# Covariant Differential Identities and Conservation Laws in Metric-Torsion Theories of Gravitation. I. General Consideration

Robert R. Lompay[1] and Alexander N. Petrov[2]
[1)] *Department of Physics, Uzhgorod National University, Voloshyna str., 54, Uzhgorod, 88000, UKRAINE*[a)]
[2)] *Moscow M. V. Lomonosov State University, Sternberg Astronomical institute, Universitetskii pr., 13, Moscow, 119992, RUSSIA*[b)]

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Arbitrary diffeomorphically invariant metric-torsion theories of gravity are considered. It is assumed that Lagrangians of such theories contain derivatives of field variables (tensor densities of arbitrary ranks and weights) up to a second order only. The generalized Klein-Noether methods for constructing manifestly covariant identities and conserved quantities are developed. Manifestly covariant expressions are constructed without including auxiliary structures like a background metric. In the Riemann-Cartan space, the following *manifestly generally covariant results* are presented: (a) The complete generalized system of differential identities (the Klein-Noether identities) is obtained. (b) The generalized currents of three types depending on an arbitrary vector field displacements are constructed: they are the canonical Noether current, symmetrized Belinfante current and identically conserved Hilbert-Bergmann current. In particular, it is stated that the symmetrized Belinfante current does not depend on divergences in the Lagrangian. (c) The generalized boundary Klein theorem (third Noether theorem) is proved. (d) The construction of the generalized superpotential is presented in details, and questions related to its ambiguities are analyzed.

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## I. INTRODUCTION

Last decades, one can see an unprecedented active development of alternative theories of gravity, which modify general relativity (GR) in various ways[1–4]. Among them there are scalar-tensor theories[5], the Einstein-Cartan theory[6], the Lovelock theory in the general form[7] as well as its special cases, such as very popular Einstein-Gauss-Bonnet gravity[8,9], metric-affine theories[10,11], supergravity[12], $f(R)$-theories[13], Chern-Simons modifications of GR[14], Lovelock-Cartan theories[15], topologically massive gravity[16], topologically massive supergravity[17], new massive gravity[18], critical gravity[19], chiral gravity[20], various topological gauge theories of gravity and supergravity[21–26], etc.

Constructing the conservation laws (CLs) and conserved quantities (CQs) in an arbitrary field theory, including gravitational theories, is a main problem. Many above listed theories, presented in the second order formalism, are the metric-torsion theories. Therefore, there is a demand in *universal* expressions for CLs and CQs. Thus, in the present paper, we consider the metric-torsion theories only. It is the main goal of the current work to construct in a *manifestly generally covariant* form and analyze differential identities and conserved quantities, existing due to a diffeomorphic invariance of metric-torsion theories of gravity in the most general formulation. Besides being self-sufficient, the present paper (Paper I[27]) is the first one in a series of the works. In the second work (Paper II[28]), we plan to apply the developed here formalism to construct and study the conserved quantities, and to examine the structure of the field equations in metric-torsion theories, which have *manifestly generally covariant Lagrangians* (see the definition below). In the third work (Paper III[29]), we plan (1) to analyze a physical and geometrical meaning of conserved quantities (with taking into account surface terms), constructed in the first and second works; (2) to apply the obtained results to study some important solutions in the Lovelock-Cartan gravity and other theories with torsion.

To avoid ambiguities, let us state the definitions utilized hereinafter. We call a theory as *generally covariant* one if it is invariant with respect to general diffeomorphisms, unlike a *gauge covariant* theory that is invariant with respect to internal gauge transformations. Hence, it is clear that for both of these types of theories conserved currents have a definite *universal* structure[30–35]. Therefore, for the sake of universality and uniformity of the presentation, we call a theory as a *gauge-invariant theory* in wide sense if it is invariant under continuous transformations, parameters of which are *functions* of spacetime points. Such transformations we call as *gauge transformations*. On the other hand, the usual gauge theories with an internal gauge group we call the gauge theories of Utiyama-Yang-Mills type. We call an expression as *manifestly generally covariant* one if it is constructed (as a rule, by contractions) from explicitly covariant quantities

[a)]Electronic mail: rlompay@gmail.com
[b)]Electronic mail: alex.petrov55@gmail.com

(tensors, spinors, covariant derivatives), which are transformed in correspondence with linear homogeneous representations of the diffeomorphism group. Thus, it is evidently that a *manifestly generally covariant* expression is a *generally covariant* one. But, the converse is generally not true. For example, pseudotensors can be interpreted as *generally covariant* quantities because their expressions hold in arbitrary coordinate systems, but they cannot be presented in a *manifestly generally covariant* form because they are non-tensorial quantities.

In metric theories of gravity a construction of energy-momentum tensors and spin tensors of pure gravitational field meets well known obstacles – ambiguities appear unavoidably. The reason is the existence of the equivalence principle. During nearly hundred year history of GR — basic metric theory of gravity — numerous variants of expressions for energy, momentum and angular momentum of gravitational field were put forth. As a rule, these expressions are *generally covariant*. However, among them there are both tensorial expressions and non-tensorial ones (for example, pseudotensors). The latter are not so desirable, therefore they or methods of their construction are usually covariantized (i.e., reconstructed into *manifestly generally covariant* form). Frequently, such a covariantization is based on including an auxiliary structure, like a background metric, see, for example, Refs.[36–38] and also recent works (Refs.[39,40]). It is impossible to present more or less complete bibliography even in GR[41], particulary one can find reviews[38,42,43]. Significantly less attention was paid to constructing *manifestly generally covariant* CQs, where auxiliary structures are not used. Concerning earlier works, *only* Komar[44] has suggested a manifestly generally covariant superpotential in GR that has been modified in Refs.[45–47] and generalized in Refs.[48–54]. In the last years, up to our knowledge, *only* manifestly generally covariant charges are constructed in asymptotically anti-de Sitter gravity (see, e.g. Refs.[55–58], and references there in).

One of the main methods for constructing conserved quantities is the procedure suggested by Noether in 1918[59,60] (for alternative methods see, e.g., review in Ref.[42] and references therein). It is well known that Noether has proved two general theorems in her seminal work[59]. The first theorem states the existence of $r$ currents $\mathbf{J}_{(a)}$, $a = \overline{1, r}$ conserved on field equations. This follows from the invariance of the action functional under transformations presenting a finite $r$-parameters Lie group, and vice versa. The method to prove the first theorem gives a recipe for constructing such currents. The second theorem – the existence of a set of differential identities between the left hand sides of the field equations of motion (the Noether identities) follows from the invariance of the action functional under the gauge transformations, and vice versa.

It is not widely known that in the same work Noether has proved the statement that has not been formulated as a separate theorem. However, sometimes it is called as the third Noether theorem, or the boundary theorem (see, e.g., Refs.[61–65]). In 1915, almost three years prior to Noether, Hilbert in his known work[66,67] has constructed the energy-momentum vector $\mathbf{J}[\boldsymbol{\xi}]$ for the system of interacting gravitational and electromagnetic fields depending on an arbitrary vector field $\boldsymbol{\xi}$. Klein, examinating this current[68], and little earlier Noether (see comments in Ref.[68]), have found that the Hilbert current transfers into a divergence of an antisymmetric tensor $\boldsymbol{\theta}[\boldsymbol{\xi}]$ if the field equations of motion hold. Thus, the current is conserved identically. Therefore, according to Klein's and Noether's opinion, the Hilbert conservation law cannot be thought as a usual conservation law for the energy-momentum. As an answer, Hilbert has supposed (see comments in Ref.[68]) that an analogous situation could take place in all the generally covariant theories. The Hilbert assumption has been proved right. Noether has generalized the properties of the Hilbert current $\mathbf{J}[\boldsymbol{\xi}]$ on arbitrary gauge-invariant theories. Combining the results of the first and second theorems, she has shown that in an *arbitrary* gauge-invariant theory the Noether current $\mathbf{J}$, constructed according to the first theorem and with the use of the Noether identities, always can be completed up to the *identically* conserved current $\mathscr{J}$. Thus, unlike $\mathbf{J}$, $\mathscr{J}$ is conserved independently of satisfying equations of motion. From here the boundary theorem follows directly: in an *arbitrary* gauge-invariant theory the Noether current $\mathbf{J}$, constructed by the first theorem for a finite (global) subgroup of a gauge group is presented as a sum of two terms: the first vanishes on the equations of motion, whereas the second is expressed through a divergence of an antisymmetric tensor $\boldsymbol{\theta}$ — superpotential. At the same time, Noether did not give a rule for the superpotential construction.

It is also not widely known that the Noether identities are not a complete system of differential identities following from a gauge invariance of a theory. Noether studied the problem in an active collaboration with Klein, who independently obtained the results analogous to Noether's[68–70]. Noether remarked that her work[59] and Klein's work[69] "were mutually influential"[59,60] (see also comments by Klein and Hilbert in Ref.[68]). In work[69], Klein, considering an example of generally covariant metric theories, obtained a *complete* system of differential identities, from which the Noether identities follow. One of his identities is in fact the boundary theorem, whereas the others give recipe for constructing a superpotential.

Later, unfortunately, the above famous results by Hilbert, Klein and Noether have been almost forgotten. The studies of the identically conserved current $\mathscr{I}$ in generally covariant theories has been re-stated by Bergman[71] 30 years later. The existence of the superpotential $\boldsymbol{\theta}$ corresponding to the identically conserved Bergmann current has been stated by Zatzkis[72]. The existence of the identically conserved current $\mathscr{J}[\boldsymbol{\xi}]$ in generally covariant theories, which depends on an arbitrary vector $\boldsymbol{\xi}$ has been rediscovered by Bergmann and Shiller[73] in a special case. In the general case it was rediscovered by Mitskievich[37,74,75], who systematically studied the generalized current and have constructed the correspondent (generalized) superpotential $\boldsymbol{\theta}[\boldsymbol{\xi}]$.

The complete Klein system of identities has been rediscovered and studied in detail for constructing CQs by Trautman[76,77] (see also Refs.[78,79],[37,74,75],[80]). Little earlier the Klein-like identities has been stated by Utiyama[81,82] in $SU(N)$-invariant gauge theories. Just the above Trautman's and Utiyama's results became the basis for studying differential identities in gauge theories of gravity and the Einstein-Cartan theory[6,83–88], in supergravity[89,90], in metric-affine theories of gravity[11]. The Klein-Noether theorem in the general form, probably independently, has been rediscovered by Francaviglia and coauthors[91–94], Julia and Silva[32,33], Barnich and Brandt[34,35]. Recently, using the jet stratification technique and the variational bi-complex technique, the theorem has been stated in a very generic case (non-closed algebras, Grassmannian fields, graduate groups) in the works by Francaviglia *et. al.*[95–97] and by Sardanashvily *et. al.*[98,99]. To finalize a short historical discourse, we remark that the conclusion that a superpotential has to exist in GR directly follows from Einstein's work of 1916[100,101].

The novelty of our results is in the following:

- *Universality.* We consider an *arbitrary* diffeomorphically invariant classical field theories, Lagrangians of which contain derivatives of field variables (tensor densities of *arbitrary*, but fixed ranks and weights) up to the second order;

- *Manifest general covariance.* We develop manifestly generally covariant formalism, first, using *initially* generally covariant expressions (without using auxiliary structures, such as a background metric); second, all of our calculations, unlike many of aforementioned works, are manifestly generally covariant at *each and every steps*;

- *The torsion field is taken into account.* A spacetime under consideration is presented by an *arbitrary Riemann-Cartan space.* Both the torsion tensor and the metric tensor are the dynamical fields, the torsion coupling in the Lagrangian can be both minimal (through connection) and nonminimal (explicit).

A technique developed in the present work to analyze diffeomorphic invariance can be directly applied to both *manifestly covariant* and *gauge invariant* studies of gauge invariance properties of *arbitrary* nature field theories given in *Riemann-Cartan spacetime*, which could be classical gauge theories or theories with a local supersymmetries.

In the most of the present-day works related to analyzing general gauge theories (see, e.g., works by Julia and Silva[32,33], by Barnich and Brandt *et. al.*[34,35], by Obukhov *et. al.*[102–104], by Baykal and Delice[105], by Sardanashvily and Giachetta *et. al.*[98,99,106–109], by Francaviglia *et al.*[95,96,110–112]), unfortunately, authors frequently use rarely known formalisms for physicists, such as the aforementioned variational bi-complex, jet stratification, and also differential form technique, etc. Unlike them, we perform all the calculations and present the final results in the usual tensorial language.

The rest of the paper is organized as follows: In section II, we suggest the general Noether relation in a *manifestly generally covariant form*, find general expressions for the generalized (depending on an arbitrary infinitesimal vector field $\delta\boldsymbol{\xi}$ — displacement field) conserved current $\mathbf{J}[\delta\boldsymbol{\xi}]$ and Noether charge $Q[\delta\boldsymbol{\xi};\Sigma]$.

In section III, we develop the *complete manifestly covariant universal* system of differential identities, which take place in an *arbitrary* diffeomorphically invariant theory of the class under consideration. Thus, the results in the second part of section II and in section III present the *covariant* generalization of the Klein approach[68–70] (see also the work[71,76,77] and, especially, books[37,75]).

In section IV based on the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$, we obtain the identically conserved current $\mathscr{J}[\delta\boldsymbol{\xi}]$. Using the latter, we prove the generalized boundary Klein-Noether theorem in the manifestly generally covariant form. After that the generalized superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ is constructed and a problem of its ambiguity is analyzed. A physical meaning of the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$ and its connection to the usual Noether current $\mathbf{J}$, a numerical value of the conserved generalized charge $Q[\delta\boldsymbol{\xi};\Sigma]$, a physical meaning of the superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ are also discussed.

In section V, utilizing the generalized Belinfante procedure, the generalized symmetrized Noether current $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ is constructed. As the result, we show that this current is a linear combination of the Lagrangian derivatives of the action functional. Thus, it vanishes on the equations of motion. This means that *divergences in the Lagrangian do not contribute to the current.* This conclusion is a wide generalization of the claims made in Refs.[113,114].

Intermediate and cumbersome calculations are left in the appendixes. In Appendix A, we review (without a proof) basic facts of the Riemann-Cartan geometry. This could be used as a introduction into the world of the Riemann-Cartan geometry, assuming the reader is fluent in more simple Riemannian geometry.

In Appendix B, basic notions of irreducible representations of a symmetric group (group of permutations) for two- and three-indexes quantities are given. Using the Young projectors, we develop a *new* technique, which is employed in the main text.

In Appendix C, some general geometrical identities are proved. They are used for a simplification of the Klein system of identities and in analyzing ambiguities in the superpotential.

In Appendix D, the technique of Appendix B is used to solve a system of equations defining a superpotential and determining its general representations.

In the paper we use the following notations: Greek indexes $\alpha$, $\beta$, …, $\mu$, $\nu$, … take values of 0, 1, …, $D$

and numerate spacetime coordinates $x \stackrel{def}{=} \{x^\alpha\}$, partial $\boldsymbol{\partial} \stackrel{def}{=} \{\partial_\alpha\} \stackrel{def}{=} \{\partial/\partial x^\alpha\}$ and covariant $\boldsymbol{\nabla} \stackrel{def}{=} \{\nabla_\alpha\}$, $\stackrel{*}{\boldsymbol{\nabla}} \stackrel{def}{=} \{\stackrel{*}{\nabla}_\alpha\}$ derivatives, and spacetime tensor components of fields also. Small Latin indexes from the middle of alphabet $i,\ j,\ \ldots,\ z$ take values of $1,\ 2,\ \ldots,\ D$ and numerate space components. Coordinate $x^0$ is a time one, whereas coordinates $\vec{x} \stackrel{def}{=} \{x^i\}$ are space ones. Capital Latin indexes $A,\ B,\ \ldots,$ are collective and numerate components of the full set of the physical fields $\boldsymbol{\Phi} \stackrel{def}{=} \{\Phi^A(x)\}$ (containing both gravitational and matter fields) and are related to $1, 2, \ldots, N$. At last, small Latin indexes from the beginning of the alphabet $a, b, \ldots, h$ numerate components of matter (non-gravitational) fields $\boldsymbol{\varphi} \stackrel{def}{=} \{\varphi^a(x)\}$ and take values of $1,\ 2,\ \ldots,\ n$.

As usual, for a twice repeated index, the Einstein summation rule is assumed. Indexes in parentheses need to be symmetrized; whereas, indexes in brackets needs to be antisymmetrized, for example,

$$A_{(\alpha\beta)} = \frac{1}{2}\left(A_{\alpha\beta} + A_{\beta\alpha}\right), \qquad A_{[\alpha\beta]} = \frac{1}{2}\left(A_{\alpha\beta} - A_{\beta\alpha}\right).$$

Two vertical lines inside the brackets () and [] mean that indexes between them do not participate in symmetrization/antisymmetrization, for example,

$$\begin{aligned} A_{(\alpha|\beta\gamma|\delta)} &= \frac{1}{2}\left(A_{\alpha\beta\gamma\delta} + A_{\delta\beta\gamma\alpha}\right),\\ A_{[\alpha|\beta|\gamma]} &= \frac{1}{2}\left(A_{\alpha\beta\gamma} - A_{\gamma\beta\alpha}\right). \end{aligned}$$

Covariant derivatives $\boldsymbol{\nabla}$ and $\stackrel{*}{\boldsymbol{\nabla}}$, and a sign convention for the curvature tensor $\mathbf{R} \stackrel{def}{=} \{R^\alpha{}_{\beta\gamma\delta}\}$ and the torsion tensor $\mathbf{T} \stackrel{def}{=} \{T^\alpha{}_{\beta\gamma}\}$ are derived in Appendix A.

The speed of light in vacuum is set to one.

## II. THE GENERAL NOETHER IDENTITY. GENERALIZED NOETHER'S CURRENT AND CHARGE

We consider a classical field theory determined by the action functional

$$I[\boldsymbol{\Phi};\Sigma_{1,2}] = \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\mathscr{L}, \tag{1}$$

in space-time $\mathcal{C}(1,D)$ (see Appendix A). Here, $dx \stackrel{def}{=} dx^0dx^1\ldots dx^D$; integration is provided over an arbitrary $(D+1)$-dimensional volume in $\mathcal{C}(1,D)$ restricted by two spacelike $D$-dimensional hypersurfaces $\Sigma_1$ and $\Sigma_2$; Lagrangian $\mathscr{L}$ is a local function of a set of field variables $\boldsymbol{\Phi}(x) = \{\Phi^A(x); A = \overline{1,N}\}$ and their first and second derivatives.

Consider a *total variation of the action* $\bar{\delta}I[\boldsymbol{\Phi};\Sigma_{1,2}]$ initiated by both the general variations of the field variables $\delta\boldsymbol{\Phi}$ and the boundary hypersurfaces $\delta\Sigma_{1,2}$:

$$\left\{\begin{array}{rcl} \boldsymbol{\Phi}(x) & \to & \boldsymbol{\Phi}'(x) = \boldsymbol{\Phi}(x) + \delta\boldsymbol{\Phi}(x);\\ \Sigma_{1,2}(x) & \to & \Sigma'_{1,2}(x) = \Sigma_{1,2}(x) + \delta\Sigma_{1,2}(x). \end{array}\right. \tag{2}$$

By the definition, one has

$$\begin{aligned} &\bar{\delta}I[\boldsymbol{\Phi};\Sigma_{1,2}] \stackrel{def}{=} I[\boldsymbol{\Phi}+\delta\boldsymbol{\Phi};\Sigma_{1,2}+\delta\Sigma_{1,2}] - I[\boldsymbol{\Phi};\Sigma_{1,2}]\\ &= \left(I[\boldsymbol{\Phi}+\delta\boldsymbol{\Phi};\Sigma_{1,2}+\delta\Sigma_{1,2}] - I[\boldsymbol{\Phi}+\delta\boldsymbol{\Phi};\Sigma_{1,2}]\right)\\ &+ \left(I[\boldsymbol{\Phi}+\delta\boldsymbol{\Phi};\Sigma_{1,2}] - I[\boldsymbol{\Phi};\Sigma_{1,2}]\right). \end{aligned} \tag{3}$$

We assume that the field variables and their derivatives vanish sufficiently fast at spatial infinity. Then, up to the first order terms in variations, for the first parenthesis in (3) we obtain

$$\begin{aligned} &\delta_\Sigma I[\boldsymbol{\Phi};\Sigma_{1,2}] \stackrel{def}{=} I[\boldsymbol{\Phi};\Sigma_{1,2}+\delta\Sigma_{1,2}] - I[\boldsymbol{\Phi};\Sigma_{1,2}]\\ &= \left(\int_{\Sigma_1+\delta\Sigma_1}^{\Sigma_2+\delta\Sigma_2} - \int_{\Sigma_1}^{\Sigma_2}\right) dx\,\sqrt{-g}\mathscr{L}\\ &= \left(\int_{\Sigma_2} - \int_{\Sigma_1}\right) d\sigma_\mu\,\mathscr{L}\delta x^\mu = \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\stackrel{*}{\nabla}_\mu\left(\mathscr{L}\delta x^\mu\right) \end{aligned} \tag{4}$$

where the generalized Gauss theorem (A42) was employed. The second parenthesis in (3) is the *functional variation of the action*:

$$\begin{aligned} &\delta_\Phi I[\boldsymbol{\Phi};\Sigma_{1,2}] \stackrel{def}{=} I[\boldsymbol{\Phi}+\delta\boldsymbol{\Phi};\Sigma_{1,2}] - I[\boldsymbol{\Phi};\Sigma_{1,2}]\\ &= \int_{\Sigma_1}^{\Sigma_2} dx\,\delta\left(\sqrt{-g}\mathscr{L}\right). \end{aligned} \tag{5}$$

We consider generally covariant theories of the most popular type, when $\delta_\Phi I$ is present always as

$$\begin{aligned} &\delta_\Phi I = \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\frac{\Delta I}{\Delta\Phi^A}\delta\Phi^A\\ &+ \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\stackrel{*}{\nabla}_\mu\left\{K^\mu|_A\delta\Phi^A + Ł^{\beta\mu}|_A\nabla_\beta\delta\Phi^A\right\}. \end{aligned} \tag{6}$$

Hereinafter, $\Delta I/\Delta\Phi^A$ is defined by the variational derivative $\delta I/\delta\Phi^A$, which is the operator of equations of motion,

$$\frac{\Delta I}{\Delta\Phi^A} \stackrel{def}{=} \frac{1}{\sqrt{-g}}\frac{\delta I}{\delta\Phi^A}, \tag{7}$$

$\mathbf{K} \stackrel{def}{=} \{K^\mu|_A\}$ and $\mathbf{L} \stackrel{def}{=} \{Ł^{\beta\mu}|_A\}$ are local functions of the field variables $\boldsymbol{\Phi}$ and their first and second derivatives, and are defined in an unique way (without ambiguities) by the Lagrangian $\mathscr{L}$. Combining the expressions (3) – (6), one finds

$$\begin{aligned} &\bar{\delta}I = \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\frac{\Delta I}{\Delta\Phi^A}\delta\Phi^A\\ &+ \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\stackrel{*}{\nabla}_\mu\left(K^\mu|_A\delta\Phi^A + Ł^{\beta\mu}|_A\nabla_\beta\delta\Phi^A + \mathscr{L}\delta x^\mu\right). \end{aligned} \tag{8}$$

We name (2) as *the symmetry transformation* (see Refs.[64,76,90,115]), if it induces the total variation of the action functional $\bar{\delta}I$ in the form

$$\bar{\delta}I = \int_{\Sigma_1}^{\Sigma_2} dx\,\sqrt{-g}\stackrel{*}{\nabla}_\mu\left(\delta\Lambda^\mu\right) \tag{9}$$

where $\delta\mathbf{\Lambda} \stackrel{def}{=} \{\delta\Lambda^\mu\}$ are infinitesimal local functions of $\mathbf{\Phi}$, $\delta\mathbf{\Phi}$ and their derivatives ($\delta\mathbf{\Lambda}$ is not a variation). Equating (8) with (9) and taking into account that the volume of integration is arbitrary, one finds the relation

$$\overset{*}{\nabla}_\mu J^\mu\left[\delta\mathbf{\Phi}, \delta x, \delta\mathbf{\Lambda}\right] + \frac{\Delta I}{\Delta\Phi^A}\delta\Phi^A \equiv 0, \tag{10}$$

which is called *the general Noether identity* (the main identity). Here,

$$\begin{aligned} &J^\mu\left[\delta\mathbf{\Phi}, \delta x, \delta\mathbf{\Lambda}\right] \\ &\stackrel{def}{=} K^\mu|_A\delta\Phi^A + Ł^{\beta\mu}|_A\nabla_\beta\delta\Phi^A + \mathscr{L}\delta x^\mu - \delta\Lambda^\mu. \end{aligned} \tag{11}$$

If equations of motion $\Delta I/\Delta\Phi^A = 0$ hold then the identity (10) transforms into the continuity equation

$$\overset{*}{\nabla}_\mu J^\mu\left[\delta\mathbf{\Phi}, \delta x, \delta\mathbf{\Lambda}\right] = 0 \tag{12}$$

where $\delta\mathbf{\Phi}$, $\delta x$ and $\delta\mathbf{\Lambda}$ denote the symmetry transformation.

Let us consider the infinitesimal diffeomorphisms

$$\left\{\begin{array}{rcl} \delta x^\mu &=& \delta\xi^\mu(x); \\ \delta\Phi^A(x) &=& \delta_\xi\Phi^A(x) \end{array}\right. \tag{13}$$

as the symmetry transformation, $\delta\boldsymbol{\xi} = \{\delta\xi^\mu(x)\}$ is an arbitrary infinitesimal vector (*displacement vector*). Hereafter, we assume the Lagrangian $\mathscr{L}$ is a generally covariant scalar. Then, one has

$$\delta\mathbf{\Lambda}[\delta\boldsymbol{\xi}] \stackrel{def}{=} \delta\mathbf{\Lambda}|_{\{\delta\mathbf{\Phi}\,=\,\delta_\xi\mathbf{\Phi};\ \delta x\,=\,\delta\boldsymbol{\xi}\}} = 0. \tag{14}$$

The variations of the field variables have the general form

$$\begin{aligned} &\delta_\xi\Phi^A(x) \\ &= \Phi_\alpha|^A\delta\xi^\alpha + \Phi_\alpha{}^\beta|^A\nabla_\beta\delta\xi^\alpha + \Phi_\alpha{}^{\beta\gamma}|^A\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha + \ldots \end{aligned} \tag{15}$$

where $\{\Phi_\alpha|^A\}$, $\{\Phi_\alpha{}^\beta|^A\}$, $\{\Phi_\alpha{}^{(\beta\gamma\ldots)}|^A = \Phi_\alpha{}^{\beta\gamma\ldots}|^A\}$ are local functions of $\mathbf{\Phi}$ and their derivatives, which are defined uniquely by the transformation properties of $\mathbf{\Phi}$. We consider only the case when $\delta_\xi\mathbf{\Phi}$ contains the first two terms on the right hand side of (15). However, the discussion can be easily extended to a more generic case, say, of metric-affine theories of gravity, for which the third term in (15) is also nonzero.

A vector

$$\mathbf{J}[\delta\boldsymbol{\xi}] \stackrel{def}{=} \mathbf{J}\left[\delta\mathbf{\Phi}, \delta x, \delta\mathbf{\Lambda}\right]|_{\{\delta\mathbf{\Phi}\,=\,\delta_\xi\mathbf{\Phi};\ \delta x\,=\,\delta\boldsymbol{\xi};\ \delta\mathbf{\Lambda}\,=\,0\}}, \tag{16}$$

whose components are obtained after the substitution of (13), (14) and (15) into (11):

$$\begin{aligned} &J^\mu[\delta\boldsymbol{\xi}] = \left\{K^\mu|_A\Phi_\alpha|^A + \mathscr{L}\delta^\mu_\alpha + Ł^{\nu\mu}|_A\nabla_\nu\Phi_\alpha|^A\right\}\delta\xi^\alpha \\ &+ \left\{K^\mu|_A\Phi_\alpha{}^\beta|^A + Ł^{\beta\mu}|_A\Phi_\alpha|^A + Ł^{\nu\mu}|_A\nabla_\nu\Phi_\alpha{}^\beta|^A\right\}\nabla_\beta\delta\xi^\alpha \\ &+ \left\{Ł^{\gamma\mu}|_A\Phi_\alpha{}^\beta|^A\right\}\nabla_\gamma\nabla_\beta\delta\xi^\alpha \end{aligned} \tag{17}$$

we will call as the *generalized Noether current*.

It is insightful to compare the results obtained here in the tensorial formalism with the corresponding results in a popular formalism of differential forms. It is impossible to perform a full comparison; fortunately, there is no needs for this. To show how the comparison could be done it is enough to show this for any particular example. The formulae (16) and (17), which are among the main formulae of the formalism, are well suited for this goal. Thus, the Noether current $D$-form $\mathbf{j}[\delta\boldsymbol{\xi}] = \mathbf{\Theta}[\mathbf{\Phi}, \delta_\xi\mathbf{\Phi}] - \delta\boldsymbol{\xi}\cdot\mathscr{L}$ constructed in works[49–52] coincides with the current (16), (17) up to a sign: $\mathbf{j}[\delta\boldsymbol{\xi}] \stackrel{def}{=} -J^\mu[\delta\boldsymbol{\xi}]\,\mathbf{d}\boldsymbol{\sigma}_\mu$. The symplectic potential $D$-form $\mathbf{\Theta}$ in the tensorial notations is presented as $\mathbf{\Theta}[\mathbf{\Phi}, \delta_\xi\mathbf{\Phi}] = -\left\{K^\mu|_A\delta\Phi^A + Ł^{\beta\mu}|_A\nabla_\beta\delta\Phi^A\right\}\Big|_{\{\delta\mathbf{\Phi}\,=\,\delta_\xi\mathbf{\Phi}\}}\mathbf{d}\boldsymbol{\sigma}_\mu$. When a detailed structure of the last expression is analyzed, one finds the exact correspondence with (17).

Next, transform the last term in the expression (17) following the formula (B11) in Appendix B 2:

$$\begin{aligned} &\left\{Ł^{\gamma\mu}|_A\Phi_\alpha{}^\beta|^A\right\}\nabla_\gamma\nabla_\beta\delta\xi^\alpha \\ &= \left\{\frac{1}{2}R^\varepsilon{}_{\alpha\kappa\lambda}Ł^{\kappa\mu}|_A\Phi_\varepsilon{}^\lambda|^A\right\}\delta\xi^\alpha \\ &+ \left\{-\frac{1}{2}T^\beta{}_{\kappa\lambda}Ł^{\kappa\mu}|_A\Phi_\alpha{}^\lambda|^A\right\}\nabla_\beta\delta\xi^\alpha \\ &+ \left\{Ł^{\gamma\mu}|_A\Phi_\alpha{}^\beta|^A\right\}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha. \end{aligned} \tag{18}$$

Taking this into account, find

$$J^\mu[\delta\boldsymbol{\xi}] = U_\alpha{}^\mu\delta\xi^\alpha + M_\alpha{}^{\beta\mu}\nabla_\beta\delta\xi^\alpha + N_\alpha{}^{\beta\gamma\mu}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha \tag{19}$$

where[116]

$$\left\{\begin{aligned} U_\alpha{}^\mu &\stackrel{def}{=} \mathscr{L}\delta^\mu_\alpha + K^\mu|_A\Phi_\alpha|^A + Ł^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha|^A + \frac{1}{2}R^\varepsilon{}_{\alpha\kappa\lambda}\Phi_\varepsilon{}^\lambda|^A\right); &(20)\\ M_\alpha{}^{\beta\mu} &\stackrel{def}{=} K^\mu|_A\Phi_\alpha{}^\beta|^A + Ł^{\beta\mu}|_A\Phi_\alpha|^A + Ł^{\kappa\mu}|_A\left(\nabla_\kappa\Phi_\alpha{}^\beta|^A - \frac{1}{2}T^\beta{}_{\kappa\lambda}\Phi_\alpha{}^\lambda|^A\right); &(21)\\ N_\alpha{}^{\beta\gamma\mu} &\stackrel{def}{=} Ł^{(\gamma|\mu}|_A\Phi_\alpha{}^{|\beta)}|^A. &(22) \end{aligned}\right.$$

Note that after the symmetrization in (22), we get

$$N_\alpha{}^{(\beta\gamma)\mu} = N_\alpha{}^{\beta\gamma\mu}. \tag{23}$$

Also, for the diffeomorphisms (13), (14) and (15) the general Noether identity (10) can be rewritten in the form:

$$\overset{*}{\nabla}_\mu J^\mu[\delta\boldsymbol{\xi}] \equiv -I_\alpha \delta\xi^\alpha - I_\alpha{}^\beta \nabla_\beta \delta\xi^\alpha \tag{24}$$

where

$$\begin{cases} I_\alpha \overset{def}{=} \dfrac{\Delta I}{\Delta \Phi^A} \Phi_\alpha|^A; & (25) \\ I_\alpha{}^\beta \overset{def}{=} \dfrac{\Delta I}{\Delta \Phi^A} \Phi_\alpha{}^\beta|^A. & (26) \end{cases}$$

Now, we define the *generalized Noether charge* as

$$Q[\delta\boldsymbol{\xi};\Sigma] \overset{def}{=} \int\limits_\Sigma d\sigma_\mu\, J^\mu[\delta\boldsymbol{\xi}] \tag{27}$$

where $\Sigma$ is a spacelike $D$-dimensional hypersurface in $\mathcal{C}(1,D)$.

Let the equations of motion $\Delta I/\Delta\Phi^A = 0$ be satisfied, then the relation (24) acquires the form of the continuity equation $\overset{*}{\nabla}_\mu J^\mu[\delta\boldsymbol{\xi}] = 0$ for the current $\mathbf{J}[\delta\boldsymbol{\xi}] = \{J^\mu[\delta\boldsymbol{\xi}]\}$. Next, if additionally the field variables and their derivatives vanish fast enough at a spatial infinity, then (24) leads to conservation of the generalized charge

$$\frac{\Delta I}{\Delta \Phi^A} = 0 \qquad \Rightarrow \qquad Q[\delta\boldsymbol{\xi};\Sigma_1] = Q[\delta\boldsymbol{\xi};\Sigma_2] \tag{28}$$

meaning that its value is the same on each of hypersurfaces $\Sigma$.

Note that the above conclusions are valid for *arbitrary* vectors $\delta\boldsymbol{\xi}$, *not* just for Killing vectors. Therefore, the aforementioned conservation laws are not connected with existence or absence of a spacetime group of motions. After series of works by Bergmann's group[73,117,118], who have studied this situation in general relativity, the conclusion was reached that the charge $Q[\delta\boldsymbol{\xi};\Sigma]$ is the *generator of infinitesimal diffeomorphisms* (13), (15). Bergmann *et al.* have utilized the canonical formalism where on the equations of motion one has

$$\begin{aligned} \bar{\delta} I|_{eq.mot.} &= \int\limits_{\Sigma_1}^{\Sigma_2} dx\, \overset{*}{\nabla}_\mu J^\mu[\delta\boldsymbol{\xi}] = \left( \int\limits_{\Sigma_2} - \int\limits_{\Sigma_1} \right) d\sigma_\mu\, J^\mu[\delta\boldsymbol{\xi}] \\ &= Q[\delta\boldsymbol{\xi};\Sigma]|_{\Sigma_2} - Q[\delta\boldsymbol{\xi};\Sigma]|_{\Sigma_1}. \end{aligned} \tag{29}$$

In the framework of the Lagrangian formalism, the same conclusion follows from the *Schwinger dynamical principle*[119] (see also Refs.[120,121]).

In the case when a spacetime has a continuous group of motion $\mathscr{K}_r$ with $r$ independent parameters, there are $r$ linearly independent Killing vector fields $\boldsymbol{\chi}_{(a)} \overset{def}{=} \{\chi_{(a)}{}^\mu; a = \overline{1,r}\}$. Then, defining infinitesimal displacement vectors $\delta\boldsymbol{\xi}$ as

$$\delta\boldsymbol{\xi}(x) = \delta\boldsymbol{\chi}(x) \overset{def}{=} \delta\varepsilon^{(a)} \boldsymbol{\chi}_{(a)}(x), \tag{30}$$

one obtains for the generalized current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (19):

$$\mathbf{J}[\delta\boldsymbol{\chi}] = \delta\varepsilon^{(a)} \mathbf{J}_{(a)} \tag{31}$$

where $\{\delta\varepsilon^{(a)}\}$ is the set of infinitesimal *constant* transformation parameters of $\mathscr{K}_r$. The quantities

$$\mathbf{J}_{(a)} \overset{def}{=} \{J_{(a)}{}^\mu\}; \tag{32}$$

$$J_{(a)}{}^\mu \overset{def}{=} U_\alpha{}^\mu \chi_{(a)}{}^\alpha + M_\alpha{}^{\beta\mu} \nabla_\beta \chi_{(a)}{}^\alpha + N_\alpha{}^{\beta\gamma\mu} \nabla_{(\gamma}\nabla_{\beta)} \chi_{(a)}{}^\alpha \tag{33}$$

are just the conserved currents constructed according to the *first Noether theorem* for the group of motions $\mathscr{K}_r$.

## III. THE KLEIN AND NOETHER IDENTITIES

Using the generalized Leibnitz rule (A34), let us open explicitly the left hand side of the general Noether identity (24) where the current is introduced in (19):

$$\begin{aligned} \overset{*}{\nabla}_\mu J^\mu[\delta\boldsymbol{\xi}] &= \left\{ \overset{*}{\nabla}_\mu U_\alpha{}^\mu \right\} \delta\xi^\alpha \\ &+ \left\{ U_\alpha{}^\beta + \overset{*}{\nabla}_\mu M_\alpha{}^{\beta\mu} \right\} \nabla_\beta \delta\xi^\alpha \\ &+ \left\{ M_\alpha{}^{\beta\gamma} + \overset{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} \right\} \nabla_\gamma \nabla_\beta \delta\xi^\alpha \\ &+ \left\{ N_\alpha{}^{\beta\gamma\delta} \right\} \nabla_\delta \nabla_\gamma \nabla_\beta \delta\xi^\alpha. \end{aligned} \tag{34}$$

Using the formula (B11), we transform the third term on the R.H.S. in the same way as in (18). Keeping in mind the property (23), one obtains

$$\begin{aligned} &\left\{ M_\alpha{}^{\beta\gamma} + \overset{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} \right\} \nabla_\gamma \nabla_\beta \delta\xi^\alpha \\ &= \left\{ \frac{1}{2} R^\varepsilon{}_{\alpha\kappa\lambda} M_\varepsilon{}^{\lambda\kappa} \right\} \delta\xi^\alpha + \left\{ -\frac{1}{2} T^\beta{}_{\kappa\lambda} M_\alpha{}^{\lambda\kappa} \right\} \nabla_\beta \delta\xi^\alpha \\ &+ \left\{ M_\alpha{}^{(\beta\gamma)} + \overset{*}{\nabla}_\mu N_\alpha{}^{(\beta\gamma)\mu} \right\} \nabla_{(\gamma}\nabla_{\beta)} \delta\xi^\alpha. \end{aligned} \tag{35}$$

Transformation of the fourth term on the R.H.S. in (34) is more complicated, which is worked out it in Appendix B 4. The finalized result is presented in (B30) as well as

$$\begin{aligned} \left\{ N_\alpha{}^{\beta\gamma\delta} \right\} \nabla_\delta \nabla_\gamma \nabla_\beta \delta\xi^\alpha &= \left\{ -\frac{1}{3} N_\kappa{}^{\lambda\mu\nu} \left( \nabla_\lambda R^\kappa{}_{\alpha\mu\nu} + \frac{1}{2} T^\sigma{}_{\mu\nu} R^\kappa{}_{\alpha\lambda\sigma} \right) \right\} \delta\xi^\alpha \\ &+ \left\{ \frac{1}{3} N_\alpha{}^{\lambda\mu\nu} \left( 2R^\beta{}_{\lambda\mu\nu} + \nabla_\lambda T^\beta{}_{\mu\nu} + \frac{1}{2} T^\sigma{}_{\mu\nu} T^\beta{}_{\lambda\sigma} \right) - N_\kappa{}^{\beta\mu\nu} R^\kappa{}_{\alpha\mu\nu} \right\} \nabla_\beta \delta\xi^\alpha \\ &+ \left\{ N_\alpha{}^{\beta\mu\nu} T^\gamma{}_{\mu\nu} \right\} \nabla_{(\gamma}\nabla_{\beta)} \delta\xi^\alpha + \left\{ N_\alpha{}^{\beta\gamma\delta} \right\} \nabla_{(\delta}\nabla_\gamma\nabla_{\beta)} \delta\xi^\alpha. \end{aligned} \tag{36}$$

Substituting (35) and (36) into formula (34), then into (24), one obtains an equivalent representation of the general Noether identity:

$$\begin{aligned}&\left\{\overset{*}{\nabla}_\mu U_\alpha{}^\mu - \frac{1}{2}M_\lambda{}^{\mu\nu}R^\lambda{}_{\alpha\mu\nu} - \frac{1}{3}N_\kappa{}^{\lambda\mu\nu}\left(\nabla_\lambda R^\kappa{}_{\alpha\mu\nu} + \frac{1}{2}T^\sigma{}_{\mu\nu}R^\kappa{}_{\alpha\lambda\sigma}\right)\right\}\delta\xi^\alpha\\
&+\left\{U_\alpha{}^\beta + \left(\overset{*}{\nabla}_\mu M_\alpha{}^{\beta\mu} + \frac{1}{2}M_\alpha{}^{\mu\nu}T^\beta{}_{\mu\nu}\right) + \frac{1}{3}N_\alpha{}^{\lambda\mu\nu}\left(2R^\beta{}_{\lambda\mu\nu} + \nabla_\lambda T^\beta{}_{\mu\nu} + \frac{1}{2}T^\sigma{}_{\mu\nu}T^\beta{}_{\lambda\sigma}\right) - N_\kappa{}^{\beta\mu\nu}R^\kappa{}_{\alpha\mu\nu}\right\}\nabla_\beta\delta\xi^\alpha\\
&+\left\{M_\alpha{}^{(\beta\gamma)} + \overset{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} + N_\alpha{}^{(\beta|\mu\nu}T^{|\gamma)}{}_{\mu\nu}\right\}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha + \left\{N_\alpha{}^{(\beta\gamma\delta)}\right\}\nabla_{(\delta}\nabla_\gamma\nabla_{\beta)}\delta\xi^\alpha \equiv -I_\alpha\delta\xi^\alpha - I_\alpha{}^\beta\nabla_\beta\delta\xi^\alpha.\end{aligned} \tag{37}$$

Notice that this identity is valid, when each function from the set $\{\delta\xi^\alpha, \partial_\beta\delta\xi^\alpha, \partial_\gamma\partial_\beta\delta\xi^\alpha, \partial_\delta\partial_\gamma\partial_\beta\delta\xi^\alpha\}$ has an arbitrary values at every world point. Then, opening (37) explicitly, one can equate to zero the coefficients in front of each function independently and obtain the system of identities. Such a system is not manifestly covariant. However, one can transfer to another set of arbitrary functions $\{\delta\xi^\alpha, \nabla_\beta\delta\xi^\alpha, \nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha, \nabla_{(\delta}\nabla_\gamma\nabla_{\beta)}\delta\xi^\alpha\}$. Because the Jacobian of the transformation is not degenerated, one can use the second set as equivalent instead to the first one. Thus, equating to zero the coefficients at the functions of the second set in identity (37), one obtain the covariant system of identities equivalent to (37):

$$\overset{*}{\nabla}_\mu U_\alpha{}^\mu - \frac{1}{2}M_\lambda{}^{\mu\nu}R^\lambda{}_{\alpha\mu\nu} - \frac{1}{3}N_\kappa{}^{\lambda\mu\nu}\left(\nabla_\lambda R^\kappa{}_{\alpha\mu\nu} + \frac{1}{2}T^\sigma{}_{\mu\nu}R^\kappa{}_{\alpha\lambda\sigma}\right) \equiv -I_\alpha; \tag{38}$$

$$\begin{aligned}&U_\alpha{}^\beta + \left(\overset{*}{\nabla}_\mu M_\alpha{}^{\beta\mu} + \frac{1}{2}M_\alpha{}^{\mu\nu}T^\beta{}_{\mu\nu}\right)\\
&\quad+\frac{1}{3}N_\alpha{}^{\lambda\mu\nu}\left(2R^\beta{}_{\lambda\mu\nu} + \nabla_\lambda T^\beta{}_{\mu\nu} + \frac{1}{2}T^\sigma{}_{\mu\nu}T^\beta{}_{\lambda\sigma}\right) - N_\kappa{}^{\beta\mu\nu}R^\kappa{}_{\alpha\mu\nu} \equiv -I_\alpha{}^\beta;\end{aligned} \tag{39}$$

$$M_\alpha{}^{(\beta\gamma)} + \overset{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} + N_\alpha{}^{(\beta|\mu\nu}T^{|\gamma)}{}_{\mu\nu} \equiv 0; \tag{40}$$

$$N_\alpha{}^{(\beta\gamma\delta)} \equiv 0. \tag{41}$$

The equations (38)–(41) present *complete manifestly covariant universal* system of differential identities, which is valid in an *arbitrary* diffeomorfically invariant field theory. Originally the system, analogous to the above, has been obtained in a non-covariant form by Klein[69] for purely metric theories of gravity. Therefore we will name system (38) – (41) as the *Klein identities.*

In Appendixes C 2 and C 3, we show that the Klein identities (38) and (39) can be rewritten in the form

$$\overset{*}{\nabla}_\mu\left(U_\alpha{}^\mu - \frac{1}{3}N_\lambda{}^{\mu\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right) - \frac{1}{2}\left(M_\lambda{}^{[\rho\sigma]} - \frac{2}{3}\overset{*}{\nabla}_\mu N_\lambda{}^{\mu[\rho\sigma]} + \frac{1}{3}N_\lambda{}^{[\rho|\mu\nu}T^{|\sigma]}{}_{\mu\nu}\right)R^\lambda{}_{\alpha\rho\sigma} \equiv -I_\alpha \tag{42}$$

and

$$\begin{aligned}&\left(U_\alpha{}^\beta - \frac{1}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right)\\
&+\overset{*}{\nabla}_\mu\left(M_\alpha{}^{[\beta\mu]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]} + \frac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}\right) + \frac{1}{2}\left(M_\alpha{}^{[\rho\sigma]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\rho\sigma]} + \frac{1}{3}N_\alpha{}^{[\rho|\kappa\lambda}T^{|\sigma]}{}_{\kappa\lambda}\right)T^\beta{}_{\rho\sigma}\\
&+\overset{*}{\nabla}_\mu\left(M_\alpha{}^{(\beta\mu)} + \overset{*}{\nabla}_\lambda N_\alpha{}^{\beta\mu\lambda} + N_\alpha{}^{(\beta|\rho\sigma}T^{|\mu)}{}_{\rho\sigma}\right) - \overset{*}{\nabla}_\mu\overset{*}{\nabla}_\lambda N_\alpha{}^{(\beta\mu\lambda)} \equiv -I_\alpha{}^\beta,\end{aligned} \tag{43}$$

respectively. At the beginning, note that due to identities (40) and (41) the last two terms on the left hand side of (43) are equal to zero. Next, subtract the divergence $\overset{*}{\nabla}_\beta$

of (43) from the identity (42), taking into account the identity (C2) where one sets

$$\theta_\alpha{}^{\beta\mu} = M_\alpha{}^{[\beta\mu]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]} + \frac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}. \quad (44)$$

After, we obtain the new identity

$$\overset{*}{\nabla}_\mu I_\alpha{}^\mu - I_\alpha \equiv 0 \quad (45)$$

that is the *Noether identity* rewritten in a manifestly covariant form. All of these mean that instead of the Klein system (38)–(41), one can use the equivalent *Klein-Noether system of identities*:

$$\left\{\begin{aligned}
&\overset{*}{\nabla}_\mu I_\alpha{}^\mu \equiv I_\alpha; && (46)\\
&\left(U_\alpha{}^\beta - \frac{1}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right) + \overset{*}{\nabla}_\mu\left(M_\alpha{}^{[\beta\mu]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]} + \frac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}\right) \\
&\qquad + \frac{1}{2}\left(M_\alpha{}^{[\rho\sigma]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\rho\sigma]} + \frac{1}{3}N_\alpha{}^{[\rho|\kappa\lambda}T^{|\sigma]}{}_{\kappa\lambda}\right)T^\beta{}_{\rho\sigma} \equiv -I_\alpha{}^\beta; && (47)\\
&M_\alpha{}^{(\beta\gamma)} + \overset{*}{\nabla}_\mu N_\alpha{}^{\beta\gamma\mu} + N_\alpha{}^{(\beta|\mu\nu}T^{|\gamma)}{}_{\mu\nu} \equiv 0; && (48)\\
&N_\alpha{}^{(\beta\gamma\delta)} \equiv 0. && (49)
\end{aligned}\right.$$

## IV. THE GENERALIZED NOETHER SUPERPOTENTIAL. THE BOUNDARY KLEIN-NOETHER THEOREM

Substituting $I_\alpha$ from (46) into the general Noether identity (24), one obtains another identity

$$\overset{*}{\nabla}_\mu \mathscr{J}^\mu[\delta\boldsymbol{\xi}] \equiv 0, \quad (50)$$

which has a meaning of the continuity equation for the current defined as $\mathscr{J}[\delta\boldsymbol{\xi}] \overset{def}{=} \{\mathscr{J}^\mu[\delta\boldsymbol{\xi}]\}$, where

$$\begin{aligned}\mathscr{J}^\mu[\delta\boldsymbol{\xi}] &\overset{def}{=} (U_\alpha{}^\mu + I_\alpha{}^\mu)\,\delta\xi^\alpha \\ &+ M_\alpha{}^{\beta\mu}\nabla_\beta\,\delta\xi^\alpha + N_\alpha{}^{\beta\gamma\mu}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha.\end{aligned} \quad (51)$$

It is evidently that the current $\mathscr{J}[\delta\boldsymbol{\xi}]$ is connected with the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (19) by the relation:

$$J^\mu[\delta\boldsymbol{\xi}] = -I_\alpha{}^\mu\delta\xi^\alpha + \mathscr{J}^\mu[\delta\boldsymbol{\xi}]. \quad (52)$$

Note that identity (50) takes place *independently of equations of motion*. Then, keeping in mind identity (C1), one should conclude that the current in (50) can be represented in the form

$$\mathscr{J}^\mu[\delta\boldsymbol{\xi}] = \overset{*}{\nabla}_\nu\theta^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2}\theta^{\rho\sigma}[\delta\boldsymbol{\xi}]T^\mu{}_{\rho\sigma} \quad (53)$$

where

$$\boldsymbol{\theta}[\delta\boldsymbol{\xi}] \overset{def}{=} \{\theta^{\mu\nu}[\delta\boldsymbol{\xi}]\}\,; \qquad \theta^{[\mu\nu]}[\delta\boldsymbol{\xi}] = \theta^{\mu\nu}[\delta\boldsymbol{\xi}] \quad (54)$$

is an antisymmetric tensor — the *generalized Noether superpotential*.[122] The formulae (52) and (53) represented in the formalism of the differential forms are equivalent (on the equations of motion) to the relation $\mathbf{j}[\delta\boldsymbol{\xi}] = \mathbf{dQ}[\delta\boldsymbol{\xi}]$[49–52], where $\mathbf{j}[\delta\boldsymbol{\xi}]$ is the Noether current $D$-form (see discussion above after formula (17)); the Noether charge $(D-1)$-form $\mathbf{Q}$ in the tensorial notations is presented as $\mathbf{Q}[\delta\boldsymbol{\xi}] = -\frac{1}{2!}\theta^{\mu\nu}[\delta\boldsymbol{\xi}]\mathbf{ds}_{\mu\nu}$.

A superpotential in (53) is not defined uniquely. Indeed, if

$$\theta'^{\mu\nu} \overset{def}{=} \theta^{\mu\nu} + \left(\overset{*}{\nabla}_\lambda\theta^{\mu\nu\lambda} + \theta^{[\mu|\rho\sigma}T^{|\nu]}{}_{\rho\sigma}\right) \overset{def}{=} \theta^{\mu\nu} + \Delta\theta^{\mu\nu} \quad (55)$$

is another superpotential where

$$\theta^{[\mu\nu\lambda]} = \theta^{\mu\nu\lambda}, \quad (56)$$

then

$$\begin{aligned}\mathscr{J}'^\mu &= \overset{*}{\nabla}_\nu\theta'^{\mu\nu} + \frac{1}{2}\theta'^{\rho\sigma}T^\mu{}_{\rho\sigma} \\
&= \left[\overset{*}{\nabla}_\nu\theta^{\mu\nu} + \frac{1}{2}\theta^{\rho\sigma}T^\mu{}_{\rho\sigma}\right] + \left[\overset{*}{\nabla}_\nu\left(\overset{*}{\nabla}_\lambda\theta^{\mu\nu\lambda} + \theta^{[\mu|\rho\sigma}T^{|\nu]}{}_{\rho\sigma}\right) + \frac{1}{2}\left(\overset{*}{\nabla}_\lambda\theta^{\rho\sigma\lambda} + \theta^{[\rho|\kappa\lambda}T^{|\sigma]}{}_{\kappa\lambda}\right)T^\mu{}_{\rho\sigma}\right] \\
&\overset{def}{=} \mathscr{J}^\mu + \Delta\mathscr{J}^\mu.\end{aligned} \quad (57)$$

However, it is easily to show (see Appendix C 4) that

$$\Delta \mathscr{J}^{\mu} \equiv 0, \tag{58}$$

therefore

$$\mathscr{J}'^{\mu}[\delta\boldsymbol{\xi}] = \mathscr{J}^{\mu}[\delta\boldsymbol{\xi}]. \tag{59}$$

Now, let us construct the superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ corresponding to the current (51). We assume that it has the form

$$\theta^{\mu\nu}[\delta\boldsymbol{\xi}] = A_\alpha{}^{\mu\nu}\delta\xi^\alpha + B_\alpha{}^{\beta\mu\nu}\nabla_\beta\delta\xi^\alpha \tag{60}$$

where coefficients $A_\alpha{}^{[\mu\nu]} = A_\alpha{}^{\mu\nu}$ and $B_\alpha{}^{\beta[\mu\nu]} = B_\alpha{}^{\beta\mu\nu}$ do not depend on $\delta\boldsymbol{\xi}$ and its derivatives. Thus, one has to find the tensors $\mathbf{A} \stackrel{def}{=} \{A_\alpha{}^{\mu\nu}\}$ and $\mathbf{B} \stackrel{def}{=} \{B_\alpha{}^{\beta\mu\nu}\}$. Substituting (60) into (53), one obtains an expression for the current $\mathscr{J}[\delta\boldsymbol{\xi}]$:

$$\begin{aligned}\mathscr{J}^{\mu}[\delta\boldsymbol{\xi}] = & \left\{\left(\stackrel{*}{\nabla}_\nu A_\alpha{}^{\mu\nu} + \frac{1}{2}A_\alpha{}^{\rho\sigma}T^\mu{}_{\rho\sigma}\right) + \frac{1}{2}B_\lambda{}^{\rho\sigma\mu}R^\lambda{}_{\alpha\rho\sigma}\right\}\delta\xi^\alpha \\ & + \left\{-A_\alpha{}^{\beta\mu} + \left(\stackrel{*}{\nabla}_\lambda B_\alpha{}^{\beta\mu\lambda} + \frac{1}{2}B_\alpha{}^{\beta\rho\sigma}T^\mu{}_{\rho\sigma} - \frac{1}{2}B_\alpha{}^{\rho\sigma\mu}T^\beta{}_{\rho\sigma}\right)\right\}\nabla_\beta\delta\xi^\alpha + \left\{-B_\alpha{}^{\beta\gamma\mu}\right\}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha.\end{aligned} \tag{61}$$

Equating this expression to the current (51), we get the system of equations defining $\mathbf{A}$ and $\mathbf{B}$:

$$\left(\stackrel{*}{\nabla}_\nu A_\alpha{}^{\mu\nu} + \frac{1}{2}A_\alpha{}^{\rho\sigma}T^\mu{}_{\rho\sigma}\right) + \frac{1}{2}B_\lambda{}^{\rho\sigma\mu}R^\lambda{}_{\alpha\rho\sigma} = U_\alpha{}^\mu + I_\alpha{}^\mu; \tag{62}$$

$$-A_\alpha{}^{\beta\mu} + \left(\stackrel{*}{\nabla}_\lambda B_\alpha{}^{\beta\mu\lambda} + \frac{1}{2}B_\alpha{}^{\beta\rho\sigma}T^\mu{}_{\rho\sigma} - \frac{1}{2}B_\alpha{}^{\rho\sigma\mu}T^\beta{}_{\rho\sigma}\right) = M_\alpha{}^{\beta\mu}; \tag{63}$$

$$-B_\alpha{}^{(\beta\gamma)\mu} = N_\alpha{}^{\beta\gamma\mu}. \tag{64}$$

A general solution of this system (see Appendix D) reads

$$\begin{aligned}A_\alpha{}^{\mu\nu} = & -M_\alpha{}^{[\mu\nu]} + \frac{2}{3}\left(\stackrel{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_\alpha{}^{\nu]\rho\sigma}\right) \\ & + \left(\stackrel{*}{\nabla}_\lambda c_\alpha{}^{\mu\nu\lambda} + c_\alpha{}^{[\mu|\rho\sigma}T^{|\nu]}{}_{\rho\sigma}\right);\end{aligned} \tag{65}$$

$$B_\alpha{}^{\lambda\mu\nu} = -\frac{4}{3}N_\alpha{}^{\lambda[\mu\nu]} + c_\alpha{}^{\lambda\mu\nu}. \tag{66}$$

where $\{c_\alpha{}^{[\lambda\mu\nu]} = c_\alpha{}^{\lambda\mu\nu}\}$ is an undefined antisymmetrical tensor.

Now, recall the ambiguity in the superpotential definition (55). We set there

$$\theta^{\mu\nu\lambda}[\delta\boldsymbol{\xi}] = C_\alpha{}^{\mu\nu\lambda}\delta\xi^\alpha. \tag{67}$$

where $\{C_\alpha{}^{[\mu\nu\lambda]} = C_\alpha{}^{\mu\nu\lambda}\}$ is an arbitrary antisymmetrical tensor. Then it is easily to find that an ambiguity presented by (55) appears in $\mathbf{A}$ and $\mathbf{B}$ in the form:

$$A'_\alpha{}^{\mu\nu} = A_\alpha{}^{\mu\nu} + \left(\stackrel{*}{\nabla}_\lambda C_\alpha{}^{\mu\nu\lambda} + C_\alpha{}^{[\mu|\rho\sigma}T^{|\nu]}{}_{\rho\sigma}\right); \tag{68}$$

$$B'_\alpha{}^{\beta\mu\nu} = B_\alpha{}^{\beta\mu\nu} + C_\alpha{}^{\beta\mu\nu}. \tag{69}$$

It is not surprisingly that the ambiguity in (65) and (66) is the same as in (68) and (69), respectively. But the latter does not contribute to the current, see (58), and consequently, it does not contribute to the charge. Then, without loss of a generality, one can set $c_\alpha{}^{\lambda\mu\nu} = 0$, after that (65) and (66) transfer to

$$A_\alpha{}^{\mu\nu} = -M_\alpha{}^{[\mu\nu]} + \frac{2}{3}\left(\stackrel{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_\alpha{}^{\nu]\rho\sigma}\right); \tag{70}$$

$$B_\alpha{}^{\lambda\mu\nu} = -\frac{4}{3}N_\alpha{}^{\lambda[\mu\nu]}. \tag{71}$$

Thus,

$$\begin{aligned}&\theta^{\mu\nu}[\delta\boldsymbol{\xi}] \\ &= \left\{-M_\alpha{}^{[\mu\nu]} + \frac{2}{3}\left(\stackrel{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_\alpha{}^{\nu]\rho\sigma}\right)\right\}\delta\xi^\alpha \\ &+ \left\{-\frac{4}{3}N_\alpha{}^{\beta[\mu\nu]}\right\}\nabla_\beta\delta\xi^\alpha.\end{aligned} \tag{72}$$

In a more simple case of a field theory in a Riemannian spacetime, a covariant formula of the *same form* (72) originally was presented at the workshop[123] (see also Ref.[38] and references there in). The difference is that the superpotential (72) is constructed without a background metric for initial variables of the theory, whereas the superpotential suggested in Ref.[123] is intended for perturbations in a curved *background* spacetime.

Substituting the expression (53) into formula (52), integrating it over a spacelike $D$-dimensional hypersurface $\Sigma$ and using the Stockes rule (A43), we rewrite the generalized Noether charge (27) in the form

$$Q[\delta\boldsymbol{\xi};\Sigma] = -\int_{\Sigma} d\sigma_\mu\, I_\alpha{}^\mu \delta\xi^\alpha + \frac{1}{2!}\oint_{\partial\Sigma} ds_{\mu\nu}\, \theta^{\mu\nu}[\delta\boldsymbol{\xi}]. \quad (73)$$

The above relation is a special case (in an integral form) of a more general statement: *the boundary Klein theorem*[69] or *the third Noether theorem*[59] (see also Refs.[32,33,63–65,76,92–94,98,99]) that reads as *in an arbitrary gauge-invariant theory the Noether current is presented by a sum of two terms, the first vanishes on equations of motion, the second is a divergence of a superpotential.* Note that the ambiguity in definition of the superpotential presented in (55) disappears in definition of the charge (73) by the Stokes theorem. It is not surprisingly because as we already know that the ambiguity does not contribute to the generalized Noether current.

The structure of the charge (73) in the Lagrangian formulation is analogous to the structure of the diffeomorphism generators in the Hamiltonian formulation of general relativity (GR). Indeed, the first term on the right hand side of (73) vanishes on equations of motion, see (26). Thus, the value of the charge (73) is defined by the second term on the right hand side (surface integral of a superpotential) only. The Hamiltonians in GR have the same property: the first its part presents integrals of constraints over hypersurface $\Sigma$ and disappears. Then the value of the Hamiltonians in GR is defined by the second part: a surface integral over the boundary $\partial\Sigma$ of $\Sigma$. An assumption that surface terms and their contributions in the Lagrangian formalism are equivalent to the correspondent ones in the Hamiltonian formalism in an *arbitrary gauge-invariant theory* has been formulated in an explicit form in Ref.[33].

In earlier works in the Hamiltonian GR, all the boundary terms were ignored which led to the problem of "zero Hamiltonian" (or "frozen formalism")[124,125]. The role of the boundary terms has been studied and clarified in Refs.[126,127] based on the requirement of well defined variation of the Hamiltonian action and well defined Poisson brackets (see also Ref.[128]). A consideration of the boundary terms in the generators of canonical transformations initiates an extension of the canonical formalism with inclusion of fields at the boundary $\partial\Sigma$. This important problem has been formulated and studied in Refs.[129–136].

## V. THE GENERALIZED SYMMETRIZED NOETHER CURRENT

From now we call the generalized Noether current $\mathbf{J}[\delta\boldsymbol{\xi}]$ (19) and the generalized superpotential $\boldsymbol{\theta}[\delta\boldsymbol{\xi}]$ (72) as the generalized *canonical* Noether current and the generalized *canonical* superpotential, respectively.

Recall that the canonical current $\mathbf{J}[\delta\boldsymbol{\xi}]$ contains derivatives of a displacement vector $\boldsymbol{\nabla}\delta\boldsymbol{\xi}$, $\boldsymbol{\nabla}\boldsymbol{\nabla}\delta\boldsymbol{\xi}$. In this section, we construct a new current $\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}]$, instead of the canonical one $\mathbf{J}[\delta\boldsymbol{\xi}]$, with the property that it does not contain derivatives of $\delta\boldsymbol{\xi}$. In other words, we search for

$$\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}] = \overset{sym}{U}{}_{\alpha}{}^{\mu}\delta\xi^\alpha. \quad (74)$$

Why is such a property important?

*First*, as we have remarked in Sec. II and in Sec. IV (see discussions after formulae (28) and (73), respectivelly), under the transition to canonical formalism the current $\mathbf{J}[\delta\boldsymbol{\xi}]$ becomes a generator of infinitesimal diffeomorphisms with the parameters $\delta\boldsymbol{\xi}$. Therefore this generator, like generators of infinitesimal canonical transformations in a field theory, *should be* proportional to parameters of transformations *only* (like in (74)), and not should be proportional to the derivatives. In the cases when derivatives appear, one has to suppress them.

*Second*, the form (74) is more compact. In the case when a spacetime belongs to a group of motion (exact or asymptotic), dynamic quantities presented by $\mathbf{J}[\delta\boldsymbol{\chi}]$ and based on Killing vectors $\delta\boldsymbol{\chi}$ of this group are constructed with using all the tensors $\mathbf{U}$, $\mathbf{M}$ and $\mathbf{N}$. At the same time, dynamic quantities based on (74), presented by $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\chi}]$, are constructed with using $\overset{sym}{\mathbf{U}} \overset{def}{=} \{\overset{sym}{U}{}_{\alpha}{}^{\mu}\}$ only. Of course, according to the Ockham's razor argument, the latter is preferred.

In Paper II[28] of the current series of works, we will show that in manifestly generally covariant theories a tensor $\mathbf{U}$ contains the canonical energy-momentum tensor (EMT): $\mathbf{t} \overset{def}{=} \{t^\mu{}_\nu\}$, and a tensor $\mathbf{M}$ contains the spin tensor (ST): $\mathbf{s} \overset{def}{=} \{s^\pi{}_{[\rho\sigma]} = s^\pi{}_{\rho\sigma}\}$. It is well known that in a general case canonical EMT is not symmetrical: $t^{\nu\mu} \neq t^{\mu\nu}$. Owing to this property, even for a field theory in Minkowski space, it is not possible to construct a *total* conserved angular momentum with using EMT only (it is necessary to use ST as well). Thus, a total conserved angular momentum is constructed as a sum of two terms representing orbital and spin momenta. For a symmetrical EMT the converse is true: a *total* conserved angular momentum is constructed by using EMT *only* (without an additional ST). Therefore, it is desirable to construct a symmetrical EMT. Originally a procedure reconstructing a canonical EMT into a symmetrical EMT in a field theory in Minkowski space (*symmetrization procedure*) has been suggested in Belinfante's works[137,138139].

In the terms of the current, under the Belinfante symmetrization procedure a spin term disappears from the explicit consideration. Therefore, the reconstruction of the canonical current $\mathbf{J}[\delta\boldsymbol{\xi}]$ into the current (74) without derivatives of $\delta\boldsymbol{\xi}$ (i.e., without the spin term) is just a generalization of the Belinfante procedure. By the requirement (74) the tensor $\overset{sym}{\mathbf{U}}$ is equal to the *generalized symmetrized* EMT $\overset{sym}{\mathbf{t}}$. However, one has to keep in mind that in a general case a symmetrized EMT need not be symmetrical (see, e.g., a symmetrized EMT for perturbation in GR on curved backgrounds in Refs.[140,141]).

Now, let us search for a *generalized symmetrized Noether current* $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$. Because a new current has to be also differentially conserved, we construct it by adding an antisymmetrical tensor $\mathscr{B}[\delta\boldsymbol{\xi}] \overset{def}{=} \left\{\mathscr{B}^{[\mu\nu]}[\delta\boldsymbol{\xi}] = \mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}]\right\}$, similarly to (53):

$$\overset{sym}{J}{}^{\mu}[\delta\boldsymbol{\xi}] \overset{def}{=} J^{\mu}[\delta\boldsymbol{\xi}] - \left(\overset{*}{\nabla}_{\nu}\mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2}\mathscr{B}^{\rho\sigma}[\delta\boldsymbol{\xi}]T^{\mu}{}_{\rho\sigma}\right) \tag{75}$$

We call this formula as a *generalized Belinfante relation*, and a tensor $\mathscr{B}[\delta\boldsymbol{\xi}]$ — as a *generalized Belinfante tensor*.

Assume that

$$\mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] = A_{\alpha}{}^{\mu\nu}\delta\xi^{\alpha} + B_{\alpha}{}^{\beta\mu\nu}\nabla_{\beta}\delta\xi^{\alpha} \tag{76}$$

where

$$A_{\alpha}{}^{[\mu\nu]} = A_{\alpha}{}^{\mu\nu}; \qquad B_{\alpha}{}^{\beta[\mu\nu]} = B_{\alpha}{}^{\beta\mu\nu} \tag{77}$$

are tensors, which are to be determined. Then,

$$\begin{aligned}&\overset{*}{\nabla}_{\mu}\mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] + \frac{1}{2}\mathscr{B}^{\rho\sigma}[\delta\boldsymbol{\xi}]T^{\mu}{}_{\rho\sigma} = \left\{\overset{*}{\nabla}_{\nu}A_{\alpha}{}^{\mu\nu} + \frac{1}{2}A_{\alpha}{}^{\rho\sigma}T^{\mu}{}_{\rho\sigma} + \frac{1}{2}B_{\lambda}{}^{\rho\sigma\mu}R^{\lambda}{}_{\alpha\rho\sigma}\right\}\delta\xi^{\alpha}\\&+\left\{-A_{\alpha}{}^{\beta\mu} + \overset{*}{\nabla}_{\nu}B_{\alpha}{}^{\beta\mu\nu} + \frac{1}{2}B_{\alpha}{}^{\beta\rho\sigma}T^{\mu}{}_{\rho\sigma} - \frac{1}{2}B_{\alpha}{}^{\rho\sigma\mu}T^{\beta}{}_{\rho\sigma}\right\}\nabla_{\beta}\delta\xi^{\alpha} + \left\{-B_{\alpha}{}^{\beta\gamma\mu}\right\}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^{\alpha}.\end{aligned} \tag{78}$$

Substituting (74), (19), (78) into (75) and equating coefficients at $\{\delta\xi^{\alpha}\}$, $\{\nabla_{\beta}\delta\xi^{\alpha}\}$, $\left\{\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^{\alpha}\right\}$, one obtains the system of equations for determining the tensors $\mathbf{A}$, $\mathbf{B}$, $\overset{sym}{\mathbf{U}}$:

$$\left\{\begin{aligned}\overset{sym}{U}{}_{\alpha}{}^{\mu} &= U_{\alpha}{}^{\mu} - \left(\overset{*}{\nabla}_{\nu}A_{\alpha}{}^{\mu\nu} + \frac{1}{2}A_{\alpha}{}^{\rho\sigma}T^{\mu}{}_{\rho\sigma}\right) - \frac{1}{2}B_{\lambda}{}^{\rho\sigma\mu}R^{\lambda}{}_{\alpha\rho\sigma}; && (79)\\ 0 &= M_{\alpha}{}^{\beta\gamma} + A_{\alpha}{}^{\beta\gamma} - \left(\overset{*}{\nabla}_{\nu}B_{\alpha}{}^{\beta\mu\nu} + \frac{1}{2}B_{\alpha}{}^{\beta\rho\sigma}T^{\mu}{}_{\rho\sigma} - \frac{1}{2}B_{\alpha}{}^{\rho\sigma\mu}T^{\beta}{}_{\rho\sigma}\right); && (80)\\ 0 &= N_{\alpha}{}^{\beta\gamma\mu} + B_{\alpha}{}^{(\beta\gamma)\mu}. && (81)\end{aligned}\right.$$

The system (80)-(81) for $\mathbf{A}$ and $\mathbf{B}$ exactly coincides with one in (63)-(64); therefore, its solution is given by (70)-(71). As a consequence of (71), one has

$$B_{\lambda}{}^{[\rho\sigma]\mu} = \frac{2}{3}N_{\lambda}{}^{\mu[\rho\sigma]}. \tag{82}$$

Substituting (70), (82) into (79), we obtain

$$\begin{aligned}\overset{sym}{U}{}_{\alpha}{}^{\mu} &= \left(U_{\alpha}{}^{\mu} - \frac{1}{3}N_{\lambda}{}^{\mu\rho\sigma}R^{\lambda}{}_{\alpha\rho\sigma}\right)\\&+\overset{*}{\nabla}_{\nu}\left[M_{\alpha}{}^{[\mu\nu]} - \frac{2}{3}\left(\overset{*}{\nabla}_{\lambda}N_{\alpha}{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_{\alpha}{}^{\nu]\rho\sigma}\right)\right]\\&+\frac{1}{2}\left[M_{\alpha}{}^{[\rho\sigma]} - \frac{2}{3}\left(\overset{*}{\nabla}_{\lambda}N_{\alpha}{}^{\lambda[\rho\sigma]} + \frac{1}{2}T^{[\rho}{}_{\varepsilon\kappa}N_{\alpha}{}^{\sigma]\varepsilon\kappa}\right)\right]T^{\mu}{}_{\rho\sigma}.\end{aligned} \tag{83}$$

Note that the right hand side of (83) exactly coincides with the left hand side of the Klein identity (47). Therefore, one can write also

$$\overset{sym}{U}{}_{\alpha}{}^{\mu} = -I_{\alpha}{}^{\mu}. \tag{84}$$

Comparing formulae (76) with (60), one finds that *the generalized Belinfante tensor coincides with the generalized canonical superpotential:*

$$\begin{aligned}&\mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] = \theta^{\mu\nu}[\delta\boldsymbol{\xi}]\\&= \left\{-M_{\alpha}{}^{[\mu\nu]} + \frac{2}{3}\left(\overset{*}{\nabla}_{\lambda}N_{\alpha}{}^{\lambda[\mu\nu]} + \frac{1}{2}T^{[\mu}{}_{\rho\sigma}N_{\alpha}{}^{\nu]\rho\sigma}\right)\right\}\delta\xi^{\alpha}\\&+\left\{-\frac{4}{3}N_{\alpha}{}^{\beta[\mu\nu]}\right\}\nabla_{\beta}\delta\xi^{\alpha}.\end{aligned} \tag{85}$$

Combining this equality with (52), (53) and (75), one finds that the *generalized symmetrized superpotential* $\overset{sym}{\boldsymbol{\theta}}[\delta\boldsymbol{\xi}] \overset{def}{=} \{\overset{sym}{\theta}{}^{[\mu\nu]}[\delta\boldsymbol{\xi}] = \overset{sym}{\theta}{}^{\mu\nu}[\delta\boldsymbol{\xi}]\}$ corresponding to the current $\overset{sym}{\mathbf{J}}[\delta\boldsymbol{\xi}]$ is equal to zero identically:

$$\overset{sym}{\theta^{\mu\nu}}[\delta\boldsymbol{\xi}] = \theta^{\mu\nu}[\delta\boldsymbol{\xi}] - \mathscr{B}^{\mu\nu}[\delta\boldsymbol{\xi}] = 0. \tag{86}$$

Recall that

$$\overset{sym}{U}{}_{\alpha}{}^{\mu} = \overset{sym}{t}{}^{\mu}{}_{\alpha}, \tag{87}$$

then relation (84) is a proof of the claim that *symmetrized EMT* $\overset{sym}{\mathbf{t}}$ *does not depend on divergences in the Lagrangian*. Indeed, the right hand side of (84) essentially is defined by the variational derivative of the action (see definition (26)), compare this also with Refs.[113,114]. Already in Ref.[142], it was stated that the Belinfante procedure applied both to the Hilbert Lagrangian and to the non-covariant Einstein Lagrangian (differed by a divergence) give the same result. An analogous statement (that divergences in Lagrangians do not influence the Belinfante operation) has been proved for perturbations on a fixed curved background in GR in Refs.[140,141] and in metric theories in the review of Petrov[38].

Finally, we stress the following: The generalized Belinfante relation (75) with accounting for formulae (74), (84) (the latter has been obtained with the use of the Klein identity (47)) and (85) coincides with the boundary Klein-Noether theorem (73). Therefore the *success* of the Belinfante approach is based on the Klein-Noether system of identities (46)-(49) only.


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## Appendix A: Main relations of the Riemann-Cartan geometry

The goal of this Appendix is to introduce main formulae of the Riemann-Cartan geometry, which are necessary in the text, and to identify notations for a reader.

Let $\mathcal{M}$ be a $(D+1)$-dimensional real manifold with a coordinate system $x \overset{def}{=} \{x^\mu\}$ defined on it. The *Riemann-Cartan geometry* is given on $\mathcal{M}$ if smooth fields

1. of a symmetric covariant tensor (*metric*)

$$\mathbf{g} \overset{def}{=} \{g_{\mu\nu}(x)\}, \qquad g_{(\mu\nu)} = g_{\mu\nu}, \tag{A1}$$

and

2. of an *affine connection* compatible with the metric (A1),

$$\boldsymbol{\Gamma} \overset{def}{=} \{\Gamma^\lambda{}_{\mu\nu}(x)\}, \tag{A2}$$

are defined on $\mathcal{M}$.

Because one sets that $\mathcal{M}$ presents a spacetime the metric tensor $\mathbf{g}$ is of the Lorentzian signature:

$$\mathrm{sign}\,\mathbf{g} = (-1, \underbrace{1, 1, \ldots, 1}_{D \text{ times}}). \tag{A3}$$

The metric determinant is denoted as

$$g \overset{def}{=} \det\{g_{\mu\nu}\}. \tag{A4}$$

The connection $\boldsymbol{\Gamma}$ is not symmetrical in lower indexes:

$$\Gamma^\lambda{}_{(\mu\nu)} \neq \Gamma^\lambda{}_{\mu\nu} \tag{A5}$$

and defines covariant derivatives of a vector $\mathbf{V} = \{V^\mu(x)\}$ and an 1-form $\mathbf{W} = \{W_\mu(x)\}$ by the rules

$$\nabla_\lambda V^\mu \overset{def}{=} \partial_\lambda V^\mu + \Gamma^\mu{}_{\alpha\lambda} V^\alpha, \tag{A6}$$

$$\nabla_\lambda W_\mu \overset{def}{=} \partial_\lambda W_\mu - \Gamma^\alpha{}_{\mu\lambda} W_\alpha. \tag{A7}$$

A compatible condition of a connection $\boldsymbol{\Gamma}$ with a metric $\mathbf{g}$ is presented as

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha{}_{\mu\lambda} g_{\alpha\nu} - \Gamma^\alpha{}_{\nu\lambda} g_{\mu\alpha} = 0 \tag{A8}$$

meaning that a *tensor of non-metricity* $\mathbf{Q} \overset{def}{=} \{Q_{\lambda,\mu\nu}(x)\} \overset{def}{=} \{\nabla_\lambda g_{\mu\nu}(x)\}$ is equal to zero. Usually a set $(\mathcal{M}, \mathbf{g}, \boldsymbol{\Gamma})$ is denoted as $\mathcal{C}(1, D)$ and is called as the *Riemann-Cartan manifold*.

The *torsion tensor* $\mathbf{T} \overset{def}{=} \{T^\lambda{}_{\mu\nu}(x)\}$ and the *curvature tensor* $\mathbf{R} \overset{def}{=} \{R^\kappa{}_{\lambda\mu\nu}(x)\}$ are defined by the relation

$$\left(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu\right) V^\lambda \overset{def}{=} -T^\alpha{}_{\mu\nu} \nabla_\alpha V^\lambda + R^\lambda{}_{\alpha\mu\nu} V^\alpha \tag{A9}$$

and are expressed through the connection as follows

$$T^\lambda{}_{\mu\nu} = -2\Gamma^\lambda{}_{[\mu\nu]}; \tag{A10}$$

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa{}_{\lambda\nu} - \partial_\nu \Gamma^\kappa{}_{\lambda\mu} + \Gamma^\kappa{}_{\alpha\mu} \Gamma^\alpha{}_{\lambda\nu} - \Gamma^\kappa{}_{\alpha\nu} \Gamma^\alpha{}_{\lambda\mu}. \tag{A11}$$

As is seen, the torsion tensor $\mathbf{T}$ is antisymmetric in lower indexes:

$$T^\lambda{}_{[\mu\nu]} = T^\lambda{}_{\mu\nu}. \tag{A12}$$

Rising or lowering indexes for a torsion tensor $\mathbf{T}$, we remark their places by coma. For example,

$$T^{\lambda,\,\mu}{}_\nu \overset{def}{=} g^{\mu\alpha} T^\lambda{}_{\alpha\nu}; \qquad T^{\lambda,\,\mu\nu} \overset{def}{=} g^{\mu\alpha} g^{\nu\beta} T^\lambda{}_{\alpha\beta}. \tag{A13}$$

A compatible condition (A8) permits to express a connection $\boldsymbol{\Gamma}$ trough both derivatives of the metric $\{\partial_\alpha g_{\mu\nu}\}$ and the torsion tensor $\{T^\lambda{}_{\mu\nu}\}$ in an unique way. Thus,

$$\Gamma^\lambda{}_{\mu\nu} = g^{\lambda\alpha} \Gamma_{\alpha,\,\mu\nu} \tag{A14}$$

where

$$\Gamma_{\alpha,\,\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) + \frac{1}{2}\left(T_{\mu,\,\alpha\nu} + T_{\nu,\,\alpha\mu} - T_{\alpha,\,\mu\nu}\right) \tag{A15}$$

and

$$T_{\lambda,\,\mu\nu} \overset{def}{=} g_{\lambda\alpha} T^\alpha{}_{\mu\nu}. \tag{A16}$$

In the Riemann-Cartan $(D+1)$-dimensional geometry, the fully covariant curvature tensor

$$R_{\kappa\lambda\mu\nu} \overset{def}{=} g_{\kappa\alpha} R^\alpha{}_{\lambda\mu\nu} \tag{A17}$$

has $(D+1)^2D^2/4$ essential components, is antisymmetrical both in the first pair of indexes:

$$R_{[\kappa\lambda]\mu\nu} = R_{\kappa\lambda\mu\nu} \tag{A18}$$

and in the second pair of indexes:

$$R_{\kappa\lambda[\mu\nu]} = R_{\kappa\lambda\mu\nu}, \tag{A19}$$

but, unlike Riemannian geometry, it is not symmetrical in *pairs* of indexes:

$$R_{\mu\nu\kappa\lambda} \neq R_{\kappa\lambda\mu\nu}. \tag{A20}$$

Tensor $\mathbf{R}$ satisfies the generalized *Ricci identities*

$$R^\kappa{}_{[\lambda\mu\nu]} \equiv \nabla_{[\lambda} T^\kappa{}_{\mu\nu]} + T^\kappa{}_{\alpha[\lambda} T^\alpha{}_{\mu\nu]} \tag{A21}$$

and the generalized *Bianchi identities*

$$\nabla_{[\lambda|} R^\alpha{}_{\beta|\mu\nu]} \equiv -R^\alpha{}_{\beta\gamma[\lambda} T^\gamma{}_{\mu\nu]}. \tag{A22}$$

We need also in the *torsion vector* $\{T_\mu\}$, *modified torsion tensor* $\overset{*}{\mathbf{T}} \overset{def}{=} \{\overset{*}{T}{}^\lambda{}_{\mu\nu}\}$ and *modified torsion vector* $\{\overset{*}{T}_\mu\}$ defined as

$$T_\mu \overset{def}{=} T^\alpha{}_{\mu\alpha}; \tag{A23}$$

$$\overset{*}{T}{}^\lambda{}_{\mu\nu} \overset{def}{=} T^\lambda{}_{\mu\nu} + \delta^\lambda_\mu T_\nu - \delta^\lambda_\nu T_\mu; \tag{A24}$$

$$\overset{*}{T}_\mu \overset{def}{=} \overset{*}{T}{}^\alpha{}_{\mu\alpha}, \tag{A25}$$

respectively, where $\delta^\mu_\nu$ is the Kronecker symbol. It is easily to find a relation between the torsion vector and the modified torsion vector:

$$\overset{*}{T}_\mu = -(D-1)T_\mu. \tag{A26}$$

The *Ricci* and *Einstein tensors*, and the *curvature scalar* are defined as usual:

$$R_{\mu\nu} \overset{def}{=} R^\alpha{}_{\mu\alpha\nu}, \tag{A27}$$

$$E_{\mu\nu} \overset{def}{=} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{A28}$$

$$R \overset{def}{=} g^{\alpha\beta} R_{\alpha\beta}. \tag{A29}$$

The first two are not symmetrical now:

$$R_{(\mu\nu)} \neq R_{\mu\nu}; \qquad E_{(\mu\nu)} \neq E_{\mu\nu}. \tag{A30}$$

Contracting the Ricci identities (A21), one easily states that an antisymmetrical part of the Ricci tensor satisfies the identity:

$$R_{[\mu\nu]} \equiv -\frac{1}{2} \overset{*}{\nabla}_\lambda \overset{*}{T}{}^\lambda{}_{\mu\nu} \tag{A31}$$

where

$$\overset{*}{\nabla}_\lambda \overset{def}{=} \nabla_\lambda + T_\lambda \tag{A32}$$

is a *modified covariant derivative.* From the definition (A32) it follows, first, that the commutator of modified covariant derivatives is presented by

$$\overset{*}{\nabla}_{[\mu}, \overset{*}{\nabla}_{\nu]} = \nabla_{[\mu}, \nabla_{\nu]} + \left(\nabla_{[\mu} T_{\nu]}\right), \tag{A33}$$

second, that for arbitrary two tensors $\mathbf{A} = \{A^{\alpha\cdots}{}_{\beta\ldots}\}$ and $\mathbf{B} = \{B^{\gamma\cdots}{}_{\delta\ldots}\}$ the *modified(!) formula of differentiating their product (the modified Leibnitz rule)*:

$$\begin{aligned} &\overset{*}{\nabla}_\mu \left(A^{\alpha\cdots}{}_{\beta\ldots} B^{\gamma\cdots}{}_{\delta\ldots}\right) \\ &= \left(\overset{*}{\nabla}_\mu A^{\alpha\cdots}{}_{\beta\ldots}\right) B^{\gamma\cdots}{}_{\delta\ldots} + A^{\alpha\cdots}{}_{\beta\ldots} \left(\nabla_\mu B^{\gamma\cdots}{}_{\delta\ldots}\right) \end{aligned} \tag{A34}$$

takes a place. It is also equivalent to

$$\begin{aligned} &\overset{*}{\nabla}_\mu \left(A^{\alpha\cdots}{}_{\beta\ldots} B^{\gamma\cdots}{}_{\delta\ldots}\right) \\ &= \left(\nabla_\mu A^{\alpha\cdots}{}_{\beta\ldots}\right) B^{\gamma\cdots}{}_{\delta\ldots} + A^{\alpha\cdots}{}_{\beta\ldots} \left(\overset{*}{\nabla}_\mu B^{\gamma\cdots}{}_{\delta\ldots}\right). \end{aligned} \tag{A35}$$

The last formula represented in the form

$$\begin{aligned} &A^{\alpha\cdots}{}_{\beta\ldots} \left(\overset{*}{\nabla}_\mu B^{\gamma\cdots}{}_{\delta\ldots}\right) \\ &= \overset{*}{\nabla}_\mu \left(A^{\alpha\cdots}{}_{\beta\ldots} B^{\gamma\cdots}{}_{\delta\ldots}\right) - \left(\nabla_\mu A^{\alpha\cdots}{}_{\beta\ldots}\right) B^{\gamma\cdots}{}_{\delta\ldots} \end{aligned} \tag{A36}$$

we call as the *formula of a differentiation by parts*, and we use it actively.

The *two times contracted* Bianchi identities (A22) acquire the form:

$$\overset{*}{\nabla}_\mu E^\mu{}_\nu \equiv -E^\mu{}_\lambda T^\lambda{}_{\mu\nu} + \frac{1}{2} \overset{*}{T}{}^{\pi,\,\rho\sigma} R_{\rho\sigma\pi\nu}. \tag{A37}$$

Thus, in the Riemann-Cartan geometry, unlike Riemannian geometry, the Einstein tensor (A28) is not conserved.

The *contorsion tensor* $\mathbf{K} = \{K^\lambda{}_{\mu\nu}\}$ defined as

$$K^\lambda{}_{\mu\nu} \overset{def}{=} g^{\lambda\alpha} K_{\alpha,\,\mu\nu}; \tag{A38}$$

$$K_{\lambda,\,\mu\nu} \overset{def}{=} \frac{1}{2}\left(T_{\mu,\,\lambda\nu} + T_{\nu,\,\lambda\mu} - T_{\lambda,\,\mu\nu}\right) \tag{A39}$$

is useful also. The *contorsion vector* defined as

$$K_\mu \overset{def}{=} K^\alpha{}_{\mu\alpha} \tag{A40}$$

is expressed trough the torsion vector (A23) as

$$K_\mu = -T_\mu . \tag{A41}$$

The *Gauss* and *Stockes formulae* are modified essentially with respect to ones in the Riemannian geometry and have the form

$$\int\limits_\Omega \mathbf{d}\boldsymbol{\omega}\, \overset{*}{\nabla}_\mu V^\mu = \oint\limits_{\partial\Omega} \mathbf{d}\boldsymbol{\sigma}_\mu\, V^\mu ; \tag{A42}$$

$$\int\limits_\Sigma \mathbf{d}\boldsymbol{\sigma}_{[\mu}\, \overset{*}{\nabla}_{\nu]} W^{\mu\nu} + \frac{1}{2}\int\limits_\Sigma \mathbf{d}\boldsymbol{\sigma}_\lambda\, T^\lambda{}_{\mu\nu} W^{\mu\nu} = \frac{1}{2!}\oint\limits_{\partial\Sigma} \mathbf{ds}_{\mu\nu}\, W^{\mu\nu}, \tag{A43}$$

respectively. Here, $\{V^\mu\}$ and $\{W^{\mu\nu}\}$ are arbitrary contravariant vector and antisymmetrical contravariant tensor, $W^{[\mu\nu]} = W^{\mu\nu}$; $\Omega$ and $\partial\Omega$ are an arbitrary $(D+1)$-dimensional domain in $\mathcal{C}(1, D)$ and its $D$-dimensional boundary; $\Sigma$ and $\partial\Sigma$ are an arbitrary $D$-dimensional hypersurface in $\mathcal{C}(1, D)$ and its $(D-1)$-dimensional boundary; the notations

$$\mathbf{d}\boldsymbol{\omega} \overset{def}{=} \frac{\sqrt{-g}}{(D+1)!}\varepsilon_{\alpha_0\alpha_1\ldots\alpha_D}\mathbf{dx}^{\alpha_0}\wedge\mathbf{dx}^{\alpha_1}\wedge\ldots\mathbf{dx}^{\alpha_D}, \tag{A44}$$

$$\mathbf{d}\boldsymbol{\sigma}_\lambda \overset{def}{=} \frac{\sqrt{-g}}{D!}\varepsilon_{\lambda\alpha_1\ldots\alpha_D}\mathbf{dx}^{\alpha_1}\wedge\mathbf{dx}^{\alpha_2}\wedge\ldots\mathbf{dx}^{\alpha_D}, \tag{A45}$$

$$\mathbf{ds}_{\mu\nu} \overset{def}{=} \frac{\sqrt{-g}}{(D-1)!}\varepsilon_{\mu\nu\alpha_1\alpha_2\ldots\alpha_{D-1}}\mathbf{dx}^{\alpha_1}\wedge\mathbf{dx}^{\alpha_2}\wedge\ldots\mathbf{dx}^{\alpha_{D-1}} \tag{A46}$$

mean $(D+1)$-, $D$- and $(D-1)$-forms of elementary volumes; $\boldsymbol{\varepsilon} \overset{def}{=} \{\varepsilon_{\alpha_0\alpha_1\ldots\alpha_D}\}$ is the fully antisymmetrical $(D+1)$-dimensional Levi-Civita symbol,

$$\varepsilon_{[\alpha_0\alpha_1\ldots\alpha_D]} = \varepsilon_{\alpha_0\alpha_1\ldots\alpha_D}, \qquad \varepsilon_{012\ldots D} = +1; \tag{A47}$$

$\mathbf{dx}^\alpha$ are the basic 1-forms, a symbol $\wedge$ means a wedge product.

## Appendix B: Irreducible representations of a symmetric group for two- and three-index quantities

To provide many of calculations we decompose tensors onto irreducible representations of a symmetric group (group of index permutations). In this Appendix, we give main properties of such representations for 2- and 3-index quantities and formulae necessary in the text.

### 1. The Young projectors for 2-index quantities

For 2-index quantities $\{A_{\ldots}{}^{\beta\gamma}\}$ one has 2 *Young diagrams* only:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \tag{B1}$$

to which the *Young projectors*:

$$\hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right) \overset{def}{=} \frac{1}{2}\left((12) + (21)\right) \tag{B2}$$

and

$$\hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) \overset{def}{=} \frac{1}{2}\left((12) - (21)\right) \tag{B3}$$

correspond. Projectors (B2) and (B3) act as follows

$$\hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right) A_{\ldots}{}^{\beta\gamma} = \frac{1}{2}\left(A_{\ldots}{}^{\beta\gamma} + A_{\ldots}{}^{\gamma\beta}\right) = A_{\ldots}{}^{(\beta\gamma)} \tag{B4}$$

and

$$\hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) A_{\ldots}{}^{\beta\gamma} = \frac{1}{2}\left(A_{\ldots}{}^{\beta\gamma} - A_{\ldots}{}^{\gamma\beta}\right) = A_{\ldots}{}^{[\beta\gamma]}. \tag{B5}$$

They are orthonormal:

$$\left\{\begin{array}{l} \hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right)\hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right) = \hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right); \\ \hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right)\hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) = \hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right); \\ \hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right)\hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) = \hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right)\hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right) = 0 \end{array}\right. \tag{B6}$$

and present themselves a full set:

$$\hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right) + \hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) = 1. \tag{B7}$$

### 2. A transformation of the expression $M_\alpha{}^{\beta\gamma}\nabla_\gamma\nabla_\beta\delta\xi^\alpha$

Using (B7), (B4) and (B5), one finds

$$\begin{aligned} M_\alpha{}^{\beta\gamma} &= 1\cdot M_\alpha{}^{\beta\gamma} \\ &= \left(\hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right) + \hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right)\right) M_\alpha{}^{\beta\gamma} \\ &= \hat{s}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}\right) M_\alpha{}^{\beta\gamma} + \hat{a}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\right) M_\alpha{}^{\beta\gamma} \\ &= M_\alpha{}^{(\beta\gamma)} + M_\alpha{}^{[\beta\gamma]}. \end{aligned} \tag{B8}$$

From here for an arbitrary vector $\delta\boldsymbol{\xi} = \{\delta\xi^\alpha(x)\}$ one has

$$\begin{aligned} &M_\alpha{}^{\beta\gamma}\nabla_\gamma\nabla_\beta\delta\xi^\alpha = M_\alpha{}^{(\beta\gamma)}\nabla_\gamma\nabla_\beta\delta\xi^\alpha + M_\alpha{}^{[\beta\gamma]}\nabla_\gamma\nabla_\beta\delta\xi^\alpha \\ &= M_\alpha{}^{\beta\gamma}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha + M_\alpha{}^{\beta\gamma}\nabla_{[\gamma}\nabla_{\beta]}\delta\xi^\alpha . \end{aligned} \tag{B9}$$

Using in the last term the formula (A9) for a commutator of covariant derivatives, one obtains

$$\nabla_{[\gamma}\nabla_{\beta]}\delta\xi^{\alpha}=-\frac{1}{2}T^{\varepsilon}{}_{\gamma\beta}\nabla_{\varepsilon}\delta\xi^{\alpha}+\frac{1}{2}R^{\alpha}{}_{\varepsilon\gamma\beta}\delta\xi^{\varepsilon}. \tag{B10}$$

Then the expression (B9) transforms into

$$\begin{aligned}M_{\alpha}{}^{\beta\gamma}\nabla_{\gamma}\nabla_{\beta}\delta\xi^{\alpha}&=\left\{\frac{1}{2}R^{\varepsilon}{}_{\alpha\kappa\lambda}M_{\varepsilon}{}^{\lambda\kappa}\right\}\delta\xi^{\alpha}\\&+\left\{-\frac{1}{2}T^{\beta}{}_{\kappa\lambda}M_{\alpha}{}^{\lambda\kappa}\right\}\nabla_{\beta}\delta\xi^{\alpha}+\left\{M_{\alpha}{}^{\beta\gamma}\right\}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^{\alpha}.\end{aligned} \tag{B11}$$

### 3. The Young projectors for 3-index quantities

For 3-index quantities $\{A_{\dots}{}^{\beta\gamma\delta}\}$ one has 4 different Young diagram:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,,\qquad \begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\,,\qquad \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\qquad\text{and}\qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\,. \tag{B12}$$

In this case, there are two *different* full orthonormal sets of the Young projectors. Their construction is carried out as follows. For the *sequence* $I$ from the beginning one provides an antisymmetrization in indexes in a column, only after that one provides a symmetrization in indexes in a line; for the *sequence* $II$, inversely, from the beginning one provides a symmetrization in indexes in a line, only after that one provides an antisymmetrization in indexes in a column. Thus

$$I\left\{\begin{aligned}&\hat{s}\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right)\overset{def}{=}\frac{1}{6}\left((123)+(132)+(312)+(321)+(231)+(213)\right); &\text{(B13)}\\&\hat{s}\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\right)\overset{def}{=}\frac{1}{3}\left((123)-(132)+(213)-(231)\right); &\text{(B14)}\\&\hat{s}\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right)\overset{def}{=}\frac{1}{3}\left((123)-(213)+(132)-(312)\right); &\text{(B15)}\\&\hat{a}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right)\overset{def}{=}\frac{1}{6}\left((123)-(132)+(312)-(321)+(231)-(213)\right) &\text{(B16)}\end{aligned}\right.$$

and

$$II\left\{\begin{aligned}&\hat{s}\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right)\overset{def}{=}\frac{1}{6}\left((123)+(132)+(312)+(321)+(231)+(213)\right); &\text{(B17)}\\&\hat{a}\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\right)\overset{def}{=}\frac{1}{3}\left((123)+(213)-(132)-(312)\right); &\text{(B18)}\\&\hat{a}\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right)\overset{def}{=}\frac{1}{3}\left((123)+(132)-(213)-(231)\right); &\text{(B19)}\\&\hat{a}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right)\overset{def}{=}\frac{1}{6}\left((123)-(132)+(312)-(321)+(231)-(213)\right). &\text{(B20)}\end{aligned}\right.$$

As a result of action of the operator $\hat{s}\,(\,\dots)$ onto a 3-index quantity, one obtains a *symmetrical* in correspondent indexes quantity, and, analogously, after action of the operator $\hat{a}\,(\,\dots)$ one obtains an *antisymmetrical* in correspondent indexes quantity.

### 4. A transformation of the expression $N_{\alpha}{}^{\beta\gamma\delta}\nabla_{\delta}\nabla_{\gamma}\nabla_{\beta}\delta\xi^{\alpha}$

To transform the expression $N_{\alpha}{}^{\beta\gamma\delta}\nabla_{\delta}\nabla_{\gamma}\nabla_{\beta}\delta\xi^{\alpha}$ we use the set of projectors $II$. Decompose the tensor $N_{\alpha}{}^{(\beta\gamma)\delta}=N_{\alpha}{}^{\beta\gamma\delta}$ onto irreducible with respect to this set parts:

$$
\begin{aligned}
N_\alpha{}^{\beta\gamma\delta} &= 1\cdot N_\alpha{}^{\beta\gamma\delta} = \left(\hat{s}\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) + \hat{a}\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\right) + \hat{a}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) + \hat{a}\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right)\right) N_\alpha{}^{\beta\gamma\delta}\\
&= \hat{s}\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) N_\alpha{}^{\beta\gamma\delta} + \hat{a}\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\right) N_\alpha{}^{\beta\gamma\delta} + \hat{a}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) N_\alpha{}^{\beta\gamma\delta} + \hat{a}\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right) N_\alpha{}^{\beta\gamma\delta}\\
&\stackrel{def}{=} a_\alpha{}^{\beta\gamma\delta} + b_\alpha{}^{\beta\gamma\delta} + c_\alpha{}^{\beta\gamma\delta} + d_\alpha{}^{\beta\gamma\delta}.
\end{aligned}
\tag{B21}
$$

Here,

$$
a_\alpha{}^{\beta\gamma\delta} \stackrel{def}{=} \hat{s}\left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\right) N_\alpha{}^{\beta\gamma\delta} = N_\alpha{}^{(\beta\gamma\delta)};
\tag{B22}
$$

$$
\begin{aligned}
b_\alpha{}^{\beta\gamma\delta} &\stackrel{def}{=} \hat{a}\left(\begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}\right) N_\alpha{}^{\beta\gamma\delta}\\
&= \frac{1}{3}\left(N_\alpha{}^{\beta\gamma\delta} + N_\alpha{}^{\gamma\beta\delta} - N_\alpha{}^{\beta\delta\gamma} - N_\alpha{}^{\delta\beta\gamma}\right)\\
&= \frac{2}{3}\left(N_\alpha{}^{\beta\gamma\delta} - N_\alpha{}^{\beta\delta\gamma}\right) = \frac{4}{3} N_\alpha{}^{\beta[\gamma\delta]};
\end{aligned}
\tag{B23}
$$

$$
c_\alpha{}^{\beta\gamma\delta} \stackrel{def}{=} \hat{a}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\right) N_\alpha{}^{\beta\gamma\delta} = N_\alpha{}^{[\beta\gamma\delta]} = 0;
\tag{B24}
$$

$$
\begin{aligned}
d_\alpha{}^{\beta\gamma\delta} &\stackrel{def}{=} \hat{a}\left(\begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}\right) N_\alpha{}^{\beta\gamma\delta}\\
&= \frac{1}{3}\left(N_\alpha{}^{\beta\gamma\delta} + N_\alpha{}^{\beta\delta\gamma} - N_\alpha{}^{\gamma\beta\delta} - N_\alpha{}^{\gamma\delta\beta}\right)\\
&= \frac{1}{3}\left(N_\alpha{}^{\delta\beta\gamma} - N_\alpha{}^{\delta\gamma\beta}\right) = \frac{2}{3} N_\alpha{}^{\delta[\beta\gamma]}.
\end{aligned}
\tag{B25}
$$

Thus,

$$
N_\alpha{}^{\beta\gamma\delta} = N_\alpha{}^{(\beta\gamma\delta)} + \frac{4}{3} N_\alpha{}^{\beta[\gamma\delta]} + \frac{2}{3} N_\alpha{}^{\delta[\beta\gamma]}.
\tag{B26}
$$

Then

$$
\begin{aligned}
&N_\alpha{}^{\beta\gamma\delta}\nabla_\delta\nabla_\gamma\nabla_\beta\delta\xi^\alpha = N_\alpha{}^{\beta\gamma\delta}\nabla_{(\delta}\nabla_\gamma\nabla_{\beta)}\delta\xi^\alpha\\
&+\frac{4}{3} N_\alpha{}^{\beta\gamma\delta}\nabla_{[\delta}\nabla_{\gamma]}\nabla_\beta\delta\xi^\alpha + \frac{2}{3} N_\alpha{}^{\delta\beta\gamma}\nabla_\delta\nabla_{[\gamma}\nabla_{\beta]}\delta\xi^\alpha.
\end{aligned}
\tag{B27}
$$

Here, to calculate 2-nd and 3-rd terms on the right hand side, firstly, we apply the formulae for the commutator of the type (A9), next, use a decomposition of a 2-index quantity of the type (B11), at last, apply again the formula (A9). After collecting similar terms we obtain

$$
\begin{aligned}
&\frac{4}{3} N_\alpha{}^{\beta\gamma\delta}\nabla_{[\delta}\nabla_{\gamma]}\nabla_\beta\delta\xi^\alpha = \left\{\frac{1}{3} N_\kappa{}^{\lambda\mu\nu} T^\sigma{}_{\mu\nu} R^\kappa{}_{\alpha\sigma\lambda}\right\}\delta\xi^\alpha\\
&+\left\{\frac{2}{3} N_\alpha{}^{\lambda\mu\nu}\left(\frac{1}{2} T^\sigma{}_{\mu\nu} T^\beta{}_{\lambda\sigma} + R^\beta{}_{\lambda\mu\nu}\right) - \frac{2}{3} N_\lambda{}^{\beta\mu\nu} R^\lambda{}_{\alpha\mu\nu}\right\}\nabla_\beta\delta\xi^\alpha + \left\{\frac{2}{3} N_\alpha{}^{\beta\mu\nu} T^\gamma{}_{\mu\nu}\right\}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha;
\end{aligned}
\tag{B28}
$$

$$
\begin{aligned}
&\frac{2}{3} N_\alpha{}^{\delta\beta\gamma}\nabla_\delta\nabla_{[\gamma}\nabla_{\beta]}\delta\xi^\alpha = \left\{\frac{1}{3} N_\kappa{}^{\lambda\mu\nu}\left(\frac{1}{2} T^\sigma{}_{\mu\nu} R^\kappa{}_{\alpha\lambda\sigma} - \nabla_\lambda R^\kappa{}_{\alpha\mu\nu}\right)\right\}\delta\xi^\alpha\\
&+\left\{\frac{1}{3} N_\alpha{}^{\lambda\mu\nu}\left(\nabla_\lambda T^\beta{}_{\mu\nu} + \frac{1}{2} T^\sigma{}_{\mu\nu} T^\beta{}_{\sigma\lambda}\right) - \frac{1}{3} N_\kappa{}^{\beta\mu\nu} R^\kappa{}_{\alpha\mu\nu}\right\}\nabla_\beta\delta\xi^\alpha + \left\{\frac{1}{3} N_\alpha{}^{\gamma\mu\nu} T^\beta{}_{\mu\nu}\right\}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha.
\end{aligned}
\tag{B29}
$$

Substituting (B28) and (B29) into (B27), we obtain finally

$$
\begin{aligned}
&N_\alpha{}^{\beta\gamma\delta}\nabla_\delta\nabla_\gamma\nabla_\beta\delta\xi^\alpha = \left\{-\frac{1}{3} N_\kappa{}^{\lambda\mu\nu}\left(\nabla_\lambda R^\kappa{}_{\alpha\mu\nu} + \frac{1}{2} T^\sigma{}_{\mu\nu} R^\kappa{}_{\alpha\lambda\sigma}\right)\right\}\delta\xi^\alpha\\
&+\left\{\frac{1}{3} N_\alpha{}^{\lambda\mu\nu}\left(2R^\beta{}_{\lambda\mu\nu} + \nabla_\lambda T^\beta{}_{\mu\nu} + \frac{1}{2} T^\sigma{}_{\mu\nu} T^\beta{}_{\lambda\sigma}\right) - N_\kappa{}^{\beta\mu\nu} R^\kappa{}_{\alpha\mu\nu}\right\}\nabla_\beta\delta\xi^\alpha\\
&+\left\{N_\alpha{}^{\beta\mu\nu} T^\gamma{}_{\mu\nu}\right\}\nabla_{(\gamma}\nabla_{\beta)}\delta\xi^\alpha + \left\{N_\alpha{}^{\beta\gamma\delta}\right\}\nabla_{(\delta}\nabla_\gamma\nabla_{\beta)}\delta\xi^\alpha.
\end{aligned}
\tag{B30}
$$

## Appendix C: Transformation of the Klein identities

### 1. Three useful identities

For arbitrary tensors $\{\theta^{[\mu\nu]} = \theta^{\mu\nu}\}$, $\{\theta_\alpha{}^{[\mu\nu]} = \theta_\alpha{}^{\mu\nu}\}$ and $\{\theta_\alpha{}^{\beta[\mu\nu]} = \theta_\alpha{}^{\beta\mu\nu}\}$ the identities

$$
\stackrel{*}{\nabla}_\mu\left[\stackrel{*}{\nabla}_\nu\theta^{\mu\nu} + \frac{1}{2}\theta^{\rho\sigma} T^\mu{}_{\rho\sigma}\right] \equiv 0;
\tag{C1}
$$

$$\overset{*}{\nabla}_\mu \left[\overset{*}{\nabla}_\nu \theta_\alpha{}^{\mu\nu} + \frac{1}{2}\theta_\alpha{}^{\rho\sigma}T^\mu{}_{\rho\sigma}\right] \equiv -\frac{1}{2}R^\lambda{}_{\alpha\rho\sigma}\theta_\lambda{}^{\rho\sigma}; \tag{C2}$$

$$\begin{aligned}\overset{*}{\nabla}_\mu &\left[\overset{*}{\nabla}_\nu \theta_\alpha{}^{\beta\mu\nu} + \frac{1}{2}\theta_\alpha{}^{\beta\rho\sigma}T^\mu{}_{\rho\sigma}\right]\\ &\equiv \frac{1}{2}\left(-R^\lambda{}_{\alpha\rho\sigma}\theta_\lambda{}^{\beta\rho\sigma} + R^\beta{}_{\lambda\rho\sigma}\theta_\alpha{}^{\lambda\rho\sigma}\right);\end{aligned} \tag{C3}$$

take a place. Let us prove, for example, the 1-st one. Using the formulae for commutators of the types (A9) and (A33), one has

$$\begin{aligned}&\overset{*}{\nabla}_\mu \left(\overset{*}{\nabla}_\nu \theta^{\mu\nu}\right) = \overset{*}{\nabla}_{[\mu}\overset{*}{\nabla}_{\nu]}\theta^{\mu\nu} =\\ &\left(\nabla_{[\mu}T_{\nu]}\right)\theta^{\mu\nu} - \frac{1}{2}T^\lambda{}_{\mu\nu}\nabla_\lambda\theta^{\mu\nu} + \frac{1}{2}R^\mu{}_{\lambda\mu\nu}\theta^{\lambda\nu} + \frac{1}{2}R^\nu{}_{\lambda\mu\nu}\theta^{\mu\lambda}\\ &= \left(\overset{*}{\nabla}_{[\mu}T_{\nu]}\right)\theta^{\mu\nu} - \overset{*}{\nabla}_\lambda\left(\frac{1}{2}T^\lambda{}_{\mu\nu}\theta^{\mu\nu}\right) + \frac{1}{2}\left(\overset{*}{\nabla}_\lambda T^\lambda{}_{\mu\nu}\right)\theta^{\mu\nu}\\ &+R_{[\mu\nu]}\theta^{\mu\nu}.\end{aligned}$$

The sum of the 1-st and 3-rd terms with taking into account (A24) is equal to

$$\frac{1}{2}\overset{*}{\nabla}_\lambda\left(\delta^\lambda_\mu T_\nu - \delta^\lambda_\nu T_\mu + T^\lambda{}_{\mu\nu}\right) = \frac{1}{2}\overset{*}{\nabla}_\lambda \overset{*}{T}{}^\lambda{}_{\mu\nu}.$$

Then, keeping in mind the identity (A31), one obtains finally

$$\overset{*}{\nabla}_\mu\left(\overset{*}{\nabla}_\nu\theta^{\mu\nu}\right) \equiv -\overset{*}{\nabla}_\lambda\left(\frac{1}{2}T^\lambda{}_{\mu\nu}\theta^{\mu\nu}\right)$$

that coincides exactly with (C1). The identities (C2) and (C3) are proved analogously.

### 2. Representation of the Klein identity (38) in the form (42)

To transfer from the formula (38) to the formula (42) it is enough in $-\frac{1}{3}N_\kappa{}^{\lambda\mu\nu}\overset{*}{\nabla}_\lambda R^\kappa{}_{\alpha\mu\nu}$ to provide a differentiation by parts and collect similar terms.

### 3. Representation of the Klein identity (39) in the form (43)

We transform (39) step by step as follows.

1. In $\overset{*}{\nabla}_\mu M_\alpha{}^{\beta\mu}$ the quantity $M_\alpha{}^{\beta\mu}$ is decomposed onto irreducible parts with correspondence to the formula (B8):

$$\overset{*}{\nabla}_\mu M_\alpha{}^{\beta\mu} = \overset{*}{\nabla}_\mu M_\alpha{}^{[\beta\mu]} + M_\alpha{}^{(\beta\mu)}.$$

2. In $\frac{1}{3}N_\alpha{}^{\lambda\mu\nu}\nabla_\lambda T^\beta{}_{\mu\nu}$ a differentiation by parts is provided:

$$\begin{aligned}&\tfrac{1}{3}N_\alpha{}^{\lambda\mu\nu}\nabla_\lambda T^\beta{}_{\mu\nu}\\ &= \overset{*}{\nabla}_\mu\left(\tfrac{1}{3}N_\alpha{}^{\mu\rho\sigma}T^\beta{}_{\rho\sigma}\right) + \tfrac{1}{2}\left(-\tfrac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\rho\sigma]}\right)T^\beta{}_{\rho\sigma}.\end{aligned}$$

3. The term $\frac{1}{3}N_\alpha{}^{\lambda\mu\nu}\left(\frac{1}{2}T^\sigma{}_{\mu\nu}T^\beta{}_{\lambda\sigma}\right)$ is rewritten in the way:

$$\frac{1}{3}N_\alpha{}^{\lambda\mu\nu}\left(\frac{1}{2}T^\sigma{}_{\mu\nu}T^\beta{}_{\lambda\sigma}\right) = \frac{1}{2}\left(\frac{1}{3}N_\alpha{}^{[\rho|\kappa\lambda}T^{|\sigma]}{}_{\kappa\lambda}\right)T^\beta{}_{\rho\sigma}.$$

4. The terms $\frac{2}{3}N_\alpha{}^{\lambda\mu\nu}R^\beta{}_{\lambda\mu\nu} - N_\lambda{}^{\beta\mu\nu}R^\lambda{}_{\alpha\mu\nu}$ are represented as

$$\begin{aligned}&\frac{2}{3}N_\alpha{}^{\lambda\mu\nu}R^\beta{}_{\lambda\mu\nu} - N_\lambda{}^{\beta\mu\nu}R^\lambda{}_{\alpha\mu\nu} = -\frac{1}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\\ &+\frac{1}{2}\left(\frac{4}{3}N_\alpha{}^{\lambda\rho\sigma}R^\beta{}_{\lambda\rho\sigma} - \frac{4}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right).\end{aligned}$$

5. The above points 1–4 are taken into account in the identity (39), and the expression

$$\begin{aligned}&\overset{*}{\nabla}_\mu\left(-\tfrac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]} + \tfrac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}\right)\\ &+\overset{*}{\nabla}_\mu\left(\overset{*}{\nabla}_\lambda N_\alpha{}^{\beta\mu\lambda} + N_\alpha{}^{(\beta|\rho\sigma}T^{|\mu)}{}_{\rho\sigma}\right)\end{aligned}$$

is added and subtracted in the left hand side of (39). Then the left hand side of (39) acquires the form:

$$\begin{aligned}&\left(U_\alpha{}^\beta - \frac{1}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right)\\ &+\overset{*}{\nabla}_\mu\left(M_\alpha{}^{[\beta\mu]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]} + \frac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}\right) + \frac{1}{2}\left(M_\alpha{}^{[\rho\sigma]} - \frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\rho\sigma]} + \frac{1}{3}N_\alpha{}^{[\rho|\kappa\lambda}T^{|\sigma]}{}_{\kappa\lambda}\right)T^\beta{}_{\rho\sigma}\\ &+\overset{*}{\nabla}_\mu\left(M_\alpha{}^{(\beta\mu)} + \overset{*}{\nabla}_\lambda N_\alpha{}^{\beta\mu\lambda} + N_\alpha{}^{(\beta|\rho\sigma}T^{|\mu)}{}_{\rho\sigma}\right) + \frac{1}{2}\left(\frac{4}{3}N_\alpha{}^{\lambda\rho\sigma}R^\beta{}_{\lambda\rho\sigma} - \frac{4}{3}N_\alpha{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right)\\ &+\overset{*}{\nabla}_\mu\left(\frac{1}{3}N_\alpha{}^{\mu\rho\sigma}T^\beta{}_{\rho\sigma}\right) - \overset{*}{\nabla}_\mu\left(-\frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]} + \frac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}\right) - \overset{*}{\nabla}_\mu\left(\overset{*}{\nabla}_\lambda N_\alpha{}^{\beta\mu\lambda} + N_\alpha{}^{(\beta|\rho\sigma}T^{|\mu)}{}_{\rho\sigma}\right).\end{aligned}$$

6. Here, the sum of the last three terms is transformed to the form

$$-\overset{*}{\nabla}_\mu\left[\overset{*}{\nabla}_\lambda\left(N_\alpha{}^{\beta\mu\lambda}-\frac{2}{3}N_\alpha{}^{\lambda[\beta\mu]}\right)+\frac{1}{2}\left(\frac{4}{3}N_\alpha{}^{\beta[\rho\sigma]}\right)T^\mu{}_{\rho\sigma}\right]$$
$$=-\overset{*}{\nabla}_\mu\left(\overset{*}{\nabla}_\lambda N_\alpha{}^{(\beta\mu\lambda)}\right)$$
$$-\overset{*}{\nabla}_\mu\left[\overset{*}{\nabla}_\lambda\left(\frac{4}{3}N_\alpha{}^{\beta[\mu\lambda]}\right)+\frac{1}{2}\left(\frac{4}{3}N_\alpha{}^{\beta[\rho\sigma]}\right)T^\mu{}_{\rho\sigma}\right]$$

where the decomposition (B26) has been used.

7. At last, setting in the identity (C3) $\theta_\alpha{}^{\beta\mu\nu}=\frac{4}{3}N_\alpha{}^{\beta[\mu\nu]}$, one can see that

$$-\overset{*}{\nabla}_\mu\left[\overset{*}{\nabla}_\lambda\left(\frac{4}{3}N_\alpha{}^{\beta[\mu\lambda]}\right)+\frac{1}{2}\left(\frac{4}{3}N_\alpha{}^{\beta[\rho\sigma]}\right)T^\mu{}_{\rho\sigma}\right]$$
$$+\frac{1}{2}\left(\frac{4}{3}N_\alpha{}^{\lambda\rho\sigma}R^\beta{}_{\lambda\rho\sigma}-\frac{4}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right)\equiv 0.$$

In the result the Klein identity (39) is represented in the equivalent form:

$$\begin{aligned}&\left(U_\alpha{}^\beta-\frac{1}{3}N_\lambda{}^{\beta\rho\sigma}R^\lambda{}_{\alpha\rho\sigma}\right)\\&+\overset{*}{\nabla}_\mu\left(M_\alpha{}^{[\beta\mu]}-\frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\beta\mu]}+\frac{1}{3}N_\alpha{}^{[\beta|\rho\sigma}T^{|\mu]}{}_{\rho\sigma}\right)+\frac{1}{2}\left(M_\alpha{}^{[\rho\sigma]}-\frac{2}{3}\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\rho\sigma]}+\frac{1}{3}N_\alpha{}^{[\rho|\kappa\lambda}T^{|\sigma]}{}_{\kappa\lambda}\right)T^\beta{}_{\rho\sigma}\\&+\overset{*}{\nabla}_\mu\left(M_\alpha{}^{(\beta\mu)}+\overset{*}{\nabla}_\lambda N_\alpha{}^{\beta\mu\lambda}+N_\alpha{}^{(\beta|\rho\sigma}T^{|\mu)}{}_{\rho\sigma}\right)-\overset{*}{\nabla}_\mu\overset{*}{\nabla}_\lambda N_\alpha{}^{(\beta\mu\lambda)}\equiv -I_\alpha{}^\beta;\end{aligned}\tag{C4}$$

#### 4. The proof of the formula (58)

The identity (C2) after using the antisymmetry property of the curvature tensor (A18) transforms to

$$\overset{*}{\nabla}_\mu\left[\overset{*}{\nabla}_\nu\theta^{\lambda\mu\nu}+\frac{1}{2}\theta^{\lambda\rho\sigma}T^\mu{}_{\rho\sigma}\right]\equiv\frac{1}{2}R^\lambda{}_{\kappa\mu\nu}\theta^{\kappa\mu\nu}. \tag{C5}$$

Using this identity for the expression $\Delta\mathscr{J}^\mu$ (57), one obtains

$$\Delta\mathscr{J}^\mu=\frac{1}{2}\left(R^\mu{}_{\kappa\nu\lambda}-\nabla_\kappa T^\mu{}_{\nu\lambda}-T^\mu{}_{\rho\kappa}T^\rho{}_{\nu\lambda}\right)\theta^{\kappa\nu\lambda}$$

Taking into account the property (56), one can write also

$$\Delta\mathscr{J}^\mu=\frac{1}{2}\left(R^\mu{}_{[\kappa\nu\lambda]}-\nabla_{[\kappa}T^\mu{}_{\nu\lambda]}-T^\mu{}_{\rho[\kappa}T^\rho{}_{\nu\lambda]}\right)\theta^{\kappa\nu\lambda}.$$

But the expression in the parenthesis is equal to zero by the Ricci identity (A21). Thus,

$$\Delta\mathscr{J}^\mu=0.$$

## Appendix D: The solution to the system of equations (62) – (64)

In this Appendix, essentially basing on the results of the Appendix B 3, we give the full solution to the system of equations

$$\begin{cases}\left(\overset{*}{\nabla}_\nu A_\alpha{}^{\mu\nu}+\frac{1}{2}A_\alpha{}^{\rho\sigma}T^\mu{}_{\rho\sigma}\right)+\frac{1}{2}B_\lambda{}^{\rho\sigma\mu}R^\lambda{}_{\alpha\rho\sigma}=U_\alpha{}^\mu+I_\alpha{}^\mu; & \text{(D1)}\\ -A_\alpha{}^{\beta\mu}+\left(\overset{*}{\nabla}_\lambda B_\alpha{}^{\beta\mu\lambda}+\frac{1}{2}B_\alpha{}^{\beta\rho\sigma}T^\mu{}_{\rho\sigma}-\frac{1}{2}B_\alpha{}^{\rho\sigma\mu}T^\beta{}_{\rho\sigma}\right)=M_\alpha{}^{\beta\mu}; & \text{(D2)}\\ -B_\alpha{}^{(\beta\gamma)\mu}=N_\alpha{}^{\beta\gamma\mu}. & \text{(D3)}\end{cases}$$

#### 1. Determination of the tensor B

Using the full set of the Young projectors $II$ (B17) – (B20), decompose the tensor $\{B_\alpha{}^{\lambda[\mu\nu]}=B_\alpha{}^{\lambda\mu\nu}\}$ onto irreducible with respect to its contravariant indexes parts:

$$B_\alpha{}^{\lambda\mu\nu}=a_\alpha{}^{\lambda\mu\nu}+b_\alpha{}^{\lambda\mu\nu}+c_\alpha{}^{\lambda\mu\nu}+d_\alpha{}^{\lambda\mu\nu}$$

where

$$a_\alpha{}^{\lambda\mu\nu}\overset{def}{=}\hat{s}\left(\boxed{1}\boxed{2}\boxed{3}\right)B_\alpha{}^{\lambda\mu\nu}=B_\alpha{}^{(\lambda\mu\nu)}=0;$$

$$d_\alpha{}^{\lambda\mu\nu} \stackrel{def}{=} \hat{a}\left(\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\right) B_\alpha{}^{\lambda\mu\nu} = \frac{2}{3}\left(B_\alpha{}^{\lambda(\mu\nu)} - B_\alpha{}^{\mu(\lambda\nu)}\right) = 0;$$

$$c_\alpha{}^{\lambda\mu\nu} \stackrel{def}{=} \hat{s}\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right) B_\alpha{}^{\lambda\mu\nu} = B_\alpha{}^{[\lambda\mu\nu]} = \frac{1}{3}\left(B_\alpha{}^{\lambda\mu\nu} + B_\alpha{}^{\mu\nu\lambda} + B_\alpha{}^{\nu\lambda\mu}\right);$$

$$b_\alpha{}^{\lambda\mu\nu} \stackrel{def}{=} \hat{s}\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right) B_\alpha{}^{\lambda\mu\nu} = B_\alpha{}^{\lambda\mu\nu} - c_\alpha{}^{\lambda\mu\nu}.$$

Consequently,

$$B_\alpha{}^{\lambda\mu\nu} = b_\alpha{}^{\lambda\mu\nu} + c_\alpha{}^{\lambda\mu\nu}. \tag{D4}$$

By the construction, the tensors $\mathbf{b} \stackrel{def}{=} \{b_\alpha{}^{\lambda\mu\nu}\}$ and $\mathbf{c} \stackrel{def}{=} \{c_\alpha{}^{\lambda\mu\nu}\}$ have the following properties of symmetry:

$$b_\alpha{}^{\lambda[\mu\nu]} = b_\alpha{}^{\lambda\mu\nu}; \tag{D5}$$

$$b_\alpha{}^{[\lambda\mu\nu]} = b_\alpha{}^{\lambda\mu\nu}; \tag{D6}$$

$$c_\alpha{}^{[\lambda\mu\nu]} = c_\alpha{}^{\lambda\mu\nu}. \tag{D7}$$

Then from (D4), (D6) and (D7) one has

$$B_\alpha{}^{(\beta\gamma)\mu} = b_\alpha{}^{(\beta\gamma)\mu}. \tag{D8}$$

Substituting this result into (D3), one obtains the equation:

$$b_\alpha{}^{(\beta\gamma)\mu} = -N_\alpha{}^{\beta\gamma\mu} \tag{D9}$$

that has to determine tensor $\mathbf{b}$.

Recall the symmetry properties (23) and (49) for the tensor $\mathbf{N}$:

$$N_\alpha{}^{(\lambda\mu)\nu} = N_\alpha{}^{\lambda\mu\nu}; \tag{D10}$$

$$N_\alpha{}^{(\lambda\mu\nu)} = 0. \tag{D11}$$

Using them and the definition (B14) of the Young projector $\hat{s}\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right)$, one finds that

$$\hat{s}\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right) N_\alpha{}^{\lambda\mu\nu} = N_\alpha{}^{\lambda\mu\nu}. \tag{D12}$$

On the other hand, by the symmetry property (D5),

$$\hat{s}\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right) b_\alpha{}^{\lambda\mu\nu} = \frac{4}{3} b_\alpha{}^{(\lambda\mu)\nu}. \tag{D13}$$

Combining the last two formulae and (D9) one obtains

$$\hat{s}\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right) b_\alpha{}^{\lambda\mu\nu} = -\hat{s}\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right) \frac{4}{3} N_\alpha{}^{\lambda\mu\nu}. \tag{D14}$$

Now, act onto both sides of this equality by the Young projector $\hat{a}\left(\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right)$ (B18). Then, taking into account (D12), (D10), (D13) and the property $b_\alpha{}^{(\lambda\mu\nu)} = 0$, one gets

$$b_\alpha{}^{\lambda\mu\nu} = -\frac{4}{3} N_\alpha{}^{\lambda[\mu\nu]}. \tag{D15}$$

Substituting this result into (D4), one finds finally

$$B_\alpha{}^{\lambda\mu\nu} = -\frac{4}{3} N_\alpha{}^{\lambda[\mu\nu]} + c_\alpha{}^{\lambda\mu\nu}. \tag{D16}$$

One needs an antisymmetrical part of this quantity:

$$\begin{aligned} B_\alpha{}^{[\lambda\mu]\nu} &= -\frac{2}{3}\left(N_\alpha{}^{[\lambda\mu]\nu} - N_\alpha{}^{\nu[\lambda\mu]}\right) + c_\alpha{}^{[\lambda\mu]\nu} \\ &= \frac{2}{3} N_\alpha{}^{\nu[\lambda\mu]} + c_\alpha{}^{\lambda\mu\nu}, \end{aligned} \tag{D17}$$

where properties (D7) and (D10) have been used.

### 2. Determination of the tensor $\mathbf{A}$

Rewriting the equation (D2) as

$$\begin{aligned} A_\alpha{}^{\mu\nu} &= -M_\alpha{}^{[\mu\nu]} - M_\alpha{}^{(\mu\nu)} + \overset{*}{\nabla}_\lambda B_\alpha{}^{(\mu\nu)\lambda} \\ &+ \left(\overset{*}{\nabla}_\lambda B_\alpha{}^{[\mu\nu]\lambda} - \tfrac{1}{2} T^\mu{}_{\rho\sigma} B_\alpha{}^{[\rho\sigma]\nu} + \tfrac{1}{2} B_\alpha{}^{\mu[\rho\sigma]} T^\nu{}_{\rho\sigma}\right), \end{aligned}$$

substituting here the expressions (D3) and (D17), and taking into account the identity (48), one obtains

$$\begin{aligned} A_\alpha{}^{\mu\nu} &= -M_\alpha{}^{[\mu\nu]} + \frac{2}{3}\left(\overset{*}{\nabla}_\lambda N_\alpha{}^{\lambda[\mu\nu]} + \frac{1}{2} T^{[\mu}{}_{\rho\sigma} N_\alpha{}^{\nu]\rho\sigma}\right) \\ &+ \left(\overset{*}{\nabla}_\lambda c_\alpha{}^{\mu\nu\lambda} + c_\alpha{}^{[\mu|\rho\sigma} T^{|\nu]}{}_{\rho\sigma}\right). \end{aligned} \tag{D18}$$

### 3. The use of the equation (D1)

Up to now only the equations (D2) and (D3) from the system (D1) – (D3) has been used. Now, turn to the equation (D1). Substitution of the expressions (D16) and (D18) into (D1), and taking into account the Klein identities (47) – (49) lead to

$$\begin{aligned} &\overset{*}{\nabla}_\mu \left[\overset{*}{\nabla}_\nu c_\alpha{}^{\lambda\mu\nu} + c_\alpha{}^{[\lambda|\rho\sigma} T^{|\mu]}{}_{\rho\sigma}\right] \\ &+ \frac{1}{2} T^\lambda{}_{\rho\sigma}\left[\overset{*}{\nabla}_\mu c_\alpha{}^{\rho\sigma\mu} + c_\alpha{}^{[\rho|\varepsilon\kappa} T^{|\sigma]}{}_{\varepsilon\kappa}\right] + \frac{1}{2} R^\varepsilon{}_{\alpha\rho\sigma} c_\varepsilon{}^{\rho\sigma\mu} = 0. \end{aligned}$$

After taking into account the identity (C3) and using the symmetry property (D7) this equality transfers to

$$\left(R^\mu{}_{[\pi\rho\sigma]} - \nabla_{[\pi} T^\mu{}_{\rho\sigma]} - T^\mu{}_{\varepsilon[\pi} T^\varepsilon{}_{\rho\sigma]}\right) c_\alpha{}^{\pi\rho\sigma} = 0.$$

The last, by the Ricci identities (A21), is satisfied identically with an *arbitrary tensor* **c**.

Thus, the general solution to the system of equations (D1) – (D3) are presented by the formulae (D16) and (D18) where $\{c_\alpha{}^{[\lambda\mu\nu]} = c_\alpha{}^{\lambda\mu\nu}\}$ is undefined tensor. In section IV, we show why without loss of a generality one can set $c_\alpha{}^{\lambda\mu\nu} = 0$.


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[116] Remark that the position of indexes in definitions of $\mathbf{M}$ (21) and $\mathbf{N}$ (22) differs from that of Refs.[39,40,143–145]. Thus $\{M_\alpha{}^{\beta\mu}\}$ and $\{N_\alpha{}^{\beta\gamma\mu}\}$ defined here correspond to $\{\hat{m}_\alpha{}^{\mu\beta}\}$ and $\{\hat{n}_\alpha{}^{\mu\beta\gamma}\}$ in the cited works. The definition of the tensor $\mathbf{U}$ (20) is also changed. Thus, the tensor $\{U_\alpha{}^\mu + I_\alpha{}^\mu\}$ corresponds to the tensor density $\{\hat{u}_\alpha{}^\mu\}$. At last, the current $\{\mathscr{J}^\mu = J^\mu + I^\mu\}$ (51) corresponds to the current $\{\hat{\imath}^\mu\}$.
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