Open problems in mathematical physics

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Abstract

We present a list of open questions in mathematical physics. After a historical introduction, a number of problems in a variety of different fields are discussed, with the intention of giving an overall impression of the current status of mathematical physics, particularly in the topical fields of classical general relativity, cosmology and the quantum realm. This list is motivated by the recent article proposing 42 fundamental questions (in physics) which must be answered on the road to full enlightenment [1]. But paraphrasing a famous quote by the British football manager Bill Shankly, in response to the question of whether mathematics can answer the Ultimate Question of Life, the Universe, and Everything, mathematics is, of course, much more important than that.

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1 Mathematical Problems

There are essentially two branches of mathematics, which in the broadest sense can be referred to as pure mathematics and applied mathematics (but there are actually three types of mathematicians; those that can count and those that cannot!). The actual mathematics (the problems, techniques and rigour) used in both are exactly the same, but perhaps pure mathematicians and applied mathematicians are motivated differently. Pure mathematics is concerned with mathematics for its own sake, and an important criterion for assessing a worthy problem is whether it leads to new developments in mathematics [inwardly directed]. Applied mathematics is also (and perhaps primarily) concerned with establishing facts of real world interest [outwardly directed]. For a more philosophical discussion on the nature of mathematics see, for example, the preface to [2] and references within.

Noted probems in mathematics have always been important and are part of the mathematical culture, both as recreation and as tests of acumen. Unlike physics, where problems are dictated by necessity and practicalities, problems in mathematics, particularly on the more pure side, have a life of their own and the opinions of central characters have always been very important and played an elevated and pivotal role. Hence the importance attached to problems espoused by famous mathematicians.

In the sixteenth century, and according to the custom of the time, mathematical challenges, a type of intellectual duel and a way of showing ones mathematical chops and gaining respect, were often made. In 1530, there was a famous contest between Niccol Tartaglia and Antonio Fiore (a student of Scipione del Ferro) on solving cubic equations. Each contestant proposed a number of problems for his rival to solve, and whoever solved the most problems would receive all of the money put up by the two contestants. Since Tartaglia had worked out a general method for solving cubic equations, he won the contest. Later, Tartaglia revealed his secret method to Gerolamo Cardano (which later led to questions of priority between Ferro and Tartaglia) when Cardano published a book on cubic equations. This subsequently led to a challenge by Tartaglia, which was eventually accepted by Cardano's student Lodovico Ferrari. Ferrari beat Tartaglia in the challenge, and Tartaglia lost both his prestige and income [3].

In 1696 John Bernoulli published a challenging problem: To find the curve connecting two points, at different heights and not on the same vertical line, along which a body acted upon only by gravity will fall in the shortest time (the curve which solves this problem is called the 'brachistochrone'). Gottfried Wilhelm Leibniz and Bernoulli were confident that only a person who knew calculus could solve this problem (and it was rumoured that this problem was set, in part, to determine what Isaac Newton knew on this topic since he had not published his results yet). Within one day of receiving the challange, Newton sent in his solution. When Bernoulli announced the winners of his contest, he named Leibniz and l'Hopital (Leibniz's student) and one anonymous winner. Bernoulli recognized the anonymous winner in public with the phrase: "we know the lion by his claw".

The twenty-three problems published by the mathematician David Hilbert in 1900 [4] are probably the most famous problems in mathematics. All of the problems were unsolved at the time of publication. Several of them have been very influential in the development of mathematics. Mathematicians and mathematical organizations have since announced several lists of problems, but these have not had the same influence as Hilbert's original problems. At the end of the millennium, which was also the centennial of Hilbert's publication of his problems, several mathematicians accepted the challenge to formulate "a new set of Hilbert problems". Most notable are Steven Smale's eighteen problems, but to date these have not garnered very much popular attention. Perhaps the twentyfirst century analogue of Hilbert's problems is the list of seven Millennium Prize Problems chosen in the year 2000 by the Clay Mathematics Institute.

1.1 Hilbert's problems

Hilbert's twenty-three problems in mathematics were published by David Hilbert in 1900 [4], and ranged over a number of topics in contemporary mathematics of the time. Some of these problems were stated precisely enough to enable a clear answer, while for others a solution to an accepted interpretation might have been possible but closely related unsolved problems exist. And some of Hilbert's problems were not formulated precisely enough in themselves, but were suggestive for more modern problems. At the time of publication the problems were all unsolved. Several of them were very influential for twentieth century mathematics; for example, the 11th and the 16th problems (H11 and H16 – see the text below and the Appendix where *all* of the problems referred to are stated) have given rise to the flourishing mathematical subdisciplines of quadratic forms and real algebraic curves. A number of problems have given rise to solutions that have garnered great acclaim including, for example, H1 and H10. And many aspects of these problems are still of great interest today.

There are two problems that are not only unresolved but may, in fact, not be resolvable by modern standards. For example, H6 concerns the axiomatization of physics and H4 concerns the foundations of geometry. H4 is generally thought to be too vague to enable a definitive answer, and there is no clear mathematical consensus on the possible relevence of Godel's second incompleteness theorem (which gives a precise sense in which such a finitistic proof of the consistency of arithmetic is unprovable). In addition, Hilbert originally included a "24th problem" (in proof theory, on a criterion for simplicity and general methods), but H24 was withdrawn from the list since it was regarded as being too vague to ever be described as solved.

Noteworthy for its appearance on the list of Hilbert problems, and Smale's list and the list of Millennium Prize Problems, is the Riemann hypothesis (H8), which asserts that all nontrivial zeros of the analytical continuation of the Riemann zeta function have a real part of 1/2. A proof or disproof of this would have far-reaching implications in number theory. H8 is still considered to be an important open problem, and has led to other important prime number problems, including Goldbach's conjecture and the twin prime conjecture, both of which remain unsolved. However, even this famous hypothesis in pure mathematics is related to the energy eigenvalues of distributions of random matrices, which is important in nuclear physics and quantum chaos [5].

1.1.1 Summary and status of Hilbert's problems

Of the clearly formulated Hilbert problems, problems H3, H7, H10, H11, H13, H14, H17, H19, H20 and H21 have a resolution that is generally accepted by consensus. On the other hand, problems H1, H2, H5, H9, H15, H18 and H22 have solutions that have been partially accepted, although there is some controversy as to whether the problems have been adequately resolved.

That leaves H8 (the Riemann hypothesis), H12 and H16 as unresolved. H6 might be considered as a problem in physics rather than in mathematics. And

H4 and H23 are too vague to ever be described as solved. The 4 unsolved problems are [4]:

- H6 Mathematical treatment of the axioms of physics.
- H8 The Riemann hypothesis.
- H12 Extend the Kronecker-Weber theorem on abelian extensions of the rational numbers to any base number field.
- H16 Describe relative positions of ovals originating from a real algebraic curve and as limit cycles of a polynomial vector field on the plane.

The other Hilbert problems are listed in the Appendix. The majority of these problems are in pure mathematics; only H19-H23 are of direct interest to physicists. The Riemann hypothesis (H8), and H12 and H16 are problems in pure mathematics in the areas of number theory and algebra (and H16 is unresolved even for algebraic curves of degree 8).

H6 concerns the axiomatization of physics. In particular, Hilbert proposed the following two specific problems: (i) the axiomatic treatment of probability with limit theorems for the foundation of statistical physics and (ii) the rigorous theory of limiting processes "which lead from the atomistic view to the laws of motion of continua." Kolmogorov's axiomatics [6] is now accepted as standard and there has been some success regarding (ii) [7]. This is indeed a problem within mathematical physics, although it is perhaps not necessarily regarded as being of prime importance in contemporary physics.

1.2 Smale's problems

Steven Smale proposed a list of eighteen unsolved problems in mathematics in 1998 [8], inspired by Hilbert's original list of problems and at the behest of Vladimir Arnold. Smale's problems S1 and S13 are Hilbert's eighth (Riemann hypothesis) and sixteenth (H8 and H16) problems, respectively, which remain unsolved.

The Poincare conjecture (S2), which asserts that in three dimensions a sphere is characterized by the fact that it is the only closed and simply-connected surface, was proved by Grigori Perelman in 2003 using Ricci flows [9]. This problem is central to the more general problem of classifying all 3-manifolds, and has many applications in modern theoretical physics.

There are 9 remaining unsolved problems:

- S3 Does P = NP?
- S4 Shub-Smale conjecture on the integer zeros of a polynomial of one variable.
- S5 Height bounds for Diophantine curves
- S8 Extend the mathematical model of general equilibrium theory to include price adjustments.
- S9 The linear programming problem: find a strongly-polynomial time algorithm which decides whether, for given a matrix A (in $\mathbb{R}^{m \times n}$) and b (in \mathbb{R}^{m}), there exists an x (in \mathbb{R}^{n}) with $Ax \geq b$.
- S10 Pugh's closing lemma (higher order of smoothness)
- S15 Do the Navier-Stokes equations in \mathbb{R}^3 always have a unique smooth solution that extends for all time?
- S16 Jacobian conjecture
- S18 Limits of intelligence.

The famous problem Does P = NP? (S3) is whether or not, for all problems for which an algorithm can verify a given solution in polynomial time (termed a non-deterministic polynomial time or NP problem), an algorithm can also find that solution quickly (a polynomial time or P problem); that is, whether all problems in NP are also in P. This is generally considered to be one of the most important open questions in mathematics and theoretical computer science and it has far-reaching consequences to other problems in mathematics, and in biology, philosophy and cryptography. A common example of a P versus NP problem is the so-called travelling salesman problem (which asks the following question: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?) It is an NP-hard problem in combinatorial optimization, important in operations research and theoretical computer science. Most mathematicians and computer scientists expect that the answer is that it is not true (i.e., $P \neq NP$). This problem also appears in the Millennium Prize list.

The problem S8 is in financial mathematics, which might be regarded as within the purview of theoretical physics. Gjerstad [10] has extended the deterministic model of price adjustment to a stochastic model and shown that when the stochastic model is linearized around the equilibrium the result is the autoregressive price adjustment model used in applied econometrics. In tests it was found that the model performs well with price adjustment data from a general equilibrium experiment with two commodities.

Problems S4, S5, S9, S10 and S16 are problems in pure mathematics. Smale also listed three additional problems in pure mathematics: the Mean value problem, the question of whether the three-sphere is a minimal set, and whether an Anosov diffeomorphism of a compact manifold topologically is the same as the Lie group model of John Franks? The solved problems are listed in the Appendix. Unlike the Hilbert problems, many of these problems have practical applications and are of relevence in physics. For example, an alternative formulation of S7 is the Thompson Problem of the distribution of equal point charges on a unit sphere governed by the electrostatic Coulomb law. Problem S18 is concerned with the fundamental problems of intelligence and learning, both from the human and machine side.

The Navier-Stokes equations describe the motion of fluids. The problem is essentially to make progress towards a well-defined mathematical theory that will give insight into these equations. Therefore S15 is truely a problem in mathematical physics and has imported applications in many branches of theoretical physics including engineering and oceanography, and even astrophysics.

Solutions of the compressible Euler equations typically develop singularities (that is, discontinuities of the basic fluid variables), in a finite time [11]. The proofs of the development of singularities are often by contradiction and consequently do not give detailed information on what occurs when the smooth solutions break down. The formation of shock waves are possible, and it is known that in some cases solutions can be physically extended beyond the time of shock formation. The extended solutions only satisfy the equations in the "weak sense". For the classical Euler equations there is a well-known theorem on the global existence of weak solutions in one (space) dimension [12], and a one dimensional class of weak solutions has recently been found in which both existence and uniqueness hold [13]. In higher (space) dimensions there are no general global existence theorems known. The question of which quantities must blow up when a singularity forms in higher dimensions has been partially addressed for classical hydrodynamics [14]. A smooth solution of the classical Euler equations has been proven to exist for all time when the initial data are small and the fluid is initially flowing outwards uniformly [15].

1.3 Millennium Prize problems

The Millennium Prize Problems are seven problems in mathematics that were proposed by the Clay Mathematics Institute in 2000 [16], with a \$1 million US prize being awarded by the Institute to the discoverer(s) of a correct solution to any of the problems. At present, the only Millennium Prize problem to have been solved is the Poincare conjecture [9]. In addition to the Poincare conjecture, three other problems, namely the Riemann hypothesis (H8), P versus NP (S3), and the existence and smoothness of the Navier-Stokes equations (S15), are also on Smale's list.

There are 3 remaining unsolved problems [16]:

- M1 The Hodge conjecture that for projective algebraic varieties, Hodge cycles are rational linear combinations of algebraic cycles.
- M2 Yang-Mills existence and mass gap.
- M3 The Birch and Swinnerton-Dyer conjecture.

Problem M2 aims to establish the existence of the quantum Yang-Mills theory and a mass gap rigorously, and is truely a problem in mathematical physics. Classical Yang-Mills theory [17] is a generalization (or analogue) of Maxwell's theory of electromagnetism in which the chromo-electromagnetic field itself carries charges. As a classical field theory, it's solutions propagate at the speed of light and so its quantum version describes massless gluons. The so-called mass gap is the problem that color confinement only allows bound states of gluons, which form massive particles. The asymptotic freedom of confinement also makes it possible that a quantum Yang-Mills theory exists without restriction to low energy scales.

Many important mathematical questions remain unsolved, including stability theorems and the proof of existence of Yang-Mills fields by methods of partial differential equations. More contemporary questions are to obtain solutions of the Yang-Mills equations on a Riemannian (or Lorentz) manifold. The Yang-Mills equations in general relativity will be discussed later.

The Birch and Swinnerton-Dyer conjecture M3 asserts that that there is a simple way to tell whether the equations defining elliptic curves have a finite or infinite number of rational solutions. This is a special case of Hilbert's tenth problem, in which it has been proven that there is no way to decide whether a given equation in the more general case even has any solutions.

2 Mathematical Physics

Not all mathematical problems are necessarily of interest to a physicist. Similarly, not all problems in physics are of a mathematical character. For example, there are many lists of problems in physics, including problems in high-energy physics/particle physics, astronomy and astrophysics, nuclear physics, atomic, molecular and optical physics, condensed matter physics, and biophysics [20, 21]. But these cannot all be regarded as problems within mathematics. Most problems of a mathematical nature are restricted to fundamental physics and particularly theoretical physics (and especially in theories such as general relativity (GR) and quantum gravity (QG)). It is perhaps illuminating to recall the quote by Werner von Braun who said that "Basic research is what I am doing when I don't know what I am doing".

Five of the most important and interestingly unsolved problems in theoretical physics in the quantum regime (in the small) and in cosmology (in the large) are commonly agreed to be the following (see, for example, [22]):

- Ph1 The Problem of Quantum Gravity.
- Ph2 The Foundational Problems of Quantum Mechanics.
- Ph3 The Unification of Particles and Forces.
- Ph4 The Tuning Problem.
- Ph5 The Problem of Cosmological Mysteries.

We shall be interested in problems which we shall refer to as problems in mathematical physics, which we shall define to mean problems that are wellformulated (i.e., well-posed) mathematical problems, which are of interest to physicists. Many such problems involve systems of partial differential equations, which are of central importance in theoretical physics.

In general, problems in mathematical physics will not include problems where the basic underlying physics is not understood (such as, for example the quantization of gravity), and although it is clear that their solution will inevitably involve a lot of mathematics (and perhaps even lead to new areas of mathematics), an explicit well-posed mathematical problem cannot be formulated. Nor do they include problems in pure mathematics where there is no clear physical application (e.g., the Riemann hypothesis). There are also questions in computational mathematics, and it is also debatable whether such problems qualify as problems in mathematical physics. The meaning of problems in mathematical physics is nicely illustrated by the set of 15 open problems proposed by mathematical physicist Barry Simon [18], which we shall discuss a little later. This paper is motivated, in part, by the recent article entitled *Life, the Universe, and everything: 42 fundamental questions* (referred to hereafter as **AL42** [1]: the actual list of questions is given in the appendix), which itself was inspired by *The Hitchhiker's Guide to the Galaxy*, by Douglas Adams. There are many questions in theoretical physics discussed in **AL42**, some of which are of relevance here and will be discussed in more detail later, including the cosmological constant problem (AL2.1), the dark energy problem (AL2.2), the regularization of quantum gravity (AL2.3), black hole entropy and thermodynamics (AL2.4), black hole information processing (AL2.5), supersymmetry and the hierarchy problems (AL3.3), and higher dimensions and the geometry and topology of internal space (AL5.1).

In this paper, I shall present a number of what I consider to be problems in mathematical physics, primarily in the current areas of theoretical and fundamental physics. Classical GR remains healthy and vigorous, in part, due to a frequent injection of fertile mathematical ideas (such as those of Hawking and Penrose and, more recently, of Schoen-Yau and Witten). By any reasonable definition of the term, it is clear that much of classical GR is "mathematical physics". GR problems have typically been under-represented in lists of problems in mathematical physics (e.g., see [18]), perhaps due to their advanced technical nature. Obviously any such list is subjective, and classical GR may well be over-represented here, but I feel at liberty to comprehensively discuss problems in GR (artistic licence?) and to present some of my own personal favorites (\mathbf{PF}) (perhaps to justify my own research interests?).

After the current more introductory and historical section, I shall discuss in more detail classical GR first, and then return to quantum theory and cosmology (and specifically discuss the 5 physics problems above) in the ensuing sections. It is the technical problems that are of interest to mathematicians. Often physicists are perhaps not as interested in the technical aspects of the problem, but more in the context and the consequences of the results. Hence, although I shall attempt to state the problems relatively rigorously, as is appropriate for mathematicians, I shall endeavor to select less technical questions, or at least describe them in as heuristic manner as possible, which may well be of more interest to physicists.

This article is written primarily for a readership with some background in mathematics and physics. However, regardless of background, the intention here is not for readers to understand each and every problem, but rather to get an overall impression of the open questions in the various fields. In particular, one aim is to outline which areas are currently exciting with unsolved problems whose potential solution might have a huge impact on the field, and consequently motivate readers (and especially young physicists) to possibly get more involved in research. Obviously this article takes it for granted that mathematics is necessarily the language of physics (that is, the so-called *unreasonable effectiveness* of mathematics in the natural sciences [19]); however, the philosophical reasons for this it is beyond the current discussion.

2.1 More on lists

There are many lists of unsolved problems in mathematics (see, for example, [23] [24]). These include many problems in applied mathematics (and hence mathematical physics), some of which have been discussed above (the regularity of the Navier-Stokes and Yang-Mills equations, and problems on turbulence). In particular, there are questions on stability (e.g., for what classes of ordinary differential equations, describing dynamical systems, does the Lyapunov second method formulated in the classical and canonically generalized forms define the necessary and sufficient conditions for the asymptotic stability of motion?), questions in ergodic theory (e.g., the Furstenberg conjecture), on actions in higher-rank groups (e.g., the Margulis conjecture), the question of whether the Mandelbrot set is locally connected, and problems in Hamiltonian flows (e.g., the Weinstein conjecture: Does a regular compact contact type level set of a Hamiltonian on a symplectic manifold carry at least one periodic orbit of the Hamiltonian flow?)

In particular, very recently the DARPA Mathematical Challenges were proposed [25], which are very heavy in applied mathematics and theoretical physics. They involve not only problems in classical fluid dynamics and the Navier-Stokes Equation (and their use in the quantitative understanding of shock waves, turbulence, and solitons), but also problems in which new methods are needed to tackle complex fluids (such as foams, suspensions, gels, and liquid crystals), and the Langlands program (see below). In addition, a number of DARPA challenges involve traditional problems in pure mathematics, such as the Riemann hypothesis (number theory), the Hodge conjecture (in algebraic geometry), and in convex optimization (e.g., whether linear algebra be replaced by algebraic geometry in a systematic way). They also include the physical consequences of Perelman's proof of Thurston's geometrization theorem and the implications for spacetime and cosmology of the Poincare conjecture in four dimensions.

Also a number of more speculative problems were proposed in an attempt to apply mathematics to new areas of interest, including the mathematics of the brain, the dynamics of networks, stochasticity in nature, problems in theoretical biology and biological quantum field theory (e.g., what are the fundamental laws of biology, can Shannon's information theory be applied to virus evolution, the geometry of genome space, what are the symmetries and action principles for biology) and the mathematics of quantum computing (algorithms and entanglement) including optimal nanostructures, and problems in theoretical computation in many dimensions. One of the most important advances in the last few years has been the use of theoretical computing and neural networks to attempt to solve all kinds of previously untractable problems.

There are also a number of interesting questions, some of which are discussed in **AL42**, which might be considered to be more metaphysics than physics, and certainly outside the realm of mathematical physics (although they may be addressed by scientists, and indeed mathematicians, in the future). These include the study of the multiverse and the anthropic principle, and emergent phenomena such as life and consciousness (the puzzle of the possible role of human consciousness in resolving questions in quantum physics is discussed by [26]). For example, in Section 7 of **AL42** the ultimate nature of reality, the reality of human experience, conscious minds and questions on the origin of complex life are broached. To this list, questions of ethics and even religion might be added. The potential for breakthroughs in theoretical, computational, experimental, and observational techniques are also discussed in **AL42**. Although such topics are outside the purview of the current article, that is not to say that mathematics might not be useful in their consideration.

2.2 Mathematical physicists

In mathematics, the Langlands program [27] constitutes a number of conjectures that relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields. DARPA proposed two challenges: (geometric Langlands and quantum physics) how does the Langlands program explain the fundamental symmetries of physics (and vice versa), and (arithmetic Langlands, topology, and geometry) what role does homotopy theory play in the classical, geometric, and quantum Langlands programs.

It has been thought for a long time that the Langlands duality ought to be related to various dualities observed in quantum field theory and string theory. The so-called Langlands dual group [27], which is essential in the formulation of the Langlands correspondence, plays an important role in the study of Sdualities in physics and was introduced by physicists in the framework of fourdimensional gauge theory [28]. Witten recently showed that Langlands duality is closely related to the S-duality of quantum field theory, which opens up exciting possibilities for both subjects [29]. Indeed, the connections between the Langlands program and two-dimensional conformal field theory give important insights into the physical implications of the Langlands duality.

Edward Witten is a theoretical physicist working in string theory, quantum gravity, supersymmetric quantum field theories, and other areas of mathematical physics. In addition to his contributions to physics, Witten's work has also significantly impacted pure mathematics. In 1990 he became the first (and so far only) physicist to be awarded a Fields Medal by the International Mathematical Union. The Fields Medal is regarded as the highest honour a mathematician can receive and, together with the Abel Prize, has often been viewed as the "Nobel Prize" for mathematics.

In a written address to the International Mathematical Union, Michael Atiyah said of Witten [30]: "Although he is definitely a physicist his command of mathematics is rivaled by few mathematicians, and his ability to interpret physical ideas in mathematical form is quite unique". As an example of Witten's work in pure mathematics, Atiyah cited his application of techniques from quantum field theory to the mathematical subject of low-dimensional topology. In particular, Witten realized that Chern-Simons theory in physics could provide a framework for understanding the mathematical theory of knots and 3-manifolds [31]. Witten was also awarded the Fields Medal, in part, for his proof in 1981 of the positive energy theorem in general relativity [32].

There are also many mathematicians who have greatly influenced physics. These include Roger Penrose and Steven Hawking (whose contributions will be discussed later). Michael Atiyah is a mathematician specializing in geometry, and was awarded the Fields Medal in 1966. He helped to lay the foundations for topological K-theory, an important tool in algebraic topology. The Atiyah–Singer index theorem [33] (in which the index is computed by topological means) is widely used in counting the number of independent solutions to differential equations. The index theorem provides a link between geometry and topology and has many applications in theoretical physics. Some of his more recent theo-

retical physics inspired work, and particularly that on instantons and monopoles, is responsible for some subtle corrections in quantum field theory.

Simon Donaldson, one of Atiyah's students, is known for his work on the topology of smooth (differentiable) four-dimensional manifolds and the Donaldson (instanton) invariant (among other things). Donaldson's work is on the application of mathematical analysis (and especially that of elliptic partial differential equations) to problems in the geometry of 4-manifolds, complex differential geometry and symplectic geometry [34]. He has used ideas from physics to solve mathematical problems, and investigated problems in mathematics which have physical applications (e.g., an application of gauge theory to four-dimensional topology [35]). Recently, Donaldson's work has included a problem in complex differential geometry regarding a conjectured relationship between the stability conditions for smooth projective varieties and the existence of Kahler-Einstein metrics with constant scalar curvature [36], which is of interest in string theory. String theory is often described as a topic within mathematics rather than in physics (in much the same way GR was fifty years ago).

The mathematician Shing-Tung Yau was awarded the Fields Medal in 1982. Yau's work is mainly in differential geometry, especially in geometric analysis. He has been active and very influential at the interface between geometry and theoretical physics (see later). Together with Schoen, Yau used variational methods to prove the positive energy theorem in GR, which asserts that (under appropriate assumptions) the total energy of a gravitating system is always positive and can vanish only when the geometry is that of flat Minkowski spacetime. It consequently establishes Minkowski space as a stable ground state of the gravitational field. As mentioned above, Witten's later simpler (re)proof [32] used ideas from supergravity theory. Yau also proved the Calabi conjecture which allows physicists to demonstrate, utilizing Calabi-Yau compactification, that string theory is a viable candidate for a unified theory of nature. Calabi-Yau manifolds are currently one of the standard tools for string theorists.

2.3 Simon's problems

Problems in mathematical physics are well formulated mathematical questions of interest to physicists. The meaning of problems in mathematical physics is nicely illustrated by the set of 15 open problems proposed by mathematical physicist Barry Simon in 1984 [18], who was awarded the American Mathematical Society's Steele Prize for Lifetime achievements in mathematics in 2016.

I shall display and briefly discuss six of these problems below. The first two questions are in fluid dynamics and have been alluded to earlier. The sixth, cosmic censorship, will be discussed later. The remaining problems are displayed in the Appendix (the citations therein are circa 1984 [18], and there has subsequently been progress on these problems). Although many of these problems involve Schrodinger operators, Simon's own field of expertise, I believe that the problems do help give a flavour of what problems constitute mathematical physics to a general physicist (for example, one who is not necessarily an expert in GR, one of the fields to be discussed below).

- BS1 Existence for Newtonian Gravitating Particles. A: Prove that the set of initial conditions which fails to have global solutions is of measure zero (some mathematicians believe that there may be an open set of initial conditions leading to non-global solutions). B: Existence of non-collisional singularities in the Newtonian N-body problem.
- BS3 Develop a comprehensive theory of the long time behavior of dynamical systems including a theory of the onset of, and of fully developed, turbulence.
- BS2 Open Questions in Ergodic Theory. Particular problems include A: Ergodicity of gases with soft cores, B: Approach to equilibrium, and C: Asymptotic Abelianness for the quantum Heisenberg dynamics.
- BS8 Formulation of the Renormalization Group and Proof of Universality. A: Develop a mathematically precise version of the renormalization transformations for ν-dimensional Ising-type systems. B: In particular, show that the critical exponents in the three dimensional Ising models with nearest neighbor coupling but different bond strengths in the three directions are independent of the ratios of these bond strengths.
- BS14 Quantum field theory remains a basic element of fundamental physics and a continual source of inspiration to mathematicians. A: give a precise mathematical construction of quantum chromodynamics, the model of strong interaction physics. B: Construct any non-trivial renormalizable but not super-renormalizable quantum field theory. C: Prove that quantum electrodynamics is not a consistent theory. D: Prove that a non-trivial lattice cutoff theories theory does not exist.
- BS15 Cosmic censorship.

Problem BS3 is very general and rather vague, and so the first problem is to formulate the really significant questions. For recent reviews of some of the more spectacular developments see [37, 38]. There has been considerable progress in understanding the onset of turbulence (e.g., see [39]), but fully developed turbulence is far from being comprehensively understood. Even the connection between turbulence and the Navier-Stokes equation is not absolutely clear [40].

Regarding BS2, the developers of statistical mechanics and thermodynamics, including Boltzmann and Gibbs, realized that from a microscopic point of view bulk systems rapidly approach equilibrium states parametrized by a few macroscopic parameters. It was originally believed that it could be proven that the classical dynamics on the constant energy manifolds of phase space is ergodic. However, the Kolmogorov–Arnold–Moser (KAM) theorem [41] is a result in dynamical systems about the persistence of quasiperiodic motions under small perturbations. An important consequence of KAM is that many classical systems will not be ergodic: there will be an invariant subset of phase space consisting of a union of invariant tori of positive total measure.

Problem BS14 concerns the question of whether quantum field theory really is a mathematical theory at all. This question remains open for any nonlinear quantum field theory in three-space plus one-time dimensions. The basic difficulty in formulating the mathematical problem is the singular nature of the nonlinear equations proposed. Physicists eventually developed sets of ad hoc rules to cancel the infinities in QFT and to calculate observable effects. These rules of renormalization were remarkably accurate in producing verifiable numbers in electrodynamics.

Fisher, Kadanoff and Wilson [42] developed the "renormalization group theory" of critical phenomena which, regarding question BS8.B above, is often claimed to "explain" universality (rather than universality being assumed). The basic idea of shifting scales as one approaches a critical point via a nonlinear map of Hamiltonians and obtaining information from the fixed points of that map has been applied in a variety of situations [37]. In some studies, the nonlinear maps are on well defined spaces and there has been considerable progress on a rigorous mathematical analysis on the Feigenbaum theory [43]. The original Wilson theory is on functions of infinitely many variables and it is far from clear how to formulate the maps in a mathematically precise way (let alone then analyze their fixed point structure); indeed, there are various no-go theorems [44] on how one might try to make a precise formulation in lattice systems.

2.4 Yau, Penrose and Bartnik

Analytical methods (and especially the theory of partial differential equations) used in the study of problems in differential geometry, and subjects related to geometry such as topology and physics, were surveyed in [45]. There was a section in [45] with 120 open questions by Yau himself (p669). Most of these problems are technical and in differential geometry (and mostly Riemannian geometry), and are old and well known (even in 1982; see original references therein). Many of the problems are not related to physics directly, and hence are not necessarily problems in mathematical physics. But some of the problems concern the Dirac equation, gravitational instantons, Kahler and Calabi manifolds and Gauss-Bonnet theory. There were 2 problems in Yang-Mills theory (Problems Y117 and Y118), and 5 problems in GR: problems Y115, Y116 & Y119 concern the topology of a geodesically complete Lorentzian 4-manifold of nonnegative Ricci curvature which contains an absolutely maximizing timelike geodesic (see later), the topology of a static stellar model, and the characterization of asymptotical flatness of a manifold in terms of a suitable decay rate of the curvature, respectively. The problems Y114 (cosmic censorship) and Y120 (the definition of total angular momentum) are also included in the list of open problems by Penrose (RP12 & RP10, respectively) in the same book [45].

The fourteen unsolved problems in classical GR presented by Roger Penrose (p631 in [45]; problems RP1 – RP14 in the Appendix), represented the status of the subject circa 1982. (An earlier list of 62 problems in GR was given by Wheeler [46]). Many of them were technical questions concerning definitions of null infinity, appropriate (conformal) properties, and conservation laws and physical quantities, necessary for the formulation of the important problems and conjectures that followed. In particular, in 1982 it was known that spherically symmetrical collapse models lead to a black hole horizon, but if the initial data is perturbed away from spherical symmetry, a so-called naked singularity could arise (from which causal curves can extend to external future infinity). But the belief was that naked singularities will not arise "generically", whence it is said that cosmic censorship holds [47, 48].

Problem RP11, which is related to problem RP4, is necessary for the statement of the cosmic censorship problem RP12, which was stated somewhat vaguely; indeed, it is a problem in itself to find a satisfactory mathematical formulation of what is physically intended [48, 49] (such as, for example, are "generic" maximally extended Ricci-flat spacetimes globally hyperbolic or necessarily have a Cauchy surface [50]).

With a suitable assumption of cosmic censorship, together with some other reasonable physical assumptions, it is possible to derive a certain sequence of inequalities [51]. Problem RP13 concerns the Penrose inequality, which generalizes RP6 and is related to RP7. The validity of these inequalities are sometimes regarded as giving some credence to cosmic censorship.

There are many other problems involving black holes which have not yet been solved, including RP14. In particular, there are many open problems generalizing vacuum results to results with matter. Generally results for the Einstein-Maxwell equations are similar to those for the pure Einstein vacuum equations, and Einstein-Maxwell analogues exist for the problems RP3, RP4, RP9 and RP11. However, the statement of problem RP14 is not true in the presence of electromagnetic fields.

There is also a list of open problems in mathematical GR by Robert Bartnik [52] (also see references within). Theoretical GR had developed to such an extent that rigorous mathematical arguments have replaced many of the formal calculations and heuristics of the past, which will yield new insights for both mathematics and physics. Many of the Bartnik [52] problems are technical and concern clarifications and motivations for important contemporary problems, and many have been noted elsewhere in this paper.

The problems are on the topics of (i) apparent horizons (RB1 - RB17), (ii) initial data sets (RB8 - RB112), (iii) uniqueness and rigidity theorems for static and stationary metrics (RB13 - RB17), (iv) approximations (RB18 - RB25), (v) maximal and prescribed mean curvature surfaces (RB26 - RB29), (vi) causality and singularities (RB30 - RB34), (vii) the initial value problem and cosmic censorship (RB35 - RB47), and (viii) quasi-local mass (RB48 - RB53).

Regarding (iv), there has been a lot of work done on constructing metrics which approximately satisfy the Einstein equations, primarily consisting of numerical computation, but also involving asymptotic expansion, linearisation and matching techniques. As noted earlier, it is debatable as to whether numerical problems are in the realm of mathematical physics. But problem RB21 concerns a rigorously proof of the Newtonian limit to the Einstein equations and problem RB20 concerns the range of validity of post-Newtonian and post-Minkowskian asymptotic expansions. Problem RB23 on whether test particles follow spacetime geodesics, is a famous problem and includes an extensive investigation of asymptotic expansions [53].

Problem RB32 in (vi) is the "Bartnik splitting conjecture": Let M be a "cosmological spacetime" satisfying the timelike convergence condition: then either M is timelike geodesically incomplete or M splits as $R \times M^3$ isometrically (and thus is static). This is essentially problem Y115 in [45], which posed the question of establishing a Lorentzian analogue of the Cheeger-Gromoll splitting theorem of Riemannian geometry [54]. The concept of geodesic completeness in Lorentzian geometry differs considerably from that of Riemannian geometry, and this question was concretely realized in the Bartnik splitting conjecture RB32 [52]. In the case of a 4D vacuum (i.e., Ricci flat) globally hyperbolic, spatially compact spacetime, if M splits it is necessarily flat and covered by $R \times T^3$, and thus for a non-vacuum "cosmological spacetime" the conjecture asserts that the spacetime either is singular or splits. The resolution of the basic Lorentzian splitting conjecture as considered in RB32 was given in [55], and can be viewed as a (rigidity) singularity theorem since the exceptional possibility that spacetime splits can be ruled out as unphysical, and hence the spacetime has an inextendible timelike geodesic which ends after a finite proper time (i.e., it is timelike geodesically incomplete and hence singular). The status of the Bartnik conjecture was discussed in [56], and more general versions of the conjecture and partial results were discussed in [57].

Regarding (vii), Bartnik stated there are many versions of cosmic censorship, but that essentially the aim is to prove a theorem showing that singularities satisfying certain conditions are not naked. In addition, problem RB43 concerns the 2-body system in Einstein gravity, which Bartnik claimed is probably the most embarassing indictment of our (lack of) understanding of the Einstein equations (however, see [58] and the discussion later). The problem in (viii) of defining the total energy of an isolated system was essentially solved in [59], but the correct definition of the energy content of a bounded region in spacetime is still not settled. Although a number of candidate definitions have been suggested, so far none of these verify all the properties expected of a quasi-local mass.

There have been a number of reviews on the global existence problem in GR, including those of [60] and [61] (also see references within). In these reviews there is an emphasis on very technical questions in differential geometric and analytical global properties of 1+3 dimensional spacetimes containing a compact Cauchy surface (and particularly the vacuum case), but they draw attention to a number of open questions in the field.

3 Open problems in General Relativity

Mathematical questions about the general properties of solutions of Einstein's field equations of GR are truly problems in mathematical physics. Problems in GR are not necessarily more important than other problems in theoretical physics, but they do often have a more well-formulated mathematical expression. They are also perhaps more difficult for a broad based physics audience to fully appreciate. Therefore, I will first review some mathematical background, which can be skipped by general readers.

In general, a smooth (or sufficiently differentiable) 4-dimensional Lorentz manifold (M, g) is considered. The Lorentzian metric, g, which defines the causal structure on M, is required to satisfy the Einstein field equations, which constitute a hyperbolic system of quasilinear partial differential equations which are, in general, coupled to other partial differential equations describing the matter content of spacetime [60]. Primarily the vacuum case (when g is Ricci flat) is considered. Physicists are then interested in the Cauchy problem in which the unknowns in the resulting Einstein vacuum constraint equations, consisting of a Riemannian metric and a symmetric tensor defined on a three-dimensional manifold (and initial data for any matter fields present), are the initial data for the remaining Einstein vacuum evolution equations.

The Einstein equations are invariant under a change of the coordinate system (general covariance or gauge freedom), which complicates the way they must be formulated in order to faciliate the study of their global properties [61]. Although the Einstein vacuum equations are not hyperbolic in the usual sense due to general covariance, the Einstein vacuum equations in spacetime harmonic coordinates constitute a quasilinear hyperbolic system and therefore the Cauchy problem is well posed and standard results imply local existence [62]. It is also possible to show that if the constraints and gauge conditions are satisfied initially, they are preserved by the evolution. For example, the global regularity and modified scattering for small and smooth initial data with suitable decay at infinity for a coupled Wave-Klein-Gordon system (a simplified version of the full Einstein-Klein-Gordon system) in 3D was studied in [63]. Analogues of the results for the vacuum Einstein equations are known for the Einstein equations coupled to many different types of matter, including perfect fluids, gases satisfying kinetic theory, scalar fields, Maxwell fields, Yang-Mills fields and various combinations of these.

The general results for perfect fluids only apply in the restricted circumstances in which the energy density is uniformly bounded away from zero (in the region of interest) [60]. The existence of global solutions for models with more exotic matter, such as stringy matter, has also been studied [64]. **Existence:** The basic local existence theorem says that, given smooth (i. e., infinitely differentiable C^{∞}) data for the vacuum Einstein equations, there exists a smooth solution of the equations (on a finite time interval) which gives rise to these data [65]. The standard global uniqueness theorem for the Einstein equations asserts that the long term solution (maximal development [62]) of any Cauchy data is unique up to a diffeomorphism which fixes the initial hypersurface and that, in an appropriate sense, the solution depends continuously on the initial data [65].

The local existence of solutions of the Einstein equations is understood quite well. However, the problem of proving general global existence theorems for the Einstein equations is beyond the reach of current mathematics [60]. The usual method for solving the Einstein equations is the conformal method [65], in which the so-called free data are chosen and the constraints then reduce to four elliptic equations. In the simplified constant mean curvature case these reduce further to a linear system of three elliptic equations, which decouple from the remaining equation which essentially reduces to the nonlinear, scalar Lichnerowicz equation.

The causal structure of a Lorentzian spacetime is conformally invariant. Friedrich derived the compactified "regular conformal field equations" from the Einstein equations, a first order symmetric hyperbolic system, which leads to well posed evolution equations and hence small data global existence results from the stability theorem for quasilinear hyperbolic equations. For example, Friedrich [66] proved global existence to the future for "small" hyperboloidal initial data (that is, data close to the standard data on a hyperboloid) in Minkowski space. It is still an open question what general conditions on initial data on an asymptotically flat Cauchy surface give a Cauchy development with regular conformal completion. Friedrich has developed an approach to this problem in which the conformal structure at spatial infinity is analyzed (see [67] for references, and [68] which points out some new obstructions to regularity; also see the more recent articles [69, 70] and references within).

Therefore, for the full 1+3 dimensional Einstein equations (without symmetries) the only global existence results known are the theorem on nonlinear stability of Minkowski space [71], the semi-global existence theorem for the hyperboloidal initial value problem [66] and the semi-global existence theorem for spatially compact spacetimes with Cauchy surface of hyperbolic type [72], which are all small data results. It has been shown that for analytic vacuum or electrovac spacetimes, with an analytic Cauchy horizon which is assumed to be ruled by closed null geodesics, there exists a nontrivial Killing field [73]. Theorems in the cases of special spacetimes with symmetries are briefly reviewed below. Since spacetimes with Killing fields are nongeneric, this result may be viewed as supporting evidence for the strong cosmic censorship (see below).

Special cases: It is possible to solve the global existence problem for the Einstein equations in special cases, such as for spacetimes with symmetry [60, 61]. For example, basic global existence theorems for spherically symmetric static solutions (which are everywhere smooth) have been proved for perfect fluids and collisionless matter (see [60] and references within). The spacetime symmetry is defined by the number and character of Killing vectors. For example, consider spacetimes with an r-dimensional Lie algebra of spacelike Killing fields. For each $r \leq 3$, there are some basic results and conjectures on global existence and cosmic censorship [61]. In the cases r = 3 (Bianchi models; see, for example, [74]) and a special case of r = 2 (polarized Gowdy models – see Refs. below), the global behavior of the Einstein equations is well understood.

For the general r = 2 case (local $U(1) \times U(1)$ G_2 symmetry), there are only partial results on the global existence problem and the cosmic censorship problem remains open [61]. The first global existence result for Gowdy spacetimes with topology $R \times T^3$ was proven in [75], and subsequently generalized for spacetimes on S^3 and $S^2 \times S^1$ in [76] (a class of "nongeneric" metrics still remains to be studied). The first result concerning global constant mean curvature foliations in vacuum Gowdy spacetimes was proven in [77]. The question of cosmic censorship for the Gowdy spacetimes may be studied by analyzing the asymptotic behavior of curvature invariants such as the Kretschmann scalar, and this has been done for the class of polarized Gowdy spacetimes [78] and in more generality [79]. The structure of the horizon and extensions in the polarized Gowdy class can be very complicated [80, 81]. In the cases r = 1 (U(1) symmetry) and r = 0 (no symmetry), the large data global existence and cosmic censorship problems are open. However, in the U(1) case there are conjectures on the general behavior which are supported by numerical evidence, and there is a small data semi-global existence result for the expanding direction [82, 83]. **Differentiability:** The technical questions relating to differentiability are important from a mathematical point of view regarding well-posedness [60]. The differentiability of the allowed initial data for the Cauchy problem for a system of partial differential equations and the differentiability properties of the corresponding solutions are related and determined by the equations themselves. For example, in the context of the Einstein constraints there is a correspondence between the regularity of the free data and the full data.

There are reasons for considering regularity conditions weaker than the natural C^{∞} condition. One motivation is that physical matter fields are not necessarily C^{∞} (so that the theorems need not apply). Another motivation for considering low regularity solutions is connected to the possibility of extending (continuing) a local existence result to a global one. It is also worth noting that there are examples which indicate that generically Cauchy horizons may be nondifferentiable [84].

There is continued interest in finding a theory for the evolution and constraint equations for metrics with low differentiability (e.g., to prove the theorems under milder differentiability assumptions such as, for instance, that the metric is of regularity $C^{1,1}$ [85] in which the first derivatives of the metric are locally Lipschitz continuous functions, which is a more natural differentiability class than C^2 in a number of physically reasonable situations). In the existence and uniqueness theorems, the assumptions on the initial data for the vacuum Einstein equations can be weakened so that initial data belong to a local Sobolev space. In spacetime harmonic coordinates, in which the Einstein vacuum equations form a quasilinear hyperbolic system, standard results show that the Cauchy problem is well posed in an appropriate Sobolev space [86], with improvements on the necessary regularity recently given in [87, 88].

Singularity theorems: The famous singularity theorems are perhaps one of the greatest theoretical accomplishments in GR and in mathematical physics more generally [89]. Penrose's theorem [90] was the first modern singularity theorem, in which the concepts of geodesic incompleteness (i.e., the existence of geodesic curves which cannot be extended in a regular manner within the spacetime and do not take all possible values of their canonical parameter) to characterize singularities, Cauchy hypersurfaces and global hyperbolicity, and closed trapped surfaces [49], were introduced, and has led to many new developments in mathematical GR. Hawking realized that closed trapped surfaces will also be present in any expanding Universe in its past, which would then inevitability lead to an initial singularity under reasonable conditions within GR [91]. This subsequently led to the singularity theorem by Hawking and Penrose [92], which states that if a convergence and a generic condition holds for causal vectors, and there are no closed timelike curves and there exists at least one of the following: a closed achronal imbedded hypersurface, a closed trapped surface, a point with re-converging light cone, then the spacetime has *incomplete causal geodesics.* It has been argued that due to the discovery of the cosmic background radiation the singularity theorems give strong evidence that

a singularity actually occurred in our past [93].

The singularity theorems of Hawking and Penrose proved the inevitability of spacetime singularities under rather general conditions [90, 92]. But the singularity theorems say little about the nature of generic singularities. It should also be pointed out that there are generic spacetimes without singularities [94]. For example, the proof of the Penrose singularity theorem does not guarantee that a trapped surface can arise in evolution. Christodoulou [95] proved for vacuum spacetimes a trapped surface can indeed form dynamically from regular initial data free of trapped surfaces. This result was generalized in [96] (for more recent work see [97]). A sequence of marginally outer trapped surfaces with areas going to zero which form an apparent horizon within a region up to the "center" of gravitational collapse for the 1+3 dimensional Einstein vacuum equations were constructed in [98]. Marginally outer trapped surfaces also play an important role in proving the positive mass theorem and the Penrose inequality [99] (see below).

There are a number of open questions, which include proving more general singularity theorems with weaker energy conditions and differentiability conditions, and determining the relationship between geodesic incompleteness and curvature (e.g., is there always a divergence of a curvature invariant) [94]. There are also a number of related open problems in cosmology. Generic spacelike singularities are traditionally referred to as being cosmological singularities (but it is not clear that this is necessarily their natural physical interpretation [89], since oscillatory singularities might also be related to the spacelike part of generic black hole singularities [94]; for example, there is evidence that the mass inflationary instability at the inner horizon of an accreting, rotating black hole is generically followed by oscillatory collapse to a spacelike singularity [100]). There is also the question of singularity resolution in GR by quantum effects and the possibility of singularity theorems in higher dimensions. We shall return to these questions later.

Perhaps the most important open problem within GR is cosmic censorship.

3.1 Cosmic censorship hypothesis

The Hawking-Penrose theorem [90, 92] implies that singularities exist. But although the well known Schwarzschild spacetime contains a singularity, it is inside the black hole event horizon and is consequently not visible to outside observers. This leads to the question of whether gravitational collapse of realistic matter produces singularities that are similar to the singularity of Schwarzschild [49], in that they are hidden inside black hole event horizons (weak cosmic censorship) and are non-timelike (strong cosmic censorship).

Penrose proposed [47] the cosmic censorship hypothesis, which roughly states that for Einstein's equations coupled to "physical" matter, no "naked singularity" will develop "generically" from nonsingular "realistic" initial conditions (Cauchy data). A naked singularity is essentially one with the property that light rays from points arbitrarily near it can escape to infinity. These singularities are much more disturbing from a physical point of view, and the question cosmic censorship effectively asks is whether the future can be theoretically predicted [93]. It cannot be conjectured that naked singularities never occur, since there are known examples. However, these examples are of high symmetry and it is conceivable that naked singularities tend to become clothed by horizons under most small perturbations. Indeed, recent results [77] tend to support the notion that naked singularities imply symmetry.

Naked singularities are known to exist in Taub-NUT spacetime [101, 81] and simply by removing regions from Minkowski spacetime. It is also known that the equations of a pressureless fluid or "dust" will lead to spurious "shell crossing" naked singularities. In particular, a central locally naked singularity forms in spherical dust Tolman-Bondi-de Sitter collapse [102] from a non-zero-measure set of regular initial data, at which the Weyl and Ricci curvature scalars diverge. The most comprehensive results known on global inhomogeneous solutions of the Einstein equations are for solutions of the spherically symmetric Einstein equations coupled to a massless scalar field with asymptotically flat initial data, where Christodoulou has proved that naked singularities can develop from regular initial data [103] and that this phenomenon is unstable with respect to perturbations of the data [104].

Consequently, we seek to formulate cosmic censorship as a precise mathematical conjecture and then find a proof or a counterexample. Theorems on maximal Cauchy developments are within the global theory of partial differential equations and are generally very difficult to prove [105]. There can be no timelike singularities in a globally hyperbolic spacetime. Thus, a method for formulating (strong) cosmic censorship is as a statement that (under suitable conditions) spacetime must be globally hyperbolic. However, an initial data set has a maximal Cauchy development, which is a globally hyperbolic spacetime, but that maximal Cauchy development may not be the complete spacetime.

There are two other particular problems that must be faced. First, a naked singularity is very difficult to accurately define mathematically. Since the Einstein equations are essentially hyperbolic, the notion of extending a solution to points which can "see" the singularity is problematic, and so we have to seek an alternative definition of a naked singularity that is more stable and can be mathematically formulated. The second problem is genericity. It is known that there are special examples of solutions in GR which, for all reasonable definitions, contain a naked singularity where the maximal development is extendible. So it is impossible to prove a general statement that says a naked singularity cannot exist. That is, without some sort of "generic condition", this version of cosmic censorship would fail. We are, of course, ultimately interested in the real process of gravitational collapse, but care must be taken not to formulate a conjecture that will be vulnerable to what a physicist might claim appears to be an artificial counterexample. Therefore, the aim is to refine the conditions of the conjecture to rule out non-physical counterexamples, but not to the extent of making cosmic censorship irrefutable [60].

There are actually two different cosmic censorship hypotheses, which are only minimally related to each other. The weak cosmic censorship hypothesis states that: For generic initial data to the evolution problem in GR, there cannot be naked singularities. This is such an open problem that the correct formulation of the statement is not even known [106]. For an extensive treatment (including a somewhat precise version) of the weak cosmic censorship conjecture see [107].

Problem P1: Prove the weak cosmic censorship conjecture.

In the case of asymptotically flat spacetimes (describing isolated systems in GR), the work of Christodoulou establishes weak cosmic censorship in the class of spherically symmetric Einstein-scalar field spacetimes [108], and also gives examples of initial data such that the Cauchy development has a naked singularity [104].

The second hypothesis is strong cosmic censorship, which states that: A generic solution to the Einstein's equation cannot be continued beyond the Cauchy horizon. For earlier surveys on the strong cosmic censorship conjecture, see [109, 80]. It is of interest to prove weak and strong cosmic censorship even for vacuum solutions of Einstein's equations (i.e., those with no matter) or, more generally, within special classes of spacetimes.

Problem P2: Let M be a 3 dimensional compact manifold. Prove that the maximal vacuum Cauchy development for generic vacuum data sets is equal to the maximal vacuum extension of M.

An alternative strategy is to search for a counterexample to cosmic censorship. If a wide class of possible counterexamples can be shown to fail, then this might even be seen as evidence for the likely validity of the conjecture. A possible counterexample for weak cosmic censorship might arise from a process in which a black hole turns into a naked singularity. For example, the Kerr metric with mass M and angular momentum J represents a black hole if $J \leq M^2$ and a naked singularity if $J > M^2$. Therefore, a naked singularity might possibly be produced by overspining a black hole. Since spinning black holes repel the particles whose angular momentum would increase their spin, such a "spin-spin repulsion" unfortunately prevents the overspinning of a black hole [89].

A plausible candidate for a vacuum counterexample to cosmic censorship (with a negative cosmological constant) has recently been proposed based on the superradiant instability of Kerr-AdS black holes [110]. Another plausible counterexample (based on a holographic model of an electrically charged localised defect) in four-dimensional Einstein-Maxwell theory with asymptotically anti– de Sitter boundary conditions was presented in [111]; smooth initial data was shown to evolve to a region of arbitrarily large curvature in a finite time that is visible to distant observers. Unlike the spherical collapse 'counterexamples' which are finely tuned, this candidate is generic [112].

Finally, we note that by considering only globally hyperbolic spacetimes, solutions with gross causality violations are excluded, while some singular behaviour is still possible. But there are exact solutions with closed timelike curves known (e.g., the Godel and NUT spacetimes). The existence of such causality violation gives rise to "existential problems of an imponderable nature" [93]. Stephen Hawking has suggested the "chronology protection conjecture" that asserts that the closed timelike curves which arise in some solutions to the equations of GR (and which imply the possibility of backwards time travel) will be ruled out by a future theory of quantum gravity.

3.2 Penrose inequality

The mathematical ideas behind the proofs of the singularity theorems have been applied to several important results in GR, such as the positive mass theorem in its original form [113] which has, in turn, led to research on the rigidity of asymptotically flat manifolds with non-negative scalar curvature.

In particular, Penrose has shown [51] that if a certain inequality involving the area of a marginally (outer) future-trapped surface (the apparent horizon) and the (ADM) mass of the initial hypersurface containing this horizon were violated, then the spacetime that results from evolving the initial data contains a naked singularity. Therefore, initial data violating this so-called *Penrose inequality* would constitute a counterexample to weak cosmic censorship, while a proof of this inequality would provide evidence in favor of weak cosmic censorship. In fact, such a proof would possibly lead to an approach for attacking the cosmic censorship conjecture using methods in partial differential equations [114] (this is discussed further in [61]).

Problem P3: Find a proof of the Penrose inequality or present a counterexample in the general case.

The Riemannian version of the Penrose inequality was recently proved [115]. The proof in the Lorentzian case is not known. Even in spherical symmetry only a weaker version (using the energy rather than the mass) is known to hold. Proofs have been given under various restrictive assumptions, such as the existence of certain foliations (e.g., the constant mean curvature time gauge [116]), and global conditions on the spacetime (see [117]).

The Penrose inequality is one of a large class of mass inequalities for spacetime manifolds [118]; for example, an analogous inequality is based on the Penrose quasi-local mass [119]. It is also of interest to find a generalisation of the Penrose inequality to initial data sets which are not time-symmetric. There also exist stronger versions of the Penrose inequality involving angular momentum, electric charge, and/or the cosmological constant [89], most of which lead to open questions. There are further refinements of the conjectures, such as the so-called Gibbons-Penrose inequality, which gives some improved lower bounds when there are multiple black holes [120]. Another inequality is Thorne's hoop conjecture [121], which exploits the physical idea that since black holes are extremely localized objects, their energy/matter content must be severely compacted in all spatial directions. Despite the difficulty in making this idea precise, the hoop conjecture has proven successful [107]. A possible mathematically viable reformulation of the conjecture has been presented in [122].

3.3 Yang–Mills equations and GR

Many important mathematical questions, including stability theorems and the proof of existence of Yang-Mills (YM) fields by methods of partial differential equations, remain unsolved. We have already discussed problem M2 on the existence of solutions of YM earlier, and there were 2 well known problems (Y117, Y118) presented in [45], the first of which is the question of whether every SU(2) Yang-Mills field is self-dual or anti-self-dual. A key contemporary question is to obtain solutions of the YM equations on a Riemannian (or Lorentzian) manifold. Recently it has been shown numerically that the static, spherically symmetric Einstein-Yang-Mills (EYM) equations have non-singular, asymptotically flat solutions [123]. Six interesting questions for EYM solutions were presented in [52] (RB17, see above).

A central feature of YM theory is the invariance of the physics under an infinite-dimensional group, in which bundles, connections and curvature play a fundamental role. It is consequently a subject of interest not only to physicists but also, particularly after the work of Atiyah, Hitchin, and Singer [124, 125, 33], to mathematicians (as discussed earlier and in [45]). The YM field equations depend on how a section of the Lie algebra valued bundle is choosen. The choice of such a section is called the choice of a gauge. In a suitable gauge, the YM equations become a quasilinear elliptic system whose highest order term is linear. Physicists are mostly interested in YM fields over \mathbb{R}^4 or \mathbb{S}^4 .

It is known that in four dimensions there exist global smooth solutions of the YM equations corresponding to rather general initial data. Global existence in Minkowski space, assuming initial data of sufficiently high differentiability, was first proven in [126] and a new proof of a local existence theorem for data of finite energy (and since energy is conserved this immediately proves global existence) was given in [127]. A global existence proof on 1+3 dimensional, globally hyperbolic spacetimes was given in [128] (see also [79]). The proof of the global existence to the future for hyperboloidal initial data close to the standard data on a hyperboloid in Minkowski space by Friedrich was later generalized to Maxwell and YM matter in [129]. However, although asymptotically flat (with regular interior) spherically symmetric and localised ("particle-like") solutions of the coupled EYM equations with gauge group SU(2) have been known for many years, their properties are still not well understood [130].

In dimensions greater than five it is known that there exist solutions which develop singularities in a finite time. Numerical evidence indicates that this type of blow-up is stable (i.e., it occurs for an open set of initial data) and that there is a critical self-similar solution separating this kind of blow-up from dispersion. There is as yet no rigorous proof of blow-up in five dimensions. In six dimensions singularities form, but apparently differently from those in five dimensions [61].

The effects found in YM theory are captured in two dimensions less by *wave* maps with values on spheres, where it is easier to prove theorems. The existence of a solution having the properties expected of the critical solution associated with singularity formation for wave maps in four dimensions has been proven in [131]. An important open question is the global existence problem for the classical wave map equation (i.e., the nonlinear σ -model, hyperbolic harmonic map equation). The wave map equation has small data global existence for spatial dimension $n \geq 2$. But global existence for large data is known only for symmetric solutions and, in particular, the global existence problem for the wave map equation is open for the case n = 2. For the case n = 1, global existence can be proved using energy estimates [61, 132]. The U(1) symmetric vacuum 1+3 case in which the Einstein equations reduce to 1+2 gravity coupled to wave map matter in the presence of a hypersurface orthogonal spacelike Killing field, is of intermediate difficulty between the full 1+3 Einstein equations and the highly symmetric Gowdy equations [133].

It is also of interest to consider other forms of matter such as, for example, self-gravitating collisionless matter models (see the reviews [60, 61, 134] and references within). There are theorems on the global existence and uniqueness of smooth solutions of the Vlasov-Poisson and the classical Boltzmann equations in Newtonian theory. Many analogues of these results have been proven in GR, including the global existence of weak solutions, the convergence to equilibrium for classical solutions starting close to equilibrium, basic existence theorems for spherically symmetric static solutions, plane and hyperbolic symmetric spacetimes and a subset of general Gowdy spacetimes, and studies of spherically symmetric collapse. Collisionless matter models are known to admit a global singularity-free evolution, and in many cases can also lead to isotropization at late times. Analytical techniques have not been applied in the general case, although numerical methods have been used to gain some insights [60, 61].

Problem P4: Prove the global existence of classical spatially inhomogeneous solutions for small initial data in collisionless matter models. Prove an existence and uniqueness theorem for general spatially homogeneous (such as Bianchi type IX) solutions of the Einstein-Vlasov equations and investigate the large initial data case.

3.4 Uniqueness and stability

Mathematically, any proof of stability requires deriving the asymptotic behaviour of solutions to the Einstein equations in GR, a highly nonlinear system of partial differential equations, which is notoriously very difficult. However, there are some special cases for which there exist proofs or which have received particular attention.

Stability of Minkowski spacetime: Minkowski spacetime is globally stable [135]. That is, if we start with a universe that is already very sparse, it is guaranteed that it will evolve asymptotically to Minkowski spacetime. The first result on the global existence (for small data) and the stability of Minkowski spacetime under the field equations of GR was due to Christodoulou and Klainerman [71, 135]. They proved that if initial data for the vacuum Einstein equations are prescribed which are asymptotically flat and sufficiently close to those induced by Minkowski spacetime on a hyperplane, then the maximal Cauchy development of this data is geodesically complete (and they further provided details on the asymptotic behaviour of the solutions). Results can also be found for any asymptotically flat spacetime where the initial matter distribution has compact support, so long as attention is confined to a suitable neighbourhood of infinity. There are recent extensions to these results by various authors (e.g., see [136]).

Uniqueness of black holes: If we conjecture that the final state of a spacetime is either Minkowski space or a black hole, we can then ask whether a black hole is the only possible stationary (steady state) solution. The problem of black hole uniqueness is not completely resolved. The study of uniqueness for non-vacuum spacetimes is colloquially known as "no-hair" theorems.

In the case where it is assumed that the spacetime has additional symmetry and is either axially symmetric or rotationally symmetric, the uniqueness of black holes is known. The uniqueness of the 4D Schwarzschild and Kerr solutions in GR was discussed in [93]. The uniqueness theorem for Schwarzschild spacetime was presented in [137, 138]. The unique stationary (nonstatic) regular predictable Ricci flat spacetime subject to certain assumptions is the Kerr solution [139]. The uniqueness theorem for the Kerr spacetime was proven in [140, 141]. In the non-vacuum case the uniqueness of the rotating electrically charged black hole solution of Kerr-Newman has not yet been generally proven (however, see [142, 143]).

We also know that black holes are unique if we assume real analyticity. If the regularity assumption is relaxed to just infinitely differentiable the result is still expected to be true. In this case there are only some partial results. For example, if only small perturbations of a stationary black hole are allowed then there are no other stationary solutions that are approximately a known black hole solution without being one, and if certain special structures on the event horizon are assumed then other stationary exteriors are not possible. **Stability of Kerr-Newman black hole:** If we assume that the known Kerr-Newman family of black holes form the unique stationary state of GR, the next problem is to prove that they are actually stable under perturbations. That is, if we start out with initial data very close to that of a Kerr-Newman black hole, does the the evolution "track" a Kerr-Newman black hole. Although there has been substantial and exciting progress made in the linearised problem [144], results for the full nonlinear problem are still elusive.

The stability of the Kerr metric was discussed in [145], and a comprehensive review was given in [146]. The aim is to show that perturbations of the Kerr (and Schwarzschild [147]) solution decay exponentially and are thus stable. Unfortunately, a mathematically rigorous understanding of the stability of the generic Kerr black hole, as well as a thorough understanding of its dynamics under arbitrary non-linear perturbations, is still lacking. However, current observational data are compatible with the predictions of GR, and suggest that the end point of mergers is a Kerr black hole. Indeed, all numerical results provide evidence that the Kerr (and Kerr-Newman) black holes are non-linearly stable (at least within a certain range of the angular momentum) [148].

Problem P5: Prove the stability of the Kerr black hole.

It is of interest to extend stability results to the case of a non-zero cosmological constant [149]. Regarding the stability of the Kerr-de Sitter family of black hole solutions, there has been recent results on nonlinear perturbations in the slowly rotating case [150]. The case of a negative cosmological constant is much more problematic, because it is not even clear if the Kerr-AdS black hole is itself stable (due to superradiance and stable trapping phenomena [151]). We shall discuss the stability of the de–Sitter and anti–de–Sitter spacetimes later. It is also of interest to study the stability of models with matter, particularly in the cosmological context (also see later). Unfortunately, even generalizations to simple inhomogeneous perfect fluids are problematic since the formation of shocks (or, in the case of dust, shell-crossings) are anticipated to occur which form a barrier to the mathematical study of the evolution of the cosmological models with known techniques. Criteria for the development of shocks (or their absence), based on the techniques of classical hydrodynamics, should be developed further.

3.5 Other problems

Curvature invariants: In [152] it was shown that the class of 4D Lorentzian manifolds that cannot be completely characterized by the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives must be of a special "degenerate Kundt form". This result, which is also believed to be true in higher dimensions [153], implies that generally a spacetime is completely characterized by its scalar curvature invariants (at least locally, in the space of Lorentzian metrics). The special Kundt class is defined by those metrics admitting a null vector that is geodesic, expansion-free, shear-free and twist-free. We recall that in the Riemannian case a manifold is always locally characterized by its scalar polynomial invariants.

It is also of interest to study (the 'inverse question') of when a spacetime can be explicitly constructed from its scalar curvature invariants. In 4D we can (partially) characterize the Petrov type of the Weyl tensor in terms of scalar curvature invariants [154]. Having determined when a spacetime is completely characterized by its scalar curvature invariants, it is also of interest to determine the minimal set of such invariants needed for this characterization.

Problem PF1: Determine when a 4D spacetime can be explicitly constructed from its scalar curvature invariants and determine the minimal set of such invariants.

Evolution of the horizon: There is much interest in determining the appropriate definition of the "boundary of a black hole". A closed oriented space-like 2-surface (normally isomorphic to S^2) in a spacetime determines two future null vector fields, normal to the surface. If the future evolutions of the surface along these directions are both area-non-increasing, the surface is future trapped, and if one of the null mean curvatures is zero, then the surface is called an "apparent horizon". It is also important to determine the evolution of the horizon and, more generally, formulate an appropriate definition of a dynamical horizon in GR. We note that much work on the evolution of apparent horizon (such as black hole evaporation) is based on a linear analysis, which to first order assumes that the horizons do not move. The true nonlinear versions of the evolution is not yet well understood. The problem of identifying and locating horizons using scalar curvature invariants has recently been studied [155].

Problem P6: Formulate an appropriate definition of a dynamical horizon.

Geodesic hypothesis: One of the postulates of GR is that point particles with negligible mass will travel along geodesics of the spacetime.

Problem P7: Prove that test particles move on spacetime geodesics.

This famous problem (RB23) was first considered by Einstein in the 1920s and is still not completely resolved (there has been an extensive investigation using asymptotic expansions — see the discussion in [53]). The main problem is how to make the process of "taking the negligible mass limit" rigorous. And for a physical object in GR, when it moves, its motion will cause "ripples" in the spacetime caused by gravitational backreaction of its own presence. In addition, while the three body problem is difficult in classical mechanics, even the two body problem in full generality is still unresolved in GR.

Newtonian limit: It is difficult to give a precise mathematical formulation of the statement that Newtonian gravitational theory is the limit of GR as the speed of light tends to infinity. Ehlers gave a definition of the Newtonian limit of GR which encodes those properties which are physically desirable [156]. However, even when a suitable definition has been given, the question still remains as to whether the definition is compatible with GR in the sense that there are general families of solutions of the Einstein equations which have a Newtonian limit with the chosen definition. Asking whether there are such families which are suitably differentiable is related to the issue of giving a mathematical justification of the so–called post-Newtonian approximation. See problems RB20 and RB21.

4 Theoretical physics problems in the Quantum realm (Ph1-Ph4)

There are a number of fundamental questions in the quantum realm, culminating in the ultimate question of whether there is a single theory (or even, more precisely, one single equation) that would unify all of nature within a so-called "theory of everything". In particular, is this theory string theory? And would this theory then give an explanation of the fundamental gauge group in the grand unification theory of the three non–gravitational forces [157] and also explain the values of all fundamental physical constants (and whether they vary over time)? In addition, are there fundamental particles that have not yet been observed and, if so, what are their properties? Let us consider the following particular problems.

The foundational problems of quantum mechanics (Ph2): These problems concern the fundamental understanding of quantum physics and especially the important role that measurement and observation play in the description of physical reality. There are currently many interpretations of quantum physics, including the classic Copenhagen interpretation, Everett's controversial "many worlds" interpretation, and even more controversial ones such as the "participatory anthropic principle".

In particular, how does the quantum interpretation of reality, which includes the superposition of states and wavefunction collapse or quantum decoherence, give rise to what we perceive? What are the actual causes of the collapse of the quantum wavefunction? Are there non-local phenomena in quantum physics and, if they do in fact exist, are they limited to the entanglement revealed in the violations of the Bell inequalities and can they be observed? What does the existence or absence of non-local phenomena imply about the fundamental structure of spacetime and how is this related to quantum entanglement? Most modern physicists who work within quantum field theory perhaps no longer consider questions of the proper interpretation of the fundamental nature of quantum physics to be of prime importance. Indeed, many may believe that the principle of decoherence is essentially an appropriate explanation; for example, interaction with the environment causes the quantum collapse.

However, dynamical models have been proposed to explain the collapse of the wave-function and perhaps provide a possible solution to the quantum measurement problem, by proposing that the Schrodinger equation is an approximation to a stochastic nonlinear dynamics (with the stochastic nonlinear aspect becoming increasingly more important when progressing from microscopic systems to macroscopic ones) [158]. In addition, as in most other physical systems, evolution in time is central to the understanding of quantum systems. The time that is used to define evolution in quantum theory is clearly part of the classical spacetime manifold. However, this perhaps suggests that the present formulation of quantum theory is incomplete and that there ought to exist a reformulation of quantum theory which does not refer to classical time. The unification of particles and forces (Ph3) and the tuning problem (Ph4): The standard model of particle physics involves eighteen different fundamental particles. It is often believed that a theory of nature should have a more fundamental method of unifying these particles. For example, string theory, which is perhaps the most well-defined approach, predicts that all particles are different vibrational modes of fundamental filaments of energy or strings. It is, of course, of great importance to determine whether or not the various particles and forces can be unified within a theory that explains them all as manifestations of a single, fundamental entity.

In the standard model of particle physics the parameters representing the eighteen particles predicted by the theory are required to be determined (i.e., measured by observations) in order for theoretical predictions to be made. However, some physicists argue that fundamental physical principles of a unified field theory should set these parameters, independent of measurement. In particular, there is the question of whether the form of the universe is inherently set by its properties (in the sense that the properties would not occur if the form is different). In the multiverse paradigm there is not just a single universe, but there are a wide range of fundamental theories (or different variants of the same theory, based on different physical parameters) and our universe is just one of the possible universes that could be created. In this paradigm the question then becomes why our particular universe has properties that appear to be so finely tuned to allow for the existence of life. This has led some scientists to turn to the anthropic principle for explaining this fine-tuning problem: this asserts that our universe must have the properties it does because if it had different properties we wouldn't be here to be able to beg the question.

Finally, the recent observation of a Higgs boson appears to complete the standard model, but with the addition of new physics in order to protect the particle mass from quantum corrections that would increase it by perhaps fourteen orders of magnitude or more. It is widely thought that the most plausible resolution of this hierarchy (or naturalness) problem is supersymmetry. However, the simplest supersymmetric models have not worked, and no convincing mechanism has yet been found to either break supersymmetry or to determine the many supersymmetric parameters (AL3.3).

The problem of quantum gravity (Ph1): There are four fundamental forces of physics. However, the standard model of particle physics includes only the three forces of electromagnetism and the strong and weak nuclear forces. An attempt to formulate a theory which unites all four forces, including gravity, into a single unified field theory is a primary goal of theoretical physics. The theory that includes both GR and the standard model of particle physics is referred to as quantum gravity (QG). Unfortunately, at present these two theories describe different scales of nature and any attempts to explore the overlaping scale has yielded incomprehensible results, such as the force of gravity (or curvature of spacetime) becoming infinite. If quantum mechanics and GR can be realized as a fully consistent theory of QG a number of natural questions arise, which include
the following: Is spacetime fundamentally continuous or discrete? Would such a theory include a force mediated by a hypothetical graviton, or would it be a product of a discrete structure of spacetime itself (such as, for example, in loop quantum gravity)? Are there possible differences from the predictions of GR at very small or very large scales (or in any other extreme circumstances) that result from a theory of QG?

Although QG effects modify GR, leading to new gravitational physics, it appears that these modifications do not significantly affect the macroscopic behavior of stellar systems and black holes. For example, a black hole that evaporates through the emission of Hawking radiation [159], perhaps the most dramatic consequence of uniting GR and quantum mechanics, does not differ significantly from a classical black hole over astrophysical timescales [160].

Many of the above problems are in theoretical physics and generally are not problems in mathematical physics. However, a lot of mathematics is utilized in *string theory:* for example, Yau proved the Calabi conjecture, which allowed physicists to show, using Calabi-Yau compactification, that string theory is a viable candidate for a unified theory of nature. In addition, there are a number of related fundamental questions in Yang-Mills theory, which we have discussed earlier. We shall next discuss some specific problems that are definitely within mathematical physics.

4.1 Instability of Anti-de Sitter spacetime

Anti-de Sitter (AdS) spacetime in any dimension is the unique maximally symmetric Lorentzian manifold with constant negative scalar curvature. AdS spacetimes are of interest in theories of QG formulated in terms of string theory (in which elementary particles are modelled not as zero-dimensional points but as one-dimensional objects called strings) or its modern extension, M-theory. Indeed, AdS spacetimes have come to play a central role in theoretical physics, primarily due to the AdS/CFT correspondence (or Maldacena gauge/gravity duality) which is the conjectured equivalence between string theory on an asymptotically AdS spacetime and a conformally invariant quantum field theory (CFT) living on the boundary of this spacetime [161, 162]. CFT are quantum field theories, including theories similar to Yang-Mills theories, that describe elementary particles.

The AdS/CFT correspondence suggests that it is possible to describe a force in quantum mechanics (like electromagnetism, the weak force or the strong force) in a certain number of dimensions with a string theory where the strings exist in an AdS spacetime with one additional dimension. The duality represents a major advance in our understanding of string theory and QG since it provides a non-perturbative formulation of string theory with certain boundary conditions. The usefulness of this strong-weak duality results from the fact that strongly coupled quantum field theories can be studied by investigating the corresponding weakly interacting gravitational theory which is mathematically more tractable. This has been used to study many aspects of nuclear and condensed matter physics (such as, for example, the modelling of non-equilibrium processes such as heavy ion collisions) by translating those problems into more mathematically tractable problems in string theory. That is, the AdS/CFT dictionary is used to translate the strongly coupled CFT to the string dual, which effectively reduces to classical AdS gravity, and the results are then utilized to produce useful information on the physics of the CFT. Unfortunately, the problem with this holographic approach is that the gravity side in the non-stationary regime is not well understood.

The AdS/CFT correspondence provides strong motivation for studying the dynamics of asymptotically AdS spacetimes. But, of course, this is an interesting problem in classical GR in its own right. AdS spacetime is different to Minkowski and de-Sitter spacetimes [163], which were proven to be nonlinearly stable a long time ago [135, 164]. It has recently been conjectured that the AdS spacetime is unstable under arbitrarily small perturbations [165]. This is related to some interesting more general mathematical problems.

The question of the global nonlinear stability of AdS was given a huge boost by the seminal work of Bizon and Rostworowski [112] following a conjectured instability by Dafermos and Holzegel [166]. While it would of course be desirable to study the nonlinear stability of AdS with no symmetry restrictions, this problem currently is analytically and numerically intractable. In [112] the analysis was restricted to spherical symmetry within pure Einstein gravity with a massless scalar field. Numerical results suggested that AdS is nonlinearly unstable to a weakly turbulent mechanism that forms an arbitrarily small black hole. whose mass is controlled by the energy of the initial data. While this nonlinear instability seems to occur for generic perturbations, there are perturbations that do not necessarily generate an instability (see [167] and references within) which, in turn, appears to lead to the existence of islands of stability [168]. Using standard perturbation theory to third order in the amplitude of the linear seed, it was shown [112] that this leads to secular growth and nonlinearities occur that can create resonances. The heuristic explanation for the mechanism which triggers the turbulent behaviour is thus the generation of secular terms by resonant four-wave interactions; it is this weak turbulence that is a driving mechanism of the instability.

There are modifications of standard perturbation theory that can capture the dynamics up to certain time scales, such as the resonant approximation [169], but rely on the spherical symmetry assumption. It is not known if any solution of the Einstein equations with a fully resonant spectrum necessarily possesses a nonlinear instability, but it is clear it is a necessary condition for the existence of the weakly turbulent instability. It is an interesting open question as to whether the nondispersive character of the linearized spectrum is essential for the turbulent instability and how generic is the turbulent instability. In order to study this beyond spherical symmetry third order perturbation theory calculations for a variety of different seeds have been performed [167, 168], and it was found that the gravitational case is more richer than the spherically symmetry case analysed in [112]. The prime question is consequently to determine the endpoint of instability of arbitrary dimensional AdS spacetimes for non-spherical perturbations [165]. Note that recently nonlinear instability was proved for the spherically symmetric Einstein-massless Vlasov system [170].

Problem P8: Determine whether the conjectured nonlinear instability in Antide Sitter spacetime, which leads to a weakly turbulent mechanism that develops a cascade towards high frequencies leading to black hole formation, behaves differently in more general models than spherically symmetric scalar field collapse. Since this is a particularly topical problem, let me discuss it in a little more detail [171]. In the case of AdS, the question of stability must be supplemented by a choice of boundary conditions at infinity and, a priori, any results may depend on this choice. A local well-posedness result to the initial boundary value problem for a large class of AdS boundary conditions was proven by Friedrich for the vacuum Einstein equations with negative cosmological constant in 4D [164], allowing local stability to be studied mathematically. In the case of reflective boundary conditions, for which there is no flux of energy across the conformal boundary acts like a mirror at which perturbations propagating outwards bounce off and return to the bulk. This leads to very complex nonlinear wave interactions in the bulk, which is extremely difficult to study even in the case of small perturbations. Consequently, it is hardly surprising that the question of the stability of AdS spacetime remains open.

For reflective boundary conditions, the problem of the linear stability of AdS reduces to a much simpler spectral problem for a certain master linear operator whose coefficients depend on the character (i.e., scalar, electromagnetic or gravitational) of the perturbations [172]. The problem of the nonlinear stability of n + 1 dimensional AdS spacetime in full generality is currently beyond the theory of partial differential equations. Thus it is natural to consider more tractable special cases. In particular, for spherically symmetric perturbations of a self-gravitating minimally coupled massless scalar field, the system of Einstein-scalar field equations with appropriate boundary conditions and compatible smooth initial data constitutes a locally well-posed initial-boundary value problem in asymptotically AdS spacetimes. Perturbative and numerical studies of the global behavior of small data solutions to this problem give evidence (first for n = 3 and later generalized to $n \ge 3$ [173]) for the conjecture that (within the model) the n + 1 dimensional AdS spacetime is unstable to the formation of a black hole for a large class of arbitrarily small perturbations [112].

4.2 Higher dimensions

Extra dimensions (beyond the familiar four of ordinary spacetime) are employed in string theory [174]. In addition, spacetime manifolds of higher dimensions are considered in some cosmological scenarios. If there are, in fact, higher dimensions, then deep questions on the structure of the internal space for our Universe arises (AL5.1). If nature has more than four spacetime dimensions, what are their size, what is the topology of Universe, and why are there 3 apparent spatial dimensions? And can we experimentally observe evidence of higher spatial dimensions?

The study of black holes in GR, and the differences between black holes in 4D and higher dimensions, is currently of great interest. At the classical level, gravity in higher dimensions exhibits a much richer structure than in 4D; for example, one of most remarkable features of 4D GR is the uniqueness of the Kerr black hole. In contrast, there exist a number of different asymptotically flat, higher-dimensional vacuum black hole solutions [175]. The uniqueness and stability of higher dimensional black holes is of paramount interest.

Problem P9: Determine the uniqueness of black holes in higher dimensions.

Problem P10: Determine the stability of higher dimensional black holes.

A number of sub-problems, including adapting the hypotheses of analyticity, non-degeneracy, and connectedness in the black-hole uniqueness theorems, and classifying all vacuum near-horizon geometries with compact cross-sections, have been proposed.

Differential geometry and geometric results have recently been developed in higher dimensions [152, 176]. In particular, even though the singularity theorems were originally proven in 4D, results in which the closed trapped surface is a codimension two trapped submanifold hold in arbitrary dimensional spacetimes. The concept of being trapped can also be associated with submanifolds of any co-dimension, so long as an appropriate curvature condition is assumed to ensure the existence of focal points to the submanifold. Closed trapped surfaces in codimension 3 and in arbitrary dimension were discussed in [177]. The positive mass theorem has recently been proven in all dimensions [178]. The question of stability in higher dimensions is more problematic. On one hand, radiative decay of solutions is stronger in higher dimensions and would enhance stability. On the other hand, there are more degrees of freedom which will generally increase the possibilities of instability. There is numerical evidence to suggest that certain types of higher dimensional black holes are in fact unstable [175].

However, the problem of cosmic censorship in higher dimensions is not well posed and very difficult. Indeed, in higher dimensions there is strong numerical evidence that cosmic censorship fails [179], and higher dimensional black holes can be unstable under gravitational perturbations. This was first shown by Gregory and Laflamme for black strings and black p-branes [180] (in 4D this instability does not exist). Convincing numerical evidence that unstable black strings pinch off in finite asymptotic time, thus resulting in a naked singularity, was presented in [179]. Since no fine-tuning of the initial data was required, this result constituted a violation of the weak cosmic censorship, albeit in spacetimes with compact extra dimensions. The black rings of [181] also suffer from various types of instabilities [182], including the Gregory–Laflamme instability.

At very large angular momenta, black holes become highly deformed and resemble black branes. The rapidly spinning Myers-Perry (higher-dimensional analogues of Kerr) black holes [183] in spacetime dimensions greater than 6 were shown to be unstable under a (Gregory–Laflamme type of) "ultraspinning instability" [184]. In particular, the end point of the axisymmetric ultraspinning instability of asymptotically flat Myers-Perry 6D black holes was studied in [185], and this instability was found to give rise to a sequence of concentric rings connected by segments of black membrane on the rotation plane which become thinner over time in the non-linear regime, resulting in the formation of a naked singularity in finite time and consequently a violation of the weak cosmic censorship conjecture.

4.3 Singularity resolution in GR by quantum effects

Einstein's theory suffers from the problem of classical singularities, which are a generic feature of spacetimes in GR. The existence of singularities indicate a breakdown of the classical theory at sufficiently large spacetime curvature, which is precisely when gravitational quantum effects are expected to be relevant. Consequently, QG is necessary for the clarification of whether the singularity theorems survive when entering the quantum regime. The question of whether a theory of QG can extend solutions of classical GR beyond the singularities was first discussed in [186].

In any analysis of the singularity theorems in the quantum realm, an important step is the weakening of the energy conditions and finding an appropriate version of the curvature conditions. For example, averaged energy conditions to deal with the quantum violations of the energy conditions have been considered. It is also necessary to go beyond semiclassical theories and take into account the quantum fluctuations of the spacetime itself, which leads to additional difficulties in seeking quantum singularity theorems. In particular, in the classical theorems the pointwise focusing of geodesics is utilised, which cannot hold exactly (despite the smallness of the fluctuations) in a quantum regime. The notion of closed trapped surface can also be generalized and adapted to quantum situations [187].

Let us discuss cosmological and black hole singularity resolution within loop quantum gravity (LQG) and string theory. LQG is a non-perturbative canonical quantization of gravity based on Ashtekar variables [188], in which classical differential geometry of GR is replaced by a quantum geometry at the Planck scale, and has been used to perform a rigorous quantization for spacetimes with symmetries. Applying the techniques of LQG to cosmological spacetimes is known as loop quantum cosmology (LQC), in which the spatial homogeneity reduces the infinite number of degrees of freedom to a finite number. Indeed, LQG reveals that singularities may be generically resolved because of the quantum gravitational effects [189]. And due to the quantum geometry, which replaces the classical differential geometry at the Planck scale, the big bang is replaced by a big bounce, when energy density reaches a maximum value of about a half of the Planck density. The existence of a viable non-singular bounce in the very early universe appears to be a generic result in all investigations of simple models of LQC, and occurs without any violation of the energy conditions or fine tuning [190].

Often a singularity in GR, such as the big bang and big crunch (to the future for contracting models) as well as black holes, is characterised by the divergence of a physical or geometrical quantity (such as a curvature invariant) and the breakdown of the geodesic evolution. However, singularities can also arise due to pathologies of the tangent bundle, for instance in conical singularities, or where there are directional singularities, defined as limit points towards which the curvature tensor diverges along some (but not necessarily all) directions. These complications led to an elaborate classification of possible singularities arising from the curvature tensors [191]. LQC attempts to resolve all singularities, including, for example, the big rip, and sudden and big freeze singularities.

In contrast to the classical theory where singularities are a generic feature, there is growing evidence in LQC that singularities may be absent. Recently various spatially homogeneous cosmological models have been studied within the context of LQC [192]. In particular, for the models that have been exactly solved at the quantum level, the dynamics of sharply peaked, semiclassical states is very well described by an effective theory that incorporates the main quantum corrections to the dynamics [193, 194]. For example, at the effective level an infinite number of bounces and recollapses occur in the positive spatial curvature FLRW model. Simple Bianchi type IX models have also been shown to be nonsingular [195, 196]. The original study [197] was improved (to solve the problems with the infrared limit [193]), and the Bianchi IX dynamics was constructed [198]. It is within the framework of the improved dynamics that solutions to the effective equations for the Bianchi IX class of spacetimes whose matter content is a massless scalar field was studied numerically within LOC in [196]. and the big bang singularity was resolved and the classical dynamics far from the bounce was reproduced.

Problem P11: Can the singularity resolution results obtained in the spatially homogeneous spacetimes in LQG be proven in a more general setting.

Symmetry reduction within LQC entails a drastic simplification, and therefore important features of the theory might be lost by restricting the symmetry prior to quantization. However, it is believed that such studies do lead to valuable hints on loop quantization and inhomogeneous spacetimes (and black holes) [192]. There is strong evidence from the numerical studies of the BKL conjecture (see later), that near the singularities the structure of the spacetime is not determined by the spatial derivatives, so that it might be hoped that singularity resolution in spatially homogeneous models would capture some aspects of the singularity resolution in more general inhomogeneous spacetimes. On the other hand, however, sometimes the limitations of LQC have been used to shed doubts on its results. For example, in [199] it is claimed that a fully covariant approach with validity beyond symmetry reduced scenarios produces physical results inequivalent to those obtained from LQC (see also [189]). Recently LQG techniques have been used to study the effects of QG in the simple Gowdy inhomogeneous models with infinite degrees of freedom [200]. And the first steps in the study of classical oscillatory singularities governed by the BKL dynamics using LQG have been taken [201].

Loop quantization of black hole spacetimes uses similar techniques as in LQC, and leads to similar results on singularity resolution [202, 203, 204, 205]. The resolution of gravitational black holes singularities has been also studied in string theory [206, 207]. Indeed, there has been significant progress on understanding black holes in string theory recently [208], and some interesting examples have been presented where gravitational singularities are resolved by higher derivative corrections to the action [209]. For example, the fundamental string in five dimensions, which is singular in the standard supergravity description, is regular after taking into account higher derivative corrections

determined by anomalies and supersymmetry [210]. In particular, singularities were resolved in string solutions of five dimensional supergravity corrected by the mixed gauge-gravitational Chern-Simons term with $AdS_3 \times S^2$ near string geometry (which can be interpreted in M-theory as M5-branes wrapped on four-cycles in a Calabi-Yau manifold) [210]. The techniques to resolve singularities can applied in more general situations, including black holes in five dimensions with different near horizon geometry, rotating black holes, and generalizations to other dimensions, including 10 and 11 dimensions, and theories with more general matter content.

4.4 Black hole information paradox

Hawking discovered that black holes are not completely black but emit a dim radiation due to quantum effects near the event horizon [159]. This result poses a fundamental theoretical problem because it appears to suggest that evaporating black holes destroy information, which is in conflict with a basic postulate of quantum mechanics that physical systems evolve in time according to the Schrodinger equation (which is fully deterministic and unitary and thus no information can be truly lost or destroyed). The apparent contradiction between Hawking's result and the unitarity postulate of quantum mechanics has become known as the *black hole information paradox* [211] (see AL2.4 and AL2.5).

In more detail, although a black hole, formed by the gravitational collapse of a body in GR, is classically stable, quantum particle creation processes will result in the emission of Hawking radiation [159] to infinity and corresponding mass loss of the black hole, eventually resulting in the complete evaporation of the black hole. Semi-classical arguments, from applying the local evolutionary laws of quantum field theory in a classical curved spacetime, strongly suggest that in the process of black hole formation and evaporation, a pure quantum state will evolve to a mixed state [212]. That is, if the black hole itself has completely disappeared then only the thermal radiation is left, and this final state would be largely independent of the initial state and would thus not suffice to deduce the initial state and information would have been lost. But this behavior is quite different from that of familiar quantum systems under unitary time evolution. There are, however, a number of natural ways to attempt to restore unitarity, including QG corrections and additional degrees of freedom, in addition to a modification of quantum mechanics itself. The black hole information paradox is really a combination of two problems: the causality paradox and the entanglement problem. For illustration, consider a spherical shell of mass collapsing to form a black hole. In the semiclassical approximation the shell passes through its horizon, and ends at a singularity. (A) After the shell passes through its horizon, light cones in the region between the shell and the horizon point inwards. If we assume that faster than light propagation is not possible, then the information in the shell is causally trapped inside the horizon. Thus this information cannot escape to infinity as the hole evaporates away. (B) The process of Hawking radiation creates entangled pairs at the horizon. But the large entanglement between the radiation near infinity and the remaining hole near the endpoint of evaporation may be beyond the semi-classical approximation.

The various proposals to resolve the information paradox include the idea that quantum fields near black hole horizons are not in fact well-described by vacuum but are instead highly excited due to new unknown physics. The set of excitations is called a black hole "firewall," and might even be sufficiently strong that spacetime fails to exist in any recognizable sense in the interior of such black holes. This has also been discussed within the context of gauge/gravity duality. In particular, the AdS/CFT correspondence [162] partially resolves the black hole information paradox since it describes how a black hole (i.e., particles on the boundary of AdS spacetime) can evolve in a unitary fashion in a manner consistent with quantum mechanics, leading to information conservation in this context (see also [213]).

Other alternatives to information loss include: (i) A black hole never actually forms in the collapse, but rather some other structure without an event horizon, such as a "fuzzball", is formed. (ii) A black hole forms in the expected manner but there are major departures from semi-classical theory and there is greatly diminished entanglement during the evaporation process. (iii) The evaporation process shuts off by the time the black hole has evaporated down to the Planck scale when QG effects become dominant, such that the resulting remnant contains all of the information that went into the black hole. (iv) The evaporation process proceeds as in the semiclassical analysis until the black hole reaches the Planck scale, whence all of the information that had been stored within the black hole then emerges in a final burst.

In another older but still plausible suggestion [214] a phenomenological description of black holes and their event horizons was introduced based upon three postulates which, when implemented in a "stretched horizon" description of a black hole, preserve free infall "through the horizon" within the full quantum theory. The notion of black hole "complementarity" is then realised, whereby an observer outside of the black hole receives the information returned from the horizon to infinity (in the form of radiation emanating from the apparent horizon, which is presumed to be outside the event horizon for a dynamical shrinking black hole), but observers inside the black hole cannot communicate with the outside. Therefore, any possible contradictions might be acceptable since they are not visible to any single distant observer and consequently there would be no resulting tension with any known experiments.

5 Problems in Cosmology (Ph5)

Cosmology is the study of the large scale behaviour of the Universe within a theory of gravity, which is usually taken to be GR. There are many open problems in theoretical cosmology. For example, what precisely is the hypothetical inflaton field and what are the details of *cosmic inflation*? Is inflation self-sustaining through the amplification of quantum-mechanical fluctuations and thus still occurring in some (distant) places in the Universe? Does it give rise to countless "bubble universes" and, if so, under what initial conditions, and does a *multiverse* exist? Cosmological inflation is generally accepted as a solution to the *horizon problem*, that the universe appears more uniform on larger scales than expected, but are other explanations possible? What is the origin and future of the universe and, in particular, is the universe heading towards some sort of final singularity? Or is it evolving towards a big bounce or is it even part of an infinitely recurring cyclic model?

Since cosmology concerns the behaviour of the Universe when the small-scale structures such as stars and galaxies can be neglected, the "Cosmological Principle" (a generalization of the Copernican principle) is often assumed to hold, which asserts that: On large scales the Universe can be well-modeled by a solution to Einstein's equations which is spatially homogeneous and isotropic. That is, a (possibly preferred) notion of cosmological time can be picked such that at every instant on large scales space looks identical in all directions (isotropy), and (spatial homogeneity) it is not possible to distinguish between any two points (which is clearly not true on the astrophysical scales of galaxies). However, it would be more satisfactory if the cosmological principle could be derived as a consequence of GR (under suitable assumptions), rather than something assumed a priori. That is, could spatial homogenization and isotropization at late times be derived as a mathematical consequence of Einstein's equations under appropriate physical conditions and for suitable initial data. This question is partially addressed within the inflationary paradigm.

Dark matter and dark energy: Perhaps the most important questions in cosmology are those concerning dark matter and dark energy. These types of matter and energy are detected by their gravitational influences, but can't be observed directly. The estimated distribution of dark matter in the Universe is based on observed galaxy rotation curves, nucleosynthesis predictions and structure formation computations [215]. Although the identity of the missing dark matter is not yet known (e.g., whether it is a particle, perhaps the lightest superpartner, or whether the phenomena attributed to dark matter is not described by some form of matter but rather by an extension of GR), it is generally believed that this problem will be solved by conventional physics. The dark energy problem is much more serious. Indeed, this problem is widely regarded as one of the major obstacles to further progress in fundamental physics [216, 217].

The cosmological constant problem was discussed comprehensively by Steven Weinberg [218]. Standard quantum field theory (QFT) predicts a huge vacuum energy density from various sources. But the equivalence principle of GR requires that every form of energy gravitates in the same way, so that it is widely believed that the vacuum energy gravitates as a cosmological constant which would then have an enormous effect on the curvature of spacetime. However, the observed effective cosmological constant is so small compared with the predictions of QFT that an unknown bare cosmological constant has to cancel this huge contribution from the vacuum to better than up to at least 120 decimal places (AL2.1). It is an extremely difficult fine-tuning problem that gets even worse when the higher loop corrections are included [219]. More recently Weinberg and others have adopted the view that, of all of the proposed solutions to this problem, the only acceptable one is the controversial anthropic bound [220].

In addition, the Universe has been accelerating in its expansion for the last few billion years [221, 222]. Within standard cosmology the cause of this acceleration is commonly called *dark energy*, which appears to have the same properties as a relatively tiny cosmological constant, an effectively repulsive gravitational force (or *levitational force*) in GR. The additional cosmological constant (coincidence) problem of explaining why it has such a specific small observed value, which is the same order of magnitude as the present mass density of the matter in the Universe, must also be faced (AL2.2). It is often speculated as to whether dark energy is a pure cosmological constant or whether dynamical models such as, for example, quintessence and phantom energy models are more appropriate. Some physicists have also proposed alternative explanations for these gravitational influences, which do not require new forms of matter and energy, but these alternatives are not popular and lead to modified gravity on large scales. The possible cause of the observed acceleration of the Universe has also been discussed within the context of backreaction and inhomogeneities (see later).

Finally, it has also been proposed that an observed "dark flow", a nonspherically symmetric gravitational pull from outside the observable Universe, is responsible for some of the observed motion of large objects such as galactic clusters in the Universe. Analyses of the local bulk flow of galaxies (as measured in the frame of the cosmic microwave background) indicate a lack of convergence to the cosmic background frame even beyond 100 Mpc [223], in contrast to standard expectations if the Universe is in fact spatially homogeneous on larger scales. Indeed, low redshift supernova data have shown that there is an anomalously high and apparently constant bulk flow of approximately 250 km/s extending all the way out to the Shapely supercluster at approximately 260 Mpc and further, a discrepancy which has been confirmed by analysis of the 6dF galaxy redshift survey [224].

However, although mathematics is very important in many of these problems, they are not problems in mathematical physics per se. In addition, numerical computations have always played an important role in physical cosmology [225], but it is not clear that such calculations are within the remit of mathematical physics. Indeed, computational cosmology within full GR is now beginning to address fundamental issues [226, 227, 228]. Let us briefly discuss some topical examples.

Studies of "bubble universes" in which our Universe is one of many, nucleat-

ing and growing inside an ever-expanding false vacuum, have been undertaken with computational cosmological tools. In particular, [229] investigated the collisions between bubbles. It is expected that initial conditions will contain some measure of inhomogeneities, and random initial conditions will not necessarily give rise to an inflationary spacetime. It has been shown that large field inflation is robust to simple inhomogeneous (and anisotropic) initial conditions with large initial gradient energies in situations in which the field is initially confined to the part of the potential that supports inflation, while it is also known that small field inflation is much less robust to inhomogeneities than its large field counterpart [230].

Using exotic matter, or alternative modified theories of gravity, can classically lead to the initial singularity being replaced by a *bounce* to an expanding universe [231]. For example, computational cosmology methods have been applied to the study of bouncing cosmologies in the ekpyrotic cosmological scenario; by studying the evolution of adiabatic perturbations in a nonsingular bounce [232], it was shown that the bounce is disrupted in regions with significant spatial inhomogeneity and anisotropy compared with the background energy density, but is achieved in regions that are relatively spatially homogeneous and isotropic. The precise properties of a cosmic bounce depend upon the way in which it is generated, and many mechanisms have been proposed for this both classically and non-classically. There are quantum gravitational effects associated with string theory [233] and LQG [197, 193]. In particular, in LQC there is such a bounce when the energy density reaches a maximum value of approximately one half of the Planck density.

Nevertheless, some precise mathematical questions in cosmology can be formulated. For example, there are questions about the generality of inflation for generic initial data (although precise statements are difficult because there are many theories of inflation and there are no natural initial conditions). But mathematical theorems are possible in the study of the stability of de Sitter spacetime. This is part of the more general question of the *stability of cosmological solutions* (namely, if a cosmological solution is perturbed a little bit by, for example, factoring in the small-scale structure, is the evolution as governed by Einstein's equations qualitatively the same in the large as the evolution of the underlying cosmological solution). This requires the study of the (late time) behaviour of a complicated set of partial differential equations around a special solution (and there are several cosmological models that are of particular interest, including the very simple Milne model [72, 234]). These are genuinely problems in mathematical physics.

We first recall that when the cosmological constant vanishes and the matter satisfies the usual energy conditions, spacetimes of Bianchi type IX recollapse and so are never indefinitely expanding. This is formalized in the closed universe recollapse conjecture [235], which was proven by Lin and Wald [236]. However, spacetimes of Bianchi type IX need not always recollapse when there is a nonzero positive cosmological constant.

5.1 Stability of de Sitter spacetime

In [237] Friedrich proved (using regular conformal field equations) a result on the stability of de Sitter spacetime: all initial data (vacuum with positive cosmological constant) near enough (in a suitable Sobolev topology) to initial data induced by de Sitter spacetime on a regular Cauchy hypersurface have maximal Cauchy developments which are geodesically complete. de Sitter spacetime is thus an attractor for expanding cosmological models with a positive cosmological constant. The result also gives additional details on the asymptotic behaviour and may be thought of as proving a form of the so-called " cosmic no hair" conjecture in the vacuum case. For more recent work see [70] and references within.

A general theorem of Wald [238] states that any spatially homogeneous model whose matter content satisfies the strong and dominant energy conditions and which expands for an infinite proper time (i.e., does not recollapse) is asymptotic to an isotropic de Sitter spacetime. This cosmic no hair theorem does not depend on the details of the matter fields, and therefore the question remains as to whether solutions corresponding to initial data for the Einstein equations with a positive cosmological constant coupled to reasonable matter exist globally in time only under the condition that the model is originally expanding. It can be shown that this is true for various matter models using the techniques of [239].

Models with a scalar field with an exponential potential are also inflationary because the rate of (volume) expansion is increasing with time, and global results are possible [240, 241]. Inflationary behaviour also arises in the presence of a scalar field with a power law potential, but occurs at intermediate times rather than at late times. Local results are then possible but are difficult; primarily this question is studied numerically.

It is of interest to know what happens to the cosmic no-hair theorem in inhomogeneous spacetimes. Some partial results are possible for a positive cosmological constant in the inhomogeneous case [242]. But even less is known for scalar field models with an exponential potential [240].

Problem P12: Prove a cosmic no-hair theorem in generic inhomogeneous spacetimes.

The potential instability of de Sitter spacetime in quantized theories has been investigated. In a semi-classical analysis of backreaction in an expanding Universe with a conformally coupled scalar field and vacuum energy it was shown that a local observer perceives de Sitter spacetime to contain a constant thermal energy density despite the dilution from expansion due to a continuous flux of energy radiated from the horizon, leading to the evolution of the Hubble rate at late times which deviates significantly from that in de Sitter spacetime, which is thus unstable [243]. This result is in apparent disagreement with the thermodynamic arguments in [244] in which it was concluded that unlike black holes de Sitter spacetime is stable. However, if de Sitter spacetime were unstable to quantum corrections and could indeed decay, it could provide an important mechanism for alleviating the cosmological constant problem and perhaps also the fine-tuning issues encountered in the extremely flat inflationary potentials that are required by observations. A de Sitter instability would certainly have a profound impact on the fate of the Universe since it rules out the possibility of an eternally exponentially expanding de Sitter spacetime as classically implied by the standard concordance model. This issue is currently unresolved.

5.2 Cosmological singularities and spikes

The singularity theorems tell us that singularities occur under very general circumstances in GR, but they say little about their nature [94]. Belinskii, Khalatnikov and Lifshitz (BKL) [245] have conjectured that within GR, for a generic inhomogeneous cosmology, the approach to the (past) spacelike singularity is vacuum dominated, local, and oscillatory (mixmaster). The associated dynamics is referred to as the BKL dynamics. In particular, due to the non-linearity of the Einstein field equations, if the matter is not a (massless) scalar field, then sufficiently close to the singularity one can *neglect all matter terms* in the field equations relative to the dynamical anisotropy. BKL checked that their assumptions were consistent with the Einstein field equations; but that doesn't necessarily mean that those assumptions hold in general physical situations. Recent numerical simulations have verified that the BKL conjecture is satisfied for special classes of spacetimes [246, 247].

To date there have been three main approaches to investigate the structure of generic singularities, including the heuristic BKL metric approach and the Hamiltonian approach. The dynamical systems approach [248], in which Einstein's field equations are recast into scale invariant asymptotically regularized dynamical systems (first order systems of autonomous ordinary differential equations and partial differential equations) in the approach towards a generic spacelike singularities. In particular, a dynamical systems formulation for the Einstein field equations without any symmetries was introduced in [249], resulting in a detailed description of the generic attractor, concisely formulated conjectures about the asymptotic dynamic behavior toward a generic spacelike singularity, and a basis for a numerical investigation of generic singularities [250].

In more detail, in order to construct the solution in a sufficiently small spacetime neighborhood of a generic spacelike singularity [251, 249] Einstein's field equations are reformulated by assuming that a small neighborhood near the singularity can be foliated with a family of spacelike surfaces such that the singularity "occurs" simultaneously, and the expansion of the normal congruence to the assumed foliation are factored out by utilizing a conformal transformation (whence the Einstein field equations split into decoupled equations for the conformal factor and a coupled system of dimensionless equations for quantities associated with the dimensionless conformal metric). Unfortunately, until recently very few rigorous mathematical statements had been made. Based on the work of Rendall [252], Ringstrom produced the first major proofs about asymptotic spatially homogeneous Bianchi type IX cosmological dynamics [253]. Notably, Ringstrom obtained the first theorems about oscillatory behavior of generic singularities for Bianchi type VIII and, more substantially, type IX models in GR. In particular, Ringstrom managed to prove that the past attractor in Bianchi type IX resides on a subset that consists of the union of the Bianchi type I and II vacuum subsets. But this theorem does not identify the attractor completely, nor determine if the Kasner map is relevant for dynamics asymptotic to the initial singularity in Bianchi type IX, and the theorem says very little about Bianchi type VIII [251] (however, see the recent work of Brehm [254]).

In the spatially homogeneous case the focus is on mathematically rigorous results. For example, it has been argued that the idea that Bianchi type IX models are essentially understood is a misconception, and what has actually been proven about type IX asymptotic dynamics was addressed in [251] (however, see [254]). In particular, all claims about chaos in Einstein's equations (especially at a generic spacelike singularity) rest on the (plausible) belief that the Kasner map (which is associated with chaotic properties) actually describes the asymptotic dynamics of Einstein's equations; however, this has not been proved rigorously to date [251]. Most importantly, the role of type IX models in the context of generic singularities not yet been rigorously established [255, 249, 251]. There remain several important open problems, including:

Problem P13: Prove that the past attractor of the Bianchi type IX dynamical system coincides with the Mixmaster attractor (as defined in [251] – the Bianchi II variety of [253]) rather than being a subset thereof.

The BKL oscillatory dynamics have been studied in simple perfect fluid models with a linear equation of state. Some matter fields can have an important effect on the dynamics near the singularity. A scalar field or stiff fluid leads to the oscillatory behaviour being replaced by monotonic behaviour and consequently to a significant reduction in the complexity of the dynamics close to the singularity [256]. Based on numerical work and the qualitative analysis of [257], the so-called exceptional Bianchi type VI_{$-\frac{1}{9}$} class B model (which has the same number of degrees of freedom as the most general Bianchi type VIII and IX class A models) has an oscillatory singularity. An electromagnetic field can lead to oscillatory behaviour which is not present in vacuum or perfect fluid models of the same symmetry type. For example, models of Bianchi types I and VI₀ with an electromagnetic field have oscillatory behaviour [258]. Oscillations can also occur in all Bianchi models in the presence of a tilting fluid [259, 260]. It is imperative to discuss generic oscillatory singularities in inhomogeneous cosmologies. In [261] qualitative and numerical support was presented for the BKL scenario in the Hubble-normalized state space context for an open set of time lines. In more generality, a heuristic physical justification of asymptotic locality may be that ultra strong gravity causes particle horizons shrink to zero size toward the singularity along each timeline, which prohibits communication between different time lines in the asymptotic limit (and may hence be referred to as asymptotic silence).

To gain further insights about general oscillatory singularities in inhomogeneous spacetimes, models with two commuting spacelike Killing vectors (socalled G_2 models) have been investigated. The BKL dynamics has been discussed in generic vacuum, spatially compact $U(1) \times U(1)$ -symmetric spacetimes with vanishing twist and in generic polarized U(1) spacetimes [61], and in twisting $U(1) \times U(1)$ -symmetric vacuum models on T^3 Gowdy models, and on $S^2 \times S^1$, S^3 and lens spaces L(p, q) [262].

The description of generic asymptotic dynamics towards a generic spacelike singularity in terms of an attractor, has resulted in mathematically precise conjectures [255, 249], and involves the possible existence of finite dimensional attractors in infinite dimensional systems [263].

Problem P14: Prove the BKL locality conjecture in the general inhomogeneous context.

Spikes: Recently, a new spike phenomenon that had not been anticipated by BKL was found in numerical simulations [246]. Since it is a general feature of solutions of partial differential equations that spikes occur it is, of course, expected that they occur in solutions of Einstein's field equations in GR. In the case of spikes, the spatial derivative terms do have a significant effect at special points. In particular, in the approach to the singularity in the mixmaster regime, a spike occurs when a particular spatial point is stuck in an old Kasner epoch while its neighbours eventually bounce to the new one. Because spikes become arbitrarily narrow as the singularity is approached, they are a challenge to the numerical simulations. Spikes are also a challenge to the mathematical treatment of spacetimes. Mathematical justification has been presented in [264]. More success has been obtained in finding exact spike solutions [265].

Spikes were originally found numerically in the context of vacuum orthogonally transitive, spatially inhomogeneous G_2 models [246, 266]. Therefore, numerical studies of G_2 and more general cosmological models have produced evidence that the BKL conjecture generally holds except possibly at isolated points (surfaces in the three-dimensional space) where spiky structures ("spikes") form [267], and the asymptotic locality part of the BKL conjecture is violated. Spikes naturally occur in a class of non-vacuum G_2 models and, due to gravitational instability, leave small residual imprints on matter in the form of matter perturbations. Particular interest has been paid to spikes formed in the initial oscillatory regime, and their imprint on matter and structure formation has been studied numerically [268].

Therefore, generic singularities are not only associated with asymptotic locality but also with non-local recurring spikes, although it is believed that a set of measure zero of timelines exhibit spike formation [250].

Problem P15: Prove the existence of spikes and determine their effect on any eventual generic singularity proofs.

There are other unresolved questions pertaining to recurring spike behavior and generic spacelike singularities. For example, are there spikes that undergo infinitely many recurring spike transitions? How, where and how often do spikes form? How does spike interference work and can spikes annihilate? Are there generic singularities without recurring spikes and are there generic singularities with a dense set of recurring spikes? Some of these issues have been discussed recently in [269].

The asymptotic dynamics of general solutions of the Einstein vacuum equations toward a generic spacelike singularity have been studied. Matter sources such as spatially homogeneous perfect fluids and simple massless scalar fields [245, 248] have been considered. It is of particular interest to determine the structural stability of generic inhomogeneous spacelike singularities, especially by including matter such as massless scalar fields and electromagnetic fields (which influence the generic spacelike singularity in different ways), and to also go beyond GR and include form fields.

Problem P16: Determine the structural stability of generic inhomogeneous spacelike singularities for general matter fields present in the early universe.

In [270], a heuristic Hamiltonian approach (closely connected to that of BKL) was used to study the dynamics of the Einstein-dilaton-p-form system in the neighborhood of a generic spacelike singularity. The asymptotic behavior of the fields was described by a billiard motion in a region of hyperbolic space bounded by straight walls (dubbed "cosmological billiards"), and a remarkable connection between the asymptotic dynamics of generic spacelike singularities and Kac-Moody algebras was revealed [270]. A link between the Hamiltonian and the dynamical systems approach to inhomogeneous cosmologies was established in [255]. More recently the fermionic sector of supergravity theories, in which the gravitino is treated classically, was studied [271]. The quantum generalization of the resulting fermionic cosmological billiards, defined by the dynamics of a quantized supersymmetric Bianchi type IX cosmological model (within simple 4D supergravity) [272], was also investigated. The hidden Kac-Moody structures were again displayed.

Isotropic singularity: Based on entropy considerations, Penrose [26] proposed the "Weyl curvature hypothesis" that asserts that the initial singularity in a cosmological model should be such that the Weyl curvature tensor tends to zero or at least remains bounded. There is some difficulty in representing this condition mathematically and it was proposed in [273] that a clearly formulated geometric condition, which on an intuitive level is closely related to the original condition, is that the conformal structure should remain regular at the singularity. Singularities of this type are known as conformal or isotropic singularities. It has been shown [274, 275] that solutions of the Einstein equations coupled to a perfect fluid satisfying the radiation equation of state with an isotropic singularity are determined uniquely by certain free data given at the singularity. The data which can be given is, roughly speaking, half as much as in the case of a regular Cauchy hypersurface. In [276] this was extended to the linear equation of state case, and can be extended to more general matter (e.g., general fluids and a collisionless gas of massless particles) [60].

Many additional questions can be asked in the context of alternative, modified theories of gravity such as, for example, the general applicability of the BKL behaviour close to the cosmological singularity. Such questions will not be included here. However, the following mathematical physics question on isotropization is relevant.

Problem P17: Are isotropic singularities typical in modified theories of gravity.

The stability of the isotropic vacuum Friedmann universe on approach to an initial cosmological singularity in gravity theories with higher–order curvature terms added to the Einstein-Hilbert Lagrangian of GR have been studied [277]. A special isotropic vacuum solution exists which behaves like the radiation-dominated Friedmann universe and is stable to anisotropic and small inhomogeneous perturbations in the past, unlike the situation in GR. An analytical solution valid for particular values of the equation of state parameter was also found such that the singularity is isotropic in a higher dimensional flat anisotropic Universe filled by a perfect fluid in Gauss-Bonnet gravity [278]. Some simple cosmological solutions of gravity theories with quadratic Ricci curvature terms added to the Einstein-Hilbert Lagrangian have also be studied [279].

5.3 Averaging Einstein's field equations

The averaging problem in GR is of fundamental importance. The gravitational field equations on large scales are obtained by averaging or coarse graining the Einstein field equations of GR. The averaging problem in cosmology is crucial for the correct interpretation of cosmological data. The so-called fitting problem is perhaps the most important unsolved problem in mathematical cosmology [280].

The spacetime or space volume averaging approach must be well posed and generally covariant. This raises important new questions in differential geometry. The formal mathematical issues of averaging tensors on a differential manifold have recently been revisited [281, 282, 283]. The coarse grained or averaged field equations need not take on the same mathematical form as the original field equations. Indeed, in the case of the macroscopic gravity approach [283, 284] the averaged spacetime is not necessarily even Riemannian. Scalar quantities can be averaged in a straightforward manner. In general, a spacetime is completely characterized by its scalar curvature invariants, and this suggests a particular spacetime averaging scheme based entirely on scalars [282]. In the approach of Buchert [285] a 1+3 cosmological spacetime splitting is employed and only scalar quantities are averaged.

The spacetime averaging procedure adopted in macroscopic gravity, which is fully covariant and gauge independent, is based on the concept of Lie-dragging of averaging regions, and it has been shown to exist on an arbitrary Riemannian spacetime with well-defined local averaged properties (however, see [281]). The averaging of the structure equations for the geometry of GR then produces the structure equations for the averaged (macroscopic) geometry and gives a prescription for the (additional) correlation functions (in the macroscopic field equations) which emerge in an averaging of the non-linear field equations [283]. **Problem PF2** Provide a rigorous mathematical definition for averaging in GR.

Although the standard spatially homogeneous and isotropic Friedmann– Lemaitre–Robertson–Walker (FLRW) model (or so-called Λ CDM cosmology) has been extremely successful in describing current observations (up to various possible anomalies and tensions [286], and particularly the existence of structures on gigaparsec scales such as the cold spot and some super-voids [287]), it requires sources of dark energy density that dominate the present Universe that have never been directly detected. In addition, the Universe is not isotropic or spatially homogeneous on local scales. An averaging of inhomogeneous spacetimes on large scales can lead to important effects. For example, on cosmological scales the dynamical behavior can differ from that in the standard FLRW model and, in particular, the expansion rate may be significantly affected [285]. Indeed, current observations of the structure of the late epoch Universe reveal a significantly complex picture in which groups and clusters of galaxies of various sizes form the largest gravitationally bound structures, which themselves form knots, filaments and sheets that thread and surround very underdense voids, creating a vast cosmic web [288]. A significant fraction of the volume of the present Universe is in voids of a single characteristic diameter of approximately 30 megaparsecs [289] and a density contrast which is close to being empty, so that by volume the present universe is void–dominated [290].

A hierarchy of steps in coarse graining is necessary to model the observed complex gravitationally bound structures on large scales [291]. In the standard FLRW cosmology it is implicitly assumed that regardless of the gravitational physics in the coarse graining hierarchy, at the final step the matter distribution can be approximated by an "effective averaged out" stress-energy tensor. However, the smallest scale on which a notion of statistical homogeneity arises is 70–120 megaparsecs [292], based on the two-point galaxy correlation function, and variations of the number density of galaxies of order 7–8% are still seen when sampling on the largest possible survey volumes [293, 294].

Problem PF3 Can averaging play an important role in cosmology. In particular, what is the largest scale that we can coarse-grain matter and geometry that obeys Einstein's equations on smaller scales such that the average evolution is still an exact solution of Einstein's equations.

After coarse graining we obtain a smoothed out macroscopic geometry (with macroscopic metric) and macroscopic matter fields, valid on larger scales. Indeed, the averaging of the Einstein field equations for local inhomogeneities on small scales can in general lead to very significant dynamical *backreaction* effects [295] on the average evolution of the Universe [285]. In addition, all deductions about cosmology are based on light paths (null geodesics) that traverse great distances (which preferentially travel through underdense regions – the voids in the real Universe). However, inhomogeneities affect curved null geodesics and can significantly alter observed luminosity distances. This leads to the following fundamental problem: although photons follow null geodesics in the local geometry, what trajectories do photons follow in the averaged macro-geometry [296]. More importantly, however, is the fact that averaging (and inhomogeneities in general) can affect the interpretation of cosmological data [297, 298, 296].

A topical but theoretically conservative approach is to treat GR as a *meso-scopic* (classical) theory applicable on those small scales on which it has actually been tested, with a local metric field (the geometry) and matter fields, whence the effective dynamical equations on cosmological scales are then obtained by averaging. In this approach, backreaction effects might offer a resolution to problems related to dark energy and dark matter.

6 Summary of problems in mathematical physics

The hardest part, perhaps, is making the number of open questions in mathematical physics add up to 42. First, we must decide whether we mean 42 in the mathematical sense (i.e., the exact number 42), or in the physics sense (i.e., a number between 40 and 44).

There are the 5 classical problems, H6, S3, S8, S15 and M2, and the related problems BS1 and BS3 (and the problem of turbulence) and the more specific problems in YM theory and their generalizations to EYM theory (e.g., Y117 and Y118). In addition, there are the problems BS2, BS8 and BS14. Most of the problems proposed by Bartnik and Penrose have been subsumed in the open problems P1 – P17. However, problems RB20, RB21, RB32 (Y115) and RB43 remain. There are 7, 4 and 6 open problems in each of GR, the quantum realm and cosmology, respectively, in the list P1 – P17. I have also listed 3 personal favorites (PF1 – PF3).

As mentioned earlier, many of the most important problems in theoretical physics are generally not problems in mathematical physics, despite the fact that a lot of mathematics is often utilized (as discussed earlier, for example, in string theory). Some of the problems which are absolutely fundamental for theoretical physics, and which almost by definition are vague and not yet well formulated, have been briefly discussed in the text. But they may (or may not) turn into bone fide problems in mathematics or mathematical physics in the future (**MPF**). For example, although important, the question to explain the anthropic reasons for the fine tuning of our Universe is not likely to to lead to an explicit problem in mathematical physics. The following important problem may well lead to a problem in mathematical physics.

Problem MPF1: Resolve the black hole information paradox.

The two most fundamental problems in theoretical physics will likely lead to problems in mathematical physics in the future (see AL1/AL2 and AL3).

Problem MPF2: The cosmological constant problem and dark energy.

Problem MPF3: Formulate a fully consistent theory of QG.

Numerical computations have always played an important role in any mature area of theoretical physics (such as GR and more recently in computational cosmology [225]). But it is not clear that, in general, such numerical problems are really problems in mathematical physics. In addition, numerical problems typically also require the "complete control" on the behaviour of gravitation in the very non-linear regime. This always concerns the technical and practical question of the "cost" due to the length of the required computation and the small numerical error necessary to ensure the solution can be trusted, which is not really a problem in mathematical physics.

There are perhaps no important open questions in numerical relativity per se. On the other hand, any important problem within GR that involves nonlinear phenomena would be an important problem for numerical relativity. Numerical work supports many of the conjectures discussed in this paper and has led to many important theoretical advances. For example, the mathematical stability of AdS spacetime has an important numerical component and cosmic censorship is supported by numerical experimentation. In addition, we have discussed the role of numerics in the understanding of spikes and the BKL dynamics, and in problems in cosmology and higher dimensional gravity. In particular, we discussed the problems of the generality of bouncing models (at a cosmological singularity) and of inflation for generic initial data. The latter problem may well lead to a genuine computationally motivated problem in mathematical physics (**CMP**), at least within a specific physical realization of inflation.

Problem CMP1: What is the generality of inflation for generic initial data.

There are also important numerical problems in relativistic astrophysics, such as in the ultra-relativistic regime of interactions, infinite-boost black hole collisions and colliding plane-fronted waves, and most importantly in black hole mergers in general. As noted earlier, the two-body problem has played and continues to play a pivotal role in gravitational physics [58]. Recent advances in numerical computations have enabled the study of the violent inspiral and merger of two compact objects (such as, for example, black holes and neutron stars), in which an enormous amount of gravitational radiation is produced.

In particular, the detection and analysis of the gravitational-wave signals generated by black hole collisions necessitate very precise theoretical predictions for use as template waveforms to be cross-correlated against the output of the gravitational-wave detectors, which is of great importance in light of recent LIGO observations [299]. The orbital dynamics and gravitational-wave emission of such systems can be modelled using a variety of analytical approximation schemes, including post Newtonian expansions, black hole perturbation theory and the effective one body approach, and this is complemented by numerical relativity near the late time coalescence where perturbative methods break down [58, 300]. **Problem CMP2:** Determine the predictions of emitted waveforms for binary black hole systems for optimal detection and parameter extraction.

Non-vacuum compact binary systems involving at least one neutron star also produce copious amounts of gravitational waves and are likely to lead to intense neutrino and electromagnetic emission that could also be detected. However, the simulation of binaries with neutron stars is complicated by the need to include nongravitational physics, and hence analytical techniques are less effective [300].

Finally, some numerical results have lead to the formulation of new problems in mathematical physics, some of which have been discussed earlier. In particular, critical phenomena in gravitational collapse within GR was discovered numerically [300]. Families of solutions to the coupled Einstein-matter equations, labeled by a continuous parameter p, were studied. The prescribed initial data depends on p, which controls the strength of the (initially imploding) matter in the ensuing gravitational interaction. For a small p, gravity is weak during the evolution and the spacetime remains regular everywhere (for example, in the case of massless radiation, the radiation will disperse to infinity). For a large p, gravity becomes sufficiently strong that some of the matter is trapped within a singular black hole. For some critical value of p, there is a "critical" (self similar) solution corresponding to the threshold of black hole formation. Evidence to date suggests that virtually any collapse model that admits black hole formation will contain such critical behaviour. Understanding these critical solutions and the ensuing critical behaviour is now an interesting problem within mathematical GR (especially in the case in which there are no symmetries).

Problem CMP3: Understand critical phenomena in gravitational collapse in GR.

In summary, and in the spirit of **AL42** [1], the final 42 open problems in mathematical physics are: H6, S3, S8, S15 and M2, BS1 and BS3, Y117 and Y118, BS2, BS8 and BS14, RB20, RB21, RB32 and RB43, and the problems P1 - 17, PF1 - 3, MPF1 - 3, and CMP1 - 3.

All of these problems are explicitly stated in the text or in the Appendix. In addition, there are many other open problems referred to in this paper.

7 Appendix: Lists

7.1 Hilbert's problems

The remaining problems are [4]:

- H1 The continuum hypothesis (that is, there is no set whose cardinality is strictly between that of the integers and that of the real numbers).
- H2 Prove that the axioms of arithmetic are consistent.
- H3 Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces that can be reassembled to yield the second?
- H4 Construct all metrics where lines are geodesics.
- H5 Are continuous groups automatically differential groups?
- H7 Is a^b transcendental, for algebraic $a \neq 0, 1$ and irrational algebraic b?
- H9 Find the most general law of the reciprocity theorem in any algebraic number field.
- H10 Find an algorithm to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.
- H11 Solving quadratic forms with algebraic numerical coefficients.
- H13 Solving 7th degree equations using algebraic functions of two parameters.
- H14 Is the ring of invariants of an algebraic group acting on a polynomial ring always finitely generated?
- H15 Rigorous foundation of Schubert's enumerative calculus.
- H17 Express a nonnegative rational function as quotient of sums of squares.
- H18 (a) Is there a polyhedron that admits only an anisohedral tiling in three dimensions? (b) What is the densest sphere packing?
- H19 Are the solutions of regular problems in the calculus of variations always necessarily analytic?
- H20 Do all variational problems with certain boundary conditions have solutions?
- H21 Proof of the existence of linear differential equations having a prescribed monodromic group.
- H22 Uniformization of analytic relations by means of automorphic functions.
- H23 Further development of the calculus of variations.

7.2 Smale's problems

Smale's solved (partially or fully) problems are [8]:

- S6 Finiteness of the number of relative equilibria in celestial mechanics.
- S7 Distribution of points on the 2-sphere.
- S11 Is one-dimensional complex-variable dynamics generally hyperbolic?
- S12 Centralizers of diffeomorphisms.
- S14 Lorenz attractor.
- S17 Solving polynomial equations in polynomial time in the average case.

7.3 AL42 problems

The problems are [1]:

- AL1 Why does conventional physics predict a cosmological constant that is vastly too large?
- AL2 What is the dark energy?
- AL3 How can Einstein gravity be reconciled with quantum mechanics?
- AL4 What is the origin of the entropy and temperature of black holes?
- AL5 Is information lost in a black hole?
- AL6 Did the universe pass through a period of inflation, and if so how and why?
- AL7 Why does matter still exist?
- AL8 What is the dark matter?
- AL9 Why are the particles of ordinary matter copied twice at higher energy?
- AL10 What is the origin of particle masses, and what kind of masses do neutrinos have?
- AL11 Does supersymmetry exist, and why are the energies of observed particles so small compared to the most fundamental (Planck) energy scale?
- AL12 What is the fundamental grand unified theory of forces, and why?
- AL13 Are Einstein's GR and standard field theory always valid?
- AL14 Is our universe stable?
- AL15 Are quarks always confined inside the particles that they compose?
- AL16 What are the complete phase diagrams for systems with nontrivial forces, such as the strong nuclear force?
- AL17 What new particles remain to be discovered?
- AL18 What new astrophysical objects are awaiting discovery?
- AL19 What new forms of superconductivity and superfluidity remain to be discovered?
- AL20 What further properties remain to be discovered in highly correlated electronic materials?

- AL21 What new topological phases remain to be discovered?
- AL22 What other new phases and forms of matter remain to be discovered?
- AL23 What is the future of quantum computing, quantum information, and other applications of entanglement?
- AL24 What is the future of quantum optics and photonics?
- AL25 Are there higher dimensions, and if there is an internal space, what is its geometry?
- AL26 Is there a multiverse?
- AL27 Are there exotic features in the geometry of spacetime, perhaps including those which could permit time travel?
- AL28 How did the universe originate, and what is its fate?
- AL29 What is the origin of spacetime, why is spacetime four-dimensional, and why is time different from space?
- AL30 What explains relativity and Einstein gravity?
- AL31 Why do all forces have the form of gauge theories?
- AL32 Why is Nature described by quantum fields?
- AL33 Is physics mathematically consistent?
- AL34 What is the connection between the formalism of physics and the reality of human experience?
- AL35 What are the ultimate limits to theoretical, computational, experimental, and observational techniques?
- AL36 What are the ultimate limits of chemistry, applied physics, and technology?
- AL37 What is life?
- AL38 How did life on Earth begin and how did complex life originate?
- AL39 How abundant is life in the universe, and what is the destiny of life?
- AL40 How does life solve problems of seemingly impossible complexity?
- AL41 Can we understand and cure the diseases that afflict life?
- AL42 What is consciousness?

7.4 Simon's problems

The remaining BS problems are [18]:

- BS4 Transport theory: At some level, the fundamental difficulty of transport theory is that it is a steady state rather than equilibrium problem, so that the powerful formalism of equilibrium statistical mechanics is unavailable. A: Fourier's heat law. B: Kubo formula.
- BS5 Heisenberg models: Lattice models of statistical mechanics (especially the Ising model) have been fruitful testing grounds for ideas in the theory of phase transitions. Four particular questions were postulated, including the proof of the Griffiths, Kelly and Sherman inequality for classical Heisenberg models.
- BS6 Existence of ferromagnetism.
- BS7 Existence of continuum phase transitions.
- BS9A/B Asymptotic completeness for short range N-body quantum systems and for Coulomb potentials.
- BS10 Quantum potential theory: Basic to atomic and molecular physics is the binding energy of a quantum mechanical system of electrons interacting with one or more nuclei. Five particular questions were posed.
- BS11 Existence of crystals: Most materials occur in a crystalline state at low temperatures. Prove the existence of crystals for ensembles of quantum mechanical atoms (even at zero temperatures) for infinite nuclear masses with an integer nuclear charge.
- BS12 Five questions on random and almost periodic potentials.
- BS13 Critical exponent for self-avoiding walks.

The two problems in Yau [45] referred to earlier are:

- Y117 Prove that any YM field on S^4 is either self-dual or antiself-dual.
- Y118 Prove that the moduli space of the self-dual fields on S^4 with a fixed Pontryagin number is connected.

7.5 Penrose's problems

The fourteen unsolved problems in classical GR by Penrose (p631 in [45]) are:

- RP1&2 Find a suitable quasi-local definition of energy-momentum in general relativity. And the more ambitious: Find a suitable quasi-local definition of angular momentum in general relativity.
- RP3&4 Find an "asymptotically simple" (essentially a spacetime in which every light ray escapes, both in past and future directions, to an asymptotically flat region) Ricci-flat spacetime which is not flat, or at least prove that such spacetimes exist. And the related problem: Are there restrictions on k for non-stationary "k-asymptotically simple" spacetimes, with non-zero mass, which are vacuum near null infinity.
- RP5 Find conditions on the Ricci tensor (e.g., satisfying the null convergence condition and the related physically reasonable weak energy condition) which ensure that the generators of past and future null infinity (i.e., the null geodesic curves lying on these curves constituting a fibration of null infinity) are infinitely long.
- RP6–8 Assuming appropriate energy conditions hold, show that if a "cut" C (a general cross section) of future (or past) null infinity can be spanned by a spacelike hypersurface, then the so-called Bondi-Sachs mass defined at C is non-negative. Does the Bondi-Sachs mass defined on cuts of future null infinity have a well-defined limit as the cuts recede into the past along this limit agreeing with the mass defined at spacelike infinity? Show that if the spacetime is assumed not to be flat everywhere in the region of an appropriate spacelike hypersurface, then the Bondi-Sachs energy-momentum, and also the energy-momentum defined at spacelike infinity, are future-timelike.
- RP9&10 In an asymptotically simple spacetime which is vacuum near null infinity and for which outgoing radiation is present and falls off suitably near spacelike and future-timelike infinities, is it necessarily the case that spacelike and future-timelike infinities are nontrivially related? This leads to: Find a good definition of angular momentum for asymptotically simple spacetimes.
- RP11 If there is no incoming nor outgoing radiation and the spacetime manifold is vacuum near future infinity (and, in some suitable sense, near spacelike infinities) is the manifold necessarily stationary near null infinity.
- RP12 Is cosmic censorship valid in classical general relativity?
- RP13 Let S be a spacelike hypersurface which is compact with boundary, the boundary consisting of a cut C of future null infinity together with a trapped surface (the horizon of the black hole). Then, assuming that the dominant energy condition is satisfied, show that there is a lower bound on the ADM mass [59] in terms of the area of S.

• RP14 Show that there is no vacuum equilibrium configuration involving more than one black hole.

7.6 Bartnik's problems

The problems are [52]:

- RB1 Given asymptotically flat initial data and a (future) trapped surface, prove the existence of (smooth, spherical) apparent horizons.
- RB2 Is there an analogy between the behaviour of minimal surfaces and the behaviour of apparent horizons.
- RB3 Prove that there is an asymptotically flat vacuum initial data set, diffeomorphic to R^3 , which contains an apparent horizon.
- RB4 Determine whether an asymptotically flat metric on R^3 with zero scalar curvature can admit a minimal 2-sphere (this is the restriction of RB3 to time-symmetric initial data).
- RB5 Give an explicit example of an apparent horizon that does not persist under the Einstein evolution.
- RB6 Find conditions on conformally flat, asymptotically flat metrics with non-negative scalar curvature which ensure that the manifold has no horizon.
- RB7 Prove the Penrose inequality.
- RB8 Determine conditions on the initial data for a compact manifold with non-constant mean curvature which ensure the Einstein equation is solvable.
- RB9 Describe suitable asymptotic conditions which enable the conformal method to be used to construct initial data sets on an asymptotically hyperbolic manifold.
- RB10 Characterise those hyperboloidal initial data arising from a spacetime with a smooth conformal null infinity.
- RB11 Classify the various kinds of smoothness properties which hyperboloidal initial data may have at conformal infinity.
- RB12 Can the space of globally hyperbolic, vacuum Einstein metrics on $R \times M^3$ have more than one connected component?
- RB13 What conditions on the stress-energy tensor are needed to show that a static, asymptotically flat metric is necessarily spherically symmetric?
- RB14 Find a local invariant characterisation of the Kerr solution amongst stationary metrics.

- RB15 If two disjoint Cauchy surfaces in an asymptotically flat (vacuum) spacetime are isometric, show that the spacetime is stationary.
- RB16 Characterize the possible types of singularities which may occur for solutions to the static and stationary vacuum Einstein equations.
- RB17 Several questions on the spherically symmetric Einstein-Yang-Mills equations.
- RB18 Prove an "approximate solution" result for the (vacuum) Einstein equations in some suitable norm that would provide a good way to evaluate approximate/asymptotic and numerical solutions.
- RB19 Show that a solution of the linearised (about Minkowski space) Einstein equations is close to a (non-flat) exact solution.
- RB20 Determine the range of validity of the post-Newtonian and post-Minkowskian asymptotic expansions.
- RB21 Prove rigorously the existence of a limit in which solutions of the Einstein equations reduce to Newtonian spacetimes.
- RB22 Prove the quadrupole radiation formula.
- RB23 Show that test particles move on spacetime geodesics.
- RB24 In what sense does a Regge spacetime (i.e., a piecewise linear manifold with piecewise linear metric [301]), and generally spacetimes constructed by numerical relativity, approximate a smooth vacuum spacetime?
- RB25 A problem encountered in numerical relativity is that of ensuring that the constraint equations are preserved by the evolution [302].
- RB26 Prove a uniqueness theorem for maximal surfaces, assuming only the dominant energy condition.
- RB27 Rigorously demonstrate the existence of a constant mean curvature hypersurface asymptotic to a given cut of future null infinity in an asymptotically flat spacetime.
- RB28 Show that there is a maximal Cauchy hypersurface of an asymptotically flat spacetime having a Cauchy surface without horizons.
- RB29 Show that a maximal surface in a "boost-type domain" is necessarily asymptotically flat and must coincide with the maximal slices.
- RB30 Is there a timelike geodesically complete inextendible Lorentz manifold satisfying an energy condition and having a partial Cauchy surface which contains a trapped surface?

- RB31 Show that a weak Cauchy surface in a globally hyperbolic spacetime satisfying suitable energy conditions cannot contain an inextendible null geodesic.
- RB32 Prove that a "cosmological spacetime" satisfying the timelike convergence condition is either timelike geodesically incomplete or it splits as $R \times M^3$ isometrically (and is thus static).
- RB33 Prove a singularity theorem assuming the dominant energy condition rather than the timelike convergence condition.
- RB34 Determine the weakest condition on the smoothness of the metric in the initial value problem for maximising geodesics to have a unique solution.
- RB35 Prove a long time existence theorem for the vacuum asymptotically flat Einstein equations in the maximal slicing gauge [114].
- RB36 Determine conditions on asymptotically flat initial data which ensure that the null infinity of the resulting solution of the initial value problem is sufficiently regular that the Penrose extended manifold exists [303].
- RB37 Show the existence of (and construct an exact solution to) the Einstein vacuum equations with positive mass which has complete smooth null infinity and regular timelike infinity.
- RB38 Find the weakest possible regularity conditions for a metric to satisfy the (distributional) vacuum Einstein equations [86].
- RB39 What are the regularity conditions for the vacuum Einstein initial value problem for geodesics which guarantee the existence of a solution, but not uniqueness.
- RB40 Demonstrate the long-time existence of so-called crushing singularities in the constant mean curvature slicing gauge for the cosmological vacuum spacetime.
- RB41 Prove that every globally hyperbolic, maximally extended spacetime solution of the Einstein or Einstein-Maxwell equations on $R \times S^3$ contains a maximal hypersurface (and thus also both a big bang and big crunch). Prove this result for special cases such as the spatially homogeneous Bianchi type IX solutions.
- RB42 Show that, in an appropriate sense, the set of spacetime metrics which are smoothly (distributionally?) extendible across compact Cauchy horizons are of "measure zero" in the set of all spacetimes.
- RB43 Find an exact solution of the Einstein equations which represents two orbiting bodies. Is the 2-body system unstable in Einstein gravity?
- RB44 Prove the Belinskii, Khalatnikov and Lifshitz conjecture.
- RB45 Show that the only solution of the vacuum (or Einstein-Maxwell) Robinson-Trautman equations on $S^2 \times R$ with positive mass is the Schwarzschild metric.
- RB46 Show that a perturbation of the Schwarzschild (and Kerr) solution decays exponentially (so that the solutions are thus stable).
- RB47 Show that a cosmological spacetime with constant mean curvature initial data having positive Ricci 3-curvature has an evolution which preserves Ricci positivity.
- RB48 Find a sensible notion of quasi-local mass that can be used in non-trivial black hole theorems.
- RB49 Show that the set of asymptotically flat 3-manifolds which satisfy the conditions of the positive mass theorem has some weak compactness property (and what regularity might be expected in the limit manifold?)
- RB50 Prove the static metric conjecture [303].
- RB51 Construct a general proof of the positive mass theorem that does not require the existence of a foliation with special properties.
- RB52 Show that the Bartnik quasi-local mass [303] is strictly positive for non-flat data and that the Penrose quasi-local mass is non-negative for reasonable data.
- RB53 Explain the relation between the various definitions of quasi-local mass.

7.7 Lists of lists

- List of acronyms used in this paper.
- Top 10 movies.
- Top 10 songs.
- Further list of lists¹.

¹ A list of the top 10 books, perhaps subdivided into popular books, popular science books and technical science books in mathematical physics, has proven more problematic to formulate.

7.7.1 Acronyms

Let us present a list of commonly used acronyms:

- AdS: Anti-de Sitter spacetime.
- BKL: Belinskii, Khalatnikov and Lifshitz.
- CFT: quantum field theory.
- CMB: cosmic microwave background.
- EYM: Einstein-Yang-Mills.
- FLRW: Friedmann–Lemaitre–Robertson–Walker.
- GR: general relativity.
- LQC: loop quantum cosmology.
- LQG: loop quantum gravity.
- QFT: quantum field theory.
- QG: quantum gravity.
- YM: Yang-Mills.
- 4D: four dimensions.

The abbreviations for the various lists are: H (Hilbert), S (Smale), M (Millennium), AL (Allen and Lidstrom), BS (Simon), Y (Yau), RB (Bartnik), RP (Penrose), Ph (theoretical physics), PF (personal favorites), P (open mathematical physics problems in the contemporary fields of GR, the quantum realm and in cosmology), MPF (mathematical physics problems in the future), and CMP (computationally motivated problems). All of the problems referred to are explicitly stated in the text or in the Appendix.

7.7.2 Top 10 movies

- Ramanujan (2014). Director: Gnana Rajasekaran.
- Einstein and Eddington (2008). Director: Philip Martin.
- Infinity (1996). Director: Matthew Broderick.
- The Theory of Everything (2014). Director: James Marsh
- A Beautiful Mind (2001). Director: Ron Howard.
- Proof (2005). Director: John Madden.
- Good Will Hunting (1997). Director: Gus Van Sant.
- Pi (1998). Director: Darren Aronofsky.
- The Imitation Game (2014). Director: Morten Tyldum.
- Fermat's Room (2007). Directors: Luis Piedrahita and Rodrigo Sopena.

7.7.3 Top 10 songs

- Eric Idle (Monty Python) Galaxy Song.
- The Bare Naked Ladies The Big Bang.
- Kate Bush Pi.
- Jack Black Math Song.
- Jarvis Cocker Quantum Theory.
- Nick Cave and The Bad Seeds Higgs Boson Blues.
- They Might Be Giants Why Does the Sun Shine?
- MC Hawking $-E = mc^2$.
- One Direction Maths Song.
- Bjork Mutual Core.
- Louis Armstrong What a Wonderful World².

All of these songs are available on You Tube. ² No list of top songs should ever exclude this song.

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