

On the Meaning of the Principle of General Covariance

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Abstract

We present a definite formulation of the Principle of General Covariance (GCP) as a Principle of General Relativity with physical content and thus susceptible of verification or contradiction. To that end it is useful to introduce a kind of coordinates, that we call quasi-Minkowskian coordinates (QMC), as an empirical extension of the Minkowskian coordinates employed by the inertial observers in flat space-time to general observers in the curved situations in presence of gravitation. The QMC are operationally defined by some of the operational protocols through which the inertial observers determine their Minkowskian coordinates and may be mathematically characterized in a neighbourhood of the world-line of the corresponding observer. It is taken care of the fact that the set of all the operational protocols which are equivalent to measure a quantity in flat space-time split into inequivalent subsets of operational prescriptions under the presence of a gravitational field or when the observer is not inertial. We deal with the Hole Argument by resorting to the idea of the QMC and show how it is the metric field that supplies the physical meaning of coordinates and individuates point-events in regions of space-time where no other fields exist. Because of that the GCP has also value as a guiding principle supporting Einstein's appreciation of its heuristic worth in his reply to Kretschmann in 1918.

I. INTRODUCTION

Since first formulated nine decades ago the question of the meaning of the GCP has been a subject of polemic and confusion. Thus Kretschmann [1] in 1917 claimed the GCP to be devoid of physical content and that given enough mathematical ingenuity any

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theory could be set in a general covariant form. Einstein [2] begrudgingly accepted the objection stating however the heuristic value the GCP had in searching for a good theory and that that was a reason to prefer General Relativity to Newtonian gravitation which -in his opinion- would only be awkwardly casted into generally covariant form. Einstein was soon proved wrong as Cartan [3] in 1923 and Friedrichs [4] in 1927 found serviceable generally covariant formulations of Newtonian gravitation theory. See also Misner *et al*(1973, ch 12) [5]. In his excellent book Fock [6] makes interesting and critical remarks about the term “general relativity” adopted by Einstein to name his theory of gravitation and the connection of the term with general covariance that, in his view, is merely a logical requirement that is always satisfiable. Fock rightly points out that though Einstein had agreed with Kretschman objection as to the physical vacuity of the GCP his agreement was rather formal, because actually to the end of his life Einstein related the requirement of general covariance to the idea of some kind of “general relativity” and with the equivalence of all frames of reference. The subject has subsequently been addressed in several ways, for example, by Anderson [7](1967), Stachel [8](1980, 1986, 2002), Norton [9](1993) and Ellis and Matravers [10](1995). All these works while attempting to clarify the formulation and meaning of the GCP in our opinion fail to give it a specific expression susceptible of physical verification or contradiction. And certainly whatever the claim about the physical content of the GCP might be that should be subject to experimental test to be confirmed or refuted. In fact, though not directly dealing with the GCP but acknowledging that it has a conceptual content far more deeper than the simple invariance under arbitrary changes of coordinates, Lusanna and Pauri [13] (2006) go a significant step beyond previous authors considering the physical individuation of space-time points by experimental procedures and revise the Hole Argument. We fully subscribe to their contention that “the gauge freedom of general relativity is unavoidably entangled with the definition-constitution of the very *stage*, space-time, where the *play* of physics is enacted” and lend further support to that idea by discussing the Hole Argument using operationally defined coordinates introduced in our formulation of the GCP. Ellis and Matravers [10] point out how physicists and astrophysicists in fact almost always use preferred coordinate systems not merely to simplify the calculations but also to help define quantities of physical interest, and that this suggests that we should reconsider and perhaps refine the dogma of general covariance. In that spirit we present in this contribution a proposal for the GCP and show that it has

two meanings: a predictive one as a principle of general relativity, that in principle may be falsifiable by resorting to experience, and a heuristic one as a guiding principle to extend the theory and probe the nature of space-time.

The plan of the paper is as follows:

- In Section II we define a principle of general relativity and take it as the GCP.
- In Section III we construct a family of coordinates which are useful to endow with physical meaning the GCP.
- Section IV presents a treatment of the Hole Argument with the help of the mentioned family of coordinates and considers the implications of the GCP on the meaning of coordinates and point-events in space-time.
- In Section V some other consequences of our formulation of the GCP are indicated and finally our conclusions regarding the meaning of the GCP are stated.

II. THE GCP AS A PRINCIPLE OF GENERAL RELATIVITY

A. Principles of Restricted Relativity (RRP) and the Principle of Special Relativity (SRP)

We will first formulate a principle of restricted relativity (RRP) with respect to a group of isometric diffeomorphisms of space-time: Let us have a generic space-time (\mathbf{M}, \mathbf{g}) , \mathbf{M} and \mathbf{g} respectively denoting the manifold and the metric. Let x^λ , $\lambda = 0, 1, 2, 3$, be the coordinates corresponding to some neighbourhood N of an arbitrary point P of \mathbf{M} and let $F(P)$ be a physical quantity defined at P and possibly on N . Let $Q[F(P, x)]$ denote the set of all the different operational protocols -but equivalent in the sense that they yield the same values- that may be used to measure $F(P)$ when working in the said coordinates x^λ . The physical meaning of $F(P)$ is given by $Q[F(P, x)]$. If there is a group \mathcal{L} of isometries of \mathbf{g} we shall say that a RRP with respect to \mathcal{L} exists if the following two conditions hold:

- (a) The equations describing the behaviour of the physical quantities are form-invariant under all the transformations induced by the elements of \mathcal{L} .

- (b) One has that if $\Lambda \in \mathcal{L}$ and $\Lambda : x \longrightarrow x'$, $\Lambda : F(P, x) \longrightarrow F'(P, x')$, then $Q[F(P, x)] = Q[F'(P, x')] \forall \Lambda \in \mathcal{L}$ and $\forall F(P)$. Note in particular that $Q[x^\lambda] = Q[x'^\lambda]$ ¹.

A RRP is a symmetry principle with a clear physical meaning contained in the above two conditions. If \mathbf{g} has no isometry there is no RRP in the defined sense. The principle of special relativity is a RRP with respect to the *proper* Poincaré group of transformations². Let (\mathbf{M}, η) be flat space-time of special relativity and let \mathcal{I} stand for the set of all the inertial observers in it. Each of these observers is supposed to be located at the origin of a non-rotating Cartesian frame at rest in his or her proper reference frame with the help of which he or she assigns the spatial coordinates to events. The time coordinate of any event is assigned by each observer by the standard criteria of special relativity. The resulting coordinates are the Minkowskian coordinates in which the metric tensor is $\eta_{\mu\nu} = \text{diag}(+1, +1, +1, -1)$. Let O and O' be any two observers of \mathcal{I} and x and x' their respective Minkowskian coordinate systems. Then the SRP may be defined more specifically as follows:

- (a) The equations describing the behaviour of the physical quantities have the same form for O and O' when expressed in terms of x and x' .
- (b) If $F(P, x)$ and $F'(P', x')$ stand for the same physical quantity respectively measured by O and O' at points P and P' using Minkowskian coordinates x and x' , $Q[F(P, x)] = Q[F'(P', x')] \forall F$ and $\forall P$ and P' where F might be defined.

B. Principle of General Relativity (GRP)

We will introduce the GRP as a generalization in two directions of the SRP as previously defined. Both restrictions that of the existence of an isometry and that of referring to measurements performed only by inertial observers will be relinquished. The set \mathcal{I} will be enlarged to the class of observers, \mathcal{K} , having world-lines as differentiable as it may be needed for the subsequent developments. The Minkowskian coordinates employed by the inertial observers of \mathcal{I} will be generalized to coordinates used by the observers of \mathcal{K} operationally defined by some of the operational protocols through which the observers of \mathcal{I} determine their

¹ Wald calls principle of special covariance to a RRP [11].

² The improper elements of the full Poincaré group should be excluded on account of the existence of phenomena that violate parity and or time reversal symmetry.

Minkowskian coordinates. These more general coordinates will be called quasi-Minkowskian (QMC). In other words, if \tilde{x}^λ and x^λ respectively are quasi-Minkowskian and Minkowskian, it should hold that $\tilde{Q}[\tilde{x}^\lambda] \subseteq Q[x^\lambda]$, where the left-hand term denotes the different procedures of measurement of \tilde{x}^λ by the observer of \mathcal{K} in question. The latter relation follows from the expectation that the set $Q[x^\lambda]$ should split into inequivalent subsets of operational prescriptions under the presence of a gravitational field or when the observer is not inertial [14]. Let us then first characterize mathematically the QMC:

Let O be an observer in a generic space-time (\mathbf{M}, \mathbf{g}) following a world-line C given by its equations $x^\lambda = f^\lambda(\tau)$, where τ is O 's proper time. O 's four-velocity is $u^\lambda = \frac{dx^\lambda}{d\tau} = \dot{f}^\lambda(\tau)$, $u^\lambda u_\lambda = -c^2$. O transports an orthonormal tetrad $e_{(\nu)}$ along his world-line whose components verify

$$e^\lambda_{(\nu)} e_{(\mu)\lambda} = \eta_{\nu\mu} \quad , \quad e^\mu_{(0)} = \frac{u^\mu}{c} . \quad (1)$$

The most general transportation law of the tetrad that conserves these conditions is given by

$$\frac{De^\mu_{(\sigma)}}{d\tau} = \frac{1}{c^2}(u^\mu a^\nu - u^\nu a^\mu)e_{(\sigma)\nu} + \frac{1}{c}\omega_\alpha u_\beta \epsilon^{\alpha\beta\mu\nu} e_{(\sigma)\nu} , \quad (2)$$

where

$$a^\nu = \frac{Du^\nu}{d\tau} , \quad \epsilon_{\alpha\beta\gamma\delta} = (-g)^{\frac{1}{2}}[\alpha\beta\gamma\delta] , \quad \epsilon^{\alpha\beta\gamma\delta} = -(-g)^{-\frac{1}{2}}[\alpha\beta\gamma\delta] ,$$

$$[\alpha\beta\gamma\delta] = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of } 0123 \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of } 0123 \\ 0 & \text{if } \alpha\beta\gamma\delta \text{ are not all different} \end{cases}$$

$$g = \det ||g_{\alpha\beta}|| ,$$

and the ω_α 's are the covariant components of a rotation vector such that $u^\alpha \omega_\alpha = 0$.

As an example, the observer O could choose a set of QMC, $\tilde{x}^\lambda = (c\tilde{t}, \tilde{x}^i)$, $i = 1, 2, 3$, in the following way: He manages to send a radar signal at his proper time τ_e such that it arrives at the point-event whose QMC coordinates are to be determined just as it happens. The signal is immediately reflected back reaching O at his proper time τ_r . O defines $\tilde{x}^0 \equiv \frac{1}{2}c(\tau_r + \tau_e)$. He also sets $d \equiv \frac{1}{2}c(\tau_r - \tau_e)$ for his "distance" to the point-event and has previously noted the direction of the emitted radar signal by recording the cosines, $\cos \phi_i$, $i = 1, 2, 3$ of the angles, ϕ_i , $i = 1, 2, 3$, that the signal ray respectively makes with each of the directions of the spatial

triad $\mathbf{e}_{(i)}, i = 1, 2, 3$. Then O defines the remaining QMC as $\tilde{x}^i \equiv d \cos \phi_i, i = 1, 2, 3$. Let v^λ the components of a 4-vector along the emitted electromagnetic signal such that $v^0 = 1$. Then $\cos \phi_i = v^\lambda e_{(i)\lambda}, i = 1, 2, 3$, and since only two of the three $\cos \phi_i$ are independent it turns out that the four \tilde{x}^λ can be expressed in terms of four independent invariants, $\tau_e, \tau_r, v^\lambda e_{(i)\lambda}, i = 1, 2$, for instance³. These invariants are directly associated to the tools used by O to label the events but, since any other set of coordinates in a small enough neighborhood of his world-line will be functionally related to the \tilde{x}^λ , any coordinates will also be so functions of the said four invariants albeit different ones.

More generally, we will impose the following mathematical conditions to characterize the QMC coordinates, $\tilde{x}^\lambda = (c\tau, \tilde{x}^i) = (c\tilde{t}, \tilde{x}^i), i = 1, 2, 3$:

1. C is described in the \tilde{x}^λ coordinates by: $\tilde{x}^i = 0, \tilde{t} = \tau$.
2. The restriction of the metric in the \tilde{x}^λ coordinates on C is: $\tilde{g}_{\mu\nu} |_{C=} = \eta_{\mu\nu}$, and $\frac{\partial \tilde{g}_{\mu\nu}}{\partial \tilde{x}^\lambda} |_{C=} = 0$, when the four-acceleration of O and the four-rotation of the tetrad vanish: $\mathbf{a} = \omega = 0$.
3. $\mathbf{e}_{(\alpha)} = \frac{\partial}{\partial \tilde{x}^\alpha} |_{C} \iff e_{(\alpha)}^\lambda(\tau) = \frac{\partial x^\lambda}{\partial \tilde{x}^\alpha} |_{C}$.
4. The \tilde{x}^λ 's become the usual Minkowskian coordinates in a neighborhood of C when $\mathbf{a} = \omega = 0$ and the curvature tensor vanishes, $R_{\alpha\beta\mu\nu} = 0$, within that neighborhood.

The \tilde{x}^λ 's are QMC and since they are so with respect to a world-line C and depend on the choice of the space triad, $\mathbf{e}_{(i)}$, they will be denoted henceforth by $\text{QMCC}\omega$ (quasi-Minkowskian coordinates relative to the world-line C and to the tetrad $\mathbf{e}_{(\alpha)}$ subject to the rotation ω). Fermi coordinates and the coordinates introduced by Lachièze-Rey [12]⁴ are other particular instances of $\text{QMCC}\omega$'s.

1. Principle of General Relativity (GRP)

Let us have a generic smooth enough space-time (\mathbf{M}, \mathbf{g}) , \mathbf{M} and \mathbf{g} respectively denoting the manifold and the metric. Let O be any observer in that space-time belonging to the

³ τ_e and τ_r are invariants once the origin for the proper time has been set which amounts to setting the origin of the time $\tilde{t} \equiv \frac{1}{c}\tilde{x}^0$.

⁴ I am indebted to Lluís Bel for bringing to my attention this reference.

class \mathcal{K} that uses any type of QMCC ω 's. If F is any -generally multicomponent- physical quantity let us denote by \tilde{F} the same quantity -or its components- in the QMCC ω , \tilde{x}^λ 's, as determined by O . Let us denote by $\tilde{Q}[\tilde{F}]$ the set of operational protocols that may be used by O to measure \tilde{F} .

We shall say that the Principle of General relativity (GRP) is verified when the following two conditions hold:

- (a) The equations describing the behaviour of the physical quantities have the same form in all sufficiently regular coordinate systems with the metric \mathbf{g} being the only quantity pertaining to space-time that can appear in those equations; moreover, the latter become on C their corresponding ones in Minkowskian coordinates in flat space-time (\mathbf{M}, η) when they are expressed in terms of the \tilde{x}^λ 's if $\mathbf{a} = \omega = 0$. (This implies the Equivalence Principle in general, but curvature dependent terms may still appear if there is no minimal coupling or higher order derivatives are involved in the equations.)
- (b) One has $\tilde{Q}[\tilde{F}] \subseteq Q[F]$, $\forall F$ and \forall QMCC ω .

Note that if \tilde{x}^λ and \tilde{x}'^λ are two different systems of QMCC ω 's for O -with common \mathbf{a}, ω and $\mathbf{e}_{(\alpha)}$ - in (\mathbf{M}, \mathbf{g}) , one has in general that $\tilde{Q}[\tilde{x}^\lambda] \neq \tilde{Q}'[\tilde{x}'^\lambda]$, and if x^λ are Minkowskian coordinates in flat space-time (\mathbf{M}, η) and $q[x^\lambda] \in Q[x^\lambda]$, there will be QMCC ω 's, $\tilde{x}^\lambda, \forall O(C, \omega)$, such that $q[x^\lambda] \in \tilde{Q}[\tilde{x}^\lambda] \subseteq Q[x^\lambda]$.

2. Principle of General Covariance (GCP)

The GCP is the above GRP.

- It implies the Equivalence Principle.
- The physical content of the GCP is thus that one given in the previous definition of the GRP.

What is called for now is to verify that this interpretation of the GCP is coherent with the theory and agrees with experience. That is therefore tantamount to testing the validity of the stated GRP. To that end we proceed to construct the QMCC ω 's in the next Section.

III. CONSTRUCTION OF THE QMCC ω 'S

The \tilde{x}^λ 's will be constructed under the assumption that the x^λ 's can be expressed as power series of the \tilde{x}^i 's with coefficients depending on τ about C . The transportation law (2) sets the following constraints on C upon the Christoffel symbols $\tilde{\Gamma}_{\mu\nu}^\rho(\tau)$ and $\tilde{\Gamma}_{\mu\nu\rho}(\tau)$:

$$\tilde{\Gamma}_{00}^0 = \tilde{\Gamma}_{000} = 0, \tilde{\Gamma}_{k0}^0 = -\tilde{\Gamma}_{0k0} = \tilde{\Gamma}_{k00} = \tilde{\Gamma}_{00}^k = \frac{1}{c^2}\tilde{a}^k, \tilde{\Gamma}_{k0}^j = \tilde{\Gamma}_{jk0} = -\frac{1}{c}\tilde{\omega}^i\tilde{\epsilon}_{0ijk}, \quad (3)$$

where the tildes always indicate that the components of the quantities are in the QMCC ω coordinates.

Noting that $\tilde{a}^0 = \tilde{\omega}^0 = 0$ and assuming that the remaining Christoffel symbols on C may be expressed as power series of the components of the 4-acceleration and rotation, condition 2 of subsection IIB allows us to put for them

$$\tilde{\Gamma}_{jk}^\alpha(\tau) = p_{jki}^\alpha(\tau)\tilde{a}^i + q_{jki}^\alpha(\tau)\tilde{\omega}^i + O_{jk}^\alpha(2), \quad (4)$$

with $p_{jki}^\alpha(\tau)$ and $q_{jki}^\alpha(\tau)$ being smooth enough but otherwise arbitrary functions of the proper time along C , τ , and $O_{jk}^\alpha(2)$ similarly standing for a second order term in the \tilde{a}^i 's and $\tilde{\omega}^i$'s. Obviously all these functions are symmetric in the indices j and k . Eq. (4) may be rewritten in terms of the components of \mathbf{a} and ω in the given coordinates x as

$$\tilde{\Gamma}_{ik}^\alpha(\tau) = A_{ik\nu}^\alpha(\tau)a^\nu + B_{ik\nu}^\alpha(\tau)\omega^\nu + O_{ik}^\alpha(2), \quad (5)$$

where the $A_{ik\nu}^\alpha$ and $B_{ik\nu}^\alpha$ are arbitrary save by being sufficiently smooth and the constraints

$$A_{ik\nu}^\alpha = A_{kiv}^\alpha, \quad B_{ik\nu}^\alpha = B_{kiv}^\alpha, \quad A_{ik\nu}^\alpha e_{(0)}^\nu = B_{ik\nu}^\alpha e_{(0)}^\nu = 0; \quad (6)$$

and $O_{ik}^\alpha(2) = O_{ki}^\alpha(2)$ is of second order in the a^ν 's and ω^ν 's, otherwise $O_{ik}^\alpha(2)$ is a function of τ as so is $\Gamma_{\rho\sigma}^\lambda(\tau)$, as both are defined on C .

Taking into account condition 3 of subsection IIB and using the transformation law for the metric connection on C we get

$$x^\lambda = f^\lambda(\tau) + e_{(k)}^\lambda\tilde{x}^k + \frac{1}{2}(e_{(\alpha)}^\lambda\tilde{\Gamma}_{ik}^\alpha(\tau) - e_{(i)}^\rho e_{(k)}^\sigma\Gamma_{\rho\sigma}^\lambda(\tau))\tilde{x}^i\tilde{x}^k + \Phi^\lambda(\tilde{x}), \quad (7)$$

with $\Phi^\lambda(\tilde{x})$ being of third order in the \tilde{x}^i 's. In order to ensure the fulfilment of condition 4 of subsection IIB we split $\Phi^\lambda(\tilde{x})$ into two terms as follows:

$$\Phi^\lambda(\tilde{x}) = \Psi^\lambda(\tilde{x}) + \Phi_{(0)}^\lambda(\tilde{x}), \quad (8)$$

with $\Psi^\lambda(\tilde{x})$ verifying

$$\Psi^\lambda|_C = \frac{\partial\Psi^\lambda}{\partial\tilde{x}^i}|_C = \frac{\partial^2\Psi^\lambda}{\partial\tilde{x}^i\partial\tilde{x}^j}|_C = 0, \quad (9)$$

and vanishing whenever \mathbf{a}, ω and the curvature tensor, \mathbf{R} , in a finite neighborhood of C , all vanish; the latter is equivalent to the vanishing of the $\tilde{\Gamma}_{\mu\nu}^\lambda$'s in that neighborhood; otherwise the Ψ^λ 's apart from being sufficiently smooth are arbitrary; it is $\Phi_{(0)}^\lambda(\tilde{x})$ that should be determined to assure that the coordinates \tilde{x} have the property specified in condition 4 of subsection II B. To that end we put

$$\Phi_{(0)}^\lambda(\tilde{x}) = \sum_{l,m,n} \frac{1}{l!m!n!} C_{lmn}^\lambda(\tau) (\tilde{x}^1)^l (\tilde{x}^2)^m (\tilde{x}^3)^n, \quad (10)$$

with

$$l \geq 0, m \geq 0, n \geq 0, l + m + n \geq 3, \text{ and } l, m, n \text{ all being integers,}$$

and the $C_{lmn}^\lambda(\tau)$'s are systematically calculated by the following algorithm: Consider the equation

$$\frac{\partial^2\Phi_{(0)}^\lambda}{\partial\tilde{x}^i\partial\tilde{x}^k} = -\frac{\partial x^\rho}{\partial\tilde{x}^i} \frac{\partial x^\sigma}{\partial\tilde{x}^k} \Gamma_{\rho\sigma}^\lambda(x(\tilde{x}^\mu)) + e_{(i)}^\beta e_{(k)}^\gamma \Gamma_{\beta\gamma}^\lambda(\tau), \quad (11)$$

where the first term on the rhs is taken as dependent, in general, on the \tilde{x}^μ 's, while the second term only depends on τ as is evaluated on C . Eq. (11) is a consequence of considering the equation for the transformation of the Christoffel symbols on C and using eqs. (7) and (8) besides setting $\tilde{\Gamma}_{\mu\nu}^\alpha = \Psi^\lambda = 0$. The $C_{lmn}^\lambda(\tau)$'s are found by using the power series for $\Phi_{(0)}^\lambda$ given in eq. (10) and taking successive derivatives of eq. (11) with respect to the \tilde{x}^j 's, taking the result on C , and doing it all along as if $\tilde{\Gamma}_{\mu\nu}^\alpha = \Psi^\lambda = 0$ at all points.

So we get, for instance,

$$C_{300}^\lambda(\tau) = -\frac{\partial}{\partial\tilde{x}^1} \left(\frac{\partial x^\rho}{\partial\tilde{x}^1} \frac{\partial x^\sigma}{\partial\tilde{x}^1} \Gamma_{\rho\sigma}^\lambda(x) \right) |_C = -4e_{(1)}^\sigma A_{11}^\rho \Gamma_{\rho\sigma}^\lambda - e_{(1)}^\rho e_{(1)}^\sigma \Gamma_{\rho\sigma,\gamma}^\lambda e_{(1)}^\gamma,$$

with $A_{ik}^\lambda = -\frac{1}{2}e_{(i)}^\rho e_{(k)}^\sigma \Gamma_{\rho\sigma}^\lambda(\tau)$, all evaluated on C at the point corresponding to τ . That way any $C_{lmn}^\lambda(\tau)$ may be expressed in terms of the $C_{l'm'n'}^\lambda$'s of lower order: $l' + m' + n' < l + m + n$, $l' \leq l$, $m' \leq m$, $n' \leq n$; the $e_{(k)}^\lambda(\tau)$, the $\Gamma_{\mu\nu}^\alpha(\tau)$'s, and the partial derivatives of the $\Gamma_{\mu\nu}^\alpha$'s with respect to the x^λ 's up to order $\leq l + m + n - 2$.

Fixing the Ψ^λ 's, $A_{ik\nu}^\alpha$'s, $B_{ik\nu}^\alpha$'s and the $O_{ik}^\alpha(2)$'s uniquely determines a set of corresponding QMCC ω . If space-time is flat and also $\mathbf{a} = \omega = 0$, it follows that $\Psi^\lambda = \tilde{\Gamma}_{\mu\nu}^\alpha(\tilde{x}) = 0$, and the entire family of the QMCC ω 's collapses to the unique usual Lorentzian coordinates corresponding to the chosen tetrad $e_{(\nu)}^\lambda$. This does not mean that if space-time is not flat

and/or if $\mathbf{a} \neq 0$, or $\omega \neq 0$, by taking $\Psi^\lambda = A_{ik\nu}^\alpha = B_{ik\nu}^\alpha = O_{ik}^\alpha(2) = 0$, the resulting QMCC ω would be Lorentzian, as these do not simply exist for non-flat space-times or in non-inertial reference frames.

IV. THE HOLE ARGUMENT

The Hole Argument (HA) is a consideration that was first raised by Einstein in a letter of November 2, 1913, to Ludwig Hopf. He was then struggling to find the gravitational field equations and intended to prove with the HA, whose final Einsteinian version was published in 1914 [15], that the theory could not be generally covariant. Obviously Einstein had discarded that implication of the HA by November, 1915, with his settling upon the correct generally covariant field equations. The HA has been widely discussed in the literature (see, for instance, Stachel [8](1980, 2002), Rovelli [16] (1991, 2008), Norton [9] (1993) and Lusanna and Pauri [13] (2006)). Here we present a treatment of the HA by resorting to the idea of the QMCC ω 's introduced in subsection IIB that we believe further clarifies the whole issue, specially by showing how it is the metric field that supplies the physical meaning of coordinates and individuates point-events in regions of space-time where no other fields exist.

Essentially the HA goes as follows: Let us assume there is an region H (the hole) where all the nongravitational fields are null. Let that region be covered by coordinates x^λ that extend to a larger coordinate patch U : $H \subset U$. Let us consider a coordinate transformation

$$\phi : x \rightarrow x', x'^\lambda = \phi^\lambda(x), \quad (12)$$

that smoothly becomes the identity transformation outside H and on its boundary. Under the transformation the components of the metric tensor change according to

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x(x')) \quad (13)$$

Let σ be the map of U into \mathfrak{R}^4 that assigns the x coordinates to the point events in U and let us suppose that $\phi \circ \sigma(H) = \sigma(H) \subset \mathfrak{R}^4$ and also that all the coordinates involved, x as well as x' , are homogeneous quantities, say everyone of them is a length value. Then the GC of the Einstein equations assures that the metric $g'_{\mu\nu}(x) \forall x \in \sigma(U)$ provides a new solution to those equations if the argument coordinates in $g'_{\mu\nu}$ and $g_{\mu\nu}$ are interpreted as designating

the same events whenever they take the same values. This is the HA that leads to different solutions for a single mass-energy distribution and hence to a supposed inadequacy of GR as a consequence of its GC. It would violate causality in an obvious sense.

We shall see now how the introduction of QMCC ω 's dispels the difficulty posed by the HA. Let us have a set of QMCC ω 's, \tilde{x} , covering H or part of it. Let the given coordinates, x , be related to the \tilde{x} by

$$x^\lambda = \chi^\lambda(\tilde{x}) \quad (14)$$

that under suitable conditions might be expressed as in eq. (7) A concrete event in H may be labelled by its coordinates \tilde{x} as these are uniquely assigned to that event by a specific operational protocol. The particular physical process that the measurement act entails individuates the corresponding event. Eqs. (12) and (14) give

$$x'^\lambda = \phi^\lambda(x) = \phi^\lambda(\chi(\tilde{x})) \equiv \rho^\lambda(\tilde{x}) \quad (15)$$

In the preceding exposition of the HA the x' in the functions $g'_{\mu\nu}$ were called x to get the *new* metric $g'_{\mu\nu}(x)$. Coherently with that we should rewrite the above equation as

$$x^\lambda = \rho^\lambda(\tilde{x}) \quad (16)$$

and since the functions χ^λ and ρ^λ are different on H , unless (12) is the identity transformation, one may not have in general the same values for the \tilde{x}^α on the rhs's of eqs. (14) and (16) if one insists on having identical values for the x^λ on the lhs's of those equations. Thus we will have in general

$$x^\lambda = \chi^\lambda(\tilde{x}_1) = \rho^\lambda(\tilde{x}_2), \quad (17)$$

with $\tilde{x}_1 \neq \tilde{x}_2$, indicating that we are dealing in general with distinct events when they are labeled by the same values in the coordinates x and x' . Since the x^λ and the x'^λ are different functions of the \tilde{x}^α which have been given a precise operational meaning, the x^λ and the x'^λ have a distinct physical interpretation. The relationships of x and x' to \tilde{x} clearly respectively depend on the functional forms of the metric tensor $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x')$ in terms of those coordinates as explicitly follows from our construction in eq. (7) and implicitly and more generally from eqs. (12), (13), (14) and (15). Therefore one has to conclude that *it is the functional form of the metric that determines the physical meaning of its coordinate arguments*. Thus in GR coordinates in space-time are physically meaningless before specifying

the metric tensor though they designate a particular point of the underlying mathematical manifold \mathbf{M} , as has been pointed out differently by Stachel [8] and Norton [17](2002). This way one clearly sees that the HA is no objection to the requirement of GC for a metric theory such as GR. The preceding conclusion has an interesting corollary: *Let us ask ourselves if it would be possible, in a metric theory such as GR, to have two **different** space-times of respective metrics $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x')$ functionally related by eq. (13) and such that the physical (operational) meanings of the coordinates x and x' were the same. The answer should be in the negative!*

There is an alternative coordinate-independent way of presenting the HA that was first pointed out by Stachel in 1980 [8]. It is essentially equivalent to the one just given, but it has customarily become the modern account of the HA and provides other insights regarding the conclusions reached at the end of the preceding paragraph, particularly concerning the physical individuation of point-events in space-time as a consequence of the metric field. We will sketch it here for completeness and to find that both descriptions complement each other illuminating part of the deep significance of the GCP.

Let $\phi : M \rightarrow M$ be a sufficiently differentiable diffeomorphic map that becomes the identity map outside H and on its boundary so that also $\phi : H \rightarrow H$. Let p be an arbitrary point belonging to H , and x its given coordinates in a certain chart containing p . Likewise let x' be the coordinates of the diffeomorphic image of p , $\phi(p)$, in another chart covering this last point that may coincide or not with the former chart. We shall also denote by eq. (12) the functional correspondence induced by the diffeomorphism between the coordinates in the two charts associated to some neighbourhoods of p and $\phi(p)$. It is well known that the *active* diffeomorphism ϕ also generates a drag-along ϕ^* from tensors at p to $\phi(p)$. In particular the drag-along metric tensor at $\phi(p)$, $\phi^*\mathbf{g}$, has components $g'_{\mu\nu}(x')$ in the x' coordinates verifying eq. (13) above, with the terms $g_{\alpha\beta}(x)$ entering its rhs being now the components of \mathbf{g} at p in the x coordinates. The GC of GR again implies that the *new* metric $\phi^*\mathbf{g}$ satisfies as well the Einstein equations. It is true that the tensors \mathbf{g} and $\phi^*\mathbf{g}$ would be the same were they attached at the same point and the different coordinates x and x' corresponded to that point, but that is not the case. One has now a relocation of the metric field over the points of H in which $\phi^*\mathbf{g}$ is at $\phi(p)$, whereas \mathbf{g} , its geometrical equivalent, was at p before the drag-along. The answer to this version of the HA has been to assert that a unique physical solution of the Einstein equations is given by the class of equivalence, $\{(M, \phi^*\mathbf{g}), \forall\phi\}$, obtained by

considering the action of all possible diffeomorphisms of the previous kind in the way just explained. That equivalence has been called *Leibniz equivalence* in the literature. Along with this emerges the idea that the point-events of space-time are only individuated by the physical entities present at them, in our case by only the metric field that is the only physical field existing in the hole. In the language of the QMCC ω coordinates, \tilde{x} : *the two manifold points p and $\phi(p)$, with the respectively attached metrics, \mathbf{g} and $\phi^*\mathbf{g}$, correspond to the same event in space-time as their coordinates, x^λ and x'^λ , are related to the same values of the individuating \tilde{x} 's by the functions $\chi^\lambda(\tilde{x})$ and $\rho^\lambda(\tilde{x})$ whose forms -as was pointed out above- are precisely determined by the components of mentioned two geometrically equivalent metrics.*

V. REMARKS AND CONCLUSIONS

Consider a specific but generic observer O of world-line C who uses a chosen set of QMCC ω , \tilde{x}^λ 's. Let P be the space-time position of O at its proper time τ_P and let δx^α the components of an infinitesimal 4-vector with origin at P in the given coordinates x^λ . One may put

$$\delta x^\alpha = \delta x^\alpha_{\parallel} + \delta x^\alpha_{\perp} ,$$

where the last two terms respectively stand for the the parallel and perpedicular parts of δx^α to \mathbf{e}_0 . The proyector on the hyperplane normal to \mathbf{e}_0 is

$$g_{\alpha\beta} + \frac{1}{c^2}u_\alpha u_\beta .$$

Then the quantity

$$dl^2 \equiv (g_{\alpha\beta} + \frac{1}{c^2}u_\alpha u_\beta)\delta x^\alpha \delta x^\beta = \delta x^\alpha_{\perp} \delta x_{\alpha\perp} , \quad (18)$$

certainly is $\delta\tilde{x}^i \delta\tilde{x}_i$ and should therefore be interpreted as the spatial distance squared measured by O at time τ_P between P and the point-event Q at the tip of the vector δx^α . This is consistent with what follows from inverting eq. (7) neglecting higher order terms in the infinitesimals.

If the observer O happens to be in free fall, momentarily at rest at τ_P and his reference tetrad does not rotate he would then measure the proper distance between P and Q , and eq. (18) above yields for the square of that

$$dl^2 = \delta_{ik} \delta\tilde{x}^i \delta\tilde{x}^k = (g_{\alpha\beta} - \frac{g_{0\alpha}g_{0\beta}}{g_{00}})\delta x^\alpha \delta x^\beta , \quad (19)$$

that is the usually accepted result [18].

The relationships of the generic given coordinates x^λ and two different sets of QMCC ω , \tilde{x}^λ and \tilde{x}'^λ , may differ at most by terms of second order in the \tilde{x}^i 's and \tilde{x}'^i 's if the observer is not in free fall (C is not a geodesic) and/or his/her choiced transported tetrad rotates, which corresponds to the freedom allowed to choose the $\tilde{\Gamma}_{ik}^\alpha(\tau)$'s via eq. (5), or by terms of third order in the same variables if the observer is in free fall and its reference tetrad is paralell transported, corresponding to the freedom allowed to choose the function Φ^λ when the space-time is not flat.

It follows from eq. (7) that the values of tensor quantities measured on the world-line C corresponding to two different sets of QMCC ω 's, \tilde{x} and \tilde{x}' , -but with the same choice of reference tetrad- should be identical. However if these quantities are measured by the observer at small $\delta\tilde{x}^k$, equivalently $\delta\tilde{x}'^k$, off his world-line C one has, for instance and with no loss of generality, for the components of the electromagnetic field tensor, $F^{\lambda\mu}$, in the two sets of coordinates:

$$\tilde{F}^{\lambda\mu} = \tilde{F}'^{\lambda\mu} + (C_{ik\nu}^\lambda \tilde{F}'^{i\mu} + C_{ik\nu}^\mu \tilde{F}'^{\lambda i}) a^\nu \delta\tilde{x}'^k + (D_{ik\nu}^\lambda \tilde{F}'^{i\mu} + D_{ik\nu}^\mu \tilde{F}'^{\lambda i}) \omega^\nu \delta\tilde{x}'^k + h.o.t.,$$

where eqs. (7) and (5) have been used and we have put $C_{ik\nu}^\lambda \equiv A_{ik\nu}^{\lambda'} - A_{ik\nu}^\lambda$ and $D_{ik\nu}^\lambda \equiv B_{ik\nu}^{\lambda'} - B_{ik\nu}^\lambda$, $A_{ik\nu}^{\lambda'}$ and $B_{ik\nu}^{\lambda'}$ being the coefficients that correspond to $\tilde{\Gamma}_{ik}^{\lambda'}(\tau)$ in its expression analogous to eq. (5) for $\tilde{\Gamma}_{ik}^\alpha(\tau)$; *h.o.t.* stands for a series of first or higher order terms in the spatial displacements $\delta\tilde{x}'^k$ and higher order terms in the a^ν and/or ω^ν when the $\delta\tilde{x}'^k$ occur only to first order. That means that in general, in the vicinity of the observer, the discrepancies between the outcomes of the different *standard* measurement methods of the same physical quantity are more sensitive to local inertial effects, when they exist, than to gravitational fields.

After all the foregoing considerations we can say that the meaning of the GCP is, at least, two-fold: On the one side, as a GRP such as we defined it in Section II, it is a really predictive physical principle like the SRP, but with the generalizations and conditions specified thereby, namely replacement of inertial observers by general ones, of Minkowskian coordinates by quasi- Minkowskian ones, the appearance of the space-time metric as a new physical tensor quantity and the splitting induced in the results of the measurements of the same physical quantities when different measurement protocols are used though they be equivalent in the absence of gravitation or inertial effects. On the other side, as has been

shown in Section IV with our discussion of the HA, it provides deep insights on how the nature of coordinates depends on the form of the gravitational fields that, consistently with that, are the entities that individuate -or, together with other physical entities that might be present, contribute to individuate- the point-events of space-time. Because of that the GCP has also value as a guiding principle supporting Einstein's appreciation of its heuristic worth in his reply to Kretschmann [2] (1918). So it would seem to favor quantum theories of gravitation without *a priori* background space-time against those theories that assume such background structure *ab initio* as has also been pointed out by Lusanna and Pauri [13] (2006).

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