Global ontologies for relativistic quantum systems and quantum field theory

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Epistemic models of nature prove to be problematic in many relativistic settings as well as in settings in which measurement procedures are ill-defined. By contrast, in ontological models of nature, measurement results are independent of the procedure used to obtain them. If we assume that all measurement results can be expressed in terms of pointer readings, then any useful ontology would need to unambiguously specify the positions of things. Proposals for such ontologies have included the lattice fermion number density as proposed by Bell, and a relational specification of position as proposed by Smolin. In this article we first show that Bell's choice of ontology is unsuitable due to the ambiguity of particle number as a result of the Unruh effect, and due to the fact that lattice fermions suffer from the problem of fermion doubling. We then show that Smolin's alternative does not unambiguously specify position. This motivates us to develop an alternative set of ontologies based on the Wheeler-DeWitt equation that does not suffer from these problems and we provide a set of constraints for the geometry, matter field, and action for such a model to be deterministic.

1 Background

Quantum mechanics is ostensibly a theory about the results of measurements that are performed 'on' systems (objects). This act of measurement appears to presuppose that something or someone (subject) must be doing the measuring. That is, quantum mechanics inherently includes epistemic elements. But, as Bell pointed out, precisely where or when to draw a distinction between subject and object is not manifest in the theory itself rendering the theory's epistemic elements ambiguous [3]. Bell suggests that good physical theories should be able to say something concrete about reality itself rather than about measurement procedures. This

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is particularly true in situations in which the very concept of measurement can be ill-defined. Thus epistemic models of nature can be problematic in some situations.

By contrast, in ontological models of nature, measurement results are independent of the procedure used to obtain them. Bell refers to these concrete, objective elements of reality as '*be*ables' in contrast to the more ambiguous term 'observables' since they are independent of observation [4, 5]. Bell's aim was to unambiguously represent the 'positions of things' which he takes as referring to the positions of instrument pointers and argues that any ontology must adequately be distillable to these positions [5]. In other words, Bell assumes that the positions of instrument pointers have a direct, one-to-one correspondence with some element of physical reality. If an instrument pointer that measures some physical quantity A gives a value of exactly a then the value a is an objective fact about the system under consideration. Thus knowledge of the exact position of an instrument pointer gives objectively true information about the universe. So, for example, if the instrument pointer corresponds to a voltage or current reading, the assumption is that this tells us something objectively true about an electric field. In order to adequately capture this, Bell proposes the lattice fermion number density as a viable beable for position. As we show in Section 2, however, this choice suffers from two problems. First, the Unruh effect renders any particle density ambiguous. In addition, lattice fermions suffer from the problem of fermion doubling. Smolin recently proposed an alternative ontology for position that is expressly non-local (though more abstract). Smolin's model, like Bell's, maintains the usual quantum dynamics, but, as we show in Section 3, it does not unambiguously specify position.

We thus propose an alternative theory of *global* beables in Section 4 that is based on solutions to the Wheeler-DeWitt equation. We begin by first reviewing the Wheeler-DeWitt formalism before describing our model. We then develop a set of constraints for the geometry, matter field, and action that would need to be satisfied in order for the model to be considered
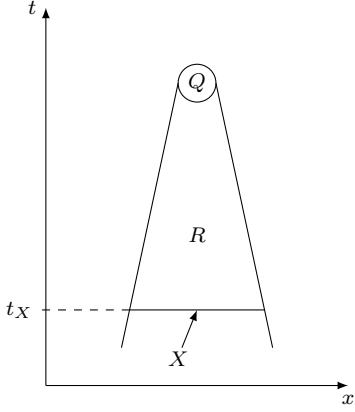
Figure 1: Beables associated with the bounded region of spacetime Q are determined by those associated with any temporal cross-section (i.e. spacelike slice) X of region R that fully encloses the past light-cone of Q.

fully deterministic. Throughout this article, we take the word 'beable' to be synonymous with the word 'ontology.'

2 Local beables

Local beables are specifically those beables that are confined to some specific bounded region of spacetime [4, 6]. Specifically such beables must be associated with some bounded set $Q = \{\mathbf{x}, t\}$ of spacetime coordinates where Q is the subset of some metric space (M, g) where g is a metric on M. Specifically beables defined for Q are assumed to be determined by those corresponding to any temporal cross section (i.e. spacelike slice) X of the spacetime region R that fully encloses the past light-cone of Q where X is a subset of R corresponding to all possible values of $\mathbf{x}$ associated with the value t_X. This is shown for a single spatial dimension, x, in Fig. 1. Bell refers to this condition as 'local determinism' and it serves as the foundation for his development of a local causal theory [4].

2.1 Lattice fermion number

Fields, of course, are not strictly local since they exist in a *region* of spacetime rather than at a single event. In other words, the spacetime region defined by Q for a field consists of a *range* of spacetime coordinate values, Δx and Δt as depicted in Fig. 1. Unambiguously determining the position of something (e.g. a pointer) however, requires specifying its exact coordinates. Bell's solution was to consider fields in terms of a lattice particle density. That is, the continuum is replaced by a

dense lattice of particles. Since ordinary matter is composed of fermions and their locations are crucial for determining various macroscopic properties including the positions of instrument pointers, Bell specifically begins with the lattice fermion number density [5]. While he does acknowledge that this is not the only possible choice, he does not elaborate on any of the other possibilities.

For simplicity, we begin by replacing the three-space continuum with a dense lattice (which, in essence, quantizes space *a priori*) while time remains continuous. The lattice points are given by $l = 1, 2, \ldots, L$ where L is assumed to be very large. The lattice point fermion number operators can be defined as

$$\hat{F}(l) \equiv \hat{\psi}^\dagger(l)\hat{\psi}(l) \tag{1}$$

where we are assuming a summation over Dirac indices and all Dirac fields. Strictly speaking, this is a number *density* and thus measures the number of fermions at a given lattice point [8]. The corresponding eigenvalues are integers

$$F(l) = 1, 2, \ldots, 4N \tag{2}$$

where N is the number of Dirac fields. That is, $F(l)$ labels a particular configuration of the Dirac fields at a given lattice point l. For example, for a single Dirac field, $N = 1$ and thus $F(l) = 1, 2, 3, 4$ corresponding to the four possible solutions to the Dirac equation. The fermion number operator for the entire lattice is simply the sum of the lattice point operators,

$$\hat{F} = \sum_l \hat{F}(l). \tag{3}$$

The fermion number configuration of the world, i.e. precisely *how* the fermions in the universe are arranged at any given instant of time t, is a list of such integers (one for each lattice point),

$$n(t) = [F(1), F(2), \ldots, F(L)]_t \tag{4}$$

where the subscript t labels the set for a given time t. This is the lattice fermion number configuration and it serves as the local beable for a theory since it is associated with definite positions in space i.e. a definite configuration of all fermions in the universe at a time t [5]. To this Bell also adds the state vector $|t\rangle$ as a *non-local* beable. The complete specification of the universe at a time t is then given by the set $T = \{n(t), |t\rangle\}$ where, in accordance with the theory of local beables, $n(t)$ is associated with some bounded set of spacetime coordinates Q. Since the configuration $n(t)$ is exact, there is no spread in the spacetime coordinates of any given fermion at time t, i.e. the position of each fermion is assumed to be known exactly. Thus Q is *local in its elements*, i.e. each element is perfectly localized.

2.2 The Unruh effect

In order to construct a fermion number operator we first briefly review the algebra of fermionic operators on an antisymmetric Fock space [13, 15]. We define the fermion creation and annihilation operators by the following relations,

$$\hat{f}_{\mathbf{k}}^{\dagger}|1\rangle = 0, \qquad \hat{f}_{\mathbf{k}}^{\dagger}|0\rangle = |1\rangle, \qquad (5)$$

$$\hat{f}_{\mathbf{k}}|1\rangle = |0\rangle, \qquad \hat{f}_{\mathbf{k}}|0\rangle = 0, \qquad (6)$$

where we note that (5) follows from the fact that fermionic Fock states can contain a maximum of a single particle. In other words, the only two possible Fock states for fermions are $|0\rangle$ and $|1\rangle$. Unlike the bosonic operators, the fermionic operator algebra is defined by the *anti*-commutators due to the antisymmetry of the space, i.e.

$$\{\hat{f}_{\mathbf{j}}, \hat{f}_{\mathbf{k}}^{\dagger}\} = \delta_{jk},$$
$$\{\hat{f}_{\mathbf{j}}), \hat{f}_{\mathbf{k}}\} = \{\hat{f}_{\mathbf{j}}^{\dagger}, \hat{f}_{\mathbf{k}}^{\dagger}\} = 0. \qquad (7)$$

Multi-particle fermionic number states can then be constructed from the vacuum state as

$$|n_0, n_1, \ldots\rangle = (f_1^{\dagger})^{n_1}(f_0^{\dagger})^{n_0}|0\rangle. \qquad (8)$$

We therefore define the fermion number operator as

$$\hat{\mathcal{F}} \equiv \int d^3k \, \hat{f}_{\mathbf{k}}^{\dagger}\hat{f}_{\mathbf{k}} \qquad (9)$$

which, in the lattice approximation, is equivalent to the discrete fermion number operator of (3), i.e.

$$\hat{\mathcal{F}} \to \hat{F} \qquad \text{(lattice approximation)}. \qquad (10)$$

For simplicity, let us consider only the lowest eigenmode in the Minkowski vacuum $|0_{\mathrm{M}}\rangle$. The expectation value of the fermion number operator gives the number of fermions that we would expect to see in the vacuum. For the lowest eigenmode this is

$$\langle 0_{\mathrm{M}}|\hat{\mathcal{F}}|0_{\mathrm{M}}\rangle = \langle 0_{\mathrm{M}}|\hat{f}_0^{\dagger}\hat{f}_0|0_{\mathrm{M}}\rangle = 0 \qquad (11)$$

which follows from (6). In other words any inertial observer in the Minkowski vacuum would not expect to detect any fermions in the lowest eigenmode.

Now consider an accelerated observer in this same Minkowski vacuum. We define coordinate transformations as

$$t = \pm \frac{e^{a\xi}}{a}\sinh(a\tau)$$
$$x = \pm \frac{e^{a\xi}}{a}\cosh(a\tau) \qquad (12)$$

where τ and ξ are the Rindler time and space respectively and a is a parameter corresponding to the proper acceleration of an arbitrarily chosen reference trajectory. The signs require that we define two coordinate patches that we will denote I ($x > 0$) and II ($x < 0$). Within each coordinate patch we define creation and annihilation operators such that, for example, in region I we have $\hat{f}_{\mathbf{k},\mathrm{I}}^{\dagger}|0_{\mathrm{I}}\rangle = |1_{\mathrm{I}}\rangle$ and $\hat{f}_{\mathbf{k},\mathrm{I}}|0_{\mathrm{I}}\rangle = 0$ where $|0_{\mathrm{I}}\rangle$ is the vacuum state in that region (patch). The restriction of $|0_{\mathrm{M}}\rangle$ to region I, however, is equivalent to a thermal state with temperature $T = a/2\pi$ [10, 16, 27, 29]. Specifically, it can be shown that [9]

$$\langle 0_{\mathrm{M}}|\hat{\mathcal{F}}_{\mathrm{I}}|0_{\mathrm{M}}\rangle = \langle 0_{\mathrm{M}}|\hat{f}_{\mathbf{k},\mathrm{I}}^{\dagger}\hat{f}_{\mathbf{k},\mathrm{I}}|0_{\mathrm{M}}\rangle$$
$$\propto \frac{1}{e^{2\pi|\mathbf{k}|/a} - 1}. \qquad (13)$$

This is known as the Unruh effect and it is clear that a non-inertial observer will see fermions in the lowest eigenmode when an inertial observer will not. This is clearly problematic for any theory of beables that seeks to unambiguously specify the position of something presumably composed of fermionic matter. On the other hand, it has also recently been shown that massive Unruh particles cannot be directly observed [18]. This might suggest that the fermion lattice number could at least *approximate* the position of objects composed of fermionic matter to a high enough degree of precision that it could serve as a beable for all practical purposes. Unfortunately, the fermion lattice number also suffers from the problem of fermion doubling.

2.3 Fermion doubling

As it turns out, naïvely assigning fermionic fields to a lattice results in the appearance of spurious fermion states. Specifically, the theory produces 2^d fermions for each original fermion where d is the number of discretized dimensions. This phenomenon is known as *fermion doubling* [7, 22, 23]. To see this, consider the continuous action of a free Dirac fermion of mass m in d dimensions,

$$S = \int d^d x \, \bar{\psi}(\gamma_{\mu}\partial^{\mu} + m)\psi \qquad (14)$$

where γ_{μ} are the usual gamma matrices. We can discretize this action on a cubic lattice by replacing the fermion field $\psi(x)$ with a discretized version ψ_x where x denotes the lattice site and the derivative is replaced by a finite difference. The action becomes

$$S = l^d \sum_{x,\mu} \frac{1}{2l}(\bar{\psi}_x \gamma_{\mu}\psi_{x+\hat{\mu}} - \psi \bar{x+\hat{\mu}}\gamma_{\mu}\psi_x)$$
$$+ l^d \sum_x m\bar{\psi}_x\psi_x \qquad (15)$$

where l is the lattice spacing and $\hat{\mu}$ is a vector of length l in the μ direction. The inverse fermion propagator in momentum space is then

$$S^{-1}(p) = m + \frac{i}{l} \sum_{\mu} \gamma_{\mu} \sin(p_{\mu}, l). \qquad (16)$$

Since the lattice spacing is finite the momenta p_{μ} must be inside the first Brillouin zone. This is usually taken to be the interval $[-\pi/l, +\pi/l]$. No problem arises if we naïvely take $l \to 0$ in (16). However, if we expand around a specific *value* for p_{μ} where one or more of the components is at the corners of the zone, i.e. at $\pm\pi/l$, then the continuum form appears again even though the sign of the gamma matrix can change. In other words, when one of the components of p_{μ} is near $\pm\pi/l$, we recover (14). Since this can happen for any of the d components, one ends up with 2^d fermion fields [7]. This clearly poses a problem for $n(t)$ as a beable. One solution would be to abandon the requirement that beables themselves be local which is something that Bell himself considered even in concert with the lattice fermion number density (configuration) since he included the state vector $|t\rangle$ as a *non*-local beable [5]. As he notes, in theories in which spacetime is quantized or emergent, local beables would be ill-defined since locality itself would be an obscure concept [6].

3 Non-local beables

Smolin's recognition of the need for a theory of non-local beables was motivated by three observations [25]. First he notes that if the metric of spacetime is a quantum operator and thus subject to the usual quantum fluctuations, then locality is merely a classical approximation. Non-locality must arise from quantum fluctuations of the metric and there are some arguments that non-locality must be present in quantum gravity at large scales [21]. Second, he notes that, if space itself is not fundamental, i.e. is emergent—something most theories of quantum gravity agree on—then locality must also be emergent. As such, space and the quantum state emerge simultaneously, each carrying some information about the non-local ontology.

3.1 Theory of *a*-local beables

Smolin notes that the only meaningful beables are those that describe relationships between elementary events or particles which is reconcilable with the primacy of pointer readings. So the hidden variables do not give a detailed description of the inner workings of, say, an electron, for example. Rather they describe the details of the relations between electrons and each other or between electrons and other fundamental entities in the universe that are ignored or not obvious when coarse-graining is applied. Since these beables are more fundamental than space itself, Smolin prefers the term *a*-local to *non*-local. This leads him to propose that the fundamental beables are relational and 'a-local' and that their fundamental description must necessarily be in a phase from which space (and quantum theory, for that matter) has yet to emerge. In fact space and quantum theory are assumed to emerge at the same time. Smolin argues that the stochasticity of quantum theory arises from our lacking control over beables that describe relationships between a system and other, distant systems in the universe [25]. Smolin's hypothesis has been expressed in a detailed dynamical theory of relational hidden variables in several different ways [1, 20, 25, 26]. Of interest to us here is Smolin's description elaborated in [24]. While Smolin originally presented it as a specifically bosonic model in [24], he expresses it in more general terms in [25].

The beables of the theory are d, $N \times N$ real symmetric matrices X_{ai}^{j} with $a = 1, \ldots, d$ and $i, j = 1, \ldots, N$. The classical, local observables corresponding to such things as pointer readings on measurement devices, are taken to be the eigenvalues λ_i^a of these matrices. In direct analogy to Bell's use of the lattice fermion number, these eigenvalues are taken as corresponding to the positions of N particles in d space. The dynamics of these matrices is given by the action

$$S = \mu \int dt \, \mathrm{Tr} \left[\dot{X}_a^2 - \omega^2 \left[X_a, X_b \right] \left[X^a, X^b \right] \right]. \qquad (17)$$

The matrices X^a are assumed to be dimensionless, ω is a frequency, and μ has units of mass·length^2. As such the parameters of the theory define an energy, $\varepsilon = \mu\omega^2$. In this case, the N particles are free. But classical interactions can be modeled by including a potential $V(\lambda)$ that is a function of the eigenvalues in the trace.

The theory is invariant under $SO(N)$ transformations

$$X^a \to U X^a U^T \qquad (18)$$

where $U \in SO(N)$, and thus they constitute the gauge transformations of the theory. As such the physical observables, corresponding to the λ_i^a, are invariant under $SO(N)$ transformations. The off-diagonal elements of X^a are the non-local hidden variables of the system. The model includes a translation symmetry that ensures that the center of mass momentum of the system is conserved and defines the potential energy in a manner such that it has its minima whenever the X^a commute with one another in which case they can be simultaneously diagonalized. This is precisely what gives the classical approximation and leads to the interpretation of the eigenvalues as labeling the positions of N identical particles in $\mathbb{R}^d$.

3.2 Noncommutativity

There is a problem with using X^a to define the beables for the theory, however. The eigenvalues of the X^a correspond to the positions of N objects in d space. So each individual X^a can be thought of as a single configuration of these particles in that space akin to Bell's lattice fermion number density (configuration). Smolin notes that these eigenvalues are specifically the classical, local observables [25] and should, thus, be distillable to pointer readings.

But, as Bell clearly notes, observables that do not all have simultaneous eigenvalues, i.e. that do not all commute, cannot be promoted to the status of beables [5]. For example, in [5] he expressly *rejects* energy density $T_{00}(x)$ as a choice of beable precisely for this reason. As he notes, the lack of simultaneous eigenvalues for all positions means we would need to devise some new manner of specifying a joint probability distribution for any pair of such observables. Thus if the eigenvalues of the X^a correspond to observables, which they should if their eigenvalues correspond to positions, then the X^a must be operators, in which case there is a limit to the precision by which we can simultaneously specify their values, i.e. any pair of non-commuting X^a must satisfy an indeterminacy relation and would not have simultaneous eigenvalues.

Recall that the purpose of a beable is to be able to say what *is* rather than what is merely *observed*. The idea is to be able to 'uncover' the hidden variables, ultimately allowing us to say something for certain about the actual state of a system or of the universe as a whole at a given time. But the potential presence of an indeterminacy relation precludes us from doing this. In effect, since the eigenvalues of X^a are associated with such things as pointer positions, anytime a pair of X^a do not commute, there must exist two such things (e.g. two pointer positions) that cannot be simultaneously known to perfect precision which violates the very essence of a beable as set forth by Bell. Another solution is needed.

4 Theory of global beables

We now describe a theory of global beables based on solutions to the Wheeler-DeWitt equation that does not suffer from the same problems as the theories discussed in the previous sections. We choose the word 'global' since possible solutions to the Wheeler-DeWitt equation include wave functions (or, more properly, functionals) of the universe. We begin by briefly reviewing the Wheeler-DeWitt formalism before introducing the theory.

4.1 Wheeler-DeWitt formalism

If we take spacetime as being foliated into spacelike submanifolds, we can decompose the metric tensor as

$$g_{\mu\nu}dx^\mu dx^\nu = \left(-\alpha^2 + \beta_k\beta^k\right)dt^2 \\ + 2\beta_k dx^k dt + \gamma_{ij}dx^i dx^j \qquad (19)$$

where α is the lapse function [1], the β_k are the shift functions, and γ_{ij} is the spatial three-metric. The usual summation convention is assumed such that Greek indices range from $0 \to 3$ while Latin indices range from $1 \to 3$. The lapse function is given as $\alpha = \left(-g^{00}\right)^{-1/2}$ where the g^{00} is the usual four-dimensional value. The shift functions are thus given as $\beta_k = g_{0i}$ where, again, these are elements of the usual four-dimensional metric. The spatial three-metric is therefore $\gamma_{ij} = g_{ij}$. In the Hamiltonian formulation that follows, the spatial three-metric serves as the set of generalized coordinates to which we can associate conjugate momenta π^{ij}. We define $\mathcal{R} = {}^{(3)}R$ where ${}^{(3)}R$ is the three-dimensional Ricci scalar. The Hamiltonian is then a constraint given by

$$\mathcal{H} = \frac{1}{2\sqrt{\gamma}}G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{\gamma}\mathcal{R} = 0 \qquad (20)$$

where $\gamma = \det\left(\gamma_{ij}\right)$ and $G_{ijkl} = \left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}\right)$ is the Wheeler-DeWitt metric on superspace which is the space of all three-geometries. We can quantize this by employing the ADM formalism [2] which allows us to turn the momenta and field variables into operators such that the Hamiltonian operator becomes

$$\hat{\mathcal{H}} = \frac{1}{2\sqrt{\gamma}}\hat{G}_{ijkl}\hat{\pi}^{ij}\hat{\pi}^{kl} - \sqrt{\gamma}\hat{\mathcal{R}} \qquad (21)$$

where, in position space, the generalized coordinates and their conjugate momenta are

$$\hat{\gamma}_{ij}(t, x^k) \to \gamma_{ij}(t, x^k); \quad \text{and}$$
$$\hat{\pi}^{ij}(t, x^k) \to -i\frac{\delta}{\delta\gamma_{ij}(t, x^k)} \qquad (22)$$

respectively. The Hamiltonian is not, however, applied to the usual wave function. Instead it is applied to a wave *functional* $\Psi(\gamma)$ of field configurations defined on the full metric, i.e. all of space-time. The Hamiltonian constraint (20) necessarily implies, then, that

$$\hat{\mathcal{H}}\Psi(\gamma) = 0 \qquad (23)$$

or, more familiarly

$$\hat{H}(x)\left|\psi\right\rangle = 0 \qquad (24)$$

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[1] We use α for the lapse function and β for the shift functions instead of the usual N in order to distinguish these from the N labeling the matrices in Smolin's theory.

which is known as the Wheeler-DeWitt equation [11, 12]. Here $\hat{H}(x)$ is a Hamiltonian constraint, of which there are technically an infinite number, and $|\psi\rangle$ is referred to as the wave function of the universe even though it is more properly a state vector. Since $\Psi(\gamma)$ is a functional of the field configurations defined on the full metric, i.e. all of spacetime, the Hamiltonian no longer determines time evolution and thus the usual Schrödinger equation does not apply because all of space and time are subsumed within the functional. Specifically $\Psi(\gamma)$ contains *all* of the information about the matter and geometry content of the universe [17].

The Hamiltonian in the Wheeler-DeWitt equation, unlike the case in typical quantum field theory or quantum mechanics, is a first class constraint on physical states which is a dynamical quantity in a constrained Hamiltonian system whose Poisson bracket with all other constraints must vanish on the constraint surface in phase space [14]. For example, due to the invariance of the wave functional under spatial diffeomorphism, the Wheeler-DeWitt equation is typically accompanied by a momentum constraint, $\hat{P}(x)|\psi\rangle = 0$. Thus, because the Hamiltonian is a first class constraint, $\{\hat{P}(x), \hat{H}(x)\} = 0$.

4.2 Global beables

We wish to construct a theory of *global* beables guided by the Wheeler-DeWitt equation that does not suffer the same fate as Bell's and Smolin's theories but that still endeavors to make the concept 'positions of things' more precise. Since we technically have an infinite number of Hamiltonian constraints due to the infinite degrees of freedom of the phase space, we do not appear to be much closer to a useful and unambiguous definition of the 'positions of things' in real, physical space. But, as it turns out, we can reduce the number of Hamiltonian constraints to just one by making a minisuperspace approximation [12, 17, 28].

In any dynamical model, positions change over time. But in the model we will consider here, time is simply a manner by which we can order the spacelike submanifolds. In other words, four-dimensional spacetime is foliated into three-dimensional spacelike surfaces ordered over the fourth dimension (time). For our model, we assume that each spacelike surface is a specific closed three-sphere on which the matter field is fixed. Transitions from one three-sphere to another take the place of time evolution in this model. For example, suppose that the total classical action for some particular metric g coupled to a scalar field ϕ is $S[g, \phi]$. The quantum-mechanical amplitude for the occurrence of a particular spacetime and thus field history is $\exp(iS[g, \phi])$. In analogy to the usual propagator $\langle x'', t''|x', t'\rangle$ and re-

calling that the spatial three-metric is $\gamma_{ij} = g_{ij}$, the transition amplitude between any pair of three-spheres is [17]

$$\langle \gamma_{ij}'', \phi''|\gamma_{ij}', \phi'\rangle = \int \delta g \delta \phi \; e^{iS[g,\phi]} \qquad (25)$$

where the integral is over all four-geometries and field configurations that match the given values on the two three-spheres. In Smolin's model, evolution occurs explicitly according to the Schrödinger equation. But in this model, there is no evolution, strictly speaking. Rather (25) more correctly describes the amplitude for a certain three-geometry and an associated field to be fixed on any pair of three-spheres. The wave functionals are then defined as

$$\Psi[\gamma_{ij}, \phi] = \int_C \delta g \delta \phi \; e^{iS[g,\phi]} \qquad (26)$$

where the integral is over a class C of spacetimes with a compact boundary on which γ_{ij} is the induced metric and ϕ is the field configuration on that boundary. One can also show that because we expect in gravity to find the field equations satisfied as identities, then

$$\int \delta g \delta \phi \; \hat{H}(x) \; e^{iS[g,\phi]} = 0 \qquad (27)$$

for any class of geometries summed over and for any intermediate three-sphere on which $\hat{H}(x)$ is evaluated. In order to specify a particular state of the universe, the details of the class C must be specified. Therefore, if the universe is in a quantum state specified by a particular state vector and corresponding wave function, then that wave function describes the correlations between observables to be expected in that state. As such, like Hartle and Hawking, we restrict the geometrical degrees of freedom to spatially homogenous, isotropic, closed universes with S^3 topology (i.e. closed three-spheres) and the matter degrees of freedom to a single, homogenous, conformally invariant scalar field with the cosmological constant assumed to be positive. Our aim is to consider what it means to specify the 'positions of things' in such a model and we refer the reader to [17] for the full mathematical formalism of the minisuperspace model.

In order to specify the 'positions of things' we first must specify both what we mean by 'position' and what we mean by 'thing.' Since Bell's aim in developing the concept of beables was to account for positions of such classical things as instrument pointers, we will assume that Bell was referring to things that were constructed of ordinary matter. As such, the 'things' in this model are represented by the matter field ϕ. In Bell's fermionic model, the three-space continuum was replaced by a dense but discrete lattice. As we saw, however, the discreteness of the lattice gives rise to the problem of

fermion doubling. In the minisuperspace model, however, we retain the three-space continuum and its form is specified by the induced metric γ_{ij} where the induced metric takes the place of the discrete lattice. Thus, specifying the 'positions of things' entails mapping the matter field *onto* the metric which amounts to jointly specifying a field configuration and a metric (i.e. a three-sphere) via a state vector $|\gamma_{ij}, \phi\rangle$ or its associated wave functional $\Psi(\gamma_{ij}, \phi)$. Thus, if the beables must specify the positions of things, then jointly specifying the field configuration ('things') and the induced metric on which the field is specified ('positions') accomplishes this task. Therefore, *the state vector $|\gamma_{ij}, \phi\rangle$ or its associated wave functional $\Psi(\gamma_{ij}, \phi)$ are the beables of the theory.*

4.3   Deterministic constraints

For a fully deterministic model, one would expect that the transition amplitude between any pair of three-spheres would be either unity or zero—such a transition is either allowed or it is not. This provides an added constraint on (25),

$$\int \delta g \delta \phi \, e^{iS[g,\phi]} \; \in \; \{0,1\} \quad \text{for all} \quad |\gamma_{ij}, \phi\rangle . \quad (28)$$

It is worth noting, here, that we are assuming that the Wheeler-DeWitt approach is fundamentally a path integral approach. The implications of such an assumption are discussed in [19].

Recall that in the theory of *local* beables, beables defined on some bounded set $Q = \{\mathbf{x}, t\}$ of spacetime coordinates were determined by those corresponding to any spacelike slice X of the spacetime region R that fully encloses the past light-cone of Q (see Section 2 and Fig. 1). Bell referred to this condition as *local determinism*. This suggests the determinism inherently includes some idea of order in the sense that time is a label of spacelike slices such that those slices must occur in a given order for the theory to be considered deterministic. For example, consider a set of beables defined on some bounded set $Q = \{\mathbf{x}, t\}$ of spacetime coordinates where the beables are assumed to be determined by those corresponding to any temporal cross section (i.e. spacelike slice) X of the spacetime region R that fully encloses the past light-cone of Q where X is a subset of R corresponding to all possible values of $\mathbf{x}$ associated with the value t_X. The beables defined on Q are, of course, also assumed to be determined by those corresponding to a different spacelike slice X' that is also a subset of R as shown for a single spatial dimension, x, in Fig. 2. This follows from the fact that where we drew X in the first place was simply restricted to intersecting the past light-cone of Q. Exactly *where* it
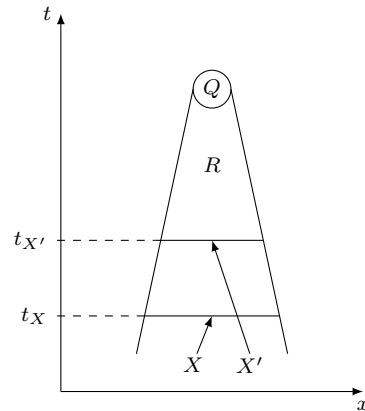

Figure 2: Beables associated with the bounded region of spacetime Q are determined by those associated with any temporal cross-section (i.e. spacelike slice) X of region R that fully encloses the past light-cone of Q. Likewise, they are determined with any spacelike slice X' that also fully encloses the past light-cone of Q.

intersected the past light-cone of Q was not specified. Thus the beables defined on Q are determined both by those defined on X as well as those defined on X'. But notice that any beables defined on the spacelike slice X' are *also* determined by those associated with X since X intersects the past light-cone of every point on X'. In other words, beables associated with X determine both the beables associated with X' as well as those associated with Q. But the converse is not true, of course. Thus determinism appears to require an ordering of the spacelike slices. In fact the ordering of the set of all such spacelike slices *defines* the past and future light-cones of Q.

In a global theory of beables based on the Wheeler-DeWitt model there is no 'past' light-cone, strictly speaking, since time is treated as a fourth dimension that we use to label and order the three-spheres. Each three-sphere is technically a set of *all* spacetime coordinates along with a metric. The metric defines the relations between the spacetime coordinates, i.e. the geometry, and thus differentiates one three-sphere from another. In analogy to local determinism, we expect that for a global theory of beables to be deterministic, *the set of all three-spheres must be ordered.* (Recall that locality has little meaning in this context since we are talking about globally defined beables and so the notion of determinism here is broader.) Thus, in general we can say that *the class of all theories for a given Hamiltonian constraint $\hat{H}(x)$ whose specification of g, ϕ, and S satisfy both (27), is deterministic if (28) holds and its set of all three-spheres is ordered.*

5   Analysis

We have proposed a theory of global beables—objectively known properties of the universe aimed at making the concept 'positions of things' more precise—that does not suffer from some of the problems inherent in several other such theories. In addition, we have proposed a set of deterministic constraints for such a theory.

It is worth noting that it is not entirely clear whether or not Smolin's theory is deterministic. The elements of X^a undergo Brownian motion as they oscillate in the potential which means that the eigenvalues must also undergo Brownian motion [25]. Strictly speaking, Brownian motion is a stochastic process and is thus not deterministic. However, in classical physics if one takes into account all of the information of the environment of some object undergoing Brownian motion then it is considered deterministic since it ignores quantum effects. Smolin expressly uses the Brownian motion of the elements of X^a as a method for introducing quantum effects which might suggest his model is not deterministic. However, he is also including hidden variables (the off-diagonal elements of X^a) which are usually invoked in order to restore determinism. By contrast, the theory we have proposed avoids any ambiguity by giving a clear set of constraints that must be satisfied in order for the theory to be considered deterministic in a manner analogous to Bell's notion of local determinism.

It is also important to note that the beables of our theory are entirely unambiguous about how to specify the 'positions of things' in the sense Bell seems to have intended. Indeed, the specification of a given state vector or wave functional in our theory is analogous to a set $T = \{n(t), |t\rangle\}$ consisting of the lattice fermion number configuration at time t and an associated state vector in Bell's theory. The main difference is that we have avoided the problems introduced by the Unruh effect and fermion doubling. While we no long have time evolution, per sé, we note that the Wheeler-DeWitt formalism still allows for a certain sense of 'dynamics' in that the usual propagator is replaced by an analogous transition amplitude.

Of course, it may be the case that spacetime really *is* quantizable and discrete in which case our theory might not hold. In addition, it is not yet clear in this model if quantum correlations can be preserved while simultaneously satisfying the determinism constraints. Likewise, the Hartle-Hawking minisuperspace model includes situations in which the universe can quantum mechanically tunnel between two states which also could potentially be a problem for determinism since it would seemingly render some three-spheres redundant. Thus one avenue of future research would be to further examine the determinism constraints of our model to determine if they are compatible with quantum correlations and tunneling. This, of course, assumes that there are values for the metric, the matter field, and the action that satisfy both (27) and (28) for a given Hamiltonian constraint. Certainly there is a trivial solution that holds when the action and the Hamiltonian constraint are both zero, but this is not particularly informative. But, if one or more non-trivial solutions exist and additionally they preserve determinism in the broadest sense, i.e. that the state of the universe at one instant determines the state of the universe at the next, then we may yet achieve Bell's goal of an unambiguous theory of beables that gives a precise accounting of the 'positions of things.'

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