Bulk geometry from entanglement entropy of CFT

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Abstract

In this paper, we compute the exact form of the bulk geometry emerging from a $(1+1)$-dimensional conformal field theory using the holographic principle. We first consider the $(2+1)$-dimensional asymptotic AdS metric in Poincare coordinates and compute the area functional corresponding to the static minimal surface γ_A and obtain the entanglement entropy making use of the holographic entanglement entropy proposal. We then use the results of the entanglement entropy for $(1+1)$-dimensional conformal field theory on an infinite line, on an infinite line at a finite temperature and on a circle. Comparing these results with the holographic entanglement entropy, we are able to extract the proper structure of the bulk metric. The analysis reveals the behaviour of the bulk metric in both the near boundary region and deep inside the bulk. The results also show the influence of the boundary UV cut-off a on the bulk metric. It is observed that the reconstructed metrics match exactly with the known results in the literature when one moves deep inside the bulk.

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Introduction

One of the most interesting and difficult challenge that theoretical physics has come across over the past few decades is to reconcile the general theory of relativity with quantum mechanics. It sounds quite obvious to go for the standard prescription of field quantization to formulate the quantum version of Einstein's general relativity, but unfortunately this programme runs into all sorts of trouble due to the dynamic background involved in Einstein's theory. This has led us to believe that the correct route leading to a complete theory of quantum gravity must be fundamentally different from the route taken for the other known fundamental interactions. This philosophy has led to the holographic idea which says that a gravitational theory has a dual picture in terms of a non-gravitational quantum field theory living on a lower dimensional spacetime, which is the boundary of the spacetime where the gravitational theory lives. Over the past couple of decades great progress has been made in our understanding of quantum theory of gravity, thanks to the remarkable AdS/CFT (gauge/gravity) correspondence [1, 2], which is intrinsically a non-perturbative approach to finding a quantum gravity theory.

The gauge/gravity correspondence realizes the holographic principle [3, 4] in the sense that the information about states in the higher dimensional gravitational system is correlated with the states of the ordinary quantum field theory (without any gravitational degrees of freedom) existing in the lower dimensional manifold. In other words, each observable in the non-gravitational quantum field theory corresponds to some observable of the gravitational theory in the bulk. The duality basically connects a weakly coupled theory with a strongly coupled theory, thus opening a window for exploring a strongly coupled quantum field theory with the help of a weakly coupled gravitational theory or vice versa. The remarkable dual nature of this conjecture has helped us to understand various paradoxes of general relativity, namely, black hole information paradox [5], gravitational singularity [6, 7], inflation theory [8], the origin of Hawking radiation [9], to name a few. The correspondence has also led to fundamental results in quantum theory, namely, entanglement in quantum field theory, complexity in quantum field theory [10, 11], energy loss of a Brownian particle in quark-gluon plasma [12], to name a few, from the well-established framework of general relativity.

Entanglement entropy (EE) is a well-studied subject in quantum mechanics [13]. The prescription to calculate EE of a quantum field theory with conformal group symmetry is known as 'replica trick' and was given in [14]. For $(1+1)$-dimensional conformal field theory (CFT), the exact results of EE can be calculated for a subsystem defined on various topologies, for instance CFT on a finite strip or CFT on a circle. In higher dimensional quantum field theory, it becomes notoriously difficult to calculate exact results of EE. Remarkably the holographic principle together with the famous Bekenstein-Hawking area law [15]-[17] is able to reproduce the EE results available in CFT. The basic principle of holographic entanglement entropy (HEE) proposed in [18] states that the EE of a subsystem (A) belonging to a (d+1)-dimensional CFT living in the boundary manifold corresponds to the area of the d-dimensional static minimal surface which belongs to a (d+2)-dimensional bulk spacetime where the gravity theory lives. This prescription to calculate HEE [19]-[22] has been extended to its covariant version in [23]. In the context of the HEE proposal, it is worth exploring how the bulk spacetime geometry emerges holographically from a CFT using the exact results for the EE of a subsystem living in the CFT. In this paper, we investigate the problem of extracting the exact form of the bulk metric through the exact results of EE of a CFT. Studies along this direction have been carried out earlier. For instance, in [24, 25], Einstein's equation in AdS space were obtained in a perturbative approach. Another nice approach to obtain the bulk metric can be seen in [26]-[28]. In [29], the bulk spacetime metric has been obtained numerically. In [30], the length of the bulk curves were obtained using boundary data. It is a well known fact that the EE can be obtained by constructing the reduced density matrix of the concerned subsystem. Hence it is worth asking how the reduced density matrix of the quantum system is holographically connected with the bulk. This was investigated in [31]. In addition to these studies, some other notable works on holographic bulk reconstruction can be found in [32]-[34]. In this paper, we follow the approach in [27] to reconstruct the bulk geometry from the results of the EE of a $(1+1)$-dimensional CFT on an infinite line, on an infinite line at a finite temperature and on a circle. A crucial input in our analysis is the holographic principle.

The paper is organized as follows. In section 1, we have discussed the basic formalism on which the subsequent analysis rests. In section 2, we obtain the exact bulk metric holographically dual to a $1+1$-dimensional CFT on an infinite line. In section 3, we holographically reconstruct the dual bulk metric for a $1+1$-dimensional CFT on an infinite line at a finite temperature. In section 4, we extract the exact structure of the bulk geometry for a $1+1$-dimensional CFT on a circle. We conclude in section 5.

1 Basic formalism

In this section, we briefly discuss the formalism to obtain the exact form of the bulk geometry on the boundary of which lies the CFT. The holographic principle states that the entanglement entropy S_A of a static subsystem A in $(d+1)$-dimensional CFT can be obtained from a d-dimensional static minimal surface γ_A in the $(d+2)$-dimensional bulk, the boundary of γ_A being given by the $(d-1)$-dimensional manifold $\partial\gamma_A$. The trick to reconstruct the bulk geometry is to use this principle together with the CFT result for the EE of the static subsystem living in the $(d+1)$-dimensional CFT. The static subsystems A considered in the subsequent analysis are CFT on the infinite line, CFT in a circle and CFT on the infinite line at a finite temperature. The EE of these theories are known from field theoretical considerations. The holographic entanglement entropy (HEE) is given by the Bekenstein-Hawking formula

$$S_A = \frac{Area(\gamma_A)}{4G_N^{(d+2)}} \tag{1}$$

where $Area(\gamma_A)$ is the area of the static minimal surface γ_A and $G_N^{(d+2)}$ is the Newton's gravitational universal constant in $(d+2)$-dimensions.
We start our analysis by considering the asymptotically AdS planar metric in Poincare coordinates in $(2+1)$-dimensions [35]

$$ds^2 = \frac{R^2}{z^2}\Bigg(-h(z)dt^2 + f(z)^2dz^2 + dx^2\Bigg) \tag{2}$$

where $z \geq 0$ denotes the bulk coordinate and $z = 0$ defines the boundary of the bulk where the CFT lives. Now the area functional of the above geometry reads (setting $dt = 0$)

$$Area(\gamma_A) = R\int dx \frac{\sqrt{[z'f(z)]^2+1}}{z} \ , \ z' = \frac{dz}{dx} \ . \tag{3}$$

To obtain the static minimal surface γ_A, we need to minimize the area functional above. To do this, the first step is to compute the Hamiltonian which reads

$$\mathcal{H} = z'\frac{d\mathcal{L}}{dz'} - \mathcal{L} = -\frac{1}{z\sqrt{[z'f(z)]^2+1}} \tag{4}$$

where the Lagrangian ($\mathcal{L}$) is given by $\mathcal{L}(z,z',x) = \frac{\sqrt{[z'f(z)]^2+1}}{z}$. Since the Hamiltonian has no explicit x dependence, hence it is a constant. To determine this constant, we use the fact that $\frac{dz}{dx}|_{z=z*} = 0$ where $z = z_*$ denotes the turning point of the minimal surface γ_A in the bulk. This yields

$$\frac{dz}{dx} = -\frac{\sqrt{z_*^2 - z^2}}{zf(z)} \ . \tag{5}$$

Substituting eq.(5) in eq.(3), we get

$$Area[\gamma_A(z_*)] = R\int dz \frac{z_* f(z)}{z\sqrt{z_*^2 - z^2}} \ . \tag{6}$$

Now substituting eq.(6) in eq.(1, we obtain the HEE to be

$$S_A = \frac{Area[\gamma_A(z_*)]}{4G_N^{(3)}} = \frac{1}{4G_N^{(3)}} \int dz \frac{z_* f(z)}{z\sqrt{z_*^2 - z^2}} \ . \tag{7}$$

Our aim in this paper is to obtain $f(z)$ by comparing this result with the CFT result for the EE employing the approach in [27]. The metric coefficient of dt^2 can then be obtained by substituting $f(z)$ in Einstein's field equations of general relativity with a negative cosmological constant

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ , \qquad \Lambda = -\frac{1}{R^2} \ . \tag{8}$$

2 (1+1)-dimensional CFT on an infinite line

In this section we would like to obtain the bulk metric corresponding to the result for the EE of the $(1+1)$-dimensional CFT on the infinite line. This reads [14]

$$S_{EE}(l) = \frac{c}{3} \log\left(\frac{l}{a}\right)$$

where $c = \frac{3R}{2G_N}$ is the central charge representing the degrees of freedom in the CFT [36], l denotes the length of the subsystem and a is the lattice spacing or the UV cut-off needed to avoid divergence. One can rewrite the above expression using the definition of central charge c as

$$S_{EE}(l) = \frac{2R}{4G_N} \log\left(\frac{l}{a}\right) \ . \tag{9}$$

To obtain the bulk geometry corresponding to this CFT, we write down the area functional (3) once again

$$\begin{aligned} Area[\gamma_A] &= R\int_{-l/2}^{+l/2} dx \frac{\sqrt{[z' f(z)]^2 + 1}}{z} \\ &= 2R\int_0^{l/2} dx \frac{\sqrt{[z' f(z)]^2 + 1}}{z} \\ &= 2R\int_a^{z_*} dz \frac{z_* f(z)}{z\sqrt{z_*^2 - z^2}} \\ &= Area[\gamma_A(z_*)] \equiv \mathcal{A}_l(z_*) \end{aligned} \tag{10}$$

where in the third line we have used eq.(5). The HEE of the subsystem A of length l therefore reads

$$S_A = \frac{\mathcal{A}_l(z_*)}{4G_N} \ . \tag{11}$$

From eq.(5), we can also write the length of the subsystem l in terms of the bulk coordinate z as

$$l = 2\int_a^{z_*} \frac{z f(z)}{\sqrt{z_*^2 - z^2}} dz \ . \tag{12}$$

To proceed further, we would like to have a relation between z_* and l. To get this relation, we note that according to the holographic principle

$$\frac{dS_{EE}(l)}{dl} = \frac{dS_A}{dl} \ . \tag{13}$$

Now using eq.(9), we have

$$\frac{dS_{EE}(l)}{dl} = \frac{2R}{4G_N}\frac{1}{l} \tag{14}$$

and using eq.(s)(10), (12), we have

$$\frac{dS_A}{dl} = \frac{1}{4G_N}\frac{d\mathcal{A}_l(z_*)}{dz_*}\frac{dz_*}{dl} = \frac{2R}{4G_N}\frac{1}{2z_*} \ . \tag{15}$$

Substituting eq.(s)(14) and (15) in eq.(13), we get

$$l = 2z_* \ . \tag{16}$$

Substituting eq.(16) in eq.(9), we can rewrite the expression for EE of the CFT in terms of the bulk coordinate as

$$S_{EE}(z_*) = \frac{2R}{4G_N}\log\left(\frac{2z_*}{a}\right) \ . \tag{17}$$

On the basis of the holographic principle mentioned earlier, we now equate eq.(s)(11) and (17) to obtain the expression of the area functional $\mathcal{A}_l(z_*)$ to be

$$\mathcal{A}_l(z_*) = 4G_N S_{EE}(z_*) \tag{18}$$

which in turn yields (using eq.(10))

$$\frac{4G_N}{2Rz_*}S_{EE}(z_*) = \int_a^{z_*} dz \frac{f(z)}{z\sqrt{z_*^2 - z^2}} \ . \tag{19}$$

Setting $\frac{4G_N}{2Rz_*}S_{EE}(z_*) = \mathcal{B}(z_*)$, $f(z)/z = m(z)$, the above equation can be recast as

$$\mathcal{B}(z_*) = \int_a^{z_*} dz \frac{m(z)}{\sqrt{z_*^2 - z^2}} \ . \tag{20}$$

The solution to this Volterra first kind (Abel type) integral equation reads [37, 38]

$$m(z) = \frac{1}{\pi}\frac{d}{dz}\int_a^{z} dz_* \frac{\mathcal{B}(z_*)2z_*}{\sqrt{z^2 - z_*^2}} \ . \tag{21}$$

Substituting $m(z)$ and $\mathcal{B}(z_*)$ in the above equation, we get

$$f(z) = \frac{2}{\pi}z\frac{d}{dz}\int_a^{z} dz_* \frac{\log(2z_*/a)}{\sqrt{z^2 - z_*^2}} \ . \tag{22}$$

We now proceed to calculate the integral in the above expression. This gives

$$\begin{aligned} I &= \int_a^{z} dz_* \frac{\log(2z_*/a)}{\sqrt{z^2 - z_*^2}} \\ &= \log 2\left[\frac{\pi}{2} - \sin^{-1}(a/z)\right] + \log z\left[\frac{\pi}{2} - \sin^{-1}(a/z)\right] - \log a\left[\frac{\pi}{2} - \sin^{-1}(a/z)\right] - \frac{\pi}{2}\log 2 \\ &\quad -\frac{i}{2}\Bigg[\sin^{-1}(a/z)^2 + 2\log\left(\sqrt{1-\frac{a^2}{z^2}} + \frac{ia}{z}\right)\log(a/z) - 2i\sin^{-1}(a/z)\log\left(\frac{2a^2 + 2iaz\sqrt{1-\frac{a^2}{z^2}}}{z^2}\right) \\ &\quad \texttt{Polylog}[2, 1 - \frac{2a^2}{z^2} - \frac{2ia\sqrt{1-\frac{a^2}{z^2}}}{z}]\Bigg] + \frac{i\pi^2}{12} \end{aligned}$$

where $\texttt{Polylog}[2\ ,\ 1-\frac{2a^2}{z^2}-\frac{2ia\sqrt{1-\frac{a^2}{z^2}}}{z}]$ represents the polylogarithmic function $\texttt{Li}_2\left(1-\frac{2a^2}{z^2}-\frac{2ia\sqrt{1-\frac{a^2}{z^2}}}{z}\right)$.
Substituting the above result in eq.(22), we get

$$
\begin{aligned}
f(z) \;=\;& 1+\frac{2}{\pi}\log(\frac{2z}{a})\frac{1}{\sqrt{1-(a/z)^2}}(a/z)-\frac{2}{\pi}\sin^{-1}(a/z)+\frac{iz}{\pi}\Bigg[-\frac{2a\sin^{-1}(a/z)}{z^2\sqrt{1-(a/z)^2}} \\
& -\frac{2i\sin^{-1}(a/z)\big[2ia\sqrt{1-(a/z)^2}+\frac{2ia^3}{z^2\sqrt{1-(a/z)^2}}\big]}{2a^2+2ia\sqrt{1-(a/z)^2}z}+\frac{4i\sin^{-1}(a/z)\big[2a^2+2ia\sqrt{1-(a/z)^2}z\big]}{2a^2z+2ia\sqrt{1-(a/z)^2}z^2} \\
& -\frac{2\log\big[\sqrt{1-(a/z)^2}+\frac{ia}{z}\big]}{z} \\
& -\frac{\big[-\frac{-2ia^3}{z^4\sqrt{1-(a/z)^2}}+\frac{4a^2}{z^3}+\frac{2ia\sqrt{1-(a/z)^2}}{z^2}\big]\log\big[\frac{2a^2}{z^2}+\frac{2ia\sqrt{1-(a/z)^2}}{z}\big]}{1-\frac{2a^2}{z^2}-\frac{2ia\sqrt{1-(a/z)^2}}{z}}\Bigg]\ . \qquad (23)
\end{aligned}
$$

In the limit $(a/z)\to 0$, which corresponds to moving deep inside the bulk, we get

$$f(z)=1$$

thereby fixing the coefficient of dz^2 in the metric (2). Hence we have

$$ds^2=\frac{R^2}{z^2}\Bigg(-h(z)dt^2+dz^2+dx^2\Bigg)\equiv g_{\mu\nu}dx^\mu dx^\nu\ ;\ \mu,\nu=0,1,2\ . \qquad (24)$$

The above metric gives the following Einstein field equations

$$G_{22} = -\frac{h'(z)}{2zh(z)}=0 \qquad (25)$$

$$G_{33} = -\frac{zh'(z)^2+2h(z)h'(z)-2zh(z)h''(z)}{4zh(z)^2}=0\ . \qquad (26)$$

Solving eq.(25) gives $h(z)=constant=\mathcal{K}$ which also satisfies eq.(26).
The exact form of the bulk geometry in the limit $(a/z)\to 0$ corresponding to the CFT on an infinite line therefore has the form

$$ds^2=\frac{R^2}{z^2}\Bigg(-\mathcal{K}dt^2+dz^2+dx^2\Bigg)\ . \qquad (27)$$

This is the well known pure AdS metric in Poincare coordinates.

3 (1+1)-dimensional CFT on an infinite line at a finite temperature

In this section we analyse the subsystem A of length l discussed in the previous section at a finite temperature T. The entanglement entropy of the CFT at a finite temperature from field theoretic considerations reads [14]

$$S_{EE}(l)=\frac{2R}{4G_N}\log\left[\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right] \qquad (28)$$

where $\beta=1/T$ represents the temperature of the CFT.
Differentiating $S_{EE}(l)$ with respect to l gives

$$\frac{dS_{EE}(l)}{dl}=\frac{2R}{4G_N}\left(\frac{\pi}{\beta}\right)\coth\left(\frac{\pi l}{\beta}\right)\ . \qquad (29)$$

Substituting eq.(s)(15, 29) in eq.(13), we get the length l of the subsystem A in terms of the turning point z_* to be

$$l = \frac{\beta}{\pi}\coth^{-1}\left(\frac{\beta}{2\pi z_*}\right) . \tag{30}$$

The expression for the EE (28) can now be recast using eq.(30) as

$$S_{EE}(z_*) = \frac{2R}{4G_N}\log\left[\frac{\beta}{\pi a}\sinh\left(\coth^{-1}\left(\frac{\beta}{2\pi z_*}\right)\right)\right] . \tag{31}$$

Substituting the above result in eq.(7), we get

$$f(z) = \frac{2}{\pi}z\frac{d}{dz}\int_a^z dz_*\frac{\log\left[\frac{\beta}{\pi a}\sinh\left(\coth^{-1}(\frac{\beta}{2\pi z_*})\right)\right]}{\sqrt{z^2-z_*^2}} . \tag{32}$$

The integral in the above expression gives

$$\begin{aligned}
I &= \int_a^z dz_*\frac{\log\left[\frac{\beta}{\pi a}\sinh\left(\coth^{-1}(\frac{\beta}{2\pi z_*})\right)\right]}{\sqrt{z^2-z_*^2}} \\
&= \log(2/a)\left[\frac{\pi}{2}-\sin^{-1}(a/z)\right]+\log z\left[\frac{\pi}{2}-\sin^{-1}(a/z)\right]-\frac{\pi}{2}\log 2 \\
&\quad +\frac{i}{2}\left[\sin^{-1}(a/z)^2+2\log\left(\sqrt{1-\frac{a^2}{z^2}}+\frac{ia}{z}\right)\log(a/z)-2i\sin^{-1}(a/z)\log\left(\frac{2a^2+2iaz\sqrt{1-\frac{a^2}{z^2}}}{z^2}\right)\right. \\
&\quad \left.\texttt{Polylog}[2,1-\frac{2a^2}{z^2}-\frac{2ia\sqrt{1-\frac{a^2}{z^2}}}{z}]\right]+\frac{i\pi^2}{12}-\frac{\pi}{2}\log\left(\frac{1}{2}+\frac{\sqrt{1-\frac{4\pi^2z^2}{\beta^2}}}{2}\right) \\
&\quad -\frac{1}{2}\sum_{n=1}^{\infty}(\frac{2\pi z}{\beta})^{2n}\frac{(\frac{a}{z})^{1+2n}\ {}_2F_1(\frac{1}{2},\frac{1}{2}+n,\frac{3}{2}+n,\frac{a^2}{z^2})}{1+2n}
\end{aligned}$$

where ${}_2F_1(\frac{1}{2},\frac{1}{2}+n,\frac{3}{2}+n,\frac{a^2}{z^2})$ is the Gauss hypergeometric function.

Using this result in eq.(32), we get

$$
\begin{aligned}
f(z) &= 1+\frac{\frac{4\pi^2z^2}{\beta^2}}{\left(\sqrt{1-\frac{4\pi^2z^2}{\beta^2}}\right)\left(1+\sqrt{1-\frac{4\pi^2z^2}{\beta^2}}\right)}+\frac{2}{\pi}\frac{(a/z)}{\sqrt{1-(a/z)^2}}\log(2z/a)-\frac{2}{\pi}\sin^{-1}(a/z) \\
&\quad -\frac{iz}{\pi}\Bigg[-\frac{2a\sin^{-1}(a/z)}{z^2\sqrt{1-(a/z)^2}}-\frac{2i\sin^{-1}(a/z)\big[2ia\sqrt{1-(a/z)^2}+\frac{2ia^3}{z^2\sqrt{1-(a/z)^2}}\big]}{2a^2+2ia\sqrt{1-(a/z)^2}z} \\
&\quad +\frac{4i\sin^{-1}(a/z)\big[2a^2+2ia\sqrt{1-(a/z)^2}z\big]}{2a^2z+2ia\sqrt{1-(a/z)^2}z^2}-\frac{2\log\big[\sqrt{1-(a/z)^2}+\frac{ia}{z}\big]}{z} \\
&\quad -\frac{\big[-\frac{-2ia^3}{z^4\sqrt{1-(a/z)^2}}+\frac{4a^2}{z^3}+\frac{2ia\sqrt{1-(a/z)^2}}{z^2}\big]\log\big[\frac{2a^2}{z^2}+\frac{2ia\sqrt{1-(a/z)^2}}{z}\big]}{1-\frac{2a^2}{z^2}-\frac{2ia\sqrt{1-(a/z)^2}}{z}}\Bigg] \\
&\quad -\sum_{n=1}^{\infty}\frac{1}{\pi}(a/z)^{1+2n}\left(\frac{2\pi z}{\beta}\right)^{2n}\frac{1}{\sqrt{1-(\frac{a}{z})^2}} \\
&\quad +\sum_{n=1}^{\infty}\frac{1}{\pi}(a/z)^{1+2n}\left(\frac{2\pi z}{\beta}\right)^{2n}\quad {}_2F_1\left(\frac{1}{2},\frac{1}{2}+n,\frac{3}{2}+n,\frac{a^2}{z^2}\right) \\
&\quad -\sum_{n=1}^{\infty}\frac{1}{\pi(1+2n)}(a/z)^{1+2n}\left(\frac{2\pi z}{\beta}\right)^{2n}\quad {}_2F_1\left(\frac{1}{2},\frac{1}{2}+n,\frac{3}{2}+n,\frac{a^2}{z^2}\right)\ .
\end{aligned}
$$

In the limit $\frac{a}{z}\to 0$, the above expression for $f(z)$ reduces to

$$
f(z)=\frac{1}{\sqrt{1-\frac{4\pi^2z^2}{\beta^2}}}\ . \tag{33}
$$

Now substituting $f(z)$ in eq.(2), we get

$$
ds^2=\frac{R^2}{z^2}\left(-h(z)dt^2+\frac{dz^2}{\sqrt{1-\frac{4\pi^2z^2}{\beta^2}}}+dx^2\right)\equiv g_{\mu\nu}dx^\mu dx^\nu\ ;\ \mu,\nu=0,1,2. \tag{34}
$$

The above metric leads to the following Einstein's field equations

$$
G_{22}=-\frac{8\pi^2zh(z)+\beta^2h'(z)-4\pi^2z^2h'(z)}{2b^2zh(z)-8\pi^2z^3h(z)}=0 \tag{35}
$$

$$
G_{33}=\frac{z(4\pi^2z^2-\beta^2)h'^2(z)-2h(z)[\beta^2h'(z)+z(4\pi^2z^2-\beta^2)h''(z)]}{4\beta^2zh(z)^2}=0. \tag{36}
$$

Solving eq.(35), we get

$$
h(z)=(Constant)\beta^2\left(1-\frac{4\pi^2z^2}{\beta^2}\right)=\mathcal{C}\left(1-\frac{4\pi^2z^2}{\beta^2}\right) \tag{37}
$$

which satisfies eq.(36). Substituting $h(z)$ in the metric (34), we obtain the final form of the bulk geometry corresponding to the CFT on an infinite line at a finite temperature T to be

$$
ds^2=\frac{R^2}{z^2}\left[-\mathcal{C}\left(1-\frac{4\pi^2z^2}{\beta^2}\right)dt^2+\frac{dz^2}{1-\frac{4\pi^2z^2}{\beta^2}}+dx^2\right]\ .
$$

Using the coordinate transformation $z = \frac{R}{r}$ and $x = \phi$, we can recast the above metric in a familiar form as

$$ds^2 = -\mathcal{C}(r^2 - r_+^2)dt^2 + \frac{R^2 dr^2}{(r^2 - r_+^2)} + r^2 d\phi^2 \tag{38}$$

which is the static BTZ black hole metric [39] with the event horizon at $r_+ = \frac{2\pi R}{\beta}$.
The above analysis represents the fact that the $2+1$-dimensional static BTZ geometry emerges holographically from the $1+1$-dimensional CFT on an infinite line at a finite temperature T.

4 (1+1)-dimensional CFT on a circle

In this section we shall obtain the exact form of the bulk geometry corresponding to the result for the EE of the $(1+1)$-dimensional CFT on a circle. From conformal field theoretic considerations, the result for the EE of a subsystem A of length l on a circle reads [14]

$$S_{EE}(l) = \frac{2R}{4G_N} \log\left[\frac{L}{\pi a} \sin\frac{\pi l}{L}\right] \tag{39}$$

where L represents the total length of the circle.
The present situation is quite different from the previous cases due to the presence of circular symmetry. This fact requires to introduce the circular symmetry in the metric ansatz (2). To do this, we shall use the basic property of a circle which is

$$\left(\frac{2\pi}{L}dx\right) = d\theta \ ; \ \theta \approx \theta + 2\pi \ . \tag{40}$$

With the above relation in mind, we take the metric ansatz in the following form

$$ds^2 = \frac{R^2}{z^2}\left[-h(z)dt^2 + f(z)^2 dz^2 + \left(\frac{L}{2\pi}\right)^2 d\theta^2\right] \ . \tag{41}$$

This leads to the following area functional

$$\begin{aligned} Area[\gamma_A] &= R\int_{-\pi l/L}^{+\pi l/L} d\theta \frac{\sqrt{[z'f(z)]^2 + (\frac{L}{2\pi})^2}}{z} \\ &= 2R\int_0^{\pi l/L} d\theta \frac{\sqrt{[z'f(z)]^2 + (\frac{L}{2\pi})^2}}{z} \ ; \ z' = \frac{dz}{d\theta} \ . \end{aligned}$$

As before, we once again write down the area functional and subsystem size l in terms of the bulk coordinate z. Hence we have

$$Area[\gamma_A(z_*)] \equiv \mathcal{A}_l(z_*) = 2R\int_a^{z_*} dz \frac{z_* f(z)}{z\sqrt{z_*^2 - z^2}} \tag{42}$$

$$l = 2\int_a^{z_*} \frac{z f(z)}{\sqrt{z_*^2 - z^2}} dz \ . \tag{43}$$

The HEE is therefore given by

$$S_A = \frac{\mathcal{A}_l(z_*)}{4G_N} = \frac{2R}{4G_N}\int_a^{z_*} dz \frac{z_* f(z)}{z\sqrt{z_*^2 - z^2}} \ . \tag{44}$$

Now it is observed from eq.(39) that

$$\frac{dS_{EE}(l)}{dl} = \left(\frac{2R}{4G_N}\right)\left(\frac{\pi}{L}\right)\cot\left(\frac{\pi}{L}\right) \tag{45}$$

which in turn gives using eq.(s)(13), (43) and (44)

$$l = \frac{L}{\pi}\cot^{-1}\left(\frac{L}{2\pi z_*}\right) .$$

With the above relation in place, we can recast the expression for the EE (39) in terms of the bulk coordinate as

$$S_{EE}(z_*) = \frac{2R}{4G_N}\log\left[\frac{L}{\pi a}\sin\left(\cot^{-1}\left(\frac{L}{2\pi z_*}\right)\right)\right] . \tag{46}$$

Substituting the above result in eq.(6), we obtain

$$f(z) = \frac{2}{\pi} z \frac{d}{dz}\int_a^z dz_* \frac{\log\left[\frac{L}{\pi a}\sin\left(\cot^{-1}(\frac{L}{2\pi z_*})\right)\right]}{\sqrt{z^2 - z_*^2}} . \tag{47}$$

We now proceed to calculate the integral in the above expression. This yields

$$\begin{aligned}
I &= \int_a^z dz_* \frac{\log\left[\frac{L}{\pi a}\sin\left(\cot^{-1}(\frac{L}{2\pi z_*})\right)\right]}{\sqrt{z^2 - z_*^2}} \\
&= \log(2\pi)\left[\frac{\pi}{2} - \sin^{-1}(a/z)\right] + \log z\left[\frac{\pi}{2} - \sin^{-1}(a/z)\right] - \frac{\pi}{2}\log 2 \\
&\quad + \frac{i}{2}\Bigg[\sin^{-1}(a/z)^2 + 2\log\left(\sqrt{1-\frac{a^2}{z^2}} + \frac{ia}{z}\right)\log(a/z) - 2i\sin^{-1}(a/z)\log\left(\frac{2a^2 + 2iaz\sqrt{1-\frac{a^2}{z^2}}}{z^2}\right) \\
&\quad \texttt{Polylog}[2, 1 - \frac{2a^2}{z^2} - \frac{2ia\sqrt{1-\frac{a^2}{z^2}}}{z}]\Bigg] + \frac{i\pi^2}{12} - \frac{\pi}{2}\log\left(\frac{1}{2} + \frac{\sqrt{1+\frac{4\pi^2 z^2}{L^2}}}{2}\right) \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty}(-1)^{1+n}(\frac{2\pi z}{L})^{2n}\frac{(a/z)^{1+2n}}{1+2n}\quad {}_2F_1(\frac{1}{2}, \frac{1}{2}+n, \frac{3}{2}+n, \frac{a^2}{z^2}) .
\end{aligned}$$

Substituting the above result in eq.(47), we get

$$
\begin{aligned}
f(z) &= 1 - \frac{\frac{4\pi^2 z^2}{L^2}}{\left(\sqrt{1+\frac{4\pi^2 z^2}{L^2}}\right)\left(1+\sqrt{1+\frac{4\pi^2 z^2}{L^2}}\right)} + \frac{2}{\pi}\frac{(a/z)}{\sqrt{1-(a/z)^2}}\log(2\pi z) - \frac{iz}{\pi}\Bigg[- \frac{2a\sin^{-1}(a/z)}{z^2\sqrt{1-(a/z)^2}} \\
&\quad - \frac{2i\sin^{-1}(a/z)\big[2ia\sqrt{1-(a/z)^2} + \frac{2ia^3}{z^2\sqrt{1-(a/z)^2}}\big]}{2a^2 + 2ia\sqrt{1-(a/z)^2}z} + \frac{4i\sin^{-1}(a/z)\big[2a^2 + 2ia\sqrt{1-(a/z)^2}z\big]}{2a^2 z + 2ia\sqrt{1-(a/z)^2}z^2} \\
&\quad - \frac{2\log\big[\sqrt{1-(a/z)^2} + \frac{ia}{z}\big]}{z} \\
&\quad - \frac{\big[-\frac{-2ia^3}{z^4\sqrt{1-(a/z)^2}} + \frac{4a^2}{z^3} + \frac{2ia\sqrt{1-(a/z)^2}}{z^2}\big]\log\big[\frac{2a^2}{z^2} + \frac{2ia\sqrt{1-(a/z)^2}}{z}\big]}{1 - \frac{2a^2}{z^2} - \frac{2ia\sqrt{1-(a/z)^2}}{z}}\Bigg] - \frac{2}{\pi}\sin^{-1}(a/z) \\
&\quad - \sum_{n=1}^{\infty}\frac{(-1)^{1+n}}{\pi}(a/z)^{1+2n}\left(\frac{2\pi z}{L}\right)^{2n}\frac{1}{\sqrt{1-(\frac{a}{z})^2}} \\
&\quad + \sum_{n=1}^{\infty}\frac{(-1)^{1+n}}{\pi}(a/z)^{1+2n}\left(\frac{2\pi z}{L}\right)^{2n}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2}+n,\frac{3}{2}+n,\frac{a^2}{z^2}\right) \\
&\quad - \sum_{n=1}^{\infty}\frac{(-1)^{1+n}}{\pi(1+2n)}(a/z)^{1+2n}\left(\frac{2\pi z}{L}\right)^{2n}\ {}_2F_1\left(\frac{1}{2},\frac{1}{2}+n,\frac{3}{2}+n,\frac{a^2}{z^2}\right)\ .
\end{aligned}
$$

In the limit $\frac{a}{z} \to 0$, the above expression for $f(z)$ reduces to

$$
f(z) = \frac{1}{\sqrt{1+\frac{4\pi^2 z^2}{L^2}}}\ . \tag{48}
$$

Now substituting $f(z)$ in eq.(41), we get

$$
ds^2 = \frac{R^2}{z^2}\left[-h(z)dt^2 + \frac{dz^2}{1+\frac{4\pi^2 z^2}{L^2}}dz^2 + \left(\frac{L}{2\pi}\right)^2 d\theta^2\right] \equiv g_{\mu\nu}dx^\mu dx^\nu\ ;\ \mu,\nu = 0,1,2\ . \tag{49}
$$

The above metric leads to the following Einstein's field equations

$$
G_{22} = \frac{8\pi^2 z h(z) - L^2 h' - 4\pi^2 z^2 h'}{2L^2 z h(z) + 8\pi^2 z^3 h(z)} = 0 \tag{50}
$$

$$
G_{33} = \frac{-z(L^2 + 4\pi^2 z^2)h^{2\prime} + h(z)[-2L^2 h'(z) + 2z(L^2 + 4\pi^2 z^2)h''(z)]}{4L^2 z h^2(z)} = 0\ . \tag{51}
$$

Solving eq.(50), we obtain

$$
h(z) = (Constant)L^2\left(1 + \frac{4\pi^2 z^2}{L^2}\right) = \mathcal{M}\left(1 + \frac{4\pi^2 z^2}{L^2}\right)
$$

which satisfies eq.(51). The exact form of the bulk geometry corresponding to the CFT on a circle therefore reads

$$
ds^2 = \frac{R^2}{z^2}\left[-\mathcal{M}\left(1 + \frac{4\pi^2 z^2}{L^2}\right)dt^2 + \frac{dz^2}{1+\frac{4\pi^2 z^2}{L^2}} + \left(\frac{L}{2\pi}\right)^2 d\theta^2\right]\ .
$$

We shall rewrite the above metric in a more familiar form by casting it in global coordinates $\frac{2\pi z}{L}\sinh\rho = 1$. This leads to

$$
ds^2 = R^2\left(-\mathcal{M}\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\theta^2\right). \tag{52}
$$

This is the well known pure AdS metric in global coordinates.

5 Conclusion

In this paper, we briefly discuss the method of bulk reconstruction with the help of the holographic prescription of computing entanglement entropy. The exact results of entanglement entropy in conformal field theory have been used to reconstruct the geometrical structure of the bulk metric. We consider planar symmetric, asymptotically AdS metric in $(2+1)$-dimension in Poincare coordinates. The choice of the $2+1$-spacetime dimensions has been made since the exact results of entanglement entropy of a conformal field theory are only available in $(1+1)$-dimensions. We start our analysis by obtaining the area functional corresponding to the static minimal surface (γ_A) and compute the entanglement entropy of the conformal field theory holographically using the Bekenstein-Hawking formula. We then compare this result with the exact result known from conformal field theoretic considerations. The formalism is applied to the exact results of entanglement entropy corresponding to three different types of subsystems of the conformal field theory. The first scenario consists of obtaining the bulk metric corresponding to the conformal field theory on an infinite line. The exact form of the metric function reveals the effect of the bulk-boundary UV cut-off (a) on the bulk metric. We observe that deep inside the bulk, the familiar pure AdS result is recovered. Once we have the exact form of the static metric, we obtain the metric coefficient corresponding to dt^2 by substituting the full metric ansatz (with the static sector now being reconstructed from the holographic prescription) in the Einstein's field equations with a cosmological constant $\Lambda = -\frac{1}{R^2}$. This then leads to the pure AdS metric in Poincare coordinates. It is observed that deep inside the bulk, the dynamics of the boundary conformal field theory is holographically related to the gravitational theory with pure AdS metric. We then carry out the same procedure for a conformal field theory on an infinite line at a finite temperature T. In this case the bulk reconstruction formalism leads to the BTZ black hole spacetime. It is quite remarkable that the BTZ black hole spacetime holographically emerges from a conformal field theory on an infinite line at a finite temperature. Finally, we carry our analysis for a conformal field theory on a circle. In this case also we are able to reconstruct the geometry using the holographic proposal and the circular symmetry of the problem. Our investigation in this paper once again confirms the validity of the AdS/CFT correspondence from the entanglement entropy view point. The current study can be extended to higher dimensions also if the exact results of the entanglement entropy from field theoretical considerations are known.

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Author contribution statement

All authors have contributed equally.

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