

Quantum features of natural cellular automata

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Abstract. Cellular automata can show well known features of quantum mechanics, such as a linear rule according to which they evolve and which resembles a discretized version of the Schrödinger equation. This includes corresponding conservation laws. The class of “natural” Hamiltonian cellular automata is based exclusively on integer-valued variables and couplings and their dynamics derives from an Action Principle. They can be mapped reversibly to continuum models by applying Sampling Theory. Thus, “deformed” quantum mechanical models with a finite discreteness scale l are obtained, which for $l \rightarrow 0$ reproduce familiar continuum results. We have recently demonstrated that such automata can form “multipartite” systems consistently with the tensor product structures of nonrelativistic many-body quantum mechanics, while interacting and maintaining the linear evolution. Consequently, the Superposition Principle fully applies for such primitive discrete deterministic automata and their composites and can produce the essential quantum effects of interference and entanglement.

1. Why cellular automata?

The *Cellular Automaton Interpretation* of Quantum Mechanics (QM) has recently been outlined by G. 't Hooft [1]. This presents an attempt to redesign the foundations of quantum theory in accordance with essentially classical concepts, such as determinism and existence of ontological states of reality. Which entails a surprising and intuitive explanation of the Born rule and the apparent collapse of quantum mechanical states in measurement processes.

Such an approach may be founded on the observation that quantum mechanical features arise in a large variety of deterministic and, loosely speaking, “classical” models. – So far, however, practically all of these models have been exceptional in that they cannot easily be generalized to cover real phenomena, incorporating interactions and relativity. Yet Cellular Automata (CA) may provide the necessary versatility, as we shall presently continue to discuss [2, 3]. For a large variety of earlier attempts in this field, see, for example, Refs. [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and further references there.

To begin with, let us recall the linearity of quantum mechanics (QM) as a crucial aspect of the unitary dynamics embodied in the Schrödinger equation. This linearity does not depend on the particular object under study, provided it is sufficiently isolated from anything else. It is reflected in the Superposition Principle and implies the possibilities of interference effects and of non-factorizable states of composite objects, *i.e.* entanglement in multipartite systems.

The linearity of QM has been questioned and nonlinear modifications have been proposed before – notwithstanding various ‘no-go’ arguments – especially in order to test experimentally the robustness of QM against such nonlinear deformations. Until now no deviations from the predictions of QM have been observed, in particular no indications of nonlinearity.

We have studied a seemingly unrelated *discrete* dynamical theory, *i.e.*, which appears to deviate substantially from quantum theory. Nevertheless, it has been shown with the help of Sampling Theory that the deterministic mechanics of the class of discrete *Hamiltonian CA* can be mapped one-to-one to continuum models pertaining to nonrelativistic QM, which are modified by the presence of a fundamental time scale [2, 3].

Perhaps surprisingly, this construction of a linear relation between CA and QM with a nonzero discreteness scale, compatible with the consistency of the Action Principle underlying the discrete dynamics on one side and the locality of the continuum description on the other, requires that both theories are linear [16].

This result has led us to consider composite objects formed from CA subsystems [17]. – Clearly, QM is special in that it is characterized not only by interference effects, like any classical wave theory would be, but also by the tensor product structures of states and observables applying for composite systems, which allow for entanglement. – It is not obvious that CA can constitute composites which conform with QM in this aspect, in the limit of negligible discreteness scale. A main obstruction could have been the fact that the state space of Hamiltonian CA is not a complex projective space, since the norm of the analogue of state vectors is not conserved by the dynamics; instead there is a conserved two-time correlation function, as we shall recall, which becomes the familiar norm only in the continuum limit.

In Section 2., we will review the earlier results concerning individual CA and briefly indicate the construction of multipartite CA – in such a way that the Superposition Principle is respected and the composition rules of CA are compatible with those of QM. This ‘outlandish’ perspective based on CA may lead to additional insight in regard to interference and entanglement and eventually to an understanding of nonrelativistic QM in accordance with Ref. [1]. Concluding remarks are presented in Section 3.

2. Natural Hamiltonian CA – action, evolution, conservation laws and composites

We describe classical Hamiltonian CA with countably many degrees of freedom presently in terms of *complex integer-valued* state variables ψ_n^α (also known as *Gaussian integers*), where $\alpha \in \mathbf{N}_0$ denote different degrees of freedom and $n \in \mathbf{Z}$ different states labelled by this discrete *clock variable*. Various equivalent forms of the action for such CA exist, as indicated earlier [2]. We will employ a particularly compact form here, which is useful for the following construction of composite CA in analogy with multipartite QM systems.

Let $\hat{H} := \{H^{\alpha\beta}\}$ denote a self-adjoint matrix of Gaussian integers that will play the role of the Hamilton operator. Furthermore, we introduce the suggestive notation $\dot{O}_n := O_{n+1} - O_{n-1}$, for any quantity O_n depending on the clock variable n . The summation convention for Greek indices, $r^\alpha s^\alpha \equiv \sum_\alpha r^\alpha s^\alpha$, will often allow us to simplify notation further by suppressing them altogether, for example, writing $\psi_n^{*\alpha} H^{\alpha\beta} \psi_n^\beta \equiv \psi_n^* \hat{H} \psi_n$.

Then, with ψ_n^α and $\psi_n^{*\alpha}$ as independent variables, the CA action \mathcal{S} is defined by:

$$\mathcal{S}[\psi, \psi^*] := \sum_n \left[\frac{1}{2i} (\psi_n^* \dot{\psi}_n - \dot{\psi}_n^* \psi_n) + \psi_n^* \hat{H} \psi_n \right] \equiv \psi^* \hat{\mathcal{S}} \psi \quad , \quad (1)$$

where the operator $\hat{\mathcal{S}}$ is a useful abbreviation, *cf.* below. In order to set up the variational principle, we introduce *integer-valued* variations δf to be applied to a polynomial g as follows:

$$\delta_f g(f) := [g(f + \delta f) - g(f - \delta f)] / 2\delta f \quad , \quad (2)$$

and $\delta_f g \equiv 0$, if $\delta f = 0$. – Note that variations of terms that are *constant, linear, or quadratic* in integer-valued variables yield analogous results as standard infinitesimal variations of corresponding expressions in the continuum. – Making use of these ingredients, we postulate the variational principle:

[*CA Action Principle*] The discrete evolution of a CA is determined by stationarity of the action under arbitrary integer-valued variations of all dynamical variables, $\delta\mathcal{S} = 0$. •

Let us point out several characteristics of this *CA Action Principle*:

i) While infinitesimal variations do not conform with integer valuedness, there is *a priori* no restriction of integer variations. Hence *arbitrary* integer-valued variations must be admitted.

ii) One could imagine contributions to the action (1) which are of higher than second order in ψ_n or ψ_n^* . However, in view of arbitrary variations $\delta\psi_n^\alpha$ and $\delta\psi_n^{*\alpha}$, such additional contributions to the action must be absent for consistency. Otherwise the number of equations of motion generated by variation of the action, according to Eq. (2), would exceed the number of variables. (A limited number of such remainder terms, which are nonzero only for some fixed values of n , could serve to encode the *initial conditions* for the evolution.)

These features of the *CA Action Principle* are essential in constructing a map between Hamiltonian CA and equivalent quantum mechanical continuum models [2]. – For curiosity, generalizations of the variations defined in Eq. (2) have also been considered, which allow higher than second order polynomial terms in the action. While leading to consistent discrete equations of motion, however, these nonlinear equations generally are beset with undesirable nonlocal features in the corresponding continuum description [16].

2.1. Equations of motion

The equations of motion are obtained by applying the *CA Action Principle* to the action \mathcal{S} of Eq. (1) with the variations as defined in Eq. (2). Thus, variations $\delta\psi_n^*$ and $\delta\psi_n$, respectively, yield discrete analogues of the Schrödinger equation and its adjoint:

$$\dot{\psi}_n = \frac{1}{i}\hat{H}\psi_n , \quad (3)$$

$$\dot{\psi}_n^* = -\frac{1}{i}(\hat{H}\psi_n)^* , \quad (4)$$

recalling that $\hat{H} = \hat{H}^\dagger$ and $\dot{\psi}_n = \psi_{n+1} - \psi_{n-1}$, *etc.* We remark that the action \mathcal{S} vanishes when evaluated for solutions of these finite difference equations.

Note that by setting $\psi_n^\alpha =: x_n^\alpha + ip_n^\alpha$, with real integer-valued variables x_n^α and p_n^α , and suitably separating real and imaginary parts of Eqs. (3)–(4), the discrete equations assume a form that resembles Hamilton's equations for a network of coupled classical oscillators [18, 19]:

$$\dot{x}_n^\alpha = h_S^{\alpha\beta} p_n^\beta + h_A^{\alpha\beta} x_n^\beta , \quad \dot{p}_n^\alpha = -h_S^{\alpha\beta} x_n^\beta + h_A^{\alpha\beta} p_n^\beta , \quad (5)$$

where we also split the self-adjoint matrix \hat{H} into real integer-valued symmetric and antisymmetric parts, respectively, $H^{\alpha\beta} =: h_S^{\alpha\beta} + ih_A^{\alpha\beta}$. These equations led us to the name *Hamiltonian CA*; this is furthermore justified by the fact that analogues of Poisson brackets and classical like observables can be introduced here [20].

2.2. Conservation laws

The time-reversal invariant equations of motion of Section 2.1. give rise to conservation laws which are in *one-to-one correspondence* with those of the related Schrödinger equation in the continuum. It is straightforward to demonstrate the following theorem.

[*Theorem A*] For any matrix \hat{G} that commutes with \hat{H} , $[\hat{G}, \hat{H}] = 0$, there is a *discrete conservation law*:

$$\psi_n^{*\alpha} G^{\alpha\beta} \dot{\psi}_n^\beta + \dot{\psi}_n^{*\alpha} G^{\alpha\beta} \psi_n^\beta = 0 . \quad (6)$$

For self-adjoint \hat{G} , defined by Gaussian integers, this relation is about real integer quantities. •

A rearrangement of Eq. (6) yields the corresponding conserved quantity $q_{\hat{G}}$ (using matrix notation, as before):

$$q_{\hat{G}} := \psi_n^* \hat{G} \psi_{n-1} + \psi_{n-1}^* \hat{G} \psi_n = \psi_{n+1}^* \hat{G} \psi_n + \psi_n^* \hat{G} \psi_{n+1} , \quad (7)$$

i.e. a real integer-valued two-point correlation function which is invariant under a shift $n \rightarrow n + m$, $m \in \mathbf{Z}$. – In particular, for $\hat{G} := \hat{1}$, the corresponding conservation law amounts to a constraint on the state variables:

$$q_{\hat{1}} = 2\text{Re } \psi_n^* \psi_{n-1} = 2\text{Re } \psi_{n+1}^* \psi_n = \text{const} . \quad (8)$$

This can be anticipated to play a similar role for discrete CA as the familiar normalization of state vectors in continuum QM. – We may also define the following symmetrized quantity:

$$\psi_n^* \hat{Q} \psi_n := \frac{1}{2} \text{Re } \psi_n^* (\psi_{n+1} + \psi_{n-1}) \equiv \frac{1}{2} \text{Re } \psi_n^{*\alpha} (\psi_{n+1}^\alpha + \psi_{n-1}^\alpha) , \quad (9)$$

which, by Eq. (8), is conserved as well.

2.3. Continuum representation

We have shown earlier how to construct an one-to-one invertible map between the dynamics of discrete Hamiltonian CA and continuum QM in presence of a fundamental time scale [2, 3, 16]. Due to this discreteness scale l , continuous time wave functions are *bandlimited*, *i.e.*, their Fourier transforms have only finite support in frequency space, $\omega \in [-\pi/l, \pi/l]$. Hence, we can apply Sampling Theory, in order to reconstruct continuous time signals, wave functions $\psi^\alpha(t)$, from their discrete samples, the CA state variables ψ_n^α , and *vice versa* [21, 22, 23].

Let us represent here the resulting mapping rules obtained through the reconstruction formula provided by *Shannon's Theorem* [21, 22]:

$$\psi_n^\alpha \longmapsto \psi^\alpha(t) , \quad (10)$$

$$\psi_{n\pm 1}^\alpha \longmapsto \exp[\mp l \frac{d}{dt}] \psi^\alpha(t) = \psi^\alpha(t \mp l) , \quad (11)$$

$$\psi^\alpha(nl) \longmapsto \psi_n^\alpha , \quad (12)$$

keeping in mind that the continuum wave function is bandlimited.

These results allow to map CA equations of motion, in particular Eqs. (3)–(4) to appropriate continuum versions. Corresponding to Eqs. (6)–(9), there exist analogous conservation laws and conserved quantities, which are found by applying the mapping rules separately to all wave function factors that appear. Thus, for example, we obtain from Eq. (9) the conserved quantity:

$$\text{const} = \psi_n^* \hat{Q} \psi_n \longmapsto \psi^*(t) \hat{Q} \psi(t) = \text{Re } \psi^*(t) \cosh[l \frac{d}{dt}] \psi(t) \quad (13)$$

$$= \psi^{*\alpha}(t) \psi^\alpha(t) + \frac{l^2}{2} \text{Re } \psi^{*\alpha}(t) \frac{d^2}{dt^2} \psi^\alpha(t) + O(l^4) , \quad (14)$$

which shows l -dependent corrections to the continuum limit, which here amounts to the usual conserved normalization $\psi^{*\alpha} \psi^\alpha = \text{const}$. Similarly, the Schrödinger equation with finite- l correction terms are obtained [2].

Up to this point, our considerations dealt with individual Hamiltonian CA. Based on these results, we will briefly indicate in the following the construction of multipartite systems.

2.4. From single to multipartite Hamiltonian CA [17]

Can discrete CA combine to form composite multipartite systems? – A positive answer to this question is crucial, if we would like the quantum features of CA that we have found, so far, to also include the full extent of the Superposition Principle. It is responsible for interference effects that we can find in single CA. However, a more dramatic consequence in QM is the possibility of entanglement in states or observables.

When scrutinizing possible constructions of multipartite CA, several requirements appear naturally. – One may wonder whether not only the *linearity* of the evolution law but also the *tensor product structure* of composite wave functions finds its analogue here. These are the fundamental ingredients of the usual continuum theory reflected in interference and entanglement. Which should be recovered in the continuum limit ($l \rightarrow 0$) of the CA picture, at least. Furthermore, when the discreteness scale l is finite, the dynamics of composites of CA which do not interact among each other should lead to *no spurious correlations* among them. This principle of “no correlations without interactions” holds in all of known physics.

However, we are facing some serious obstacles which seem to prevent satisfying these requirements, when trying to form composites of Hamiltonian CA.

The want-to-be discrete time derivative introduced before, $\dot{O}_n := O_{n+1} - O_{n-1}$, for any quantity O_n depending on the clock variable n , which appears all over in the CA equations of motion and conservation laws, does not obey the *Leibniz rule*:

$$[A_n \dot{B}_n] = \dot{A}_n \frac{B_{n+1} + B_{n-1}}{2} + \frac{A_{n+1} + A_{n-1}}{2} \dot{B}_n \neq \dot{A}_n B_n + A_n \dot{B}_n . \quad (15)$$

Similar observations can be expected for other definitions one might come up with. Ignoring this for a moment, consider a trial multi-CA equation of motion analogous to the single-CA Eq. (3):

$$\dot{\Psi}_n = \frac{1}{i} \hat{H}_0 \Psi_n , \quad (16)$$

where \hat{H}_0 may describe a block-diagonal Hamiltonian in the absence of interactions among the CA. Then, through Eq. (15), the required *factorization* of Eq. (16) is hindered on the left-hand side, since unphysical correlations will be produced among the components of a factorized wave function, such as

$$\Psi_n^{\alpha\beta\gamma\dots} = \psi_n^\alpha \phi_n^\beta \kappa_n^\gamma \dots , \quad (17)$$

or for superpositions of such factorized terms.

For a bipartite system, for example, we obtain immediately: $\dot{\Psi}_n^{\alpha\beta} = \dot{\psi}_n^\alpha (\phi_{n+1}^\beta + \phi_{n-1}^\beta)/2 + \psi \leftrightarrow \phi \neq \dot{\psi}_n^\alpha \phi_n^\beta + \psi_n^\alpha \dot{\phi}_n^\beta$. – Furthermore, applying the mapping rules of Section 2.3, before taking the limit $l \rightarrow 0$, we find that the bilinear terms here do not converge to the appropriate QM expression. Which should be $\partial_t(\psi^\alpha \phi^\beta) = (\partial_t \psi^\alpha) \phi^\beta + \psi^\alpha \partial_t \phi^\beta$, in order to allow the decoupling of two subsystems that do not interact.

This latter problem is a general one of nonlinear terms in the equations of motion of discrete CA, which we discussed before [16]: *The linear map provided by Shannon’s Theorem does not commute with the multiplication implied by the nonlinearities.* This follows from the explicit reconstruction formula (or any variant thereof that is linear) [2, 21, 22]. Also on the right-hand side of Eq. (16) we find this obstruction, when trying to map the equation with a factorized wave function to its continuous time description.

2.4.1. The many-time formulation It can be observed that the difficulties we pointed out arise from the implicit assumption that the components of a multipartite CA are *synchronized* to the extent that they share a common clock variable n . In Ref. [17], we have considered a radical way out of the impasse encountered by resorting to a *many-time* formalism. This means giving

up synchronization among parts of the composite CA by introducing a set of clock variables, $\{n(1), \dots, n(m)\}$, one for each one out of m components.

It may be surprising to encounter this in the present nonrelativistic context, since the many-time formalism has been introduced by Dirac, Tomonaga, and Schwinger in their respective formulations of relativistically covariant many-particle QM or quantum field theory, where a global synchronization cannot be maintained [24, 25, 26].

We replace here the single-CA action of Eq. (1) by an integer-valued multipartite-CA action:

$$\mathcal{S}[\Psi, \Psi^*] := \Psi^* \left(\sum_{k=1}^m \hat{\mathcal{S}}_{(k)} + \hat{\mathcal{I}} \right) \Psi, \quad (18)$$

with $\Psi := \Psi_{n_1 \dots n_m}^{\alpha_1 \dots \alpha_m}$ and, correspondingly, Ψ^* as independent *Gaussian integer* variables; the self-adjoint operator $\hat{\mathcal{I}}$ incorporates interactions between different CA; whereas $\hat{\mathcal{S}}_{(k)}$ is as introduced in Eq. (1), with the subscript $_{(k)}$ indicating that it acts *exclusively* on the pair of indices pertaining to the k -th single-CA subsystem:

$$\Psi^* \hat{\mathcal{S}}_{(k)} \Psi := \sum_{\{n_k\}} [(\text{Im } \Psi_{\dots n_k \dots}^* \dot{\Psi}_{\dots n_k \dots}^{\alpha_k \dots} + \Psi_{\dots n_k \dots}^* H_{(k)}^{\alpha_k \beta_k} \Psi_{\dots n_k \dots}^{\beta_k \dots})], \quad (19)$$

with summation over *all* clock variables (summation over twice appearing Greek indices remains understood); the $\dot{}$ -operation, however, acts only with respect to the explicitly indicated n_k , $\dot{f}(n_k) := f(n_k + 1) - f(n_k - 1)$, while the single-CA Hamiltonian, $\hat{H}_{(k)}$, requires a matrix multiplication, as before.

Obviously, we can apply the *CA Action Principle* also to the present situation with the generalized action of Eq. (18). This results in the following discrete equations of motion:

$$\sum_{k=1}^m \dot{\Psi}_{\dots n_k \dots}^{\alpha_k \dots} = \frac{1}{i} \left(\sum_{k=1}^m H_{(k)}^{\alpha_k \beta_k} \Psi_{\dots n_k \dots}^{\beta_k \dots} + \mathcal{I}^{\dots \alpha_k \dots \beta_1 \dots \beta_m} \Psi_{\dots n_k \dots}^{\beta_1 \dots \beta_m} \right), \quad (20)$$

together with the adjoint equations; here the interaction $\hat{\mathcal{I}}$, like $\hat{H}_{(k)}$, is assumed to be independent of the clock variables and the $\dot{}$ -operation acts only with respect to n_k in the k -th term on the left-hand side.

We have verified in Ref. [17] that the many-time formulation avoids the problems of a single-time multi-CA equation, such as Eq. (16), which we discussed.

In particular, in the absence of interactions with each other, between CA subsystems, $\hat{\mathcal{I}} \equiv 0$, *no unphysical correlations* are introduced among independent CA subsystems.

Furthermore, *continuous multi-time equations* corresponding to Eqs. (20) are obtained by applying the mapping rules given in Section 2.3. to the discrete equations. We find no problem of incompatibility between multiplication according to nonlinear terms *vs.* linear mapping according to *Shannon's Theorem*, since a separate mapping is applied for each one of the clock variables. This effectively replaces $n_k \rightarrow t_k$, $k = 1, \dots, m$, where t_k is a continuous real time variable. In this way, a *modified multi-time Schrödinger equation* is obtained:

$$\sum_{k=1}^m \sinh \left[l \frac{d}{dt_k} \right] \Psi_{\dots t_k \dots}^{\alpha_k \dots} = \frac{1}{i} \left(\sum_{k=1}^m H_{(k)}^{\alpha_k \beta_k} \Psi_{\dots t_k \dots}^{\beta_k \dots} + \mathcal{I}^{\dots \alpha_k \dots \beta_1 \dots \beta_m} \Psi_{\dots t_k \dots}^{\beta_1 \dots \beta_m} \right), \quad (21)$$

where an overall factor of two from the left-hand side has been absorbed into the matrices on the right. By construction, here Ψ is bandlimited with respect to each variable t_k .

Performing the continuum limit, $l \rightarrow 0$, we arrive at the multi-time Schrödinger equation (one power of l^{-1} providing the physical dimension of $\hat{H}_{(k)}$ and $\hat{\mathcal{I}}$) considered by Dirac and

Tomonaga [24, 25]. However, when l is fixed and finite, modifications in the form of powers of ld/dt_k arise on its left-hand side, similarly as in single CA case before.

In the present nonrelativistic context, it may be appropriate to identify $t_k \equiv t$, $k = 1, \dots, m$, in which case the operator on the left-hand side of Eq. (21), for $l \rightarrow 0$, can be simply replaced by d/dt . This results in the usual (single-time) *many-body Schrödinger equation*.

Finally, the study of the *conservation laws* of the multipartite CA equations of motion can be performed along the lines of Section 2.2. and analogous results have been obtained [17].

2.4.2. Remarks on the Superposition Principle in composite CA The equivalent discrete or continuous many-time equations (20) and (21) are both linear in the CA wave function Ψ . Therefore, superpositions of solutions of these equations also present solutions and the *Superposition Principle* does indeed hold for multipartite Hamiltonian CA.

As in the case of single CA, this entails the fact that these discrete systems – with all variables, parameters, *etc.* presented by Gaussian integers – can produce *interference* effects as in quantum mechanics. Even more interesting, their composites can also show *entanglement*, which is deemed an essential feature of QM. This follows from the form of the equations of motion, which allow for superpositions of factorized states.

For example, in the bipartite case ($k = 1, 2$), assuming that the individual CA are characterized by two degrees of freedom ($\alpha_k = 0, 1$), a time dependent analogue of the totally antisymmetric *Bell state* is given by:

$$\Psi \propto \psi_{n_1}^{\alpha_1=0} \psi_{n_2}^{\alpha_2=1} - \psi_{n_1}^{\alpha_1=1} \psi_{n_2}^{\alpha_2=0} \quad , \quad (22)$$

which may be a solution of appropriate discrete equations of motion.

We conclude with a word of warning concerning the freely used expressions here that we borrowed from QM, such as “wave functions” and “states”, in particular. They usually invoke the notion of vectors in a *Hilbert space*, which turns into a complex projective space upon normalization of the vectors. However, as has become obvious in Section 2.2., see Eqs. (8)–(9), and which can be seen similarly in the multipartite CA case [17], as long as the CA are truly discrete ($l \neq 0$), the normalization (squared) of vectors is not among the conserved quantities, hence not applicable, but is replaced by a conserved two-time correlation function instead.

Furthermore, contrary to first impression, perhaps, the envisaged space of states is *not* a Hilbert space, since it fails in two respects: the vector-space and completeness properties are missing. – First of all, the relevant *Gaussian integers* are not complete. Hence the completeness property of the space of states is lacking, which is built with these integers as underlying scalars. Secondly, the integers *only* featuring in all aspects of the CA do not form a field but a commutative ring (for the multiplication of vectors by such scalars there is no multiplicative inverse, such as exists, *e.g.*, for rational, real, or complex numbers). Therefore, we do not have a vector space over a field, as usual in QM. It has to be replaced by a more general structure. This is known as a module over a ring, presently a module over the commutative ring of Gaussian integers. This allows the construction of a linear space endowed with an integer-valued scalar product, *i.e.* a *unitary space*. Taking its incompleteness into account, then, the space of states in the presented CA theory can be classified as a *pre-Hilbert module over the commutative ring of Gaussian integers*.

In any case, we conclude that superpositions of states, interference effects, and entanglement, as in QM, all can be found already on the “primitive” level of the presently considered natural Hamiltonian CA, discrete single or multipartite systems which are characterized by (complex) integer-valued variables and couplings.

3. Conclusion

This presents a brief review of earlier work which has demonstrated surprising quantum features arising in integer-valued, hence “natural”, *Hamiltonian cellular automata* [2, 3, 16, 17, 20].

The study of this particular class of CA is motivated by ’t Hooft’s *Cellular Automaton Interpretation of QM* [1] and various recent attempts to construct models which may eventually lead to demonstrating that the essential features of QM can all be understood to emerge from pre-quantum deterministic dynamics.

The single CA we have considered allow practically for the first time to reconstruct quantum mechanical models with nontrivial Hamiltonians in terms of such systems with a finite discreteness scale. – Furthermore, we have recently extended this study by describing *multipartite systems*, analogous to many-body QM. Not only is this useful for the construction of more complex models *per se* (especially with a richer structure of energy spectra), but it is also necessary, in order to extend the *Superposition Principle* of QM to a description at the CA level. We find that it can be introduced already there to the fullest extent, compatible with a tensor product structure of multipartite states, which entails the possibilities of their *interference* and *entanglement*.

Surprisingly, we have been forced – in our approach employing Sampling Theory for the map between CA and an equivalent continuum picture – to introduce a many-time formulation, which only appeared in relativistic quantum mechanics before, as introduced by Dirac, Tomonaga, and Schwinger [24, 25, 26]. This points towards a crucial further step in these developments, which is still missing, namely a CA model of *interacting quantum fields*. Without the possibility of interacting multipartite CA with quantumlike features, as described here, it is hard to envisage a CA picture of dynamical fields spread out in spacetime. Last not least, a detailed model is still lacking in which the Born rule can be related quantitatively to the “ordinary” statistical mechanics of deterministic dynamical degrees of freedom.

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