

ON THE EMERGENCE OF THE STRUCTURE OF PHYSICS

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ABSTRACT. We consider Hilbert’s problem of the axioms of Physics at a qualitative or conceptual level. This issue is more pressing than ever as we seek to understand how both GR and quantum theory could emerge from some deeper theory of quantum gravity, and in this regard I have previously proposed a ‘principle of self-duality’ or ‘quantum Born reciprocity’ as a key structure. Here I outline recent work around the idea of quantum spacetime as motivated by this non-standard philosophy, including a new toy model of gravity on a spacetime consisting of four points forming a square.

1. INTRODUCTION

To think about the axioms of physics we will start with a non-standard philosophy of ‘relative realism’ whereby reality is more like pure mathematics, created by decisions to work within certain axioms or assumptions [17, 22, 24]. To the extent that we are not conscious of this, to that extent we experience the reality created by those axioms. To the extent that we *are* aware, we transcend that level of ‘reality’ but the fact that those axioms were possible and all the substructure they contain is an element of a larger reality in which that was just one path we could have taken. This makes reality relative to your point of view which is not necessarily a bad thing given the Copenhagen interpretation of quantum mechanics.

In this point of view, Physical Reality as we know it should be characterised or in some sense created by the decision to adopt certain axioms or assumptions. The difference with most mathematical subjects is that we don’t *apriori* know what the axioms are but are working backwards to find them. My thesis in [17, 22, 24] was that if we eventually succeed we will in fact uncover a characterisation of what it is to be a physicist. And knowing this, one can anticipate that one of the central axioms of Physics should be rooted in the scientific method, which I see as a dual relationship between theory and experiment.

This in turn can be expressed mathematically as duality between an abstract structure and its realisation or representation, a theme that runs throughout mathematics as shown in Figure 1 taken from [17, 19]. The arrows here are meant to be inclusion functors between categories of structures, loosely interpreted. The familiar case here is that of an Abelian group G . Its set of representations itself forms a group \hat{G} and $G \subseteq \hat{G}$ says that from mathematical point of view one is free to

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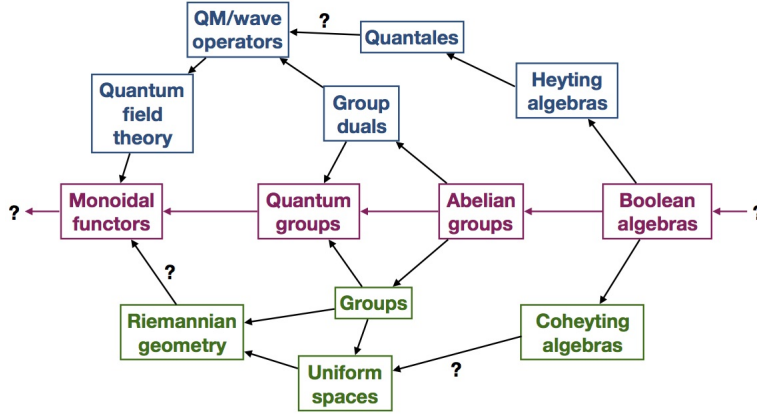


FIGURE 1. An axiom of Physics is the search for a self-dual structure in a self-dual category.

reverse which is the abstract structure and which is its representation. For example in Physics, G could be position space \mathbb{R}^n then \hat{G} in a suitable setting would be momentum space \mathbb{R}^n , a self-dual example in the self-dual category. The *principle of representation-theoretic self duality*[17] or ‘generalised Mach principle’ is the idea that Physics should admit a reversal of which parts are structure and which parts are representation, for example which is position and which is momentum. This need not result in the same theory but merely a dual theory. The strong version is that the dual theory should have the same form but possibly with different values of parameters. From this point of view, Boolean algebra with its de Morgan duality is arguably the ‘birth’ of physics[19], while the next self-dual category beyond Abelian groups is Hopf algebras or ‘quantum groups’. Thus I argued in my 1988 PhD thesis that constructing noncommutative noncocommutative Hopf algebras could be seen as a toy model of constructing elements of quantum gravity, and used this to obtain one of the two main classes of such true quantum groups, the *bicrossproduct* ones. This was around the same time as V.G. Drinfeld introduced the other (and more famous) class of q -deformed quantum groups coming from quantum integrable systems. I will say more about bicrossproducts shortly.

Quantum groups here are a big enough category to include nonAbelian groups and their Fourier duality. If G is a compact Lie group, say, its function algebra $C(G)$ and its group convolution algebra $C^*(G)$ can be completed to mutually dual Kac or Hopf-von Neumann algebras. At the algebraic level we have coordinate algebras $\mathbb{C}[G]$ and enveloping algebras $U(\mathfrak{g})$ as essentially dual. Traditionally one has to do non-Abelian Fourier transform categorically but in the language of Hopf algebras it becomes quantum Fourier transform, for example $\mathbb{C}[G] \rightarrow U(\mathfrak{g})$ (indicating a suitable completion that includes exponentials). Here $U(\mathfrak{g})$ is regarded as a ‘coordinate algebra’ of a noncommutative space. We will come to the physics of this shortly but for the moment we continue along the self-dual axis in Figure 1. Here in the search for the ‘next’ self-dual category, I found in 1990 the following duality construction $(\mathcal{C} \rightarrow \mathcal{V})^\circ = \mathcal{C}^\circ \rightarrow \mathcal{V}$ for functors between monoidal categories[18, 19]. Here a monoidal category \mathcal{C} means there is a \otimes product which is associative up

$$\begin{array}{ccc}
V \otimes F(X) \otimes F(Y) & \cong & V \otimes F(X \otimes Y) \\
(\lambda_V)_X \downarrow & f_{X,Y} & \downarrow (\lambda_V)_{X \otimes Y} \\
F(X) \otimes V \otimes F(Y) & & \\
(\lambda_V)_Y \downarrow & & \\
F(X) \otimes F(Y) \otimes V & \cong & F(X \otimes Y) \otimes V \\
& f_{X,Y} &
\end{array}$$

FIGURE 2. Representation of a monoidal category underlying Drinfeld-Majid centre construction.

to an associator cocycle and \mathcal{V} another one, for example Vector Spaces, in which we construct our representations. The objects of \mathcal{C}^0 are pairs (V, λ_V) where V is an object of \mathcal{V} and $\lambda_V \in \text{Nat}(V \otimes F, F \otimes V)$ is a natural transformation such that the diagram in Figure 2 commutes. Here λ_V is a collection of morphisms $(\lambda_V)_X : V \otimes F(X) \rightarrow F(X) \otimes V$ for all $X \in \mathcal{C}$ which are functorial in the sense of compatible with any morphisms $X \rightarrow Y$ in \mathcal{C} and the condition in the figure says that it ‘represents’ the tensor product of \mathcal{C} as composition in \mathcal{V} . Note that the monoidal functor F comes equipped with an associated natural isomorphism f in the sense of functorial isomorphisms $f_{X,Y} : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ for all objects X, Y in \mathcal{C} , which we use. One has $\mathcal{C} \subseteq \mathcal{C}^{00}$ and the construction generalises both group and Hopf algebra duality. The tensor product of two ‘representations’ is just

$$(\lambda_{V \otimes W})_X = ((\lambda_V)_X \otimes \text{id})(\text{id} \otimes (\lambda_W)_X)$$

where we move W past $F(X)$ then V past $F(X)$. By a theorem of Mac Lane for monoidal categories, we suppress the associator between tensor products as these can be inserted afterwards.

Example 1. *If G is a finite group and \mathcal{C} the category of G -graded vector spaces, we can tensor product such spaces by the product of gradings in the group, obtaining a monoidal category. We take F the functor that forgets the grading, then \mathcal{C}^0 has objects vector spaces V equipped with natural isomorphisms $(\lambda_V)_X : V \otimes F(X) \rightarrow F(X) \otimes V$ sending $(\lambda_V)_X(v \otimes x_g) = x_g \otimes v \triangleleft g$ for some right action of G on V . One can check that this meets the requirements above. Thus, \mathcal{C}^0 is essentially the category of representations of G .*

This connects our monoidal duality to non-Abelian group duality. The latter also includes Hopf algebra duality when appropriately formulated. A genuinely new example of considerable interest these days in topological quantum field theory is the following.

Example 2. *(Drinfeld-Majid centre.) A special case of the above construction is when $\mathcal{V} = \mathcal{C}$ and $F = \text{id}$. This case was found independently by Drinfeld but the definitions and proofs are identical to our \mathcal{C}^0 construction just leaving out the symbol F . In this case there is a tautological braiding if we assume the λ are isomorphisms.*

Note that Drinfeld came across the preprint of [18] in the library at Harvard and wrote to me that he was aware of but had not published this special case and that

it was braided. Hence it is not the case that \mathcal{C}^0 was a generalisation of Drinfeld's work, we simply came to essentially the same construction for different reasons. My reason was the principle of representation-theoretic self-duality while Drinfeld's was I believe to generalise his famous double construction for quantum groups in [9]. Some of my own follow-up work was [19].

How is this dualism reflected in Physics? One setting already alluded to and which we have called *quantum Born reciprocity (QBR)*, is Fourier duality between position and momentum space and its generalisations. In the Abelian group case this is just wave particle duality, but it also works in the nonAbelian case. If the universe is spatially $S^3 = SU_2$ (and it might be) then spatial momentum is the representations of this. These form a category but one can also see Fourier transform at the Hopf algebra level from $\mathbb{C}[SU_2]$ to (a completion of) $U(su_2)$ where the latter is regarded as a quantum momentum space $[p_i, p_j] = \frac{\hbar}{l_c} \epsilon_{ijk} p_k$. Here l_c is the cosmological curvature scale and p_i are left-invariant vector fields. Dually, if the momentum space of some system were to be the nonAbelian group SU_2 then the Fourier dual would be the quantum spacetime with relations $[x_i, x_j] = \iota \lambda_P \epsilon_{ijk} x_k$ where λ_P is a length scale and the $U(su_2)$ generators are now regarded as position coordinates. This as we will see shortly is thought to be the case in some models of 3D quantum gravity.

In my PhD thesis[15, 16] I took this point of view and the above self-duality principle as a motivation to look for self-dual type Hopf algebras, and constructed these in the 'bicrossproduct' form $\mathbb{C}[M] \blacktriangleleft U(\mathfrak{g})$ with dual $U(\mathfrak{m}) \blacktriangleright \mathbb{C}[G]$ associated to any local factorisation of a Lie group $X = G \rtimes M$. These were originally thought of these as quantum phase space but since 1994 in [26] I have also thought of them as quantum Poincaré groups acting on $U(\mathfrak{m})$ and $U(\mathfrak{g})$ respectively as auxiliary quantum spacetimes. In fact there are different covariant systems with equivalent data related by *semidualisation*[15, 16] (where one Hopf algebra is systematically replaced by its dual),

$$\mathbb{C}[M] \blacktriangleleft U(\mathfrak{g}), U(\mathfrak{m}) \Leftrightarrow U(\mathfrak{r}), \mathbb{C}[M]; \quad U(\mathfrak{m}) \blacktriangleright \mathbb{C}[G], U(\mathfrak{g}) \Leftrightarrow U(\mathfrak{r}), \mathbb{C}[G]$$

where \mathfrak{r} is the Lie algebra of X and in each pair we give the (possibly quantum) symmetry group and the (possibly quantum) spacetime algebra on which it acts. The relevant factorisation for 3D quantum gravity is $SL(2, \mathbb{C}) = SU_2 \rtimes H_3$ where $H_3 = \mathbb{R}^2 \rtimes \mathbb{R}$ is the group of upper-triangular matrices in the Iwasawa decomposition. Focusing on the first two pairs, we have the top line of Figure 3 where on the top left $U(h_3)$ is the quantum spacetime

$$(1) \quad [x_i, t] = \iota \lambda_P x_i$$

for $i = 1, 2$ which is the 3D version of the Majid-Ruegg quantum spacetime [26]. In its Poincaré quantum group the momentum is commutative because its 'enveloping algebra' is the commutative Hopf algebra $\mathbb{C}[H_3]$ but curved as H_3 is non-Abelian. The semidual of this on the top right is a classical model of a particle on H_3 as curved positions space with its classical $U(so_{1,3}) = U(su_2) \rtimes U(h_3)$ symmetry containing $U(h_3)$ as the translational momentum. So the roles of position and momentum are swapped between the two models – an example of QBR. It is also striking that the model on the right is classical (a particle on a curved space H_3) while the other model is quantum, so QBR interchanges classical and quantum.

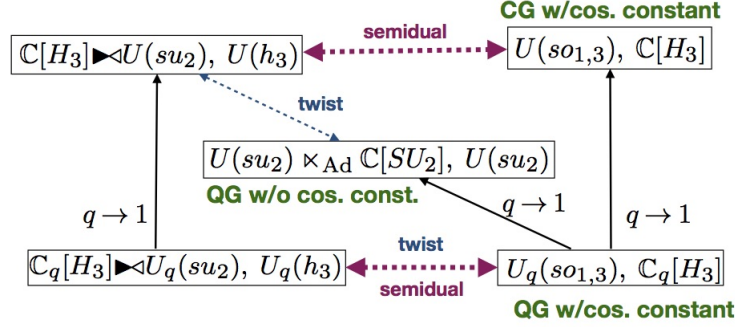


FIGURE 3. Quantum Born reciprocity expressed as semidualisation in 3D quantum gravity and its limits.

In fact this picture q -deforms [27] as we show on the bottom line of Figure 3, where the model on the bottom right is thought to encode quantum gravity with cosmological constant via an expression of the form $q = e^{-\lambda_P/l_c}$. Its QBR-dual model on the bottom left when $q \neq 1$ is at some algebraic level isomorphic to two copies of $\mathbb{C}_q[H_3] = U_q(su_2)^{cop}$ acting on $U_q(h_3) = \mathbb{C}_q[SU_2]^{op}$ up to some technicalities, i.e. a classical but q -deformed particle on a 3-sphere, and this is related by a categorical equivalence (a Drinfeld twist) to the model on the right. In other words, 3D quantum gravity with cosmological constant is in some sense *self-dual* under QBR. Finally, we can take $q \rightarrow 1$ in different ways and the first one, on the outer right, is $\lambda_P \rightarrow 0$ (so a classical but curved model). Alternatively we can send $l_c \rightarrow \infty$ and this is the model in the centre of the figure encoding 3D quantum gravity without cosmological constant (to see this one should write $U_q(so_{1,3}) = U_q(su_2) \rtimes U_q(h_3) = U_q(su_2) \rtimes \mathbb{C}_q[SU_2]^{op}$ up to some technicalities and then take the limit). The diagonal twist equivalence between this conventional version of 3D quantum gravity with $U(su_2)$ quantum spacetime and the one we began with (on top left) was recently shown by P. Osei and the author[25]. More details of the point of view for 3D quantum gravity are in [27].

If we leave the self-dual axis then the dual structure is not of the same type but is still a structure. If we take a more categorical view of group duals then the dual of a compact Lie group comes down to quantisation (for example of coadjoint orbits) or geometrically to diagonalising the natural Laplacian or wave operator. Compact Lie groups are the simplest examples of Riemannian manifolds and we can similarly think of that the ‘dual’ of the latter coming down to quantum mechanics or wave operators. Recall that quantum mechanics in nice cases can be seen as the non-relativistic limit of the Klein-Gordon spacetime Laplacian for fields where $e^{-imc^2t/\hbar}$ is factored out. In fact a Riemannian or pseudo-Riemannian manifold cannot be totally reconstructed from the Laplacian alone but if we use the Dirac operator the one has Connes’ reconstruction of a spin manifold from an abstractly defined ‘spectral triple’ in the commutative case[8]. Either way, if one extends these ideas from classical to quantum field theory then logically an element of this should be that it corresponds to *quantum Riemannian geometry*, where spacetime coordinates become noncommutative. This should then be taking us towards the self-dual axis as shown Figure 1. Interestingly, [6] have now constructed quantum field theories on

curved spacetimes as a functor from the monoidal category of globally hyperbolic spacetimes to noncommutative C^* -algebras with tensor product, which is some kind of arrow from QFT to monoidal functors in Figure 1. Such functors are not self-dual under $(\)^\circ$ but then this is not yet quantum gravity. At any rate, one could speculate in this context that Einstein's equation might eventually emerge as the classical limit of a self-duality condition as it equates the Einstein tensor from the geometry side to the expectation value of the stress energy tensor from the quantum field theory side, probably requiring both to be generalised so as to be in a self-dual setting. I do not know the final framework for this but the monoidal category dual may be a step in the right direction.

In summary, we were led on philosophical grounds to the view that spacetime should be both curved and 'quantum' in the sense of a noncommutative coordinate algebra, as a consequence of a deep self-duality principle for quantum gravity[17, 22, 24]. We call this aspect the *quantum spacetime hypothesis*. It's a prediction which, if confirmed, would be on a par with the discovery of gravity and indeed dual to it. What it could entail mathematically is our next topic.

2. AXIOMS OF QUANTUM RIEMANNIAN GEOMETRY

I will recap a constructive approach to this from my own work in the last decade (much of it with Edwin Beggs) rather than the better-known 'Dirac operator' approach of Connes expressed in the axioms of a spectral triples. The two approaches have recently begun to come together with our programme of *geometric realisation* of Connes' spectral triples[4].

In fact all main approaches have in common (in our case as starting point) the notion of differential forms (Ω, d) over a possibly noncommutative algebra A as a differential graded algebra. This means

$$\Omega = \bigoplus_n \Omega^n, \quad d : \Omega^n \rightarrow \Omega^{n+1}, \quad d^2 = 0$$

where $\Omega^0 = A$ and d obeys a graded-Leibniz rule with respect to the graded product \wedge . We assume that Ω is generated by A, Ω^1 in which case one may focus on (Ω^1, d) first and construct higher differential forms as a quotient of the tensor algebra of this over A . Here

$$d : A \rightarrow \Omega^1, \quad d(ab) = (da)b + a(db), \quad (adb)c = a((db)c)$$

where Ω^1 has an associative multiplication from the left and the right by A (one says that Ω^1 is an A -bimodule) and d is a derivation.

The next ingredient is a left connection,

$$\nabla : \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1, \quad \nabla(a\omega) = da \otimes \omega + a\nabla\omega$$

which is a *bimodule connection*[10, 29] if there also exists a bimodule map σ such that

$$\sigma : \Omega^1 \otimes_A \Omega^1 \rightarrow \Omega^1 \otimes_A \Omega^1, \quad \nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes da).$$

The map σ if it exists is unique, so this is not additional data but a property that some left connections have. In the classical case where $A = C^\infty(M)$, if X is a vector field then a connection ∇ defines a covariant derivative $\nabla_X : \Omega^1 \rightarrow \Omega^1$ by evaluating X against the left output of ∇ (this also works with care in the quantum case).

However, we consider all such covariant derivatives together by leaving ∇ as a 1-form valued operator on 1-forms. The curvature and torsion of a left connection, see for example [20], are

$$R_{\nabla} = (d \otimes \text{id} - \text{id} \wedge \nabla) \nabla : \Omega^1 \rightarrow \Omega^2 \otimes_A \Omega^1, \quad T_{\nabla} = \wedge \nabla - d : \Omega^1 \rightarrow \Omega^2.$$

Incidentally, all the same ideas except for the torsion hold for any vector bundle, which we axiomatize via its space of sections E as a left module over A , and $\nabla_E : E \rightarrow \Omega^1 \otimes_A E$. In the bimodule case, if E, F are bimodules and ∇_E, ∇_F are bimodule connections then the tensor product $E \otimes_A F$ has a bimodule connection

$$\nabla_{E \otimes F}(e \otimes f) = \nabla_E e \otimes f + (\sigma_E \otimes \text{id})(e \otimes \nabla_F f)$$

and a certain $\sigma_{E \otimes F}$. This makes the collection of such pairs (E, ∇_E) into a monoidal category with morphisms usually taken as bimodule maps that intertwine the connections. There is a forgetful functor from this to the category of bimodules over A , so this is an example of monoidal functor (in the sense of Figure 1) associated to any manifold and to any algebra more generally.

Next we consider a Riemannian metric $g = g^1 \otimes_A g^2 \in \Omega^1 \otimes_A \Omega^1$ (sum of such terms understood). We want this to non-degenerate in the sense of a bimodule map $(\ , \) : \Omega^1 \otimes_A \Omega^1$ that is inverse, $(\omega, g^1)g^2 = \omega = g^1(g^2, \omega)$ for all $\omega \in \Omega^1$. In this case $(\omega, ag^1)g^2 = (\omega a, g^1)g^2 = \omega a = (\omega, g^1)g^2 a$ tells us that $[a, g] = 0$ for all a , i.e. g has to be central[3]. So even though we are doing noncommutative geometry and do not assume that 1-forms commute with functions, we will need the metric to be central. This is a significant constraint on quantum spacetime in the noncommutative case which is invisible classically. We also usually want the metric to be quantum symmetric in the sense $\wedge(g) = 0$.

Finally, we want ∇ to be metric compatible. There is a weak notion that makes sense for any left connection, namely it is ‘weak quantum Levi-Civita’ if it is torsion free and

$$coT_{\nabla} = (d \otimes \text{id} - \text{id} \wedge \nabla)g \in \Omega^2 \otimes_A \Omega^1$$

vanishes. This cotorsion tensor was introduced in [20] and classically says that $\nabla_{\mu} g_{\nu\rho} - \nabla_{\nu} g_{\mu\rho} = 0$. In the case of a bimodule connection we can do better and we say this is *quantum Levi-Civita (QLC)* if it is torsion free and $\nabla g = 0$ where ∇ extends to $\Omega^1 \otimes_A \Omega^1$ by the tensor product formula.

Usually one wants A to be a $*$ -algebra and for this to extended as a graded-involution to Ω commuting with d , and for $g^{\dagger} = g$, $\nabla \circ * = \sigma \circ \dagger \circ \nabla$ where $(\omega \otimes_A \eta)^{\dagger} = \eta^* \otimes_A \omega^*$. These reality conditions in a self-adjoint basis (if one exists) would ensure that the metric and connection coefficients are real at least in the classical limit. This completes our lighting review.

By now there are many specific quantum Riemannian geometries constructed, for example on the quantum sphere $\mathbb{C}_q[S^2]$, see [21], on the quantum spacetime (1), see [3], on the functions on the permutation group $\mathbb{C}(S_3)$, and on its dual $\mathbb{C}S_3$, see [28], each with natural differential structure, quantum metric and QLC’s or weak QLCs according to the model. The Ricci tensor is only partially understood because to follow the usual trace contraction one would need a lifting map $i : \Omega^2 \rightarrow \Omega^1 \otimes_A \Omega^1$, which is additional data. The Dirac operator is also only partially understood needing both a ‘spinor’ bundle with connection compatible with ∇ and a ‘Clifford

action'. At least for $\mathbb{C}_q[S^2]$ one can come close to the axioms of a Connes spectral triple at least at an algebraic level before any functional analysis completions[4].

3. POISSON-RIEMANNIAN GEOMETRY

Our motivation has been that quantum geometry deforms classical geometry by order λ_P corrections, as a measure of some quantum gravity effects. The semiclassical theory of which the above is a quantisation was recently worked out by Beggs and the author in [5]. This theory is to aspects of quantum gravity as classical mechanics is to quantum mechanics, except the deformation parameter is not \hbar , so a kind of 'classical quantum gravity'. One could imagine other applications, including to quantum mechanics if the phase space also has a natural Riemannian structure, so we will just call the parameter λ (and take conventions where it is imaginary).

The first layer of this is of course the Poisson structure, a tenet of mathematical physics since the early days of quantum mechanics being to 'quantise' functions $C^\infty(M)$ on a manifold to a noncommutative algebra A . We suppose that

$$a \bullet b = ab + O(\lambda)$$

where we denote the $C^\infty(M)$ product by juxtaposition and the A product by \bullet . We assume all expressions can be expanded in λ and equated order by order. In this case

$$a \bullet b - b \bullet a = \lambda\{a, b\} + O(\lambda^2)$$

defines a map $\{, \}$ and the assumption of an associative algebra quickly leads to the necessary feature that this is a Lie bracket (i.e. antisymmetric and satisfies the Jacobi identity, making $C^\infty(M)$ into a Lie algebra) and $\hat{a} := \{a, \}$ is a ('Hamiltonian') vector field associated to a function a . It is known that every such Poisson bracket can be quantised to an associative algebra at least at some formal level[14]. The second layer is to find a differential structure Ω^1 deforming the classical $\Omega^1(M)$. One can similarly analyse the data for this by defining a map ∇ by

$$a \bullet (db) - (db) \bullet a = \lambda \nabla_{\hat{a}} db + O(\lambda^2).$$

The assumption of a left action and the Leibniz rule for d , requires at order λ that

$$(2) \quad \nabla_{\hat{a}}(bdc) = \{a, b\}dc + b \nabla_{\hat{a}} dc, \quad d\{a, b\} = \nabla_{\hat{a}} db - \nabla_{\hat{b}} da$$

(these follow easily from $[a, b \bullet dc] = [a, b] \bullet dc + b \bullet [a, dc]$ and $d[a, b] = [da, b] + [a, db]$). The first condition of (2) says that ∇ is a covariant derivative along Hamiltonian vector fields \hat{a} and the second is an additional 'Poisson-compatibility'. The first part of (2) applies similarly for any bundle and can be formulated as ∇ a contravariant or Lie-Rinehart connection[13], while the second part was observed in [12, 2]. Finally, the associativity of left and right actions on a bimodule gives

$$R_{\nabla}(\hat{a}, \hat{b}) := \nabla_{\hat{a}} \nabla_{\hat{b}} - \nabla_{\hat{b}} \nabla_{\hat{a}} - \nabla_{\widehat{\{a, b\}}} = 0$$

(this follows from the Jacobi identity $[a, [db, c]] + [db, [c, a]] + [c, [a, db]] = 0$). So a zero curvature Poisson-compatible partially-defined connection is what we strictly need.

In [5] we make two convenient variations. First of all we are *not* going to require zero curvature because the effect of curvature is visible only at order λ^2 , so we do not really need this in the order λ theory. If there is curvature then it will not be possible

to have an associative differential calculus of classical dimension on A but this is actually a situation that one frequently encounters in noncommutative geometry. We can either absorb this in a higher dimensional associative differential structure or we can live with nonassociative differentials at order λ^2 . Strictly speaking, the same applies to the Poisson bracket obeying the Jacobi identity not being strictly needed at order λ^2 in which case we would have A itself being non-associative. Secondly, for simplicity, we are going to make the assumption that $\nabla_{\hat{a}}$ is indeed the restriction of an actual connection ∇ . This will allow us to speak more freely of geometric concepts such as the contorsion tensor. In fact this assumption is not critical; if the Poisson tensor in these coordinates is $\omega^{\mu\nu}$ then we are in most formulae making use only of the combination $\nabla^\mu := \omega^{\mu\nu}\nabla_\nu$ rather than the full covariant derivatives ∇_μ themselves. It means that our data has redundant ‘auxiliary modes’ that do not affect the quantum differential structure at order λ , a situation not unfamiliar from other situations such as gauge theory. There is also the matter of extending from Ω^1 to forms of all degree but this turns out[5] to impose no further conditions.

The third layer is the construction of a quantum metric and the natural data for this will be a classical metric g on M . As one might guess the metric compatibility of ∇ is just that $\nabla g = 0$. To avoid confusion we will write $\widehat{\nabla}$ for the classical Levi-Civita connection of g and we let S be the contorsion tensor of ∇ whereby $\widehat{\nabla} = \nabla + S$. It is well-known in general relativity that a metric compatible connection is determined by its torsion tensor T or equivalently a cotorsion tensor S antisymmetric in its outer indices when all indices are lowered. Hence under our simplifying assumption the data for ∇ can be thought of as T or S . In this case Poisson compatibility of ∇ can be written as[5],

$$(3) \quad \widehat{\nabla}_\gamma \omega^{\alpha\beta} + S^\alpha_{\delta\gamma} \omega^{\delta\beta} + S^\beta_{\delta\gamma} \omega^{\alpha\delta} = 0.$$

The fourth layer is more specialised as it is specifically the quantisation data for a bimodule quantum Levi-Civita connection (one could be happy with something weaker) and comes down to the identity

$$(4) \quad \widehat{\nabla}_\rho \mathcal{R}_{\mu\nu} + S^\beta_{\alpha\nu} H^\alpha_{\beta\rho\mu} - S^\beta_{\alpha\mu} H^\alpha_{\beta\rho\nu} = 0$$

where the curvature R of ∇ combines with the contorsion to define

$$(5) \quad H^\alpha_{\beta\mu\nu} = g_{\beta\gamma} \omega^{\gamma\rho} (\nabla_\rho S^\alpha_{\mu\nu} + R^\alpha_{\nu\mu\rho}), \quad \mathcal{R}_{\mu\nu} = \frac{1}{2} (H^\alpha_{\alpha\mu\nu} - H^\alpha_{\alpha\nu\mu}).$$

The latter is called the *generalised Ricci 2-form* associated to our classical data. In summary, the field equations of Poisson-Riemannian geometry come down to[5]:

- (0) A metric $g_{\mu\nu}$ and an antisymmetric bivector $\omega^{\mu\nu}$ typically obeying the Poisson bracket Jacobi identity;
- (1) A metric compatible connection ∇ at least along Hamiltonian vector fields;
- (2) Poisson-compatibility of ∇ given in the fully defined case by (3);
- (3) The optional quantum Levi-Civita condition (4).

These equations can be quite restrictive, particularly if one also wants to preserve a symmetry.

Example 3. (*Quantizing the Schwarzschild black hole*[5]) *Solving the above equations for the Schwarzschild metric in polar coordinates t, r, θ, ϕ , and asking to preserve rotational symmetry leads to a unique Poisson tensor ω and unique ∇ up to*

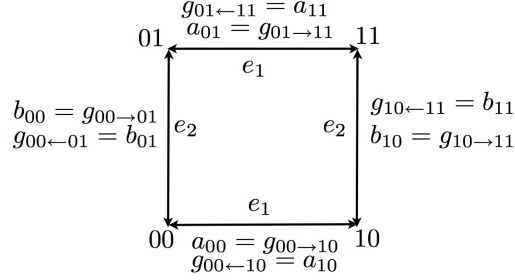


FIGURE 4. Metric coefficients a, b are interpreted as arrow lengths in our formulation of discrete gravity on a square.

auxiliary modes. This has r, t, dr, dt central (unquantised) and for each r, t one has a radius r ‘nonassociative fuzzy sphere’

$$[z_i, z_j] = \lambda \epsilon_{ijk} z_k, \quad [z_i, dz_j] = \lambda z_j \epsilon_{imn} z_m dz_n.$$

to order λ in coordinates where $\sum_i z_i^2 = 1$. Here ∇ on S^2 is the Levi-Civita connection with constant curvature, hence Ω^1 is not associative at order λ^2 .

This uniqueness result was extended to generic static spherically symmetric spacetimes in [11].

4. QUANTUM GRAVITY ON A SQUARE GRAPH

The mathematics of quantum Riemannian geometry is simply more general than classical Riemannian geometry and includes discrete[23] as well as deformation examples. What is significant in this section is that whatever we find *emerges* from little else but the axioms applied to a square graph as ‘manifold’.

Let X be a discrete set and $A = \mathbb{C}(X)$ functions on it as our ‘spacetime algebra’. It is an old result that its possible 1-forms and differential (Ω^1, d) are in 1-1 correspondence with directed graphs with X as the set of vertices. Here Ω^1 has basis $\{\omega_{x \rightarrow y}\}$ over \mathbb{C} labelled by the arrows of the graph and differential $df = \sum_{x \rightarrow y} (f(y) - f(x))\omega_{x \rightarrow y}$. In this context a quantum metric

$$g = \sum_{x \rightarrow y} g_{x \rightarrow y} \omega_{x \rightarrow y} \otimes \omega_{y \rightarrow x} \in \Omega^1 \otimes_{\mathbb{C}(X)} \Omega^1$$

requires weights $g_{x \rightarrow y} \in \mathbb{R} \setminus \{0\}$ for every edge and for every edge to be bi-directed (so there is an arrow in both directions). Taking all weights to be 1 is the so-called ‘Euclidean metric’ [23]. The calculus over \mathbb{C} is compatible with complex conjugation on functions $f^*(x) = \overline{f(x)}$ and $\omega_{x \rightarrow y}^* = -\omega_{y \rightarrow x}$. Finding a QLC for a metric depends on how Ω^2 is defined and one case where there is a canonical choice of this is X a group and the graph a Cayley graph generated by right translation by a set of generators. Previously a QLC was found for some specific groups and the ‘Euclidean metric’ but here we give a first calculation for a reasonably general class of metrics.

We take $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ with its canonical 2D calculus given by a square graph with vertices 00,01,10,11 in an abbreviated notation as shown in Figure 4. The graph is regular and there are correspondingly two basic 1-forms

$$e_1 = \omega_{00 \rightarrow 10} + \omega_{01 \rightarrow 11} + \omega_{10 \rightarrow 00} + \omega_{11 \rightarrow 01}, \quad e_2 = \omega_{00 \rightarrow 01} + \omega_{10 \rightarrow 11} + \omega_{01 \rightarrow 00} + \omega_{11 \rightarrow 10}$$

with relations and differential

$$e_i f = (R_i f) e_i, \quad df = (\partial^1 f) e_1 + (\partial^2 f) e_2$$

where $R_1 f$ shifts by 1 mod 2 (i.e. takes the other point) in the first coordinate, similarly for R_2 , and $\partial^i = R_i - \text{id}$. The exterior algebra is the usual Grassmann algebra on the e_i (they anticommute). The general form of a quantum metric and its inverse are

$$g = a e_1 \otimes e_1 + b e_2 \otimes e_2, \quad (e_1, e_1) = \frac{1}{R_1 a}, \quad (e_2, e_2) = \frac{1}{R_2 b}, \quad (e_1, e_2) = (e_2, e_1) = 0$$

where the a, b are no-where vanishing functions. With the standard $*$ structure $e_i^* = -e_i$, the metric obeys the reality property if a, b are real valued. In terms of the graph their 8 values are equivalent to the values of g on the 8 arrows as shown in Figure 4. It is natural here to focus on the symmetric case where the metric weight assigned to an edge does not depend on the direction of the arrow. This means $\partial^1 a = \partial^2 b = 0$ and we assume this now for simplicity. In this case we find a 1-parameter family of torsion free metric compatible or ‘quantum Levi-Civita’ connections:

$$\nabla e_1 = (1 + Q^{-1}) e_1 \otimes e_1 + (1 - \alpha)(e_1 \otimes e_2 + e_2 \otimes e_1) - \frac{b}{a}(R_2 \beta - 1) e_2 \otimes e_2.$$

$$\nabla e_2 = -\frac{a}{b}(R_1 \alpha - 1) e_1 \otimes e_1 + (1 - \beta)(e_1 \otimes e_2 + e_2 \otimes e_1) + (1 - Q) e_2 \otimes e_2.$$

$$\sigma(e_1 \otimes e_1) = -Q^{-1} e_1 \otimes e_1 + \frac{b(R_2 \beta - 1)}{a} e_2 \otimes e_2, \quad \sigma(e_2 \otimes e_2) = Q e_2 \otimes e_2 + \frac{a(R_1 \alpha - 1)}{b} e_1 \otimes e_1$$

$$\sigma(e_1 \otimes e_2) = \alpha e_2 \otimes e_1 + (\alpha - 1) e_1 \otimes e_2, \quad \sigma(e_2 \otimes e_1) = \beta e_1 \otimes e_2 + (\beta - 1) e_2 \otimes e_1$$

where Q, α, β are functions on the group defined as

$$Q = (q, q^{-1}, q^{-1}, q) = q^\chi, \quad \alpha = \left(\frac{a_{01}}{a_{00}}, 1, 1, \frac{a_{00}}{a_{01}} \right), \quad \beta = \left(1, \frac{b_{10}}{b_{00}}, \frac{b_{00}}{b_{10}}, 1 \right)$$

when we list the values on the points in the same binary sequence as above. Here q is a free parameter and $\chi(i, j) = (-1)^{i+j} = (1, -1, -1, 1)$ is a function on $\mathbb{Z}_2 \times \mathbb{Z}_2$. If we write σ as a matrix $\sigma^{i_1 i_2}_{j_1 j_2}$ where the multi-indices are in order 11, 12, 21, 22, is

$$(6) \quad \sigma = \begin{pmatrix} -Q^{-1} & 0 & 0 & \frac{a(R_1 \alpha - 1)}{b} \\ 0 & \alpha - 1 & \beta & 0 \\ 0 & \alpha & \beta - 1 & 0 \\ \frac{b(R_2 \beta - 1)}{a} & 0 & 0 & Q \end{pmatrix}.$$

What we have coming out of the axioms is a field of such ‘generalised braiding’ matrices because the entries here are functions on the group. The eigenvalues are $-1, \alpha\beta, -Q^{-1}, Q$ as functions on the group. Notice that these ‘generalised braidings’ have a broadly 8-vertex form normally associated with quantum integrable systems but here arising out of nothing but the requirements of quantum Riemannian geometry applied to a square graph.

The Laplacian for the above QLC's are computed as

$$\Delta f = (\cdot, \cdot) \nabla (\partial_i f e_i) = -\frac{2}{a} \partial_1 f - \frac{2}{b} \partial_2 f + \partial_i f (\cdot, \cdot) \nabla e_i = \left(\frac{Q^{-1} - R_2 \beta}{a} \right) \partial_1 f - \left(\frac{Q + R_1 \alpha}{b} \right) \partial_2 f$$

using our formula for ∇ , the connection property, and $\partial_i^2 = -2\partial_i$. The curvatures are given by

$$\begin{aligned} R_{\nabla} e_1 &= \left(Q^{-1} R_1 \alpha - Q \alpha + (1 - \alpha)(R_1 \beta - 1) + \frac{R_2 a}{a} (R_2 \beta - 1)(R_2 R_1 \alpha - 1) \right) \text{Vol} \otimes e_1 \\ &\quad + \left(Q^{-1}(1 - \alpha) + \alpha(R_2 \alpha - 1) + Q^{-1} \frac{R_1 b}{a} (\beta^{-1} - 1) \right) + \frac{b}{a} (R_2 \beta - 1) R_2 \beta \text{Vol} \otimes e_2 \end{aligned}$$

where $\text{Vol} = e_1 \wedge e_2$, and a similar formula for $R_{\nabla} e_2$ interchanging e_1, e_2 ; R_1, R_2 ; α, β ; a, b and $Q, -Q^{-1}$ (so that Vol also changes sign). One can discern contributions from $q \neq 1$ and from a, b non-constant. The connection reality condition comes down to

$$(7) \quad |q| = 1$$

so that in particular the function $Q - Q^{-1}$ is pointwise imaginary.

Next we find the Ricci tensor defined by a lifting map i , for which in our case there is a canonical choice $i(\text{Vol}) = \frac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1)$. If we write $R_{\nabla} e_i = \rho_{ij} \text{Vol} \otimes e_j$ then

$$\text{Ricci} = ((\cdot, \cdot) \otimes \text{id})(\text{id} \otimes i \otimes \text{id})(\text{id} \otimes R_{\nabla})(g) = \frac{1}{2} \begin{pmatrix} -R_2 \rho_{21} & -R_2 \rho_{22} \\ R_1 \rho_{11} & R_1 \rho_{12} \end{pmatrix}$$

as the matrix of coefficients on the left in our tensor product basis. Applying (\cdot, \cdot) again, we have scalar curvature

$$S = \frac{1}{2} \left(-\frac{R_2 \rho_{21}}{a} + \frac{R_1 \rho_{12}}{b} \right)$$

which is invariant under the interchange above. For the simplest case where $q \neq 1$ and a, b are constant, the QLCs and their curvature reduce to

$$\nabla e_1 = (1 + Q^{-1})e_1 \otimes e_1, \quad \nabla e_2 = (1 - Q)e_2 \otimes e_2$$

$$R_{\nabla} e_1 = -(Q - Q^{-1})\text{Vol} \otimes e_1, \quad R_{\nabla} e_2 = (Q - Q^{-1})\text{Vol} \otimes e_2$$

as the intrinsic quantum Riemannian geometry of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with its rectangular metric. This has

$$\text{Ricci} = \frac{Q - Q^{-1}}{2} (e_1 \otimes e_2 + e_2 \otimes e_1), \quad S = 0$$

which we see is quantum symmetric but does not obey the same reality condition as the metric if we impose (7) needed for the connection to obey its 'reality' condition.

The general Ricci curvature is more complicated but for $q = 1$, say, it has values

$$(8) \quad \text{Ricci}_{q=1} = \frac{1}{2} \begin{pmatrix} \frac{1}{b} \left(-\frac{\partial_2 a}{\alpha} + \chi \frac{\partial_1 b}{\beta} \right) & -\frac{\partial_1 b}{b} \left(\alpha + \frac{1}{\alpha} - \chi - 2 \right) \\ -\frac{\partial_2 a}{a} \left(\beta + \frac{1}{\beta} - \chi - 2 \right) & \frac{1}{a} \left(-\frac{\partial_2 a}{\alpha} + \chi \frac{\partial_1 b}{\beta} \right) \end{pmatrix}$$

for the matrix of coefficients. This is in general neither quantum symmetric nor real in the sense of the metric. For the scalar curvature the general formula is

$$S = -\frac{1}{4ab} \left((3 + q + (1 - q)\chi) \frac{\partial_2 a}{\alpha} + (1 - q^{-1} - (3 + q^{-1})\chi) \frac{\partial_1 b}{\beta} \right).$$

Finally, it is not obvious what measure we should use to integrate either of these but if we take measure $\mu = |ab| = ab$ (we assume for now the a, b are positive edge lengths, i.e. the theory has Euclidean signature) and sum over $\mathbb{Z}_2 \times \mathbb{Z}_2$ then we have

$$(9) \quad \int S = \sum_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mu S = (a_{00} - a_{01})^2 \left(\frac{1}{a_{00}} + \frac{1}{a_{01}} \right) + (b_{00} - b_{10})^2 \left(\frac{1}{b_{00}} + \frac{1}{b_{10}} \right).$$

independently of q . We consider this action as some kind of energy of the metric configuration. If we took other measures such as $\mu = 1$ or $\mu = \sqrt{|g|} = \sqrt{|ab|}$ then we would not have invariance under q so the action would not depend only on the metric but on the choice of ∇ .

Next we Fourier transform on $\mathbb{Z}_2 \times \mathbb{Z}_2$ to write our results in ‘momentum space’. We have

$$1, \quad \phi(i, j) = (-1)^i = (1, 1, -1, -1), \quad \psi(i, j) = (-1)^j = (1, -1, 1, -1), \quad \phi\psi = \chi$$

$$\partial_1\phi = -2\phi, \quad \partial_2\phi = 0, \quad \partial_1\psi = 0, \quad \partial_2\psi = -2\psi$$

as the plane waves and given the conditions we imposed on a, b , we can expand these in terms of four real momentum space coefficients as

$$a = k_0 + k_1\psi, \quad b = l_0 + l_1\phi.$$

Then some computation gives the Scalar curvature for $q = 1$ as

$$S = \frac{2}{(k_0^2 - k_1^2)(l_0^2 - l_1^2)} \left((l_0 - l_1)(k_1(k_0 + k_1) - l_1(k_0 - k_1)), (k_0 + k_1)(l_1(l_0 + l_1) - k_1(l_0 - l_1)), \right. \\ \left. (k_0 - k_1)(k_1(l_0 + l_1) - l_1(l_0 - l_1)), (l_0 + l_1)(l_1(k_0 + k_1) - k_1(k_0 - k_1)) \right).$$

With measure $\mu = ab$ as above, this gives

$$\int S = 8 \left(\frac{k_0 k_1^2}{k_0^2 - k_1^2} + \frac{l_0 l_1^2}{l_0^2 - l_1^2} \right).$$

To analyse this we define $k = k_1/k_0$ with $|k| < 1$ corresponding to $a > 0$ at all points and similarly for $l = l_1/l_0$ and fix $k_0, l_0 > 0$ as the average values of a, b so that we can focus on fluctuations about these as controlled by k, l . In this case the action becomes

$$(10) \quad \int S = 8 \left(\frac{k_0 k^2}{1 - k^2} + \frac{l_0 l^2}{1 - l^2} \right) = 8k_0(k^2 + k^4 + k^6 \dots) + 8l_0(l^2 + l^4 + l^6 + \dots).$$

This has a ‘bathtub’ shape with coupling constants k_0, l_0 and a minimum at $k = l = 0$, which makes sense as a measure of the energy of the gravitational field. The k, l are not momentum variables but the relative amplitude of the unique allowed non-zero momentum in each direction.

In the Minkowski version, we require say $a < 0, b > 0$ everywhere. We suppose $k_0 < 0, l_0 > 0$ as the average values and require $|k_1| < -k_0, |l_1| < l_0$ to maintain the sign. We define k, l as before for the relative fluctuations and regard $\tilde{k}_0 = -k_0, \tilde{l}_0$ as coupling constants. Now $\mu = |ab| = -ab$ for our measure, giving

$$\int S = 8 \left(\frac{\tilde{k}_0 k^2}{1 - k^2} - \frac{l_0 l^2}{1 - l^2} \right) = 8\tilde{k}_0(k^2 + k^4 + k^6 \dots) - 8l_0(l^2 + l^4 + l^6 + \dots).$$

In either case, if we ignore higher order terms then we have S quadratic in k, l as for a massless free field in a universe with only one momentum in each direction.

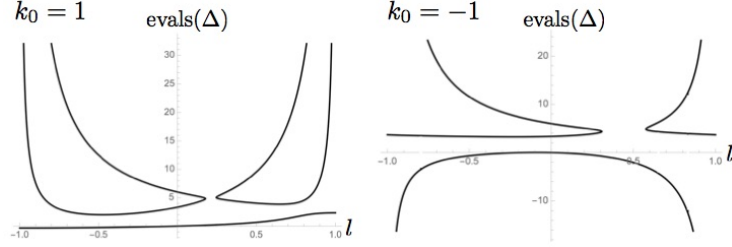


FIGURE 5. Nonzero eigenvalues of the Laplacian vs l at fixed $k = 0.5$, $q = l_0 = 1$ and $k_0 = \pm 1$ for the two signatures.

The higher terms correspond to quartic and higher derivatives in the action from this point of view.

Finally, we can add matter using the Laplacian above. However, this Laplacian does depend on q . For example, one can check in the momentum parametrization that

$$\Delta_{k_0, l_0, q; k, l} \sim \Delta_{l_0, k_0, -q; l, -k}$$

in the sense of the same eigenvalues. In other words, the theory with a, b swapped is the same but has the negated value of q . These eigenvalues are mostly real when q is real, leading to $q = \pm 1$ as the natural choices. We plot the three nonzero eigenvalues in Figure 5 for $q = 1$ and the two signatures, at a typical value $k = 0.5$. The cross-section passes a narrow region in the k, l plane where two of the eigenvalues become complex but otherwise they are positive. The remaining mainly small eigenvalue is zero at $l = 0$ and $q = 1$ or $k = 0$ and $q = -1$ among possibly other zeros.

In principle, one can proceed to consider ‘functional integrals’ over any of our parametrizations of the metric field. Thus for gravity we can consider integrals of the form

$$\int_{-1}^1 \int_{-1}^1 dk dl e^{\int S}$$

(this converges when we use the ι in the action, otherwise we have to renormalise due to divergence at the endpoints), and similar integrations against functions of k, l to extract expectation values of operators. If we add matter to the action via the Laplacian then we will have a q -dependence as discussed. We should also in the full theory integrate over the k_0, l_0 rather than treating them as constants as we have above.

5. CONCLUSIONS

Section 1 was philosophical in nature as a brief introduction to a principle of ‘representation-theoretic self-duality’ as an ‘axiom of physics’[17, 22, 24] that has motivated many of my works. We saw how this at an abstract level was one route to the discovery of the ‘centre’ of a monoidal category, while as ‘quantum Born reciprocity’ it led to the discovery of an early class of quantum groups. We also explained how the big picture leads one to the *quantum spacetime hypothesis*.

Sections 2 and 3 was a brief outline of a formulation of such quantum spacetimes with curvature, using a bimodule approach developed mostly with Edwin Beggs[2, 3, 5, 4]. Section 4 then proceeded with a new application to a discrete model, namely quantum Riemannian geometry on a square. Unlike lattice approximations used in physics, we do not consider the model as an approximation but rather as an exact finite geometry[23]. We found a 1-parameter family of quantum Levi-Civita connections for every bidirectional metric and an Einstein-Hilbert action as a measure of the energy in the gravitational field and independent of the connection parameter.

We also found that the ‘generalised braiding’ σ [10, 29] emerging in our case from nothing other than the axioms of quantum Riemannian geometry applied to a square graph has a strong resemblance to the 8-vertex solutions[1] of the Yang-Baxter equations in the theory of quantum integrable systems. Our σ does not obey the braid relations other than the trivial case (this is usually tied to flatness of the connection[23]) but has a similar flavour.

Note that while I have focussed on my own path, reflected also in the bibliography, there are by now many other works on quantum spacetime which I have not had room to cover.

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