# Counting Vacua in Random Landscapes 

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#### Abstract

It is speculated that the correct theory of fundamental physics includes a large landscape of states, which can be described as a potential which is a function of $N$ scalar fields and some number of discrete variables. The properties of such a landscape are crucial in determining key cosmological parameters including the dark energy density, the stability of the vacuum, the naturalness of inflation and the properties of the resulting perturbations, and the likelihood of bubble nucleation events. We codify an approach to landscape cosmology based on specifications of the overall form of the landscape potential and illustrate this approach with a detailed analysis of the properties of N -dimensional Gaussian random landscapes. We clarify the correlations between the different matrix elements of the Hessian at the stationary points of the potential. We show that these potentials generically contain a large number of minima. More generally, these results elucidate how random function theory is of central importance to this approach to landscape cosmology, yielding results that differ substantially from those obtained by treating the matrix elements of the Hessian as independent random variables.


[^0]
## I. INTRODUCTION

For over a decade, considerations motivated by flux-compactified string vacua [1] 5] have suggested that fundamental physics may be specified within a landscape, a highly complex, multidimensional scalar potential. From this perspective the search for a unique theory of everything yields an apparent theory of anything: a vast range of possible configurations of "low energy" physics. An immediate corollary of this development is that if such a landscape emerges from fundamental physics, then anthropic reasoning may be central to understanding the observed properties of our universe.

Unfortunately, the string landscape itself appears to be so complex that quantitative explorations of its properties are computationally intractable. However, an alternative perspective is to treat the landscape as a realization of an $N$-dimensional random function $V(\bar{\phi})$ drawn from a specified distribution $\square^{\top}$ This distribution is fixed by the hypothesized architecture of the landscape. The critical observation underpinning this approach is that in many scenarios the tools of random function / random matrix theory will yield the distributions of key cosmological observables within this landscape. Physically, this approach is reasonable if we have grounds to believe that a landscape potential is a superposition of many largely independent terms, rather than (for instance) an almost periodic function with strong long-range correlations.

An early step in this direction was taken in Ref. [6], which posited that the ensemble of Hessian matrices associated with the stationary points in a generic landscape could be described by a set of symmetric matrices with elements chosen from independent identical Gaussian distributions. This ensemble is called the Gaussian Orthogonal Ensemble (GOE). From this perspective it appears that relative to saddle points, minima are superexponentially rare, as they correspond to large fluctuations from the Wigner semi-circle eigenvalue distribution [7]. Similar arguments were made about the critical points in a general four-dimensional $\mathcal{N}=1$ supergravity [8].

Conversely, Battefeld et al. [9] gave a semi-analytic treatment of the properties of Hessian matrices associated with minima of a "softly bounded" landscape, looking at Hessians derived directly from explicitly constructed random functions, in this case finite sums of

[^1]Fourier terms. Stability and distribution of the vacuum energies of this theory was studied in [10]. These potentials are naturally bounded below and while minima are outnumbered by saddles, they are not super-exponentially rare. Treating the potential as a random function also facilitates analyses of the vacuum stability, and the likelihood of tunneling has been studied in polynomial 1114 and bounded Fourier landscapes 10 .

More generally, in the large- $N$ limit, distributions of parameters associated with random functions and random matrices are often well-defined, potentially transforming the complexity of the landscape into a predictive tool. For simpler models with many non-interacting fields this approach has led to predictions for the mass-spectrum of N-flation [15] and the perturbation spectra of many-field scenarios [16, 17]. However, the key insight of this paper is to elucidate how a full understanding of the properties of critical points in a generic landscape with nontrivial couplings between the fields will require random function theory and not just random matrix theory, which implicitly treats the ensemble of Hessian matrices at extrema as uncorrelated matrices with uncorrelated elements.

The primary goal of this paper is to fully develop random function theory as a tool for understanding landscape models, extending the methods of Ref. [18] to $N$ dimensions and understanding the approach to the large- $N$ limit [19]. In doing so we categorize correlations between elements of the Hessian matrices, and elucidate the ways in which properties of extrema are correlated with the value of the potential. These correlations can be partly understood on purely topological grounds and highlight the information discarded by analyses which treat elements of the Hessian as independent and identically distributed variables.

As an illustrative example, perhaps the two simplest possible architectures are i) a set of uncoupled, self-interacting fields and ii) a potential $V(\bar{\phi})$ which is an $N$-dimensional isotropic, Gaussian random field, where $\bar{\phi}$ denotes the $N$ independent scalar fields, and $V$ is a map from $\mathbb{R}^{N}$ to $\mathbb{R}$. The distribution of possible values of the cosmological constant, $p(\Lambda)$, in a specific realization of the landscape is synonymous with the distribution of values of $V(\bar{\phi})$ at minima of the potential. The expected value of $\Lambda$ will effectively be the convolution of $p(\Lambda)$ with a selection function whose form is only loosely defined and which will depend on complex questions of measure and anthropic selection [20].

In this paper we focus on simple Gaussian random landscapes, and find that the relative numbers of extrema and saddles are roughly but not exactly binomial. The analysis here is effectively an $N$-dimensional generalization of the approach taken in the now classic
treatment of the theory of fluctuations in the density profile of the early universe of Bond, Bardeen, Kaiser and Szalay [18], or BBKS. We also provide arguments that this may be due to topological constraints and hence applicable beyond random Gaussian potentials.

A large set of cosmological parameters is associated with the properties of the landscape and we begin with a survey of these observables and the analogous properties of $V(\bar{\phi})$ in Section III. In Section III] we provide several topological hints on the relative number of different types of stationary points and in Section IV we provide the formalism for calculating relative number of different types of stationary points. In Section $V$ we present the results of our calculation for up to 100 fields and compare our results with the large-N limit results obtained in [19. Finally we conclude in Section VI.

## II. COSMOLOGICAL PROPERTIES AND LANDSCAPE ARCHITECTURE

As is now well-established, an inflationary phase in the early universe can resolve the initial conditions problems faced by simple models of the hot big bang [21|23]. However there is no unique mechanism to drive the accelerated expansion associated with inflation [24], and several hundred different models have been proposed and examined [25]. Typically these models are specified by the effective potential of the inflaton field(s). If it is assumed that the inflationary potential is contained somewhere in the overall landscape, the "typical" inflationary mechanism in a landscape will be associated with the expectation values of derivatives of the fields on candidate inflationary trajectories.

Let us begin by surveying the range of observables that may be associated with a landscape potential, and the properties of the potential that determine them:

- The distribution of vacuum energies, $p(\Lambda)$, in the landscape (see e.g. Ref. [26]). In a given pocket [27] this corresponds to local minima of the landscape, with the value of the vacuum energy / dark energy density $\Lambda=V(\bar{\phi})$ at each minimum.
- The stability of the vacuum as a function of $\Lambda$ [28, 29]. The local vacuum in a landscape potential is typically metastable due to bubble nucleation via quantum tunneling [10, 11, 13, 30 32] and will be unstable if there is a noticeable probability of decay within cosmologically relevant time scales.
- Bubble collision [33-35]. Collision rates depend on the nucleation rate which is a
function of $V(\bar{\phi})$ at local minima and the surrounding barrier heights.
- The likelihood of slow roll inflation. Slow roll inflation requires sufficiently long, flat "plateaus" or "valleys" in $V(\bar{\phi})$, and that $V_{, i}$ and $V_{, i j}$ in downhill directions in the potential are parametrically small.
- Primordial perturbation spectrum. Expectation values for the spectral index $n_{s}$ and tensor to scalar ratio $r$ (and correlations between them) are derived from the expected values of $V_{, i j}$ and $V_{, i}$ along inflationary trajectories.

While the landscape may be almost arbitrarily complex, its overall form may be motivated by a handful of fundamental physical principles. The heart of this proposal is to use these principles to identify the overall architecture of the landscape. Such an architecture can lead to specification of a detailed probability distribution from which $V(\bar{\phi})$, the potential energy function of the landscape, can be drawn.

To illustrate this approach, the simplest landscape architecture we can imagine consists of $N$ fields, $\phi_{1}, \phi_{2}, \ldots, \phi_{N}$ with self-interaction potentials $V_{i}\left(\phi_{i}\right)$ and no mutual interactions, so that the landscape potential $V$ is

$$
\begin{equation*}
V(\phi)=\sum_{i=1}^{N} V_{i}\left(\phi_{i}\right), \tag{1}
\end{equation*}
$$

which each $V_{i}$ is to be chosen probabilistically. Even without specifying the probability distribution for each $V_{i}$ we know that the number of maxima and minima cannot differ by more than one, which would be negligible if the number of stationary points is large. If the $V_{i}\left(\phi_{i}\right)$ are periodic, then the number of maxima and minima must match exactly. For a given stationary point, the probability that $k$ of the eigenvalues of the corresponding Hessian are positive is exactly 6]

$$
\begin{equation*}
P_{k}=2^{-N}\binom{N}{k} . \tag{2}
\end{equation*}
$$

We thus immediately deduce that the ratio of the number of minima to stationary points is $1: 2^{N}$ for a landscape that consists purely of uncoupled, self-interacting fields. The central limit theorem implies that, for sufficiently large $N$, the distribution of vacuum energies would be a Gaussian [10].

Conversely assuming that all fields have similar mutual- and self-interactions and that the overall landscape potential is a combination of many individual terms motivates a landscape
architecture that consists of a Gaussian random function,

$$
\begin{equation*}
V(\bar{\phi})=U(\bar{\phi}) \tag{3}
\end{equation*}
$$

where $\bar{\phi}$ denotes the $N$ scalar fields and $U(\bar{\phi})$ is a Gaussian random function. If we stipulate that there are no preferred directions or positions in the landscape then the correlation function will naturally be rotationally invariant:

$$
\begin{equation*}
\left\langle U\left(\bar{\phi}_{1}\right) U\left(\bar{\phi}_{2}\right)\right\rangle=N f\left(\frac{\left(\bar{\phi}_{1}-\bar{\phi}_{2}\right)^{2}}{N}\right) . \tag{4}
\end{equation*}
$$

Likewise, given that we aim to investigate the implications of a given landscape architecture for the cosmological constant problem, it is natural to stipulate that the mean of $U$ is zero. A more sophisticated architecture arises from assuming that $U(\bar{\phi})$ is associated with new physics at a very high but sub-Planckian scale $M$ (e.g. string or GUT-scale physics) and that Planck-scale operators induce an extra correlation at large VEVs, so that

$$
\begin{equation*}
V(\bar{\phi})=V_{0}+\frac{1}{2} m^{2} \bar{\phi} \cdot \bar{\phi}+U(\bar{\phi}) \tag{5}
\end{equation*}
$$

where $U$ is again a Gaussian random function and we have also added an arbitrary offset $V_{0}$ for generality. If $|\bar{\phi}|$ can approach Planckian values, then $m^{2} M_{p}^{2} \sim|U| \sim M^{4}$ at the "edge" of the landscape, or

$$
\begin{equation*}
m=M \frac{M}{M_{\mathrm{P}}} \tag{6}
\end{equation*}
$$

so that

$$
\begin{equation*}
\langle U(\bar{\phi})\rangle=0, \quad\left\langle U^{2}(\bar{\phi})\right\rangle=\mathcal{O}\left(M^{8}\right) . \tag{7}
\end{equation*}
$$

By hypothesis, $M$ is significantly smaller than the Planck mass and we also require the typical correlation length of $U(\bar{\phi})$ to be much less than $M_{\mathrm{P}}$. In this case we will naturally expect $U$ to contain a large number of extrema.

In this paper, our goal is to understand the properties of the critical points in this landscape. Any specific critical point of a function $V$ is characterized by the Hessian

$$
\begin{equation*}
\zeta_{i j}=\frac{\partial^{2} V}{\partial \phi_{i} \partial \phi_{j}} . \tag{8}
\end{equation*}
$$

In the absence of any other information, it may be tempting to assume that the Hessian is a symmetric random matrix [6]. Denoting the (ordered) eigenvalues of $\zeta\left(\phi_{\mathrm{st}}\right)$ by $\lambda_{1} \geq$ $\lambda_{2} \geq \ldots \geq \lambda_{N}$ one might naively assume that because each eigenvalue is equally likely to be
positive or negative the likelihood that $\phi_{\mathrm{st}}$ is a vacuum (local minimum) is $(1 / 2)^{N}$. However, the joint probability distribution for the $\lambda_{i}$ is

$$
\begin{equation*}
P\left(\lambda_{1}, \ldots, \lambda_{N}\right) \propto \prod_{i<j}\left(\lambda_{i}-\lambda_{j}\right) \tag{9}
\end{equation*}
$$

and the likelihood that all the eigenvalues are positive scales as $e^{-\alpha N^{2}}$ [6, 7]. However, as we will describe in detail below, the Hessians of random functions are not random matrices with independent and identically distributed components. As we show below, the expected fraction of minima is much closer to the binomial form $2^{-N}$ than $e^{-\alpha N^{2}}$.

## III. TOPOLOGY AND MORSE THEORY

One way to approach the landscape is to assume that the Hessian matrices are random, symmetric matrices drawn from the GOE. If we wish to apply the standard results of random matrix theory to the Hessians derived from a given random function, it is necessary for the individual elements of the Hessian to be drawn from independent and identical distributions. However, the Hessians of a random function are not uncorrelated; consider two elements of the Hessian; $\zeta_{i j}$ and $\zeta_{k l}$; given than mixed derivatives commute, working from the definition of the Hessian it follows that

$$
\begin{equation*}
\partial_{k} \partial_{l} \zeta_{i j}=\partial_{i} \partial_{j} \zeta_{k l} \tag{10}
\end{equation*}
$$

Consequently, the elements of the Hessian matrices derived from a specified function are not independent of one another.

In the next sections we will directly compute the fraction of extremal points of an N dimensional Gaussian random function which are actual minima. However, we can also gain significant insight from global, topological arguments based on Morse Theory. To start with, consider a two-dimensional periodic landscape. Let's $N_{\max }, N_{\min }$ and $N_{\text {saddle }}$ denote the number of maxima, minima and saddle points of this function. From Morse theory we know

$$
\begin{equation*}
N_{\min }-N_{\text {saddle }}+N_{\max }=\chi_{\text {torus }}=0 \tag{11}
\end{equation*}
$$

where $\chi_{\text {torus }}$ is the Euler characteristic of the torus. This result holds for any function on torus and it must be satisfied case by case and not an average. For random Gaussian functions the symmetry of $U \rightarrow-U$ ensures $N_{\max }=N_{\min }$ which combined with (11) gives

$$
\begin{equation*}
N_{\text {saddle }}=2 N_{\max }=2 N_{\min } \tag{12}
\end{equation*}
$$



FIG. 1. Top left and right, the contours for $\partial V / \partial \phi_{1}=0$ and $\partial V / \partial \phi_{2}=0$, bottom the two graphs super-imposed. Sign of $\partial^{2} U / \partial \phi_{1}^{2}$ and $\partial^{2} U / \partial \phi_{2}^{2}$ are shown by black and red plus and minus signs. It is clear that the stationary points of the potential which are located at the black filled squares have alternating signs for diagonal elements of the Hessian $\zeta$.

We calculated the same quantities using a GOE

$$
\begin{equation*}
N_{\mathrm{saddle}}=2(1+\sqrt{2}) N_{\min } \approx 4.82 N_{\min } \tag{13}
\end{equation*}
$$

Notice that to derive (12) we used an average symmetry of $U \rightarrow-U$ which does not hold in general. Therefore, (12) will not be valid for each realization of the potential. A counter example is shown in Figure 2 .

This result generalizes to higher numbers of dimensions. If $N_{i}$ is the number of stationary points with $i$ negative eigenvalues in the Hessian, we get

$$
\begin{equation*}
\sum_{i=0}^{N}(-1)^{i} N_{i}=\chi(\mathcal{M}) \tag{14}
\end{equation*}
$$



FIG. 2. One can add a maximum and a saddle point to a straight slope without creating any new minimum illustrating that 122 is valid only statistically and not for all individual realizations of the potential.
where $\chi(\mathcal{M})$ is the Euler characteristic of the manifold $\mathcal{M}$. It is easy to show that the random Gaussian matrices in even number of dimensions do not satisfy this requirement.

We can give a heuristic argument that there are important correlations between elements of the Hessian at adjacent stationary points. This argument is illustrated in Fig.1. A stationary point of the potential is given by $\nabla U(\phi)=0$. Let's first look at the hyper-surfaces $\partial U / \partial \phi_{1}=0$ which is shown on the top left panel. Along the straight horizontal line, the potential is a one-dimensional function, and therefore the sign of the second derivative along the line must change at the stationary points (blue dots). Consequently, the corresponding Hessian element $\zeta_{11}$ must change sign along this line. Moreover, the sign of $\zeta_{11}$ must be constant on each hypersurface. It can only change if there are horizontal lines where $\zeta_{11}$ vanishes at all blue dots, which is highly unlikely, or if the topology of the contours change. Similarly for the hypersurfaces $\partial U / \partial \phi_{2}=0$ shown in the top right panel must have alternating signs for $\zeta_{22}$. The stationary points lie on the intersections of these contours which are shown as black filled squares in the lower panel. Therefore the sign structure of the Hessian diagonal elements $\zeta_{i i}$ is very similar to the one for sum of independent potentials presented in (1). Moreover, by a change of coordinate which mixes the diagonal and non diagonal elements, there will be correlations between the non-diagonal elements of the adjacent stationary points. This strongly suggests that the Hessian cannot be modeled with a random Gaussian matrix and there are always important correlation between the Hessian at different stationary points. This is only a heuristic argument and by no means a proof
but it seems to be a generic situation and should hold statistically and our results in the later sections suggest that this argument is plausible. Moreover, this statement applies to any function not just random functions.

## IV. COUNTING STATIONARY POINTS OF RANDOM GAUSSIAN FIELDS

Here we start from a random Gaussian potential as defined in (4). Without making any assumption about the Hessian, we directly calculate the number of different types of stationary points by closely following the methods of BBKS [18]. Because these potentials are translationally invariant we end up with the number density of these stationary points. Let's first rewrite (4) in Fourier space and define the power spectrum $P(k)$

$$
\begin{equation*}
\left\langle U\left(\bar{\phi}_{1}\right) U\left(\bar{\phi}_{2}\right)\right\rangle=\frac{1}{(2 \pi)^{N}} \int d^{N} k P(k) e^{i \bar{k} \cdot\left(\bar{\phi}_{1}-\bar{\phi}_{2}\right)} . \tag{15}
\end{equation*}
$$

Because $U$ is a Gaussian random field, all the odd moments of the distribution vanish. We use the following notation for the even moments and gradient :

$$
\begin{equation*}
\sigma_{n}^{2}=\frac{1}{(2 \pi)^{N}} \int d^{N} k\left(k^{2}\right)^{n} P(k), \quad \eta_{i}=\frac{\partial U}{\partial \phi_{i}} . \tag{16}
\end{equation*}
$$

We denote the eigenvalues of the $\zeta$, the Hessian, as $\lambda_{1}, \ldots, \lambda_{N}$ and to specify them unambiguously we choose $\lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{N}$. We use a variant of the Kac formula [36] to compute the number of stationary points with $P$ positive eigenvalues $(P \geq 1)$ in a region,

$$
\begin{equation*}
N_{P}=\int d^{N} \phi \delta^{N}\left(\eta_{i}\right)\left|\operatorname{det}\left(\zeta_{i j}\right)\right| \theta_{H}\left(\lambda_{P}\right) \tag{17}
\end{equation*}
$$

where $\delta^{N}$ is the $N$-dimensional Dirac delta and $\theta_{H}$ is the Heaviside step function. The ability of this expression to "count" extrema follows by noting that it is nonzero only at stationary points and at those points the Jacobian is cancelled by the change of variables in $\delta$ function, and its overall value would be $\pm 1$ depending on the number of negative eigenvalues. Similarly, the number of maxima is given by

$$
\begin{equation*}
N_{0}=\int d^{N} x \delta^{N}\left(U_{, i}\right) \operatorname{det}\left(U_{, i j}\right) \theta_{H}\left(-\lambda_{1}\right) \tag{18}
\end{equation*}
$$

The functions $U, \eta_{i}$ and $\zeta_{i j}$ are Gaussian variables with zero mean and all we need is their standard deviation. It is now easy to show that the only nonzero two-point functions are

$$
\begin{align*}
\langle U(\bar{\phi}) U(\bar{\phi})\rangle & =\sigma_{0}^{2} \\
\left\langle\eta_{i}(\bar{\phi}) \eta_{j}(\bar{\phi})\right\rangle & =\frac{1}{N} \delta_{i j} \sigma_{1}^{2} \\
\left\langle U(\bar{\phi}) \zeta_{i j}(\bar{\phi})\right\rangle & =-\frac{1}{N} \delta_{i j} \sigma_{1}^{2} \\
\left\langle\zeta_{i j}(\bar{\phi}) \zeta_{k l}(\bar{\phi})\right\rangle & =\frac{\sigma_{2}^{2}\left(\delta_{i j} \delta_{k l}+\delta_{i l} \delta_{j k}+\delta_{i k} \delta_{j l}\right)}{N(N+2)} \tag{19}
\end{align*}
$$

To calculate the total number of stationary points of a specific type from (23) we define the following vector

$$
\begin{align*}
\alpha_{i}= & \left\{U, \eta_{1}, \eta_{2}, \ldots, \eta_{N}, \zeta_{11}, \zeta_{22}, \ldots, \zeta_{N N}, \zeta_{N-1, N}, \zeta_{N-2, N},\right. \\
& \left.\ldots, \zeta_{1 N}, \zeta_{N-2, N-1}, \ldots, \zeta_{1, N-1}, \ldots, \zeta_{12}\right\} . \tag{20}
\end{align*}
$$

This vector has $\frac{1}{2} N(N+3)+1$ elements and we define

$$
\begin{equation*}
M_{i j} \equiv\left\langle\alpha_{i} \alpha_{j}\right\rangle, \text { and } K_{i j} \equiv M^{-1} \tag{21}
\end{equation*}
$$

Because $\alpha_{i}$ have normal distribution, their joint probability distribution is given by

$$
\begin{equation*}
p\left(\alpha_{i}\right)=\frac{\sqrt{\operatorname{det} K}}{(2 \pi)^{N(N+3) / 4}} e^{-Q} \text { where } Q=\frac{1}{2} \alpha K \alpha . \tag{22}
\end{equation*}
$$

It is easy to check that (22) gives the correlation functions given in (19). In order to get the total number of a given type of stationary point we evaluate the integral given in (23) and (18) using this probability distribution. For example, for $P>0$, we get

$$
\begin{equation*}
N_{P}=\int d^{N} x \prod_{i=1}^{1+N(N+3) / 2} d \alpha_{i} \delta^{N}\left(\eta_{i}\right) \operatorname{det}\left(\zeta_{i j}\right) \theta_{H}\left(\lambda_{P}\right) p\left(\alpha_{i}\right) \tag{23}
\end{equation*}
$$

The most general real symmetric matrix $\zeta_{i j}$ can be written as

$$
\zeta=R^{T}\left(\begin{array}{ccc}
\lambda_{1} & \ldots & 0  \tag{24}\\
\vdots & \ddots & \vdots \\
0 & \ldots & \lambda_{N}
\end{array}\right) R
$$

where $R$ is a rotation matrix that diagonalizes it. One way to evaluate these integrals is to parametrize the Hessian as in (24) and write the matrix $R$ in terms of $N(N-1) / 2$ Euler angles. Adding the $N$ eigenvalues, we recover the $N(N+1) / 2$ independent parameters which


FIG. 3. Ratio of saddles to minima for a 3-d function with spherical cutoff; clearly centered around 3.05 .
specify a symmetric matrix. This approach is tractable for $N=2$ and 3 and we present explicit results in Appendices A and B. In particular, for $N=2$ the ratio of Hessians with zero, one and two positive eigenvalues is $1: 2: 1$ while for $N=3$ the analogous ratio is $1: 3.05: 3.05: 1$, showing that the number of relative number of minima and saddles is close to (and exact, for $N=2$ ) the result for uncoupled potentials.

For comparison, we computed the expectations for the eigenvalue distribution for matrices drawn from the GOE, by integrating over the relevant measure. For $N=3$ we get these ratios

$$
\begin{equation*}
n_{\mathrm{rel}}=\left\{1,1+\frac{8}{\sqrt{2} \pi-4}, 1+\frac{8}{\sqrt{2} \pi-4}, 1\right\}=\{1,19.06,19.06,1\} \tag{25}
\end{equation*}
$$

Separately, we directly generated numerical realizations of random functions, by generating realizations of band-limited Fourier sums. We looked at two cases, a spherical cutoff including all modes with $|\bar{k}|$ less than a fixed cutoff and a Gaussian, scale invariant power spectrum ${ }^{2}$ Averaging over multiple realizations we find:

$$
\begin{array}{ll}
n_{\text {rel }}=\{1 ., 3.056,3.057,1.0008\}, & \text { Spherical } \\
n_{\text {rel }}=\{1 ., 3.055,3.055,1.0003\}, & \text { Gaussian } \tag{26}
\end{array}
$$

The distribution of the relative number of the saddle points to minima and maxima for an scale invariant power spectrum for the $N=3$ case is shown in Figure 3 .

[^2]
## V. DENSITY OF STATIONARY POINTS FOR GENERAL $N$

For $N>3$ it turns out to be more convenient to use the techniques of BBKS [18], generalized to $N$ dimensions. We will need the integration measure which is the Jacobian of the transformation from the Hessian $\zeta_{i j}$ to the Euler angles and eigenvalues. We obtain this by noting that if we have a metric (inner product) on this space, the square root of its determinant gives the integration measure. Following the overall approach of BBKS, we use the following inner product on the space of symmetric $N \times N$ matrices

$$
\begin{equation*}
S_{1} \cdot S_{2}=\operatorname{Tr}\left(S_{1} S_{2}\right) \tag{27}
\end{equation*}
$$

It is obvious that $\operatorname{Tr}\left[\mathrm{S}_{1}\left(\mathrm{~S}_{2}+\alpha \mathrm{S}_{3}\right)\right]=\operatorname{Tr}\left(\mathrm{S}_{1}\right)+\alpha \operatorname{Tr}\left(\mathrm{S}_{2}\right), \operatorname{Tr}\left(\mathrm{S}_{1} \mathrm{~S}_{2}\right)=\operatorname{Tr}\left(\mathrm{S}_{2} \mathrm{~S}_{2}\right)$ and $\operatorname{Tr}\left(\mathrm{S}_{1} \mathrm{~S}_{1}\right)=$ $\sum \lambda_{\mathrm{i}}^{2} \geq 0$ and hence this is a valid metric. If a matrix $S$ can be diagonalized by an orthogonal matrix $R$, the matrix $S+d S$ will be diagonalized by $R+d R$ and if $R^{\dagger} S R=$ $\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{N}\right)$, the eigenvalues would change by $d \lambda$. Using $R^{\dagger} R=I$

$$
\begin{equation*}
R^{\dagger} d R+d R^{\dagger} R=0 \tag{28}
\end{equation*}
$$

which translates to $d R^{\dagger}=-R^{\dagger} d R R^{\dagger}$. Therefor $\otimes^{3}$

$$
\begin{align*}
d S=R^{\dagger} d \lambda R+R^{\dagger} \lambda d R+d R^{\dagger} \lambda R & =R^{\dagger} d \lambda R+R^{\dagger} \lambda d R-R^{\dagger} d R R^{\dagger} \lambda R \\
& =R^{\dagger}\left(d \lambda+\left[\lambda, d R R^{\dagger}\right]\right) R \tag{29}
\end{align*}
$$

From here we get

$$
\begin{equation*}
d s^{2}=\operatorname{Tr}\left(\mathrm{dS}{ }^{2}\right)=\operatorname{Tr}\left(\mathrm{d} \lambda^{2}+\left[\lambda, \mathrm{dRR}^{\dagger}\right]^{2}+2 \mathrm{~d} \lambda\left[\lambda, \mathrm{dRR}^{\dagger}\right]\right) . \tag{30}
\end{equation*}
$$

Because $d \lambda$ and $\lambda$ are both diagonal they commute and the last term vanishes, leaving

$$
\begin{equation*}
d s^{2}=\operatorname{Tr}\left(\mathrm{dS}^{2}\right)=\operatorname{Tr}\left(\mathrm{d} \lambda^{2}+\left[\lambda, \mathrm{dRR}^{\dagger}\right]^{2}\right) \tag{31}
\end{equation*}
$$

Noticing that (28) shows that $R^{\dagger} d R$ is an antisymmetric matrix the trace in (31) is given by

$$
\begin{equation*}
d s^{2}=\sum\left(d \lambda_{i}\right)^{2}+\sum_{i \neq j}\left(\lambda_{i}-\lambda_{j}\right)^{2} \omega_{i j}^{2} \tag{32}
\end{equation*}
$$

[^3]where $\omega_{i j}=\left(d R R^{\dagger}\right)_{i j}$ depends only on elements of the rotation group which are given by Euler angles. There are $N+N(N-1) / 2=N(N+1) / 2$ orthogonal vectors in (32) and hence these form the orthonormal set we were looking for. Therefore the volume element is
\[

$$
\begin{equation*}
d \mathrm{Vol}=\prod_{i \leq j} d \zeta_{i j}=\prod_{i=1}^{N} d \lambda_{i} \prod_{i \neq j}\left(\lambda_{i}-\lambda_{j}\right) \Omega\left(\alpha_{k}\right) \prod_{k=1}^{\frac{1}{2} N(N-1)} d \alpha_{k}=F\left(\alpha_{i}\right) \prod_{i=1}^{N} d \lambda_{i} \prod_{i \neq j}\left(\lambda_{i}-\lambda_{j}\right) \tag{33}
\end{equation*}
$$

\]

Because of spherical symmetry, the only dependence on the Euler angles $\alpha_{i}$ would be through a normalization factor which is irrelevant for the calculation of the relative number of different types of stationary points. To evaluate $Q$ in (22) we find the inverse of the $M$ in (21) and we later evaluate $Q$ on the surface $\eta_{i}=0$. The nonzero elements of $K$ are

$$
\begin{align*}
K_{U U} & =\frac{\sigma_{2}^{2}}{\sigma_{0}^{2} \sigma_{2}^{2}-\sigma_{1}^{4}} \\
K_{\eta_{i} \eta_{j}} & =\frac{N}{\sigma_{1}^{2}} \\
K_{U, \zeta_{i j}} & =\frac{\sigma_{1}^{2}}{\sigma_{0}^{2} \sigma_{2}^{2}-\sigma_{1}^{4}}, \\
K_{\zeta_{i j} \zeta_{k l}} & =\frac{\sigma_{1}^{2}}{\sigma_{0}^{2} \sigma_{2}^{2}-\sigma_{1}^{4}} \delta_{i j} \delta_{k l}+\frac{N(N+2)}{2 \sigma_{2}^{2}}\left(2 \delta_{i k} \delta_{j l}-\delta_{i l} \delta_{j k}\right) . \tag{34}
\end{align*}
$$

We evaluate $Q$ on the surface $\eta_{i}=0$

$$
\begin{equation*}
Q=\frac{1}{\sigma_{0}^{2} \sigma_{2}^{2}-\sigma_{1}^{4}}\left(\frac{1}{2} \sigma_{2}^{2} U^{2}+\sigma_{1}^{2} U \operatorname{Tr} \zeta+\frac{(\mathrm{N}+2) \sigma_{1}^{4}-\mathrm{N} \sigma_{0}^{2} \sigma_{2}^{2}}{4 \sigma_{2}^{2}}(\operatorname{Tr} \zeta)^{2}\right)+\frac{N(N+2)}{4 \sigma_{2}^{2}} \operatorname{Tr} \zeta^{2} \tag{35}
\end{equation*}
$$

From here we can easily show that at a constant $U$

$$
\begin{align*}
\left\langle\zeta_{i j}\right\rangle_{U} & =-\frac{\sigma_{1}^{2}}{N \sigma_{0}^{2}} U \delta_{i j} \\
\left\langle\zeta_{i j} \zeta_{k l}\right\rangle_{U} & =\left(-\frac{\sigma_{1}^{4}}{N^{2} \sigma_{0}^{2}}+\frac{\sigma_{1}^{4} U^{2}}{N^{2} \sigma_{0}^{4}}\right) \delta_{i j} \delta_{k l}+\frac{\sigma_{2}^{2}}{N(N+2)}\left(\delta_{i j} \delta_{k l}+\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right) \tag{36}
\end{align*}
$$

Similar results for a very specific power spectrum was obtained in [37] which looked at the distribution of minima for a power spectrum given by

$$
\begin{equation*}
\left\langle U\left(\bar{\phi}_{1}\right) U\left(\bar{\phi}_{2}\right)\right\rangle \propto e^{-\left|\phi_{1}-\phi_{2}\right|^{2} / \Lambda_{h}^{2}} \tag{37}
\end{equation*}
$$

where $\Lambda_{h}$ is the correlation length. It is easy to check that in this special case the term which has $\left((N+2) \sigma_{1}^{4}-N \sigma_{0}^{2} \sigma_{2}^{2}\right) \operatorname{Tr} \zeta^{2}$ in (35) vanishes and then the Hessian can be thought as the sum of a GOE plus a coefficient times the identity matrix. However, as it is clear from (35) this is not true in general. To see it more clearly, if one could write $\zeta=W+p(U) I$ where $W$ is chosen form a GOE, then the term proportional to $(\operatorname{Tr} \zeta)^{2}$ in 35 would not appear.

Therefore, except for very special cases, the Hessian is not the sum of a matrix from GOE and a matrix proportional to identity. Interestingly, the second equation in (36) shows that even when the potential is zero the elements of the Hessian are not generically drawn from a random orthogonal matrix. Because in this paper we are only interested in the number of stationary points and not their energy we integrate over the potential to get the distribution of $\zeta$. The probability distribution of the Hessians is give by

$$
\begin{equation*}
P(\zeta) \propto e^{-\tilde{Q}} \tag{38}
\end{equation*}
$$

where

$$
\begin{equation*}
\tilde{Q}=\frac{N}{4 \sigma_{2}^{2}}\left[(N+2) \sum \operatorname{Tr} \zeta^{2}-(\operatorname{Tr} \zeta)^{2}\right] \tag{39}
\end{equation*}
$$

The last ingredient we would need for an explicit calculation of the number of stationary points of a given kind is $\operatorname{det} M$ which is given by

$$
\begin{equation*}
\operatorname{det} M=\frac{2^{N-1}}{(N+2)}\left(\frac{\sigma_{1}^{2}}{N}\right)^{N}\left(\frac{\sigma_{2}^{2}}{N(N+2)}\right)^{N(N+1) / 2}=G(N) \sigma_{1}^{2 N} \sigma_{2}^{N(N+1)} \tag{40}
\end{equation*}
$$

The density of stationary points of a given type (using the appropriate $\theta$ function) is given by

$$
\begin{align*}
n_{i} & =\frac{1}{(2 \pi)^{N(N+3) / 2} \sqrt{\operatorname{det} M}} \int \prod d \lambda_{i} \prod \lambda_{i} \prod_{i>j}\left(\lambda_{i}-\lambda_{j}\right) e^{-\tilde{Q}} \theta_{H} \int \prod d \alpha_{i} F\left(\alpha_{i}\right) \\
& =\frac{F_{N}}{(2 \pi)^{N(N+3) / 2} \sqrt{\operatorname{det} M}} \int \prod d \lambda_{i} \prod \lambda_{i} \prod_{i>j}\left(\lambda_{i}-\lambda_{j}\right) e^{-\tilde{Q}} \theta_{H}, \tag{41}
\end{align*}
$$

where $F(N)$ is the normalization factor we get by integrating over all Euler angles which we can ignore in what follows. The number of stationary points depends on a geometrical factor from Euler angles and the only dependence on the power spectrum is through

$$
\begin{equation*}
n_{i} \propto \frac{\sigma_{2}^{N}}{\sigma_{1}^{N}} \tag{42}
\end{equation*}
$$

This dependence does not change the relative number of stationary points. It is fascinating to notice that the relative number of stationary points is independent of the power spectrum and depends only on the ratios of integrals over the $\lambda_{i}$. We cannot evaluate this integral analytically for arbitrary $N$, but we can make substantial numerical progress. For $N \leq 9$ we used the implementation of VEGAS [38], an adaptive Monte Carlo method, in the GNU Scientific Library (GSL) [39] with sample sizes of $10^{7}$ to $10^{9}$ points to get percent-level precision. However, for $N \geq 10$, the GSL implementation failed with underflow errors but

| $N$ | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n_{0} \sigma_{1}^{N} /\left(F_{N} \sigma_{2}^{N}\right)$ | $3.2 \times 10^{-11}$ | $1.5 \times 10^{-15}$ | $2.1 \times 10^{-20}$ | $1.1 \times 10^{-25}$ | $2.3 \times 10^{-31}$ | $2.0 \times 10^{-37}$ |

TABLE I. Numerical values for the density of minima for $N=4, \ldots, 9$.

| $N$ | $\beta_{0}$ | $\beta_{1}$ | $\beta_{2}$ | $\beta_{3}$ | $\beta_{4}$ | $\beta_{5}$ | $\beta_{6}$ | $\beta_{7}$ | $\beta_{8}$ | $\beta_{9}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 1 | 4.04 | 6.08 | 4.04 | 1.00 |  |  |  |  |  |
| 5 | 1 | 5.36 | 11.08 | 11.08 | 5.36 | 1.00 |  |  |  |  |
| 6 | 1 | 6.62 | 17.45 | 23.68 | 17.45 | 6.614 | 1.00 |  |  |  |
| 7 | 1 | 8.09 | 26.2 | 45.3 | 45.3 | 26.2 | 8.1 | 1.02 |  |  |
| 8 | 1 | 9.28 | 36.0 | 76. | 96.6 | 76. | 36. | 9.28 | 1.00 |  |
| 9 | 1 | 10.9 | 49.1 | 123. | 192. | 192. | 123. | 49.3 | 10.9 | 1.00 |

TABLE II. Numerical results of $\beta_{i}$ for $N>3$. We see this is very close to the ratios we get for independent potentials described in (1). These numbers are accurate within one percent error.
a purpose-written implementation of Metropolis-Hastings 40, 41] algorithm allowed us to reliably evaluate these integrals for up to $N=100$. Detailed results for $N=50$ are presented in Appendix []$^{4}$ Finally, we verified this approach for $N \leq 50$ using the PolyChord [42] - this package is designed to calculate Bayesian evidence which (for a suitable choice of likelihood function) is mathematically equivalent to the problem faced here.

We express our densities relative to $n_{0}$, the density of minima for each value of $N$, and write $n_{i}=n_{0} \beta_{i}$. We show the results of $n_{0}$ for $N \leq 10$ in Table Iand the corresponding $\beta_{i}$ 's in Table II. The values computed and presented in Table $\mathbb{\square}$ contain an overall geometric factor that scales as the relative volume of the $N$-sphere which accounts for the very small numerical values, and this term cancels from the ratios found in Table I. The $\beta_{i}$ 's are relatively close to a binomial distribution, which is the exact result for landscape that is a sum of independent potentials in (1). For contrast, the $\beta_{i}$ that would be expected if the distribution of extrema was controlled by the relatives numbers of same-sign eigenvalues in matrices drawn from the GOE are shown Table III. These numbers are dramatically different from both the explicit results we obtained for a Gaussian random function and the pure binomial distribution.

The properties of stationary points of random Gaussian fields in the large- $N$ limit have

[^4]| $N$ | $\beta_{0}$ | $\beta_{1}$ | $\beta_{2}$ | $\beta_{3}$ | $\beta_{4}$ | $\beta_{5}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 1 | 4.83 | 1 |  |  |  |
| 3 | 1 | 19.0 | 19.0 | 1 |  |  |
| 4 | 1 | 72.0 | 261.1 | 72.0 | 1 |  |
| 5 | 1 | 268.7 | 3299. | 3299. | 268.7 | 1 |

TABLE III. The values of $\beta_{i}$ 's for a GOE. We see that this is not a good model for the distribution of stationary points of the landscape and for even number of dimensions it also does not satisfy the criteria from Morse theory.
been extensively studied, see for example Ref. [19] and the references within. For landscape potentials $N$ is typically assumed to be $\mathcal{O}(100)$ and our next goal is to check the rate at which results for finite $N$ approach the large- $N$ limit. Adopting the notation Ref. [19], the Hessian of each stationary point of the potential can have between 0 to $N$ negative eigenvalues. Denote the number of stationary points whose Hessian has $N \alpha$ negative eigenvalues by $\mathcal{N}(\alpha)$. By definition, $\alpha$ is in the range [0,1] and we express $\mathcal{N}(\alpha)$ in terms of "complexity" $\Sigma(\alpha)$ via

$$
\begin{equation*}
\mathcal{N}(\alpha)=e^{N \Sigma(\alpha)} \tag{43}
\end{equation*}
$$

The complexity was calculated in [19] to be

$$
\begin{equation*}
\Sigma(\alpha)=-\frac{\bar{\lambda}^{2}}{4 f^{\prime \prime}(0)}+\text { normalization constant } \tag{44}
\end{equation*}
$$

where $\bar{\lambda}$ is defined by

$$
\begin{equation*}
\frac{2}{\pi} \int_{\bar{\lambda} / 2 \sqrt{f^{\prime \prime}(0)}}^{1} d y \sqrt{1-y^{2}}=\alpha \tag{45}
\end{equation*}
$$

Here $f$ is the correlation function defined in (4). We plotted the $\Sigma(\alpha)$ in Fig.4. The center of $\Sigma(\alpha)$ fits well with a quadratic. We can calculate its width for $\alpha$ close to $1 / 2$ (small $\bar{\lambda}$ ).

$$
\begin{equation*}
\bar{\lambda}^{2}=\pi^{2} f^{\prime \prime}(0)\left(\alpha-\frac{1}{2}\right)^{2} \tag{46}
\end{equation*}
$$

The complexity near the center is

$$
\begin{equation*}
\Sigma(\alpha)=-\frac{\pi^{2}}{4}\left(\alpha-\frac{1}{2}\right)^{2} \tag{47}
\end{equation*}
$$

If one assumes a binomial distribution, this coefficient would be 2 instead of $\pi^{2} / 4$. This shows that the binomial distribution for large- $N$ values is not a good approximation. Our
results at the center of the distribution are consistent with [19] but the discrepancy grows for small value of $\alpha$, corresponding to local minima. In Fig. 4 we compare the complexity obtained from our exact results with the large- $N$ limit results of [19] and the binomial distribution for $N=10,50$ and 100. The binomial approximation overestimates the density of minima relative to saddles, while the large- $N$ results of [19] underestimates these values for finite $N$. However, in all cases the likelihood of a given extremum being a minimum decreases exponentially with $N$, rather than super-exponentially as would be the case if the Hessians were drawn from the Gaussian Orthogonal ensemble of random, symmetric matrices. This result is also consistent with the observations of Ref. [10]. Finally, we plot the ratio of minima to stationary points computed from our evaluations of the (41) in Fig. 5 and it seems that it fits well by a line on a $\log$ scale.

## VI. DISCUSSION AND CONCLUSIONS

We have taken steps towards quantifying expectations for the properties of landscape potentials embedded in theories of high energy physics, including string theory. Our overall approach is to begin with the architecture of the landscape, specifying expectations for its global properties. This paper focuses on an apparently simple question; the relative numbers of minima and saddle points in generic landscapes. For $N=2$ we obtained strong results from Morse theory for general functions. At larger values of $N$ we begin from the default assumption that the landscape can be modeled as a Gaussian random function and generalize methods used by Bardeen, Bond, Kaiser and Szalay to analyze primordial density fluctuations [18] to treat this problem in $N$ dimensions.

Our results demonstrate that for Gaussian random fields, saddles outnumber minima by a factor of roughly $2^{N}$. This is in contrast to analyses that treat the Hessians associated with extrema as random matrices, which suggest that the ratio is closer to $\exp \left(-\alpha N^{2}\right)$ where $\alpha$ is a positive constant of order unity. The discrepancy arises because Hessian matrices associated with a random function are not composed of independent and identically distributed elements. Consequently, while the present work shares the fundamental philosophy of Ref. [6] that a sufficiently complex landscape can be modelled as a random distribution, the analysis must focus on the underlying function, and not the individual Hessians. Note also that for the Gaussian random functions the relative number of stationary points is


FIG. 4. Graphs of $\Sigma(\alpha)$ defined in (43) obtained from a binomial distribution in black (top curve), our exact calculation in blue (the middle curve) and the large- $N$ calculation of Bray-Dean in red (the bottom curve) for $N=10,50$ and 100 . We see that for small $N$ the binomial is a good approximation and while the $N=100$ case is close to the large $N$ limit [19] there is still a significant mismatch, when $\alpha$ is close to 0 or 1 .


FIG. 5. Probability of finding a minimum among different stationary points as a function of $N$ the number of fields. It is very well approximated by $\ln P=1.32-0.87 N$ or $P \propto 2.39^{-N}$.
independent of the power spectrum and depends only the ratios of integrals over the $\lambda_{i}$.
The analysis in this paper has established generic expectations for the relative numbers of different types of extrema in Gaussian random function in many dimensions. A Gaussian random function with zero mean is arguably one of simplest possible specifications of the landscape architecture, and the distribution of potential cosmological constants $p(\Lambda)$ can be obtained in the large- $N$ limit [19]. We have shown that the total number of minima is only exponentially suppressed by increasing $N$, but $p(\Lambda \mid \Lambda \approx 0)$ will also depend strongly on the values of $\sigma_{1}$ and $\sigma_{2}{ }^{[5}$ We will save the full analysis of this scenario for future work.

Beyond a pure Gaussian random function, a more physically realistic landscape architecture might include both a random function and an overall potential arising from Planck-scale operators which is dominant at large values of $\bar{\phi}$. In this case, $p(\Lambda \mid \Lambda \approx 0)$ will depend on the overall position in the landscape. Conversely, the layering phenomenon described in Ref 43] implies that the minima of $V$ will all be low-lying for many possible landscape architectures in which case $p(\Lambda \mid \Lambda \approx 0)$ will be super-exponentially small and the putative landscape cannot supply a single minimum in which $\Lambda$ is consistent with observations or even our

[^5]existence. Interestingly, in this situation the specific multiverse associated with the assumed form $V(\bar{\phi})$ would generate a strong prediction for $\Lambda$, and underlying hypothesis could be rejected with confidence. As a consequence, considerations of landscape architectures will - at the very least - provide a sandbox for exploring circumstances in which we can draw reliable inferences about multiverse scenarios, even in the presence of anthropic selection. From this starting point we can then consider landscape architectures which are physically reasonable while preserving, so far as possible, the overall brevity of the underlying specification.

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## Appendix A: Random Gaussian ensemble of two fields

In this case the vector $\alpha$ has a simple form $\alpha_{i}=\left\{\eta_{1}, \eta_{2}, \zeta_{11}, \zeta_{22}, \zeta_{12}\right\}$ and we have

$$
M=\left(\begin{array}{ccccc}
\frac{\sigma_{1}^{2}}{2} & 0 & 0 & 0 & 0  \tag{A1}\\
0 & \frac{\sigma_{1}^{2}}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{3 \sigma_{2}^{2}}{8} & \frac{\sigma_{2}^{2}}{8} & 0 \\
0 & 0 & \frac{\sigma_{2}^{2}}{8} & \frac{3 \sigma_{2}^{2}}{8} & 0 \\
0 & 0 & 0 & 0 & \frac{\sigma_{2}^{2}}{8}
\end{array}\right), \quad K=\left(\begin{array}{ccccc}
\frac{2}{\sigma_{1}^{2}} & 0 & 0 & 0 & 0 \\
0 & \frac{2}{\sigma_{1}^{2}} & 0 & 0 & 0 \\
0 & 0 & \frac{3}{\sigma_{2}^{2}} & -\frac{1}{\sigma_{2}^{2}} & 0 \\
0 & 0 & -\frac{1}{\sigma_{2}^{2}} & \frac{3}{\sigma_{2}^{2}} & 0 \\
0 & 0 & 0 & 0 & \frac{8}{\sigma_{2}^{2}}
\end{array}\right)
$$

We can write the Hessian matrix in terms of two eigenvalues and a single Euler angle

$$
\begin{equation*}
\left\{\zeta_{11}, \zeta_{12}, \zeta_{22}\right\}=\left\{\lambda_{2} \sin ^{2} \theta+\lambda_{1} \cos ^{2} \theta,\left(\lambda_{1}-\lambda_{2}\right) \sin \theta \cos \theta, \lambda_{1} \sin ^{2} \theta+\lambda_{2} \cos ^{2} \theta\right\} \tag{A2}
\end{equation*}
$$

Jacobian of this transformation is given by

$$
\begin{equation*}
J=\left|\frac{\partial\left\{\zeta_{11}, \zeta_{12}, \zeta_{22}\right\}}{\partial\left\{\lambda_{1}, \lambda_{2}, \theta\right\}}\right|=\left(\lambda_{1}-\lambda_{2}\right) \tag{A3}
\end{equation*}
$$

Because $\tilde{Q}$ in 22 is rotationally invariant, we evaluate it at $\theta=0$.

$$
\begin{equation*}
\tilde{Q}=\frac{6\left(\eta_{1}^{2}+\eta_{2}^{2}\right) \sigma_{2}^{2}+\left(9 \lambda_{1}^{2}-6 \lambda_{2} \lambda_{1}+9 \lambda_{2}^{2}\right) \sigma_{1}^{2}}{6 \sigma_{1}^{2} \sigma_{2}^{2}} \tag{A4}
\end{equation*}
$$

The $\delta$ functions in (23) sets $\eta_{1}=\eta_{2}=0$ which leads to

$$
\begin{equation*}
\tilde{Q}=\frac{3 \lambda_{1}^{2}-2 \lambda_{2} \lambda_{1}+3 \lambda_{2}^{2}}{2 \sigma_{2}^{2}} . \tag{A5}
\end{equation*}
$$

The probability density of $\alpha_{i}$ simplifies immensely

$$
\begin{equation*}
p\left(\alpha_{i}\right)=\frac{1}{(2 \pi)^{5 / 2}} \frac{16}{\sigma_{1}^{2} \sigma_{2}^{3} \sqrt{3}} \exp \left(-\frac{3 \lambda_{1}^{2}-2 \lambda_{2} \lambda_{1}+3 \lambda_{2}^{2}}{2 \sigma_{2}^{2}}\right) . \tag{A6}
\end{equation*}
$$

Now we can calculate the number of minima, maxima and saddle points (keeping in mind $\lambda_{1} \geq \lambda_{2}$ and $\operatorname{det}\left(\zeta_{i j}\right)=\lambda_{1} \lambda_{2}$ and a factor of $\frac{1}{2}$ for the double counting in rotation group).

$$
\begin{align*}
n_{i} & =\frac{1}{2} \int_{0}^{2 \pi} d \theta \int_{-\infty}^{\infty} d \lambda_{1} \int_{-\infty}^{\lambda_{1}} d \lambda_{2} \lambda_{1} \lambda_{2}\left(\lambda_{1}-\lambda_{2}\right) \frac{1}{(2 \pi)^{5 / 2}} \frac{16}{\sigma_{1}^{2} \sigma_{2}^{3} \sqrt{3}} \exp \left(-\frac{3 \lambda_{1}^{2}-2 \lambda_{2} \lambda_{1}+3 \lambda_{2}^{2}}{2 \sigma_{2}^{2}}\right) \theta_{H} \\
& =\frac{1}{(2 \pi)^{3 / 2}} \frac{8 \sigma_{2}^{2}}{\sigma_{1}^{2} \sqrt{3}} \int_{-\infty}^{\infty} d \lambda_{1} \int_{-\infty}^{\lambda_{1}} d \lambda_{2} \lambda_{1} \lambda_{2}\left(\lambda_{1}-\lambda_{2}\right) \exp \left(-\frac{3 \lambda_{1}^{2}-2 \lambda_{2} \lambda_{1}+3 \lambda_{2}^{2}}{2}\right) \theta_{H} \tag{A7}
\end{align*}
$$

where $\theta_{H}$ is the appropriate $\theta$ function. It is easy to check that this integral vanishes without a $\theta$ function as expected form the Morse theory. We get

$$
\begin{equation*}
\left\{n_{\min }, n_{\text {saddle }}, n_{\max }\right\}=\frac{\sigma_{2}^{2}}{24 \pi \sigma_{1}^{2}}\{1,2,1\} \tag{A8}
\end{equation*}
$$

This is coincidently a binomial distribution for the relative number of maxima, minima and saddle points. The relative numbers from a random Gaussian ensemble is

$$
\begin{equation*}
\left\{n_{\min }, n_{\text {saddle }}, n_{\max }\right\} \propto\{1,2(1+\sqrt{2}), 1\} \tag{A9}
\end{equation*}
$$

## Appendix B: $\mathbf{N}=3$

In this case we define the following vector

$$
\begin{equation*}
\alpha=\left\{\eta_{1}, \eta_{2}, \eta_{3}, \zeta_{11}, \zeta_{22}, \zeta_{33}, \zeta_{23}, \zeta_{13}, \zeta_{12}\right\} \tag{B1}
\end{equation*}
$$

In this basis the matrices $M$ and $K$ are given by

$$
M=\left(\begin{array}{ccccccccc}
\frac{\sigma_{1}^{2}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{B2}\\
0 & \frac{\sigma_{1}^{2}}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sigma_{1}^{2}}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sigma_{2}^{2}}{5} & \frac{\sigma_{2}^{2}}{15} & \frac{\sigma_{2}^{2}}{15} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sigma_{2}^{2}}{15} & \frac{\sigma_{2}^{2}}{5} & \frac{\sigma_{2}^{2}}{15} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sigma_{2}^{2}}{15} & \frac{\sigma_{2}^{2}}{15} & \frac{\sigma_{2}^{2}}{5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_{2}^{2}}{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_{2}^{2}}{15} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_{2}^{2}}{15}
\end{array}\right), K=\left(\begin{array}{ccccccccc}
\frac{3}{\sigma_{1}^{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3}{\sigma_{1}^{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{3}{\sigma_{1}^{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{6}{\sigma_{2}^{2}} & -\frac{3}{2 \sigma_{2}^{2}} & -\frac{3}{2 \sigma_{2}^{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{3}{2 \sigma_{2}^{2}} & \frac{6}{\sigma_{2}^{2}} & -\frac{3}{2 \sigma_{2}^{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{3}{2 \sigma_{2}^{2}} & -\frac{3}{2 \sigma_{2}^{2}} & \frac{6}{\sigma_{2}^{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{\sigma_{2}^{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{\sigma_{2}^{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{15}{\sigma_{2}^{2}}
\end{array}\right) .
$$

The most general rotation in three dimension is given in terms of the Euler angles $\xi, \beta$ and $\gamma$, where $0 \leq \xi, \gamma<2 \pi$ and $0 \leq \beta<\pi$. The rotation is given by

$$
R_{\text {Euler }}=\left(\begin{array}{ccc}
\cos \xi \cos \beta-\cos \beta \sin \xi \sin \gamma & \cos \gamma \sin \xi+\cos \xi \cos \beta \sin \gamma \sin \beta \sin \gamma  \tag{B3}\\
-\cos \beta \cos \gamma \sin \xi-\cos \xi \sin \gamma & \cos \xi \cos \beta \cos \gamma-\sin \xi \sin \gamma \cos \gamma \sin \beta \\
\sin \xi \sin \beta & -\cos \xi \sin \beta & \cos \beta
\end{array}\right)
$$

Again we rewrite the matrix of second derivatives in terms of the eigenvalues and rotation angles

$$
\left(\begin{array}{ccc}
\xi_{11} & \xi_{12} & \xi_{13}  \tag{B4}\\
\xi_{12} & \xi_{22} & \xi_{23} \\
\xi_{13} & \xi_{23} & \xi_{33}
\end{array}\right)=R_{\text {Euler }}\left(\begin{array}{ccc}
\lambda_{1} & 0 & 0 \\
0 & \lambda_{2} & 0 \\
0 & 0 & \lambda_{3}
\end{array}\right) R_{\text {Euler }}^{T}
$$

We can easily calculate the Jacobian using Mathematica to get

$$
\begin{equation*}
J=\left|\frac{\partial\left\{\zeta_{11}, \zeta_{22}, \zeta_{33}, \zeta_{23}, \zeta_{13}, \zeta_{23}\right\}}{\partial\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}, \xi, \beta, \gamma\right\}}\right|=\left(\lambda_{1}-\lambda_{2}\right)\left(\lambda_{1}-\lambda_{3}\right)\left(\lambda_{2}-\lambda_{3}\right) \sin \beta \tag{B5}
\end{equation*}
$$

To get the distribution of $\alpha_{i}$ vectors, we again calculate $\tilde{Q}$

$$
\begin{equation*}
\tilde{Q}=\frac{3\left(2 \zeta_{11}^{2}-\left(\zeta_{22}+\zeta_{33}\right) \zeta_{11}+2 \zeta_{22}^{2}+2 \zeta_{33}^{2}+5\left(\zeta_{12}^{2}+\zeta_{13}^{2}+\zeta_{23}^{2}\right)-\zeta_{22} \zeta_{33}\right)}{2 \sigma_{2}^{2}}+\frac{3\left(\eta_{1}^{2}+\eta_{2}^{2}+\eta_{3}^{2}\right)}{2 \sigma_{1}^{2}} \tag{B6}
\end{equation*}
$$

We evaluate this at $\eta_{1}=\eta_{2}=\eta_{3}=0$ because of the delta functions and we use the spherical symmetry to set $\xi=\beta=\gamma=0$ which makes $\zeta_{12}=\zeta_{13}=\zeta_{23}=0$ and $\zeta_{11}=\lambda_{1}, \zeta_{22}=\lambda_{2}$ and $\zeta_{33}=\lambda_{3}$. This gives

$$
\begin{equation*}
\tilde{Q}=\frac{3}{4 \sigma_{2}^{2}}\left(5 \sum \lambda_{i}^{2}-\left(\sum \lambda_{i}\right)^{2}\right) . \tag{B7}
\end{equation*}
$$

From here the density of different type of minima is given by

$$
\begin{align*}
N_{i}= & \frac{5^{5 / 2} 3^{9 / 2}}{2 \sigma_{1}^{3} \sigma_{2}^{6}(2 \pi)^{9 / 2}} \int_{0}^{2 \pi} d \alpha \int_{0}^{2 \pi} d \gamma \int_{0}^{\pi} d \beta \sin \beta \int d \eta_{1} d \eta_{2} d \eta_{3} \delta\left(\eta_{1}\right) \delta\left(\eta_{2}\right) \delta\left(\eta_{3}\right) \\
& \int d \lambda_{1} d \lambda_{2} d \lambda_{3} e^{-\tilde{Q}}\left(\lambda_{1}-\lambda_{2}\right)\left(\lambda_{1}-\lambda_{3}\right)\left(\lambda_{2}-\lambda_{3}\right) \lambda_{1} \lambda_{2} \lambda_{3} \theta_{H} \\
= & 9\left(\frac{15}{2 \pi}\right)^{5 / 2} \frac{\sigma_{2}^{3}}{\sigma_{1}^{3}} \int \Pi\left(\lambda_{i} d \lambda_{i}\right)\left(\lambda_{1}-\lambda_{2}\right)\left(\lambda_{1}-\lambda_{3}\right)\left(\lambda_{2}-\lambda_{3}\right) \exp \left[-\frac{3}{4}\left(5 \sum \lambda_{i}^{2}-\left(\sum \lambda_{i}\right)^{2}\right)\right] \theta_{H} . \tag{B8}
\end{align*}
$$

Again $\theta_{H}$ determines the type of stationary point we want to calculate. Again we can show that without the $\theta$ function this integral vanishes as expected from Morse theory. Now we evaluate the density of different type of stationary points.

$$
\begin{align*}
& n_{0}=n_{3}=\frac{29 \sqrt{15}-18 \sqrt{10}}{450 \pi^{2}} \frac{\sigma_{2}^{3}}{\sigma_{1}^{3}} \\
& n_{1}=n_{2}=\frac{29 \sqrt{15}+18 \sqrt{10}}{450 \pi^{2}} \frac{\sigma_{2}^{3}}{\sigma_{1}^{3}} . \tag{B9}
\end{align*}
$$

The ratio $n_{1} / n_{0}$ is approximately 3.05 which is close to but still distinct from the value of 3 that we would expect from binomial distribution.

## Appendix C: Distribution of different types of stationary points for $N=50$

In this section we present the data for the chance of finding a stationary point with $n$ negative eigenvalues from a set of stationary points. Because we expect $P(i)=P(50-i)$ from the symmetry $U \rightarrow-U$ we only show the data for $n$ up to 25 in Table IV.
[1] R. Bousso and J. Polchinski, JHEP 06, 006 (2000), arXiv:hep-th/0004134 [hep-th].
[2] J. L. Feng, J. March-Russell, S. Sethi, and F. Wilczek, Nucl. Phys. B602, 307 (2001), arXiv:hep-th/0005276 [hep-th].
[3] T. Banks, M. Dine, P. J. Fox, and E. Gorbatov, JCAP 0306, 001 (2003), arXiv:hepth/0303252 [hep-th].
[4] J. Polchinski, in The Quantum Structure of Space and Time (2006) pp. 216-236, arXiv:hepth/0603249 [hep-th].
[5] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, JHEP 06, 060 (2007), arXiv:hepth/0601001 [hep-th].

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $P(n)$ | $8.43 \times 10^{-19}$ | $1.07 \times 10^{-16}$ | $5.69 \times 10^{-15}$ | $1.77 \times 10^{-13}$ | $3.65 \times 10^{-12}$ | $5.59 \times 10^{-11}$ | $6.51 \times 10^{-10}$ |
| $n$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $P(n)$ | $6.11 \times 10^{-9}$ | $4.67 \times 10^{-8}$ | $2.98 \times 10^{-7}$ | $1.64 \times 10^{-6}$ | $7.73 \times 10^{-6}$ | $3.18 \times 10^{-5}$ | $1.14 \times 10^{-4}$ |
| $n$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $P(n)$ | $3.64 \times 10^{-4}$ | $1.04 \times 10^{-3}$ | $2.62 \times 10^{-3}$ | $5.95 \times 10^{-3}$ | $1.22 \times 10^{-2}$ | $2.28 \times 10^{-2}$ | $3.83 \times 10^{-2}$ |
| $n$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| $P(n)$ | $5.81 \times 10^{-2}$ | $8.00 \times 10^{-2}$ | 0.101 | 0.117 | 0.122 | 0.116 | 0.101 |

TABLE IV. The values of $P(n)$ defined above for $N=50$. From here we can reconstruct $\Sigma(\alpha)$ shown in Fig.4. These numbers are obtained from a Monte-Carlo integration which is accurate to 1 percent.
[6] A. Aazami and R. Easther, JCAP 0603, 013 (2006), arXiv:hep-th/0512050 [hep-th].
[7] D. S. Dean and S. N. Majumdar, Phys. Rev. Lett. 97, 160201 (2006), arXiv:cond-mat/0609651 [cond-mat]
[8] D. Marsh, L. McAllister, and T. Wrase, JHEP 03, 102 (2012), arXiv:1112.3034 [hep-th].
[9] D. Battefeld, T. Battefeld, and S. Schulz, JCAP 1206, 034 (2012), arXiv:1203.3941 [hep-th].
[10] A. Masoumi and A. Vilenkin, JCAP 1603, 054 (2016), arXiv:1601.01662 [gr-qc].
[11] B. Greene, D. Kagan, A. Masoumi, D. Mehta, E. J. Weinberg, and X. Xiao, Phys. Rev. D88, 026005 (2013), arXiv:1303.4428 [hep-th].
[12] A. Aravind, D. Lorshbough, and S. Paban, Phys. Rev. D89, 103535 (2014), arXiv:1401.1230 [hep-th].
[13] M. Dine and S. Paban, JHEP 10, 088 (2015), arXiv:1506.06428 [hep-th].
[14] M. Dine, (2015), arXiv:1512.08125 [hep-th].
[15] R. Easther and L. McAllister, JCAP 0605, 018 (2006), arXiv:hep-th/0512102 [hep-th].
[16] R. Easther, J. Frazer, H. V. Peiris, and L. C. Price, Phys. Rev. Lett. 112, 161302 (2014), arXiv:1312.4035 [astro-ph.CO].
[17] L. C. Price, H. V. Peiris, J. Frazer, and R. Easther, Phys. Rev. Lett. 114, 031301 (2015), arXiv:1409.2498 [astro-ph.CO].
[18] J. M. Bardeen, J. R. Bond, N. Kaiser, and A. S. Szalay, Astrophys. J. 304, 15 (1986).
[19] A. J. Bray and D. S. Dean, Phys. Rev. Lett. 98, 150201 (2007).
[20] A. De Simone, A. H. Guth, M. P. Salem, and A. Vilenkin, Phys. Rev. D78, 063520 (2008), arXiv:0805.2173 [hep-th].
[21] A. H. Guth, Phys. Rev. D23, 347 (1981).
[22] A. D. Linde, Second Seminar on Quantum Gravity Moscow, USSR, October 13-15, 1981, Phys. Lett. B108, 389 (1982).
[23] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982).
[24] P. A. R. Ade et al. (Planck), (2015), arXiv:1502.02114 [astro-ph.CO],
[25] J. Martin, C. Ringeval, and V. Vennin, Phys. Dark Univ. 5-6, 75 (2014), arXiv:1303.3787 [astro-ph.CO].
[26] S. Weinberg, Phys. Rev. D61, 103505 (2000), arXiv:astro-ph/0002387 [astro-ph].
[27] A. H. Guth, Phys. Rept. 333, 555 (2000), arXiv:astro-ph/0002156 [astro-ph].
[28] T. D. Lee and G. C. Wick, Phys. Rev. D9, 2291 (1974).
[29] S. Hawking and I. Moss, Physics Letters B 110, 35 (1982).
[30] S. R. Coleman, Phys. Rev. D15, 2929 (1977), [Erratum: Phys. Rev.D16,1248(1977)].
[31] C. G. Callan, Jr. and S. R. Coleman, Phys. Rev. D16, 1762 (1977).
[32] S. R. Coleman and F. De Luccia, Phys. Rev. D21, 3305 (1980).
[33] S. Chang, M. Kleban, and T. S. Levi, JCAP 0904, 025 (2009), arXiv:0810.5128 [hep-th],
[34] A. Aguirre and M. C. Johnson, Rept. Prog. Phys. 74, 074901 (2011), arXiv:0908.4105 [hep-th],
[35] S. M. Feeney, M. C. Johnson, D. J. Mortlock, and H. V. Peiris, Phys. Rev. Lett. 107, 071301 (2011), arXiv:1012.1995 [astro-ph.CO],
[36] S. O. Rice, Bell System Technical Journal 23, 282 (1944).
[37] T. C. Bachlechner, JHEP 04, 054 (2014), arXiv:1401.6187 [hep-th].
[38] G. P. Lepage, Journal of Computational Physics 27, 192 (1978).
[39] B. Gough, GNU Scientific Library Reference Manual - Third Edition, 3rd ed. (Network Theory Ltd., 2009).
[40] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, J. Chem. Phys. 21, 1087 (1953).
[41] W. K. Hastings, Biometrika 57, 97 (1970).
[42] W. J. Handley, M. P. Hobson, and A. N. Lasenby, Mon. Not. Roy. Astron. Soc. 450, L61 (2015), arXiv:1502.01856 [astro-ph.CO].
[43] A. Auffinger, G. B. Arous, and J. Černý, Communications on Pure and Applied Mathematics 66, 165 (2013).


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[^1]:    ${ }^{1}$ This does not imply that the form of the landscape is arbitrary, but rather that it consists of a sufficient number of largely uncorrelated terms that it can be treated probabilistically and in this scenario the most natural choice of distribution will be Gaussian, motivated by the central limit theorem.

[^2]:    ${ }^{2}$ We also analyzed a "Cartesian" landscape in which $k_{1}$, $k_{2}$ and $k_{3}$ are summed over, up to a fixed cutoff. This function is anisotropic in $k$ and does satisfy the criteria that define the random functions analyzed here. The corresponding ratios were $1: 3: 3: 1$, which is to be expected since there is more high-frequency power in directions where $\left|k_{1}\right| \sim\left|k_{2}\right| \sim\left|k_{3}\right|$ as the Hessian matrices in a diagonal basis are dominated by the diagonal terms.

[^3]:    ${ }^{3}$ We have corrected a typo in Appendix B of BBKS.

[^4]:    ${ }^{4}$ This code was implemented in Power-BASIC and run on several desktop machines.

[^5]:    ${ }^{5}$ We may be able to extend our numerical methods to evaluate $p(\Lambda)$, but the integrand will contain $\sigma_{1}$ and $\sigma_{2}$ as well as $\Lambda$ so these computations will be less trivial than those performed here.

