

Lectures on Constrained Systems

Ghanashyam Date

The Institute of Mathematical Sciences

C.I.T. Campus, Tharamani, Chennai, 600 113, India.

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Preface

These lecture notes have been prepared as a basic introduction to the theory of constrained systems which is how basic forces of nature appear in their Hamiltonian formulation. Only a basic familiarity of Lagrangian and Hamiltonian formulation of mechanics is assumed.

The First chapter makes some introductory remarks indicating the context in which various types of constrained systems arise. It distinguishes constrained systems for which the equations of motion are uniquely specified from those for which the equations of motion are *not* uniquely determined. The focus of the lectures is on the latter types of constraints. The notations that will be used is introduced here.

In the second chapter, the general features are introduced in the familiar example of source-free electrodynamics.

In the third chapter, features seen in electrodynamics are abstracted in the simpler context of systems with finitely many degrees of freedom. How constraints arise in a Lagrangian and in a Hamiltonian formulation is discussed.

In the fourth chapter, the Dirac's algorithm for discovering constraints and their classification is discussed in a sufficiently general context.

In the fifth chapter, a special class of constrained systems for which the Hamiltonian itself is a constraint is discussed. Such systems arise in the context of general relativity and lead to a host of issues of interpretation of 'dynamics'.

A set of exercises is also included for practice in the last chapter.

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Contents

1	Introductory Remarks	4
2	Hamiltonian Formulation of Electrodynamics	8
3	Constrained Lagrangian and Hamiltonian Systems	12
4	Dirac-Bergmann theory of Constrained Hamiltonian Systems	17
5	Systems with the Hamiltonian as a constraint	24
6	Exercises	27

Chapter 1

Introductory Remarks

Let us begin by recalling some elementary understanding of what one means by mechanics. There are two distinct parts: (a) kinematics and (b) dynamics. Kinematics specifies the (possible) *states* of a given system while dynamics specifies how a system evolves from one state to another state. The central problem one wants to solve is to find the state of system at some time t given its state at an (earlier) time t_0 . While there are various types of ‘state spaces’ possible, we will be dealing with those systems for which the state space is a *manifold* (finite or infinite dimensional) and an evolution which is of the continuous variety.

There are three distinct tasks before us (i) devise a *framework* or a calculational prescription which will take as input a state space, some quantity reflecting/encoding a dynamics and give us a corresponding rule to evolve any given state of the system. A familiar example is the Lagrangian framework wherein the state space is described in terms of positions and velocities, dynamics is encoded in terms of a Lagrangian function and the variational principle leads to the Euler-Lagrange equations of motion. (ii) Use physical, qualitative analysis of a given physical system to associate a particular state space and a particular Lagrangian (say) with the physical system and (iii) devise methods to obtain generic evolutions in sufficiently explicit terms.

The last task is where one will make ‘predictions’ and is the most relevant in applications (eg engineering). Here the issues such as sensitivity to initial conditions etc play an important role. This task can be addressed only after the first two are specified. This is *not* the aspect we will discuss in these lectures. Some of it will be discussed in the NLDL part of the course.

To be definite, let us consider the Lagrangian framework. Thus we will have some *configuration space*, Q , which is some n dimensional manifold with coordinates $q^i, i = 1, 2, \dots, n$. The dynamics is encoded in a function $L(q^i, \dot{q}^i, t)$ and the equations of motion are the Euler-Lagrange equations of motion,

$$\delta S = \delta \int dt L(q, \dot{q}) = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad \forall i. \quad (1.1)$$

Questions: How general can Q be? How general can L be? Can we *always* obtain equations of motion via the variational principle?

Generally, one takes Q to be an n -dimensional manifold. However, this *need not* be \mathbb{R}^N , although usually it is. For instance one may begin with n particles moving in 3 dimensions,

so that Q is $3n$ dimensional, but there could be restrictions such as distance between any two particles is always fixed (rigid body) or that particles must have velocities tangential to the surface of a ball. In the latter case, the relevant Q would be the 2-sphere which is a compact manifold without boundary. In both cases the *dimension* of the relevant configuration space is smaller than what we began with. We could also imagine confining the particles to a 3 dimensional box in which case the relevant configuration space is a bounded portion of \mathbb{R}^{3n} . In short, the possible motion of the system may be *constrained*. A majority of applied mechanics problems have to deal with such constrained systems and there is a massive recent treatise on such systems [1]. Typically, such constrained systems are described by specifying relations $f_\alpha(t, \vec{r}, \vec{v}) = 0$. These relations are required to be *functionally independent*, mutually consistent, and valid for *all possible forces*. Some terminology that you might have come across classifies the constraints as: (1) Relations independent of velocities are termed *positional/holonomic/finite constraints*; (2) Velocity dependent ones are called *velocity constraints* which are further divided into *scleronomous* (time independent), *rheonomous* (time dependent), *holonomic* (integrable), *non-holonomic* (non-integrable). Constrained systems of this variety, serve to restrict the configuration space and also possible motions, but the *Euler-Lagrange equations are always solvable for accelerations, so that dynamics is uniquely specified*. We will *not* be dealing with this type of constrained systems.

The constrained systems that we will deal with arise in situations where the well motivated choices of the configuration space and the Lagrangian, do *not* specify the equations of motion uniquely. In the Lagrangian framework, these correspond to the so called *singular* Lagrangians. Analysis of such systems is better carried in the Hamiltonian framework wherein one arrives at an understanding of *gauge theories* and we will be essentially focusing on such *constrained Hamiltonian systems*.

From the point of view of usual applications of classical mechanics, such systems would appear quite exotic and possibly ‘irrelevant’. However *all* the four fundamental interactions that we know of, when cast in a Lagrangian or Hamiltonian framework, *precisely* correspond to the kind of constrained systems we will discuss. An understanding of these basic forces at the classical structural level is crucial also for constructing/understanding the corresponding quantum theories. In the context of general relativity, the constrained nature of the theory throws up challenging conceptual and interpretational issues. With these reasons as primary motivations, we will discuss constrained systems.

We will also use certain notations which are introduced below.

1. I will freely use terms such as ‘manifold’, ‘tensors’ etc. Here is a very rapid, heuristic introduction to these terms.

The idea of a *manifold* (actually a differentiable manifold), is invented to be able to do *differential and integral calculus* on arbitrarily complicated spaces such as (say) the surface of an arbitrarily shaped balloon. The basic definition of differentiation involves taking differences of values of a function at two nearby points eg., $f(x+h) - f(x)$, dividing it by h and taking the limit $h \rightarrow 0$. This is fine when x, h etc are numbers. But when we go to the surface of a balloon, we do not know how to implement such definitions eg how to take the ‘difference of two points’ on the surface. The way the idea of differentiation is captured is to *assign* numbers to points and then use them in the usual manner. This assignment can be thought of as pasting small pieces of a graph paper on the surface, and reading off the numbers corresponding to the points just below the numbers. This pasting gives a set of *local coordinates* in that portion

of the surface. Clearly there is huge freedom in the choice of the pieces of graph paper as well as in the pasting. Thus, while local coordinates can be introduced, the arbitrariness must be respected. Changing these assignments is called *local coordinate transformations*. Likewise, quantities such as tiny arrows stuck on the surface can be described in terms of *components* of the arrows with respect to the coordinate axes provided by the graph paper, When the graph paper is changed, these components change, but *not* the arrows themselves. Consequently, the two sets of components must be related to the local coordinate transformations in a specific manner so that both the sets refer to the same arrow. The arrows are an example of a *vector field* which can be thought of as a collection of their components, transforming in a specific manner. The components are denoted as quantities with arbitrary number of *upper and lower indices*. These must transform as,

$$(X')_{j_1, \dots, j_n}^{i_1, \dots, i_m}(y(x)) = \frac{\partial y^{i_1}}{\partial x^{r_1}} \dots \frac{\partial y^{i_m}}{\partial x^{r_m}} \frac{\partial x^{s_1}}{\partial y^{j_1}} \dots \frac{\partial x^{s_m}}{\partial y^{j_m}} (X)_{s_1, \dots, s_n}^{r_1, \dots, r_m}(x) \quad (1.2)$$

Objects represented by such indexed quantities have a meaning *independent* of the choice of local coordinates and are called *Tensors of contravariant rank m and covariant rank n*. Equations involving tensors (correctly matched index distribution) are *covariant* (or form invariant) with respect to local coordinate transformations. This is all that we will need to know.

2. In the context of Lagrangian framework, the generalized coordinates will be denoted by q^i . These are not necessarily Cartesian and are to be thought of as arbitrary local coordinates on the configuration space manifold. Consequently, we will occasionally also comment on whether various equations/conditions are ‘covariant/invariant’ under coordinate transformations. For these purposes, tensor notation will be used freely. Lagrangian will always be taken to be a function of generalized coordinates and velocities and independent of time.
3. In the context of Hamiltonian framework, the generalized *canonical* coordinates will be denoted as q^i, p_i . It will be convenient to subsume them as $2n$ coordinates ω^μ . The phase space will be denoted as Γ which is a $2n$ dimensional manifold. On this there is a distinguished antisymmetric rank 2 tensor, $\Omega_{\mu\nu}(\omega)$ which satisfies further properties namely,

$$\partial_\lambda \Omega_{\mu\nu} + \text{cyclic} = 0. \quad (1.3)$$

The antisymmetric matrix $\Omega_{\mu\nu}$ is assumed to be *invertible* and its inverse is denoted as $\Omega^{\mu\nu}$, $\Omega^{\mu\lambda} \Omega_{\lambda\nu} = \delta_\nu^\mu$.

Such an antisymmetric tensor is called a *symplectic form*. The coordinates in which the matrix takes the block off-diagonal form

$$\Omega^{\mu\nu} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \Omega_{\mu\nu} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad (1.4)$$

are the *canonical coordinates* and these are guaranteed to exist by Darboux theorem.

4. The usual Poisson bracket of two functions $f(\omega), g(\omega)$ get expressed as,

$$\{f(\omega), g(\omega)\} = \Omega^{\mu\nu} \frac{\partial f}{\partial \omega^\mu} \frac{\partial g}{\partial \omega^\nu} \quad (1.5)$$

The (1.3) condition implies the Jacobi identity of Poisson brackets.

For every function f on the phase space, one can associate the so-called *Hamiltonian* vector field, $v_f^\mu := \Omega^{\mu\nu} \partial_\nu f$. Such Hamiltonian vector fields generate *infinitesimal canonical transformations* as,

$$\omega^\mu \rightarrow \omega'^\mu := \omega^\mu + \epsilon v_f^\mu(\omega). \quad (1.6)$$

It is easy to check that the symplectic form is invariant under these coordinate transformations (and hence these are called canonical transformations). In particular, the Hamiltonian function generates time dependent changes in coordinates which is nothing but the dynamical evolution. Hence, dynamical evolution can be viewed as a “continuous unfolding of canonical transformations”. Taking $\epsilon := \delta t$, $f = H$ implies the Hamilton’s equations of motion,

$$\frac{d\omega^\mu}{dt} = \Omega^{\mu\nu} \frac{\partial H}{\partial \omega^\nu} \quad (1.7)$$

Note that these are not just any system of first order, ordinary differential equation, but have a specific form involving the antisymmetric tensor $\Omega^{\mu\nu}$ because of which H is constant along solutions of equations of motion. Furthermore, one also has the Liouville theorem regarding conservation of phase space volumes.

We will begin our discussion by taking the example of source free electrodynamics.

Chapter 2

Hamiltonian Formulation of Electrodynamics

We will begin with the usual Maxwell equations, put them in the four dimensional relativistic form, arrive at an action formulation from which we will go to the Lagrangian and Hamiltonian form. To be able to use the four dimensional relativistic tensor notation, we need to choose a set of conventions.

Maxwell Equations:

$$\vec{\nabla} \cdot \vec{E} = \rho \quad , \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad (2.1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad , \quad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \quad (2.2)$$

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j} \quad (2.3)$$

Conventions:

- Write all quantities without derivatives and with arrows as quantities with *upper* indices i.e. as *contravariant tensor*, eg, $\vec{a} \leftrightarrow (a^1, a^2, a^3) \leftrightarrow a^i$ and vector derivatives as derivatives with *lower* indices (*covariant tensor*), eg, $\vec{\nabla} \leftrightarrow (\partial_1, \partial_2, \partial_3) \leftrightarrow \partial_i$.

Thus, electric field, magnetic field and vector potential are contravariant tensors while the gradient operator is a covariant quantity. The cross product explicitly gives a contravariant quantity.

- Quantities with upper indices are related to those with lower indices by, $a^i = -a_i$, $a^0 = a_0$ i.e. $a^\mu := \eta^{\mu\nu} a_\nu$ with $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) = \eta_{\mu\nu}$.
- Cross products are expressed as, $(\vec{\nabla} \times \vec{a})^i := \epsilon^{ij}{}_k \partial_j a^k$, where, $\epsilon^{123} = 1 = -\epsilon_{123}$. It then follows that,

$$\epsilon^{ij}{}_k \epsilon^k{}_{mn} = -(\delta_m^i \delta_n^j - \delta_n^i \delta_m^j) . \quad (2.4)$$

Maxwell equations now become,

$$\partial_i E^i = \rho \quad , \quad \epsilon^{ij}{}_k \partial_j B^k - \frac{\partial E^i}{\partial t} = j^i \quad (2.5)$$

$$\partial_i B^i = 0 \quad , \quad \epsilon^{ij}{}_k \partial_j E^k + \frac{\partial B^i}{\partial t} = 0 \quad (2.6)$$

Define, $j^0 := \rho$, $F^{0i} := -E^i$, $F^{ij} := -\epsilon^{ij}{}_k B^k$. It is easy to check that

$$(2.5) \quad \rightarrow \quad \partial_\nu F^{\nu\mu} = j^\mu \quad , \quad (2.7)$$

$$(2.6) \quad \rightarrow \quad \partial^\mu F^{\nu\lambda} + \text{cyclic} = 0 = \partial_\mu F_{\nu\lambda} + \text{cyclic} \quad (2.8)$$

It follows that $B^i = \frac{1}{2}\epsilon^i{}_{jk} F^{jk}$ and the identification of the electric field matches with the usual definition with $A^\mu \leftrightarrow (A^0 := \Phi, \vec{A})$.

This allows us to think of Maxwell equations as tensor equations involving 4 dimensional tensors. If one transforms the coordinates x^μ as $x'^\mu := \Lambda^\mu{}_\nu x^\nu$ and transforms the F, j quantities by tensor rules (index-by-index action), then evidently the Maxwell equations are *form invariant* or *covariant* for *all* invertible matrices $\Lambda^\mu{}_\nu$. Remember though that we have also used specific rules to relate the upper and the lower indexed quantities. These rules must also be respected by the primed observer (coordinates) i.e. $\eta_{\mu\nu}$ must also be *invariant* under the coordinate transformations and this restricts the $\Lambda^\mu{}_\nu$ to be the familiar Lorentz transformation matrices. We thus also see the Lorentz covariance of Maxwell equations. Incidentally, operating by ∂_μ on (2.8) and using (2.7) one can deduce the wave equation, $\square F_{\mu\nu} = 0$ and invariance of $\square := \eta^{\mu\nu} \partial_\mu \partial_\nu$ is precisely the requirement of Lorentz invariance.

Note that the 4-tensor notation, is just a compact notation to write Maxwell equations and the physical property of electrodynamics being Lorentz covariant is encoded by the invariance of \square which in turn requires invariance of η . The compact notation helps to keep track of the Lorentz covariance property.

Now we would like to see if these equations can be obtained from an action principle and we all know that the answer is of course yes. For our purposes, it is sufficient to consider the source free case and we will now take $j^\mu = 0$.

If $F_{\mu\nu}$ are treated as basic variables describing electrodynamics, then we have 8 first order partial differential equations for six quantities. Usual equations of Lagrangian framework are second order (in t) equations for configuration space variables. However one notices that (2.8) can be *identically* solved by putting $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. This is always possible to do locally. Now the remaining 4 equations become 4 second order equations for the 4 quantities A_μ . At least now the number of equations equals the number of unknowns. However, the definition of F in terms of A does *not* determine A uniquely (not even up to a constant); $A'_\mu = A_\mu + \partial_\mu \Lambda$ gives the same F . Therefore, although we have correct number of equations, these equations do not suffice to determine the candidate dynamical variables, A , uniquely. Modulo this observation, let us go ahead any way by thinking of $A_\mu(x^\alpha)$ as ‘configurations space variables’.

It is a very easy exercise to check that if we define an action,

$$S[A(x)] := \int dt \int d^3x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) := \int d^4x \mathcal{L}(A_\mu, \partial_\alpha A_\beta) \quad , \quad (2.9)$$

then its stationarity condition gives the Maxwell equations expressed in terms of A . This of course is an example of a *field system* i.e. a dynamical system with *infinitely many degrees of freedom*, $A_\mu(t, x^i)$ with μ, x^i serving as labels. To proceed further, let us introduce some

working rules and notations: derivatives of the Lagrangian density etc will be denoted as $\frac{\delta \mathcal{L}(x)}{\delta A_\mu(x')}$ with the rules,

$$\frac{\delta A_\mu(x)}{\delta A_\nu(x')} = \delta_\mu^\nu \delta^3(x^i - x'^i) \quad , \quad \frac{\delta \partial_\nu A_\mu(x)}{\delta A_\lambda(x')} = \delta_\mu^\lambda \partial_\nu \delta^3(x^i - x'^i) \quad . \quad (2.10)$$

Now let us try to get to a Hamiltonian formulation. Our basic configuration space variables are $A_\mu(t, x^i)$ while the Lagrangian is $L = \int d^3x (-1/4) F_{\mu\nu} F^{\mu\nu}$. The conjugate momenta are given by,

$$\pi^\mu(x) := \int d^3x' \frac{\delta \mathcal{L}(x)}{\delta \partial_0 A_\mu(x')} = -F^{0\mu}(x) \quad . \quad (2.11)$$

Clearly, $\pi^0 = 0$ identically while $\pi^i = -F^{0i}$. Therefore, $\pi_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0$. This allows us to express the ‘velocities’ $\partial_0 A_i = -\pi_i + \partial_i A_0$ which is needed in getting to the Hamiltonian form. However, *we cannot do so for $\partial_0 A_0$!*

The canonical Hamiltonian is defined as,

$$H_c = \int d^3x \left[\pi^\mu \partial_0 A_\mu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \quad (2.12)$$

$$\begin{aligned} &= \int d^3x \left[\pi^i (\partial_0 A_i - \partial_i A_0 + \partial_i A_0) - \frac{1}{2} (F^{0i})^2 + \frac{1}{4} F_{ij} F^{ij} + \pi^0 \partial_0 A_0 \right] \\ &= \int d^3x \left[-\frac{1}{2} \pi^i \pi_i + \frac{1}{4} F_{ij} F^{ij} - A_0 (\partial_i \pi^i) + \pi^0 \partial_0 A_0 \right] \end{aligned} \quad (2.13)$$

In the last equation, a partial integration has been done. The first two terms are the usual electromagnetic field energy density, $\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}$. The last term depends only on $\pi^0, \partial_0 A_0$ and is *not* a function of phase space variables, it contains a velocity. If we naively consider the Hamilton’s equation of motion for A_0 , we get an identity, implying that the time dependence of A_0 is undetermined. The equation of motion for π^0 however leads to $\partial_0 \pi_0 = \partial_i \pi^i$. Observe that *if* we drop the last term by setting $\pi^0 = 0$ at some initial time, and *require* this condition to hold for all times, *then* we will need $\partial_i \pi^i = 0$ for all time which is just one of the Maxwell equations (the Gauss Law). This is also suggested by the Legendre transformation involved in going from the Lagrangian to the Hamiltonian – we are required to use the definition of the momenta. Hence, in the Hamiltonian we should not have the last term. Since $\pi_0 = 0$ from definition *at all times*, we must also obtain $\partial_0 \pi_0 = 0$ from the Hamilton’s equation of motion. Thus it is self consistent to interpret the dropping of the last term as *imposition of a constraint*, $\pi^0 = 0$ on the phase space of (A^μ, π_μ) and the Gauss law following as a consistency condition. Since A_0 remains undetermined, we will replace A_0 in the third term by an *arbitrary* function of time, $\lambda(t)$. Since the Gauss law is also independent of positions/velocities, we will interpret it also as a constraint, $\chi := \partial_i \pi^i$. We will refer to $\pi^0 = 0$ as a *primary constraint* and $\chi = 0$ as a *secondary constraint*, because it is needed for the primary constraint to hold for all times.

Let us consider now the canonical transformations generated by the two constraints. It follows,

$$\delta_\epsilon A_\mu(x) = \left\{ A_\mu(x), \int d^3x' \epsilon(x') \pi^0(x') \right\} = \delta^0_\mu \epsilon(x) \quad (2.14)$$

$$\delta_\eta A_\mu(x) = \left\{ A_\mu(x), \int d^3x' \eta(x') \partial_i \pi^i(x') \right\} = -\delta^i_\mu \partial_i \eta(x) \quad (2.15)$$

$$\delta \pi(x) = 0 \quad (2.16)$$

Clearly, $\delta_\epsilon A_0(x) = \epsilon(x)$, $\delta_\eta A_i(x) = -\partial_0 \eta(x)$. Choosing $\epsilon(x) := -\partial_0 \eta(x)$, reveals that the canonical transformations generated by the constraints are nothing but the usual *gauge transformations*.

Let us summarize.

1. Maxwell equations can be written in a manifestly relativistic form.
2. These can be derived from relativistic action principle treating $A_\mu(t, x^i)$ as generalized coordinates. In the Lagrangian formulation, the ‘matrix of second derivatives’,

$$\frac{\delta^2 \mathcal{L}}{\delta(\partial_0 A_\mu(x)) \delta(\partial_0 A_\nu(x'))} := \mathcal{M}^{\mu\nu}(x, x') = (-\eta^{\mu\nu} + \eta^{0\mu} \eta^{0\nu}) \delta^3(x - x'), \quad (2.17)$$

is *non-invertible*.

3. In the Hamiltonian formulation, there are *constraints*. These generate canonical transformations which are the familiar *gauge transformations*.
4. The canonical Hamiltonian has the form which contains the constraints with arbitrary functions of time as coefficients.
5. We also know that while manifest special relativistic formulation requires us to use 4 components of the vector potential as basic configuration space variables (i.e. 4 degrees of freedom per point), physically there are only 2 degrees of freedom (the two polarizations). Thus the constraints inferred above have something to do with identifying *physical degrees of freedom*.

We will keep these in mind and try to abstract these features in a general framework. On the one hand we will *simplify* by restricting to *finitely many* degrees of freedom and at the same time generalize to arbitrarily complicated systems.

Chapter 3

Constrained Lagrangian and Hamiltonian Systems

Consider a dynamical system with finitely many degrees of freedom, described in a Lagrangian framework. Let Q be an n -dimensional manifold with local coordinates q^i serving as generalized coordinates. Let $L(q, \dot{q})$ be the Lagrangian function of the dynamical system and $S[q(t)] := \int dt L(q, \dot{q})$ being the action. The Variational Principle leads to the Euler-Lagrange equations of motion,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i} \quad i = 1, 2, \dots, n, \quad (3.1)$$

$$\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right) \ddot{q}^j := M_{ij}(q, \dot{q}) \ddot{q}^j = \frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \dot{q}^j := Q_i(q, \dot{q}). \quad (3.2)$$

Usually, the matrix M_{ij} is invertible which allows us to solve for the accelerations, $\ddot{q}^i = (M^{-1})^{ij} Q_j$. This implies that given position and velocity at an instance, one can always determine the dynamical trajectory at other instances.

If on the other hand the matrix M_{ij} is *not* invertible, then let $1 \leq r \leq n$, denote its rank. Then there exist $(n - r)$ independent h_a^i , vectors satisfying $Mh = 0$ and we cannot solve for *all* the accelerations *uniquely*. The general solution for the accelerations will be, $\ddot{q}^i = \underline{\ddot{q}}^i + \sum_{a=1}^{n-r} \alpha_a h_a^i$ and the first term is the solution of the inhomogeneous equation. Furthermore, $Mh = 0$ implies $h^T M = 0 = h_a^i Q_i(q, \dot{q}) = 0$ for $a = 1, \dots, (n - r)$. Thus on the one hand the equations of motion are *not* uniquely specified and on the other hand there are $(n - r)$ relations among the $2n$ positions and velocities. If these relations are *not* satisfied identically, then in particular, they correspond to *restrictions on the initial data*, $q^i(0), \dot{q}^i(0)$.

As an example, consider a system with a one dimensional configuration space with coordinate q and with a Lagrangian, $L = f(q)\dot{q} - V(q)$. Clearly, $M = 0$ and $Q = -V'(q)$. The acceleration is given by $\ddot{q} = \alpha h$ and $hQ = 0 \Rightarrow V'(q) = 0$. If $V = 0$ then the equation is satisfied identically while for non-zero V , the position must be at an extremum of V (which may not exist!).

Let us see what implications are there for a corresponding Hamiltonian framework. First observation is that for a singular Lagrangian, the definition of conjugate momentum $p_i := \frac{\partial L}{\partial \dot{q}^i}$ *cannot* be inverted to solve for the velocities in terms of the momenta. This is essentially the

inverse function theorem namely, $y^i = f^i(x)$ can be inverted to get $x^i = g^i(y)$ provided $\frac{\partial f^i}{\partial x^j}$ is an invertible matrix. (Strictly speaking, the n momenta are n functions of the $2n$ positions and velocities and hence one should be invoking the *implicit function theorem*, such fine prints are ignored here.)

Let us define the Hamiltonian by $H := p_i \dot{q}^i - L(q, \dot{q})$ and consider its variation,

$$\delta H = \dot{q}^i \delta p_i + \left(p_i - \frac{\partial L}{\partial \dot{q}^i} \right) \delta \dot{q}^i - \frac{\partial L}{\partial q^i} \delta q^i \quad (3.3)$$

If we treat the q, \dot{q}, p all as independent variables, then the ‘Hamiltonian’ is clearly a function of all of these. Let us *use* the definition of momenta (i.e. restrict to a $2n$ dimensional sub-manifold of the $3n$ dimensional space). On this sub-manifold, the middle term vanishes and the Hamiltonian becomes a function *only* of positions and momenta in the sense that it varies only when q^i, p_i are varied. This is *independent* of whether velocities can be solved for in terms of momenta and positions. The next step in the usual case of non-singular Lagrangians is when one infers the Hamilton’s equations of motion as,

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = - \frac{\partial H}{\partial q^i} \quad (3.4)$$

This assumes that (a) the variations $\delta p_i, \delta q^i$ are *independent* and (b) Hamiltonian is a function *only* of p_i, q^i . Both these statements fail for singular Lagrangians.

To see this, notice that variation of the Hamiltonian being given in terms of variations of momenta and positions, depends on evaluating the variation of the Hamiltonian on the sub-manifold defined by $p_i = \frac{\partial L}{\partial \dot{q}^i}$. If the variations are also to respect this condition, we must have

$$\delta p_i = M_{ij} \delta \dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \delta q^j \quad (3.5)$$

$$h_a^i \delta p_i = 0 + h_a^i \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \delta q^j \quad (3.6)$$

which immediately shows that the variations $\delta p_i, \delta q^i$ are *not independent*. Consequently, one cannot deduce the Hamilton’s equations of motion. We have already noted that for singular Lagrangians, the velocities cannot be eliminated in favour of momenta and positions. The variations being not independent means that there are relations among the phase space variables p_i, q^i , namely, $\phi_a(q, p) = 0$.

Let us summarize.

A Lagrangian system is said to be *singular* if the matrix M_{ij} of second derivatives of the Lagrangian with respect to the velocities, is singular. This has the consequences that (a) the accelerations are not determined uniquely i.e. contain arbitrary functions of time and (b) it could imply relations among velocities and positions which may not even be consistent. It is possible to isolate the accelerations which are determined uniquely. However we will carry out such analysis in the context of a Hamiltonian framework. The property of being a singular Lagrangian is *independent* of any choice of local coordinates. Generically, singular Lagrangian systems are also called *constrained systems* [3].

In a Hamiltonian formulation obtained from a Lagrangian, the singular nature of the Lagrangian manifests as certain relations among the phase space variables. We will therefore

define a *constrained Hamiltonian System* to be a system with a $2n$ dimensional phase space manifold Γ with local coordinates $(q^i, p_i) \leftrightarrow \omega^\mu$, on which is given a Hamiltonian function, $H_0(q^i, p_i) := H_0(\omega)$ together with a set of relations $\phi^a(\omega) = 0, a = 1, \dots, k (< 2n)$. These relations are referred to as *Primary Constraints*. By definition, the k constraints are *functionally independent* i.e. the k differentials, $d\phi^a(\omega)$ (or the k vectors, $\partial_\mu \phi^a$) are linearly independent. The sub-manifold defined by $\phi^a = 0$ will be denoted by Σ and is called the *constraint surface*. It is a $2n - k$ dimensional sub-manifold of Γ . Note that a constraint surface is *not a phase space* in general, i.e. does not have a symplectic form (eg when k is odd).

We will focus entirely on the constrained Hamiltonian systems and analyze various possibilities of types of such systems. The aim will be to have a *procedure for obtaining equations of motion* in the *Hamiltonian form* paying attention to the constraints. This means that we want to have a variational principle for paths in Γ , which will lead to ‘Hamilton’s equations of motion’ for some suitable Hamiltonian function and such that the possible dynamical trajectories (in Γ) either remain confined to the constraint surface, Σ or never intersect it. In effect, we can continue to work in the given phase space and use a *new* Hamiltonian function so that dynamics effectively respects the constraints. This is achieved by using the method of Lagrange multipliers.

Introduce k Lagrange multipliers, λ_a and define a new Hamiltonian function $H := H_0 + \lambda_a \phi^a$, which matches with the given Hamiltonian H_0 on the constraint surface, Σ . Defining an action,

$$S[\omega(t), \lambda(t)] := \int dt \left[\frac{1}{2} \omega^\mu \Omega_{\mu\nu} \dot{\omega}^\nu - H_0(\omega) - \lambda_a(t) \phi^a(\omega) \right], \quad (3.7)$$

and invoking its stationarity, $\delta S = 0$, leads to the equations of motion,

$$\frac{d\omega^\mu}{dt} = \Omega^{\mu\nu} \left(\frac{\partial H_0}{\partial \omega^\nu} + \lambda_a \frac{\partial \phi^a}{\partial \omega^\nu} \right) \quad \text{and} \quad \phi^a(\omega) = 0, \quad a = 1, \dots, k. \quad (3.8)$$

Thus we obtain equations of motion in a Hamiltonian form and also the constraint equations. We have succeeded in having a new Hamiltonian dynamics defined for trajectories in Γ . We have now to ensure that the trajectories are such that if an initial point is on the constrained surface, then the whole trajectory also remain on the constrained surface. If this property can be ensured, then it also follows that no trajectory can enter and leave Σ , since the equations are first order.

Observe that if the value of ϕ^a for any given a is preserved under evolution (i.e. $\dot{\phi}^a = 0$ along a trajectory), then the trajectory is confined to the $2n - 1$ dimensional hyper-surface $\phi^a = \text{constant}$. However we only need the trajectory to be confined to Σ , so the value of each constraint need *not* be preserved *exactly*. Therefore we need *not* have $\dot{\phi}^a = 0$ along *all* trajectories but *only along those trajectories which lie in* Σ . This is ensured by requiring that the Poisson bracket of the constraints with the Hamiltonian H be *weakly zero*. Since Poisson brackets can be evaluated at any point of the phase space, we can evaluate these at points on the constraint surface and *weak* equality/equations refer to Poisson brackets being evaluated at points of Σ and are denoted by ‘ \approx ’ [2]. Strong equations/equalities are valid in a *neighbourhood of* Σ and are denoted by the usual ‘=’. In particular a strongly vanishing function vanishes weakly and so do *all its partial derivatives*.

$$\frac{d\phi^a}{dt} = \{\phi^a, H_0\} + \lambda_b \{\phi^a, \phi^b\} \approx 0 \quad (3.9)$$

These k conditions ensure that a trajectory beginning on the constrained surface remains on the constraint surface and our goal is reached *provided* (3.9) *holds on* Σ .

There are several possibilities now [2].

If the $k \times k$ matrix of Poisson brackets of the constraints is *non-vanishing*, then the consistency conditions can be viewed as a matrix equation at each point on Σ for the Lagrange multipliers λ_a . Since this matrix is antisymmetric, it is *non-singular* only if when k is even. In this case, all Lagrange multipliers are determined and we do have a Hamiltonian dynamics whose trajectories either lie on the constraint surface or never intersect it. Generically however the matrix is singular. As seen in the context of singular Lagrangian, this means that (a) some multipliers are necessarily undetermined and (b) some linear combinations of $\{\phi^a, H_0\}$ must vanish on Σ . Again we have several possibilities. Either (i) $\{\phi^a, H_0\} \approx 0, \forall a$ and *all* linear combinations are weakly zero, or (ii) the linear combinations vanish weakly *provided some further functions vanish* in which case we refer to these as *secondary constraints*, or (iii) there are *no* points of Σ at which the linear combinations vanish in which case we say that the Hamiltonian system is *inconsistent*.

It could also happen that the matrix is zero on the constraint surface. This could happen for instance, if $\{\phi^a, \phi^b\} = C^{ab}_c \phi^c$. (Recall the Maxwell example). In such a case, there is no equation for the Lagrange multipliers and *all* Lagrange multipliers are undetermined. This case can be thought of as a special case of rank of the matrix being zero.

If we encounter the inconsistent case, we throw away our formulation of the system and start all over again. In the case (i), we have reached our goal but have to live with some undetermined Lagrange multipliers (and hence evolution). This will turn out to be the most interesting case. In the case (ii), we have to now demand that the Poisson bracket of secondary constraints with the Hamiltonian H must vanish on Σ . Once again we will encounter similar cases as above and we have to repeat the analysis – we will either satisfy the conditions identically with some Lagrange multipliers determined or encounter *tertiary constraints* or encounter inconsistency. Since the total number of constraints cannot be larger than $2n$ (else no initial condition will be left!), the process must terminate. Barring inconsistent systems, we will eventually end up with some Lagrange multipliers being determined, some undetermined and with possibly additional constraints $\chi_A \approx 0, A = 1, \dots, l$.

Note: We began by requiring the trajectories to be confined to Σ and found as a consistency requirement that the goal cannot be attained *for all points of* Σ . We need to restrict further to a sub-manifold $\Sigma' \subset \Sigma$, due to the secondary constraints. Thus, *the dynamics defined by H on Γ , will correspond to a constrained dynamics relative to Σ' defined by all constraints being weakly zero*. The dimension of Σ' is of course $2n - k - l$. Notice that even if we began by requiring consistency condition to hold on Σ , extending it to hold on Σ' , does not contradict the previous condition since Σ' is a sub-manifold of Σ . The weak/strong equation now refer to Σ' .

To summarize: Beginning with a phase space Γ , a Hamiltonian H_0 and a set of primary constraints ϕ^a , we *can* construct a Hamiltonian dynamics on Γ such that its trajectories are either confined to the constraint surface Σ or avoid it. The construction reveals the possibility of further constraints as well as the dynamics being *not completely determined*. In the next lecture, we will consider a suitable classification of constraints and obtain a corresponding classification of constrained Hamiltonian systems.

Here are some elementary examples.

1. $H_0 = \frac{p^2}{2m}$, $\phi(q) = q - q_0$: $\{\phi, H\} = \frac{p}{m} \approx 0$, $p \approx 0$, is a secondary constraint;
2. $H_0 = \frac{p^2}{2m}$, $\phi(q) = q^2$: $\{\phi, H\} = \frac{2qp}{m} \approx 0$ condition holds;
3. $H_0 = ap$, $a \neq 0$, $\phi(q) = q$: $\{\phi, H\} = a \approx 0$ inconsistency.

Chapter 4

Dirac-Bergmann theory of Constrained Hamiltonian Systems

Our consistent constrained Hamiltonian system is specified by,

$$H = H_0 + \sum_{a=1}^k \lambda_a \phi^a \quad , \quad \phi^a \approx 0 \quad , \quad \chi^A \approx 0, \quad A = 1, \dots, l, \quad k + l < 2n. \quad (4.1)$$

where ‘ \approx ’ means evaluation on Σ defined by the primary constraints ($\phi^a \approx 0$) and the secondary constraints ($\chi^A \approx 0$). The constraints also have to satisfy,

$$\{\phi^a, H_0\} + \{\phi^a, \phi^b\} \lambda_b \approx 0 \quad , \quad \{\chi^A, H_0\} + \{\chi^A, \phi^b\} \lambda_b \approx 0. \quad (4.2)$$

Thus we have $k + l$ equations for k Lagrange multipliers and the system is naively, over-determined.

Let rank of the $(k + l) \times k$ matrix of Poisson brackets of the constraints be $K \leq k$. This means that K is the maximum number of linearly independent rows and columns of the matrix. Thus there exist $k - K$ independent relations among the k columns of the matrix i.e. $\exists \xi_b^{(\alpha)}$, $\alpha = 1, \dots, k - K$ numbers such that

$$\{\phi^a, \phi^b\} \xi_b^{(\alpha)} \approx 0 \quad , \quad \{\chi^A, \phi^b\} \xi_b^{(\alpha)} \approx 0. \quad (4.3)$$

Now define two sets of linear combinations of the primary constraints namely,

$$\tilde{\phi}^\alpha := \xi_a^{(\alpha)} \phi^a \quad , \quad \tilde{\phi}^{\alpha'} := \eta_a^{(\alpha')} \phi^a \quad , \quad \alpha = 1, \dots, k - K \quad \text{and} \quad \alpha' = 1, \dots, K. \quad (4.4)$$

The new set of K vectors $\eta_a^{(\alpha')}$ are so chosen that the the set of constraints $\tilde{\phi}^\alpha, \tilde{\phi}^{\alpha'}$ are functionally independent. Now it is clear that

$$\{\phi^a, \tilde{\phi}^\alpha\} \approx 0 \quad , \quad \{\chi^A, \tilde{\phi}^\alpha\} \approx 0. \quad (4.5)$$

Thus the $k - K$ new combinations, $\tilde{\phi}^\alpha$ of primary constraints have a weakly vanishing Poisson bracket with *all* the constraints. Such constraints are termed *first class constraints*. Constraints which do not have this property are termed *second class constraints*. With the help of the ξ 's we have regrouped the primary constraints into primary first class and primary second class constraints.

We would like to do the same for secondary constraints. Observe that linear combination of second class constraints will again satisfy the consistency conditions and to such combinations could be added any combination of primary constraints without affecting these equations. To maintain the functional independence of the secondary constraints we need to ensure that the linear combinations are also functionally independent. Thus, consider the combinations,

$$\tilde{\chi}^A := S^A_B \chi^B + S^A_\alpha \phi^\alpha + S^A_{\alpha'} \phi^{\alpha'} \quad (4.6)$$

with S so chosen that S^A_B is a non-singular matrix and $\tilde{\chi}^A$ have weakly vanishing Poisson bracket with *all* constraints for a *maximal number* of values of A . Let this number be $L \leq l$. Thus, we divide the combinations $\tilde{\chi}$ into the first class combinations, $\tilde{\chi}^\sigma, \sigma = 1, \dots, L$ and the second class combinations $\tilde{\chi}^{\sigma'}, \sigma' = 1, \dots, l - L$.

The result of these manipulations is that (a) we can write $\lambda_a \phi^a = \tilde{\lambda}_\alpha \tilde{\phi}^\alpha + \tilde{\lambda}_{\alpha'} \tilde{\phi}^{\alpha'}$ and (b) the consistency conditions can be simplified as,

$$\{\tilde{\phi}^\alpha, H_0\} \approx 0 \quad , \quad \{\tilde{\chi}^\sigma, H_0\} \approx 0 \quad (4.7)$$

$$\{\tilde{\phi}^{\alpha'}, H_0\} + \tilde{\lambda}_{\beta'} \{\tilde{\phi}^{\alpha'}, \tilde{\phi}^{\beta'}\} \approx 0 \quad , \quad \{\tilde{\chi}^{\sigma'}, H_0\} + \tilde{\lambda}_{\beta'} \{\tilde{\chi}^{\sigma'}, \tilde{\phi}^{\beta'}\} \approx 0 \quad (4.8)$$

The first set of equations involve only the $k - K + L$ first class constraints and no Lagrange multipliers while the second set of $K + l - L$ equations involve only the K Lagrange multipliers, $\tilde{\lambda}_{\alpha'}$. The $k - K$ Lagrange multipliers, $\tilde{\lambda}_\alpha$ have dropped out of the equations and will remain *undetermined*. Once again we have more equations than unknown, but because of the separation into first and second class constraints, we are now guaranteed that the $(K + l - L) \times K$ matrix of Poisson brackets of $\tilde{\phi}^{\alpha'}$ and $\tilde{\chi}^{\sigma'}$ with $\tilde{\phi}^{\beta'}$ has the maximal rank K . For, if it did not, there would exist further linear combinations which will weakly Poisson commute with all constraints and by construction we have obtained the maximum number of first class constraints. We will now solve for the Lagrange multipliers $\tilde{\lambda}_{\alpha'}$ explicitly.

Define the matrix Δ of Poisson brackets of the second class constraints as,

$$\Delta := \begin{pmatrix} \{\tilde{\phi}^{\alpha'}, \tilde{\phi}^{\beta'}\} & \{\tilde{\phi}^{\alpha'}, \tilde{\chi}^{\sigma'}\} \\ \{\tilde{\chi}^{\rho'}, \tilde{\phi}^{\beta'}\} & \{\tilde{\chi}^{\rho'}, \tilde{\chi}^{\sigma'}\} \end{pmatrix} := \begin{pmatrix} \Delta^{\alpha'\beta'} & \Delta^{\alpha'\sigma'} \\ \Delta^{\rho'\beta'} & \Delta^{\rho'\sigma'} \end{pmatrix} \quad (4.9)$$

This is an antisymmetric square matrix of order $K + l - L$. This *must be non-singular*. For, if it is singular, there will exist non-trivial linear combination of the second constraints $\tilde{\phi}^{\alpha'}, \tilde{\chi}^{\sigma'}$, which will weakly Poisson commute with all the second class constraints (and it automatically commutes with the first class constraints), implying that we have additional first class constraint. The non-singularity also requires that the total number of second class constraints, $K + l - L$, must be an *even integer*. Let its (weak) inverse be the matrix C ,

$$C := \begin{pmatrix} C_{\alpha'\beta'} & C_{\alpha'\sigma'} \\ C_{\rho'\beta'} & C_{\rho'\sigma'} \end{pmatrix} \quad (4.10)$$

The equation $C\Delta \approx \mathbb{1}$ translates into,

$$C_{\alpha'\beta'} \Delta^{\beta'\gamma'} + C_{\alpha'\sigma'} \Delta^{\sigma'\gamma'} = \delta_{\alpha'}^{\gamma'} \quad (4.11)$$

$$C_{\rho'\beta'} \Delta^{\beta'\tau'} + C_{\rho'\sigma'} \Delta^{\sigma'\tau'} = \delta_{\rho'}^{\tau'} \quad (4.12)$$

$$C_{\alpha'\beta'} \Delta^{\beta'\tau'} + C_{\alpha'\sigma'} \Delta^{\sigma'\tau'} = 0 \quad (4.13)$$

$$C_{\rho'\beta'} \Delta^{\beta'\gamma'} + C_{\rho'\sigma'} \Delta^{\sigma'\gamma'} = 0 \quad (4.14)$$

The equations for the Lagrange multipliers become,

$$\{\tilde{\phi}^{\beta'}, H_0\} + \Delta^{\beta'\gamma'} \tilde{\lambda}_{\gamma'} \approx 0 \quad , \quad \{\tilde{\chi}^{\sigma'}, H_0\} + \Delta^{\sigma'\gamma'} \tilde{\lambda}_{\gamma'} \approx 0 \quad (4.15)$$

Multiply the first one by $C_{\alpha'\beta'}$, second one by $C_{\alpha'\sigma'}$, add the two and use (4.11) to solve for $\tilde{\lambda}_{\alpha'}$. One gets,

$$\tilde{\lambda}_{\alpha'} \approx -C_{\alpha'\beta'} \{\tilde{\phi}^{\beta'}, H_0\} - C_{\alpha'\sigma'} \{\tilde{\chi}^{\sigma'}, H_0\} \quad (4.16)$$

Similar manipulation using (4.14) equation leads to,

$$C_{\rho'\beta'} \{\tilde{\phi}^{\beta'}, H_0\} + C_{\rho'\sigma'} \{\tilde{\chi}^{\sigma'}, H_0\} \approx 0 \quad (4.17)$$

We can now write the Hamiltonian explicitly using the solution (4.16) and express the evolution of any function on the phase space as its Poisson bracket with H . We will write it in a more convenient form.

$$\begin{aligned} \frac{d}{dt} f(\omega(t)) &\approx \{f, H_0\} + \lambda_\alpha \{f, \tilde{\phi}^\alpha\} \\ &\quad - \{f, \tilde{\phi}^{\alpha'}\} C_{\alpha'\beta'} \{\tilde{\phi}^{\beta'}, H_0\} - \{f, \tilde{\phi}^{\alpha'}\} C_{\alpha'\sigma'} \{\tilde{\chi}^{\sigma'}, H_0\} \\ &\quad - \{f, \tilde{\chi}^{\rho'}\} C_{\rho'\beta'} \{\tilde{\phi}^{\beta'}, H_0\} - \{f, \tilde{\chi}^{\rho'}\} C_{\rho'\sigma'} \{\tilde{\chi}^{\sigma'}, H_0\} \end{aligned} \quad (4.18)$$

The last line, which is weakly zero due to (4.17), has been added to get a more symmetric final expression.

Now let us denote all the second class constraints, $(\tilde{\phi}^{\alpha'}, \tilde{\chi}^{\rho'})$ by ξ^m , $m = 1, \dots, K+l-L$. Then, the nonsingular matrix Δ is just the matrix $\Delta^{mn} = \{\xi^m, \xi^n\}$, C_{mn} is its weak inverse as before and the last group of four terms in (4.18) are conveniently expressed as $-\{f, \xi^m\} C_{mn} \{\xi^n, H_0\}$ so that finally we obtain,

$$\begin{aligned} \frac{d}{dt} f(\omega(t)) &\approx \{f, H_0\} + \sum_{\alpha} \lambda_{\alpha} \{f, \phi^{\alpha}\} - \{f, \xi^m\} (\Delta^{-1})_{mn} \{\xi^n, H_0\} , \\ &\approx \{f, H_0 + \sum_{\alpha} \lambda_{\alpha} \phi^{\alpha}\}^* \quad \text{where,} \end{aligned} \quad (4.19)$$

$$\begin{aligned} \Delta_{mn} &:= \{\xi^m, \xi^n\} && \text{and} \\ \{f, g\}^* &:= \{f, g\} - \{f, \xi^m\} (\Delta^{-1})_{mn} \{\xi^n, g\} && \text{(Dirac Bracket) .} \end{aligned} \quad (4.20)$$

In this final expression, we have removed the $\tilde{}$, the primary first class constraints are denoted by ϕ^α while all second class constraints are denoted by ξ^m .

Several remarks are in order.

1. The first step in the analysis of constrained systems was to obtain the full set of constraints starting with a given Hamiltonian H_0 and a set of primary constraints defining the constraint surface $\Sigma \subset \Gamma$. In order to ensure that we get the final form of evolution equations to be a Hamiltonian form, we used a modified Hamiltonian H and thought of the system as thought it were *un-constrained* in the sense the variations of the phase space coordinates were *independent*. To make contact with the constrained nature of the system, we required that the un-constrained dynamics be such that its trajectories either lie in Σ or never intersect it. This lead us to discovering possible, additional constraints. Note that restrictions on the trajectories was with reference to

the sub-manifold Σ and hence only the primary constraints played a role in subsequent analysis i.e. we did *not* add to the Hamiltonian, terms corresponding to the secondary constraints. The secondary constraints however do reveal that the required segregation of trajectories holds only with respect $\Sigma' \subset \Sigma$, defined by vanishing of all constraints.

2. The next step was essentially an exercise in linear algebra. We did this to solve explicitly for those Lagrange multipliers which could be solved for. This was facilitated by regrouping the set of all constraints into first class and second class constraints. The final result reveals that evolution could be *arbitrary* if there are *primary, first class constraints* due to the undetermined λ_α . The most compact expression for the Hamiltonian evolution was obtained using the *Dirac brackets*.
3. We now define *first class variables* as those functions on Γ whose Poisson brackets with *all* constraints are weakly zero. By the consistency condition (4.2), the Hamiltonian H is a first class variable and of course so are the first class constraints. The Hamiltonian H_0 may or may not be a first class variable. It's Poisson bracket with first class constraints is of course weakly zero from (4.7). It is easy to check that sums and products of first class variables is again first class and so are the Poisson brackets of first class variables ¹.
4. As noted already, in the presence of primary first class constraints, evolution equation for a generic function f , contains the arbitrary Lagrange multipliers, λ_α . From (4.19), it follows that *evolution of first class variables is independent of λ_α and is entirely governed by H_0* . This justifies why first class variables are singled out.
5. The Dirac brackets have been introduced as a convenient compact notation. However it has many interesting properties, namely,
 - (a) $\{f, g + h\}^* \approx \{f, g\}^* + \{f, h\}^*$ (addition);
 - (b) $\{f, \mu g\}^* \approx \mu \{f, g\}^*$ (scalar multiplication);
 - (c) $\{f, gh\}^* \approx \{f, g\}^* h + g \{f, h\}^*$ (Leibniz);
 - (d) $\{f, g\}^* \approx -\{g, f\}^*$ (antisymmetry);
 - (e) $\{f, \{g, h\}^*\}^* + \text{cyclic} \approx 0$ (Jacobi identity).

Thus it has all the properties of the usual Poisson bracket. In addition, one has

- (a) $\{f, g\}^* \approx \{f, g\}$ for any first class f and $\forall g$;
- (b) $\{\xi^i, g\}^* \approx 0, \forall g$ and any second class constraint ξ^i .

Recall that weak equations involving Poisson brackets mean that the Poisson brackets are *first computed in a neighbourhood of Σ and then evaluated on Σ* . This rule is necessary since a weakly vanishing function need *not* have a weakly vanishing Poisson bracket (since some of the partial derivatives ‘off’ Σ may not be zero). This applies to the second class constraints as well. However, Dirac bracket of a second class constraint with *any* function is weakly zero. Therefore, if we use Dirac brackets for writing the equations of motion (as shown in (4.19)), then we can set the second class constraints to be zero *before* computing the Dirac brackets. This is equivalent to reducing the phase

¹Functions on the phase space of an unconstrained system are generally called *observables* while in the context of a gauge system, the first class observables are all called as *Dirac observables*.

space dimension from $2n$ to $(2n - \text{the number of second class constraints})$. Thus, *second class constraints correspond to redundant degrees of freedom which can be ignored by setting them to zero.*

6. We could now focus on systems that do *not* have any second class constraints either a priori or after eliminating them via the Dirac bracket procedure. One is effectively left systems with only first class constraints. As noted earlier, the evolution of a dynamical variable in such systems is in general, arbitrary and only variables whose evolution is *not* arbitrary are the first class variables for whom the Dirac brackets are same as the Poisson brackets.

Note that it could happen that there are *no* first class constraints left. In such a case, we have just the usual types of systems. Thus, the net conclusion of the analysis is:

Generically, consistent Hamiltonian systems are systems with first class constraints with the special case of no first class constraints. Hamiltonian systems with at least one first class constraint are termed *gauge theories*. We will now focus on these exclusively.

Let Γ be a $2n$ dimensional phase on which is given a first class Hamiltonian function H_0 and a set of $k < 2n$ first class constraints, ϕ^a whose vanishing defines the constraint surface Σ . The total Hamiltonian governing time evolutions is given by $H := H_0 + \sum_a \lambda_a \phi^a$. By virtue of being first class, we have the following relations:

$$\{\phi^a, \phi^b\} \approx 0 \leftrightarrow \{\phi^a, \phi^b\} = C^{ab}_c(\omega)\phi^c \quad \text{and} \quad \{H_0, \phi^a\} \approx 0 \leftrightarrow \{H_0, \phi^a\} = D^a_b(\omega)\phi^b. \quad (4.22)$$

The evolution equation for any function f on Γ is given by,

$$\frac{d}{dt}f(\omega(t)) \approx \{f(\omega), H_0 + \sum_a \lambda_a \phi^a(\omega)\}|_{\omega=\omega(t)} \quad (4.23)$$

Now observe that (a) if we make an arbitrary *diffeomorphism* i.e. a mapping of the manifold Γ on to itself preserving differential structure, the manifold is unchanged (by definition). However, the symplectic form would change in general; (b) if we specialize the diffeomorphisms to those which preserve the symplectic form, then the restricted diffeomorphisms are the familiar *canonical transformations*. All such (continuously connected to identity) transformations can be generated by arbitrary functions on Γ , by the rule: $\delta_{\epsilon g} \omega^\mu := \epsilon \Omega^{\mu\nu} \partial_\nu g(\omega)$. All of these however do *not* preserve the constraint surface; (c) To preserve the constraint surface, the function must preserve the constraints defining the surface i.e. must be a first class function. Thus, *all* first class functions and in particular the first class constraints, do preserve Σ . While constraints preserve the Hamiltonian, other first class functions need not. Those first class functions which do preserve the Hamiltonian as well are said to generate *symmetry transformations*. By contrast, the transformations generated by the first class constraints are distinguished as *gauge transformations*.

Thus, the diffeomorphisms generated by first class constraints, preserve the entire structure of the constrained Hamiltonian system (i.e. manifold, symplectic structure, constraint surface and the Hamiltonian) and are termed *gauge transformations*. First class functions (which are not constraints) are automatically invariant under these transformations. Provided they Poisson commute with the Hamiltonian, they generate *symmetry transformations*. This distinction among the set of all first class function, comes about for the following reason.

Recall that λ_a are undetermined, arbitrary functions that appear in the equations of motions. Therefore, beginning from any initial condition $\hat{\omega} \in \Sigma$, the actual trajectories will depend on the *choice* made for the λ_a 's. If we identified points on Σ as representing physical states of the system, we would lose determinism – a given state does *not* uniquely determine the future state. We need to identify physical states of the system differently.

Consider infinitesimal evolutions for two different choices of λ_a 's. We will have,

$$\omega'(\delta t) = \hat{\omega} + \delta t\{\omega, H(\lambda', \omega)\}|_{\hat{\omega}} \quad , \quad \omega(\delta t) = \hat{\omega} + \delta t\{\omega, H(\lambda, \omega)\}|_{\hat{\omega}} \quad , \quad (4.24)$$

which implies that,

$$\delta\omega(\delta t) = \delta t\{\omega, \sum_a \delta\lambda_a \phi^a\}|_{\hat{\omega}} = \sum_a (\delta t \delta\lambda_a) \{\omega, \phi^a\}|_{\hat{\omega}} \quad , \quad (4.25)$$

which is nothing but the infinitesimal transformation generated by the first class constraints!

Thus, if we arbitrarily choose the λ_a and consider a trajectory evolved from some $\hat{\omega}$ then another trajectory evolved by a different choice of λ_a from the same initial point, would be obtained by making a gauge transformation. Clearly, if we identified points in Σ which are related by gauge transformations as being ‘physically the same’, then we regain determinism in the sense that *physical states* evolve into *unique* physical states. Thus *the apparent dynamical indeterminism implied by the first class constraints appearing in the Hamiltonian, can be resolved by defining equivalence classes of points of Σ under transformations generated by the constraints (i.e. gauge transformations) to represent the physical states of the system.* Notice that this identification of physical states with equivalence classes under gauge transformations involves only those first class constraints which appear in the Hamiltonian since only these have a bearing on the dynamical evolution.

Since there are k first class constraints, the set of points of Σ which are related by gauge transformations is parameterized by k parameters and hence the *set of gauge equivalence classes* is parameterized by $2n - k - k$ parameters. This space is called the *Reduced Phase Space*. This space can be made explicit by introducing additional k ‘constraints’ (now called as *gauge fixing conditions* – $\chi_a(\omega) \approx 0$). These are required to be such that the matrix of Poisson brackets of the ϕ^a, χ_b constraints is non-singular. Demanding their preservation in time fixes the Lagrange multipliers and hence the name.

Having clarified the identification of physical states, the definition of physical observables and notions of symmetry obviously must be formulated for physical states. The observables must have unique evolutions since by definition these are supposed to be functions of physical states. Only first class functions satisfy this property and only these qualify to be termed as physical observables. Likewise, notion of symmetry should refer to invariance with respect to transformations of *physical states*, the generators of infinitesimal symmetries must map the entire gauge equivalence classes among themselves and of course preserve the evolution. Clearly these again have to be first class functions and must additionally Poisson commute with the Hamiltonian.

To summarize:

1. Hamiltonian systems with at least one first class constraint, *require* identification of physical states *not with individual points* of the constraint surface *but with gauge equivalence classes of points* of the constraint surface.

2. In view of the above, it is common to refer to the original phase space as the *kinematical phase space*, Γ_{kin} . The constraint sub-manifold of Γ_{kin} is Σ . The physical state space (or reduced phase space) is denoted as $\Gamma_{\text{phys}} := \Sigma / \sim$ where \sim refers to the gauge transformations. Note that the physical state space is *not* a sub-manifold of Σ .
3. Although both the constraints and ‘conserved quantities’ serve to confine the trajectories the two are distinguished by the fact that constraint impose limitation on possible initial conditions (restriction to Σ) as well as force a non-trivial identification of physical states to ensure determinism of dynamics. Notions of conserved quantities, symmetries become meaningful *only after* this identification. Also conserved quantities do not impose any *ab initio* limitation on the possible initial conditions but only on a subsequent trajectory.

Observe that in the light of the discussion of symmetry and gauge transformations, a first class Hamiltonian H_0 generates a *symmetry transformation*, namely time translations. However there are theories in which $H_0 = 0$ and H is made up entirely of first class constraints. Now the ‘time evolution’ itself becomes a gauge transformation and hence ‘no evolution’ of physical states. Next chapter discusses this case.

Chapter 5

Systems with the Hamiltonian as a constraint

Consider special types of gauge theories in which $H_0 = 0$ i.e. the Hamiltonian is entirely made up of first class constraints. The prime physical example of such a system is the dynamics of inhomogeneous cosmological space-times within the context of Einstein's theory of General Relativity ¹.

As is well known, Einstein's general relativistic theory of gravity (GR) has a four dimensional manifold on which is defined some metric tensor (of Minkowskian signature) field which makes it in to a space-time. The metric tensor is not a fixed entity, as in the case of special relativity (the Minkowski space-time), but is a dynamical entity i.e. is determined in conjunction with the (interacting) matter distribution on the manifold. The equation determining the metric is the Einstein equation, which is a set of 10, local, partial differential equations of order 2, for the 10 components of the metric tensor, $g_{\mu\nu}$. It is non-trivial fact that these equations admit a well-posed initial value formulation i.e. (i) one can take the 4 dimensional manifold as $\mathbb{R} \times \Sigma_3$, (ii) specify two symmetric tensor fields, g_{ij} , and its time derivative, \dot{g}_{ij} on Σ_3 , then the space-time can be determined for other times provided the 'initial data' satisfies certain conditions. Furthermore, the system of equations can be cast in the form of constrained Hamiltonian system, with Hamiltonian given entirely by first class constraints. There are 4 sets of constraints (per point, since one is dealing with a field theory), three of which, called *vector or diffeomorphism constraints* and the remaining one called the *scalar or Hamiltonian constraint*. The vector constraints generate usual diffeomorphisms of Σ_3 while the scalar constraint generates evolution of the 'spatial manifold', Σ_3 in the 4-manifold constructing a solution (space-time) of the Einstein equation. The upshot is that *solution space-times of Einstein equation can be viewed as phase space Hamiltonian trajectories in the gravitational phase space with initial data satisfying a set of first class constraints* and with the Hamiltonian given as linear combination of these constraints. The constraints in the Hamiltonian formulation of general relativity, reflect the 4-diffeomorphism invariance of GR [4].

This is pretty complicated to deal with in general, however a simplification is possible. If we restrict ourselves do the dynamics of only a special class of 3-geometries, namely, those metric tensors whose dependence on spatial coordinates (coordinates on Σ_3) is completely

¹For space-times corresponding to compact objects, typically asymptotically flat space times, there is a 'true Hamiltonian', the analogue of H_0 , generating asymptotic time translations.

fixed in a particular manner and only time dependence is to be determined, then GR simplifies to a Hamiltonian system for a finitely many degrees of freedom (6 for general homogeneous geometries, 3 for so-called diagonalized models and 1 for homogeneous, isotropic geometry of the Friedmann-Robertson-Walker (FRW) cosmology) with only the *single* Hamiltonian constraint remaining. We will not need to take any specific model to illustrate the issues.

We can also *construct* systems in which Hamiltonian is the single (and hence first class) constraint. To see this, let us begin with a usual *un-constrained* Hamiltonian system with a phase space $\hat{\Gamma}$ and a Hamiltonian $H_0(\omega)$. Let us extend this phase space to Γ by adding two more conjugate variables, τ, π . Let $\phi(\tau, \pi, \omega) := H_0(\omega) - \pi$ be chosen as the Hamiltonian on Γ and take it as a constraint as well, i.e. $H := \lambda\phi \approx 0$. Clearly, any evolution (with respect to T) generated by the Hamiltonian is a gauge transformation and therefore unphysical. Functions which are insensitive to the evolution are the first class ones (Dirac observables), f . Let us look for Dirac observables.

$$\{f, H\} \approx 0 \Rightarrow \{f(\tau, \pi, \omega), H_0(\omega)\} - \frac{\partial f}{\partial \tau} \approx 0. \quad (5.1)$$

where we have used the definition of Poisson bracket in the second equation.

It is clear that functions which are independent of τ, π and satisfying the usual un-constrained evolution equation in $\hat{\Gamma}$, *are* Dirac observables of the extended constrained dynamics. Functions depending *only* on τ are not Dirac observables while those depending only on π are Dirac observables. Constants with respect to the τ -dynamics, also are Dirac observables.

With this construction, we see that usual unconstrained dynamics can be viewed (albeit trivially) as a constrained dynamics in a bigger phase space. Furthermore, all functions on the unconstrained phase space, evolving by the un-constrained dynamics *are* Dirac observables of the constrained dynamics. The Dirac observables however are *constants* with respect to the T -evolution.

Consider now the trajectories of the constrained dynamics. One finds,

$$\frac{d}{dT}\tau = -\lambda, \quad \frac{d}{dT}\pi = 0, \quad \frac{d}{dT}\omega = \lambda\{\omega, H_0(\omega)\} = \lambda\frac{\partial}{\partial \tau}\omega. \quad (5.2)$$

The last equality is deduced from (5.1) with $f = \omega$. We can define a new ‘time’, T' by the equation $dT' = \lambda dT$, so that

$$\frac{d}{dT'}\tau = -1, \quad \frac{d}{dT'}\pi = 0, \quad \frac{d}{dT'}\omega = \frac{\partial}{\partial \tau}\omega. \quad (5.3)$$

Note that T evolution is generated by $\lambda\phi$ while T' evolution is generated by ϕ . Starting with some initial values at $T' = 0$, one generates the T' -trajectories. All the points along these are related by gauge transformations generated by ϕ . Thus the equivalence classes are in one-to-one and on-to correspondence with the points $\tau = 0, \pi = \hat{\pi}, \omega = \hat{\omega}$. The gauge orbits which lie on the constraint surface satisfy $\hat{\pi} = H(\hat{\omega})$ and hence these orbits are completely determined by points in $\hat{\Gamma}$. Thus the reduced phase space in this case is just the un-constrained phase space $\hat{\Gamma}$.

This simple construction brings out a few points. There are *two* notions of ‘evolution’ (i) the T or (T') evolution, which is some times called an *external time evolution* and (ii) the τ evolution which is correspondingly called an *internal time evolution*. From the point of view of the constrained system, τ is just one of the degrees of freedom which is singled out because

the constraint had a particularly simple additive form. The τ evolution can thus be thought of as evolution of a set of degrees of freedom with respect to a singled out degree of freedom. The internal time evolution is thus also called a *relational evolution* while the singled out degree of freedom is called a *clock degree of freedom*. The arbitrary function, λ is also called a *lapse function* and its arbitrariness corresponds to the freedom of re-parameterizing the external time. The Hamiltonian systems resulting from homogeneous cosmologies of GR, typically get presented in the form of the constrained system (the constraint however has a different form in general). The terminology used above is inherited from the GR context. One can in fact do a more general and systematic analysis of the notions of external and internal dynamics which my student Golam Hossain and I have carried out for finite dimensional systems.

While classically, constrained Hamiltonian systems are interesting in their own right, they become more challenging at the quantum level. As all the fundamental interactions of standard model and its extensions are gauge (field) theories, one has to face these challenges. In the perturbative analysis, one needs to ‘fix a gauge’, in order that propagators for gauge fields can be defined and then has to show that the final observable scattering cross-sections are indeed gauge invariant. When a quantum theory of gravity is attempted, the understanding of semiclassical approximation becomes quite complicated especially in a non-perturbative approach.

Chapter 6

Exercises

1. Check that the coordinate transformations generated by Hamiltonian vector fields leave the symplectic form *invariant*.
2. Check that (1.3) property is needed to prove the Jacobi identity for Poisson brackets.
3. For the Maxwell theory, check that the secondary constraint holds for all times *without* having to require any further constraint. Furthermore, the Poisson bracket of the two constraints is also zero.
4. Show that the Hamilton's equations of motion can be identified with the Maxwell equations.
5. For the Maxwell theory, we have already shown that the canonical transformations generated by the first class constraints are indeed the usual gauge transformations (hence in fact the name). Show, by direct computation, that $F_{\mu\nu}$ are first class functions.
6. Consider a massive, relativistic particle with action $S = m_0 \int d\tau \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$. Carry out the constraint analysis and give examples of first class functions.
7. Consider the four dimensional phase space with coordinates (q^1, q^2, p_1, p_2) . Consider two constraints $\phi(q, p) := p_1^2 + p_2^2 + (q^1)^2 + (q^2)^2 - R^2$ and $\chi(q, p) := p_2$. Let $H_0(q, p)$ be some suitable Hamiltonian (not given explicitly) such that these two constraints are preserved [5].
 - (a) Identify the constraint surface.
 - (b) Compute the expression for the Dirac bracket.

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