

The Big Bang and the Quantum

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This short review is addressed to cosmologists.¹

General relativity predicts that space-time comes to an end and physics comes to a halt at the big-bang. Recent developments in loop quantum cosmology have shown that these predictions cannot be trusted. Quantum geometry effects can resolve singularities, thereby opening new vistas. Examples are: The big bang is replaced by a quantum bounce; the ‘horizon problem’ disappears; immediately after the big bounce, there is a super-inflationary phase with its own phenomenological ramifications; and, in presence of a standard inflaton potential, initial conditions are naturally set for a long, slow roll inflation independently of what happens in the pre-big bang branch.

PACS numbers: 04.60.Kz,04.60Pp,98.80Qc,03.65.Sq

I. INTRODUCTION

At this conference we heard of the spectacular progress that has occurred in observational cosmology in recent years. We also learned about the upcoming missions that are poised to provide new data to further constrain or even rule out leading theoretical models. These advances have been and continue to be the engines that drive contemporary cosmology. They have brought to forefront the astonishing success of the Friedmann, Lemaître, Robertson, Walker (FLRW) models, and perturbations thereof. Indeed, it appears that the rich data that we now have, and are likely to accumulate in the near future, would be adequately described by these simple applications of general relativity and quantum field theory on the resulting cosmological backgrounds.

However these theories are conceptually incomplete. They assume that the universe begins with a big bang at which matter densities and space-time curvature become infinite. With inflationary scenarios there were initial hopes that perhaps the big-bang singularity could be avoided because the inflaton fails to satisfy the strong energy condition often used in the singularity theorems of general relativity. However, Borde, Guth and Vilenkin [1] soon established that this hope was misplaced. Inflation can be eternal in the future but, if we go back in time using Einstein’s equations, one again finds that the space-time ends and physics simply comes to a halt at the big bang. But all our experience with fundamental physics suggests that this cannot be the situation in the real world. This must be a prediction of a theory that has been pushed well beyond the domain of its validity. To know what *really* happened near the putative big bang, we must work with a genuine unification of general relativity and quantum physics, an unification which does not pre-suppose that space-time

¹ Plenary talk at the *Invisible Universe* conference held in Paris in July 2009. To appear in the Proceedings, edited by J. M. Alimi et al (AIP Publications)

is a smooth continuum, and which can encompass the rich non-linear structure of strong gravity that lies beyond the scope of perturbation theory.

But the burden on this desired unification is heavy. In the context of cosmology there is a long list of fundamental questions that must be satisfactorily addressed by such a theory. Here is an illustrative list encompassing some of the contemporary issues.

- If general relativity is transcended, how close to the putative big bang does a smooth space-time of Einstein's make sense? In particular, can one show from first principles that this approximation is valid at the onset of inflation?
- Is the big-bang singularity naturally resolved by the quantum version of Einstein's equations? Or, is some external input such as a new principle or a boundary condition at the big bang essential? An outstanding example of such an external input is the Hartle-Hawking proposal [2].
- Is the quantum evolution across the 'singularity' deterministic? One needs a fully non-perturbative framework to answer this question in the affirmative. In the pre-big-bang [3] and ekpyrotic/cyclic [4, 5] scenarios, for example, so far the answer is in the negative because these theories pre-suppose the space-time continuum of general relativity and this approximation fails at the big-bang.
- If the singularity is resolved, what is on the 'other side'? Is there just a 'quantum foam', far removed from any classical space-time, or, is there another large, classical universe?

Such questions are fundamental and must be faced squarely because the resolution of classical singularities can profoundly shift the paradigm underlying contemporary cosmology. This in itself makes it imperative that we understand the quantum nature of the big bang. But there could also be another rich pay-off: the new paradigm may enable us to address open issues that are observationally significant. For example, if there is a classical pre-big-bang branch to the universe, the horizon problem would disappear and the observed large scale homogeneity could be simply a consequence of the fact that even the most distant parts of the universe would have been in causal contact in the past. If there is a pre-big-bang branch, we would not have to specify the initial conditions for perturbations on the singularity, where the applicability of current theories is least reliable. There would be more natural ways of specifying these conditions which, in turn, may well lead to small, potentially observable deviations from current predictions. Finally, our experience with general relativity itself suggests that, once the physics of the Planck regime is well understood, we may be handed with novel predictions of central importance to the next generation of astrophysicists and cosmologists.

Loop quantum gravity (LQG) [6–8] is well suited to embark on this mission because it does not pre-suppose a classical space-time –it is background independent. At a fundamental level, everything, *including geometry*, is described in the paradigm of quantum physics [9–12]. Classical space-times emerge only on coarse graining of semi-classical quantum states. Finally, since the approach is fully non-perturbative, it is well suited for the strong field regime near the putative big bang.

In this chapter I will discuss loop quantum cosmology (LQC), the application of the principles of LQG to cosmology [13, 14]. Initial ideas appeared in [15, 16] and were developed in detail for a variety of cosmological models in [17–30, 32]. *In all cases, the big-bang and big-crunch singularities are resolved in a direct physical sense.* The resulting Planck scale physics has been explored using analytical and numerical solutions to the quantum Einstein

equations as well as effective equations which capture the leading quantum corrections. Singularity theorems are avoided *not* because one uses matter violating energy conditions. Indeed, in the models that have been studied in most detail, *all energy conditions hold*. The theorems are inapplicable because quantum geometry effects modify Einstein equations themselves.

Physically, quantum geometry gives rise to a new repulsive force. This force is *utterly* negligible under normal circumstances. It is only when the matter density becomes about 1% of the Planck density or curvature approaches $1/\ell_{\text{Pl}}^2$ that the repulsive force—and hence the deviation from classical general relativity—becomes significant. But then the repulsive force rises *very* quickly, overwhelms classical attraction and causes a quantum bounce. The density and curvature start falling and once they are below the scales just mentioned, the force again becomes negligible and classical general relativity again becomes an excellent approximation.

Immediately to the future of the bounce there is a robust phase of super-inflation which is not encountered in general relativity [33, 34]. But it is short lived and in absence of a suitable inflaton potential it does not yield a sufficient number of e-foldings. However, in presence of suitable potentials—such as $m^2\phi^2$ —super-inflation *funnels the phase space trajectories to initial conditions which virtually guarantee a slow roll inflation with 60 or more e-foldings* [35]. This is in striking contrast to what happens in general relativity where it has been argued [50] that the probability of N e-folding decreases as e^{-3N} . The super-inflationary phase is also likely to have other phenomenological consequences—such as production of gravitational waves—that are being analyzed.

The article is organized as follows. Section II lays out the conceptual setting and section III provides a bird’s eye view of LQC through illustrative results. We will see that not only has LQC answered many of the long standing questions but it has also opened new vistas. Section IV summarizes the origin of the novel predictions and places them in a broader perspective.

II. CONCEPTUAL SETTING

To set the stage, let me use the simplest cosmological models: FLRW space-times with a massless scalar field. These models are instructive because in classical general relativity all their solutions have a big-bang (and/or big-crunch) singularity. Therefore, a quantum resolution of these singularities is non-trivial. Furthermore, it is not difficult to incorporate potentials, additional matter fields and anisotropies.

Figure 1 illustrates classical dynamics for $k=0$ and $k=1$ models without a cosmological constant. ϕ is the massless scalar field while $v \sim a^3$ denotes the *physical* volume of the universe in the $k=1$ case and of a fixed fiducial cell in the $k=0$ case. If $k=0$ there are two classes of trajectories. In one the universe begins with a big-bang and expands continuously, while in the other, it contracts continuously into a big crunch. In the $k=1$ case, the universe begins with a big bang, expands to a maximum volume and then undergoes a recollapse to a big crunch singularity. Now, in quantum gravity, one does not have a single space-time in the background but rather a probability amplitude for various space-time geometries. Therefore, unlike in the classical theory, one cannot readily use, e.g., the proper time along a family of preferred observers as a clock. However, along each dynamical trajectory, the massless scalar field ϕ is monotonic. Therefore, it serves as a good clock or ‘emergent time’ with respect to which the physical degrees of freedom—the matter density or the volume,

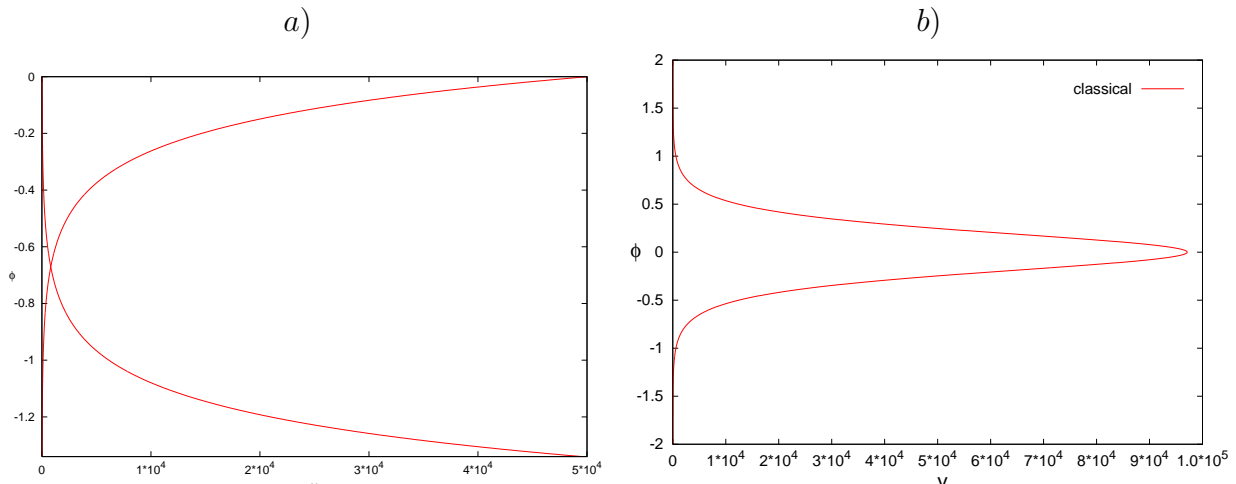


FIG. 1: Dynamics of FLRW universes with zero cosmological constant and a massless scalar field. Classical trajectories are plotted in the $v - \phi$ plane, where $v \sim a^3$ denotes the volume and ϕ the scalar field. Following the convention in general relativity, the (emergent) time variable ϕ is plotted along the y-axis. *a)* $k=0$ trajectories. *b)* A $k=1$ trajectory. In the $k=0$ case, v is the physical volume of a fiducial cell/box; $v \sim a^3$ where a is the scale factor. The final physical results are of course insensitive to the choice of this cell.

anisotropies, curvature scalars and other matter fields, if any— evolve. Note incidentally that, in the $k=1$ case, because of the classical recollapse, volume (or, the scale factor) is double-valued along any dynamical trajectory. Therefore it cannot serve as a global clock variable, while the scalar field does fulfill this role.

If one has a full quantum theory, one can proceed as follows. Choose a classical solution, i.e. a dynamical trajectory in the v, ϕ plane. The momentum $p_{(\phi)}$ conjugate to ϕ is a constant of motion. Let us suppose its value on our trajectory is $p_{(\phi)} = p_{(\phi)}^*$. Next, choose a point (v^*, ϕ^*) on the trajectory where the matter density and space-time curvature are low. This point describes the state of the FLRW universe at a late time when general relativity is expected to be valid. At the ‘time’ $\phi = \phi^*$ construct a wave packet which is sharply peaked at $v = v^*$ and $p_{(\phi)} = p_{(\phi)}^*$ and evolve it backward and forward in (the scalar field) time. We are then led to two questions:

i) *The infrared issue:* Does the wave packet remain peaked on the classical trajectory in the low curvature regime? Or, do quantum geometry effects accumulate over the cosmological time scales, causing noticeable deviations from classical general relativity? In particular, in the $k=1$ case, is there a recollapse and if so for large universes does the value V_{\max} of maximum volume agree with that predicted by general relativity [37]?

ii) *The ultraviolet issue:* What is the behavior of the quantum state when the curvature grows and enters the Planck regime? Is the big-bang singularity resolved without any extra input? Or, do we need to supplement dynamics with a new principle, such as the Hartle-Hawking ‘no boundary proposal’ [2]? What about the big-crunch?

By their very construction, perturbative and effective descriptions have no problem with the first requirement. However, physically their implications can not be trusted at the Planck scale and mathematically they generally fail to provide a deterministic evolution across

the putative singularity. Since the non-perturbative approaches often start from deeper ideas, it is conceivable that they could lead to new structures at the Planck scale which modify the classical dynamics and resolve the big-bang singularity. But once unleashed, do these new quantum effects naturally ‘turn-off’ sufficiently fast, away from the Planck regime? The universe has had some *14 billion years* to evolve since the putative big bang and even minutest quantum corrections could accumulate over this huge time period leading to observable departures from dynamics predicted by general relativity. Thus, the challenge to quantum gravity theories is to first create huge quantum effects that are capable of overwhelming the extreme gravitational attraction produced by matter densities of some 10^{94} gms/cc near the big bang, and then switching them off with extreme rapidity as the matter density falls below this Planck scale. This is a huge burden!

The question then is: How do various approaches fare with respect to these questions? The older quantum cosmology —the Wheeler-DeWitt (WDW) theory— passes the infra-red test with flying colors. But unfortunately the state follows the classical trajectory into the big bang (and in the $k=1$ case also the big crunch) singularity. The singularity is not resolved because *expectation values of density and curvature continue to diverge* in epochs when their classical counterparts do [18, 19].

For a number of years, the failure of the WDW theory to naturally resolve the big bang singularity was taken to mean that quantum cosmology cannot, by itself, shed significant light on the quantum nature of the big bang. Indeed, for systems with a finite number of degrees of freedom we have the von Neumann uniqueness theorem which guarantees that quantum kinematics is unique. The only freedom we have is in factor ordering and this was deemed insufficient to alter the status-quo provided by the WDW theory.

The situation changed dramatically in LQG. In contrast to the WDW theory, a well established, rigorous kinematical framework *is* available in full LQG [6–9]. Furthermore, this framework is uniquely singled out by the requirement of diffeomorphism invariance (or background independence) [38, 39]. If one mimics it in symmetry reduced models, one finds that a key assumption of the von-Neumann theorem is violated. As a result, *one is led to new quantum mechanics* [16]! This quantum theory is inequivalent to the WDW theory already at the kinematic level. Quantum dynamics built in this new arena agrees with the WDW theory in ‘tame’ situations but differs dramatically in the Planck regime, leading to a natural resolution of the big bang and the big crunch singularities.

III. LOOP QUANTUM COSMOLOGY

This section is divided into three parts. In the first I consider the FLRW models without a cosmological constant in some detail, in the second I summarize the more general situation, and in the third I discuss the phenomenon of super inflation and its surprising implications for inflationary scenarios.

A. Singularity resolution in the FLRW Models

The main LQC results can be summarized as follows [17–26].

Let us first consider the $k=0$ model without a cosmological constant. Following the strategy outlined in section II, let us fix a point at a late time on the trajectory corresponding to an expanding classical universe, construct a Gaussian wave function which is sharply

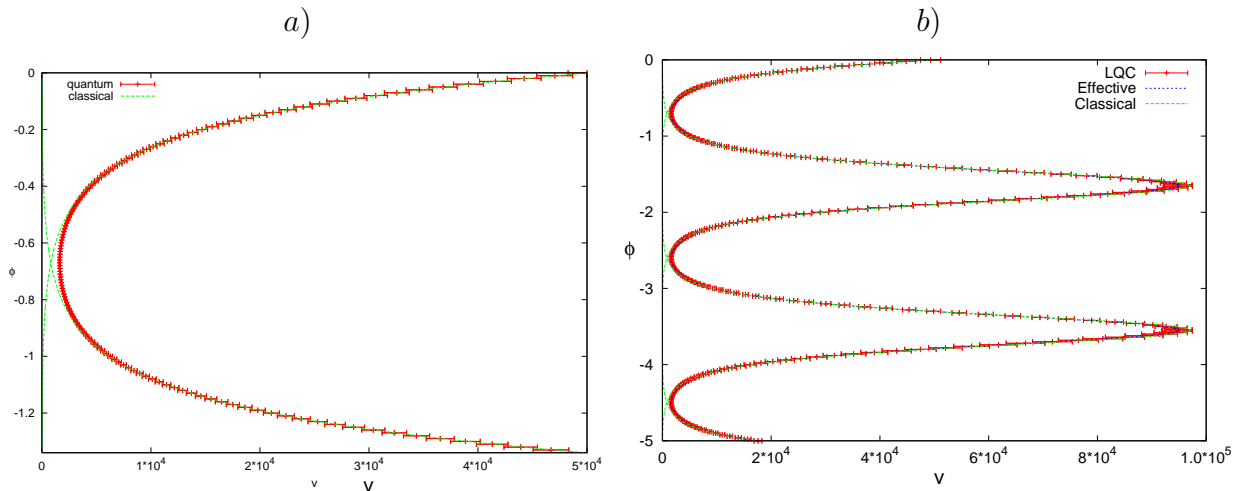


FIG. 2: In the LQC evolution of models under consideration, the big bang and big crunch singularities are replaced by quantum bounces. Expectation values and dispersion of the volume operator are compared with the classical trajectory and the trajectory from effective Friedmann dynamics. The classical trajectory deviates significantly from the quantum evolution at the Planck scale and evolves into singularities. The effective trajectory provides an excellent approximation to quantum evolution at all scales. *a)* The $k=0$ case. In the backward evolution, i.e., as ϕ decreases, the wave function follows our post big-bang branch at low densities and curvatures but undergoes a quantum bounce at matter density $\rho \sim 0.41\rho_{\text{Pl}}$ and joins on to the classical trajectory that was contracting to the future. *b)* The $k=1$ case. The quantum bounce occurs again at $\rho \sim 0.41\rho_{\text{Pl}}$. Since the big bang and the big crunch are replaced by quantum bounces and the classical re-collapse survives, the evolution undergoes cycles. In these simulations, $p_{(\phi)}^* = 5 \times 10^3$, $\Delta p_{(\phi)}/p_{(\phi)}^* = 0.018$, and $v^* = 5 \times 10^4$.

peaked at that point, and evolve it using the LQC Hamiltonian constraint. One then finds the following [19].

- The wave packet remains sharply peaked on the classical trajectory so long as the matter density ρ remains below 1% of the Planck density ρ_{Pl} . Thus, as in the WDW theory, the LQC evolution meets the infra-red challenge successfully.

- Let us evolve the quantum state back in time, toward the singularity. In the classical solution scalar curvature and the matter energy density keep increasing and eventually diverge at the big bang. The situation is very different in LQC. As mentioned in section I, once the density and curvature enter the Planck scale *quantum geometry effects become dominant creating an effective repulsive force which rises very quickly, overwhelms the classical gravitational attraction, and causes a bounce thereby resolving the big bang singularity.* (See Fig 2.) Numerical simulations show that the density acquires its maximum value $\rho_{\text{max}} \approx 0.41\rho_{\text{Pl}}$ at the bounce point.

- Although in the Planck regime the peak of the wave function deviates very substantially from the general relativistic trajectory of figure 1, it follows an effective trajectory with very small fluctuations. This effective trajectory was derived using techniques from geometric quantum mechanics [40, 41]. The effective equations it satisfies incorporate the leading corrections from quantum geometry which modify the left hand side of Einstein's equations.

However, to facilitate comparison with the standard form of Einstein's equations, one moves this correction to the right side through an algebraic manipulation. Then, one finds that the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho. \quad (3.1)$$

of classical general relativity is replaced by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_{\text{crit}}}\right). \quad (3.2)$$

Here, as usual 'dot' refers to the derivative with respect to *proper time* and $\rho_{\text{crit}} = \sqrt{3}/32\pi^2\gamma^3 G^2 \hbar$, where γ is the Barbero-Immirzi parameter of LQG (whose value $\gamma \sim 0.24$ is determined by the black hole entropy calculations in LQG). By plugging in numbers one finds $\rho_{\text{crit}} \approx 0.41\rho_{\text{Pl}}$. Thus, $\rho_{\text{crit}} \approx \rho_{\text{max}}$, found in numerical simulations. Furthermore, one can show analytically [22] that the eigenvalues of the density operator on the *physical* quantum states are bounded above by ρ_{sup} , also given by $\rho_{\text{sup}} = \sqrt{3}/32\pi^2\gamma^3 G^2 \hbar$. Thus, *there is an excellent match between the quantum theory which provides ρ_{sup} [22], the effective equations which provide ρ_{crit} [19, 41] and numerical simulations which provide ρ_{max} [19].*

- In classical general relativity the right side, $8\pi G\rho/3$, of the Friedmann equation is positive, whence \dot{a} cannot vanish; the universe either expands forever from the big bang or contracts forever ending in the big crunch. In the LQC effective equation, on the other hand, \dot{a} vanishes when $\rho = \rho_{\text{crit}}$ at which a quantum bounce occurs: To the past of this event, the universe contracts while to the future, it expands. This is possible because the LQC correction ρ/ρ_{crit} *naturally* comes with a negative sign. This is non-trivial. In the standard brane world scenario, for example, Friedmann equation also receives a ρ/ρ_{crit} correction but it comes with a positive sign (unless one artificially makes the brane tension negative) whence the singularity is not resolved.

- Consider the standard inflationary scenario for the $m^2\phi^2$ potential with phenomenologically determined values of m . Then the standard initial conditions at the onset of inflation are such that the quantum correction ρ/ρ_{crit} is of the order 10^{-11} , and hence completely negligible. Thus, LQC calculations provide *an a priori justification* for using classical general relativity during inflation.

In the closed, $k=1$ model, the situation is similar but there are two additional noteworthy features [20]. Although they are not important from phenomenological considerations, they reveal surprising properties of the domain of applicability of classical general relativity.

- To start with, classical general relativity is again an excellent approximation to the LQC evolution till matter density ρ becomes about 1% of the Planck density ρ_{Pl} but, as the density increases further, the two evolutions start diverging rapidly. Again, quantum geometry effects become dominant creating an effective repulsive force which rises very quickly, overwhelms the classical gravitational attraction, and causes a bounce thereby resolving both the big bang and the big crunch singularities. Surprisingly these considerations apply *even to universes whose maximum radius a_{max} is only $23\ell_{\text{Pl}}$* . For these universes, general relativity is a very good approximation in the range $8\ell_{\text{Pl}} < a < 23\ell_{\text{Pl}}$! The matter density acquires its minimum value ρ_{min} at the recollapse. The classical prediction $\rho_{\text{min}} = 3/8\pi G a_{\text{max}}^2$ is correct to one part in 10^5 . It is rather astonishing that general relativity is so accurate even in situations where one would have expected quantum corrections to dominate.

- On the other hand there are also situations in which one would have at first expected general relativity valid, where quantum corrections dominate! Recall that the volume of the closed universe acquires its minimum value V_{\min} at the quantum bounce. V_{\min} scales linearly with $p_{(\phi)}$. Consequently, V_{\min} can be *much* larger than the Planck size. Consider for example a quantum state describing a universe which attains a maximum radius of a megaparsec. Then the quantum bounce occurs when the volume reaches the value $V_{\min} \approx 5.7 \times 10^{16} \text{ cm}^3$, *some* 10^{115} *times the Planck volume*. A more realistic universe would have a much larger maximum radius and hence $p_{(\phi)}$ and then V_{\min} would be much larger! Although it is at first surprising that quantum effects can dominate when the universe is so large, a moment's reflection shows that deviations from the classical behavior are triggered when the density or curvature reaches the Planck scale. The size of the volume by itself is not the relevant factor for quantum gravity effects.

Thus, in the simplest $k=0$ and $k=1$ FLRW models, classical singularities are replaced by quantum bounces and LQC provides a rather detailed picture of the physics in the Planck regime. The situation is essentially the same in the open, $k=-1$ model [21]. (Although the detailed analysis in [21] has certain drawbacks, they can be overcome using techniques introduced in [29].) Furthermore, the singularity resolution does not cause infra-red problems: There is close agreement with classical general relativity away from the Planck scale. The ultraviolet-infrared tension is avoided because, although quantum geometry effects are truly enormous in the Planck regime, *they die extremely quickly*.

B. Generalizations

In this sub-section I will discuss the fate of singularities and Planck scale physics in models with increasing generality.

- *Beyond the big bang and big crunch singularities:* In section III A I focused on the big bang and big crunch singularities where the scale factor goes to zero and the matter density and curvature diverges at a finite proper time. But for general matter, new types of singularities can arise even with $k=0$ and $\Lambda = 0$. For example, in the ‘big-rip’ singularity, the scale factor *diverges* in a finite proper time along with energy density and pressure. Another possibility is the ‘sudden death’ singularity where the curvature and/or its derivatives diverge at a *finite* values of the scale factor (and proper time). In the spatially homogeneous, isotropic context, if matter has equation of state $p = p(\rho)$, there is a good control on the type of singularities that can arise [42]. From a physical perspective, we can divide them into two broad categories: strong singularities (for example, the big bang and the big crunch) where observers would be crushed and tame, weak ones (such as those resulting from shell-crossing) through which they would be able to propagate. All possible occurrences have been recently analyzed using effective equations of LQC [27]. The result is that all strong singularities are resolved but not necessarily the weak ones. Since physics does not stop at the weak singularities, they are widely regarded as ‘harmless’. An interesting situation arises in the context of a phantom model in which general relativity predicts the occurrence of a big-rip singularity. If one uses the effective equations of LQC, the energy density remains bounded but the pressure and the rate of change of the Hubble parameter diverge [43]. At first this might seem as a problem for LQC. However, a closer examination [27] shows that this is a weak singularity beyond which geodesics can be extended. Thus, in this case the quantum geometry effects tame a strong singularity and render it harmless.

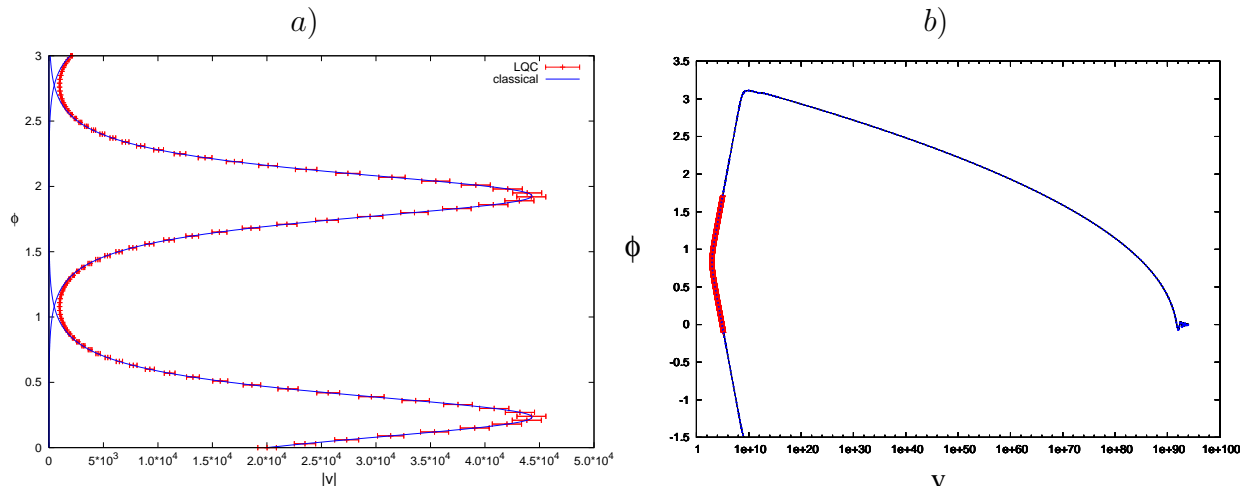


FIG. 3: Again, expectation values of the volume operator are plotted on the x-axis and emergent time ϕ on the y-axis. *a)* Negative cosmological constant. In LQC, the big bang and big crunch singularities are resolved and the universe also undergoes a classical re-collapse. Therefore, the qualitative behavior is similar to that in $k=1$, $\Lambda = 0$ model of Fig. 2.b. *b)* Inflation with $m^2\phi^2$ potential. LQC reproduces the general relativistic inflation captured by the long, horizontal part of the curve that slopes down. However, if we evolve backward in time using LQC, one obtains the near vertical segment on the left showing that the big-bang is replaced by a quantum bounce. Near the bounce, the potential plays a negligible role in the LQC dynamics.

- *Negative cosmological constant:* The $k=0$, $\Lambda < 0$ model is discussed in [24]. In this case, in classical general relativity the situation is analogous to that in the $k=1$ model, discussed above. The universe starts out with a big bang, expands to a maximum value of the scale factor, at which the positive energy density of matter is exactly balanced by the negative energy density in the cosmological constant. If the universe were to expand further, the total energy density would become negative. But since the Friedmann equation (3.1) of classical general relativity does not allow the total energy density to become negative, the universe recollapses. In LQC the model has been analyzed in detail both by solving the quantum Einstein's equations numerically and by analyzing the solutions of effective equations. Again, for states which are sharply peaked Gaussians at late times, the effective equations provide an accurate account of full quantum dynamics, including the Planck regime. The quantum corrected Friedmann equation again has the form (3.2) where ρ now includes also the energy density in the cosmological constant; the big bang and the big crunch are replaced by quantum bounces. The qualitative picture is thus similar to that in the $k=1$, $\Lambda = 0$ models. The evolution is (approximately) cyclic. The total density is maximum ($\rho = \rho_{\text{crit}}$) at the quantum bounce, the universe then expands, and the total density decreases. When it vanishes, there is a classical re-collapse, the universe starts contracting, and the total density increases till it reaches the maximum value, ρ_{crit} when there is again a quantum bounce. The value of ρ_{crit} is the same as in the $\Lambda = 0$ case. (See Fig 3.a)

- *Positive cosmological constant:* In classical general relativity, the situation in the $k=0$, $\Lambda > 0$ case is similar to that in the $k=0$, $\Lambda = 0$ case, discussed in section III A. But the classical trajectory which starts out at the big-bang now expands to an infinite volume (so that the energy density ρ_ϕ in the scalar field goes to zero) at a *finite* value ϕ_{max} of the

emergent time ϕ . This opens up an interesting possibility in LQC. Because the evolution in the emergent time ϕ is unitary in LQC, one can continue it beyond the point at which the density goes to zero. States which are semi-classical in the low ρ_ϕ regime again follow effective trajectories which now naturally extend beyond ϕ_{\max} . Since ρ_ϕ remains bounded, it is convenient to draw these trajectories in the ρ_ϕ - ϕ plane. They agree with the classical trajectories in the low ρ_ϕ regime and the extension is just the analytical continuation of classical trajectories. It would be very interesting to work out physical ramifications of this unforeseen feature of LQC dynamics. Finally, in this case, LQC predicts that the cosmological constant has a maximum possible value. As one would expect the bound is given by the Planck scale. Therefore it is not of phenomenological interest. However, it is conceptually interesting that quantum geometry effects imply that Λ cannot even in principle be arbitrarily large.

- *Inflationary scenarios:* The LQC framework has also been extended to incorporate an inflaton. Quantum Einstein's equations have been numerically solved for the $m^2\phi^2$ potential [26]. LQC admits wave functions that are sharply peaked at a standard inflating trajectory. This is just as one would expect because in the standard scenario inflation occurs well away from the Planck regime where general relativity is an excellent approximation to LQC. When such a wave function is evolved *backward* in time, one again finds that, rather than following the classical trajectory into the big bang, the wave function bounces (see Fig. 3.b)). Again, effective equations provide an excellent approximation to the full quantum equations. Numerical simulations as well as the effective equations show that the potential has negligible effect near the big bounce. All the principal features of the LQC evolution reported above are recovered, including the value of ρ_{crit} .

- *Anisotropies and gravitational waves:* Inclusion of anisotropies leads to Bianchi models. In these models, the Weyl curvature is not zero because, physically, they admit gravitational waves. Detailed LQC analysis has been carried out in Bianchi I and II models at the analytical level [28–30] (and the method extends to all class A Bianchi models; Bianchi IX, for example, is discussed in [31]). It establishes that the big bang singularity is resolved. Effective equations again show that the matter density ρ_{matt} is bounded above. However, as one might expect, the bound now is lower than ρ_{crit} because there is also some energy in the gravitational waves that is not captured in matter density. In the isotropic models, space-time curvature is completely encoded in the Ricci scalar. In the anisotropic case, we also have non-trivial Weyl curvature. Consequently, there is not just one bounce. Rather, any time a curvature invariant (more precisely, shear) enters the Planck regime, quantum geometry effects intervene and dilute it, thereby avoiding the singularity that would have occurred had one used general relativity. Thus, the Planck regime is much richer. However, outside this regime, general relativity is again an excellent approximation to LQC. Finally, as in the isotropic case, matter can be chosen to satisfy all energy conditions. The singularity theorems are transcended because Einstein's equations are modified just in the right fashion by LQC.

- *Beyond homogeneity:* In mathematical general relativity, the so called Gowdy cosmological models have drawn much attention because they admit *local, inhomogeneous degrees of freedom* and yet are sufficiently simple to be tractable. These models have been studied in detail using LQC [32]. One can construct a quantum theory of Gowdy models using conventional field theoretical methods which do not take into account quantum geometry of LQC. One then finds that the singularity is not resolved. On the other hand, using LQC to treat just the homogeneous degrees of freedom already suffices to resolve singularities, even

if the inhomogeneous modes are treated using more conventional techniques rooted in Fock quantization. The general expectation is that the ultra-violet behavior would further improve if all modes are treated using loop quantum gravity. But already this ‘hybrid’ scheme introduced by the Madrid group has given some confidence that *the singularity resolution is not tied to the homogeneous models*.

- *General space-like singularities:* In general relativity there is a conjecture due to Belinskii, Lifshitz and Khalatnikov (BKL) which says that as one approaches space-like singularities in general relativity, ‘spatial derivatives of basic fields become sub-dominant relative to the time derivatives’ and dynamics at any spatial point is well approximated by that of homogeneous models [44]. Bianchi I dynamics plays a dominant role and from time to time there are transitions from one Bianchi I solution to another, mediated by a Bianchi II solution. By now there is considerable support for this conjecture both from rigorous mathematical and numerical investigations [45]. This, in turn, provides support for the hope that the lessons on the quantum nature of singularities we learned from the Bianchi I and Bianchi II models may be valid much more generally. In particular, it suggests that the quantum geometry effects of LQG may well resolve *generic* space-like, strong curvature singularities of classical general relativity. These are precisely the singularities of direct interest to cosmology. There is now a formulation of the BKL conjecture in the framework that underlies LQG [46]. Together with the quantization procedure introduced in the Bianchi II models [29]—which, as remarked above, is directly applicable to a wide class of homogeneous models—this form of the BKL conjecture should enable one to obtain concrete results for general space-like singularities.

C. Super-inflation and inflation

So far, I have focused on singularity resolution and the new physics in the Planck regime. It turns out that LQC has unforeseen implications also to the inflationary scenarios and phenomena that occur away from the Planck regime. Because of space limitation, I will outline only one of these and mention a few other avenues that are being pursued both by cosmology and LQC communities.

1. Inflaton and super-inflation

In this sub-section, I will list some basic predictions of LQC which are rather surprising. These new features arise in the Planck regime but are important for setting the initial conditions for slow roll inflation at much lower energies. For simplicity and definiteness, in the detailed discussion I use the $k=0$, $\Lambda = 0$ isotropic cosmologies and assume that the kinetic energy of the inflaton is positive.

As we saw, LQC is generally formulated using volume $v \sim a^3$ of a fiducial cell (rather than the scale factor a), and its conjugate momentum b [22, 23]. On solutions to Einstein’s equations, $b = \gamma H$ where, as before, $\gamma \approx 0.24$ is the Barbero-Immirzi parameter of LQC and $H = \dot{a}/a$ is the Hubble parameter. However, LQC modifies the Einstein dynamics and on solutions to the effective equations we have

$$H = \frac{1}{2\gamma\lambda} \sin 2\lambda b \approx \frac{0.94}{\ell_{\text{Pl}}} \sin 2\lambda b \quad (3.3)$$

where $\lambda^2 \approx 5.17\ell_{\text{Pl}}^2$ is the ‘area-gap’, the smallest non-zero eigenvalue of the area operator. In LQC b ranges over $(0, \pi/\lambda)$ and general relativity is recovered in the limit $\lambda \rightarrow 0$.

As I already mentioned, quantum geometry effects modify the geometric, left side of Einstein’s equations. In particular, the LQC Friedmann equation first emerges as

$$\frac{\sin^2 \lambda b}{\gamma^2 \lambda^2} = \frac{8\pi G}{3} \rho \equiv \frac{8\pi G}{3} \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right). \quad (3.4)$$

To compare with the standard Friedmann equation (3.1), it is often convenient to rewrite it in the form (3.2) using (3.3). By inspection it is clear from Eqs (3.2), (3.3) and (3.4) that, away from the Planck regime —i.e., when $\lambda b \ll 1$ or, $\rho \ll \rho_{\text{Pl}}$ — we recover classical general relativity. However, modifications in the Planck regime are drastic. Recall that in general relativity, the Hubble parameter H is large throughout the Planck regime and diverges at the singularity. By contrast, in LQC H *vanishes* at the bounce (because $\dot{a} = 0$ there) and Eq.(3.3) implies that it has a *finite*, maximum value, $H = 0.94$. Second, Eq. (3.4) implies that the density ρ is bounded by $\rho_{\text{crit}} \approx 0.41\rho_{\text{Pl}}$. Third, if the potential $V(\phi)$ is bounded below, say $V \geq V_o$, then it follows from (3.4) that $\dot{\phi}^2$ is bounded by $2(\rho_{\text{crit}} - V_o)$. Fourth, if the potential grows unboundedly for large $|\phi|$, then $|\phi|$ is also bounded. For example, for $V = m^2\phi^2/2$, we have $m|\phi|_{\text{max}} = 1.41\sqrt{\rho_{\text{crit}}}$. Finally, the derivative with respect to proper time of b and the Hubble parameter are now given by

$$\dot{b} = -4\pi\gamma G \dot{\phi}^2 \quad \text{and} \quad \dot{H} = -4\pi G (\cos 2\lambda b) \dot{\phi}^2 \quad (3.5)$$

As a consequence, b *decreases monotonically in every solution*, starting from $b = \pi/\lambda$ and ending with $b = 0$ (the bounce occurs at the mid-point, $b = \pi/2\lambda$). When the potential is bounded below, $|\dot{H}|$ is also bounded by $10.3/\ell_{\text{Pl}}^2$. Thus, a large number of physical quantities which are unbounded in general relativity cannot exceed certain finite, maximum values in LQC.

In particular, these results imply that if the potential is bounded below, the matter density and curvature can not diverge anywhere on LQC space-times. One can also show that a solution where our fixed fiducial cell has a finite volume initially cannot evolve to a configuration where the volume becomes zero. Thus, *the LQC solutions are everywhere regular irrespective of whether one focuses on matter density, curvature or the scale factor*. Finally, every solution undergoes precisely one bounce at $b = \pi/2\lambda$ where the Hubble parameter vanishes because of Eq. (3.3) and the density reaches its maximum value ρ_{crit} because of Eq. (3.2).

Let me now turn to *super-inflation* (see [33] and especially [34]). At the bounce point, we have $b = \pi/2\lambda$ whence (3.3) implies that $H = 0$ and it follows from (3.5) that \dot{H} is positive. The universe expands to the future of the bounce point and contracts in its past. Eq (3.5) shows that \dot{H} continues to be positive till b has decreased to $b = \pi/4\lambda$ at which point it vanishes (and then it becomes and remains negative as v increases monotonically to infinity). Thus, *every LQC solution has a super-inflationary phase from $b = \pi/2\lambda$ to $b = \pi/4\lambda$* . Eqs (3.3) and (3.5) imply that at the beginning of this phase, the Hubble parameter H vanishes and \dot{H} is very large, while at the end of this phase H assumes its maximum value and \dot{H} vanishes. Note that the occurrence of this phase is universal; it exists even if the inflaton potential is zero! *This is a robust feature of LQC that has no analog in general relativity*. It is natural to ask if this quantum geometry driven super-inflation could be a substitute for the standard inflation. If so, we would not have to invoke *any* potential! Unfortunately the

answer is in the negative. Detailed considerations show that in absence of a potential, this phase is very short lived in proper time and therefore does not yield enough e-foldings to be a viable substitute for the standard inflation.

However, it was recently realized that in presence of the ‘standard’ potentials used in inflation, this phase has an unforeseen consequence: it naturally funnels dynamical trajectories to initial conditions that *lead to a long slow roll inflation with at least 68 e-foldings* [35].

2. LQC and inflation

Inflationary scenarios have had an extraordinary success especially in accounting for structure formation. This success brings added urgency to a long standing question: Does a sufficiently long, slow roll inflation require fine tuning of initial conditions or does it occur generically in a given theoretical paradigm? (See e.g. [47–50]). Such a slow roll requires that initially the inflaton must be correspondingly high-up in the potential. How did it get there? Is it essential to invoke some rare quantum fluctuations to account for the required initial conditions because the a priori probability for their occurrence is low? Or, is a sufficiently long, slow roll inflation robust in the sense that it is realized in ‘almost all’ dynamical trajectories of the given theory?

To make these questions precise, one needs a stream-lined framework to calculate probabilities of various occurrences *within a given theory*. A mathematically natural framework to carry out this analysis was introduced over two decades ago (see, e.g., [51–53]). It invokes Laplace’s principle of indifference [54] to calculate the *a priori* probabilities for various occurrences. More precisely, the idea is to use (a flat probability distribution $P(s) = 1$ and) the canonical Liouville measure $d\mu_L$ on the space \mathbb{S} of solutions s of the theory under consideration to calculate the *relative volumes* in \mathbb{S} occupied by solutions with desired properties [51]. In our case, then, the a priori probability is given by the *fractional* Liouville volume occupied by the sub-space of solutions in which a sufficiently long, slow roll inflation occurs. Further physical input can provide a sharper probability distribution $P(s)$ and a more reliable likelihood than the ‘bare’ a priori probability. However, a priori probabilities can be directly useful if they are very low or very high. In these cases, it would be an especially heavy burden on the fundamental theory to come up with the physical input that significantly alters them.

The task, then, is to calculate the *a priori* probability that there is inflation with at least 68 e-foldings. As in general relativity, this can be done by introducing the natural Liouville measure on the space \mathbb{S} of solutions and calculating the fractional Liouville volume of \mathbb{S} occupied by solutions with adequate number of e-foldings. In general relativity, it has been argued [50] that the resulting a priori probability for obtaining 68 or more e-foldings is suppressed by a factor larger than e^{-204} . What is the situation in LQC? Now the equations of motion are

$$\ddot{v} = \frac{24\pi G}{\rho_{\text{crit}}} v [(\rho - V)^2 + V(\rho_{\text{crit}} - V)] \quad \text{and} \quad \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0 \quad (3.6)$$

where $3H = 3\dot{a}/a = \dot{v}/v$, and v, ϕ are, in addition, subject to the LQC-modified Friedmann equation (3.2). It is simplest to write the natural Liouville measure on the space \mathbb{S} of

solutions in terms of values ϕ_B, v_B , of the scalar field and the volume at the bounce:

$$d\hat{\mu}_L = \frac{\sqrt{3\pi}}{\lambda} [1 - F_B]^{\frac{1}{2}} d\phi_B dv_B \quad (3.7)$$

where $F_B = V(\phi_B)/\rho_{\text{crit}}$ is the fraction of the total density that is in the potential energy at the bounce. Furthermore, in LQC $|\phi|$ takes values in the *bounded* interval $(0, \phi_{\text{max}} = 0.9\sqrt{\rho_{\text{Pl}}/m})$. The question now is: With respect to this measure, what is the fractional volume in \mathbb{S} occupied by solutions with at least 68 e-folds of slow roll inflation?

Recall that the Hubble parameter H takes its maximum value at the end of super-inflation and the friction term in the equation of motion for ϕ is given by H/m^2 . The phenomenologically preferred value of m is of the order $10^{-6} M_{\text{Pl}}$. Therefore, at the end of super-inflation, the friction term is necessarily large and initial conditions are naturally set for a long slow roll. To calculate the probabilities, let us use the phenomenological value of m . One can perform detailed analytical calculations using usual approximations and check their validity via high precision numerical simulations. One finds that 68 or more e-foldings are guaranteed if the fraction $F(B)$ is larger than 1.4×10^{-11} [35]. This in turn implies that the a priori probability of a slow roll with at least 68 e-foldings is greater than 0.99. This conclusion is quite robust: One can change the inflaton mass by a couple of orders of magnitude or add a quartic term to the potential (with phenomenological bounds on the coupling constant). The probability remains greater than 0.99.

Thus, even when one uses the same methods that were used for general relativity in [50], the conclusion in LQC is *opposite* of that in [50]. This comes about because, unlike in general relativity, $\phi, \dot{\phi}$ are bounded in LQC; all solutions are singularity free and undergo a quantum bounce; and the bounce is followed by a period of super-inflation which funnels dynamical trajectories to initial conditions that are well suited for a long slow roll.

There are several other phenomenological implications of super inflation, and more generally, of the LQC quantum corrections to Einstein's equations, particularly on scalar and tensor perturbations (see e.g. [55–57]). I discussed some of these in my talk. Unfortunately, space limitation does not allow me to discuss these developments here.

IV. OUTLOOK

Singularities of general relativity are perhaps the most promising gates to physics beyond Einstein. They provide a fertile conceptual and technical ground in our search of a new paradigm in cosmology as well as fundamental physics. Consider some of the deepest conceptual questions we face today: the issue of the Beginning and the end End, the arrow of time, and the puzzle of black hole information loss. Their resolutions hinge on the true nature of space-time singularities. In my view, considerable amount of contemporary confusion about such questions arises from our explicit or implicit insistence that singularities of general relativity are true boundaries of space-time; that we can trust causal structure all the way to these singularities; that notions such as event horizons are absolute even though changes in the metric in a Planck scale neighborhood of the singularity can move event horizons dramatically or even make them disappear altogether [58].

LQG is well suited to address the issue of the fate of classical singularities in quantum gravity because it is fully non-perturbative and does not pre-suppose that we have a smooth

geometry all the way up to the big-bang and big crunch [6–8]. Therefore LQC has been used to address many long standing cosmological questions in detail [13, 14, 23]. The scalar field serves as emergent time. Strong curvature singularities of classical general relativity are either resolved [19–26] or converted to harmless weak singularities by the quantum geometry effects [27]. In particular, the big bang and the big crunch are naturally replaced by quantum bounces. On the ‘other side’ of the bounce there is again a large universe. General relativity is an excellent approximation to quantum dynamics once the matter density falls below one percent of the Planck density. Thus, LQC successfully meets both the ‘ultra-violet’ and ‘infra-red’ challenges. Furthermore results obtained in a number of models using distinct methods re-enforce one another. One is therefore led to take at least the qualitative findings seriously: *Big bang is not the Beginning nor the big crunch the End*. Quantum space-times could be vastly larger than what general relativity had us believe!

How can the quantum space-times of LQC manage to be significantly larger than those in general relativity when those in the WDW theory are not? Main departures from the WDW theory occur due to *quantum geometry effects* of LQG. There is no fine tuning of initial conditions, nor a boundary condition at the singularity, postulated from outside. Furthermore, matter can satisfy all the standard energy conditions. Why then does the LQC singularity resolution not contradict the standard singularity theorems of Penrose, Hawking and others? These theorems are inapplicable because *the left hand side* of the classical Einstein’s equations is modified by the quantum geometry corrections of LQC. What about the more recent singularity theorems that Borde, Guth and Vilenkin [1] proved in the context of inflation? They are not tied to Einstein’s equations. But, motivated by the eternal inflationary scenario, they assume that the expansion remains positive if we recede in the past along any geodesic. Because of the pre-big-bang contracting phase, this assumption is violated in the LQC effective theory.

A qualitative picture that emerges is that the non-perturbative quantum geometry corrections create a ‘*repulsive*’ force. While this force is negligible under normal conditions, it dominates when curvature approaches the Planck scale and can halt the collapse that would classically have led to a singularity. In this respect, there is a curious similarity with the situation in the stellar collapse where a new repulsive force comes into play when the core approaches a critical density, halting further collapse and leading to stable white dwarfs and neutron stars. This force, with its origin in the Fermi-Dirac statistics, is *associated with the quantum nature of matter*. However, if the total mass of the star is larger than, say, 5 solar masses, classical gravity overwhelms this force. The new repulsive force of LQC is *associated with the quantum nature of geometry*. It comes into play only near Planck densities but is then so strong that it can counter the classical, gravitational attraction, irrespective of how large the collapsing mass is. It is this force that prevents the formation of singularities.

At first one might think that, since quantum gravity effects concern only a tiny region, whatever they may be, their influence on the global properties of space-time should be negligible whence they would have almost no bearing on the issue of the Beginning and the End. However, as we saw, once the singularity is resolved, vast new regions appear on the ‘other side.’ New possibilities open up that were totally unforeseen in general relativity [35]. First, matter density, curvature scalars, and the Hubble parameter *are all bounded*. Second, even in absence of a potential, there is a robust super-inflationary phase immediately after the quantum bounce [33, 34]. As we discussed in section III C this novel phase has interesting consequences. In particular, it naturally leads to the phenomenon of *Hubble funneling*: on every solution the Hubble parameter is driven to its largest possible value at the end of

super-inflation. As a consequence, if there is a potential such as $m^2\phi^2$, initial conditions are naturally set for a long slow roll inflation [35]. One does not have to appeal to rare quantum fluctuations to explain how the inflaton managed to get sufficiently high in the potential to seed a slow roll inflation leading to 68 or more e-foldings. Finally, *these conclusions are insensitive to what happened in the pre-big-bang branch.*

Another major direction of current work concerns perturbations that seed structure formation. In the new paradigm provided by LQC, one is led to re-examine the issue of initial conditions for these perturbations. We no longer have to specify them at a singularity. It is most natural to set them in the infinite past where matter density and curvatures tend to zero, whence it makes direct physical sense to speak of vacuum fluctuations. The LQC paradigm leads us to re-asses the old questions from entirely new angles (see, e.g., [55–57]). This opens up a wealth of interesting challenges and LQC provides novel ideas and tools to meet them.

Acknowledgments: I would like to thank Alex Corichi, Tomasz Pawłowski, Param Singh, David Sloan, Victor Taveras, Kevin Vandersloot and Edward Wilson-Ewing for collaboration and discussions. This work was supported in part by the NSF grant PHY0854743, the Alexander von Humboldt Foundation, The George A. and Margaret M. Downsborough Endowment and the Eberly research funds of Penn State.

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