck length

$$l_P = \left(\frac{G\hbar}{c^3}\right)^{1/(n-2)}$$

which is $n = 4$ becomes

$$l_P = \left(\frac{G\hbar}{c^3} \right)^{1/2} .$$

Thus the adimensional exponent which appears in Feynman functional integral in four dimensions is

$$\frac{1}{16\pi l_P^2} (S_H + \text{boundary terms}) + \frac{1}{\hbar} S_{\text{matter}} .$$

The Planck length can be obtained also equating one half of the Schwarzschild radius (in four dimensions)

$$\frac{Gm^2}{\frac{r_s}{2}} = mc^2 \quad \text{i.e.} \quad \frac{r_s}{2} = \frac{Gm}{c^2}$$

to the Compton wave length

$$r_c = \frac{\hbar}{mc} .$$

Solving in m the equation $r_s/2 = r_c$ gives the Planck mass

$$m_P = \left(\frac{\hbar c}{G} \right)^{1/2} = 1.21 \cdot 10^{19} \text{ GeV}/c^2 = 2.17 \cdot 10^{-8} \text{ kg}$$

whose Compton wave length is the Planck length

$$l_P = \frac{\hbar}{m_P c} = 1.62 \cdot 10^{-35} \text{ m} .$$

One can also obtain a complete set of units exploiting the electric charge. From

$$\frac{e^2}{r} = G \frac{m^2}{r}$$

we have

$$m = \sqrt{\frac{e^2}{G}}$$

where \hbar does not appear. Such m is related to the Planck mass as follows

$$m = \sqrt{\frac{e^2}{G}} = m = \sqrt{\frac{e^2 \hbar c}{\hbar c G}} = \frac{1}{\sqrt{137}} m_P .$$

5.8 Coupling of the Dirac field to the gravitational field

First we consider the Dirac Lagrangian in Minkowski space.

We saw how under a Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{5.15}$$

we have with

$$\Psi(x) = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

$$\Psi'(x') = \begin{pmatrix} \tilde{A} & 0 \\ 0 & A \end{pmatrix} \Psi(x) \equiv \mathcal{A}\Psi(x)$$

and we shall write $\mathcal{A} = \mathcal{A}[\Lambda]$ where \mathcal{A} is defined up to a sign.

We recall that

$$A\sigma_\mu x^\mu A^+ = \sigma_\mu x'^\mu = \sigma_\mu \Lambda^\mu{}_\nu x^\nu$$

or

$$A\sigma_\mu A^+ = \sigma_\nu \Lambda^\nu{}_\mu \quad (5.16)$$

and similarly

$$\tilde{A}\tilde{\sigma}_\mu \tilde{A}^+ = \tilde{\sigma}_\nu \Lambda^\nu{}_\mu \quad (5.17)$$

with $\tilde{\sigma}_\mu = (\sigma_0, -\sigma_m)$ and $\tilde{A} = (A^+)^{-1}$.

$\Psi^+\Psi$ is not an invariant; instead $\Psi^+\gamma^0\Psi = -i(\psi^+\phi + \phi^+\psi) \rightarrow -i(\psi^+A^{-1}A\phi + \phi^+A^+A^{-1}\psi)$ is invariant. It is usual to define

$$\bar{\Psi} = i\Psi^+\gamma^0 \equiv \Psi^+\gamma^4$$

and we have

$$\bar{\Psi}'(x') = \bar{\Psi}(x)\mathcal{A}^{-1}.$$

We recall that

$$\gamma_\mu = i \begin{pmatrix} 0 & \tilde{\sigma}_\mu \\ \sigma_\mu & 0 \end{pmatrix}$$

and from Eq.(5.16,5.17) we have

$$\mathcal{A}\gamma_\mu\mathcal{A}^{-1} = \gamma_\nu\Lambda^\nu{}_\mu. \quad (5.18)$$

In general for a vector v we have

$$\mathcal{A}\gamma^\mu\mathcal{A}^{-1}v_\mu = \mathcal{A}\gamma_\mu\mathcal{A}^{-1}v^\mu = \gamma_\nu\Lambda^\nu{}_\mu v^\mu = \gamma_\nu v'^\nu = \gamma^\nu v'_\nu \quad (5.19)$$

from which follows the invariance of the Lagrangian

$$L = \bar{\Psi}(\gamma^\mu\partial_\mu + m)\Psi.$$

The invariance of the mass term is trivial while for the kinetic term we have under global Lorentz transformations

$$\bar{\Psi}\gamma^\mu\partial_\mu\Psi = \bar{\Psi}\mathcal{A}^{-1}\mathcal{A}\gamma^\mu\mathcal{A}^{-1}\partial_\mu\mathcal{A}\Psi = \bar{\Psi}'\gamma^\mu\partial'_\mu\Psi'$$

having used Eq.(5.19).

We want now to write a Lagrangian invariant under diffeomorphisms and under local Lorentz transformations $SO(3, 1)$. Let us consider the Lagrangian

$$\bar{\Psi}e^\mu_a\gamma^a(\partial_\mu + \omega_\mu)\Psi + m\bar{\Psi}\Psi. \quad (5.20)$$

being ω_μ the components of a 4×4 matrix valued 1-form. We recall that the vierbeins e^μ_a are defined by $e^a = e^\mu_a dx^\mu$, being e^a the forms dual to \mathbf{e}_a ; the e^μ_a appearing in Eq.(5.20) are the inverse of e^μ_a i.e. $e^\mu_a e^\nu_b = \delta^{\mu\nu}$. The procedure is to treat the Ψ as scalars under diffeomorphisms; then we have that the above written Lagrangian is invariant under diffeomorphisms, due to the contraction $e^\mu_a \partial_\mu$. Let us now consider local $SO(3, 1)$ transformations (no diffeomorphism) $\mathbf{e}_a = \mathbf{e}'_b \Lambda^b_a$, which give $v'^a = \Lambda^a_b v^b$ (cfr. Eq.(5.15)), where Λ now depends on x . We recall that

$$\Gamma' = \Lambda d\Lambda^{-1} + \Lambda \Gamma \Lambda^{-1} \quad (5.21)$$

and

$$\Psi'(x') = \mathcal{A}[\Lambda]\Psi(x).$$

We have

$$\begin{aligned} \bar{\Psi}e^\mu_a\gamma^a(\partial_\mu + \omega_\mu)\Psi + m\bar{\Psi}\Psi &= \bar{\Psi}'e'^\mu_a\mathcal{A}\gamma^a\mathcal{A}^{-1}(\partial_\mu + \mathcal{A}\partial_\mu\mathcal{A}^{-1} + \mathcal{A}\omega_\mu\mathcal{A}^{-1})\Psi' + m\bar{\Psi}'\Psi' = \\ &= \bar{\Psi}'e'^\mu_a\gamma^a(\partial_\mu + \mathcal{A}\partial_\mu\mathcal{A}^{-1} + \mathcal{A}\omega_\mu\mathcal{A}^{-1})\Psi' + m\bar{\Psi}'\Psi' \end{aligned}$$

due to Eq.(5.19). Thus we have that the Lagrangian (5.20) is invariant iff the 1-form ω transforms as follows

$$\omega' = \mathcal{A}d\mathcal{A}^{-1} + \mathcal{A}\omega\mathcal{A}^{-1}. \quad (5.22)$$

Any 1- form which transforms according to Eq.(5.22) does the job. It is important however that one can find a ω with the above properties, without introducing a new dynamical field. We recall that Γ^a_b belongs to the algebra of $SO(3, 1)$ due to the metric compatibility

$$0 = d\eta - \eta\Gamma - \Gamma^T\eta = -\eta\Gamma - \Gamma^T\eta.$$

Moreover we saw that there is (up to the sign) a one to one correspondence between Λ and \mathcal{A} . Thus with $\Lambda = I + \varepsilon\rho$ and $\mathcal{A}[\Lambda] = \mathcal{A}[I + \varepsilon\rho] = I + \varepsilon a$

$$(I + \varepsilon\rho) \leftrightarrow (I + \varepsilon a)$$

induced by Eqs.(5.16,5.17) which as we are in the neighborhood of the origin, sets a one-to-one correspondence between ρ and a . a is a linear function of ρ whose space is $n(n-1)/2 = 6$ dimensional (as we are referring to the frames \mathbf{e}_a we shall use latin indices)

$$a = \sum_a^b \rho^a_b = \sum_{ab} \rho^{ab}.$$

We construct ω as follows

$$\omega = \Sigma_a^b \Gamma^a_b = \Sigma_{ab} \Gamma^{ab} \quad (5.23)$$

i.e. ω are defined as the representative in the representation $(1/2, 0) \oplus (0, 1/2)$ of the Lie algebra elements Γ . We must now prove that under the transformation (5.21) of Γ the so defined ω transforms according to (5.22) i.e.

$$\omega' = \Sigma_a^b (\Lambda d\Lambda^{-1} + \Lambda \Gamma \Lambda^{-1})^a_b = \mathcal{A} d\mathcal{A}^{-1} + \mathcal{A} \omega \mathcal{A}^{-1}.$$

This simply follows from the fact that $\mathcal{A} \in (1/2, 0) \oplus (0, 1/2)$ is a (double valued) representation of $\Lambda \in SO(3, 1)$.

In fact if Λ and $\Lambda + d\Lambda$ are two nearby transformations of $SO(3, 1)$ then from

$$\Lambda(\Lambda^{-1} + d\Lambda^{-1}) = I + \Lambda d\Lambda^{-1}$$

we have that $\Lambda d\Lambda^{-1} \in so(3, 1)$, i.e. it belongs to the algebra of $SO(3, 1)$. Then we have

$$\begin{aligned} \mathcal{A}[\Lambda(\Lambda^{-1} + d\Lambda^{-1})] &= \mathcal{A}[I + \Lambda d\Lambda^{-1}] = I + \Sigma_a^b (\Lambda d\Lambda^{-1})^a_b = \\ &= \mathcal{A}[\Lambda](\mathcal{A}[\Lambda^{-1}] + d\mathcal{A}^{-1}) = I + \mathcal{A}[\Lambda]d\mathcal{A}^{-1} = I + \mathcal{A}[\Lambda]d\mathcal{A}^{-1} \end{aligned}$$

from which

$$\Sigma_a^b (\Lambda d\Lambda^{-1})^a_b = \mathcal{A}[\Lambda]d\mathcal{A}^{-1}$$

Similarly

$$\Lambda(I + \varepsilon\Gamma)\Lambda^{-1} = I + \varepsilon\Lambda\Gamma\Lambda^{-1} \in SO(3, 1)$$

and thus $\Lambda\Gamma\Lambda^{-1} \in so(3, 1)$. Thus

$$\begin{aligned} \mathcal{A}[\Lambda(I + \varepsilon\Gamma)\Lambda^{-1}] &= \mathcal{A}[I + \varepsilon\Lambda\Gamma\Lambda^{-1}] = I + \varepsilon\Sigma_a^b (\Lambda\Gamma\Lambda^{-1})^a_b = \\ &= \mathcal{A}[\Lambda]\mathcal{A}[I + \varepsilon\Gamma]\mathcal{A}[\Lambda^{-1}] = \mathcal{A}[\Lambda](I + \varepsilon\Sigma_a^b \Gamma^a_b)\mathcal{A}[\Lambda^{-1}] = \mathcal{A}[\Lambda](I + \varepsilon\omega)\mathcal{A}[\Lambda^{-1}] = I + \varepsilon\mathcal{A}\omega\mathcal{A}^{-1} \end{aligned}$$

from which

$$\Sigma_a^b (\Lambda\Gamma\Lambda^{-1})^a_b = \mathcal{A}\omega\mathcal{A}^{-1}.$$

Summing the two contributions we have Eq.(5.22).

We compute now explicitly the matrices Σ_{ab} . Writing for the infinitesimal Lorentz transformation $I + \varepsilon\rho$, $\mathcal{A}[I + \varepsilon\rho] = I + \varepsilon\Sigma_{ab}\rho^{ab}$, with $\Sigma_{ab} = -\Sigma_{ba}$, keeping in mind that $\rho^{ab} = -\rho^{ba}$ we have from Eq.(5.18)

$$\mathcal{A}[I + \varepsilon\rho]\gamma_c \mathcal{A}[I + \varepsilon\rho]^{-1} = \gamma_d (I + \varepsilon\rho)^d_c$$

i.e.

$$[\Sigma_{ab}\rho^{ab}, \gamma_c] = \gamma_d \rho^{df} \eta_{fc}$$

or

$$[\Sigma_{ab}, \gamma_c] = \frac{1}{2}(\gamma_a \eta_{bc} - \gamma_b \eta_{ac}).$$

Such equation is simply solved by

$$\Sigma_{ab} = \frac{1}{8}[\gamma_a, \gamma_b].$$

as it is checked by using the Clifford algebra of the γ 's. The explicit representation of the Σ_{ab} is

$$\Sigma_{ij} = \frac{1}{4} \begin{pmatrix} \sigma_i \sigma_j & 0 \\ 0 & \sigma_i \sigma_j \end{pmatrix}, \quad \Sigma_{i0} = \frac{1}{4} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}.$$

Notice that the two dimensional traces of these Σ_{ab} are zero so that $I + \varepsilon \Sigma_{ab} \rho^{ab} \in (1/2, 0) \oplus (0, 1/2)$. As near the identity the correspondence between the Lorentz transformations and the matrices \mathcal{A} is a bijection, our solution for the Σ_{ab} is the unique solution.

Multiplying by $2i$ due to the different normalization of the generators, we have the same result as [WeinbergQFT] I, p. 217 (where the sum in the analog of Eq.(5.23) is done for $a > b$ while here the sums are on all a and b). ω is called the spin connection and has to be considered as the fundamental connection.

Summarizing the gauging of the Dirac Lagrangian is given by

$$\begin{aligned} \bar{\Psi} (\gamma^a \partial_a \Psi + m \Psi) &\rightarrow \bar{\Psi} (\gamma^a e_a^\mu (\partial_\mu + \omega_\mu) \Psi + m \Psi) \\ &\equiv \bar{\Psi} (\gamma^a e_a^\mu D_\mu \Psi + m \Psi) \equiv \bar{\Psi} (\gamma^a D_a \Psi + m \Psi) = L_s \end{aligned}$$

with obvious definitions of D_μ and D_a and

$$S_M = \int L_s \epsilon = \int e L_s d^4 x = \int L d^4 x$$

is the invariant Dirac action. Performing an integration by parts and using the fact that our connection Γ is metric compatible one proves that the action S_M is hermitean.

5.9 The field equations

We shall put $\kappa = 8G\pi/c^2$. The total action becomes

$$\frac{1}{2\kappa} S_{EC} + S_M + \text{b.t.}$$

where, for $n = 4$

$$S_{EC} = \frac{1}{2} \int R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}.$$

Variation with respect to e^d gives

$$R^{ab} \wedge e^c \varepsilon_{abcd} = \kappa t_d$$

where we set

$$\delta S_M = - \int t_d \wedge \delta e^d$$

which is always possible. Variation with respect to Γ^{ab} gives

$$De^c \wedge e^d \varepsilon_{abcd} = S^c \wedge e^d \varepsilon_{abcd} = s_{ab} \in \Lambda^3 \quad (5.24)$$

where

$$\kappa \delta S_M = - \int \delta \Gamma^{ab} \wedge s_{ab}.$$

Due to $\Gamma^{ba} = -\Gamma^{ab}$, $s_{ba} = -s_{ab}$. We see that in presence of fermions $s_{ab} \neq 0$. In fact in the case of the Dirac field we have

$$s_{ab} = -\kappa e \bar{\Psi} \gamma^c e_c^\mu \Sigma_{ab} \Psi \varepsilon_{\mu\nu\lambda\rho} \frac{1}{3!} dx^\nu \wedge dx^\lambda \wedge dx^\rho$$

i.e. the source of the torsion is given by the spin content. From this we see that in the Einstein-Cartan formulation in presence of fermions the torsion is not zero. We shall now solve Eq.(5.24). The procedure has an immediate extension to n dimensions even if we work with $n = 4$. Writing

$$S^c = \frac{1}{2} S_{ij}^c e^i \wedge e^j$$

and

$$s_{ab} = \frac{1}{3!} s_{ab}^k \varepsilon_{kmnp} e^m \wedge e^n \wedge e^p$$

and multiplying by e^h we reach

$$\frac{1}{2} S_{ij}^c \delta_{abc}^{ijh} = s_{ab}^h$$

i.e

$$S_{ab}^h + S_{ca}^c \delta_b^h - S_{cb}^c \delta_a^h = s_{ab}^h.$$

Taking the trace we have

$$S_{cb}^c + S_{cb}^c - 4S_{cb}^c = s_{cb}^c$$

from which

$$S_{cb}^c = -\frac{1}{2} s_{cb}^c$$

and

$$S_{ab}^h = s_{ab}^h + \frac{1}{2} (\delta_b^h s_{ca}^c - \delta_a^h s_{cb}^c).$$

Thus the torsion is confined in the region where the matter field is different from zero; we have a non propagating torsion. On the other hand theories in which the gravitational action contains quadratic or higher terms in the Riemann tensor can possess propagating torsion.

In order to find the connection Γ^a_b , which by assumption is metric compatible, we shall define

$$\Gamma^a_b = \Gamma[e]^a_b + K^a_b$$

being $\Gamma[e]^a_b$ the Levi-Civita connection and K^a_b the contorsion tensor 1-form. K^a_b is a true tensor being the difference of two connections. To compute $\Gamma[e]$ we could start from $g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu$, substitute in

$$\Gamma[e]^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\alpha'} (\partial_\beta g_{\alpha'\gamma} + \partial_\gamma g_{\beta\alpha'} - \partial_{\alpha'} g_{\beta\gamma})$$

and perform the transformation (4.3) with $\Omega^\mu_a = e^\mu_a$ to reach the value in our orthonormal frame. It is instructive to follow a more direct procedure. Vanishing torsion for $\Gamma[e]$ means

$$0 = de^a + \Gamma[e]^a_b \wedge e^b.$$

Setting

$$de^a = \frac{1}{2} E^a_{cb} e^c \wedge e^b$$

we have

$$0 = E_{acb} + \Gamma[e]_{abc} - \Gamma[e]_{acb}.$$

Rotating the indices twice and subtracting from the first two the last one, keeping in mind that due to metric compatibility $\Gamma[e]_{bac} = -\Gamma[e]_{abc}$, we reach

$$\Gamma[e]_{abc} = \frac{1}{2} (E_{cba} + E_{abc} - E_{bac}).$$

We must determine now the contorsion from S^a that we already know.

$$S^a = de^a + \Gamma[e]^a_b \wedge e^b + K^a_b \wedge e^b = K^a_b \wedge e^b$$

Setting as usual

$$S^a = \frac{1}{2} S^a_{ij} e^i \wedge e^j; \quad K^a_b = K^a_{bi} e^i$$

we have

$$S^a_{ij} = K^a_{ji} - K^a_{ij}$$

and lowering one index

$$S_{aij} = K_{aji} - K_{aij}.$$

Keeping in mind that metric compatibility imposes $K_{aij} = -K_{iaj}$ one can solve in K with the usual method of rotation of indices and finally we obtain

$$\begin{aligned} K_{ija} &= \frac{1}{2}(S_{aij} + S_{iaj} - S_{jai}) \\ \Gamma_{abc} &= \Gamma[e]_{abc} + \frac{1}{2}(S_{cab} + S_{acb} - S_{bca}) \end{aligned} \quad (5.25)$$

which correctly is antisymmetric in a, b .

Having solved completely the connection Γ in terms of $\Gamma[e]$ and K which is given in terms of the torsion source s_{ab} we can go back to the first equation

$$R^{ab} \wedge e^c \varepsilon_{abcd} = \kappa t_d .$$

We have

$$R(\Gamma[e] + K) = R(\Gamma[e]) + D[e]K + K \wedge K$$

where $R(\Gamma[e])$ is the usual metric curvature. Thus the equation has now become

$$(R(\Gamma[e]) + D[e]K + K \wedge K)^{ab} \wedge e^c \varepsilon_{abcd} = \kappa t_d. \quad (5.26)$$

Such equation has to be supplemented by the one obtained by varying the matter field Ψ i.e. the Dirac equation

$$\gamma^a D_a \Psi + m \Psi = 0. \quad (5.27)$$

The independent variables in the system (5.26,5.27) are the vierbeins e^a_μ and the Dirac field Ψ .

Given a solution to equations (5.26,5.27) we can apply to it an arbitrary diffeomorphism transformation and an arbitrary local Lorentz rotation to obtain again a solution. Obviously is solving the system Eqs.(5.26,5.27) one can apply a gauge fixing procedure.

By multiplying Eq.(5.26) on the right by e^f we have

$$(R(\Gamma[e]) + D[e]K + K \wedge K)^{ab} \wedge e^c \wedge e^f \varepsilon_{abcd} = \kappa t_d \wedge e^f .$$

Then isolating the $R(\Gamma[e])$ term (which is the metric Riemann tensor) and proceeding as in the discussion of the pure Einstein- Cartan action, Eq.(5.26) can be rewritten as

$$G_d^f[e] = \kappa T(ef f)_d^f$$

where $T(ef f)_d^f$ is an effective symmetric energy momentum tensor, with $\nabla[e]_f T(ef f)_g^f = 0$, which contains also the four fermion term originating from $K \wedge K$. However T^{ab} cannot be obtained from the variation of the action with respect to g_{ab} because g_{ab} does not

appear in the action, being e_μ^a and Γ_μ^{ab} the fundamental gravitational variables. Still is better to work directly with the system (5.26,5.27).

We saw that in the first order Einstein-Cartan approach when the gravitational field is coupled with the Dirac field the connection acquires torsion.

However this is not the only way to couple fermions to gravity. One could as well consider the torsionless Levi-Civita connection $\Gamma[e]$ and couple invariantly the fermion to gravity via [2]

$$\bar{\psi}\gamma^\rho\partial_\rho\psi \rightarrow \bar{\psi}\gamma^\rho(\partial_\rho + \Sigma_{ab}\Gamma[e]_\rho^{ab})\psi$$

and as such this way of proceeding can be called a second order approach. The action is still invariant under diffeomorphisms and under local Lorentz transformations. The two Lagrangians differ by a four-fermion term. We shall see in Chapter 11 that the formulation of supergravity is simplest in the first order approach in which torsion is present.

References

- [1] A. Trautman, Bull. Acad. Pol. Sci. Ser. Math. Astron. Phys. “On the Einstein-Cartan equations” 20 (1972) 185, 503, 895
 [2] P. van Nieuwehuizen, “Supergravity”, Physics Reports 68 (1981) 189

5.10 The generalized energy- momentum tensor

We saw that if the coupling of matter to gravity can be achieved by means of the metric tensor $g_{\mu\nu}$ the energy momentum tensor can be defined by the relation

$$\delta S_M = \frac{1}{2} \int d^n x \sqrt{-g} \mathbf{T}^{\mu\nu} \delta g_{\mu\nu}$$

or

$$\mathbf{T}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} = \mathbf{T}^{\nu\mu}$$

where we used the boldface for reason of clearness. By definition $\mathbf{T}^{\mu\nu}$ is symmetric and we found that on-shell

$$\nabla_\mu \mathbf{T}^{\mu\nu} = 0 .$$

Einstein equations take the form

$$G^{\mu\nu} = \kappa \mathbf{T}^{\mu\nu}$$

where we could also add the cosmological constant.

If gravity is described by the vierbeins e_μ^a as it is unavoidable in presence of fermions the energy momentum tensor is defined by

$$\frac{\delta S_M}{\delta e_\mu^a(x)} = e T_a^\mu(x) .$$

The mixed tensor $T_a^\mu(x)$ will be the fundamental object in the following. In boson theory if we introduce artificially e_μ^a through $g_{\mu\nu} = e_\mu^a \eta_{ab} e_\nu^b$ we have

$$T_a^\lambda = \frac{1}{2} (\mathbf{T}^{\lambda\nu} \eta_{ab} e_\nu^b + \mathbf{T}^{\mu\lambda} \eta_{ab} e_\mu^b)$$

and

$$T^{\lambda a} \equiv T_b^\lambda \eta^{ba} = \frac{1}{2} (\mathbf{T}^{\lambda\nu} e_\nu^a + \mathbf{T}^{\mu\lambda} e_\mu^a) .$$

If we multiply by e_a^ρ we have

$$T^{\lambda a} e_a^\rho = \frac{1}{2} (\mathbf{T}^{\lambda\rho} + \mathbf{T}^{\rho\lambda}) = \mathbf{T}^{\lambda\rho} .$$

The action in the fermionic case (and trivially in the bosonic case) is invariant under local Lorentz transformations

$$\delta e_\mu^a = \alpha^a_b e_\mu^b, \quad \text{with } \alpha^{ab} = -\alpha^{ba} .$$

Then on the equations of motion we have

$$0 = \delta S_M = \int e T_a^\mu \alpha^a_b e_\mu^b d^n x = \int e T^{ba} \alpha_{ab} d^n x$$

i.e. on the equations of motion the antisymmetric part of $T^{ba} \equiv e^b_\mu T^\mu_c \eta^{ca}$ has to vanish.

Thus we have the symmetry of T^{ba} as a consequence of local Lorentz invariance.

We examine now the consequences of invariance under diffeomorphisms. Under a diffeomorphisms we have (Lie derivative of a covariant vector)

$$\delta_\xi e_\mu^a = \xi^\nu \partial_\nu e_\mu^a + \partial_\mu \xi^\nu e_\nu^a . \quad (5.28)$$

But we can rewrite Eq.(5.28) as

$$\delta_\xi e_\mu^a = \xi^\nu \nabla[e]_\nu e_\mu^a + \nabla[e]_\mu \xi^\nu e_\nu^a$$

where $\nabla[e]_\nu$ is the covariant derivative calculated by means of the metric compatible connection $\Gamma[e]_{\lambda\nu}^\mu$ because in the Lie derivative we can replace the derivatives with covariant derivatives provided the connection is torsionless. Then, ignoring on the equations of motion the variation of the matter fields, we have

$$\delta_\xi S_M = \int d^n x e T_a^\mu (\xi^\nu \nabla[e]_\nu e_\mu^a + e_\nu^a \nabla[e]_\mu \xi^\nu) .$$

Integrating by parts the second term, using the metric compatibility of $\Gamma[e]$ we have

$$\delta_\xi S_M = \int d^n x e (T^\mu_a \xi^\nu \nabla[e]_\nu e^a_\mu - \xi^\nu \nabla[e]_\mu T^\mu_\nu) \quad (5.29)$$

where we defined from the fundamental T^μ_a , $T^\mu_\nu = T^\mu_a e^a_\nu$.

We examine now the first term. We recall that

$$\mathbf{e}_a = \mathbf{u}_\mu e^\mu_a$$

so that

$$\Gamma[e]^a_b = e^a_\mu de^\mu_b + e^a_\mu \Gamma[e]^\mu_\lambda e^\lambda_b = -de^a_\mu e^\mu_b + e^a_\mu \Gamma[e]^\mu_\lambda e^\lambda_b$$

i.e.

$$\Gamma[e]^a_b e^b_\mu = -\nabla[e] e^a_\mu$$

and thus the first term becomes

$$- \int d^n x e T^{ba} \xi^\nu \Gamma[e]_{ab\nu} . \quad (5.30)$$

As we have that $\Gamma[e]_{ab\nu}$ is antisymmetric in a, b , and T^{ab} symmetric in a, b , which is a consequence of local Lorentz invariance, Eq.(5.30) vanishes. Then from Eq.(5.29) it follows that

$$\nabla[e]_\mu T^{\mu\nu} = 0$$

which is a consequence of the combined invariances under local Lorentz transformations and diffeomorphisms. Such symmetry and covariant conservation property of the energy momentum tensor are consequences of the invariance at the classical level. Such properties can be violated at the quantum level giving rise to the so called gravitational anomalies.

References

- [1] L. Alvarez-Gaumé, E. Witten, "Gravitational Anomalies" Nucl. Phys. B234 (1984) 269
- [2] R. Bertlmann, "Anomalies in Quantum Field Theory" Oxford University Press 1996.

5.11 Einstein-Eddington affine theory

It is remarkable that one can provide an affine formulation of gravity, i.e. a formulation in which the basic variable is an affine connection and no metric is present at the beginning. We start with a symmetric connection (absence of torsion) $\Gamma^\lambda_{\mu\nu}$ from which the Ricci tensor can be readily computed. No metric is present. The action is

$$S_{EE} = \int L_{EE} d^4 x = k \int \sqrt{|\det R_{(\mu\nu)}|} d^4 x.$$

where $R_{(\mu\nu)}$ denotes the symmetrized Ricci tensor and k is a constant. Such an action is diffeomorphism invariant because

$$\det R'_{(\rho\sigma)} = \det R_{(\mu\nu)} \det \frac{\partial x^\mu}{\partial x'^\rho} \det \frac{\partial x^\nu}{\partial x'^\sigma}$$

while

$$d^4 x' = d^4 x \det \frac{\partial x'^\mu}{\partial x^\nu}.$$

The equations of motion are

$$0 = \int \frac{\partial L_{EE}}{\partial R_{(\mu\nu)}} \delta R_{\mu\nu} = \int \frac{\partial L_{EE}}{\partial R_{(\mu\nu)}} (\nabla_\lambda \delta \Gamma_\nu - \nabla_\nu \delta \Gamma_\lambda)^\lambda_\mu d^4 x$$

Define now the symmetric $(2, 0)$ tensor $g^{\mu\nu}$

$$\sqrt{|g|} g^{\mu\nu} \equiv k \frac{\sqrt{|\det R_{(\cdot)}|}}{2} [R_{(\cdot)}^{-1}]^{\mu\nu} = \frac{\partial L_{EE}}{\partial R_{(\mu\nu)}} \quad (5.31)$$

where g is the determinant of the inverse of $g^{\mu\nu}$ which will be denoted by $g_{\mu\nu}$ i.e. by definition $g^{\mu\lambda} g_{\lambda\nu} = \delta_\nu^\mu$. $g^{\mu\nu}$ is a $(2, 0)$ tensor as seen by taking the determinant of both sides of Eq.(5.31). The equations of motion now take the form

$$0 = \int g^{\mu\nu} \sqrt{|g|} (\nabla_\lambda \delta \Gamma_\nu - \nabla_\nu \delta \Gamma_\lambda)^\lambda_\mu d^4 x$$

and we are faced exactly with the Palatini problem of Section 5.3; we deduce that $\nabla_\lambda g_{\mu\nu} = 0$ from which Γ is the unique metric compatible torsionless connection. But now the associated Ricci tensor is symmetric and going back to Eq.(5.31) we have

$$\frac{2}{k} g_{\mu\nu} = R_{\mu\nu}$$

which is equivalent to

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$$

i.e. pure Einstein theory with a cosmological constant with $\Lambda = 2/k$. Coupling to matter, if attainable through the $g^{\mu\nu}$, presents no difficulties.

The symmetry constraint on the connections $\Gamma_{\nu\lambda}^\mu$ has also been relaxed in the attempt to interpret the antisymmetric part of the $\Gamma_{\nu\lambda}^\mu$ as the field-strength tensor of the electromagnetic field [2].

References

- [1] G. Magnano, "Are there metric theories of gravity other than General Relativity?" XI Italian Congress of General Relativity and Gravitation, (1994) 213; e-Print Archive: gr-qc/9511027 and references within.
- [2] A.T. Filippov "An old Einstein-Eddington generalized gravity and modern ideas of branes and cosmology" arXiv:1011.2445v1 [gr-qc] and references within.

Chapter 6

Submanifolds

6.1 Introduction

Given a 4-dimensional manifold with lorentzian metric let us consider a 3-dimensional sub-manifold Σ . The tangent space of Σ at any point is 3-dimensional. Let us consider the normal to Σ at a point, i.e. the vector which is orthogonal to all tangent vectors at p . Such a vector always exists and is unique up to a multiplying factor. In fact if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a base of the tangent space at p let us consider a further independent vector \mathbf{v}_0 and construct $g^{0\mu}\mathbf{v}_\mu$, where $g_{\mu\nu} = (\mathbf{v}_\mu, \mathbf{v}_\nu)$. The scalar product with \mathbf{v}_j is $\delta_j^0 = 0$. Any vector orthogonal to the tangent space at p is $a^\mu\mathbf{v}_\mu$ with $a^\mu g_{\mu j} \equiv a_j = 0$. Thus the only freedom is the value of a_0 .

If the normal at any point of Σ is time-like that surface is said space-like. If it is space-like the surface is said time-like. If it is light-like the surface is said to be null.

Now we prove that the intrinsic metric of a space like surface is positive definite. That of a time-like surface is of signature $-++$. The metric of a null surface is degenerate.

In fact for a space-like surface the representation of g in the base \mathbf{n}, \mathbf{v}_j is $g_{00} < 0, g_{0j} = 0$ from which it follows that the 3×3 -matrix $(\mathbf{v}_j, \mathbf{v}_k)$ has signature $+++$. Similarly for a time-like surface the signature of the matrix $(\mathbf{v}_j, \mathbf{v}_k)$ is $-++$. For a null-surface we have $0 = (\mathbf{n}, \mathbf{n}) = (g^{0\mu}\mathbf{v}_\mu, g^{0\nu}\mathbf{v}_\nu) = g^{00}$ i.e. $\mathbf{n} = g^{0j}\mathbf{v}_j$, $\mathbf{n} \in T(p)$ and at the same time orthogonal to $T(p)$, giving $g^{0j}g_{jl} = 0$.

6.2 Extrinsic curvature

Given a surface Σ embedded in an n dimensional manifold M , i.e. a $n - 1$ dimensional manifold embedded in M , let us consider the unit vector n normal to the surface Σ .

We construct the tensor $h_{ab} = g_{ab} \mp n_a n_b$ for $n_a n^a = \pm 1$. Such a tensor is the metric tensor on the surface and h_b^a is a projection on the tangent space of the surface.

We define extrinsic curvature of the surface the tensor, belonging to the tangent space on the $n - 1$ dimensional manifold

$$K_{ab} = h_a^{a'} h_b^{b'} \nabla_{a'} n_{b'} = h_a^{a'} \nabla_{a'} n_b.$$

The last equality is due to metric compatibility of ∇ and to $n^a \nabla n_a = 0$. For definiteness we work with $n^a n_a = -1$.

In order to compute $\nabla_a n_b$ we need an extension of n_b outside to surface, but due to the projection $h_a^{a'}$ the defined K_{ab} does not depend on such extension.

We want to prove that $K_{ab} = K_{ba}$. Let $\rho = 0$ be the equation defining the surface. n_a is given by

$$n_a = \lambda \nabla_a \rho$$

being λ a proper normalizing function to have $n^2 = -1$.

We have

$$\nabla_c \lambda = \lambda^2 n^a \nabla_a \nabla_c \rho.$$

Then K_{ab}

$$\begin{aligned} K_{ab} &= \lambda h_a^c (n^d n_b \nabla_c \nabla_d \rho + \nabla_c \nabla_b \rho) = \\ &= \lambda (n^d n_b \nabla_a \nabla_d \rho + n_a n_b n^c n^d \nabla_c \nabla_d \rho + \nabla_a \nabla_b \rho + n^c n_a \nabla_c \nabla_b \rho) \end{aligned}$$

which is symmetric under the exchange of a and b , due to the absence of torsion i.e. $\nabla_a \nabla_b \rho = \nabla_b \nabla_a \rho$.

There is an important relation between the extrinsic curvature and a Lie derivative of the metric h_{ab}

$$\begin{aligned} L_n h_{ab} &= n^c \partial_c h_{ab} + h_{cb} \partial_a n^c + h_{ac} \partial_b n^c \\ &= n^c \nabla_c h_{ab} + h_{cb} \nabla_a n^c + h_{ac} \nabla_b n^c = \\ &= n^c \nabla_c (n_a n_b) + \nabla_a n_b + \nabla_b n_a = \\ &= n^c n_a \nabla_c n_b + n^c n_b \nabla_c n_a + \nabla_a n_b + \nabla_b n_a = K_{ab} + K_{ba} = 2K_{ab}. \end{aligned}$$

References

- [1] [HawkingEllis] Chap. 2

6.3 The trace- K action

Using the concept of extrinsic curvature we can give a covariant form of the gravitational action. The problem is that to add to the Hilbert action a covariant boundary term whose variation cancel the contribution

$$\int_{\partial V} v^\lambda \Sigma_\lambda \quad (6.1)$$

which appears in Eq.(5.5). We start from the contribution to Eq.(6.1) of the top surface Σ_2 (see Fig 7.1).

$$\int_{\Sigma_2} \sqrt{-g} v^\lambda \varepsilon_{\lambda\mu_2\dots\mu_n} \frac{dx^{\mu_2} \wedge \dots \wedge dx^{\mu_n}}{(n-1)!} .$$

One should not confuse the surfaces Σ_1 , Σ_2 , Σ_t with the differential form Σ_λ appearing e.g. in eqs.(6.1, 5.5) and others.

Using $\sqrt{-g} = \sqrt{h} N$, $N > 0$ we can rewrite the above as (see eq(5.5))

$$\int_{\Sigma_2} N \sqrt{h} v^0 d^{n-1}x = - \int_{\Sigma_2} \sqrt{h} v^\lambda n_\lambda d^{n-1}x \quad (6.2)$$

where $n_\lambda = (-N, 0, 0, 0)$ is the normal to the space-like surface Σ_2 . n is time-like and being $n^0 = g^{0\mu} n_\mu = \frac{1}{N} > 0$ this is called the future-pointing normal. We recall that

$$v^\lambda n_\lambda = \delta \Gamma^\lambda_{\mu\nu} g^{\mu\nu} n_\lambda - \delta \Gamma^\nu_{\mu\nu} g^{\mu\lambda} n_\lambda . \quad (6.3)$$

Let us compute the variation of

$$\int_{\Sigma_2} \sqrt{h} K d^{n-1}x = \int_{\Sigma_2} \sqrt{h} h^\mu_\nu \nabla_\mu n^\nu d^{n-1}x .$$

In the variational procedure we have on ∂V , $\delta g_{\mu\nu} = 0$ and thus also $\delta n_\nu = 0$, $\delta n^\nu = 0$ on ∂V but not necessarily outside ∂V . We have

$$\begin{aligned} \delta \int_{\Sigma_2} \sqrt{h} h^\mu_\nu \nabla_\mu n^\nu d^{n-1}x &= \int_{\Sigma_2} \sqrt{h} h^\mu_\nu (\partial_\mu \delta n^\nu + \delta \Gamma^\nu_{\rho\mu} n^\rho) d^{n-1}x \\ &= \int_{\Sigma_2} \sqrt{h} (\delta \Gamma^\mu_{\rho\mu} n^\rho + n_\rho \delta \Gamma^\rho_{\nu\mu} n^\nu n^\mu) d^{n-1}x \end{aligned} \quad (6.4)$$

because on ∂V , $\delta n^\nu = 0$ and as a consequence $\partial_\mu \delta n^\nu = c n_\mu$. We can also write, using metric compatibility

$$\begin{aligned} \delta \int_{\Sigma_2} \sqrt{h} K d^{n-1}x &= \delta \int_{\Sigma_2} \sqrt{h} h^{\mu\nu} \nabla_\mu n_\nu d^{n-1}x \\ &= \int_{\Sigma_2} \sqrt{h} h^{\mu\nu} (\partial_\mu \delta n_\nu - \delta \Gamma^\rho_{\nu\mu} n_\rho) d^{n-1}x = \int_{\Sigma_2} \sqrt{h} (-g^{\mu\nu} \delta \Gamma^\rho_{\nu\mu} n_\rho - n_\rho \delta \Gamma^\rho_{\nu\mu} n^\nu n^\mu) d^{n-1}x . \end{aligned} \quad (6.5)$$

Addition of the term (6.4) and (6.5) gives minus (6.2). Thus

$$2 \int_{\Sigma_2} \sqrt{h} K d^{n-1}x$$

will cancel in the variation, the contribution to Eq.(6.1) of Σ_2 and similarly

$$-2 \int_{\Sigma_1} \sqrt{h} K d^{n-1}x$$

will cancel the contribution to Eq.(6.1) of Σ_1 .

We come now to the contribution of the mantle B in Eq.(6.1). Denoting by u_λ the outward-pointing unit normal to B we have

$$g_{\mu\nu} = \gamma_{\mu\nu} + u_\mu u_\nu .$$

The extrinsic curvature of B is

$$\Theta_{\mu\nu} = \gamma_\mu^\alpha \nabla_\alpha u_\nu, \quad \Theta = \gamma_\mu^\alpha \nabla_\alpha u^\mu .$$

Working as before we have

$$\int_B \sqrt{-\gamma} v^\lambda u_\lambda d^{m-1}x = -2\delta \int_B \sqrt{-\gamma} \Theta d^{m-1}x .$$

We can now write the trace- K action [1]

$$S_K = S_H + 2 \left(\int_{\Sigma_2} - \int_{\Sigma_1} \right) \sqrt{h} K d^3x + 2 \int_B \sqrt{-\gamma} \Theta d^3x$$

which has a form invariant under diffeomorphisms.

Remark: The above given trace- K action is complete for orthogonal boundaries i.e. when on the two dimensional surface where B meets Σ_t we have $n^\lambda u_\lambda = 0$. Otherwise as discovered in [2] and treated in detail in [1,3] one has to add two contributions belonging to two 2-dimensional sub-manifolds.

References

- [1] J.D. Brown, S.R. Lau, J.W. York “Action and energy of the gravitational field” gr-qc/0010024
- [2] G. Hayward, “Gravitational action for spacetimes with non smooth boundaries” Phys. Rev. D 47 (1993) 3275
- [3] S.W. Hawking and C.J. Hunter, “The gravitational Hamiltonian in the presence of non-orthogonal boundaries” Class. Quant. Grav. 13 (1996) 2735

6.4 The boundary term in the Palatini formulation

With the developed tools we can give a covariant form for the boundary terms in the Palatini action [1].

The boundary term in the variation of $\Gamma_{\mu\nu}^\lambda$ was

$$\int d^n x \partial_\lambda (\sqrt{-g} g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda - \sqrt{-g} g^{\lambda\mu} \delta \Gamma_{\mu\nu}^\nu).$$

Let us compute it first on Σ_2 where it gives

$$- \int_{\Sigma_2} \sqrt{h} (n_\lambda g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda - n_\lambda g^{\lambda\mu} \delta \Gamma_{\mu\nu}^\nu) d^{n-1} x. \quad (6.6)$$

where n_λ is the future-pointing time-like unit normal defined in the previous section.

Working exactly as for the trace- K action we find that the variation of

$$- \int_{\Sigma_2} \sqrt{h} (h_\nu^\mu \nabla_\mu n^\nu + h^{\mu\nu} \nabla_\mu n_\nu) d^{n-1} x \quad (6.7)$$

using the absence of torsion, i.e. the symmetry of the connection in the lower indices $\mu\nu$, is exactly minus (6.6). The only difference is that, as we do not assume at the start metric compatibility, we cannot identify Eq.(6.7) with twice the integral of the trace of the exterior curvature. Moreover we can write

$$h_\nu^\mu \nabla_\mu n^\nu + h^{\mu\nu} \nabla_\mu n_\nu = \nabla_\mu n^\mu + g^{\mu\nu} \nabla_\mu n_\nu + n^\mu \nabla_\mu (n^\nu n_\nu) = \nabla_\mu n^\mu + g^{\mu\nu} \nabla_\mu n_\nu.$$

Similarly one deals with Σ_1 and B . Summarizing we have the complete Palatini action

$$S_P = \int_V \sqrt{-g} g^{\mu\nu} R_{\mu\nu} d^n x - \left(\int_{\Sigma_2} - \int_{\Sigma_1} \right) \sqrt{h} (\nabla_\nu n^\nu + g^{\mu\nu} \nabla_\mu n_\nu) d^{n-1} x \quad (6.8)$$

$$+ \int_B \sqrt{-\gamma} (\nabla_\nu u^\nu + g^{\mu\nu} \nabla_\mu u_\nu) d^{n-1} x \quad (6.9)$$

where u_μ is the outward-pointing unit normal to B and in the second line $\gamma_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$, $u^\mu u_\mu = 1$.

References

[1] Yu.N. Obukhov, "The Palatini principle for manifolds with boundary" *Class. Quantum Grav.* 4 (1987) 1085

6.5 The boundary term in the Einstein-Cartan formulation

The developed formalism allows to compute also the covariant boundary terms in the Einstein-Cartan formulation [1]

From Eq.(5.11) we recall that the left-over term in deriving the equation of motion by varying Γ is

$$\frac{1}{(n-2)!} \int d(\delta\Gamma^{ab} \wedge e^{a_3} \wedge \dots \wedge e^{a_n} \varepsilon_{aba_3\dots a_n}) .$$

Not to overburden the notation we shall deal with the case $n = 4$, the extension to all n being trivial. We have

$$\frac{1}{2} \int_V d(\delta\Gamma^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}) = \frac{1}{2} \int_{\partial V} \delta\Gamma^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd} .$$

Such expression can be rewritten in the form (6.11) below.

For completeness we give here the derivation. In general for a three form ω we have

$$\int_{\partial V} \omega = \frac{1}{3!} \int_{\partial V} \omega_{\mu_2\mu_3\mu_4} dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} = \int_{\partial V} e \Omega^\beta \varepsilon_{\beta\mu_2\mu_3\mu_4} dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4}$$

where

$$e\Omega^\beta = -\frac{1}{3!} \varepsilon^{\beta\mu_2\mu_3\mu_4} \omega_{\mu_2\mu_3\mu_4} .$$

Thus referring to the surface Σ_2 we have

$$\begin{aligned} \frac{1}{3!} \int_{\Sigma_2} e \Omega^\beta \varepsilon_{\beta\mu_2\mu_3\mu_4} dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} &= - \int_{\Sigma_2} \sqrt{h} n_\beta \Omega^\beta d^3x = \\ &= \frac{1}{3!} \int_{\Sigma_2} \frac{\sqrt{h}}{e} n_\beta \varepsilon^{\beta\mu_2\mu_3\mu_4} \omega_{\mu_2\mu_3\mu_4} d^3x . \end{aligned} \quad (6.10)$$

where $n_\mu = (-N, 0, 0, 0)$, $N > 0$. In the present case we have

$$\omega = \frac{1}{3!} \omega_{\mu_2\mu_3\mu_4} dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4} = \frac{1}{2} \delta\Gamma_{\quad f}^{ab} e^f \wedge e^c \wedge e^d \varepsilon_{abcd} .$$

Substituting into (6.10) and using

$$\varepsilon^{\beta\mu_2\mu_3\mu_4} e_{\mu_2}^f e_{\mu_3}^c e_{\mu_4}^d = e e_k^\beta \varepsilon^{kfcd}$$

we obtain for such a contribution

$$\int_{\Sigma_2} \sqrt{h} n_k \delta\Gamma_{\quad f}^{ab} \delta_{ab}^{kf} d^3x = 2 \int_{\Sigma_2} \sqrt{h} n_a \delta\Gamma_b^{ab} d^3x \quad (6.11)$$

where we took into account the antisymmetry in a, b of $\delta\Gamma_c^{ab}$.

Consider now the variation of

$$2 \int_{\Sigma_2} \sqrt{h} h_a^\mu \nabla_\mu n^a d^3x . \quad (6.12)$$

where $h_a^\mu = h_\nu^\mu e_a^\nu = e_a^\mu + n^\mu n_a$

Following the same procedure as above we have for the variation

$$2 \delta \int_{\Sigma_2} \sqrt{h} h_a^\mu \nabla_\mu n^a d^3x = 2 \int_{\Sigma_2} \sqrt{h} (\delta \Gamma_{ba}^a n^b + n^\mu \delta \Gamma_\mu^{ab} n_a n_b) d^3x = -2 \int_{\Sigma_2} \sqrt{h} n_a \Gamma_b^{ab} d^3x$$

which is minus the boundary contribution (6.11). Also one notices that in Eq.(6.12) due to metric compatibility we have $h_a^\mu \nabla_\mu n^a = e_a^\mu \nabla_\mu n^a$. Similarly one deals with the contribution of Σ_1 and of B . Finally the complete action for Einstein-Cartan formulation is

$$S_{EC} = \frac{1}{2} \int_V R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd} - 2 \left(\int_{\Sigma_2} - \int_{\Sigma_1} \right) \sqrt{h} e_a^\mu \nabla_\mu n^a d^3x + 2 \int_B \sqrt{-\gamma} e_a^\mu \nabla_\mu u^a d^3x$$

where $n_\mu = (-N, 0, 0, 0)$ and u_μ is the unit outward-pointing normal to B and again in the last term $\gamma_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$.

References

[1] Yu.N. Obukhov, "The Palatini principle for manifolds with boundary" *Class. Quantum Grav.* 4 (1987) 1085

6.6 The Gauss and Codazzi equations

Given a surface Σ embedded in an n dimensional manifold M , i.e. an $n - 1$ dimensional manifold embedded in M , there are important relations between the curvature R in n dimensions, the intrinsic curvature \mathcal{R} of the $n - 1$ dimensional surface Σ and the extrinsic curvature of the $n - 1$ dimensional surface Σ embedded in the n dimensional manifold M . By D here we shall understand the covariant derivative in the sub-manifold, with zero torsion and compatible with the metric h_{ab} induced by g_{ab} on the sub-manifold Σ i.e. the pull back of g_{ab} to Σ .

The main equation to be used is

$$(D_a D_b - D_b D_a) v^c = \mathcal{R}^c{}_{c'ab} v^{c'}$$

from which it follows

$$(D_a D_b - D_b D_a) \omega_c = -\mathcal{R}^c{}_{cab} \omega^{c'}. \quad (6.13)$$

We recall that the covariant derivative D can be computed from the covariant derivative ∇ on the n -dimensional manifold through a projection procedure. For example

$$D_a T_{cd}^b = h_a^{a'} h_{b'}^b h_c^{c'} h_d^{d'} \nabla_{a'} T_{c'd'}^{b'}$$

A result which is useful in proving the following relations is

$$h_a^{a'} h_b^{b'} \nabla_{a'} h_b^c = h_a^{a'} h_b^{b'} \nabla_{a'} (n^c n_{b'}) = n^c K_{ab}. \quad (6.14)$$

Then from Eq.(6.13)

$$-\mathcal{R}_{cab}^d = -h_a^{a'} h_c^{c'} h_b^{b'} h_{d'}^d R_{c'a'b'}^{d'} - K_{ac} K_b^d + K_{bc} K_a^d$$

or

$$\mathcal{R}_{dcab} = \Pi_\Sigma(R_{dcab}) + K_{ac} K_{db} - K_{bc} K_{da}$$

where Π_Σ denotes the projection on Σ . This is the Gauss equation.

Taking two traces and keeping in mind the (anti)-symmetry of the Riemann tensor (see Section 4.18) in the indices one gets

$$\mathcal{R} + K^2 - \text{Tr}(KK) = 2n^b n^c G_{cb} \quad (6.15)$$

where G_{cb} is the Einstein tensor.

To prove the Codazzi equations take

$$D_a K_b^c = h_c^c h_b^{b'} h_a^{a'} \nabla_{a'} K_{b'}^{c'} \equiv h_c^c h_b^{b'} h_a^{a'} \nabla_{a'} (h_{b'}^{b''} \nabla_{b''} n^{c'}).$$

Using Eq.(6.14)we find

$$D_a K_b^c = K_{ab} h_c^c n^{b'} \nabla_{b'} n^{c'} + h_c^c h_b^{b'} h_a^{a'} \nabla_{a'} \nabla_{b'} n^{c'}.$$

Contracting c with b

$$D_a K = K_{ab} n^{b'} \nabla_{b'} n^b + h_c^c h_a^{a'} \nabla_{a'} \nabla_{b'} n^c.$$

Contracting a with c

$$D_a K_b^a = K_{ab} n^{b'} \nabla_{b'} n^a + h_c^c h_b^{b'} \nabla_{a'} \nabla_{b'} n^c$$

we have

$$\begin{aligned} D_a K_b^a - D_b K &= h_c^c h_b^{b'} (\nabla_{a'} \nabla_{b'} n^c - \nabla_{b'} \nabla_{a'} n^c) = h_c^c h_b^{b'} R_{c'a'b'}^c n^c \\ &= h_b^{b'} R_{cb'} n^c = h_b^d G_{cd} n^c. \end{aligned} \quad (6.16)$$

These are the Codazzi equations.

The Gauss and Codazzi equations relate intrinsic and extrinsic properties of an $n - 1$ -dimensional surface to geometric properties of the n -dimensional space in which such a surface is embedded. The vanishing of Eq.(6.15) for any surface i.e. for any n , tells us the $G^{ab} = 0$. For a two dimensional surface embedded in three dimensions $G^{ab} = 0$ tells

us that the three dimensional space is flat. For an $n - 1$ -dimensional surface embedded in an n -dimensional space with $n \geq 4$ $G^{ab} = 0$ tells us that the space is Ricci flat.

If we have at our disposal only one normal

$$n_a G^{ab} n_b = 0, \quad h_b^a G^{bc} n_c = 0$$

are equivalent to

$$n_a G^{ab} n_b = 0, \quad G^{ac} n_c = 0$$

i.e. in the ADM reference system $G^{00} = 0$, $G^{j0} = 0$.

References

- [1] [HawkingEllis] Chap. 2
- [2] [Wald] Chap. 10

Chapter 7

The hamiltonian formulation of gravity

7.1 Introduction

The motivations for a hamiltonian formulation of gravity are:

1. The need to define a Cauchy problem.
2. Separation of physical from gauge degrees of freedom.
3. A hamiltonian formulation is a road to quantization as done in non relativistic quantum mechanics and also in relativistic field theory.
4. Give a framework to perform numerical calculations like black hole scattering, black hole coalescence, emission of gravitational waves.

One assumes that space-time can be foliated in space like surfaces Σ_t which will be labeled by a real parameter t . Space-like surface means that all vectors belonging to the tangent space of Σ_t are space-like vectors; this is equivalent to saying that the vector n_a normal to Σ_t is a time-like vector-field.

One introduces a vector field called “time flow” which generates the diffeomorphisms which map Σ_{t_0} into Σ_{t_0+t} and thus such that $t^a \nabla_a t = 1$. From its definition we see that the t^a is not unique. The metric on Σ_t is the one induced by the metric g_{ab} on M i.e. the pull-back of g_{ab} to Σ_t .

One can decompose t^a in normal and tangential part

$$t^a = N^a - n^a t^b n_b \equiv N^a + N n^a.$$

The space metric is given by the tensor

$$h_{ab} = g_{ab} + n_a n_b; \quad n^a n_a = -1.$$

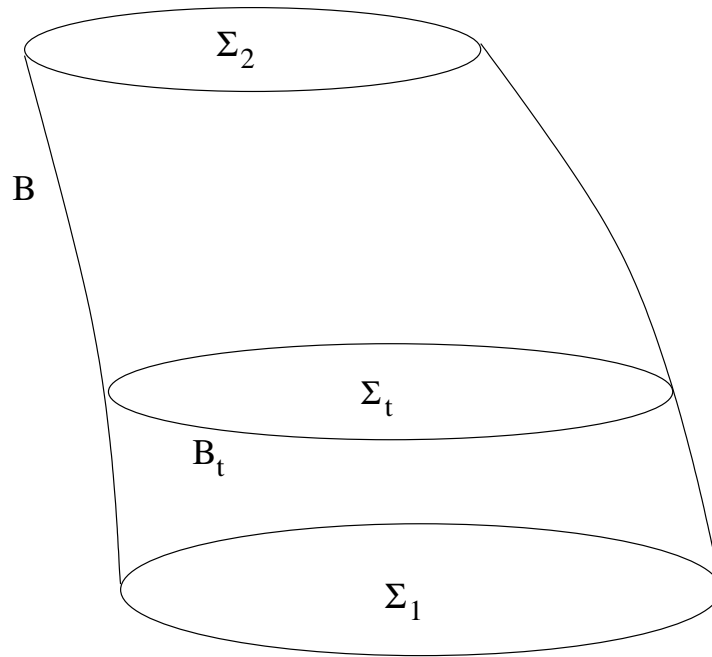


Figure 7.1: Space-time foliation

In a general frame, vectors in Σ_t are described by four (non independent) components.

The time flow vector field can be used to set up a special coordinate system $(t, [\phi_t^{-1}(p)]^j)$ i.e. the ADM coordinate system. In such a coordinate system, under ϕ_t the x^j do not vary and thus $t^\mu = (1, 0, 0, 0)$. Being N^μ tangent to Σ_t we have $N^\mu = (0, N^j)$ and being n^μ orthogonal to all tangent vectors of Σ_t we have $n_\mu = (-N, 0, 0, 0)$ and thus

$$0 = N^j + Nn^j; \quad n^j = -\frac{N^j}{N}$$

$$-1 = -Nn^0 + 0; \quad n^0 = \frac{1}{N} .$$

As $n_j = 0$ we have $h_{ij} = g_{ij}$. Being n^μ orthogonal to all tangent vectors of Σ_t $(0, \dots, \dots)$ we have

$$0 = n^\mu g_{\mu i} = \frac{1}{N} g_{0i} - \frac{N^j}{N} h_{ij}$$

from which $g_{0j} = N^i h_{ij} = N_j$.

$$-1 = n^\mu n_\mu = \frac{1}{N^2} g_{00} + h_{ij} \frac{N^i N^j}{N^2} - 2 \frac{N^i}{N} g_{i0} \frac{1}{N} .$$

Summarizing

$$g_{\mu\nu} = \begin{pmatrix} N^j N_j - N^2 & N_n \\ N_m & h_{mn} \end{pmatrix}$$

which is the ADM metric and in such coordinate system

$$t^\mu = (1, 0, 0, 0); \quad N^\mu = (0, N^j); \quad n_\mu = (-N, 0, 0, 0); \quad n^\mu = \left(\frac{1}{N}, -\frac{N^j}{N}\right).$$

Sometimes the inverse of $g_{\mu\nu}$ is needed; it is given by

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^m}{N^2} \\ \frac{N^m}{N^2} & h^{mn} - \frac{N^m N^n}{N^2} \end{pmatrix}.$$

7.2 Constraints and dynamical equations of motion

If Einstein equations are satisfied (in absence of matter it means that the space is Ricci-flat $G_{ab} = 0$) we have

$$\mathcal{R} + K^2 - \text{Tr}(KK) = 2n_a n_b G^{ab} = 0 \quad (I)$$

and

$$D_a K_b^a - D_b K = h_{bc} G^{cd} n_d = 0. \quad (II)$$

Due to $2K_{ab} = L_n h_{ab}$ the four relations contain only first order time derivative while the system of Einstein equations are (linear) in second order time derivative. Thus one naively expects that as initial conditions we must give both h_{ab}, N^i, N and $\dot{h}_{ab}, \dot{N}^i, \dot{N}$ but such initial conditions are not arbitrary but subject to (I) and (II). On the other hand if Einstein equations should predict uniquely the time development of the metric, we would have a violation of invariance under diffeomorphisms, as given a solution any other metric given by a diffeomorphism of the first, must also be a solution.

Is remarkable that the validity of the single scalar equation, which relates the intrinsic and extrinsic curvature

$$\mathcal{R} + K^2 - \text{Tr}(KK) = 0$$

on all space like surfaces is equivalent to Einstein equations i.e. to the whole general relativity, in absence of matter. In fact the validity of (I) for all time-like n implies the $G_{ab} = 0$ i.e. that space time is Ricci flat.

However given a foliation of the space time n is not arbitrary. Recalling that $h_b^a = \delta_b^a + n^a n_b$ we have that the validity of

$$\mathcal{R} + K^2 - \text{Tr}(KK) = 0$$

and

$$D_a K_b^a - D_b K = 0.$$

due to the invertibility of g_{ab} imply the four relations $G^{ab}n_b = 0$, which in the ADM frame assume the form $G^{\mu 0} = 0$. These are the 4 Einstein equations which do not contain second order time derivatives and as such are called instantaneous equations. The other 6 are $G^{ij} = 0$ and they do contain second order time derivative as will be shown in developing the hamiltonian approach.

Thus in the initial conditions of the Cauchy problem we cannot choose the metric h_{ij} , N^j , N and the time derivative of the metric \dot{h}_{ij} , \dot{N}^j , \dot{N} arbitrarily but such initial conditions are subject to (I) and (II).

7.3 The Cauchy problem

for the Maxwell and Einstein equations in vacuum

There is a strict analogy between the Cauchy problem for the Maxwell equations in the A description and the Cauchy problem for the Einstein equations

1. A_μ are 4 fields
2. $A \rightarrow A + d\Lambda$ is a transformation which leaves unchanged the field strength $F_{\mu\nu}$.
3. $\partial_\mu F^{\mu\nu} = 0$ is a system of 4 equations, which is second order in the time derivatives.
4. Not all equations are of second order in the time derivative:

$$\partial_j F^j{}_0 = \partial_j(\partial^j A_0 - \partial_0 A^j) = 0$$

is first order and contains no time derivative of A_0 .

5. $\partial_\mu F^\mu{}_j = 0$ are 3 equations containing second order time derivatives

$$\partial_\mu F^\mu{}_j = \partial_0(\partial^0 A_j - \partial_j A^0) + \partial_i(\partial^i A_j - \partial_j A^i) = 0 \quad (7.1)$$

No \ddot{A}_0 appears.

To proceed in time we need \dot{A}^j and to extract \ddot{A}^j from Eq.(7.1) we need to know on Σ , A^i and also $\partial_0 A^0$.

A consequence of Eq.(7.1) is

$$\partial_0(\partial_\mu F^{\mu 0}) = 0.$$

Thus the position of the Cauchy problem in A_μ is

- I. Choose A_0 as an arbitrary function of space time.
- II. Choose $A_j(\mathbf{x}, 0)$ and $\dot{A}_j(\mathbf{x}, 0)$ subject to the condition

$$(\partial_j \partial^j A_0 - \partial_j \dot{A}^j)(\mathbf{x}, 0) = 0 \quad (7.2)$$

but otherwise arbitrary.

III. Solve the 3 second order equations for A_j

$\partial_\mu F^\mu{}_j = 0$ i.e.

$$\ddot{A}^j = \partial^j \dot{A}_0 + \partial_i (\partial^i A^j - \partial^j A^i). \quad (7.3)$$

As a consequence of Eq.(7.3), Eq(7.2) is satisfied at all times.

The separation of the gauge degree of freedom is most clearly seen in the hamiltonian formulation.

Starting from the lagrangian

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4} (\partial_i A_j - \partial_j A_i) (\partial^i A^j - \partial^j A^i) + \frac{1}{2} (\partial_0 A_j - \partial_j A_0) (\partial^0 A^j - \partial^j A^0)$$

let us compute the momenta conjugate to A_j

$$E^j = \frac{\partial L}{\partial \dot{A}_j} = \partial^0 A^j - \partial^j A^0$$

and rewrite the action in hamiltonian form

$$S = \int (E^j \dot{A}_j - H) d^4x \quad (7.4)$$

$$= \int \left(E^j \dot{A}_j + \frac{1}{2} E^j E_j + \frac{1}{4} (\partial_i A_j - \partial_j A_i) (\partial^i A^j - \partial^j A^i) + A_0 \partial_j E^j \right) d^4x \quad (7.5)$$

where a space integration by parts has been performed. We see that A_0 appears without derivatives, i.e. as a Lagrange multiplier. Its variation give the constraint

$$\partial_j E^j = 0. \quad (7.6)$$

The variation of E^j gives

$$\dot{A}_j = -E_j + \partial_j A_0 \quad (7.7)$$

while variation of A_j give

$$\dot{E}^j = -\partial_i (\partial^i A^j - \partial^j A^i). \quad (7.8)$$

Antisymmetry in the indices in the last equation gives

$$\partial_j \dot{E}^j = 0. \quad (7.9)$$

To solve the equations of motion choose A_0 as an arbitrary function of space and time. At $t = 0$ choose A_j and E^j arbitrarily but with E^j subject to Eq.(7.6). Using eqs.(7.7,7.8) propagate the fields A_j and E^j in time. As a consequence of Eq.(7.8) the constraint equation (7.6) will be satisfied at all times.

Similarly for the Einstein equations we have

1. h_{ij} , N , N^j are 10 fields.
2. The physical content of the solution is unchanged under the transformation $x'^{\mu}(x)$ (diffeomorphisms) given by 4 arbitrary functions.
3. $G^{\mu\nu} = 0$ is a system of 10 equations, which is second order in time derivatives.
4. Not all equations contain second order time derivatives:
 $G^{00} = 0$ and $G^{0i} = 0$ contain only first order time derivatives.
5. The other 6 equations $G^{ij} = 0$ contain second order time derivatives

No \ddot{N} , \ddot{N}^a appear in the equations. These two statements will be proven in the following section.

Thus the position of the Cauchy problem in h_{ij} , N , N^a is

- I. Choose N , N^a as arbitrary functions of space time.
- II. Choose $h_{ij}(\mathbf{x}, 0)$ and $\dot{h}_{ij}(\mathbf{x}, 0)$ subject to

$$G^{00}(\mathbf{x}, 0) = 0; \quad G^{0j}(\mathbf{x}, 0) = 0 \quad (7.10)$$

but otherwise arbitrary.

- III. Solve the 6 second order equations for h_{ij}

$$G^{ij} = 0. \quad (7.11)$$

As a consequence of Eqs.(7.11), Eqs.(7.10) are satisfied at all times; also this statement will be proven in the following section.

Again the separation of the diffeomorphism from the physical degrees of freedom is most clearly seen in the hamiltonian formalism to which we turn in the next section.

7.4 The action in canonical form

In what follows we shall refer to the somewhat simpler case of the boundary B orthogonal to Σ_t , i.e. $n^\mu u_\mu = 0$ on B , being u_μ the normal to B , and n^μ as usual the normal to Σ_t .

Using the Gauss equation (6.15) we can rewrite the action as ($D = n - 1$)

$$\begin{aligned} S &= \int_V \sqrt{-g} dt d^D x (\mathcal{R} + K^2 - \text{Tr}(KK) - 2n^\mu R_{\mu\nu} n^\nu) + 2 \int_B \sqrt{-\gamma} d^D x \Theta \\ &+ 2 \left(\int_{\Sigma_2} - \int_{\Sigma_1} \right) \sqrt{h} d^D x K \end{aligned} \quad (7.12)$$

being Θ the extrinsic curvature of B and $\sqrt{-\gamma} d^D x$ the area element of B . Using

$$n^\mu R_{\mu\nu} n^\nu = n^\nu \nabla_\mu \nabla_\nu n^\mu - n^\nu \nabla_\nu \nabla_\mu n^\mu =$$

$$= -\nabla_\mu n^\nu \nabla_\nu n^\mu + \nabla_\mu (n^\nu \nabla_\nu n^\mu) + \nabla_\nu n^\nu \nabla_\mu n^\mu - \nabla_\nu (n^\nu \nabla_\mu n^\mu) \quad (7.13)$$

and transforming the two divergence terms in surface integrals, the action becomes

$$S = \int dt \int_{\Sigma_t} \sqrt{-g} d^D x (\mathcal{R} - K^2 + \text{Tr}(KK)) + 2 \int_B \sqrt{-\gamma} d^D x (\Theta + u_\mu (n^\mu K - a^\mu))$$

where a^μ is the acceleration vector $a^\mu = n^\lambda \nabla_\lambda n^\mu$; moreover due to $u_\mu n^\mu = 0$ the part proportional to K vanishes. Notice that the two surface terms due to Σ_2 and Σ_1 in action (7.12) have been canceled by the contribution of the two divergence terms in Eq.(7.13) computed on Σ_2 and Σ_1 . The surface contribution on B , i.e. the last integral in the previous equation, can be expressed in terms of the extrinsic curvature k of $B_t = B \cap \Sigma_t$ as a sub-manifold of Σ_t .

In fact under our working hypothesis $u_\mu n^\mu = 0$, u_μ coincides with the unit vector r_μ belonging Σ_t and orthogonal to B_t . The metric on B_t is given by

$$\sigma_{\mu\nu} = h_{\mu\nu} - u_\mu u_\nu$$

and thus the extrinsic curvature of B_t is

$$k_{\mu\nu} = \sigma_\mu^\alpha D_\alpha r_\nu = \sigma_\mu^\alpha D_\alpha u_\nu = h_\mu^\beta \nabla_\beta u_\nu - u^\alpha u_\mu h_\alpha^\beta \nabla_\beta u_\nu$$

whose trace is given by

$$k = k_\mu^\mu = \nabla_\mu u^\mu + n^\beta n_\mu \nabla_\beta u^\mu = \Theta - u_\mu a^\mu.$$

Thus we obtain the action in Lagrangian form $S = \int L dt$ i.e.

$$S = \int dt \left(\int_{\Sigma_t} \sqrt{h} d^D x N (\mathcal{R} - K^2 + \text{Tr}(KK)) + 2 \int_{B_t} dx^{D-1} \sqrt{-\gamma} k \right). \quad (7.14)$$

$\sqrt{-\gamma}$ can also be written as $N\sqrt{\sigma}$, being $\sqrt{\sigma}$ the area element of B_t . We notice that the extrinsic curvature of Σ_t , $K_{\mu\nu} = h_\mu^\alpha \nabla_\alpha n_\nu$ appears quadratically in the above formula.

The remarkable feature of Eq.(7.14) is that the top and bottom boundary terms have disappeared and thus it is of the form $\int L dt$ where L is the Lagrangian of the system.

Thus we can develop from it the hamiltonian formalism.

We shall need now a more explicit form of the extrinsic curvature tensor.

$$\begin{aligned} K_{ab} &= \frac{1}{2} L_n h_{ab} = \frac{1}{2} (n^c \nabla_c h_{ab} + h_{cb} \nabla_a n^c + h_{ac} \nabla_b n^c) = \\ &= \frac{1}{2N} (N n^c \nabla_c h_{ab} + h_{cb} \nabla_a (N n^c) + h_{ac} \nabla_b (N n^c)) \end{aligned}$$

because $h_{ac}n^c = 0$. Substitute now $Nn^a = t^a - N^a$ to obtain

$$K_{ab} = \frac{1}{2N}(L_t h_{ab} - L_N h_{ab})$$

which lying in T_Σ can be rewritten as

$$K_{ab} = \frac{1}{2N}(h_a^{a'} h_a^{a'} L_t h_{a'b'} - h_a^{a'} h_a^{a'} L_N h_{a'b'}) = \frac{1}{2N}(h_a^{a'} h_a^{a'} L_t h_{a'b'} - D_a N_b - D_b N_a).$$

Taking into account that $N^a \in T_\Sigma$ i.e. $N^a n_a = 0$ and going over to ADM coordinate system where $t^a = (1, 0, 0, 0)$, $n_i = 0$ we have

$$K_{ij} = \frac{1}{2N}(\partial_0 h_{ij} - D_i N_j - D_j N_i). \quad (7.15)$$

These are all the significant components of the tensor $K_{ab} \in T_\Sigma$ as all the other (redundant) components can be obtained from $n^a K_{ab} = 0$. Moreover in forming scalars on Σ i.e. contracting with vectors or tensors belonging to T_Σ only the components K_{ij} play a role as for such vectors or tensors we have $t^{0\mu} = 0$.

Notice that of the variables h_{ij}, N^i, N no time derivatives of N^i and N appear in the action and as such they appear as Lagrange multipliers. We compute with the usual rules the conjugate momenta to h_{ij}

$$\pi^{ij}(t, x) = \frac{\delta L}{\delta \dot{h}_{ij}(t, x)} = \sqrt{h}(K^{ij} - K h^{ij}) \quad (7.16)$$

which can be inverted in \dot{h}_{ij} using Eq.(7.15). Beware that π^{ij} is not a tensor on Σ_t but a tensorial density. Write now L in hamiltonian form and perform one integration by parts in $d^D x$ to obtain

$$S_H = \int dt \int_{\Sigma_t} d^D x (\pi^{ij} \dot{h}_{ij} - NH - N^i H_i) + 2 \int dt \int_{B_t} \sqrt{\sigma} d^{D-1} x (Nk - r_i \frac{\pi^{ij}}{\sqrt{h}} N_j) \quad (7.17)$$

with

$$H = \sqrt{h} \left(-\mathcal{R} + \frac{\text{Tr}(\pi\pi)}{h} - \frac{\pi^2}{(D-1)h} \right), \quad H_i = -2\sqrt{h} D_j \left(\frac{\pi^j_i}{\sqrt{h}} \right)$$

r_i being the outward pointing unit normal to B_t as a sub-manifold of Σ_t and k the extrinsic curvature of intersection of Σ_t with B considered as a sub-manifold of Σ_t .

Variation with respect to N^i and with respect to N give

$$H_i = 0, \quad H = 0.$$

These are the constraints and they are equivalent to $G^{i0} = 0$ and $G^{00} = 0$ as it easily checked using Eq.(7.16).

Variation with respect to π^{ij} give

$$\dot{h}_{ij} = \frac{2N}{\sqrt{h}}(\pi_{ij} - \frac{\pi}{D-1}h_{ij}) + D_i N_j + D_j N_i .$$

The hard part is $\frac{\delta S}{\delta h_{ij}}$ which are the real dynamical equations of motion and correspond to combinations of $G^{ij} = 0$.

$$\begin{aligned} \dot{\pi}^{ij} &= -N\sqrt{h}(\mathcal{R}^{ij} - \frac{1}{2}\mathcal{R}h^{ij}) + \frac{N}{2\sqrt{h}}h^{ij}(\text{Tr}(\pi\pi) - \frac{1}{D-1}\pi^2) \\ &- \frac{2N}{\sqrt{h}}(\pi^{ik}\pi_k^j - \frac{1}{D-1}\pi\pi^{ij}) + \sqrt{h}(D^i D^j N - h^{ij}D^k D_k N) \\ &+ \sqrt{h}D_k(\frac{1}{\sqrt{h}}N^k\pi^{ij}) - \pi^{ki}D_k N^j - \pi^{kj}D_k N^i. \end{aligned} \quad (7.18)$$

In performing such a calculation one has to keep in mind that the true canonical variables are h_{ij} and π^{ij} and thus e.g. π_i^j has to be understood as $\pi^{jn}h_{ni}$. The variation of $\sqrt{h} \mathcal{R}$ is well known from the analogous variation in the Hilbert action through the Palatini identity

$$h^{ij}\delta\mathcal{R}_{ij} = D_b(D_c\delta h^{cb} - \frac{1}{2}D^b\delta h_c^c) - \frac{1}{2}D_b D^b\delta h_c^c = D_b D_c\delta h^{cb} - D_b D^b\delta h_c^c$$

but when one integrates by parts one has to keep in mind that contrary to what happens for the Hilbert action, the N which stays in front of H gives rise to the additional contribution in (7.17). The details of the calculation are given in the Appendix.

7.5 The Poisson algebra of the constraints

It is of interest to compute the Poisson algebra of the constraints. The result is

$$\{H_i(\mathbf{x}), H_j(\mathbf{x}')\} = H_i(\mathbf{x}')\partial_j\delta(\mathbf{x}, \mathbf{x}') - H_j(\mathbf{x})\partial'_i\delta(\mathbf{x}, \mathbf{x}')$$

$$\{H_i(\mathbf{x}), H(\mathbf{x}')\} = H(\mathbf{x})\partial_i\delta(\mathbf{x}, \mathbf{x}')$$

$$\{H(\mathbf{x}), H(\mathbf{x}')\} = H^l(\mathbf{x})\partial_l\delta(\mathbf{x}, \mathbf{x}') - H^l(\mathbf{x}')\partial'_l\delta(\mathbf{x}, \mathbf{x}')$$

In smeared form with

$$\mathcal{H}[\mathbf{N}, N] = \int (H(\mathbf{x})N(\mathbf{x}) + H_i(\mathbf{x})N^i(\mathbf{x}))d^D x$$

we have

$$\{\mathcal{H}[\mathbf{N}_1, 0], \mathcal{H}[\mathbf{N}_2, 0]\} = \mathcal{H}[[\mathbf{N}_1, \mathbf{N}_2], 0] \quad (7.19)$$

$$\{\mathcal{H}[\mathbf{N}_1, 0], \mathcal{H}[0, N]\} = \mathcal{H}[0, \mathcal{L}_{\mathbf{N}_1} N] \quad (7.20)$$

$$\{\mathcal{H}[0, N], \mathcal{H}[0, N']\} = \mathcal{H}[h^{ij}(ND_j N' - N' D_j N), 0] . \quad (7.21)$$

It is simpler to work with the smeared form of the P.B. . The derivation of the above reported algebra is given in the Appendix.

One notices that:

- 1) Eqs.(7.19,7.20) give a canonical representation of the space diffeomorphisms.
- 2) The only P.B. which contains explicitly the metric is (7.21).
- 3) The P.B. algebra of the constraints closes. This implies that as

$$\dot{F} = \{F, \mathcal{H}\}$$

we have that if the constraints are initially zero they remain conserved equal to zero in virtue of the equations of motion.

References

- [1] C.J. Isham, “Canonical quantum gravity and the problem of time” Lectures at NATO Advanced Study Institute, Salamanca 1992, p. 157.
- [2] J.Pullin, “Canonical quantization of general relativity: the last 18 years in a nutshell” gr-qc/0209008
- [3] C. Rovelli, “Quantum gravity” Cambridge University Press.
- [4] T. Thiemann, “Modern canonical quantum general relativity” Cambridge University Press, (2007)

7.6 Fluids

We consider only very simple fluids. They will be useful to introduce the energy conditions, which play a very important role in various instances, like the positive energy problem and the problem of closed time like curves.

1. Dust

$$T^{\lambda\nu} = \mu U^\lambda U^\nu; \quad U^\lambda = \text{four velocity field}, \quad U_\lambda U^\lambda = -1. \quad (7.22)$$

From Bianchi identities

$$\nabla_\lambda(\mu U^\lambda) U^\nu + \mu U^\lambda \nabla_\lambda U^\nu = 0. \quad (7.23)$$

Multiply by U_ν

$$\nabla_\lambda(\mu U^\lambda) = \nabla_\lambda J^\lambda = 0 \quad (7.24)$$

we obtain the conservation of “charge”

$$Q = \int_\Sigma \sqrt{h} d^3x n_\lambda(\mu U^\lambda) = \int_\Sigma \mu U^\lambda \Sigma_\lambda .$$

Eq.(7.23) now becomes

$$\mu U^\lambda \nabla_\lambda U^\nu = 0$$

i.e. the integral lines of the field U^λ are geodesic curves and the dust follows a geodesic motion. In this case for the structure (7.22) Einstein equations determine the motion of the particles of the fluid.

Eq.(7.24) gives

$$U^0 \nabla_0 \mu + \mu \nabla_0 U^0 = -\nabla_j(\mu U^j)$$

which combined with the three equations

$$U^0 \nabla_0 U^j = -U^i \nabla_i U^j$$

and recalling that U^0 is a function of U^j form a system of four differential equation of the first order in the time derivative which determine the time evolution of μ and U^j .

2. Perfect fluid: It is assumed that the energy momentum $T^{\lambda\nu}$ is constructed out of a four velocity field U^λ and a set of scalar functions.

One imposes isotropy i.e. that in the rest frame and in an orthonormal system in the tangent space, T^{ab} is invariant under $SO(3)$ rotations

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$$

with $R \in SO(3)$, i.e.

$$\Lambda T \Lambda^T = T.$$

We have

$$\Lambda^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & R^T \end{pmatrix}.$$

Through Schur lemma this imposes T^{ab} to be of the form

$$T^{ab} = \begin{pmatrix} \mu & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

where μ is the energy density in the rest frame and p is the pressure. Written in covariant form using the four velocity we have

$$T^{\lambda\nu} = c_1 U^\lambda U^\nu + c_2 g^{\lambda\nu}$$

i.e.

$$T^{\lambda\nu} = (\mu + p)U^\lambda U^\nu + g^{\lambda\nu}p. \quad (7.25)$$

The dust is given by the equation of state $p = 0$. Developing $\nabla_\lambda T^{\lambda\nu}$ and multiplying by U_ν we have

$$0 = \nabla_\lambda((\mu + p)U^\lambda) - U^\lambda \nabla_\lambda p = (\mu + p)\nabla_\lambda U^\lambda + U^\lambda \nabla_\lambda \mu. \quad (7.26)$$

We look for a scalar ρ , density function, such that the current ρU^λ is conserved as a consequence of Eq.(7.26) i.e.

$$\nabla_\lambda(\rho U^\lambda) = U^\lambda \nabla_\lambda \rho + \rho \nabla_\lambda U^\lambda = \alpha (U^\lambda \nabla_\lambda \mu + (\mu + p)\nabla_\lambda U^\lambda)$$

It is satisfied by

$$\nabla_\lambda \rho = \alpha \nabla_\lambda \mu; \quad \rho = \alpha(\mu + p).$$

Setting $\rho = \rho(\mu)$ (isentropic fluid) we have

$$\frac{\rho'}{\rho} = \frac{1}{\mu + p}.$$

where the prime denotes the derivative with respect to μ . From the above we have also $p = p(\mu)$. Thus

$$\rho(\mu) = k \exp \int^\mu \frac{dx}{x + p(x)}$$

and for such ρ we have

$$\nabla_\nu(\rho U^\nu) = 0$$

and

$$\int_\Sigma \rho U^\lambda \Sigma_\lambda = Q \quad \text{conserved}.$$

From Eq.(7.26) we obtain

$$U^0 \nabla_0 \mu + (\mu + p)\nabla_0 U^0 = -U^i \nabla_i \mu - (\mu + p)\nabla_i U^i \quad (7.27)$$

to which we add

$$(\mu + p)U^0 \nabla_0 U^j + U^j U^0 \nabla_0 p = -(\mu + p)U^k \nabla_k U^j - U^j U^k \nabla_k p - \nabla^j p \quad (7.28)$$

where U^0 is determined by $U^\mu g_{\mu\nu} U^\nu = -1$. The set of eqs.(7.27,7.28) form a system of four differential equations of first order in the time derivative for the four unknown μ and U^j which determine their time development.

References

[1] [HawkingEllis] Chap. 3

7.7 The energy conditions

1. The weak energy condition (WEC)

It states that for any time like vector v^μ and thus also for any light-like vector the following inequality holds for the energy momentum tensor

$$v^\mu T_{\mu\nu} v^\nu \geq 0.$$

The meaning is that any observer which isolates as small region of space should associate to the included matter a positive energy.

2. The dominant energy condition (DEC)

It states that the WEC holds and in addition for any time-like and thus also for any light-like vector v^μ the vector

$$T^{\mu\nu} v_\nu$$

should be time-like or null. The meaning of such a condition is that any observer which isolates as small region of space should associate to the included matter a time like or light like energy momentum vector.

3. The strong energy condition (SEC)

It states that for any time-like and thus also for any light-like vector v^μ the following inequality holds

$$(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)v^\mu v^\nu \geq 0 .$$

In four dimensions, in absence of cosmological constant it becomes

$$v^\mu R_{\mu\nu} v^\nu \geq 0 .$$

Such condition is satisfied for most of the physical fields but does not hold generally.

The energy conditions are distinguishing features of gravity as compared to gauge theories where there is no a priori restriction on the currents.

The WEC and the DEC are the basic inputs for proving the positivity of the energy of a gravitational system and for establishing theorems with regard to the absence of closed time like curves (CTC) or the chronology protection.

References

- [1] [HawkingEllis] Chap. 4

7.8 The cosmological constant

The cosmological constant Λ provides in Einstein equations an “effective energy momentum tensor” with $\mu + p = 0$ and $p = -\Lambda$, whose canonical form is

$$T^{ab} = \frac{1}{k} \begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & -\Lambda \end{pmatrix}$$

Such “effective energy momentum tensor” satisfies (barely) the weak and dominant energy condition only for $\Lambda > 0$. On the other hand as $\mu + p = 0$ the velocity terms disappear in Eq.(7.25) and thus it can hardly be interpreted as a fluid. Presently the value of Λ is found positive; of the energy content of the universe 70% is assigned to the cosmological constant; of the remaining 30%, 4% to ordinary matter out of which only one fourth directly visible and the remaining 26% to “dark matter” i.e. a form of matter we have not yet accessed. The present value of the cosmological constant is

$$(c\hbar)^3 \frac{c^4}{8\pi G} \Lambda = (3 \cdot 10^{-3} \text{ eV})^4, \quad \Lambda l_P^2 = 3.9 \cdot 10^{-123}$$

Chapter 8

The energy of a gravitational system

8.1 The non abelian charge conservation

There is an analogy between the problem of conservation of the non-abelian charge in Yang-Mills theory and the problem of definition of energy in gravitation.

We already saw in Section 4.27 that from the YM equations $D * F = *J$ it follows $0 = DD * F = D * J = 0$, explicitly from

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] = J^\nu$$

it follows

$$0 = D_\nu J^\nu = \partial_\nu J^\nu + [A_\nu, J^\nu].$$

This is the covariant conservation of the covariant current J^ν , which does not imply a conserved quantity.

It is possible to derive a conservation law as follows: write

$$\partial_\mu F^{\mu\nu} = J^\nu - [A_\mu, F^{\mu\nu}] \equiv j^\nu.$$

Then we have $\partial_\nu j^\nu = 0$ from which the conserved charge can be obtained

$$Q = \int_\Sigma j^0 d^3x.$$

However j^μ is not a covariant quantity in contrast to J^ν . We can also write

$$Q = \int_\Sigma \partial_\mu F^{\mu 0} d^3x = \int_\Sigma \partial_j F^{j0} d^3x = \int_{\partial\Sigma} u_j F^{j0} d^2x$$

being $\partial\Sigma$ the boundary of Σ which will be taken to infinity. u_i is the outward-pointing unit normal. As a result if we consider a gauge transformation U arbitrary but constant at space infinity we have that Q under such gauge transformation, transform covariantly $Q \rightarrow U(\infty)QU(\infty)^{-1}$ due to the covariance property of $F^{\mu\nu}$.

We turn now to the energy of a system which interacts gravitationally.

8.2 The background Bianchi identities

In this chapter we shall work with the second order formalism; this means that the connection will be the metric compatible torsionless Levi-Civita connection. Later in Section 8.4 vierbeins will occur and we recall that with the Levi-Civita connection we have $\nabla e^a = 0$ as we discussed in Chapter 4. Starting from Einstein equations

$$\mathcal{G}^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu} \quad (8.1)$$

where $\kappa = 8\pi G_N/c^3$ with G_N Newton's constant, we consider a background metric $\bar{g}_{\mu\nu}$ satisfying Einstein's equations in absence of matter

$$\bar{\mathcal{G}}^{\mu\nu} \equiv \bar{R}^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}\bar{R} + \Lambda\bar{g}^{\mu\nu} = 0 \quad (8.2)$$

From the contracted Bianchi identities we have for any $g_{\mu\nu}$

$$\nabla_\mu (R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu}) = 0 \quad (8.3)$$

Put $g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}$. ε is a formal parameter which at the end will be put equal to 1; it will be useful to characterize the powers of $h_{\mu\nu}$ which intervene in an expression. We consider both $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$ as tensors under diffeomorphisms.

We have e.g.

$$\Gamma_{\mu\nu\rho} = \frac{1}{2}(\bar{g}_{\mu\nu,\rho} + \bar{g}_{\mu\rho,\nu} - \bar{g}_{\nu\rho,\mu}) + \frac{\varepsilon}{2}(h_{\mu\nu,\rho} + h_{\mu\rho,\nu} - h_{\nu\rho,\mu})$$

and raising the index with $g^{\lambda\mu}$ we have

$$\Gamma_{\nu\rho}^\lambda = \frac{1}{2}\bar{g}^{\lambda\mu}(\bar{g}_{\mu\nu,\rho} + \bar{g}_{\mu\rho,\nu} - \bar{g}_{\nu\rho,\mu}) + O(\varepsilon) = \bar{\Gamma}_{\nu\rho}^\lambda + O(\varepsilon). \quad (8.4)$$

We have

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = (\bar{R}^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}\bar{R} + \Lambda\bar{g}^{\mu\nu}) + \varepsilon K^{\mu\nu} + O(\varepsilon^2) = \varepsilon K^{\mu\nu} + O(\varepsilon^2)$$

$K^{\mu\nu}$ depends both on $\bar{g}_{\mu\nu}$ and $h_{\mu\nu}$ and it is a tensor under diffeomorphisms.

Moreover due to (8.4) we have $\nabla_\mu = \bar{\nabla}_\mu + O(\varepsilon)$. Then identity (8.3) tells us that the following identity holds

$$\bar{\nabla}_\mu K^{\mu\nu} = 0 \quad (8.5)$$

which is called the background Bianchi identity [1]. This is an *identity* which holds for any $h_{\mu\nu}$.

We notice that $\bar{\Gamma}_{\nu\rho}^\lambda$ and $\bar{R}_{\mu\nu}$ are defined in terms of the metric $\bar{g}_{\mu\nu}$ in particular we have $\bar{R}_\nu^\mu = \bar{g}^{\mu\mu'}\bar{R}_{\mu'\nu}$ and $\bar{R}^{\mu\nu} = \bar{g}^{\mu\mu'}\bar{R}_{\mu'\nu'}\bar{g}^{\nu'\nu}$.

The we can write Einstein' equation (8.1) as ($\varepsilon = 1$)

$$K^{\mu\nu} = \kappa T^{\mu\nu} - \mathcal{G}^{\mu\nu} + K^{\mu\nu} \equiv 8\pi\kappa(T^{\mu\nu} + t^{\mu\nu}) = 8\pi\kappa\mathcal{T}^{\mu\nu}$$

where $t^{\mu\nu}$ is minus the non linear part in $h_{\mu\nu}$ of the l.h.s. of (8.1) divided by $8\pi\kappa$ and is called the energy momentum tensor of the gravitational field, while $\mathcal{T}^{\mu\nu}$ is called the total energy momentum tensor. As a consequence of (8.5) we have

$$\bar{\nabla}_\mu \mathcal{T}^{\mu\nu} = 0 \quad (8.6)$$

which is the background covariant conservation of the total energy momentum tensor.

It is of interest the case in which the background metric $\bar{g}_{\mu\nu}$ possesses Killing vectors, in different words, when we have isometries i.e. diffeomorphisms ϕ for which $\phi^*\bar{g}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x)$. It means that

$$\phi^*(g_{\mu\nu} - \bar{g}_{\mu\nu})(x) = \phi^*(g_{\mu\nu})(x) - \phi^*(\bar{g}_{\mu\nu})(x) = \phi^*(g_{\mu\nu})(x) - \bar{g}_{\mu\nu}(x)$$

and thus the difference between $g_{\mu\nu}$ and an invariant (under ϕ) background, transforms like a tensor under the group of the isometry transformations of $\bar{g}_{\mu\nu}$. E.g. for $\bar{g}_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ we have that

$$g_{\mu\nu} - \eta_{\mu\nu}$$

is a tensor under Lorentz transformations.

If $\bar{\xi}_\nu$ is a Killing vector field of the background we have $\bar{\nabla}_\mu \bar{\xi}_\nu + \bar{\nabla}_\nu \bar{\xi}_\mu = 0$ and as a consequence due to (8.6) we have

$$\bar{\nabla}_\mu (\mathcal{T}^{\mu\nu} \bar{\xi}_\nu) = 0$$

i.e. a conserved quantity

$$Q = \int_\Sigma \mathcal{T}^{\mu\nu} \bar{\xi}_\nu \sqrt{-\bar{g}} \varepsilon_{\mu\alpha\beta\gamma} \frac{dx^\alpha \wedge dx^\beta \wedge dx^\gamma}{3!} = \int_\Sigma \mathcal{T}^{\mu\nu} \bar{\xi}_\nu \Sigma_\mu$$

provided $\mathcal{T}^{\mu\nu} \bar{\xi}_\nu$ vanishes sufficiently quickly at space infinity as to make the surface integral at space infinity to vanish and the integral over Σ convergent.

In general there is no problem for $T^{\mu\nu}$ which can be assumed to vanish exactly outside a compact space support; for $t^{\mu\nu}$ to vanish sufficiently quickly at space infinity $\bar{g}_{\mu\nu}$ must be a sufficiently accurate background for $g_{\mu\nu}$ at space infinity [2]. Given an isometry transformation ϕ , connected to the identity we can compute the push-forward $\phi_*\xi(\lambda)$ of a Killing vector $\xi(\lambda)$. This will be a linear combination of the Killing vectors of $\bar{g}_{\mu\nu}$ which

form a finite dimensional linear space [3]. Then we shall have for the associated conserved quantity

$$Q[\phi_*\xi(\lambda)] = \int_{\Sigma} \mathcal{T}^{\mu\nu} \bar{\phi}_* \xi_{\nu}(\lambda) \Sigma_{\mu} = \xi(\rho) A^{\rho}_{\lambda}(\phi) .$$

Now we specialize to the case $\Lambda = 0$ and the background metric chosen to be the Minkowski $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Let $\xi(\lambda)$ be one of the four Killing vector of space-time translations

$$\xi(\lambda) = \frac{\partial}{\partial x^{\lambda}} .$$

Then from

$$\partial_{\mu}(\mathcal{T}^{\mu\nu}\xi(\lambda)_{\nu}) = 0$$

provided the matter (not really a problem) and the gravitational energy momentum tensor vanish sufficiently rapidly at space infinity, we have the four conserved quantities

$$P_{\lambda} \equiv \int_{\Sigma} \mathcal{T}^{\mu\nu} \xi(\lambda)_{\nu} \Sigma_{\mu}$$

where Σ is any space-like surface (e.g. a space-like plane). Let us perform a coordinate transformation which is an isometry of the background, in our case a Poincaré transformation $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$. The new Killing vectors are given by

$$\xi'(\lambda) = \frac{\partial}{\partial x'^{\lambda}} = \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial}{\partial x^{\rho}} = \xi(\rho) (\Lambda^{-1})^{\rho}_{\lambda} .$$

Then we have

$$P'_{\lambda} = P_{\rho} (\Lambda^{-1})^{\rho}_{\lambda}$$

i.e. the P_{λ} transform like the covariant components of a four vector. This is the total energy momentum four vector of the gravitational system.

If Σ is provided by the plane $x^0 = \text{const}$, the components of the Killing vector $\xi(\lambda)$ are $\xi^{\nu}(\lambda) = \delta^{\nu}_{\lambda}$ we can also write

$$P_{\lambda} \equiv \int_{\Sigma} \mathcal{T}^{\mu}_{\nu} \xi^{\nu}(\lambda) \Sigma_{\mu} = \int_{\Sigma} \mathcal{T}^0_{\lambda} dx^1 dx^2 dx^3$$

References

- [1] L.F. Abbott and S. Deser, “Stability of gravity with a cosmological constant” Nucl.Phys. B195 (1982) 76
- [2] [WeinbergGC] Chap.7
- [3] [WeinbergGC] Chap.13

8.3 The superpotential

We now work out the superpotential in the case of Minkowski background. We recall that Einstein's equations can be written as

$$-\frac{1}{4}R^{\alpha\beta}{}_{\mu\nu}\delta^{\mu\nu\rho}{}_{\alpha\beta\lambda} = \kappa T_{\lambda}^{\rho}.$$

Then after defining

$$\hat{\Gamma}_{\beta\nu}^{\alpha} = \eta^{\alpha\alpha'}\Gamma_{\alpha'\beta\nu}$$

which is linear in $h_{\mu\nu}$ we have for the linear part \hat{G}_{λ}^{ρ} of the Einstein tensor

$$\hat{G}_{\lambda}^{\rho} = -\frac{1}{4}\hat{R}^{\alpha\beta}{}_{\mu\nu}\delta^{\mu\nu\rho}{}_{\alpha\beta\lambda} = \partial_{\mu}\hat{Q}^{\mu\rho}{}_{\lambda} = \kappa \mathcal{T}_{\lambda}^{\rho}$$

being the superpotential $\hat{Q}^{\mu\rho}{}_{\lambda}$ defined by

$$\hat{Q}^{\mu\rho}{}_{\lambda} = \frac{1}{2}\hat{\Gamma}^{\alpha\beta}{}_{\nu}\delta^{\nu\mu\rho}{}_{\alpha\beta\lambda}.$$

Due to the antisymmetry of $\hat{Q}^{\mu\rho}{}_{\lambda}$ we have $\partial_{\rho}\mathcal{T}_{\lambda}^{\rho} = 0$. After defining the two-form

$$\hat{Q}_{\lambda} = \hat{Q}_{\lambda}^{\sigma\alpha}\varepsilon_{\sigma\alpha\mu\nu}\frac{dx^{\mu}\wedge dx^{\nu}}{4} = \frac{1}{2}\hat{Q}_{\lambda}^{\sigma\alpha}\hat{S}_{\sigma\alpha}$$

the conserved P_{λ} can be written as

$$\kappa P_{\lambda} = \kappa \int_{\Sigma} \mathcal{T}_{\lambda}^{\rho}\Sigma_{\rho} = \int_{\Sigma} \partial_{\mu}\hat{Q}^{\mu\rho}{}_{\lambda}\Sigma_{\rho} = - \int_{\Sigma} d\hat{Q}_{\lambda}.$$

In fact

$$- \int_{\Sigma} d\hat{Q}_{\lambda} = - \int_{\Sigma} \partial_{\rho}\hat{Q}_{\lambda}^{\sigma\alpha}\varepsilon_{\sigma\alpha\mu\nu}\frac{dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu}}{4} = \int_{\Sigma} \partial_{\alpha}Q_{\lambda}^{\alpha\sigma}\Sigma_{\sigma}$$

where we used

$$\frac{1}{2!}\varepsilon_{\sigma\alpha\mu\nu}dx^{\rho}\wedge dx^{\mu}\wedge dx^{\nu} = \frac{1}{3!}(\delta_{\alpha}^{\rho}\varepsilon_{\sigma\mu\nu\kappa} - \delta_{\sigma}^{\rho}\varepsilon_{\alpha\mu\nu\kappa})dx^{\mu}\wedge dx^{\nu}\wedge dx^{\kappa}. \quad (8.7)$$

But now P_{λ} can be computed as a surface integral

$$\kappa P_{\lambda} = - \int_{\Sigma} d\hat{Q}_{\lambda} = - \int_{\partial\Sigma} \hat{Q}_{\lambda} = - \int_{\partial\Sigma} \hat{Q}_{\lambda}^{\sigma\alpha}\varepsilon_{\sigma\alpha\mu\nu}\frac{dx^{\mu}\wedge dx^{\nu}}{4} = -\frac{1}{2}\int_{\partial\Sigma} \hat{Q}_{\lambda}^{\sigma\alpha}\hat{S}_{\sigma\alpha} \quad (8.8)$$

i.e. the energy momentum four vector can be computed in terms of a two dimensional integral on the 2-sphere at infinity.

Saying that the energy is positive is the same as saying that given any future directed time like four vector u^{λ} we have

$$\kappa u^{\lambda}P_{\lambda} = -\frac{1}{2}\int_{\partial\Sigma} u^{\lambda}\hat{Q}_{\lambda}^{\sigma\alpha}S_{\sigma\alpha} = - \int_{\partial\Sigma} u^{\lambda}\hat{Q}_{\lambda} \leq 0. \quad (8.9)$$

It is useful to give Eq.(8.8) a covariant expression. Neither $\Gamma_{\beta\nu}^\alpha$ nor $\Gamma_{\beta'\nu}^\alpha g^{\beta'\beta}$ are linear in $h_{\mu\nu}$ but they are equivalent to $\hat{\Gamma}_{\beta\nu}^\alpha$ and $\hat{\Gamma}^{\alpha\beta}_\nu$ in the integral at infinity as the correction terms vanish more quickly and thus give a zero contribution [2]. Moreover we can subtract from $\Gamma_{\beta\nu}^\alpha$ zero written as $\Gamma_{\beta\nu}^\alpha(\text{vacuum})$ in the Minkowskian metric $g_{\mu\nu} = \eta_{\mu\nu}$ with the advantage that

$$\Delta\Gamma_{\beta\nu}^\alpha$$

is a true tensor under all diffeomorphisms. As asymptotically we have $e = \sqrt{-g} = 1$, as far as the computation of $u^\lambda P_\lambda$ is concerned, we can replace in (8.9) the form $u^\lambda \hat{Q}_\lambda$ with the covariant two-form

$$Q \equiv \frac{1}{2} u^\lambda Q^{\sigma\alpha} \epsilon_{\sigma\alpha\mu\nu} \frac{1}{2} dx^\mu \wedge dx^\nu = \frac{1}{2} u^\lambda Q^{\sigma\alpha} S_{\sigma\alpha} \quad (8.10)$$

where $\epsilon_{\alpha\beta\gamma\delta} = \sqrt{-g} \varepsilon_{\alpha\beta\gamma\delta}$.

$$Q^{\sigma\alpha}_\lambda = \frac{1}{2} \Delta\Gamma_{\beta'\nu}^\zeta g^{\beta'\beta} \delta_{\zeta\beta\lambda}^{\nu\sigma\alpha}$$

and u^λ is a time-like future directed vector field, constant at space infinity. Summarizing the conservation law takes the form

$$0 = \int_V ddQ = \int_{\Sigma_2} dQ - \int_{\Sigma_1} dQ$$

with

$$- \int_\Sigma dQ = - \int_{\partial\Sigma} Q = - \int_{\partial\Sigma} u^\lambda Q^{\sigma\alpha} dS_{\sigma\alpha} = - \frac{1}{2} \int_{\partial\Sigma} u^\lambda \Delta\Gamma_{\nu}^{\zeta\beta} \delta_{\zeta\beta\lambda}^{\nu\sigma\alpha} S_{\sigma\alpha} = \kappa u^\lambda P_\lambda \quad (8.11)$$

which is a fully covariant expression.

[1] L.F. Abbott and S. Deser “Stability of gravity with a cosmological constant” Nucl.Phys. B195 (1982) 76

[2] [WeinbergGC] Chap.7

[3] J. Nester, “A new gravitational energy expression with a simple positivity proof” Phys. Lett. 83 A (1981) 241

8.4 The positive energy theorem

In the proof of the theorem we follow [3] giving more calculational details. We start noticing that given any spinor ψ , $u^\lambda = i\bar{\psi}\gamma^\lambda\psi$ is a time future directed four vector. In fact $u^0 = i\bar{\psi}\gamma^0\psi = \psi^+\psi > 0$ in all reference frames.

We now show that we can replace in (8.11) the integrand

$$u^\lambda Q^\sigma{}_\lambda = u^\lambda \frac{1}{2} \Delta \Gamma^{\zeta\beta}{}_\nu \delta^\nu{}_{\zeta\beta\lambda}$$

with [2][3]

$$E^{\sigma\alpha} = \epsilon^{\sigma\alpha\delta\beta} (\bar{\psi} \gamma_5 \gamma_\delta \nabla_\beta \psi - \nabla_\beta \bar{\psi} \gamma_5 \gamma_\delta \psi) \quad (8.12)$$

where ψ is a properly chosen spinor.

We recall that $\gamma_5 = \gamma^1 \gamma^2 \gamma^3 \gamma^4 = i \gamma^1 \gamma^2 \gamma^3 \gamma^0$ and $\epsilon^{\sigma\alpha\delta\beta} = e^{-1} \varepsilon^{\sigma\alpha\delta\beta}$ and we notice that (8.12) is a fully covariant expression.

ψ will be chosen as

$$\psi = \psi_c + \phi$$

with ψ_c a constant spinor such that

$$i \bar{\psi}_c \gamma^\lambda \psi_c = u^\lambda$$

and

$$\phi = O\left(\frac{1}{r}\right). \quad (8.13)$$

$E^{\sigma\alpha}$ differs from $u^\lambda Q^\sigma{}_\lambda$ by a divergence and by a term

$$O\left(\frac{1}{r^3}\right).$$

In fact treating first the ∂_β part of ∇_β we have

$$\bar{\psi} M \partial_\beta \psi = \bar{\psi} M \partial_\beta \phi = \partial_\beta (\bar{\psi} M \phi) - (\partial_\beta \bar{\psi}) M \phi$$

while

$$-\partial_\beta \bar{\psi} M \psi = -\partial_\beta \bar{\psi} M \phi = -\partial_\beta (\bar{\psi} M \phi) + \bar{\psi} M \partial_\beta \phi$$

with $(\partial_\beta \bar{\psi}) M \phi$ and $\bar{\psi} M \partial_\beta \phi$ both $O(1/r^3)$. The divergence term being the boundary of the boundary zero i.e. $\partial \partial \Sigma = \emptyset$ gives zero contribution; in fact

$$\int_{\partial \Sigma} \epsilon^{\sigma\alpha\delta\beta} \partial_\beta Y \epsilon_{\sigma\alpha\mu\nu} dx^\mu \wedge dx^\nu = -2 \int_{\partial \Sigma} \delta_{\mu\nu}^{\delta\beta} \partial_\beta Y dx^\mu \wedge dx^\nu = 4 \int_{\partial \Sigma} d(Y dx^\delta) = 0$$

and the term $O(1/r^3)$ integrated on the two dimensional surface at infinity gives also a vanishing contribution.

Thus we are left with the spin-connection terms. Using $[\gamma_a, \gamma_b]/4 = \sigma_{ab}$ we have

$$\begin{aligned} & \frac{1}{2} \epsilon^{\sigma\alpha\delta\beta} (\bar{\psi} \gamma_5 \gamma_\delta \sigma_{tz} \psi + \bar{\psi} \sigma_{tz} \gamma_5 \gamma_\delta \psi) \Gamma^{tz}{}_\beta = \frac{1}{2} \epsilon^{\sigma\alpha\delta\beta} \bar{\psi} \gamma_5 \{\gamma_\delta, \sigma_{tz}\} \psi \Gamma^{tz}{}_\beta \\ &= \frac{1}{2} \epsilon^{\sigma\alpha\delta\beta} \bar{\psi} \gamma_5 \{\gamma_d, \sigma_{tz}\} \psi e_\delta^d e_\zeta^t e_\tau^\zeta \Gamma^{\tau\zeta}{}_\beta = -\frac{1}{2} \epsilon^{\sigma\alpha\delta\beta} i \bar{\psi} \gamma^\lambda \psi \epsilon_{\delta\tau\zeta\lambda} \Gamma^{\tau\zeta}{}_\beta \\ &= \frac{1}{2} \delta_{\tau\zeta\lambda}^{\sigma\alpha\beta} i \bar{\psi} \gamma^\lambda \psi \Gamma^{\tau\zeta}{}_\beta = \frac{u^\lambda}{2} \Gamma^{\tau\zeta}{}_\beta \delta_{\tau\zeta\lambda}^{\beta\sigma\alpha} = u^\lambda Q^\sigma{}_\lambda. \end{aligned} \quad (8.14)$$

The great advantage of the obtained expression is that we can compute the two dimensional integral as a three dimensional integral of a divergence. In fact, to prove the positivity of $-u^\lambda P_\lambda$ it is necessary to go back to a three dimensional integral; only in this way we can have a relation with the energy momentum tensor of matter which can be the only cause of the positivity of the energy through the energy conditions. Summarizing the present situation is

$$\begin{aligned}
8\pi\kappa u^\lambda P_\lambda &= - \int_{\partial\Sigma} E^{\sigma\alpha} S_{\sigma\alpha} = - \int_{\partial\Sigma} E^{\sigma\alpha} \epsilon_{\sigma\alpha\mu\nu} \frac{dx^\mu \wedge dx^\nu}{2} \\
&= -\frac{1}{2} \int_{\Sigma} \partial_\rho (E^{\sigma\alpha} \epsilon_{\sigma\alpha\mu\nu}) dx^\rho \wedge dx^\mu \wedge dx^\nu = -\frac{1}{2} \int_{\Sigma} \nabla_\rho E^{\sigma\alpha} \epsilon_{\sigma\alpha\mu\nu} dx^\rho \wedge dx^\mu \wedge dx^\nu \\
&= -2 \int_{\Sigma} \nabla_\alpha E^{\sigma\alpha} \epsilon_{\sigma\mu\nu\kappa} \frac{dx^\mu \wedge dx^\nu \wedge dx^\kappa}{3!} = -2 \int_{\Sigma} \nabla_\alpha E^{\sigma\alpha} \Sigma_\sigma
\end{aligned} \tag{8.15}$$

where in the second line we used the zero torsion of the connection and in the last line we used (8.7).

We compute now the divergence of $E^{\sigma\alpha}$.

Being $\nabla_\alpha \epsilon^{\sigma\alpha\delta\beta} = 0$ and noticing that due to the absence of torsion

$$\epsilon^{\sigma\alpha\delta\beta} \nabla_\alpha e_\delta^d = 0$$

we have

$$\begin{aligned}
\nabla_\alpha E^{\sigma\alpha} &= \frac{1}{2} \epsilon^{\sigma\alpha\delta\beta} \bar{\psi} \gamma_5 \gamma_\delta [\nabla_\alpha, \nabla_\beta] \psi - \frac{1}{2} \epsilon^{\sigma\alpha\delta\beta} [\nabla_\alpha, \nabla_\beta] \bar{\psi} \gamma_5 \gamma_\delta \psi + \\
&\quad + 2\epsilon^{\sigma\alpha\delta\beta} \nabla_\alpha \bar{\psi} \gamma_5 \gamma_\delta \nabla_\beta \psi
\end{aligned}$$

First we examine the curvature terms $[\nabla_\alpha, \nabla_\beta]$. Recalling that $\gamma_5 = i\gamma^1\gamma^2\gamma^3\gamma^0$ we have

$$\frac{1}{4} \epsilon^{\sigma\alpha\delta\beta} \bar{\psi} \{\gamma_5 \gamma_\delta, \sigma_{\tau\zeta}\} \psi R_{\alpha\beta}^{\tau\zeta} = \frac{1}{4} \epsilon^{\sigma\alpha\delta\beta} i\bar{\psi} \gamma^\lambda \psi \epsilon_{\lambda\delta\tau\zeta} R_{\alpha\beta}^{\tau\zeta} = \frac{1}{4} u^\lambda R_{\alpha\beta}^{\tau\zeta} \delta_{\tau\zeta}^{\alpha\beta\sigma} = -8\pi\kappa u^\lambda T_\lambda^\sigma = -8\pi\kappa V^\sigma$$

with V^σ , due to the dominant energy is a non-space like vector with $V^0 < 0$ (cfr. Section 7.7).

This term substituted in Eq.(8.15)

$$\int_{\Sigma} \nabla_\alpha E^{\sigma\alpha} d\Sigma_\sigma$$

gives

$$-8\pi\kappa \int_{\Sigma} V^\sigma \epsilon_{\sigma\mu\nu\kappa} \frac{dx^\mu \wedge dx^\nu \wedge dx^\kappa}{3!} \geq 0. \tag{8.16}$$

The remainder is

$$2\epsilon^{\sigma\alpha\delta\beta} \nabla_\alpha \bar{\psi} \gamma_5 \gamma_\delta \nabla_\beta \psi$$

which recalling that

$$\gamma_5 \gamma_\delta \epsilon^{\sigma\alpha\delta\beta} = -i \{ \gamma^\sigma, \sigma^{\alpha\beta} \}$$

becomes

$$-2i \nabla_\alpha \bar{\psi} (\gamma^\sigma \sigma^{\alpha\beta} + \sigma^{\alpha\beta} \gamma^\sigma) \nabla_\beta \psi.$$

We have, using $x^0 = \text{const}$ on three dimensional spatial integration surface

$$\begin{aligned} & -2i \nabla_j \bar{\psi} \{ \gamma^0, \sigma^{jk} \} \nabla_k \psi = -4 \nabla_j \psi^+ \sigma^{jk} \nabla_k \psi \\ & = -\nabla_j \psi^+ (\gamma^j \gamma^k - \gamma^k \gamma^j) \nabla_k \psi = 2 \nabla_j \psi^+ \nabla_k \psi \eta^{jk} - 2 \nabla_j \psi^+ \gamma^j \gamma^k \nabla_k \psi. \end{aligned} \quad (8.17)$$

There is some freedom in choosing a ψ satisfying the property of becoming a constant spinor at space infinity with the correction ϕ behaving as (8.13). One can exploit such a freedom to go over to the Witten gauge [2],[4]

$$\gamma^k \nabla_k \psi = 0.$$

Then the last term in Eq.(8.17) vanishes and one is left with a positive contribution which summed to (8.16) gives a total positive result and thus from Eq.(8.15) $8\pi\kappa u^\lambda P_\lambda < 0$ which is the statement of the positive energy theorem.

References

- [1] R. Schoen , S. T. Yau, "On the proof of the positive mass conjecture in general relativity" *Comm. Math. Phys.* 65 (1976) 45; *Phys. Rev. Lett.* 43 (1979) 1459
- [2] E. Witten, "A new proof of the positive energy theorem" *Comm. Math. Phys.* 80 (1981) 381
- [3] J. Nester, "A new gravitational energy expression with a simple positivity proof" *Phys. Lett.* 83 A (1981) 241
- [4] T. Parker, C. Taubes, "On Witten's proof of the positive energy theorem" *Comm. Math. Phys.* (1982)

Chapter 9

The linearization of gravity

9.1 Introduction

First we want to understand the field theoretical description of particle of spin 2 in Minkowski space. Given a symmetric tensor of order 2 in four dimensions, it has 10 independent components. The trace h^μ_μ is a scalar field while $\partial_\mu h^\mu_\nu$ is a vector field. We want to write down an equation whose solutions satisfy the requirements $h^\mu_\mu = 0$ and $\partial_\mu h^\mu_\nu = 0$; we are then left with $10 - 1 - 4 = 5$ independent components which describe the 5 possible values $-2, -1, 0, 1, 2$ of the helicity of a massive spin-2 particle.

We get inspiration from the linearized Einstein equations

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R = \kappa T_{\mu\nu}$$

with $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ and expanding in κ

$$R = d\Gamma \in \Lambda^2_{(1,1)}$$

or explicitly

$$R^\alpha_{\beta\mu\nu} = (\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu)^\alpha_\beta = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu}.$$

We immediately have the linearized Bianchi identities

$$dR = dd\Gamma = 0$$

which written explicitly take the form

$$R^\alpha_{\beta[\mu\nu,\rho]} = 0.$$

By double contraction we obtain the linearized contracted Bianchi identities

$$0 = 2\partial_\rho R^\rho_\nu - \partial_\nu R.$$

The explicit form of $R_{\alpha\beta\mu\nu}$ is

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\nu\alpha,\beta\mu} - h_{\beta\nu,\alpha\mu} - h_{\alpha\mu,\beta\nu} + h_{\beta\mu,\alpha\nu}) \quad (9.1)$$

while

$$R_{\beta\nu} - \frac{1}{2}\eta_{\beta\nu}R = \frac{1}{2}(h^{\alpha}{}_{\nu,\alpha\beta} - h_{\beta\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\beta\nu} + h^{\alpha}{}_{\beta,\alpha\nu}) - \frac{1}{2}\eta_{\beta\nu}(h^{\alpha\gamma}{}_{,\alpha\gamma} - h^{\gamma}{}_{\gamma,\alpha}{}^{\alpha}) \equiv G_{\beta\nu}. \quad (9.2)$$

Under the linearized form of the diffeomorphism

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu} \quad (9.3)$$

we have

$$\delta\Gamma_{\alpha\beta\nu} = \xi_{\alpha,\beta\nu}$$

or better

$$\delta\Gamma_{\alpha\beta} = d(\xi_{\alpha,\beta}) \in \Lambda^1$$

and thus we have

$$\Lambda_{0,2}^2 \ni \delta R_{\alpha\beta} = dd(\xi_{\alpha,\beta}) = 0$$

i.e. the Riemann tensor is invariant and a fortiori $G_{\beta\nu}$ is invariant.

9.2 The Fierz-Pauli equation

Before discussing the particle content of the equation $G_{\beta\nu} = 0$ we want to modify it to include a mass term. We consider the equation

$$G_{\beta\nu} + \frac{m^2}{2}(h_{\beta\nu} - c \eta_{\beta\nu} h^{\alpha}{}_{\alpha}) = 0. \quad (9.4)$$

By taking the divergence and recalling the linearized contracted Bianchi identities we have for $m^2 \neq 0$

$$h^{\beta}{}_{\nu,\beta} - c h^{\alpha}{}_{\alpha,\nu} = 0. \quad (9.5)$$

But also the trace of Eq.(9.4) has to be zero i.e.

$$-2h^{\beta\alpha}{}_{,\beta\alpha} + 2h^{\beta}{}_{\beta,\alpha}{}^{\alpha} + m^2(h^{\beta}{}_{\beta} - 4c h^{\alpha}{}_{\alpha}) = 0$$

By taking the divergence of Eq.(9.5) we have

$$h^{\beta}{}_{\nu,\beta}{}^{\nu} - c h^{\alpha}{}_{\alpha,\nu}{}^{\nu} = 0$$

Choose then the free parameter c equal to 1. In this way from Eq.(9.4) we obtain

$$-3m^2 h^\alpha{}_\alpha = 0$$

i.e. we accomplish the condition $h^\alpha{}_\alpha = 0$. Substituting then in Eq.(7.13) we obtain also $h^\beta{}_{\nu,\beta} = 0$.

The above equation can be obtained through variation of the following Lagrangian

$$L = L_G + \frac{m^2}{4}(h^\alpha{}_\beta h^\beta{}_\alpha - h^\alpha{}_\alpha h^\beta{}_\beta) \quad (9.6)$$

where

$$L_G = \frac{1}{2}(\partial_\lambda h^\lambda{}_\nu \partial^\nu h^\mu{}_\mu - \partial_\lambda h^\lambda{}_\nu \partial_\mu h^{\mu\nu}) + \frac{1}{4}(\partial_\lambda h_{\mu\nu} \partial^\lambda h^{\mu\nu} - \partial_\lambda h^\mu{}_\mu \partial^\lambda h^\nu{}_\nu).$$

In order to understand the mass, substitute in Eq.(9.4) $h^\alpha{}_\alpha = 0$ and $h^\alpha{}_{\beta,\alpha} = 0$ and thus

$$-\frac{1}{2}h_{\beta\nu,\alpha}{}^\alpha + \frac{m^2}{2}h_{\beta\nu} = 0$$

i.e. the mass-shell is given by

$$k^2 + m^2 = 0$$

after setting $h_{\mu\nu} = \varepsilon_{\mu\nu} e^{ik \cdot x}$ with $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$, $\varepsilon^\mu{}_\mu = 0$ and $k^\mu \varepsilon_{\mu\nu} = 0$, for $k^2 + m^2 = 0$. The propagator can be extracted from the Lagrangian (9.6) and is given by

$$\begin{aligned} G_{\mu\nu\alpha\beta}^m &= \frac{1}{k^2 + m^2} [\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \eta_{\mu\nu} \eta_{\alpha\beta} \\ &+ \eta_{\mu\alpha} \frac{k_\nu k_\beta}{m^2} + \eta_{\nu\beta} \frac{k_\mu k_\alpha}{m^2} + \eta_{\mu\beta} \frac{k_\nu k_\alpha}{m^2} + \eta_{\nu\alpha} \frac{k_\mu k_\beta}{m^2} \\ &+ \frac{4}{3} (\frac{1}{2} \eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}) (\frac{1}{2} \eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{m^2})]. \end{aligned} \quad (9.7)$$

Notice that on the mass-shell we have the following relation for the residues $\text{Res } G_{\mu\alpha\beta}^\mu = 0$, $\text{Res } k^\mu G_{\mu\nu\alpha\beta} = 0$.

As k is time-like we can choose a frame where $k^0 = m$, $k^j = 0$. Then due to $k^\mu \varepsilon_{\mu\nu} = 0$ we have $\varepsilon_{0\nu} = \varepsilon_{\mu 0} = 0$ and ε_{ij} is a three dimensional symmetric traceless tensor and as such describes a massive particle of spin 2.

9.3 The massless case

Due to the invariance of the linearized Riemann tensor under the gauge transformation $h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$ we have that

$$G_{\mu\nu}(\xi_{\alpha,\beta} + \xi_{\beta,\alpha}) \equiv 0$$

which means that the equation

$$G_{\mu\nu}(h_{\alpha\beta}) = \kappa t_{\mu\nu} \quad (9.8)$$

is not invertible.

On the other hand due to the linearized Bianchi identities we have that the source $t_{\mu\nu}$ is subject to the restriction

$$t^\mu{}_{\nu,\mu} = 0 .$$

Let us consider a solution of

$$G_{\mu\nu}(h_{\alpha\beta}) = 0$$

and perform on a gauge transformation such that $h'^\alpha{}_\alpha = 0$. This is always possible because set $n = h^\alpha{}_\alpha$, the equation

$$2\xi^\alpha{}_{,\alpha} + n = 0$$

is solved by

$$\xi_\mu = -\frac{1}{2}\partial_\mu \frac{1}{\partial^2} n.$$

Substituting now into Eq.(9.2) we have

$$h'^\alpha{}_{\nu,\beta\alpha} - h'_{\beta\nu,\alpha}{}^\alpha + h'^\alpha{}_{\beta,\alpha\nu} - \eta_{\beta\nu} h'^{\alpha\gamma}{}_{,\alpha\gamma} = 0 .$$

Taking the trace we have

$$h'^{\beta\alpha}{}_{,\beta\alpha} = 0 \quad (9.9)$$

We look now for a further gauge transformation such that $h''^\alpha{}_{\beta,\alpha} = 0$ i.e. defined $N_\beta = h'^\alpha{}_{\beta,\alpha}$ we need to solve

$$0 = N_\beta + \partial^2 \xi_\beta + \xi^\alpha{}_{,\alpha\beta} .$$

Notice that as a consequence of Eq.(9.9) $N^\alpha{}_{,\alpha} = 0$. Then a solution of the above equation is given by

$$\xi_\nu = -\frac{1}{\partial^2} N_\nu$$

and still we have $h''^\alpha{}_\alpha = 0$. The equation $G_{\beta\nu} = 0$ now becomes

$$h''{}_{\beta\nu,\alpha}{}^\alpha = 0$$

and thus after setting $h_{\mu\nu} = \varepsilon_{\mu\nu}(k)e^{ik \cdot x}$ the mass shell is given by $k^2 = 0$. We can go over to the Lorentz frame such that $k^\mu = (\kappa, 0, 0, \kappa)$. Imposition of $k^\mu \varepsilon_{\mu\nu} = 0$ give now $\varepsilon_{00} = -\varepsilon_{03} = -\varepsilon_{33} = c_0$ and we have $\varepsilon_{0j} = -\varepsilon_{3j} = c_j$. We are still free to perform a gauge transformation with $\xi^\mu{}_{,\mu} = 0$ i.e. $k^\mu \xi_\mu(k) = 0$, because this does not alter $\varepsilon^\mu{}_\mu = 0$ and $k^\mu \varepsilon_{\mu\nu} = 0$. In fact

$$k^\mu (k^\mu \xi_\nu(k) + k^\mu \xi_\nu(k)) = k^2 \xi_\nu(k) + k_\nu k^\mu \xi_\mu(k) = 0.$$

Thus we perform the gauge transformation with $\xi_0(k) = \xi_3(k) = c_0/2\kappa$ and $\xi_j(k) = c_j/\kappa$ and find that all elements of $\varepsilon_{\mu\nu}$ are zero except for ε_{mn} with m and n equal to 1 or 2. Keeping in mind that $\varepsilon_{11} = -\varepsilon_{22}$ we see that the quantum states are in one to one correspondence with the complex field. Two independent base vectors are given by

$$\varepsilon_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \quad \text{and} \quad \varepsilon_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

which under rotation around the 3 axis

$$R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

transform like

$$R \varepsilon_+ R^T = e^{-2i\theta} \varepsilon_+ \quad R \varepsilon_- R^T = e^{2i\theta} \varepsilon_-$$

and thus represent the two states of helicity 2 and -2 .

We come now to the propagator in the massless case. We already saw that $G_{\mu\nu}(h_{\alpha\beta})$ is invariant under the gauge transformation (9.3) i.e. $G_{\mu\nu}(\xi_{\alpha,\beta} + \xi_{\beta,\alpha}) = 0$ for any ξ_α . This means that the equation

$$G_{\mu\nu}(h_{\alpha\beta}) = \kappa t_{\mu\nu} \tag{9.10}$$

which can have solutions only for $t^\mu{}_{\nu,\mu} = 0$, has no unique solution i.e. there is no inverse of the operator $G_{\mu\nu}$. We want to examine if by restricting the $h_{\alpha\beta}$ such inverse exists. Given $h_{\alpha\beta}$ we look for a gauge transformation ξ_α such that $h'^\alpha{}_{\beta,\alpha} = 0$. Define $h^\alpha{}_{\beta,\alpha} = N_\beta$ and solve

$$\partial^2 \xi_\beta + \partial_\beta \xi^\alpha{}_{,\alpha} = -N_\beta$$

This is solved by

$$\xi_\nu = \frac{1}{\partial^2} (-N_\nu + \frac{1}{2} \partial_\nu N^\alpha{}_{,\alpha})$$

which shows that the considered gauge fixing is attainable and complete. Eq.(9.8) now becomes in Fourier transform

$$\frac{1}{2} k^2 \varepsilon_{\beta\nu}(k) + \frac{1}{2} k_\beta k_\nu \varepsilon^\alpha{}_{,\alpha}(k) - \frac{1}{2} \eta_{\beta\nu} k^2 \varepsilon^\alpha{}_{,\alpha}(k) = t_{\beta\nu}(k)$$

which has the unique solution

$$\varepsilon_{\beta\nu}(k) = \frac{1}{k^2} (2\eta_{\beta\alpha} \eta_{\nu\gamma} + \frac{k_\beta k_\nu}{k^2} \eta_{\alpha\gamma} - \eta_{\nu\beta} \eta_{\alpha\gamma}) t^{\alpha\gamma}.$$

It is simpler, but equivalent, to use the harmonic gauge fixing

$$h^\alpha{}_{\beta,\alpha} - \frac{1}{2} h^\alpha{}_{\alpha,\beta} = 0$$

which is also attainable and complete and given by

$$\xi_\alpha = -\frac{1}{\partial^2} N_\alpha$$

with

$$N_\alpha = h^\beta_{\alpha,\beta} - \frac{1}{2} h^\beta_{\beta,\alpha}.$$

Then Eq.(9.8) becomes

$$\frac{1}{2} k^2 \varepsilon_{\beta\nu}(k) - \frac{k^2}{4} \varepsilon^\alpha_\alpha(k) \eta_{\beta\nu} = \kappa t_{\beta\nu}(k)$$

which has the unique solution

$$\varepsilon_{\beta\nu}(k) = \frac{1}{k^2} (\eta_{\beta\alpha} \eta_{\nu\gamma} + \eta_{\nu\alpha} \eta_{\beta\gamma} - \eta_{\beta\nu} \eta_{\alpha\gamma}) \kappa t^{\alpha\gamma}(k).$$

i.e. the propagator in the harmonic gauge is

$$G_{\mu\nu\alpha\beta} = \frac{1}{k^2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\nu\alpha} \eta_{\mu\beta} - \eta_{\mu\nu} \eta_{\alpha\beta}). \quad (9.11)$$

A general gauge fixing can be obtained as follows [1]. Construct 4 (linear) functions C_μ of $h_{\mu\nu}$ and its derivatives which are not invariant under the gauge transformations $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ i.e.

$$\delta C_\mu = M_{\mu\nu} \xi^\nu \neq 0 \quad \text{for } \xi^\nu \neq 0.$$

This implies the invertibility of $M_{\mu\nu}$. Add now $\frac{1}{2} C^\mu C_\mu$ to the Lagrangian L appearing in the original invariant action. Consider a solution of the Euler-Lagrange equation derived from

$$\int (L + \frac{1}{2} C^\mu C_\mu) d^4x \quad (9.12)$$

i.e. from

$$\delta S + \int C^\mu \delta C_\mu d^4x = 0.$$

On such solution the action (9.12) is stationary, in particular stationary under a gauge transformation of $h_{\mu\nu}$ i.e. we must have

$$\int C^\mu M_{\mu\nu} \xi^\nu d^4x = 0 \quad \text{for any } \xi^\nu.$$

Thus due to the invertibility of $M_{\mu\nu}$ on such solutions we have $C^\mu = 0$. Thus we have achieved two goals: 1) The solution of the equations of motion derived from (9.12) are stationary point of the original action S . 2) Such solution satisfy $C^\mu = 0$.

References

- [1] G. 't Hooft, "Quantum gravity: A fundamental problem and some radical ideas" Cargese Summer School 1979, 323

9.4 The VDVZ discontinuity

We want now to compute the one graviton exchange between two massive sources in the static limit, i.e. when the space momenta of the two bodies go to zero. The two sources will be described by the energy momentum tensor $T_j^{\mu\nu} = m_j U_j^\mu U_j^\nu$ $j = 1, 2$ in the limit $U_j \rightarrow (1, 0, 0, 0)$. If we denote by g_m the coupling constant in the massive case we have, from Eq.(9.7) for the exchange of a massive graviton for small momentum transfer

$$g_m^2 m_1 m_2 G_{0000}^m(k) = \frac{g_m^2 m_1 m_2}{k^2 + m^2} \left(1 + \frac{1}{3}\right).$$

To fit Newton's law $1/m$ must be much larger than the radius of the solar system and $G_N = 4g_m^2/3$, being G_N Newton's constant.

From the exchange of a massless graviton we have, from Eq.(9.11) denoting by g the new coupling constant

$$g^2 m_1 m_2 G_{0000}(k) = \frac{g^2 m_1 m_2}{k^2}$$

and to have agreement we need

$$G_N = \frac{4g_m^2}{3} = g^2$$

We examine next the deflection of light by m_1 . The light ray is described by the energy momentum tensor $T^{\mu\nu} = E U^\mu U^\nu$ with $U^\mu = (1, 0, 0, 1)$, and as expected from electromagnetism we have $T_\mu^\mu = 0$. In the massive case we have

$$2 \frac{g_m^2 m_1 E}{k^2 + m^2}$$

while in the massless case

$$2 \frac{g^2 m_1 E}{k^2}$$

Thus also in the limit $m \rightarrow 0$ the effect of the exchange of a massive graviton is 3/4 of the one due to the exchange of a massless graviton. This is the van Dam-Veltman-Zacharov discontinuity. Experiments on light deflection agree with the exchange of a massless graviton, i.e. with general relativity. A less simple computation gives for the exchange of a massive graviton an advance of the perihelion of Mercury which is 2/3 of the value for the exchange of a zero mass graviton i.e. 2/3 of the value predicted by general relativity.

References

- [1] H. van Dam, M. Veltman, "Massive and massless Yang-Mills and gravitational fields", Nucl. Phys. B22 (1970) 397
- [2] V.I Zakharov, "Linearized gravitation theory and the graviton mass", JETP Lett. 12 (1970) 312

051208

9.5 Classical general relativity from field theory in Lorentz space

We outline here an argument which has developed over a number of years which asserts that the quantum field theory in Minkowski space, of a massless spin 2 particle contains, as low energy limit, classical general relativity.

Initially such argument was developed in the second order formalism. A notable simplification of the argument has been achieved by using the first order (Palatini) formalism. The idea is that all classical forces are mediated at the quantum level by the exchange of a particle of proper mass and proper spin. First we review why one chooses a massless spin 2 particle.

The simplest choice is the exchange of a spin 0 particle. The simplest coupling of such a scalar field ϕ with “mass”, i.e. with the energy momentum tensor is through the trace of such a tensor, i.e. $\kappa\phi T^\mu_\mu$. However such coupling, being the trace of the energy momentum tensor of the Maxwell field zero, would decouple electromagnetism from gravity and thus we would not observe deflection of light. One can investigate other relativistically invariant couplings in which light is coupled to a spin zero gravity, with the result that the deflection of light by the sun depends both on the photon momentum and photon polarization [5]; both effects are not observed.

The exchange of spin 1 particle gives rise to repulsion of equal “charge” particle, i.e. if two particles are attracted by the same source they repel each other.

Fermionic fields lack a classical limit. Thus the next simplest choice is a spin 2 field and as gravity is classically compatible with infinite range the choice falls on a massless particle. Moreover we saw in the previous section that if the particle is of spin 2 in order not to conflict with experiment the mass has to be exactly zero. We already know how to describe it via the massless Fierz-Pauli equation.

We recall that the Lagrangian is given by

$$L_G = -\frac{1}{4}[2 \partial_\mu h^\mu_\nu \partial_\nu h^\alpha_\alpha - 2 \partial_\alpha h^\alpha_\nu \partial_\beta h^{\beta\nu} + \partial_\mu h_{\alpha\beta} \partial^\mu h^{\alpha\beta} - \partial_\mu h^\alpha_\alpha \partial^\mu h^\beta_\beta] .$$

We need a Lagrangian because to develop a quantum theory an action is needed either directly in the path integral approach, or indirectly to build up the hamiltonian formalism. The purpose now is to build a Lorentz invariant, interacting, consistent field theory of a massless spin 2 particle.

As gravity is coupled to mass the simplest model is described by the Lagrangian

$$L_G + \kappa h^{\mu\nu} T_{\mu\nu}^M + L_M \tag{9.13}$$

where L_M is the free Lagrangian the matter field and T^M is the matter energy momentum tensor derived from L_M . We have at this stage a Lorentz invariant model whose equation of motion for $h_{\mu\nu}$ are obtained by varying $h_{\mu\nu}$

$$\hat{G}_{\mu\nu} = \kappa T_{\mu\nu}^M$$

being $\hat{G}_{\mu\nu}$ the linearized Einstein tensor. Such equation however is untenable because due to the linearized Bianchi identities we have identically $\partial_\mu \hat{G}^\mu{}_\nu = 0$ and thus $\partial_\mu T^{M\mu}{}_\nu = 0$. Thus the matter energy momentum tensor is locally conserved, but in presence of gravitational interaction this cannot be true: the energy momentum four vector of a planet is not constant as the interaction provides an exchange of energy and momentum between the matter field and the gravitational field. Adding to $T_{\mu\nu}^M$ the energy momentum tensor of the free gravitational field is still not sufficient because also the interaction term contributes to the generation of the total conserved energy momentum tensor of the theory. Thus the modification of (9.13) must be of the form

$$L = L_G + \kappa h_{\mu\nu} T_{\mu\nu}^M + L_M + \kappa L_I(\kappa, h_{\mu\nu}, \phi)$$

where L_I must perform the following miracle: the variation of the action w.r.t. $h_{\mu\nu}$ must give the equation

$$\hat{G}_{\mu\nu} = \kappa {}^t T_{\mu\nu}$$

where ${}^t T_{\mu\nu}$ now is the total, conserved, energy momentum tensor derived from L . Such problem is not of easy solution.

Once we have realized that a non linear interaction term of the gravitational field with itself is necessary, we see that the problem subsists even in absence of matter. Thus from now on we shall forget about the matter field and the problem now is

$$L = L_G(h_{\mu\nu}) + \kappa L_I(\kappa, h_{\mu\nu})$$

and we need

$$\frac{\delta L_I(h_{\mu\nu})}{\delta h_{\mu\nu}} = T^{\mu\nu} [L_G + \kappa L_I]$$

One can attack the problem by expanding in powers of κ .

$$L = L_G(h_{\mu\nu}) + \kappa L_I^{(1)}(h_{\mu\nu}) + \kappa^2 L_I^{(2)}(h_{\mu\nu}) + \dots$$

and imposing

$$\sum_n \kappa^n \frac{\delta L_I^{(n)}(h)}{\delta h_{\mu\nu}} = \kappa T^{\mu\nu} [L_G(h) + \sum_n \kappa^n L_I^{(n)}(h)]$$

$$\frac{\delta L_I^{(1)}}{\delta h_{\mu\nu}} = T^{\mu\nu}[L_G]$$

$$\frac{\delta L_I^{(n)}}{\delta h_{\mu\nu}} = T^{\mu\nu}[L_I^{(n-1)}] .$$

We are working in the second order formulation and such a program has been carried through by Gupta [1] and by Thirring [3].

In general given the Lagrangian

$$L_G + \kappa L_I$$

where L_G is quadratic and L_I at least cubic if

$$\hat{G}^{\mu\nu} = \frac{\delta L_G}{\delta h_{\mu\nu}}$$

satisfies the identity (linearized Bianchi identity)

$$\partial_\mu \hat{G}^{\mu\nu} = 0$$

and we have a problem of consistency [6]. In fact given the exact equations

$$\hat{G}^{\mu\nu}(h) + \kappa G_I^{\mu\nu}(h) = 0$$

let us expand the solution in power series of κ

$$h = h_0 + \kappa h_1 + \kappa^2 h_2 + \dots$$

We have

$$\hat{G}^{\mu\nu}(h_0) = 0 \tag{9.14}$$

$$\hat{G}^{\mu\nu}(h_1) + G_I^{\mu\nu}(h_0) = 0$$

.....

$$\hat{G}^{\mu\nu}(h_{n-1}) + H_{n-1}^{\mu\nu}(h_0, h_1, \dots, h_{n-2}) = 0 \tag{9.15}$$

$$\hat{G}^{\mu\nu}(h_n) + H_n^{\mu\nu}(h_0, h_1, \dots, h_{n-1}) = 0$$

.....

From the linearized Bianchi identity we have that h_0 , in addition to (9.14) must satisfy the restriction

$$\partial_\mu G_I^{\mu\nu}(h_0) = 0$$

and similarly h_0, h_1, \dots, h_{n-1} in addition (9.15) have to satisfy the further restrictions

$$\partial_\mu H_n^{\mu\nu}(h_0, h_1, \dots, h_{n-1}) = 0$$

which may well be inconsistent unless L_I is very special. This is what happens in gravity and in Yang-Mills theory.

Instead of following this path, we shall employ here the first order approach following Deser [4] and Deser and Boulware [5]; the great advantage is that due to the implicit definition of the connections a single iteration will be sufficient to solve the problem.

We start with some technical remarks about the variational derivation of Einstein equations. Usually one obtains them by varying the action

$$\frac{1}{2\kappa} \int \sqrt{-g} g^{\mu\nu} R_{\mu\nu} d^n x + \int L_M d^n x$$

w.r.t. $g^{\mu\nu}$. One could as well vary w.r.t. $g_{\mu\nu}$ but we can vary also w.r.t. $\bar{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$. The variation of the first term gives simply $R_{\mu\nu}$ while we notice that

$$\frac{\delta L_M}{\delta g^{\mu\nu}} = \frac{\delta L_M}{\delta \bar{g}^{\mu\nu}} \sqrt{-g} - \frac{1}{2} \sqrt{-g} \frac{\delta L_M}{\delta \bar{g}^{\rho\sigma}} g^{\rho\sigma} g_{\mu\nu}$$

from which defining

$$\sqrt{-g} \tau_{\mu\nu} \equiv 2 \frac{\delta L_M}{\delta \bar{g}^{\mu\nu}}$$

we have

$$T_{\mu\nu} = \tau_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tau$$

or

$$\tau_{\mu\nu} = T_{\mu\nu} - \frac{1}{n-2} g_{\mu\nu} T \quad (9.16)$$

Thus we obtain Einstein equations in the form

$$R_{\mu\nu} = \kappa \tau_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{n-2} g_{\mu\nu} T \right).$$

We give now the first order formulation of the free gravitational field.

$$L_G = \bar{h}^{\mu\nu} (\Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\alpha\nu,\mu}^\alpha) + \eta^{\mu\nu} (\Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta). \quad (9.17)$$

Here the fundamental independent fields are the *Lorentz tensors* $\bar{h}^{\mu\nu}$ and the connection $\Gamma_{\mu\nu}^\alpha$ symmetric in the lower indices and indices are raised by $\eta^{\mu\nu}$. Notice that changing in Eq.(9.17) $\bar{h}^{\mu\nu}$ into $\eta^{\mu\nu} + \bar{h}^{\mu\nu}$ changes that Lagrangian by a irrelevant divergence. Variation w.r.t. $\bar{h}^{\mu\nu}$ gives

$$\Gamma_{\mu\nu,\alpha}^\alpha - \frac{1}{2} \Gamma_{\alpha\mu,\nu}^\alpha - \frac{1}{2} \Gamma_{\alpha\nu,\mu}^\alpha = 0. \quad (9.18)$$

Notice the necessary symmetrization as starting from the first order formalism we do not know a priori that the Ricci tensor is symmetric. Variation w.r.t. $\Gamma_{\mu\nu}^\alpha$ gives

$$-\eta^{\rho\nu} \Gamma_{\rho\alpha}^\mu - \eta^{\rho\mu} \Gamma_{\rho\alpha}^\nu = \bar{h}^{\mu\nu}{}_{,\alpha} - \eta^{\mu\nu} \frac{\bar{h}_{\beta,\alpha}^\beta}{n-2} \quad (9.19)$$

from which the Γ 's can be computed and using (9.19) we obtain from (9.18), in four dimensions

$$2R_{\mu\nu}^L \equiv \square \bar{h}_{\mu\nu} - \bar{h}_{\mu, \nu\alpha}^\alpha - \bar{h}_{\mu, \nu\alpha}^\alpha - \frac{1}{2}\eta_{\mu\nu}\square \bar{h}_\alpha^\alpha = 0 . \quad (9.20)$$

These are not the equations which can be obtained contracting the linearized Riemann tensor (9.1) which would give

$$2R_{\mu\nu}^L = -\square h_{\mu\nu} - h_{\alpha, \mu\nu}^\alpha + h_{\mu, \nu\alpha}^\alpha + h_{\nu, \mu\alpha}^\alpha . \quad (9.21)$$

However setting

$$\bar{h}_{\mu\nu} = -(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_\alpha^\alpha)$$

Eq.(9.20) becomes Eq.(9.21). This is not surprising as the Lagrangian (9.17) is just the quadratic part of the Palatini Lagrangian.

We want now to perform the first step i.e. adding to the Lagrangian (9.17) the term $\bar{h}^{\mu\nu}\tau_{\mu\nu}$ where $\tau_{\mu\nu}$ is the combination (9.16) of energy momentum tensor derived from the free Lagrangian (9.17).

We know the general rule how to compute the energy momentum tensor, even if historically such a procedure came after general relativity, but it is logically independent of it. The procedure is to render the Lagrangian invariant under diffeomorphisms by properly introducing a metric $f_{\mu\nu} = \eta_{\mu\nu} + \psi_{\mu\nu}$ whose covariant metric density $\bar{f}^{\mu\nu} = \sqrt{-f}f^{\mu\nu}$ will be written as $\bar{f}^{\mu\nu} = \eta^{\mu\nu} + \bar{\psi}^{\mu\nu}$, and changing derivatives into covariant derivatives. Then one has to take the functional derivative w.r.t. $\bar{\psi}^{\mu\nu}$ and then go back to flat space.

The first replacement in (9.17) is $\eta^{\mu\nu} \rightarrow \eta^{\mu\nu} + \bar{\psi}^{\mu\nu}$. The functional derivative w.r.t. $\bar{\psi}^{\mu\nu}$ gives as contribution to $\tau_{\mu\nu}$

$$\Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\beta - \Gamma_{\beta\mu}^\alpha \Gamma_{\alpha\nu}^\beta .$$

The derivative of the *tensors* $\Gamma_{\mu\nu}^\alpha$ have to be replaced by the covariant derivatives; calling $C_{\beta\mu}^\alpha$ the metric torsionless connections generated by $f_{\mu\nu}$ we have e.g.

$$\Gamma_{\mu\nu, \beta}^\alpha \rightarrow \Gamma_{\mu\nu, \beta}^\alpha + C_{\gamma\beta}^\alpha \Gamma_{\mu\nu}^\gamma - C_{\mu\beta}^\gamma \Gamma_{\gamma\nu}^\alpha - C_{\nu\beta}^\gamma \Gamma_{\mu\gamma}^\alpha .$$

However as at the end we must go over to the flat limit the result is the same as using for C the linearized expression i.e.

$$C_{\beta\mu}^\alpha = \frac{1}{2}\eta^{\alpha\gamma}[\psi_{\beta\gamma, \mu} + \psi_{\gamma\mu, \beta} - \psi_{\beta\mu, \gamma}] .$$

As a result the related contributions to $\tau_{\mu\nu}$ are of divergence type i.e. $\partial_\alpha H_{\mu\nu}^\alpha$. On flat space the energy momentum tensor is defined up to divergences and we shall drop the last contribution. For a detailed discussion see [4]. Then the action becomes

$$L_G = (\eta^{\mu\nu} + \bar{h}^{\mu\nu})(\Gamma_{\mu\nu, \alpha}^\alpha - \Gamma_{\mu\alpha, \nu}^\alpha) + (\eta^{\mu\nu} + \bar{h}^{\mu\nu})(\Gamma_{\alpha\beta}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\mu\alpha}^\beta \Gamma_{\nu\beta}^\alpha)$$

which is exactly the Palatini Lagrangian; the theory has acquired full invariance under diffeomorphisms. The reason why in the first order formalism one step is sufficient while in the second order formalism and infinite sequence of iterations are necessary, is that in the first order formalism the connection is implicitly defined by the action and thus a single change in the action corresponds to an infinite number of changes in the expression of Γ in terms of the metric.

References

- [1] S.N. Gupta, ‘Quantization of Einstein’s gravitational field: general treatment’, Proc. Phys. Soc. (London) A65 161 (1952) 608; ‘Quantum Field Theory in Terms of Ordered Products’, Rev. Mod. Phys. 29 (1957) 334
- [2] E. Corinaldesi, ‘Quantum field theory and the two-body problem’, Nuovo Cim. 1 (1955) 1289; ‘The Two-body Problem in the Theory of the Quantized Gravitational Field’, Proc. Phys. Soc. (London) A69 (1956) 189
- [3] W. Thirring, Fortschritte der Physik, ‘Lorentz-invariante Gravitationstheorien’, 7 (1959) 79; Ann. Phys. (N.Y.), ‘An alternative approach to the theory of gravitation’, 16 (1961) 96
- [4] S. Deser, ‘Selfinteraction and gauge invariance’, J. Gen. Rel. Grav. 1(1970) 9
- [5] S. Deser, D. G. Boulware, ‘Classical General Relativity Derived from Quantum Gravity’ Ann. Phys. (N.Y.) 89 (1975) 193
- [6] R.M. Wald, ‘Spin-2 Fields and General Covariance’, Phys. Rev. D 33 (1986) 3613

Chapter 10

Quantization in external gravitational fields

10.1 Introduction

The quantization of fields in external gravitational field is a well defined procedure when the gravitational field possesses a time like Killing vector field.

More generally the presence of a Killing vector field allows to derive from the equation $\nabla_\mu T^{\mu\nu} = 0$ a true conservation law

$$\nabla_\mu (T^{\mu\nu} \xi_\nu) = 0$$

i.e. the quantity

$$E = \int_\Sigma d^D x \sqrt{\gamma} n_\mu T^{\mu\nu} \xi_\nu$$

provided the fields decrease sufficiently fast at space infinity, is conserved. n_μ is the (future directed) normal to the space-like surface Σ and γ the determinant of the metric induced on Σ . In an ADM coordinate system

$$E = - \int_\Sigma d^D x \sqrt{h} N T^{0\nu} \xi_\nu.$$

We shall consider first the special situation in which there exists a time like Killing vector field which is surface orthogonal. We recall the Frobenius criterion for surface orthogonality of a vector field: Defined $\xi = \xi_\mu dx^\mu$ necessary and sufficient condition for being ξ_μ (locally) surface orthogonal is

$$d\xi = \xi \wedge \theta$$

being θ a one form.

Starting from a surface Σ_0 orthogonal to the time like Killing vector field ξ and which intersects all the integral lines of the Killing vector field, one can construct a foliation of space time in terms of space like surfaces, orthogonal to the Killing vector field. One produces the foliation by applying the group of diffeomorphisms ϕ_t , generated by ξ_μ , to Σ_0 , symbolically $\Sigma_t = \phi_t \Sigma_0$. By ϕ_t all vectors tangent to Σ_0 are pushed forward to vectors tangent to Σ_t and we have for v belonging the tangent space of Σ_0

$$\xi^\mu(0, x) g_{\mu\nu}(0, x) v^\nu(0, x) = 0 .$$

By definition of ϕ_t , $\xi(0, x)$ is pushed forward to $\xi(t, x)$, and being ϕ_t an isometry we have

$$\xi^\mu(t, x) g_{\mu\nu}(t, x) v^\nu(t, x) = \xi^\mu(0, x) g_{\mu\nu}(0, x) v^\nu(0, x) = 0 .$$

Moreover we can choose as time-flow vector just ξ^μ . In fact by definition of tangent vector we have

$$\xi^\mu \nabla_\mu t = 1 .$$

The conserved quantity is

$$E = \int_\Sigma d^D x \sqrt{\gamma} n_\mu T^{\mu\nu} \xi_\nu$$

which in the ADM coordinate system becomes

$$E = - \int_\Sigma d^D x \sqrt{h} N T^0_0$$

being there the time-flow vector $t^\mu = \xi^\mu = (1, 0, 0, 0)$ (cfr. Section 7.1) and the last one can also be rewritten as

$$E = \int_\Sigma d^D x \frac{\sqrt{h}}{N} T_{00} .$$

In the ADM coordinate system the orthogonality of ξ to Σ_t gives $\xi^\mu g_{\mu j} = g_{0j} = N_j = 0$ and thus also $g^{0j} = 0$. Moreover being $\xi^\mu = t^\mu$ a Killing vector the $g_{\mu\nu}$ is independent of time.

We shall refer to an hermitean scalar field. The K.G. equation becomes

$$g^{00} \partial_0 \partial_0 \phi + \frac{1}{\sqrt{-g}} \partial_l \sqrt{-g} g^{ln} \partial_n \phi - m^2 \phi = 0$$

or

$$-\partial_0 \partial_0 \phi = K \phi$$

with

$$K = \frac{1}{(-g^{00})} \left[-\frac{1}{\sqrt{-g}} \partial_l \sqrt{-g} g^{ln} \partial_n + m^2 \right] .$$

K is hermitean and positive in the measure

$$(-g^{00})\sqrt{-g} d^D x. \quad (10.1)$$

In fact

$$\int \phi_2^* K \phi_1 (-g^{00})\sqrt{-g} d^D x = \int \sqrt{-g} (\partial_l \phi_2^* g^{lk} \partial_k \phi_1 + m^2 \phi_2^* \phi_1) d^D x.$$

According to Friedrich's theorem there is always a self adjoint extension with the same lower bound. For a self-adjoint extension of K the set $\phi_n(\mathbf{x}, \omega)$

$$\omega^2 \phi_n(\omega, \mathbf{x}) = K \phi_n(\omega, \mathbf{x})$$

where the index n takes into account degeneracy, is complete. Being K real we can choose a real orthonormal basis of $\phi_n(\omega, \mathbf{x})$. From the action

$$\begin{aligned} S &= \int dt \int d^D x (-g^{00})\sqrt{-g} \frac{1}{2} \left(\dot{\phi}^2 - \frac{1}{(-g^{00})} (g^{ij} \partial_i \phi \partial_j \phi + m^2 \phi^2) \right) = \\ &= \int dt \int d^D x L = \int dt \mathcal{L} \end{aligned}$$

we have

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \frac{\partial L}{\partial \dot{\phi}} = (-g^{00})\sqrt{-g} \dot{\phi}$$

and we can compute the hamiltonian

$$H = \int d^D x (\pi \dot{\phi} - L) = \int d^D x \frac{1}{2} \left(\frac{\pi^2}{(-g^{00})\sqrt{-g}} + \sqrt{-g} (g^{kl} \partial_k \phi \partial_l \phi + m^2 \phi^2) \right)$$

which agrees with

$$E = \int_{\Sigma} d^D x \frac{\sqrt{h}}{N} T_{00}$$

derived in Section 5.8 from a different procedure, i.e. $\sqrt{-g} T_{\mu\nu}$ as twice the variational derivative of the action with respect to $g^{\mu\nu}$. We could proceed to quantization by imposing the equal time commutation relations

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y}).$$

It is simpler to compute the Lagrangian (action) using the mode expansion

$$\phi(\mathbf{x}, t) = \sum_n \int d\omega q_n(\omega, t) \phi_n(\mathbf{x}, \omega)$$

with the orthonormality relation

$$\int (-g^{00})\sqrt{-g} d^D x \phi_m(\mathbf{x}, \omega) \phi_n(\mathbf{x}, \omega') d\mathbf{x} = \delta_{mn} \delta(\omega - \omega')$$

(real field) reaching for the Lagrangian

$$L = \sum_n \int d\omega \frac{1}{2} [\dot{q}_n^2(\omega, t) - \omega^2 q_n^2(\omega, t)].$$

We can compute the conjugate momenta $p_n(\omega, t) = \dot{q}_n(\omega, t)$ and the hamiltonian

$$\sum_n \int d\omega \frac{1}{2} [p_n^2(\omega, t) + \omega^2 q_n^2(\omega, t)]$$

and thus we have an infinity of harmonic oscillators. Canonical quantization is obtained by imposing

$$[q_m(\omega, t), p_n(\omega', t)] = i\delta_{mn} \delta(\omega - \omega').$$

Setting

$$a_n(\omega, t) = \frac{p_n - i\omega q_n}{\sqrt{2\omega}}$$

$$a_n^+(\omega, t) = \frac{p_n + i\omega q_n}{\sqrt{2\omega}}$$

one has the commutation relations

$$[a_m(\omega, t), a_n^+(\omega', t)] = \delta_{mn} \delta(\omega - \omega')$$

and

$$H = \sum_n \int d\omega \omega \frac{1}{2} [a_n^+(\omega, t)a_n(\omega, t) + a_n(\omega, t)a_n^+(\omega, t)]$$

or

$$H = \sum_n \int d\omega \omega [a_n^+(\omega, t)a_n(\omega, t) + \frac{1}{2}\delta(0)]$$

which gives to a^+a the particle number interpretation and to ω the meaning of energy of the quantum. The infinity is discarded through normal ordering. The field now can be written

$$\phi = \sum_n \int (f_n(\omega, \mathbf{x})a_n(\omega, t) + \bar{f}_n(\omega, \mathbf{x})a_n^+(\omega, t))d\omega$$

with

$$f_n(\omega, \mathbf{x}) = \frac{i}{\sqrt{2\omega}}\phi_n(\omega, \mathbf{x}).$$

The equation of motion $\dot{F} = \frac{1}{i}[F, H]$ gives

$$\dot{a}_n^+(\omega, t) = i\omega a_n^+(\omega, t)$$

and thus

$$a^+(\omega, t) = e^{i\omega t}a^+(\omega, 0).$$

10.2 The scalar product of two solutions

A different path to introduce the measure (10.1) is the following. Consider two solutions ϕ and f of the K.G. equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0.$$

We have

$$\begin{aligned} 0 &= \bar{f}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) - m^2\bar{f}\phi - \phi\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\bar{f}) + m^2\bar{f}\phi = \\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}(\bar{f}g^{\mu\nu}\partial_\nu\phi - \phi g^{\mu\nu}\partial_\nu\bar{f})) \end{aligned} \quad (10.2)$$

By integrating the r.h.s. of Eq.(10.2) on the volume V bounded by two space like surfaces Σ_2 and Σ_1 and a mantle at space infinity we have the result

$$\begin{aligned} 0 &= \int_V \partial_\mu(\sqrt{-g}(\bar{f}g^{\mu\nu}\partial_\nu\phi - \phi g^{\mu\nu}\partial_\nu\bar{f}))d^n x = \\ &= \int_{\Sigma_2} (\bar{f}g^{\mu\nu}\partial_\nu\phi - \phi g^{\mu\nu}\partial_\nu\bar{f})\Sigma_\mu - \int_{\Sigma_1} (\bar{f}g^{\mu\nu}\partial_\nu\phi - \phi g^{\mu\nu}\partial_\nu\bar{f})\Sigma_\mu \end{aligned} \quad (10.3)$$

with

$$\Sigma_\mu = \frac{1}{(n-1)!}\epsilon_{\mu\sigma_1\dots\sigma_{n-1}}dx^{\sigma_1} \wedge \dots \wedge dx^{\sigma_{n-1}}$$

thus defining an invariant scalar product (not positive definite). This holds also for a time dependent $g_{\mu\nu}$. In the case in which the metric is stationary $g_{\mu\nu}(\mathbf{x})$ and static $g_{0i}(\mathbf{x}) = 0$ choosing the space section $x^0 \equiv t = \text{const}$, for solutions of the form $\phi(x) = e^{i\omega t}\varphi(\mathbf{x})$ and $f(x) = e^{i\omega't}f_s(\mathbf{x})$ we have defining

$$(f, \phi) = i \int_\Sigma (\bar{f}g^{\mu\nu}\partial_\nu\phi - \phi g^{\mu\nu}\partial_\nu\bar{f})\Sigma_\mu \quad (10.4)$$

$$(f, \phi) = e^{i(\omega-\omega')t}(\omega + \omega') \int (-g^{00})\sqrt{-g}\bar{f}_s(\mathbf{x})\varphi(\mathbf{x})d^{n-1}x \quad (10.5)$$

which due to the constancy in time makes (f, ϕ) to vanish for $\omega \neq \omega'$. For $\omega = \omega'$ we have

$$(f, \phi) = 2\omega \int (-g^{00})\sqrt{-g}\bar{f}_s(\mathbf{x})\varphi(\mathbf{x})d^{n-1}x \quad (10.6)$$

obtaining the positive definite invariant conserved scalar product, on the positive frequency solutions, which differs from the one induced by the measure (10.1) by the multiplicative factor 2ω . Notice that for $\phi(x) = e^{i\omega t}\varphi(\mathbf{x})$ and $f(x) = e^{-i\omega't}f_s(\mathbf{x})$ $\omega > 0$, $\omega' > 0$ (f, ϕ) is always zero.

References

[1] [Wald] Chap. 14

[2] N.D. Birrel and P.C.W. Davies “Quantum fields in curved space” Cambridge University press, Chap. 3

10.3 The scalar product for the general stationary metric

The application of Eq.(10.3) to periodic solutions with positive frequencies in the stationary but non static case leads to the scalar product

$$i \int_{\Sigma_2} \sqrt{-g} (2\omega i \phi^* g^{00} \phi + \phi^* g^{0k} \nabla_k \phi - \phi g^{0k} \nabla_k \phi^*) dx^{n-1}$$

which is not positive definite. Here below we treat the general stationary non static case [1].

First we prove that $-g^{00} > 0$ and $g^{kl} > 0$. The time flow vector $\frac{\partial}{\partial t}$ i.e. $t^\mu = (1, 0, 0, 0)$ is always assumed to be time like, i.e. in the ADM metric $N^2 - N^j N_j > 0$ which implies $N^2 > 0$ and thus $-g^{00} = \frac{1}{N^2} > 0$.

We have

$$g^{ij} = h^{ij} - \frac{N^i N^j}{N^2} .$$

Then by Schwarz inequality we have

$$v_i g^{ij} v_j = v_i h^{ij} v_j - \frac{(v_i N^i)(N^j v_j)}{N^2} \geq v_i h^{ij} v_j - \frac{(v_i h^{ij} v_j)(N_i h^{ij} N_j)}{N^2} = v_i h^{ij} v_j - \frac{(v_i h^{ij} v_j)(N^i N_i)}{N^2} \geq 0 .$$

Let us consider the “action” functional of two fields ϕ_1 and ϕ_2 (not necessarily an hermitean action)

$$S = - \int \sqrt{-g} d^n x (g^{\mu\nu} \partial_\mu \phi_2^* \partial_\nu \phi_1 + m^2 \phi_2^* \phi_1) .$$

Variation w.r.t. ϕ_2^* and ϕ_1 give the two equations

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi_1) - m^2 \phi_1 = 0$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi_2^*) - m^2 \phi_2^* = 0 .$$

From the variation of S w.r.t. $g_{\mu\nu}$ we obtain a symmetric $\mathbf{T}^{\mu\nu}$ which on the equations of motion is covariantly conserved

$$\nabla_\mu \mathbf{T}^{\mu\nu} = 0 .$$

If ξ^μ is a Killing vector we have the conserved current $\mathbf{T}_\nu^\mu \xi^\nu$

$$\nabla_\mu(\mathbf{T}_\nu^\mu \xi^\nu) = 0 .$$

If ξ^μ is time-like we choose it as time flow vector $t^\mu = \xi^\mu$. Then we have the conserved quantity

$$\mathbf{E} = \int \sqrt{-g} \mathbf{T}_0^0 d^{n-1}x = \int \sqrt{-g} g^{0\beta} \mathbf{T}_{\beta 0} d^{n-1}x = \text{const}$$

where explicitly

$$\mathbf{T}_0^0 = -g^{00} \partial_0 \phi_2^* \partial_0 \phi_1 + g^{kl} \partial_k \phi_2^* \partial_l \phi_1 + m^2 \phi_2^* \phi_1$$

i.e. the conserved quantity is

$$\int d^{n-1}x (-g^{00}) \sqrt{-g} [\partial_0 \phi_2^* \partial_0 \phi_1 + \phi_2^* K \phi_1] = (\partial_0 \phi_2, \partial_0 \phi_1) + (\phi_2, K \phi_1) \equiv \langle \phi_2, \phi_1 \rangle \quad (10.7)$$

where K is given again by

$$K = -\frac{1}{(-g^{00})\sqrt{-g}} \partial_m \sqrt{-g} g^{ml} \partial_l + \frac{m^2}{(-g^{00})} .$$

Eq.(10.7) defines a positive definite scalar product on the solutions of the KG equation.

For a direct derivation of the above equation from the equation of motion, without any appeal to the variational procedure, notice that the equation

$$g^{00} \partial_0^2 \phi = -\frac{1}{\sqrt{-g}} \partial_m \sqrt{-g} g^{ml} \partial_l \phi - \frac{1}{\sqrt{-g}} \partial_0 \sqrt{-g} g^{0l} \partial_l \phi - \frac{1}{\sqrt{-g}} \partial_m \sqrt{-g} g^{m0} \partial_0 \phi + m^2 \phi$$

taking into account the time independence of $g_{\mu\nu}$ can be written as

$$-\partial_0^2 \phi = K \phi + A \partial_0 \phi . \quad (10.8)$$

K is hermitean and positive definite in the metric $(-g^{00})\sqrt{-g} d^{n-1}x = \frac{\sqrt{\hbar}}{N} d^{n-1}x$ and

$$A = -\frac{1}{(-g^{00})\sqrt{-g}} (\sqrt{-g} g^{0m} \partial_m + \partial_m \sqrt{-g} g^{m0})$$

is anti-hermitean, again in the metric $(-g^{00})\sqrt{-g} d^{n-1}x$. Multiplying the equation (10.8) for ϕ_1 by $\partial_0 \phi_2^*$, performing the same by exchanging 1 with 2 and adding we obtain

$$-\partial_0(\partial_0 \phi_2^* \partial_0 \phi_1) = \partial_0 \phi_2^* K \phi_1 + \partial_0 \phi_1 K \phi_2^* + \partial_0 \phi_2^* A \partial_0 \phi_1 + \partial_0 \phi_1 A \partial_0 \phi_2^* .$$

Integrating in $(-g^{00}\sqrt{-g}) d^{n-1}x$ we obtain

$$-\partial_0(\partial_0 \phi_2, \partial_0 \phi_1) = \partial_0(\phi_2, K \phi_1)$$

i.e.

$$\langle \phi_2, \phi_1 \rangle = (\partial_0 \phi_2, \partial_0 \phi_1) + (\phi_2, K \phi_1)$$

is a conserved positive definite scalar product on the solutions of the KG equation. This coincides with (10.7) i.e. the Ashtekar and Magnon scalar product [1].

For periodic solutions $\phi = \varphi e^{i\omega t}$ the above becomes

$$\langle \phi_2, \phi_1 \rangle = e^{i(\omega_1 - \omega_2)t} (\omega_2 \omega_1 (\varphi_2, \varphi_1) + (\varphi_2, K \varphi_1))$$

implying for $\omega_1 \neq \omega_2$

$$\omega_2 \omega_1 (\varphi_2, \varphi_1) + (\varphi_2, K \varphi_1) = 0$$

i.e. for $\omega_1 \neq \omega_2$ we have orthogonality in the positive definite scalar product

$$\omega_2 \omega_1 (\varphi_2, \varphi_1) + (\varphi_2, K \varphi_1) .$$

Notice that for positive frequency solutions $\phi = \varphi e^{i\omega t}$ we have

$$\omega^2 \varphi = K \varphi + \omega i A \varphi . \quad (10.9)$$

For the negative frequency solution $\phi = \varphi_1 e^{-i\omega t}$ we have

$$\omega^2 \varphi_1 = K \varphi_1 - \omega i A \varphi_1 \quad (10.10)$$

satisfied by $\varphi_1 = \bar{\varphi}$. One should prove completeness for the solutions of Eq.(10.9) [2].

For $\phi_2 = \phi_1 = \phi$ we have the energy

$$E = \int d^{n-1}x (-g^{00} \sqrt{-g}) (\partial_0 \phi^* \partial_0 \phi + \phi^* K \phi)$$

so that with normalization 1 for \langle, \rangle for the modes ϕ_n we obtain with the decomposition

$$\phi = \sum_{\omega_n > 0} \mathbf{c}_n \varphi_n e^{-i\omega_n t} + \mathbf{c}_n^+ \bar{\varphi}_n e^{i\omega_n t}$$

$$E = \sum_{\omega_n > 0} (\mathbf{c}_n^+ \mathbf{c}_n + \mathbf{c}_n \mathbf{c}_n^+)$$

i.e.

$$\mathbf{c}_n = \mathbf{a}_n \sqrt{\frac{\omega_n}{2}} \quad , \quad \mathbf{c}_n^+ = \mathbf{a}_n^+ \sqrt{\frac{\omega_n}{2}}$$

with $\mathbf{a}_n^+ \mathbf{a}_n$ the occupation number operator.

The scalar product \langle, \rangle obviously applies also to the static case giving rise to the normalization for $\phi = e^{i\omega t} \varphi$

$$\langle \phi, \phi \rangle = \omega^2 (\varphi, \varphi) + (\varphi, K \varphi) = 2\omega^2 (\varphi, \varphi) .$$

References

- [1] A. Ashtekar and A. Magnon, "Quantum fields in curved space-time", Proc. R. Soc. Lond. A. 346 (1975) 375
- [2] B.S. Kay, "Linear spin-zero quantum fields in external gravitational and scalar fields", Commun. math. Phys. 62 (1978) 62

10.4 Schwarzschild solution and its maximal extension

Imposing a surface orthogonal time-like Killing vector field and $SO(3)$ symmetry of the space sections one reaches the form

$$ds^2 = -e^\nu c^2 dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

from which

$$G_{rr} = \frac{1 - e^\lambda + r\nu'}{r^2}; \quad G_{tt} = \frac{e^{-\lambda+\nu}}{r^2}(-1 + e^\lambda + r\lambda').$$

$G_{rr} = G_{tt} = 0$ give

$$\nu' + \lambda' = 0; \quad r\lambda' = 1 - e^\lambda.$$

The second solved gives

$$e^\lambda = \frac{1}{1 + \frac{1}{kr}}$$

while the first gives

$$e^\nu = \alpha e^{-\lambda} = \alpha \left(1 + \frac{1}{kr}\right).$$

Normalizing time so as to have asymptotically Minkowski metric and comparing with the already derived

$$g_{00} = -1 - \frac{2\phi}{c^2}$$

for weak fields, one has

$$\frac{1}{k} = -\frac{2GM}{c^2}.$$

Then

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right)c^2 dt^2 + \frac{1}{1 - \frac{2MG}{c^2 r}} dr^2 + r^2 d\Omega^2.$$

The length $r_s = \frac{2GM}{c^2}$ is the Schwarzschild radius. For the sun $r_s = 3 \text{ km}$. We recall that precession of Mercury is $43''$ per century and the deflection of grazing light $1.75''$. All general relativistic effect are of order (Schwarzschild radius)/radius.

The following remarkable Birkoff theorem exists: All solution with an $SO(3)$ isometry in absence of matter are locally equivalent to the Schwarzschild metric [1]; no stationarity assumption is made.

Despite the metric is singular at $r = \frac{2GM}{c^2}$ all invariant polynomials remain finite at the Schwarzschild singularity. Instead computation of the square of the Riemann tensor gives

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{6r_s^2}{r^6}$$

and as such the sub-manifold $r = 0$ is a singular sub-manifold.

To eliminate the Schwarzschild singularity one goes over to the Kruskal coordinates. In the following by M we shall understand GM/c^2 and by t we shall understand ct . One starts by going over to light-like coordinates. The equation for radial light ray

$$\dot{t}^2 = \left(\frac{\dot{r}}{1 - \frac{2M}{r}} \right)^2$$

is solved by

$$\pm t = r + 2M \ln\left(\frac{r}{2M} - 1\right) + \text{const} \equiv r_* + \text{const} .$$

We see that infinite time t is needed for a light-pulse to reach the horizon. With

$$u = t - r_*; \quad v = t + r_*$$

the metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du dv + r^2 d\Omega^2$$

r being now defined by

$$e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1\right) = e^{\frac{v-u}{4M}} .$$

Go over to

$$V = e^{\frac{v}{4M}} > 0; \quad U = -e^{-\frac{u}{4M}} < 0$$

to have the metric

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dU dV + r^2 d\Omega^2$$

where

$$e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1\right) = -UV$$

and up to now $-UV > 0$. The extension occurs by realizing that $r \geq 0$ is uniquely defined in a larger range for $-UV$ i.e. for $-VU > -1$ being the l.h.s. monotonically increasing in r from -1 to ∞ . This remark completes the Kruskal maximal extension of the Schwarzschild metric.

Without adding anything new, it is useful to change again coordinates

$$X = \frac{M}{2}(V - U); \quad T = \frac{M}{2}(V + U)$$

the metric becomes

$$ds^2 = 32M \frac{e^{-\frac{r}{2M}}}{r} (-dT^2 + dX^2) + r^2 d\Omega^2$$

where

$$-UV = \frac{1}{M^2}(X^2 - T^2)$$

is restricted to be ≥ -1 . The remaining is a non physical region as $r = 0$ is a real singularity as there time geodesics are incomplete and the square of the Riemann tensor diverges. For a discussion of regions I, II, III, IV see [1], [2]. The relation to the original coordinates is

$$\begin{aligned} \frac{t}{2M} &= 2 \operatorname{arctanh} \frac{T}{X} \\ e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1 \right) &= \frac{X^2 - T^2}{M^2}. \end{aligned}$$

References

[1] [HawkingEllis] Chap. 5

[2] [Wald] Chap. 6

10.5 Proper acceleration of stationary observers in Schwarzschild metric and surface gravity

The Minkowskian formula for the proper acceleration can be written as

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} = u^\lambda \frac{\partial u^\mu}{\partial x^\lambda}$$

is covariantly generalized for a field of four velocities u^μ to

$$a^\mu = u^\lambda \nabla_\lambda u^\mu$$

with norm

$$g^2 = a^\mu a_\mu \geq 0$$

being a^μ space-like. If the metric possesses a time like Killing vector we can ask for the proper acceleration of observers moving along the integral lines of the Killing vector field. If we set $\xi^\mu \xi_\mu = -V^2$ we have

$$u^\mu = \frac{\xi^\mu}{V}$$

and the covariant acceleration is given by

$$a_\nu = u^\lambda \nabla_\lambda u_\nu = \frac{\xi^\lambda}{V^2} \nabla_\lambda \xi_\nu + \frac{\xi^\lambda}{V} \xi_\nu \nabla_\lambda \frac{1}{V} = \frac{\xi^\lambda}{V^2} \nabla_\lambda \xi_\nu = -\frac{\xi^\lambda}{V^2} \nabla_\nu \xi_\lambda = \nabla_\nu \ln V$$

where we used the fact that the norm of the Killing vector is constant along the integral lines of the Killing vector field.

For Schwarzschild we have

$$\xi^\mu \xi_\mu = -\left(1 - \frac{2M}{r}\right)$$

from which

$$g = \frac{1}{\left(1 - \frac{2M}{r}\right)^{1/2}} \frac{M}{r^2}$$

which diverges on the horizon.

If we multiply the proper acceleration g by the norm of the Killing vector normalized to 1 at space infinity we obtain

$$\sqrt{-\xi^\mu \xi_\mu} g = \frac{M}{r^2} \quad (10.11)$$

Eq.(10.11) computed at the horizon is a finite quantity and is called the surface gravity

$$\kappa = \frac{M}{r_s^2} = \frac{1}{4M} . \quad (10.12)$$

An interpretation of κ is the following [1] consider an inextensible massless rope of length L

$$L = \int_{r_1}^{r_2} \sqrt{g_{rr}} dr = \int_{r_1}^{r_2} \frac{1}{\sqrt{1 - \frac{2M}{r}}} dr$$

which holds the unit mass placed at r_1 against falling to decreasing values of r . If the observer at r_2 pulls the rope by dl_2 , the end of the rope at r_1 moves by dl_1 given by

$$\frac{dl_1}{\sqrt{1 - \frac{2M}{r_1}}} = \frac{dl_2}{\sqrt{1 - \frac{2M}{r_2}}} .$$

The force the observer O_1 has to apply in order to keep the mass at fixed $r = r_1$ is $g(r_1)$.

If we equate the work done by O_2 by the energy gained by O_1 we have

$$f_2 = g \frac{dl_1}{dl_2} = g \frac{\sqrt{1 - \frac{2M}{r_1}}}{\sqrt{1 - \frac{2M}{r_2}}} .$$

Taking r_2 to infinity and $r_1 = r_s$ we have the surface gravity.

References

[1] [Wald] Chap. 6 problem 6.4.b

10.6 The accelerated detector

The detector is given by a point particle with discrete energy levels E_0, E_1, \dots coupled to the scalar field $\phi(x)$ by the interaction $c(\tau) m(\tau) \phi(x(\tau))$. τ is the proper time while $c(\tau)$ is an exposure function which is equal to 1 for $-\Omega/2 < \tau < \Omega/2$ and goes to zero smoothly outside that interval.

Using perturbation theory and working in the interaction picture we have for the transition amplitude from E_0 to E and for the field from the Minkowski vacuum $|0_M\rangle$ to the state $|\psi\rangle$

$$\begin{aligned} A &= i \int_{-\infty}^{\infty} c(\tau) d\tau \langle E|m(\tau)|E_0\rangle \langle \psi|\phi(x(\tau))|0_M\rangle \\ &= i \int_{-\infty}^{\infty} c(\tau) d\tau \langle E|m(0)|E_0\rangle e^{i(E-E_0)\tau} \langle \psi|\phi(x(\tau))|0_M\rangle . \end{aligned}$$

The total transition probability from E_0 to E , found by summing $\bar{A}A$ over all $|\psi\rangle$, is

$$P = |\langle E|m(0)|E_0\rangle|^2 \int_{-\infty}^{\infty} d\tau c(\tau) \int_{-\infty}^{\infty} d\tau' c(\tau') e^{-i(E-E_0)(\tau-\tau')} W(x(\tau), x(\tau'))$$

being $W(x, x')$ the Wightman function

$$W(x, x') = \langle 0_M|\phi(x)\phi(x')|0_M\rangle .$$

Due to the spectral condition $E_n \geq 0$ we have

$$\begin{aligned} W(x, x') &= \sum_n \langle 0_M|\phi(x)|n\rangle \langle n|\phi(x')|0_M\rangle = \sum_n e^{-iE_n(t-t')} \langle 0_M|\phi(\mathbf{x}, 0)|n\rangle \langle n|\phi(\mathbf{x}', 0)|0_M\rangle = \\ &= \lim_{\varepsilon \rightarrow +0} \sum_n e^{-iE_n(t-t'-i\varepsilon)} \langle 0_M|\phi(\mathbf{x}, 0)|n\rangle \langle n|\phi(\mathbf{x}', 0)|0_M\rangle . \end{aligned}$$

In the massless case one has

$$W(x, x') = \frac{1}{4\pi^2} \frac{1}{(\mathbf{x} - \mathbf{x}')^2 - (t - t' - i\varepsilon)^2} . \quad (10.13)$$

For the inertial trajectory one has

$$P = -|\langle E|m(0)|E_0\rangle|^2 \int_{-\infty}^{\infty} c(\tau') d\tau' \int_{-\infty}^{\infty} c(\tau) d\tau e^{-i(E-E_0)(\tau-\tau')} \frac{1}{4\pi^2(\tau - \tau' - i\varepsilon)^2} .$$

The integral in $d\tau$ for $\Omega \rightarrow \infty$ converges to a finite limit and, as $E - E_0 > 0$, such a limit can be computed by closing the integration contour in τ in the lower half plane. Due to the absence of poles in the lower half plane, the result is zero. This is an expected result.

We consider now a uniformly accelerated motion (hyperbolic motion) written as

$$t = \frac{1}{g} \sinh g\tau, \quad x = \frac{1}{g} \cosh g\tau$$

where τ is the proper time and $a_\mu a^\mu = g^2$. We find

$$(\mathbf{x} - \mathbf{x}')^2 - (t - t' - i\varepsilon)^2 = -\frac{4}{g^2} \sinh^2\left(g\frac{\tau - \tau' - i\varepsilon}{2}\right).$$

Now use the formula

$$\frac{1}{\sinh^2 \pi y} = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(y + ik)^2}$$

which gives

$$-\frac{g^2}{4 \sinh^2(g\Delta\tau/2 - i\varepsilon)} = -\sum_{k=-\infty}^{\infty} \frac{1}{(\Delta\tau - i\varepsilon + 2i\pi k/g)^2}.$$

Again for $\Omega \rightarrow \infty$ the integral in $d\tau$ converges to a finite limit and such a limit can be computed by closing the integration contour in the lower half plane. Only $k = 1, 2, \dots$ contribute giving the result

$$P = \int c(\tau') d\tau' \frac{1}{2\pi} |\langle E | m(0) | E_0 \rangle|^2 \frac{(E - E_0)}{e^{2\pi(E-E_0)/g} - 1}. \quad (10.14)$$

The factor $\int c(\tau') d\tau'$ behaves like Ω for $\Omega \rightarrow \infty$ which is the effective time of exposure of the detector.

Eq.(10.14) shows that the detector behaves as immersed in a thermal radiation of temperature $T = g/(2\pi)$ where we ascribe the prefactor to the particular type of coupling of our detector to the field ϕ . Taking into account \hbar and c (which were put equal to 1) we have

$$\frac{\hbar\omega_0}{k_B T} = \frac{2\pi c \omega_0}{g}$$

i.e.

$$k_B T = \frac{g\hbar}{2\pi c} \quad (10.15)$$

giving for $g = 10m/sec^2$ the temperature $T = 3.84 \cdot 10^{-20} K$.

10.7 Elements of causal structure of space time

We give below the definitions of some special sets which are related to the causal structure of space-time. Some of them will be used in the following.

Future Cauchy development $D^+(S)$ of a closed set S : the points such that any backward directed inextendible non space like trajectory starting from them hits S .

Similarly one defines the past Cauchy development $D^-(S)$ of a closed set S .

Partial Cauchy surface: a space-like *hypersurface* with no inextendible non-spacelike curve intersecting it more than once.

Global Cauchy surface: a partial Cauchy surface with the property $D^+(S) \cup D^-(S) = M$ [1]

A space which admits a global Cauchy surface is called globally hyperbolic.

Chronological future of an event $I^+(p)$: the events which can be reached starting from p with a time like curve.

Causal future of an event $J^+(p)$: the events which can be reached starting from p with a non space like curve.

Similarly one defines the chronological and causal past.

Given a world line $x(\lambda)$, i.e. a time like curve we define “Future event horizon” $H^+(x(\lambda))$ the boundary of the past of the world line, being the past of the world line the union of the past light cones of the events belonging to the world line. I.e. $H^+(x(\lambda)) = \text{boundary}(\cup_{\lambda} J^-(x(\lambda)))$ It is the boundary of the events by which the observer can be influenced.

Past event horizon $H^-(x(\lambda))$ for a world line (observer) is the boundary of the future of the world line $H^-(x(\lambda)) = \text{boundary}(\cup_{\lambda} J^+(x(\lambda)))$. It is the boundary of the events which the observer can influence.

Future null infinity \mathcal{F}^+ : (qualitatively) the endpoints of the light-like geodesics which reach the asymptotic region [2]

Event horizon H : the boundary of the boundary of the causal past of the future null infinity $H = \text{boundary}(J^-(\mathcal{F}^+))$

Space which is future asymptotically predictable from a partial Cauchy surface S : if $\mathcal{F}^+ \subset \overline{D^+(S)}$. Similar definition for a space which is past asymptotically predictable.

Examples

1. Spacelike hyperboloid in Minkowski is a partial but not global Cauchy surface.
2. Anti de Sitter space admits a foliation in spacelike surfaces but has no global Cauchy surface as given any space like surface there are geodesics which never intersect it [3]
3. Horizons for a stationary Schwarzschild observer.
4. Horizons for a stationary Rindler observer.

The favorable situation is the following:

- 1) Asymptotically, for large negative and positive times there exist, time like Killing vector fields. This will allow a particle interpretation of the quantum fields.
- 2) There exists a partial Cauchy surface S for which the space is future and past asymptotically predictable.

This will allow to perform predictions both at the classical and quantum level.

At large negative times we can expand the field as

$$\phi = \sum_i \mathbf{a}_i f_i + \mathbf{a}_i^+ \bar{f}_i$$

and at large positive times we can expand the field as

$$\phi = \sum_i \mathbf{b}_i p_i + \mathbf{b}_i^+ \bar{p}_i .$$

We propagate back in time the solutions p_i to the partial Cauchy surface S and we propagate forward the solutions f_j and \bar{f}_j to the same partial Cauchy surface S . Due to the completeness of the f_j , \bar{f}_j we can express p_i as a superposition of f_j and \bar{f}_j

$$p_i = \sum_j \alpha_{ij} f_j + \beta_{ij} \bar{f}_j$$

and due to the linearity of the problem such relation holds for all times with constant coefficients α and β . By equating now the two expressions of the same field ϕ we can extract the relation between the operators \mathbf{a}, \mathbf{a}^+ and \mathbf{b}, \mathbf{b}^+ .

References

- [1] [HawkingEllis] Chap. 6 p.201 and following
- [2] [HawkingEllis] Chap. 9 p.310
- [3] [HawkingEllis] Chap. 6 p.133

10.8 Rindler metric

Perform on Minkowski the transformation

$$x = z \cosh(a\eta); \quad t = z \sinh(a\eta)$$

where a has the dimension of length⁻¹ and η the dimension of a length.

$$0 < z = (x^2 - t^2)^{1/2}; \quad -\infty < a\eta = \operatorname{arctanh}\left(\frac{t}{x}\right) < \infty$$

$$ds^2 = -z^2 a^2 d\eta^2 + dz^2$$

This space is not geodesically complete. With $z = a^{-1}e^{a\xi}$ can be taken to a form conformal to Minkowski.

$$ds^2 = e^{2a\xi}(d\xi^2 - d\eta^2)$$

The motion with constant (proper) acceleration $a^\mu a_\mu = g^2$ in Minkowski space is given by

$$\frac{du^\mu}{d\tau} u_\mu = a^\mu u_\mu = 0.$$

where for the four velocity we have $u^\mu u_\mu = -1$.

In the rest frame $u^i = 0$ and thus $a^0 = 0$ and then $a^\mu a_\mu = a^i a_i = g^2 = \text{const}$. Then we have

$$-u^0 a^0 + u^1 a^1 = 0; \quad -u^0 u^0 + u^1 u^1 = -1; \quad -a^0 a^0 + a^1 a^1 = g^2.$$

From the first equation $a^0 = f u^1$, $a^1 = f u^0$. Substitute in the third equation to obtain $-f^2(u^1)^2 + f^2(u^0)^2 = f^2 = g^2$ and thus, choosing $f = g$

$$a^0 = g u^1 = g \frac{dx^1}{d\tau}, \quad a^1 = g u^0 = g \frac{dx^0}{d\tau}.$$

The solution with $u^0 > 0$ is

$$u^0 = \cosh g(\tau - \tau_0), \quad u^1 = \sinh g(\tau - \tau_0).$$

Then properly normalizing the origin of space and time

$$x^0 = \frac{1}{g} \sinh g(\tau - \tau_0), \quad x^1 = \frac{1}{g} \cosh g(\tau - \tau_0).$$

This shows that the motion of Rindler observers (particles) with $z = \text{const}$. is a uniformly accelerated motion with acceleration $g = 1/z$.

“Rigidity” of Rindler space: The Rindler space is locally rigid. By this we mean that the distance of two nearby points $z_2 = \text{const}$, $z_1 = \text{const}$, $\Delta z = z_2 - z_1$ as measured by a fixed

observer $x = \text{const}$, contracts according the Lorentz contraction law. In fact the speed of the point is given by

$$V = \tanh(a\eta)$$

while measuring the distance in the rest frame means solving the equations

$$\Delta x = \Delta z \cosh(a\eta) + z \sinh(a\eta) \Delta(a\eta)$$

$$0 = \Delta z \sinh(a\eta) + z \cosh(a\eta) \Delta(a\eta)$$

and thus

$$\Delta x = \Delta z \left(\cosh(a\eta) - \frac{\sinh^2(a\eta)}{\cosh(a\eta)} \right) = \Delta z \frac{1}{\cosh(a\eta)} = \Delta z (1 - (\tanh(a\eta))^2)^{1/2} = \Delta z (1 - V^2)^{1/2}.$$

We have something like an accelerating skyscraper; at $\eta = 0$ we have $\Delta x = z_2 - z_1 = \Delta z$ and the skyscraper is at rest. At different times one measures the correct Lorentz contraction. The skyscraper cannot extend below indefinitely, i.e. for negative z , as the acceleration is diverging for $z = 0$. The acceleration of the n -th floor is $1/n$. The time like Killing vector field for the Rindler metric is $\frac{\partial}{\partial \eta}$ whose square norm is $-V^2 = -a^2 z^2$. Applying the formula for an observer which follows the Killing integral lines, given by $z = \text{const}$. we have

$$a_\eta = 0, \quad a_z = \nabla_z \log(az) = \frac{1}{z}$$

and we re-obtain the expression for the acceleration.

10.9 Eigenfunctions of K in the Rindler metric

From

$$ds^2 = dz^2 - z^2 a^2 d\eta^2$$

we have in the non zero mass case

$$K = -a^2 z^2 \frac{\partial^2}{\partial z^2} - a^2 z \frac{\partial}{\partial z} + m^2 a^2 z^2 = -a^2 \left(z \frac{\partial}{\partial z} \right)^2 + m^2 a^2 z^2 = -a^2 \left(\frac{\partial}{\partial \ln(az)} \right)^2 + m^2 a^2 z^2.$$

while in the ξ, η coordinates takes the form

$$K = -\left(\frac{\partial}{\partial \xi} \right)^2 + m^2 e^{2a\xi}.$$

For zero mass the right-moving field and left-moving field in (1+1) dimensions are local independent quantum fields. Moreover for zero mass the K.G. equation is conformal invariant and in the ξ, η coordinates takes the form

$$\left(\left(\frac{\partial}{\partial \xi} \right)^2 - \left(\frac{\partial}{\partial \eta} \right)^2 \right) \phi = 0.$$

The right moving solutions are

$$e^{\pm i\omega(\xi-\eta)} = e^{\pm i\omega(a^{-1} \log(az)-\eta)} = e^{\pm i\omega a^{-1} \log(a(x-t))}$$

and thus the right-moving field is expanded as

$$\phi(\xi, \eta) = \int d\omega \left(\frac{e^{i\omega(\xi-\eta)}}{\sqrt{2\omega}} \mathbf{b}_\omega + \frac{e^{-i\omega(\xi-\eta)}}{\sqrt{2\omega}} \mathbf{b}_\omega^+ \right)$$

or using the z, η coordinates

$$\phi(z, \eta) = \int d\omega \left(\frac{e^{i\omega(a^{-1} \log(az)-\eta)}}{\sqrt{2\omega}} \mathbf{b}_\omega + \frac{e^{-i\omega(a^{-1} \log(az)-\eta)}}{\sqrt{2\omega}} \mathbf{b}_\omega^+ \right).$$

In the z, η coordinates the metric is $(-g^{00})\sqrt{-g}dz = a^{-1}z^{-2} z dz = a^{-1}d \ln(az)$ and

$$\frac{a^{-1}}{2\pi} \int e^{\pm i\omega(a^{-1} \log(az)-\eta)} e^{\mp i\omega'(a^{-1} \log(az)-\eta)} d \ln(az) = \delta(\omega - \omega').$$

The form

$$e^{\pm i\omega a^{-1} \ln(a(x-t))}$$

clarifies the nature of the solution; we have infinite oscillations when we approach the horizon.

10.10 The Bogoliubov transformation

We have already pointed out that in presence of a partial Cauchy surface, as at large negative times the solutions f_ω, \bar{f}_ω are a complete set, it must be possible to express the outgoing solutions p_ω in terms of f_ω and \bar{f}_ω . I.e. using the discrete notation

$$p_i = \sum_j \alpha_{ij} f_j + \beta_{ij} \bar{f}_j$$

and thus

$$\mathbf{a}_j = \sum_i \mathbf{b}_i \alpha_{ij} + \mathbf{b}_i^+ \bar{\beta}_{ij} \equiv (\mathbf{b}A + \mathbf{b}^+ \bar{B})_j$$

$$\mathbf{a}_j^+ = \sum_i \mathbf{b}_i^+ \bar{\alpha}_{ij} + \mathbf{b}_i \beta_{ij} \equiv (\mathbf{b}B + \mathbf{b}^+ \bar{A})_j$$

in matrix notation with

$$A_{ij} = \alpha_{ij}; \quad B_{ij} = \beta_{ij}.$$

The \mathbf{b}, \mathbf{b}^+ will obey the commutation relations

$$[\mathbf{b}_i, \mathbf{b}_j^+] = \delta_{ij}.$$

Then we have

$$\delta_{jk} = [\mathbf{a}_j, \mathbf{a}_k^+] = (A^T \bar{A} - B^+ B)_{jk}$$

i.e.

$$A^T \bar{A} - B^+ B = I \quad (10.16)$$

and from $[\mathbf{a}_j, \mathbf{a}_k] = 0$ we have

$$A^T \bar{B} - B^+ A = 0.$$

Thus using as illustration a finite dimensional space

$$\mathcal{A} = \begin{pmatrix} A^T & B^+ \\ B^T & A^+ \end{pmatrix} \in U(N, N)$$

i.e.

$$\mathcal{A} S \mathcal{A}^+ = S \quad \text{with} \quad S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

It follows that the inverse is

$$\mathcal{A}^{-1} = \begin{pmatrix} \bar{A} & -\bar{B} \\ -B & A \end{pmatrix} \in U(N, N) \quad (10.17)$$

from which in addition to Eq.(10.16) we have

$$\bar{A} A^T - \bar{B} B^T = I$$

giving rise respectively to the sum rules

$$\sum_i |\alpha_{ij}|^2 - |\beta_{ij}|^2 = 1$$

and

$$\sum_i |\alpha_{ji}|^2 - |\beta_{ji}|^2 = 1. \quad (10.18)$$

If the state Φ is such that

$$\mathbf{a}_i \Phi = 0$$

i.e. no incoming particle, we have for the mean value of the number of outgoing particles

$$(\Phi, \mathbf{b}_i^+ \mathbf{b}_i \Phi) = \sum_j |\beta_{ij}|^2.$$

10.11 Quantum field theory in the Rindler wedge

In this section we shall work out the Bogoliubov transformation for the passage from Minkowski to Rindler space for a massless field. The advantage of treating the massless field is that all computations can be performed exactly.

In the massless case the scalar field splits in a Lorentz invariant way into right-moving and left-moving fields. $\phi(x) = \phi_R(x) + \phi_L(x)$. In fact with

$$\phi_R(x) = \int_0^\infty \left(e^{ik(x-t)} \frac{\mathbf{a}_R(k)}{\sqrt{2\omega}} + e^{-ik(x-t)} \frac{\mathbf{a}_R^+(k)}{\sqrt{2\omega}} \right) dk$$

$$\phi_L(x) = \int_{-\infty}^0 \left(e^{ik(x+t)} \frac{\mathbf{a}_L(k)}{\sqrt{2\omega}} + e^{-ik(x+t)} \frac{\mathbf{a}_L^+(k)}{\sqrt{2\omega}} \right) dk$$

with $\omega = |k|$ we see that under the Lorentz transformation

$$x = x' \cosh \alpha - t' \sinh \alpha$$

$$t = -x' \sinh \alpha + t' \cosh \alpha$$

the two fields transform independently. E.g.

$$\phi_R(x) = \int_0^\infty \left(e^{ike^\alpha(x'-t')} \frac{\mathbf{a}_R(k)}{\sqrt{2\omega}} + e^{-ike^\alpha(x'-t')} \frac{\mathbf{a}_R^+(k)}{\sqrt{2\omega}} \right) dk \quad (10.19)$$

$$= \int_0^\infty \left(e^{ik'(x'-t')} \frac{\mathbf{a}'_R(k')}{\sqrt{2\omega'}} + e^{-ik'(x'-t')} \frac{\mathbf{a}'_R^+(k')}{\sqrt{2\omega'}} \right) dk' \quad (10.20)$$

with $\mathbf{a}'_R(k') = \mathbf{a}_R(e^{-\alpha}k')$ and similarly for the left-moving field. Thus in the massless case one can treat the two fields independently. It corresponds to the fact that in two dimensions for a massless particle it has an absolute meaning to say that the particle is moving in the right or left direction.

Here to stay in touch with the general formalism we shall not perform this splitting and work with the full field ϕ . We shall exploit only the Rindler wedge $z \geq 0$.

We can extract $\mathbf{b}^+(k)$ and $\mathbf{b}(k)$ from $\phi(z, \eta)$ by computing the space Fourier transform of $\phi(z, 0)$ and $\dot{\phi}(z, 0)$ where the dot stays for the derivative w.r.t. the Rindler time η . We use the notation $\xi = a^{-1} \log(az)$ and $\omega = |k|$. We have

$$\phi(z, 0) = \int dk \left(e^{ik\xi} \frac{\mathbf{b}(k)}{\sqrt{2\omega}} + e^{-ik\xi} \frac{\mathbf{b}^+(k)}{\sqrt{2\omega}} \right)$$

$$\dot{\phi}(z, 0) = \int dk \left(-i\sqrt{\frac{\omega}{2}} e^{ik\xi} \mathbf{b}(k) + i\sqrt{\frac{\omega}{2}} e^{-ik\xi} \mathbf{b}^+(k) \right).$$

Thus

$$\mathbf{b}^+(k) = \frac{1}{2\pi} \left(\sqrt{\frac{\omega}{2}} \int \phi(\xi, 0) e^{ik\xi} d\xi - i\sqrt{\frac{1}{2\omega}} \int \dot{\phi}(\xi, 0) e^{ik\xi} d\xi \right) \quad (10.21)$$

which result can be obtained also from the scalar product (10.6). $\mathbf{b}(k)$ is obtained by hermitean conjugation.

On the Rindler wedge the field ϕ and the standard Minkowski field

$$\phi_M(x, t) = \int dk' \left(\frac{\mathbf{a}(k')}{\sqrt{2\omega'}} e^{i(k'x - \omega't)} + \frac{\mathbf{a}^+(k')}{\sqrt{2\omega'}} e^{-i(k'x - \omega't)} \right)$$

coincide and the Minkowski vacuum is characterized by $\mathbf{a}(k)|0_M\rangle = 0$. $\dot{\phi}_M(x, 0)$ is computed by using

$$\left. \frac{\partial}{\partial \eta} \right|_{\eta=0} = az \left. \frac{\partial}{\partial t} \right|_{t=0} \quad \text{and} \quad x = z \cosh a\eta$$

where $\partial/\partial\eta$ means the derivative at fixed ξ and $\partial/\partial t$ means the derivative at fixed x .

Keeping in mind that at $\eta = 0$ we have $x = z$ and inserting into Eq.(10.21) we have

$$\begin{aligned} \mathbf{b}^+(k) &= \frac{\sqrt{\omega}}{4\pi a} \int_{-\infty}^{\infty} d \log(az) e^{i\frac{k}{a} \log az} \int \left(\frac{\mathbf{a}(k')}{\sqrt{\omega'}} e^{ik'z} + \frac{\mathbf{a}^+(k')}{\sqrt{\omega'}} e^{-ik'z} \right) dk' \quad (10.22) \\ &- \frac{i}{4\pi a \sqrt{\omega}} \int_{-\infty}^{\infty} az d \log(az) e^{i\frac{k}{a} \log az} \int (-i\mathbf{a}(k')\sqrt{\omega'} e^{ik'z} + i\mathbf{a}^+(k')\sqrt{\omega'} e^{-ik'z}) dk'. \end{aligned}$$

Thus we need the integrals

$$I) \quad \int_0^{\infty} e^{i\kappa \log \zeta} e^{i\kappa' \zeta} \zeta^{-p} d\zeta; \quad \kappa' > 0$$

$$II) \quad \int_0^{\infty} e^{i\kappa \log \zeta} e^{-i\kappa' \zeta} \zeta^{-p} d\zeta; \quad \kappa' > 0$$

where we have defined $\kappa = k/a$ and $\zeta = az$ and p can take the values 0 or 1. Integrals I and II are computed by rotation of the integration in the complex plane; for I one uses $\zeta = e^{i\frac{\pi}{2}}\rho$ and for II one uses $\zeta = e^{-i\frac{\pi}{2}}\rho$ to obtain

$$I) \quad \int_0^{\infty} e^{i\kappa \log \zeta} e^{i\kappa' \zeta} \zeta^{-p} d\zeta = e^{i\frac{\pi}{2}(1-p)} e^{-\frac{\pi\kappa}{2}} (\kappa')^{-1-i\kappa+p} \Gamma(i\kappa + 1 - p); \quad \kappa' > 0$$

$$II) \quad \int_0^{\infty} e^{i\kappa \log \zeta} e^{-i\kappa' \zeta} \zeta^{-p} d\zeta = e^{i\frac{\pi}{2}(p-1)} e^{\frac{\pi\kappa}{2}} (\kappa')^{-1-i\kappa+p} \Gamma(i\kappa + 1 - p); \quad \kappa' > 0.$$

Substituting now in Eq.(10.22) we obtain

$$\frac{\mathbf{b}^+(k > 0)}{\sqrt{2\omega}} = \frac{\Gamma(\frac{i\omega}{a})}{2\pi a} \left(e^{\frac{\pi\omega}{2a}} \int_0^{\infty} \frac{\mathbf{a}^+(k')}{\sqrt{2\omega'}} \left(\frac{k'}{a} \right)^{-i\frac{\omega}{a}} dk' + e^{-\frac{\pi\omega}{2a}} \int_0^{\infty} \frac{\mathbf{a}(k')}{\sqrt{2\omega'}} \left(\frac{k'}{a} \right)^{-i\frac{\omega}{a}} dk' \right) \quad (10.23)$$

$$\frac{\mathbf{b}^+(k < 0)}{\sqrt{2\omega}} = \frac{\Gamma(-\frac{i\omega}{a})}{2\pi a} \left(e^{\frac{\pi\omega}{2a}} \int_{-\infty}^0 \frac{\mathbf{a}^+(k')}{\sqrt{2\omega'}} \left(-\frac{k'}{a} \right)^{i\frac{\omega}{a}} dk' + e^{-\frac{\pi\omega}{2a}} \int_{-\infty}^0 \frac{\mathbf{a}(k')}{\sqrt{2\omega'}} \left(-\frac{k'}{a} \right)^{i\frac{\omega}{a}} dk' \right) \quad (10.24)$$

and $\mathbf{b}(k)$ is obtained by hermitean conjugation.

The two equations (10.23)(10.24) show explicitly the splitting of the field in right and left moving parts.

The mean value on the Minkowski vacuum, defined by $\mathbf{a}(k)|O_M\rangle = 0$, of the number operator in the Rindler space $\mathbf{b}^+(k)\mathbf{b}(k)$, $k \geq 0$, is given by

$$\frac{1}{2\omega}\langle 0_M|\mathbf{b}^+(k)\mathbf{b}(k)|0_M\rangle = \frac{1}{4\pi^2 a^2}\Gamma\left(\frac{i\omega}{a}\right)\Gamma\left(-\frac{i\omega}{a}\right)e^{-\frac{\pi\omega}{a}}\int_0^\infty\frac{dk'}{2\omega'}.$$

Recalling now the relation

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)} \quad (10.25)$$

we obtain

$$\langle 0_M|\mathbf{b}^+(k)\mathbf{b}(k)|0_M\rangle = \frac{1}{\pi a}\frac{1}{e^{2\pi\omega/a}-1}\int_0^\infty\frac{dk'}{2\omega'}. \quad (10.26)$$

The UV divergence is due to the continuum normalization of the vectors and can be avoided by working with wave packets as will be done in section (10.13) in full generality. The frequency ω which appears in Eq.(10.26) is the frequency measured w.r.t. the coordinate time η as it appears in the decomposition of the Rindler field. To relate it to the frequency measured by a Rindler stationary observer i.e. one which moves with $z = \text{const.}$ one has to relate the time η with the proper time τ of the observer

$$\omega d\eta = \omega_o d\tau$$

with $d\tau = za d\eta$. Thus $\omega = az \omega_o$ and we have

$$\langle 0_M|\mathbf{b}^+(k)\mathbf{b}(k)|0_M\rangle = \frac{1}{\pi a}\frac{1}{e^{2\pi\omega_o z}-1}\int_0^\infty\frac{dk'}{2\omega'} \quad (10.27)$$

which recalling that the acceleration of the observer is $g = 1/z$ agrees with the result of the accelerated detector Eq.(10.15).

10.12 Rindler space with a reflecting wall

We shall consider in this section a simple two dimensional example [1] which reproduces many of the features of Hawking's theory [2]. We place a reflecting wall at a fixed distance $x = c$ in the Minkowski coordinates. This problem is interesting in several respects: It gives a concrete example of a geometry which has only asymptotic time-like Killing vector fields at the Rindler time $\eta = -\infty$ and $\eta = +\infty$ and the reflecting wall also simulate the reflection of the partial wave modes by the center of the collapsing star as examined by Hawking [2]. It reproduces well the physical situation for which the Hawking radiation

is a late time effect and it shows how such radiation is emitted at the boundary of the horizon.

In fact the relevant mathematics turns out to be identical to that of the Schwarzschild-Kruskal case. Not to overburden the notation we shall set the parameter a of the previous section with the dimension of a length⁻¹ equal to 1. Thus e.g. $\log z$ has to be read as $\log(az)$.

Due to the presence of the reflecting wall, Lorentz invariance is broken and thus the scalar field non longer can be invariantly decomposed in left and right moving fields.

We will start with a field which at large negative Rindler times describes incoming particles i.e. at large negative Rindler times can be written, using for clarity the discrete notation, as

$$\phi = \sum_j \mathbf{a}_j f_j + \mathbf{a}_j^+ \bar{f}_j$$

with f_j reducing at large negative Rindler times to superposition of left-moving negative frequency mode

$$\frac{1}{\sqrt{2\omega}} e^{-i\omega(\eta+\ln z)} = \frac{1}{\sqrt{2\omega}} e^{i\omega \ln(x+t)} .$$

Thus \mathbf{a}_j and \mathbf{a}_j^+ are the annihilation and creation operator for the incoming particles. The same field will also be described by

$$\phi = \sum_m \mathbf{b}_m p_m + \mathbf{b}_m^+ \bar{p}_m + \sum_n \mathbf{c}_n q_n + \mathbf{c}_n^+ \bar{q}_n$$

where at large positive Rindler times p_m becomes a superposition of right-moving negative frequency modes

$$\frac{1}{\sqrt{2\omega_m}} e^{-i\omega_m(\eta-\ln z)} = \frac{1}{\sqrt{2\omega_m}} e^{i\omega_m \ln(x-t)} \quad \text{for } x > t \quad \text{and } 0 \quad \text{for } x < t$$

and thus \mathbf{b}_m and \mathbf{b}_m^+ are the annihilation and creation operators of the particles which flow to $+\infty$. q_i and \bar{q}_i are the modes representing the particles which cross the horizon $x = t$. We shall not need the explicit form of the q_i . We can repeat the procedure of section 10.10, where now the pair p_m, q_n replaces the p_i of section (10.10), and the translation of Eq.(10.17) to the present notation is

$$\mathbf{b}_m = \sum_j \bar{\alpha}_{mj} \mathbf{a}_j - \bar{\beta}_{mj} \mathbf{a}_j^+ \tag{10.28}$$

$$\mathbf{c}_n = \sum_j \bar{\gamma}_{nj} \mathbf{a}_j - \bar{\eta}_{nj} \mathbf{a}_j^+ \tag{10.29}$$

from which we can compute \mathbf{b}_m^+ and \mathbf{c}_n^+ by hermitean conjugation. From the commutation relations we obtain the sum rules

$$\sum_j |\alpha_{mj}|^2 - |\beta_{mj}|^2 = \delta_{mm} = 1 \quad (10.30)$$

and

$$\sum_j |\gamma_{nj}|^2 - |\eta_{nj}|^2 = \delta_{nn} = 1. \quad (10.31)$$

Hawking's method [2] to compute the Bogoliubov coefficients, is to consider the wave packet p_m (which at large positive time is a superposition of right-moving negative frequency modes) and to propagate it backward in time. It is reflected by the wall and it emerges as a superposition of left-moving modes with negative *and* positive frequency.

As suggested by the notation it is much better to work with discrete modes than with modes with continuum normalization. The explicit decomposition of the field in discrete modes will be given in the next section. Here we still use the continuum normalization which is formally simpler even if it is more difficult to be interpreted physically. The resolution in wave packets is given in the next section.

In order to satisfy the boundary condition on the reflecting wall $x = c$, the solution

$$\frac{1}{\sqrt{2\omega}} e^{i\omega \ln(x-t)} \quad \text{for } x > t \quad \text{and} \quad 0 \quad \text{for } x < t$$

has to be matched with

$$\int_0^\infty \left(\alpha_{\omega\omega'} \frac{1}{\sqrt{2\omega'}} e^{-i\omega' \ln(x+t)} + \beta_{\omega\omega'} \frac{1}{\sqrt{2\omega'}} e^{i\omega' \ln(x+t)} \right) d\omega'$$

with the condition

$$\frac{1}{\sqrt{2\omega}} e^{i\omega \ln(c-t)} \theta(c-t) = \int_0^\infty \left(\alpha_{\omega\omega'} \frac{1}{\sqrt{2\omega'}} e^{-i\omega' \ln(c+t)} + \beta_{\omega\omega'} \frac{1}{\sqrt{2\omega'}} e^{i\omega' \ln(c+t)} \right) d\omega'. \quad (10.32)$$

To find $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ we need to invert such an integral representation. We can do it in a workable fashion only for small $c - t$. This is the region of interest to analyze the late time radiation as the radiation emitted from $x = c$ and time t reaches the Rindler observer placed in z at Rindler time $\eta = -\frac{1}{a} \ln \frac{c-t}{z}$. This aspect will be further clarified in the next section where we give the explicit resolution of the field in discrete modes (wave packets).

For small $c - t$ we have

$$\frac{c-t}{2c} = \frac{2c - (t+c)}{2c} \approx \ln(2c) - \ln(t+c).$$

Taking the logarithm and setting $\tau = \ln(c + t)$ and $\tau_0 = \ln(2c)$ for small $c - t$ we have

$$\ln(c - t) \approx \tau_0 + \ln(\tau_0 - \tau).$$

Thus

$$\frac{1}{\sqrt{2\omega}} e^{i\omega(\ln(\tau_0 - \tau) + \tau)} \theta(\tau_0 - \tau) = \int_0^\infty \left(\alpha_{\omega\omega'} \frac{1}{\sqrt{2\omega'}} e^{-i\omega'\tau} + \beta_{\omega\omega'} \frac{1}{\sqrt{2\omega'}} e^{i\omega'\tau} \right) d\omega'$$

from which inverting

$$2\pi \frac{\beta_{\omega\omega'}}{\sqrt{2\omega'}} = \frac{1}{\sqrt{2\omega}} \int_{-\infty}^{\tau_0} e^{i\omega \ln(\tau_0 - \tau)} e^{i\omega\tau_0} e^{-i\omega'\tau} d\tau .$$

We recall that $\omega \geq 0$, $\omega' \geq 0$ always.

The integral is one of the integrals already computed in the previous section. Again the integral is performed by rotation in the complex plane putting this time $\tau = -i\rho$ and we have

$$\begin{aligned} 2\pi \frac{\beta_{\omega\omega'}}{\sqrt{2\omega'}} &= i \frac{e^{i(\omega - \omega')\tau_0}}{\sqrt{2\omega}} \int_0^\infty (i\rho)^{i\omega} e^{-\omega'\rho} d\rho = i \frac{e^{i(\omega - \omega')\tau_0}}{\sqrt{2\omega}} e^{-\frac{\pi\omega}{2}} (\omega')^{-1-i\omega} \int_0^\infty x^{i\omega} e^{-x} dx = \\ &= i \frac{e^{i(\omega - \omega')\tau_0}}{\sqrt{2\omega}} e^{-\frac{\pi\omega}{2}} (\omega')^{-1-i\omega} \Gamma(1 + i\omega) . \end{aligned} \quad (10.33)$$

The performed analysis of the solution p_n at large negative Rindler times in terms of the f_j gives to $\alpha(\omega, \omega')$ and $\beta(\omega, \omega')$ the meaning of the Bogoliubov coefficients. Using the formula (10.25) we have

$$\langle O_M | \mathbf{b}^+(\omega) \mathbf{b}(\omega) | O_M \rangle = \frac{1}{\pi} \frac{1}{e^{2\pi\omega} - 1} \int_0^\infty \frac{d\omega'}{2\omega'} \quad (10.34)$$

which agrees with the result (10.26). The divergence of the integral on the r.h.s. at 0 is an infrared divergence due to the use of fields of zero mass. The divergence at infinity is due to the continuum normalization of the modes and has a physical interpretation. In the next section this problem will be solved by going over to discrete p_m modes.

References

- [1] P.C.W. Davies, "Scalar particle production in Schwarzschild and Rindler metrics" J. Phys. A: Math. Gen. 8 (1975) 609
- [2] S.W. Hawking, "Particle creation by black holes", Comm. Math. Phys. 43 (1975) 199
- [3] N.D. Birrel and P.C.W. Davies, "Quantum fields in curved space" paragraphs 4.5, 8.1.
- [4] [Wald] 14.2, 14.3.
- [5] R.M. Wald, "Quantum field theory in curved space-time and black hole thermodynamics"

10.13 Resolution in discrete wave packets

From Eq.(10.33) we see that the U.V. behavior of $|\beta_{\omega\omega'}|^2$ as a function of ω' is $\text{const.}/\omega'$ which makes the integral (10.27) logarithmically divergent for large ω' . This has nothing to do with the divergences of quantum field theory but it simply a kinematical problem. The meaning of such a divergence is that the amount of radiating energy measured by the observer with $z = \text{const.}$ in an infinite amount of time is infinite. This is a reflection of the fact that the Unruh-Hawking radiation is not a transient phenomenon but a permanent one i.e. it lasts for infinite time.

In order to understand this feature in detail, following Hawking [1] we shall give a resolution of the quantum field ϕ for large positive times (the out field) in discrete wave packets instead of a continuous spectral resolution.

In general given a continuous complete set of vectors $|\omega\rangle$, $0 < \omega < \infty$ one can construct the discrete set

$$|j, n\rangle = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{2\pi i n \omega / \varepsilon} |\omega\rangle d\omega .$$

It is immediately checked that

$$\langle j', n' | j, n \rangle = \delta_{n'n} \delta_{j'j} .$$

In addition the discrete set $|j, n\rangle$ is complete; in fact

$$\begin{aligned} \int \phi(\omega) |\omega\rangle d\omega &= \sum_j \int_{j\varepsilon}^{(j+1)\varepsilon} \phi(\omega) |\omega\rangle d\omega = \sum_j \sum_n \int_{j\varepsilon}^{(j+1)\varepsilon} \frac{e^{2\pi i \omega n / \varepsilon}}{\sqrt{\varepsilon}} c_{jn} |\omega\rangle d\omega \\ &= \sum_j \sum_n c_{jn} |j, n\rangle \end{aligned} \quad (10.35)$$

with

$$c_{jn} = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{-2\pi i \omega n / \varepsilon} \phi(\omega) d\omega .$$

Similarly in our case we have

$$p_{jn}(\eta, z) = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{2\pi i n \omega / \varepsilon} p_\omega(\eta, z) d\omega .$$

To understand the meaning of such wave packet we notice that for large positive n , it is peaked at the retarded time

$$\ln(x - t) = \eta - \ln z = 2\pi n / \varepsilon$$

as it is seen by the stationary phase method or from the explicit computation of Eq.(10.36) below.

Thus for an observer at $z = \text{const}$, large n means large coordinate time η and large proper time for the Rindler observer.

Then we have

$$\int_0^\infty (\mathbf{b}_\omega p_\omega + \mathbf{b}_\omega^+ \bar{p}_\omega) = \sum_{jn} (\mathbf{b}_{jn} p_{jn} + \mathbf{b}_{jn}^+ \bar{p}_{jn})$$

and using (10.35)

$$\mathbf{b}_{jn} = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{-2\pi i n \omega / \varepsilon} \mathbf{b}_\omega d\omega, \quad \mathbf{b}_{jn}^+ = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{2\pi i n \omega / \varepsilon} \mathbf{b}_\omega^+ d\omega.$$

Starting now from the continuous version of (10.28) we used in (10.22)

$$\mathbf{b}_\omega = \int_0^\infty (\bar{\alpha}_{\omega\omega'} \mathbf{a}_{\omega'} - \bar{\beta}_{\omega\omega'} \mathbf{a}_{\omega'}^+) d\omega'$$

we have for the \mathbf{b}_{jn}

$$\mathbf{b}_{jn} = \int_0^\infty (\bar{\alpha}_{jn\omega'} \mathbf{a}_{\omega'} - \bar{\beta}_{jn\omega'} \mathbf{a}_{\omega'}^+) d\omega'$$

with

$$\alpha_{jn\omega'} = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} d\omega e^{2\pi i n \omega / \varepsilon} \alpha_{\omega, \omega'}$$

$$\beta_{jn\omega'} = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} d\omega e^{2\pi i n \omega / \varepsilon} \beta_{\omega, \omega'}.$$

From Eq.(10.33)

$$\beta_{\omega\omega'} = i \frac{e^{i(\omega-\omega')\tau_0}}{2\pi\sqrt{\omega\omega'}} e^{-\frac{\pi\omega}{2}} (\omega')^{-i\omega} \Gamma(1+i\omega)$$

we have

$$\beta_{jn\omega'} = i \frac{e^{-i\omega'\tau_0}}{2\pi\sqrt{\varepsilon}\sqrt{\omega'}} \int_{j\varepsilon}^{(j+1)\varepsilon} d\omega e^{2\pi i n \omega / \varepsilon} \frac{e^{i\omega\tau_0} e^{-\frac{\pi\omega}{2}} (\omega')^{-i\omega} \Gamma(1+i\omega)}{\sqrt{\omega}}.$$

For a narrow frequency interval ε and large n (large arrival times) we have

$$\beta_{jn\omega'} \approx i \frac{e^{-i\omega'\tau_0} e^{-\frac{\pi\tilde{\omega}}{2}}}{2\pi\sqrt{\varepsilon\tilde{\omega}\omega'}} \Gamma(1+i\tilde{\omega}) e^{i\Delta(j+1/2)\varepsilon} 2 \frac{\sin \frac{\Delta\varepsilon}{2}}{\Delta} \quad (10.36)$$

with $\tilde{\omega} = (j+1/2)\varepsilon$ and $\Delta = 2\pi n/\varepsilon - \ln \omega' + \tau_0$.

Thus $|\beta_{jn\omega'}|^2$ for large n (large arrival times) will be peaked at exponentially large values of ω' . This justifies the intuitive reasoning of section (10.12) about the dominance in the Fourier expansion of the small values of $c-t$. Moreover taking the square of Eq.(10.36) we see that the large ω' behavior is

$$|\beta_{jn\omega'}|^2 \sim \frac{1}{\omega' \ln^2 \omega'}$$

which integrated in $d\omega'$ converges at infinity. Thus, after curing the infrared divergence, the number of quanta which reaches the observer in a finite time is finite.

The result is interesting in several respects. By going over to discrete modes, we found the UV divergence in Eq.(10.26) was due to the unphysical continuum normalization. We found that if the Rindler observer tunes his measuring apparatus to a certain frequency range, he measures a steady flux of particles and that the flux of such particles depends on their energy (frequency) according to Planck's law. The flux is permanent. The divergence obtained in Eq.(10.26) in the continuum formulation was an attempt of the formalism to include the fact that in an infinite time the observer detects an infinite number of particles in a given frequency interval.

In the next section we shall consider a more general decomposition in wave packets.

References

- [1] S.W. Hawking, "Particle creation by black holes", *Comm. Math. Phys.* 43 (1975) 199, paragraphs 1, 2.

10.14 Hawking radiation in 4-dimensional Schwarzschild space

We give here the complete treatment of Hawking radiation for a field obeying a linear equation of motion (free field) in 4-dimensional Schwarzschild space following [1]. One decomposes the field $\phi(\mathbf{x}, t)$ in partial waves exploiting the rotational symmetry of the problem. We shall be concerned only with the $l = 0$ wave, the extension to higher waves being completely similar. For spherical symmetry one has

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu = \frac{1}{r^2} \partial_r r^2 \left(1 - \frac{r_s}{r}\right) \partial_r - \left(1 - \frac{r_s}{r}\right)^{-1} \partial_t^2.$$

We shall build the most general solution of the K.G. equation by superposing solutions periodic in time. It is preferable to write $\phi = \frac{\psi}{r} e^{\pm i\omega t}$ and go over to the new coordinates $r_* = r + r_s \log\left(\frac{r}{r_s} - 1\right)$. With a simple computation one reaches for $m = 0$ the equation

$$-\frac{d^2}{dr_*^2} \psi(r_*) + V(r) \psi(r_*) = \omega^2 \psi(r_*) \quad (10.37)$$

where

$$V(r) = \frac{r_s}{r^3} \left(1 - \frac{r_s}{r}\right)$$

and r has to be thought as function of r_* . As

$$\frac{dr_*}{dr} = \frac{1}{1 - \frac{r_s}{r}}$$

never vanishes for $r > r_s$, r is well defined. In case of $m^2 \neq 0$ and $l \neq 0$, $V(r)$ goes over to

$$V(r) = \left(1 - \frac{r_s}{r}\right) \left(\frac{r_s}{r^3} + \frac{l(l+1)}{r^2}\right) + m^2 \left(1 - \frac{r_s}{r}\right).$$

For simplicity we shall use $m^2 = 0$ but there is no real difficulty in working with $m^2 \neq 0$. The shape of V , for $l = 0$ is depicted in fig.10. V vanishes on the horizon $r = r_s$, $r_* = -\infty$ while at $r_* = +\infty$, i.e. $r = \infty$ it goes over to m^2 . For $m^2 = 0$, V reaches its maximum at $r = 4r_s/3$ and its value at that point is $V(r_M) = 27/(256r_s^2)$. The eigenvalue equations (10.37) due to the nature of V has continuum spectrum bounded from below by 0.

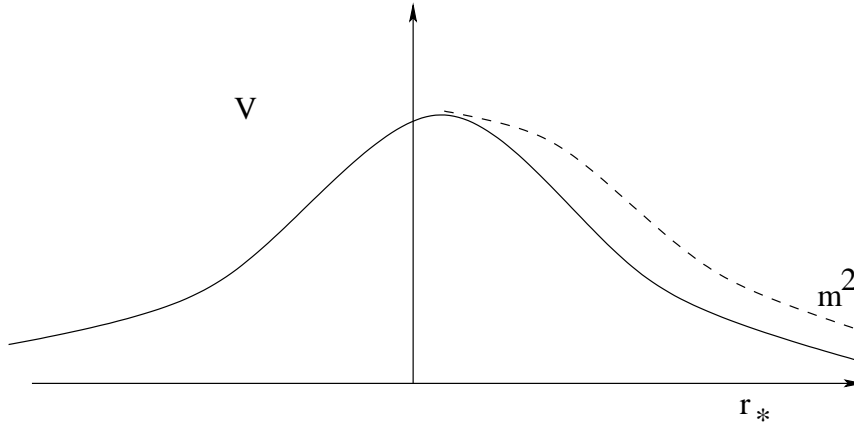


Figure 10.1: Effective Schwarzschild potential

The detector placed in a region \mathcal{O} of 4-dimensional space will be described by the positive operator Q^+Q where

$$Q = \int \phi(x)h(x)\sqrt{-g}d^4x$$

and \mathcal{O} is the support of the smooth function h . Thus the detector occupies a finite region of 3-dimensional space and is switched on for a finite time. Given the state Φ describing the system the result of the measurement is

$$(\Phi, Q^+Q\Phi)$$

which we have to compute. There is no need in this treatment to introduce creation and destruction operators, at any rate for clearness sake we point out that on the Fock vacuum $\mathbf{a}(\mathbf{k})\Phi_M = 0$ we have

$$(\Phi_M, Q^+Q\Phi_M) = \int \frac{|\tilde{h}(\mathbf{k}, \omega)|^2}{2\omega} d\mathbf{k}; \quad \omega = +\sqrt{\mathbf{k}^2 + m^2}$$

where

$$h(\mathbf{x}, t) = \frac{1}{(2\pi)^4} \int \tilde{h}(\mathbf{k}, k_0) e^{i(\mathbf{k}\mathbf{x} - k_0 t)} d^4 k .$$

In the limit in which $h(\mathbf{x}, t) = h(\mathbf{x})$ we have $(\Phi_M, Q^+ Q \Phi_M) = 0$ because in this case the support of $\tilde{h}(\mathbf{k}, k_0)$ is in $k_0 = 0$ while $\omega \geq m > 0$ for $m > 0$.

Otherwise in general the result will be non zero (and positive) also on the vacuum. This is due to the fact that the abrupt switching on and off of the detector will produce particle also out of the Minkowski vacuum. At the end we will specify $h(\mathbf{x}, t)$ so as to avoid such a production.

In Schwarzschild coordinates the stellar collapse takes an infinite time. The reason is that the Schwarzschild time t_c of the collapse is related to the Kruskal coordinates by (see Section 10.4)

$$\frac{t}{r_s} = 2 \operatorname{arctanh} \frac{T_c}{X_c} .$$

But as at the collapse $T_c = X_c$ we have $t = +\infty$. We recall also the definition of the null coordinates

$$v = t + r_*; \quad u = t - r_*$$

$v = \text{const.}$ and $u = \text{const.}$ represent incoming and outgoing null geodesics. It will be useful to go over to the new coordinates

$$r; \quad \tau = v - r = t + r_* - r.$$

For τ fixed and $r \rightarrow r_s$ we have $t \rightarrow +\infty$, $v \rightarrow \text{finite}$ and $u \rightarrow +\infty$. Recalling that

$$U = -e^{-\frac{u}{4M}}$$

and that the horizon is given by $U = 0$ we see that the horizon is crossed at a finite value of τ .

Even though not strictly necessary, the space support \mathcal{O} of h will be chosen at a large value of r (where the metric can be taken as Minkowski) and we shall consider a sequence of Q^T which are the Q translated by T along the Schwarzschild Killing time like vector $\frac{\partial}{\partial t}$. The reason is that we are interested in the measurements at large times after the gravitational collapse. Due to the linear field equations obeyed by ϕ , Q^T can be expressed in terms of ϕ and its time derivative on an arbitrary partial Cauchy surface for which the exterior Schwarzschild region is future asymptotically predictable as we shall see explicitly below.

As already saw, given two partial Cauchy surfaces Σ_1 and Σ_0 and an f solution of

$$\frac{1}{\sqrt{-g}} \partial_\mu (g^{\mu\nu} \partial_\nu f(x)) - m^2 f(x) = (g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2) f(x) = 0 \quad (10.38)$$

we have

$$\int_{\Sigma_1} (\phi(x) \overleftrightarrow{\partial}_\mu f(x)) g^{\mu\nu} \Sigma_\nu = \int_{\Sigma_0} (\phi(x) \overleftrightarrow{\partial}_\mu f(x)) g^{\mu\nu} \Sigma_\nu \quad (10.39)$$

and we shall choose for Σ_0 the partial Cauchy surface $\tau = 0$. The idea is to bring back the computation of Q^T , which is extended in time to the single partial Cauchy surface $\tau = 0$.

We start from the computation of Q writing it as sum (integral) of contributions at constant Schwarzschild time t .

$$Q = \int \phi(x) h(x) \sqrt{-g} d^4x = \int dt_0 \int \phi(\mathbf{x}, t_0) h(\mathbf{x}, t_0) \sqrt{-g} d\mathbf{x} . \quad (10.40)$$

We write

$$\int \phi(\mathbf{x}, t_0) h(\mathbf{x}, t_0) \sqrt{-g} d\mathbf{x} = \int \phi(\mathbf{x}, t_0) \partial_0 f_{t_0}(\mathbf{x}, t_0) \sqrt{-g} d\mathbf{x} \quad (10.41)$$

where ∂_0 means derivative w.r.t. the fourth argument and $f_{t_0}(\mathbf{x}, t)$ is a solution of the Eq.(10.38) with the following initial conditions

$$\partial_0 f_{t_0}(\mathbf{x}, t_0) = h(\mathbf{x}, t_0), \quad f_{t_0}(\mathbf{x}, t_0) = 0 .$$

We have

$$\int \phi(\mathbf{x}, t_0) h(\mathbf{x}, t_0) \sqrt{-g} d\mathbf{x} = \int (\phi(\mathbf{x}, t_0) \overleftrightarrow{\partial}_0 f_{t_0}(\mathbf{x}, t_0)) \sqrt{-g} d\mathbf{x}$$

which using (10.39) can be rewritten as

$$\int_{\Sigma_0} (\phi(x) \overleftrightarrow{\partial}_\mu f_{t_0}(x)) g^{\mu\nu} \Sigma_\mu$$

i.e. as an integral on Σ_0 (the surface $\tau = 0$). Then we have

$$Q = \int_{\Sigma_0} (\phi(x) \overleftrightarrow{\partial}_\mu f(x)) g^{\mu\nu} \Sigma_\mu$$

where

$$f(x) = \int f_{t_0}(x) dt_0 .$$

We shall be interested in Q^T given by

$$\int \phi(x) h(\mathbf{x}, t - T) \sqrt{-g} d^4x .$$

This is obtained from the solution $f_{t_0}^T(\mathbf{x}, t)$ with

$$\partial_0 f_{t_0}^T(\mathbf{x}, t_0) = h(\mathbf{x}, t_0 - T), \quad f_{t_0}^T(\mathbf{x}, t_0) = 0$$

and thus

$$Q^T = \int_{\Sigma_0} (\phi(x) \overleftrightarrow{\partial}_\mu f^T(x)) g^{\mu\nu} \Sigma_\nu$$

where

$$f^T(x) = \int f_{t_0}^T(x) dt_0$$

and Σ_0 again is the surface given by $\tau = 0$. Notice that this is possible because according Huygens principle $f^T(x)$ propagates back in time completely to the partial Cauchy surface Σ_0 and nowhere outside it.

In the new coordinates r, τ we have

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)d\tau^2 + 2\frac{r_s}{r}drd\tau + \left(1 + \frac{r_s}{r}\right)dr^2 + r^2d\Omega^2.$$

The inverse of this metric is easily computed and we have in particular

$$g^{\tau\tau} = -\left(1 + \frac{r_s}{r}\right); \quad g^{\tau r} = -\frac{r_s}{r}; \quad g^{rr} = 1 - \frac{r_s}{r} \quad (10.42)$$

so that

$$g^{\tau\tau}\partial_\tau + g^{\tau r}\partial_r = -\left(1 + \frac{r_s}{r}\right)\partial_\tau + \frac{r_s}{r}\partial_r \equiv D$$

while

$$\Sigma_\tau = \epsilon_{\tau r \theta \phi} dr \wedge d\theta \wedge d\phi = r^2 \sin \theta \, dr \wedge d\theta \wedge d\phi.$$

Taking into account also the contribution of Q^{T+} we have the exact result

$$(\Phi, Q^{T+} Q^T \Phi) = \quad (10.43)$$

$$\int_{\Sigma_0} \int_{\Sigma_0} (\Phi, \phi(x_1)\phi(x_2)\Phi) \overleftrightarrow{D}_1 \overleftrightarrow{D}_2 \bar{f}^T(x_1) f^T(x_2) r_1^2 dr_1 d\Omega_1 r_2^2 dr_2 d\Omega_2$$

This result is immediately extended to $m^2 \neq 0$ and to $l \neq 0$.

A complete set of solutions of Eq.(10.37) is given by the $\psi_+(\pm\omega, r_*)$ with the following asymptotic behavior at large r_* (where our detector is placed)

$$\psi_+(r^*, \pm\omega) = e^{\pm i\omega r^*}; \quad r_* \simeq +\infty$$

while an alternative complete set is classified by the behavior of the solution at $r_* \simeq -\infty$

$$\psi_-(r^*, \pm\omega) = e^{\pm i\omega r^*}; \quad r_* \simeq -\infty.$$

The $\psi_+(r^*, \pm\omega)$ form a complete set. Thus we can write

$$h(r^*, t) = \int_0^\infty d\omega [\psi_+(r^*, \omega) \tilde{h}(\omega, t) + \psi_+(r^*, -\omega) \tilde{h}(-\omega, t)]$$

where the above equation defines $\tilde{h}(\pm\omega, t)$

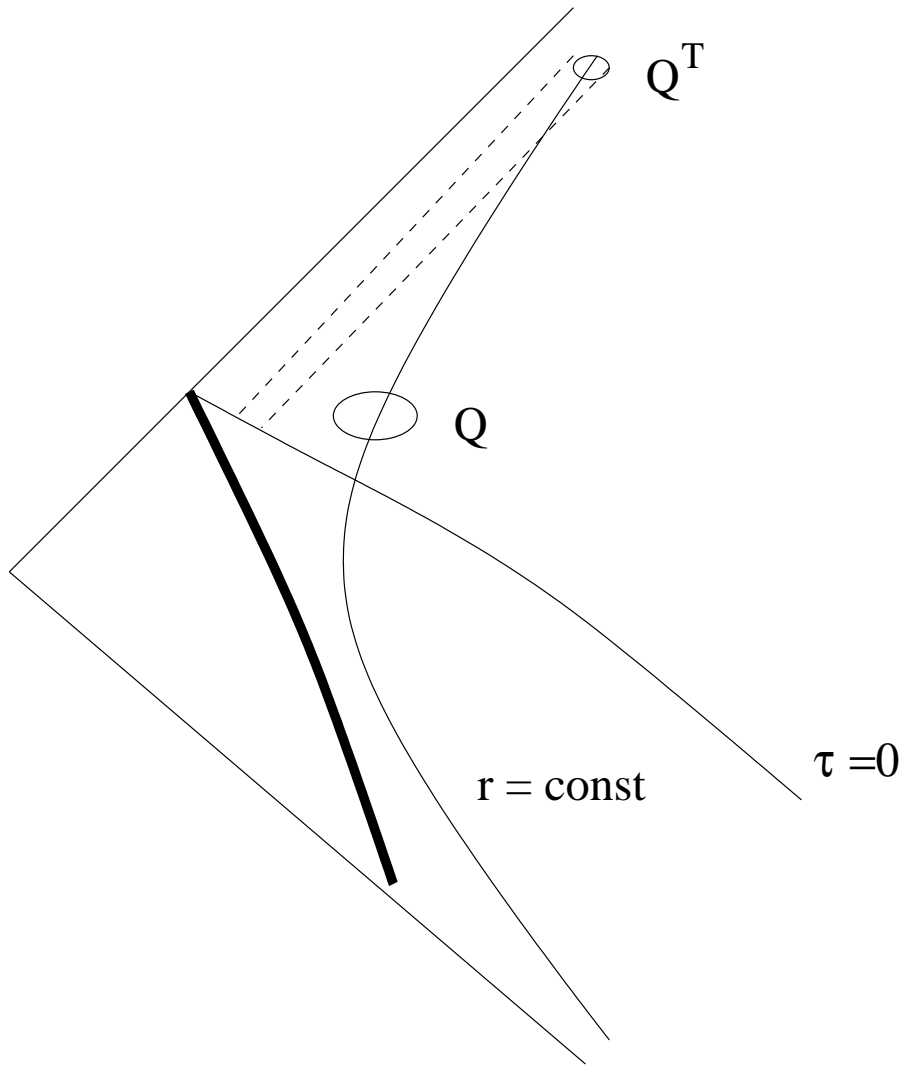


Figure 10.2: Gravitational collapse

The explicit form of $f_{t_0}(r_*, t)$ is easily obtained from the above Fourier transform

$$f_{t_0}(r_*, t) = i \int_0^\infty \frac{d\omega}{2\omega} \left[\psi_+(r^*, \omega) \tilde{h}(\omega, t_0) (e^{-i\omega(t-t_0)} - e^{i\omega(t-t_0)}) - \psi_+(r^*, -\omega) \tilde{h}(-\omega, t_0) (e^{i\omega(t-t_0)} - e^{-i\omega(t-t_0)}) \right]. \quad (10.44)$$

In fact as we have

$$f_{t_0}(r^*, t_0) = 0 \quad \text{and} \quad \partial_0 f_{t_0}(r^*, t_0) = h(r^*, t_0).$$

$f_{t_0}^T(r^*, t)$ is simply obtained by replacing in Eq.(10.44) $t - t_0$ with $t - t_0 - T$ and $f^T(r^*, t)$ is obtained by integrating in t_0 .

$$f^T(r^*, t) = \int dt_0 f_{t_0}^T(r^*, t).$$

The result is with obvious notation for $\tilde{h}(\pm\omega, \pm\omega)$

$$f^T(r^*, t) = i \int_0^\infty \frac{d\omega}{2\omega} \left[\psi_+(r^*, \omega)(\tilde{h}(\omega, \omega)e^{-i\omega(t-T)} - \tilde{h}(\omega, -\omega)e^{i\omega(t-T)}) \right. \\ \left. - \psi_+(r^*, -\omega)(\tilde{h}(-\omega, -\omega)e^{i\omega(t-T)} - \tilde{h}(-\omega, \omega)e^{-i\omega(t-T)}) \right]. \quad (10.45)$$

Of the four solutions appearing in (10.45) the first and the third move initially to increasing values of r^* (right-moving) while the second and the fourth move initially to decreasing values r^* (left-moving). We are now interested in f^T on the partial Cauchy surface $\tau = 0$ for large positive T . We see from the form of the exponential that increasing T corresponds to propagating the solutions backward in time. In so doing the “left-moving” solution move right without encountering the potential and for large T will meet the partial Cauchy surface $\tau = 0$ at large r . Thus such wave packets correspond to particle coming from the space infinity and not from the black hole and we are not interested in them. Thus we are left with

$$f^T(r^*, t) = i \int_0^\infty \frac{d\omega}{2\omega} [\psi_+(r^*, \omega)\tilde{h}(\omega, \omega)e^{-i\omega(t-T)} - \psi_+(r^*, -\omega)\tilde{h}(-\omega, -\omega)e^{i\omega(t-T)}]. \quad (10.46)$$

Let us consider first the term

$$i \int_0^\infty \frac{d\omega}{2\omega} \psi_+(r^*, \omega)\tilde{h}(\omega, \omega)e^{-i\omega(t-T)}. \quad (10.47)$$

We recall that for increasing t the wave packet described by it moves to $+\infty$ as it does not encounter any potential. On the other hand taking T positive and large corresponds to evolving the same wave packet backward in time. In the process part of it will be reflected by the potential barrier and part will be transmitted with a transmission coefficient given by the identity

$$D(\omega)\psi_-(r^*, \omega) = \psi_+(r^*, \omega) - B(\omega)\psi_+(r^*, -\omega).$$

The computation of $D(\omega)$ is an elementary problem in quantum mechanics even if one does not succeed in performing the computation analytically. As for large negative r^* the $\psi_-(r^*, \omega)$ becomes $e^{i\omega r^*}$ (with coefficient 1) we have that at large positive T the contribution becomes

$$i \int_0^\infty \frac{d\omega}{2\omega} D(\omega)\tilde{h}(\omega, \omega)e^{i\omega(r^*-t+T)}. \quad (10.48)$$

In exactly the same way one deals with second term in Eq.(10.46) and calling $f_-^T(r^*, t)$ the limit of $f(r^*, t)$ for large positive T we have

$$f_-^T(r^*, t) = i \int_0^\infty \frac{d\omega}{2\omega} [D(\omega)\tilde{h}(\omega, \omega)e^{i\omega(r^*-t+T)} - D(-\omega)\tilde{h}(-\omega, -\omega)e^{-i\omega(r^*-t+T)}] \quad (10.49)$$

and we can thus write for large T

$$\begin{aligned} \langle \Phi, Q^{T+} Q^T \Phi \rangle = & \quad (10.50) \\ \int_{\Sigma_0} \int_{\Sigma_0} (\Phi, \phi(x_1) \phi(x_2) \Phi) \overleftrightarrow{D}_1 \overleftrightarrow{D}_2 \bar{f}_-^T(x_1) f_-^T(x_2) r_1^2 dr_1 d\Omega_1 r_2^2 dr_2 d\Omega_2 . \end{aligned}$$

The difference between Eq.(10.43) and Eq.(10.50) is that while Eq.(10.43) is exact Eq.(10.50) holds for large positive T ; however the advantage of Eq.(10.50) is that f_-^T is simply known through (10.49) in terms of the transmission coefficient $D(\omega)$. To understand the physics of the problem it is useful to notice that

$$f_-^T(r^*, t) = F(-u + T) = F(2r^* - \tau - r + T)$$

which near the horizon becomes

$$f_-^T(r^*, t) \simeq F(2r_s \ln \frac{r - r_s}{r_s} - \tau + r_s + T)$$

It shows that for large T the f_-^T is concentrated in an exponentially small region around the horizon and thus the result is determined by the short distance behavior of the Wightman function $(\Phi, \phi(x_1) \phi(x_2) \Phi)$ (see fig. 10.2).

The following simplifying features intervene in the computation of Eq.(10.50)

$$\begin{aligned} D &= -(1 + \frac{r_s}{r}) \partial_\tau + \frac{r_s}{r} \partial_r \\ Du &= \frac{\partial u}{\partial r}, \quad Dv = -\frac{\partial v}{\partial r} = -1 \\ f_-^T(r^*, t) &= F(-u + T), \quad Df_-^T = \frac{\partial f_-^T}{\partial r} \end{aligned}$$

which allow for large T two integrations by parts and we reach

$$\langle Q^{T+} Q^T \rangle = \int_{\Sigma_0} \int_{\Sigma_0} [\hat{D}_1 \hat{D}_2 (\Phi, \phi(x_1) \phi(x_2) \Phi)] \bar{f}_-^T(x_1) f_-^T(x_2) r_1^2 dr_1 d\Omega_1 r_2^2 dr_2 d\Omega_2$$

with

$$\hat{D} = 2 \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial r} \right) .$$

We shall assume that the short distance behavior of the Wightman function on $\tau = 0$ which is a regular region of space time not casually connected with the crossing of the horizon by the surface of the star, is given by the usual free field value. Arguments [2] can be given also to support that fact that the short distance behavior of such a function is independent of the state. The covariant translation of Eq.(10.13) using the metric (10.42) is

$$(\Phi, \phi(x_1) \phi(x_2) \Phi) \approx \frac{1}{4\pi^2 \sigma_\varepsilon}$$

with σ given at short distances by $(x_1 - x_2)^\mu g_{\mu\nu} (x_1 - x_2)^\nu$. Near the horizon it assumes the form

$$\sigma_\varepsilon = 2(r_1 - r_2)(r_1 - r_2 + \tau_1 - \tau_2 - i\varepsilon) + r_s^2 \theta_{12}^2$$

θ_{12} being the angle between the directions 1 and 2. One obtains on the surface $\tau = 0$

$$\frac{1}{4\pi^2} \hat{D}_1 \hat{D}_2 \frac{1}{\sigma_\varepsilon} = -\frac{8}{\pi^2} \frac{(r_1 - r_2 - i\varepsilon)}{[2(r_1 - r_2 - i\varepsilon)^2 + r_s^2 \theta_{12}^2]^3}.$$

Integrating over $d\Omega_1, d\Omega_2$ one obtains

$$-\frac{4}{r_s^2} \frac{1}{(r_1 - r_2 - i\varepsilon)^2}$$

which is the most singular part, plus a term which is irrelevant for $T \rightarrow \infty$. Thus we reached the result

$$\langle Q^{T+} Q^T \rangle = -4r_s^2 \int_{r_s}^{\infty} \frac{1}{(r_1 - r_2 - i\varepsilon)^2} \bar{F}\left(2r_s \ln \frac{r_1 - r_s}{r_s} + r_s + T\right) F\left(2r_s \ln \frac{r_2 - r_s}{r_s} + r_s + T\right) dr_1 dr_2.$$

Using the new variables $\rho_1 = \frac{r_1 - r_s}{r_s} e^{\frac{T}{r_s} + 1}$ $\rho_2 = \frac{r_2 - r_s}{r_s} e^{\frac{T}{r_s} + 1}$ it can be rewritten as

$$\langle Q^{T+} Q^T \rangle = -4r_s^2 \int_0^{\infty} \frac{1}{(\rho_1 - \rho_2)^2} \bar{F}(2r_s \ln \rho_1) F(2r_s \ln \rho_2) d\rho_1 d\rho_2.$$

Now we go over to the variable $y_1 = \ln \rho_1, y_2 = \ln \rho_2$ to obtain

$$\langle Q^{T+} Q^T \rangle = -4\pi r_s \int_{-\infty}^{\infty} \frac{1}{\sinh^2 \frac{y - i\varepsilon}{2}} \frac{|D(\omega)|^2 |\tilde{h}(\omega, \omega)|^2}{\omega^2} e^{-2r_s \omega y} dy.$$

Such an integral can be computed by using the identity

$$\frac{1}{(\sinh \pi z - i\varepsilon)^2} = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(z + ik - i\varepsilon)^2}$$

as done in section (10.6) obtaining from the first part of Eq.(10.49)

$$\langle Q^{T+} Q^T \rangle = 64\pi^2 r_s^2 \int_0^{\infty} \frac{|D(\omega)|^2}{e^{4\pi r_s \omega} - 1} |h(\omega, \omega)|^2 \frac{d\omega}{\omega}. \quad (10.51)$$

To this we should add the contribution in Eq.(10.49) of $h(-\omega, -\omega)$. This gives

$$-64\pi^2 r_s^2 \int_0^{\infty} \frac{|D(-\omega)|^2}{e^{-4\pi r_s \omega} - 1} |h(-\omega, -\omega)|^2 \frac{d\omega}{\omega}.$$

which can be added to (10.51) to obtain

$$\langle Q^{T+} Q^T \rangle = 64\pi^2 r_s^2 \int_{-\infty}^{\infty} \frac{|D(\omega)|^2}{e^{4\pi r_s \omega} - 1} |h(\omega, \omega)|^2 \frac{d\omega}{\omega}. \quad (10.52)$$

The factor $|D(\omega)|^2$ is the gray body factor, which in the free field theory poses no problem to be computed; one has to keep in mind that the height of the potential V is of the order of magnitude $1/r_s^2$ which is the order of magnitude of the ω^2 of the typical quantum emitted from the black hole. Thus $|D(\omega)|^2$ plays an important role. In an interacting theory the computation of such a factor poses a real problem.

The variable T has disappeared in the integration process; this mean that the Hawking radiation is a “permanent” phenomenon; the violation of energy conservation has to be ascribed to the fact that we are working in an external field. At the end the energy has to be supplied by the black hole itself. The factor $|h(\omega, \omega)|^2$ is just the description of the counter. However the $h(\omega, \omega)$ of a physical counter should have support of on positive values of ω [3] and we go back to Eq.(10.51).

In this case $h(x)$ cannot be of compact support in space time but one has to widen the class of test function beyond those of compact support.

Finally the key point in producing the Planck factor is the short distance behavior of the two point Wightman function; any modification to it would imply a change in the emission spectrum.

We compare now the obtained formula with the Rindler spectrum. There the temperature was given by the argument of the exponential

$$e^{2\pi\omega/g} = e^{2\pi\omega c/g} = e^{2\pi\omega\hbar c/g\hbar} = e^{\omega\hbar/k_B T}$$

i.e. $k_B T = g\hbar/2\pi c$. We can now compute the surface gravity for a black hole i.e. the force exerted at the horizon on a unit mass as measured by an observer at infinity. We found (see Eq.(10.11)) that it is given by

$$g = \frac{MG_N}{(r_s)^2} = \frac{G_N M c^4}{4M^2 G_N^2} = \frac{c^4}{4G_N M}$$

and thus we expect a temperature $k_B T = \hbar c^3/8\pi G_N M$ as found in Eq.(10.51).

Thus, according to Hawking’s treatment, black holes are not eternal but they evaporate. The lifetime of a black hole of mass M can be estimated to be

$$t = \frac{5120\pi G^2 M^3}{\hbar c^4} \tag{10.53}$$

with a typical initial wave length of the radiation of $\lambda/(2\pi) \approx 4\pi r_s$ being r_s the radius of the black hole.

However for macroscopic black holes such lifetime is much larger than the age of our universe. For a black hole of the mass of the sun $M = 1.989 \cdot 10^{30} \text{ kg}$ the lifetime is $5 \cdot 10^{74} \text{ s}$. For a black hole of $M = 200 \cdot 000 \text{ kg}$ we have a lifetime of 0.67 s with an initial temperature

of 50 *TeV*. For a black hole of the Planck mass ($M = 1.21 \cdot 10^{19} \text{GeV}/c^2 = 2.17 \cdot 10^{-8} \text{ kg}$) the lifetime is of $8.6 \cdot 10^{-40} \text{ s}$ of the order of magnitude of the Planck time.

References

- [1] K. Fredenhagen, R. Haag, “On the derivation of Hawking radiation associated with the formation of a black hole”, *Comm. Math. Phys.* 127 (1990) 273
- [2] R. Haag, “Local quantum physics”, II edition, Chap.VIII, Springer 1992
- [3] R. Haag, “Local quantum physics”, II edition, Chap.VI, Springer 1992

Chapter 11

N=1 Supergravity

11.1 The Wess-Zumino model

We first discuss the operation of charge conjugation for the Dirac field. Given the Dirac equation

$$(\gamma^\mu \partial_\mu + m)\psi = 0$$

(we always use γ^j hermitean and γ^0 antihermitean, thus $\gamma^4 \equiv i\gamma^0$ hermitean) we look for a unitary operator C such that

$$\psi_c \equiv C\psi C^+ = A\psi_T^+ \equiv A\psi^* \quad (11.1)$$

which again satisfies the Dirac equation. We have

$$(\gamma^{*\mu} \partial_\mu + m)\psi_T^+ = 0$$

and thus A must be such that

$$A\gamma^{*\mu}A^{-1} = \gamma^\mu . \quad (11.2)$$

There exists an A which does the job i.e. $A = \gamma^2$ in the representation adopted in Chapter 1. A is unique up to a factor because if $B = XA$ also satisfies (11.2) X commutes with all the gammas and by Schur lemma X is a multiple of the identity c , being our representation of the gammas irreducible. If we require that C^2 brings ψ back to itself apart a phase factor, we must have

$$C(c\gamma^2\psi_T^+)C^+ = cc^*\psi = u\psi$$

with u phase factor, from which $c^*c = 1$. Redefining the ψ by a phase factor we can write simply

$$C\psi C^+ = \gamma^2\psi_T^+ \quad (11.3)$$

The current $i\bar{\psi}\gamma^\mu\psi$, which is hermitean, under C changes sign

$$Ci\bar{\psi}\gamma^\mu\psi C^+ = i\bar{\psi}_c\gamma^\mu\psi_c = -i\bar{\psi}\gamma^\mu\psi$$

as it is easily verified, taking into account the anticommutative nature of ψ .

We define a Majorana spinor as a spinor which coincides with its charge conjugate $\psi_c = C\psi C^+ = \gamma^2\psi_T^+ = \psi$. The relation with the form of the equation for a Majorana particle given in Chapter 1 is the following. Given the Dirac equation in the form

$$\begin{pmatrix} \sigma^\mu p_\mu & 0 \\ 0 & \tilde{\sigma}^\mu p_\mu \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = mc \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} \quad (11.4)$$

for a Majorana spinor from (11.3) we have

$$-i\sigma_2\phi_T^+ = \psi$$

equivalent to

$$i\sigma_2\psi_T^+ = \phi$$

and thus the lower pairs of equation in (11.4) can be written as

$$\tilde{\sigma}^\mu p_\mu\phi = mc\psi = -imc\sigma_2\phi_T^+ \equiv -imc\sigma_2\phi^*$$

which is Eq.(1.16) while the upper pair of equations in (11.4) is equivalent to the above. In dealing with Majorana spinors it is useful to adopt the so called Majorana representation of the gamma matrices: γ^i will be again hermitean and γ^0 antihermitean but now all will be real.

$$\begin{aligned} \gamma^1 &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \equiv -i\gamma^4. \\ \gamma_5 &= \gamma^1\gamma^2\gamma^3\gamma^4 = i \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}. \end{aligned} \quad (11.5)$$

The following relations valid in the Majorana representation will be useful

$$\gamma_T^j = \gamma^j; \quad \gamma_T^0 = -\gamma^0; \quad \gamma_T^4 = -\gamma^4; \quad \gamma_{5T} = -\gamma_5.$$

In the Majorana representation the matrix A of (11.1) is simply the identity and thus a Majorana spinor in the Majorana representation is described by an hermitean spinor.

To get acquainted with supersymmetry we examine now the Wess-Zumino model. In such a model we have two boson fields, a scalar A and a pseudoscalar B and one Majorana fermion λ , all massless.

The Lagrangian of the Wess-Zumino model is

$$L_{WZ} = -\frac{1}{2}\partial_\mu A\partial^\mu A - \frac{1}{2}\partial_\mu B\partial^\mu B - \frac{1}{2}\bar{\lambda}\not{\partial}\lambda.$$

As supersymmetry transformation is given by

$$\delta A = \frac{1}{2}\bar{\epsilon}\lambda, \quad \delta B = -\frac{i}{2}\bar{\epsilon}\gamma_5\lambda, \quad \delta\lambda = \frac{1}{2}(\not{\partial}A + i\gamma_5\not{\partial}B)\epsilon$$

where ϵ is a constant real classical anticommuting spinor. Notice that $\bar{\epsilon}\lambda$ and $-i\bar{\epsilon}\gamma_5\lambda$ are hermitean fields. The variation of the Lagrangian is

$$\delta L = \delta_A L + \delta_B L$$

with

$$\begin{aligned} \delta_A L &= -\frac{1}{2}\partial^\mu A \bar{\epsilon}\partial_\mu\lambda - \frac{1}{4}\bar{\lambda}\not{\partial}\not{\partial}A \epsilon + \frac{1}{4}\bar{\epsilon}\not{\partial}A\not{\partial}\lambda. \\ \delta_B L &= \frac{i}{2}\partial^\mu B \bar{\epsilon}\gamma_5\partial_\mu\lambda + \frac{i}{4}\bar{\lambda}\gamma_5\not{\partial}\not{\partial}B \epsilon - \frac{i}{4}\bar{\epsilon}\gamma_5\not{\partial}B\not{\partial}\lambda. \end{aligned}$$

Thus the variation is not zero but is equal to a divergence; i.e. it is immediately shown that

$$\delta_A L = \partial_\mu K^\mu$$

with

$$K^\mu = -\frac{1}{4}\bar{\epsilon}\gamma^\mu\not{\partial}A\lambda$$

and similarly one finds that

$$\delta_B L = \partial_\mu K_5^\mu$$

with

$$K_5^\mu = -\frac{i}{4}\bar{\epsilon}\gamma^\mu\gamma_5\not{\partial}B\lambda.$$

Thus the action is invariant.

The chiral doublets $A+iB$, $(1+\gamma_5)\lambda$ and $A-iB$, $(1-\gamma_5)\lambda$ under a SUSY transformation, transform into each other

$$\begin{aligned} \delta(A+iB) &= \frac{1}{2}\bar{\epsilon}(1+\gamma_5)\lambda \\ \delta(A-iB) &= \frac{1}{2}\bar{\epsilon}(1-\gamma_5)\lambda \\ \delta(1+\gamma_5)\lambda &= \frac{1}{2}(1+\gamma_5)\not{\partial}(A+iB)\epsilon \\ \delta(1-\gamma_5)\lambda &= \frac{1}{2}(1-\gamma_5)\not{\partial}(A-iB)\epsilon. \end{aligned}$$

One notices that

- 1) the SUSY transformations relate bosons with fermions.
- 2) Irreducible representation must involve at least a boson and a fermion.
- 3) ϵ is a (constant) spinor under Lorentz transformation if we have to maintain invariance under Lorentz transformations.
- 4) In $n = 4$ the boson field has dimension $A \sim B \sim l^{-1} \sim m$ while the fermion $\lambda \sim m^{3/2} \sim l^{-3/2}$ and thus from $\delta A = \frac{1}{2}\bar{\epsilon}\lambda$ we have $\epsilon \sim m^{-1/2} \sim l^{1/2}$
- 5) In $\delta\lambda = \text{boson} \times \epsilon$ the gap in dimensions is filled by the derivative $\partial \sim m \sim l^{-1}$.

Most important is the following result: The commutator of two SUSY transformations is a (global) space time translation. In a sense a SUSY transformation is the square root of a space time translation.

We have in fact

$$\begin{aligned}\delta_2(\delta_1 A) &= \delta_2\left(\frac{1}{2}\bar{\epsilon}_1\lambda\right) = \frac{1}{4}\bar{\epsilon}_1(\not{\partial}A + i\gamma_5\not{\partial}B)\epsilon_2 \\ \delta_1(\delta_2 A) &= \delta_1\left(\frac{1}{2}\bar{\epsilon}_2\lambda\right) = \frac{1}{4}\bar{\epsilon}_2(\not{\partial}A + i\gamma_5\not{\partial}B)\epsilon_1 = \frac{1}{4}\bar{\epsilon}_1(-\not{\partial}A + i\gamma_5\not{\partial}B)\epsilon_2\end{aligned}$$

and thus

$$(\delta_2\delta_1 - \delta_1\delta_2)A = \frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu A.$$

Similarly

$$(\delta_2\delta_1 - \delta_1\delta_2)B = \frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu B$$

and thus the commutator is a space time translation by $\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2$. Now the question is

$$(\delta_2\delta_1 - \delta_1\delta_2)\lambda = \frac{1}{4}\partial_\mu(\bar{\epsilon}_2\lambda)\gamma^\mu\epsilon_1 - (1 \leftrightarrow 2) + \frac{1}{4}\partial_\mu(\bar{\epsilon}_2\gamma_5\lambda)\gamma_5\gamma^\mu\epsilon_1 - (1 \leftrightarrow 2) = \frac{1}{2}(\bar{\epsilon}_1\gamma^\mu\epsilon_2)\partial_\mu\lambda? \quad (11.6)$$

Not quite, as we shall see.

To proceed we shall need the Fierz rearrangement identity: Given spinors $\lambda, \chi, \psi, \phi$ we have

$$(\bar{\lambda}\chi)(\bar{\psi}\phi) = -\frac{1}{4}\sum_j(\bar{\lambda}O^j\phi)(\bar{\psi}O^j\chi) \quad (11.7)$$

with

$$O^j = \{I, \gamma^a, 2i\sigma^{ab}, i\gamma_5\gamma^a, \gamma_5\} \quad a, b = 1, 2, 3, 4, \quad a < b.$$

The O^j are the 16 independent Dirac 4×4 matrices, with

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \equiv \frac{1}{2}\gamma^{ab}.$$

O^j are hermitean and orthonormal in the trace norm

$$\frac{1}{4}\text{Tr}(O^j O^l) = \delta^{jl}.$$

O^j being a complete orthonormal set we have for any matrix M

$$M = \frac{1}{4} \sum_j O^j \text{Tr}(O^j M).$$

Given the matrix $M^\gamma_\beta = \delta^\gamma_\delta \delta^\alpha_\beta$ we have

$$\delta^\gamma_\delta \delta^\alpha_\beta = \frac{1}{4} \sum_j (O^j)^\gamma_\beta (O^j)^\alpha_\delta$$

because $\text{Tr}(O^j M) = (O^j)^\alpha_\delta$. Multiplying the above by the spinors $\bar{\lambda}_\alpha, \chi^\beta, \bar{\psi}_\gamma, \phi^\delta$, and keeping in mind anticommutativity we obtain Eq.(11.7). Eq.(11.7) is immediately generalized to

$$(\bar{\lambda} M \chi)(\bar{\psi} N \phi) = -\frac{1}{4} \sum_j (\bar{\lambda} M O^j N \phi)(\bar{\psi} O^j \chi) \quad (11.8)$$

by letting $\bar{\lambda} \rightarrow \bar{\lambda} M$ and $\phi \rightarrow N \phi$.

We shall now prove by using the Fierz transformation that Eq.(11.6) holds on the equation of motion for λ (i.e. “on shell”).

For Majorana spinors (classical fields or operators) we have

$$\begin{aligned} \bar{\epsilon}_2 \epsilon_1 &= \bar{\epsilon}_1 \epsilon_2 \\ \bar{\epsilon}_2 \gamma^a \epsilon_1 &= -\bar{\epsilon}_1 \gamma^a \epsilon_2 \\ \bar{\epsilon}_2 \gamma_5 \gamma^a \epsilon_1 &= \bar{\epsilon}_1 \gamma_5 \gamma^a \epsilon_2 \\ \bar{\epsilon}_2 \gamma_5 \epsilon_1 &= \bar{\epsilon}_1 \gamma_5 \epsilon_2 \\ \bar{\epsilon}_2 \sigma^{ab} \epsilon_1 &= -\bar{\epsilon}_1 \sigma^{ab} \epsilon_2. \end{aligned} \quad (11.9)$$

We go back now to Eq.(11.6). With $\bar{\chi}$ a dummy spinor

$$\begin{aligned} &\frac{1}{4} (\bar{\chi} \gamma^\mu \epsilon_1)(\bar{\epsilon}_2 \partial_\mu \lambda) - \frac{1}{4} (\bar{\chi} \gamma^\mu \gamma_5 \epsilon_1)(\bar{\epsilon}_2 \gamma_5 \partial_\mu \lambda) = \\ &= -\frac{1}{16} (\bar{\epsilon}_2 O^j \epsilon_1)(\bar{\chi} \gamma^\mu O^j \partial_\mu \lambda) + \frac{1}{16} (\bar{\epsilon}_2 O^j \epsilon_1)(\bar{\chi} \gamma^\mu \gamma_5 O^j \gamma_5 \partial_\mu \lambda). \end{aligned}$$

where in the first we used (11.8) with $M = \gamma^\mu$, $N = I$ and in the second again (11.8) with $M = \gamma^\mu \gamma_5$ and $N = \gamma_5$.

Due to the relations (11.9) and antisymmetrization only $O^j = \gamma^\mu$ and $O^j = 2i\sigma^{\mu\nu}$ contribute and we are left with

$$\begin{aligned} &-\frac{1}{16} (\bar{\epsilon}_2 \gamma^\nu \epsilon_1)(\bar{\chi} \gamma^\mu \gamma_\nu \partial_\mu \lambda) + \frac{1}{4} (\bar{\epsilon}_2 \sigma^{\alpha\beta} \epsilon_1)(\bar{\chi} \gamma^\mu \sigma^{\alpha\beta} \partial_\mu \lambda) \\ &+ \frac{1}{16} (\bar{\epsilon}_2 \gamma^\nu \epsilon_1)(\bar{\chi} \gamma^\mu \gamma_5 \gamma_\nu \gamma_5 \partial_\mu \lambda) - \frac{1}{4} (\bar{\epsilon}_2 \sigma^{\alpha\beta} \epsilon_1)(\bar{\chi} \gamma^\mu \gamma_5 \sigma^{\alpha\beta} \gamma_5 \partial_\mu \lambda). \end{aligned}$$

The second and the fourth term cancel, while the first and the third sum together and we are left with

$$-\frac{1}{4}(\bar{\epsilon}_2\gamma^\nu\epsilon_1)(\bar{\chi}\partial_\nu\lambda) + \frac{1}{8}(\bar{\epsilon}_2\gamma^\nu\epsilon_1)(\bar{\chi}\gamma_\nu\not{\partial}\lambda) - (1 \leftrightarrow 2) = \frac{1}{2}(\bar{\epsilon}_1\gamma^\nu\epsilon_2)(\bar{\chi}\partial_\nu\lambda) - \frac{1}{4}(\bar{\epsilon}_1\gamma^\nu\epsilon_2)(\bar{\chi}\gamma_\nu\not{\partial}\lambda).$$

Removing the dummy spinor χ Eq.(11.6) becomes

$$\frac{1}{2}(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\partial_\nu\lambda - \frac{1}{4}(\bar{\epsilon}_1\gamma^\nu\epsilon_2)\gamma_\nu\not{\partial}\lambda$$

and thus it is the same translation on λ as on A and B provided we work on shell i.e. for $\not{\partial}\lambda = 0$.

11.2 Noether currents

Suppose that under a global i.e. $\varepsilon(x) = \text{const.}$ transformation the Lagrangian varies by a divergence i.e.

$$\delta L = \varepsilon \partial_\mu K^\mu.$$

Thus the action is invariant and the transformation is a symmetry transformation. We want to compute how the action varies when ε is made space-time dependent. We have

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial \phi} \varepsilon \phi_\varepsilon + \frac{\partial L}{\partial \partial_\mu \phi} \partial_\mu (\varepsilon \phi_\varepsilon) = (\text{via eq. motion}) \\ &= \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial \partial_\mu \phi} \varepsilon \phi_\varepsilon \right) = \varepsilon \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial \partial_\mu \phi} \phi_\varepsilon \right) + \frac{\partial L}{\partial \partial_\mu \phi} \phi_\varepsilon \partial_\mu \varepsilon. \end{aligned} \quad (11.10)$$

We know that for $\partial_\mu \varepsilon = 0$ we have

$$\delta L = \varepsilon \partial_\mu K^\mu$$

i.e. the conserved current

$$J_N^\mu = \frac{\partial L}{\partial \partial_\mu \phi} \phi_\varepsilon - K^\mu$$

and

$$\partial_\mu \left(\frac{\partial L}{\partial \partial_\mu \phi} \phi_\varepsilon \right) - \partial_\mu K^\mu = 0.$$

Coming back to Eq.(11.10) for non constant ε we have

$$\delta L = \varepsilon \partial_\mu K^\mu + \frac{\partial L}{\partial \partial_\mu \phi} \phi_\varepsilon \partial_\mu \varepsilon$$

and thus

$$\delta I = \int d^4x \delta L = \int d^4x \left(\varepsilon \partial_\mu K^\mu + \frac{\partial L}{\partial \partial_\mu \phi} \phi_\varepsilon \partial_\mu \varepsilon \right) = \int d^4x \left(\varepsilon \partial_\mu K^\mu + K^\mu \partial_\mu \varepsilon + J_N^\mu \partial_\mu \varepsilon \right) = \int d^4x \partial_\mu \varepsilon J_N^\mu$$

i.e.

$$\delta I = \int d^4x \partial_\mu \varepsilon J_N^\mu$$

where J_N^μ is the Noether current.

Thus given a theory invariant under a global transformation the obtained result gives a hint on how to render the theory invariant under local transformations. In fact $\partial_\mu \varepsilon$ is the first term in the variation of a gauge field (1-form) under a gauge transformation. Thus it is suggested to couple the previous theory to a gauge field A_μ described by a gauge invariant Lagrangian L_A and coupled to the field ϕ by the term

$$- \int d^4x J_N^\mu A_\mu.$$

11.3 $N = 1$ supergravity

If we want to render a theory which is invariant under global supersymmetry transformations, invariant under local supersymmetry transformation the suggestion is to couple it to a gauge field with the nature of a spinor 1-form, i.e. $\psi_\mu dx^\mu$. We have already discussed in Section 1.15 such a structure which describes, going over to irreducible representations, a spin 3/2 particle, which will be the gravitino.

However it is immediately checked that the Rarita-Schwinger Lagrangian in presence of mass is not invariant under the gauge transformation

$$\psi_\mu \rightarrow \psi_\mu + \partial_\mu \epsilon.$$

Thus we must deal with the massless Rarita-Schwinger field. In order to put to zero the two unwanted spin 1/2 components of the field we have to impose a gauge condition. We shall choose $\gamma^\mu \psi_\mu = 0$ which we proved to be always attainable. Such condition implies through the equations of motion $\partial_\mu \psi^\mu = 0$. In fact from

$$\varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma = 0$$

we have with $\varepsilon^{0123} = -1$

$$0 = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \gamma_5 (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \partial_\rho \psi_\sigma = -i(\gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho) \partial_\rho \psi_\sigma = -2i(\gamma^\rho \gamma^\sigma - \eta^{\rho\sigma}) \partial_\rho \psi_\sigma = 2i \partial_\mu \psi^\mu.$$

We saw in Chapter 5 how one can formulate gravity both in the second and first order formalism. The formalism which provides the simplest proof of the invariance of $N = 1$ gravity under supersymmetric transformations is the so called 1.5 formalism which we are going to explain.

We start from the first order action

$$S_{SG} = S_{EC} + S_{RS} \quad (11.11)$$

with

$$S_{EC} = -\frac{1}{8} \int R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd} = -\frac{1}{4} \int e \, d^4x \, R_{\mu\nu}^{ab} e^\mu_a e^\nu_b$$

($\varepsilon_{0123} = 1$) and S_{RS} is the covariant transcription of the massless Rarita-Schwinger action

$$S_{RS} = \frac{i}{2} \int d^4x \, \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \gamma_5 D_\rho \psi_\sigma$$

where $\gamma_\nu = \gamma_a e^a_\nu$ and for the Levi-Civita antisymmetric tensor we have according to section (4.25) $\varepsilon^{0123} = -1$. For computational reasons the constants in front of the Einstein-Cartan action is different from the one adopted in Section (5.4). The only thing that matters is the ratio between the Einstein-Cartan and Rarita-Schwinger terms.

$$D_\rho = \partial_\rho + \omega_\rho$$

with as in Section 5.8

$$\omega_\rho = \Gamma_\rho^{ab} \Sigma_{ab} \equiv \omega_\rho^{ba} \Sigma_{ab}$$

and

$$\Sigma_{ab} = \frac{1}{8} (\gamma_a \gamma_b - \gamma_b \gamma_a) \equiv \frac{1}{4} \gamma_{ab}$$

and thus

$$\omega_\rho = \frac{1}{4} \Gamma_\rho^{ab} \gamma_{ab} .$$

We already know that S_{EC} is invariant under local Lorentz rotations and diffeomorphisms. By construction S_{RS} is invariant under local Lorentz rotations. With regard to diffeomorphisms, due to the presence of the antisymmetric symbol we have

$$\varepsilon^{\alpha\beta\gamma\delta} \bar{\psi}'_\alpha \gamma'_\beta \gamma_5 D'_\gamma \psi'_\delta = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \frac{\partial x^\rho}{\partial x'^\gamma} \frac{\partial x^\sigma}{\partial x'^\delta} \varepsilon^{\alpha\beta\gamma\delta} \bar{\psi}_\mu \gamma_\nu \gamma_5 D_\rho \psi_\sigma = \det \left(\frac{\partial x}{\partial x'} \right) \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \gamma_5 D_\rho \psi_\sigma$$

which makes the Rarita-Schwinger action invariant.

The torsion equation is

$$\begin{aligned} \delta_\Gamma S_{EC} + \delta_\Gamma S_{RS} &= 0 \\ \delta_\Gamma S_{EC} &= -\frac{1}{8} \int D \delta \Gamma^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd} = -\frac{1}{4} \int \delta \Gamma^{ab} \wedge S^c \wedge e^d \varepsilon_{abcd} = \\ &= -\frac{1}{8} \int e \, d^4x \, \delta \Gamma_r^{ab} S_{mn}^c \varepsilon_{abcd} \varepsilon^{rmnd} = -\frac{1}{8} \int e \, d^4x \, \delta \Gamma_r^{ab} S_{ms}^c \delta_{abc}^{rms} \end{aligned}$$

while

$$\delta_\Gamma S_{RS} = \frac{i}{8} \int e \, d^4x \, \delta \Gamma_r^{ab} \bar{\psi}_m \gamma_n \gamma_5 \gamma_{ab} \psi_s \varepsilon^{mnr s}$$

and thus

$$S_{mn}^c \delta_{abc}^{rmn} = -i\bar{\psi}_m \gamma_5 \gamma_n \gamma_{ab} \psi_s \varepsilon^{mnr s} = -\frac{i}{2} \bar{\psi}_m \gamma_5 \{\gamma_n, \gamma_{ab}\} \psi_s \varepsilon^{mnr s}$$

where the last expression is obtained by summing the second to the transposed and dividing by 2.

Using the identity

$$\gamma_5 \{\gamma_n, \gamma_{ab}\} = -2i \varepsilon_{nabc} \gamma^c \quad (11.12)$$

the above equation becomes

$$S_{mn}^c \delta_{abc}^{rmn} = \bar{\psi}_m \gamma^c \psi_n \delta_{abc}^{rmn}$$

trivially solved by

$$S_{mn}^c \equiv D_m e_n^a - D_n e_m^a = \bar{\psi}_m \gamma^c \psi_n \quad (11.13)$$

or

$$S^c = \bar{\psi}_\mu \gamma^c \psi_\nu \frac{dx^\mu \wedge dx^\nu}{2}.$$

But we know the solution of the torsion in terms of the torsion source to be unique and thus the found solution is the unique solution.

Recalling an old result of Section 5.9 we have

$$\Gamma_{abc} = \Gamma[e]_{abc} + \frac{1}{2}(S_{cab} + S_{acb} - S_{bca}). \quad (11.14)$$

The so called 1.5 formalism consists in the following: Substitute the expression (11.14), where now Γ is function of the e_μ^a and of the field ψ_μ , in the R^{ab} of the L_{EC} i.e.

$$R^{ab} = d\Gamma^{ab} + (\Gamma \wedge \Gamma)^{ab}$$

and in the RS action. Thus the 1.5 formalism is really a *second order formalism with a properly chosen Lagrangian*. We can forget now how we arrived to the written Lagrangian. The real point is to show that such a Lagrangian is invariant under 1) diffeomorphisms; 2) local Lorentz transformations; 3) properly defined local supersymmetry transformations. The invariance under 1) and 2) has already been proved. We come now to the local supersymmetry transformations.

The transformations are given by

$$\delta_Q e_\mu^a = \bar{\epsilon} \gamma^a \psi_\mu \quad (11.15)$$

and

$$\delta_Q \psi_\mu = D_\mu \epsilon \equiv (\partial_\mu + \frac{1}{4} \gamma_{ab} \Gamma_\mu^{ab}) \epsilon \quad (11.16)$$

where the last transformation is expected from the reasoning on the Noether currents. The great advantage of the 1.5 formalism is the following: In varying the action (11.11) under (11.15,11.16) we can ignore the variations of the connection induced by such variations because the connection solves the stationarity equation.

The variation of S_{EC} under a variation of e_μ^a is given by

$$\delta S_{EC} = -\frac{1}{4} \int R^{ab} \wedge e^c \wedge \delta e^d \varepsilon_{abcd} = -\frac{1}{4} \int R^{ab} \wedge e^c \wedge e^f \varepsilon_{abcd} \delta e_\mu^d e_f^\mu \quad (11.17)$$

$$= \frac{1}{2} \int e G_b^a e_\mu^a \delta e_\mu^b d^4x = \frac{1}{2} \int e G_b^a \bar{\varepsilon} \gamma^b \psi_a d^4x \quad (11.18)$$

and as already discussed we have ignored the variation induced on Γ .

With regard to the variation of the RS action we have

$$\begin{aligned} \delta_Q L_{RS} &= \delta_Q \left(\frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_a e_\nu^a \gamma_5 D_\rho \psi_\sigma \right) = \\ &= \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \delta_Q (\bar{\psi}_\mu) \gamma_a e_\nu^a \gamma_5 D_\rho \psi_\sigma + \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_a \delta_Q (e_\nu^a) \gamma_5 D_\rho \psi_\sigma + \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_a e_\nu^a \gamma_5 D_\rho \delta_Q \psi_\sigma. \end{aligned}$$

Performing an integration by parts in the first term we obtain for the previous expression

$$\begin{aligned} &- \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_a (D_\mu e_\nu^a) \gamma_5 D_\rho \psi_\sigma - \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_a e_\nu^a \gamma_5 D_\mu D_\rho \psi_\sigma \\ &+ \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_a \delta_Q (e_\nu^a) \gamma_5 D_\rho \psi_\sigma + \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_a e_\nu^a \gamma_5 D_\rho \delta_Q \psi_\sigma. \end{aligned} \quad (11.19)$$

The first and the third term due to the torsion constraint (11.13) cancel i.e. we have

$$- \varepsilon^{\mu\nu\rho\sigma} (\bar{\psi}_\mu \gamma^a \psi_\nu) (\bar{\varepsilon} \gamma_a \gamma_5 D_\rho \psi_\sigma) + 2 \varepsilon^{\mu\nu\rho\sigma} (\bar{\varepsilon} \gamma^a \psi_\nu) (\bar{\psi}_\mu \gamma_a \gamma_5 D_\rho \psi_\sigma) = 0 \quad (11.20)$$

as can be proved by using the Fierz identity (11.7) on the second term of Eq.(11.20) with $\bar{\lambda} \rightarrow \bar{\varepsilon}$, $\chi \rightarrow \gamma^a \psi_\nu$, $\bar{\psi} \rightarrow \bar{\psi}_\mu$ and $\phi \rightarrow \gamma_a \gamma_5 D_\rho \psi_\sigma$ keeping into account of the presence of the antisymmetric symbol $\varepsilon^{\mu\nu\rho\sigma}$ and of the properties (11.9). This is the only point in the proof where the Fierz rearrangement occurs.

Thus the variation of the RS action, exploiting the antisymmetry of the ε symbol, and taking into account the spinor nature of ε and on the ψ_σ , is given by

$$\begin{aligned} \delta_Q L_{RS} &= \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \gamma_5 [D_\rho, D_\sigma] \varepsilon - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_\nu \gamma_5 [D_\mu, D_\rho] \psi_\sigma \\ &= \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_\nu \gamma_5 \frac{1}{4} \gamma_{ab} \varepsilon R_{\rho\sigma}^{ab} - \frac{i}{4} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_\nu \gamma_5 \frac{1}{4} \gamma_{ab} \psi_\sigma R_{\mu\rho}^{ab} \\ &= \frac{i}{16} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_5 \gamma_\nu \gamma_{ab} \psi_\sigma R_{\mu\rho}^{ab} + \frac{i}{16} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_5 \gamma_{ab} \gamma_\nu \psi_\sigma R_{\mu\rho}^{ab} = \frac{i}{16} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_5 \{ \gamma_\nu, \gamma_{ab} \} \psi_\sigma R_{\mu\rho}^{ab} \\ &= \frac{i\varepsilon}{16} \varepsilon^{mnr s} \bar{\varepsilon} \gamma_5 \{ \gamma_n, \gamma_{ab} \} \psi_s R_{mr}^{ab}. \end{aligned} \quad (11.21)$$

Using again the identity (11.12) we have

$$\delta L_{RS} = \frac{e}{8} \varepsilon^{mars} \bar{\epsilon} \gamma^b \psi_s R_{mr}^{cd} \varepsilon_{acdb} = \frac{e}{8} R_{mr}^{cd} \delta_{cdb}^{mrs} \bar{\epsilon} \gamma^b \psi_s = -\frac{e}{2} G_b^s \bar{\epsilon} \gamma^b \psi_s$$

which cancels the variation of S_{EC} .

In conclusion we have proven the invariance of S_{SG} also under local supersymmetry transformations.

References

- [1] P. van Nieuwenhuizen, "Supergravity", Phys.Rep.68 (1981) 189
- [2] Y. Tani, "Introduction to supergravity in diverse dimension" hep-th/9802138

Chapter 12

Appendix

Derivation of the hamiltonian equations of motion and of the Poisson algebra of the constraints

We give here the details of the derivation of the equations of motion and of the Poisson algebra of the constraints in the hamiltonian formulation of general relativity. We saw in Section 7.4 that the action in hamiltonian form is

$$\begin{aligned} S_H &= \int dt \int_{\Sigma_t} d^D x (\pi^{ij} \dot{h}_{ij} - NH - N^i H_i) + 2 \int dt \int_{B_t} \sqrt{\sigma} d^{D-1} x (Nk - r_i \frac{\pi^{ij}}{\sqrt{h}} N_j) \\ &= \int dt L \end{aligned} \quad (12.1)$$

with

$$H = \sqrt{h} \left(-\mathcal{R} + \frac{\text{Tr}(\pi\pi)}{h} - \frac{\pi^2}{(D-1)h} \right), \quad H_i = -2\sqrt{h} D_j \left(\frac{\pi^j_i}{\sqrt{h}} \right)$$

r_i being the outward pointing unit normal to B_t as a sub-manifold of Σ_t and k the extrinsic curvature of intersection of Σ_t with B considered as a sub-manifold of Σ_t . The equations of motion are obtained by setting to zero the variations w.r.t. the independent variables which are N , N^i , π^{ij} , h_{ij} . We recall that π^i_j has to be understood as $\pi^{il} h_{lj}$ and thus it depends on h_{ij} and similarly for π_{ij} and that π^{ij} is not a tensor in D dimensions but a tensorial density.

We recall that in the hamiltonian dynamics the variation with respect to the ‘‘coordinates’’ h_{ij}, N, N_i has to be performed with vanishing variation at the boundary, while non such restriction can be imposed on the variation of the conjugate momenta π^{ij} .

The variations w.r.t. N and N^i give rise to the constraints

$$H = 0, \quad H_i = 0$$

To compute the variation w.r.t. π^{ij} and h_{ij} it is better to revert to the form

$$S_H = \int dt \int_{\Sigma_t} d^D x (\pi^{ij} \dot{h}_{ij} - NH - 2\pi^{ij} D_i N_j) + 2 \int dt \int_{B_t} \sqrt{\sigma} d^{D-1} x N k \quad (12.2)$$

The variation w.r.t. π^{ij} gives

$$0 = \frac{\delta L}{\delta \pi^{ij}} = \dot{h}_{ij} - \frac{2N}{\sqrt{h}} \left(\pi_{ij} - \frac{\pi h_{ij}}{D-1} \right) - D_i N_j - D_j N_i . \quad (12.3)$$

Equation (12.3) is just the inversion of Eq.(7.16).

We come now to the variation w.r.t. h_{ij} . Repeating in our D -dimensional case the procedure applied in Section 5.2 for the n -dimensional case, we have

$$\delta(N\sqrt{h}\mathcal{R}) = -\delta h_{ij} N \sqrt{h} (\mathcal{R}^{ij} - \frac{h^{ij}}{2} \mathcal{R}) + N \sqrt{h} D_i v^i \quad (12.4)$$

with

$$v^i = \delta \Gamma_{kl}^i h^{kl} - \delta \Gamma_{kl}^l h^{ki}$$

where the connections appearing here are the connections on Σ i.e. computed in terms of the metric h_{ij} .

The change in the connection induced by the change of the metric can be computed by the following general procedure.

Let D_l be the covariant derivative with the metric h_{ij} and $D_l + \Delta \Gamma_l$ the covariant derivative with the metric $\bar{h}_{ij} = h_{ij} + \Delta h_{ij}$. Metric compatibility gives

$$0 = (D_l + \Delta \Gamma_l) \bar{h}_{ij} = D_l \Delta h_{ij} - \Delta \Gamma_{il}^k \bar{h}_{kj} - \Delta \Gamma_{jl}^k \bar{h}_{ik} \equiv D_l \Delta h_{ij} - \Delta \Gamma_{jil} - \Delta \Gamma_{ijl} .$$

Then exploiting zero torsion i.e. the symmetry $\Delta \Gamma_{ijl} = \Delta \Gamma_{ilj}$ we have the exact formula

$$\Delta \Gamma_{ij}^l = \frac{1}{2} \bar{h}^{lk} (D_j \Delta h_{ik} + D_i \Delta h_{kj} - D_k \Delta h_{ij})$$

and for infinitesimal Δh_{ij}

$$\delta \Gamma_{ij}^l = \frac{1}{2} h^{lk} (D_j \delta h_{ik} + D_i \delta h_{kj} - D_k \delta h_{ij}) \quad (12.5)$$

giving

$$v^i = D_k \delta h^{ik} - h^{ik} h^{mn} D_k \delta h_{mn} .$$

Substituting we have

$$\int N \sqrt{h} D_i v^i d^D x = \int \partial_i (N \sqrt{h} v^i) dx^D - \int \sqrt{h} (D_k \delta h^{ik} - h^{ik} h^{mn} D_k \delta h_{mn}) D_i N d^D x .$$

The first contribution is canceled by the variation of the boundary term in (12.2) as it happens in the n -dimensional case of Section 6.3, while the second, integrated by parts, gives

$$\int \sqrt{h} \delta h_{ij} (D^i D^j N - h^{ij} D^l D_l N) d^D x . \quad (12.6)$$

We come now to the term

$$- \frac{N}{\sqrt{h}} \left(\pi^{ij} h_{il} h_{jm} \pi^{lm} - \frac{(\pi^{ij} h_{ij})^2}{D-1} \right) .$$

The dependence on h_{ij} here is algebraic (no derivatives are present) and the variation is easily computed to be

$$\delta h_{ij} \left(\frac{N h^{ij}}{2\sqrt{h}} (\pi^{kl} \pi_{kl} - \frac{\pi^2}{D-1}) - \frac{2N}{\sqrt{h}} (\pi^{ik} \pi_k^j - \frac{\pi \pi^{ij}}{D-1}) \right) . \quad (12.7)$$

With regard to the term

$$- 2\pi^{ij} D_i N_j$$

using Eq.(12.5) for the variation of D_i and using the diffeomorphism constraint $H_i = 0$ we have

$$\delta h_{ij} \left(\sqrt{h} D_l \left(\frac{N^l \pi^{ij}}{\sqrt{h}} \right) - \pi^{lj} D_l N^i - \pi^{li} D_l N^j \right) . \quad (12.8)$$

Summing (12.4) after using (12.6), to (12.7) and (12.8) we obtain Eq.(7.18) of Section 7.4.

We come now to the algebra of constraints.

$$\{A, B\} = \int d^D y \left(\frac{\delta A}{\delta h_{ij}(y)} \frac{\delta B}{\delta \pi^{ij}(y)} - \frac{\delta A}{\delta \pi^{ij}(y)} \frac{\delta B}{\delta h_{ij}(y)} \right) .$$

It is simpler to work with the constraints H_i and H weighted by test functions (C_0^∞ functions of $x^1 \dots x^D$) which we shall denote by $N^i(x)$, $N(x)$ and $N^i(x)$, $N'(x)$ like e.g.

$$\left\{ \int N^i(x) H_i(x) d^D x, \int N^l(x') H_l(x') d^D x' \right\} .$$

We begin computing

$$\begin{aligned} & \left\{ h_{ij}(x), \int N^l(y) H_l(y) d^D y \right\} = -2 \frac{\delta}{\delta \pi^{ij}(x)} \int d^D y \sqrt{h} N^l D_k \left(\frac{\pi_l^k}{\sqrt{h}} \right) \\ & = 2 \frac{\delta}{\delta \pi^{ij}(x)} \int d^D y \pi^{ml}(y) h_{lc}(y) D_m N^c = h_{il}(x) D_j N^l + h_{jl}(x) D_i N^l = \mathcal{L}_{\mathbf{N}} h_{ij}(x) . \end{aligned}$$

In addition

$$\begin{aligned} & \{\pi^{ij}(x), \int N^l(y) H_l(y) d^D y\} = 2 \frac{\delta}{\delta h_{ij}(x)} \int d^D y N^l \sqrt{h} D_k \left(\frac{\pi_l^k}{\sqrt{h}} \right) \\ & = -2 \frac{\delta}{\delta h_{ij}(x)} \int d^D y h_{nl} \pi^{mn} D_m N^l . \end{aligned}$$

Under a variation δh_{ij} of h_{ij} we have using the identity (12.5)

$$\delta(-2 \int d^D y h_{nl} \pi^{mn} D_m N^l) = -2 \int d^D y (\delta h_{nl} \pi^{mn} D_m N^l + \frac{1}{2} \pi^{ms} N^l D_l \delta h_{sm})$$

which after an integration by parts gives

$$\begin{aligned} & \int d^D y \delta h_{mn} \sqrt{h} (N^l D_l \frac{\pi^{mn}}{\sqrt{h}} - \frac{\pi^{ln}}{\sqrt{h}} D_l N^m - \frac{\pi^{lm}}{\sqrt{h}} D_l N^n + \frac{\pi^{mn}}{\sqrt{h}} D_l N^l) \\ & = \int d^D y \delta h_{mn} \mathcal{L}_N \pi^{mn} . \end{aligned}$$

The reason for the last equality is that π^{ij} is not a tensor in D dimensions but a tensorial density. Thus using

$$\mathcal{L}_N \sqrt{h} = \sqrt{h} D_l N^l \quad (12.9)$$

we have

$$\mathcal{L}_N \pi^{ij} = \sqrt{h} \mathcal{L}_N \frac{\pi^{ij}}{\sqrt{h}} + \frac{\pi^{ij}}{\sqrt{h}} \mathcal{L}_N \sqrt{h} = \sqrt{h} \mathcal{L}_N \frac{\pi^{ij}}{\sqrt{h}} + \pi^{ij} D_l N^l .$$

Thus we found that $\int d^D y N^l(y) H_l(y)$ generates the space diffeomorphisms on h_{ij} and on π^{ij} and thus on any functional of h_{ij} and π^{ij} , in particular on the constraints H_i and H . For the H_i we have

$$\mathcal{L}_N H_i = \sqrt{h} \mathcal{L}_N \frac{H_i}{\sqrt{h}} + H_i D_l N^l = N^l \partial_l H_i + H_l \partial_i N^l + H_i \partial_l N^l .$$

Summarizing we found

$$\{H_i(x), \int d^D y N^l H_l\} = N^l \partial_l H_i + H_l \partial_i N^l + H_i \partial_l N^l$$

which can be rewritten as

$$\{H_i(x), H_j(x')\} = H_i(x') \partial_j \delta(x, x') - H_j(x) \partial'_i \delta(x, x') .$$

With regard to

$$\{H(x), \int d^D y N^l H_l\} = \mathcal{L}_N H(x)$$

we notice that H is a scalar density and thus

$$\mathcal{L}_N H(x) = \sqrt{h} N^l \partial_l \frac{H}{\sqrt{h}} + \frac{H}{\sqrt{h}} \mathcal{L}_N \sqrt{h} = N^l \partial_l H + H \partial_l N^l$$

from which we have

$$\{H(x), H_i(x')\} = -H(x')\partial'_i\delta(x', x) .$$

The result is that the Poisson brackets with $\int d^D y N^l H_l$ is a canonical representation of the algebra of the diffeomorphisms in D dimensions.

We come now to the P.B.

$$\left\{ \int d^D x N(x)H(x), \int d^D x' N'(x')H(x') \right\} .$$

We note that h_{ij} intervenes in H both in algebraic way and also in non algebraic way i.e. through derivatives. On the other hand π^{ij} intervenes in H only in algebraic way. Moreover in the computation of the P.B. the algebraic-algebraic variation contribute always zero because for

$$\frac{\delta H(x)}{\delta h_{ij}(y)} = f_{h_{ij}}(x)\delta(x, y), \quad \frac{\delta H(x)}{\delta \pi^{ij}(y)} = f_{\pi^{ij}}(x)\delta(x, y)$$

we have

$$\begin{aligned} & \int d^D y (f_{h_{ij}}(x)\delta(x, y)f_{\pi^{ij}}(x')\delta(x', y)N(x)N'(x') \\ & - f_{\pi^{ij}}(x)\delta(x, y)f_{h_{ij}}(x')\delta(x', y)N(x)N'(x'))d^D x d^D x' = 0 . \end{aligned}$$

Thus only the non algebraic contribution of the variation of h_{ij} combined with the (algebraic) contributions of the variation of π^{ij} contribute. The non algebraic contribution of the variation of h_{ij} is given by Eq.(12.6)

$$- \int d^D x \sqrt{h} \delta h_{ij} (D^i D^j N - h^{ij} D^l D_l N) \quad (12.10)$$

and thus we have

$$\begin{aligned} & \int d^D y \frac{\delta \int d^D x N H}{\delta h_{ij}(y)} \frac{\delta \int d^D x' N' H}{\delta \pi^{ij}(y)} \\ & = - \int d^D y \sqrt{h} (D^i D^j N - h^{ij} D^l D_l N) 2 \left(\frac{\pi_{ij}}{\sqrt{h}} - \frac{\pi h_{ij}}{(D-1)\sqrt{h}} \right) N' \\ & = 2 \int d^D y \sqrt{h} D_i N D_j \left(\frac{\pi^{ij}}{\sqrt{h}} N' \right) . \end{aligned}$$

Summing the second part of the P.B. (i.e. subtract exchanging N with N') and integrating by parts we have

$$\begin{aligned} & \left\{ \int d^D x N(x)H(x) \int d^D x' N'(x')H(x') \right\} \\ & = 2 \int d^D y \sqrt{h} (N' D_i N - N D_i N') D_j \left(\frac{\pi^{ij}}{\sqrt{h}} \right) = \int d^D y (-N' D_i N + N D_i N') H^i \end{aligned}$$

equivalent to

$$\{H(x), H(x')\} = H^i(x)\partial_i\delta(x, x') - H^i(x')\partial'_i\delta(x, x') .$$