

# Loop Quantum Cosmology and Spin Foams

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Loop quantum cosmology (LQC) is used to provide concrete evidence in support of the general paradigm underlying spin foam models (SFMs). Specifically, it is shown that: i) the physical inner product in the timeless framework equals the transition amplitude in the deparameterized theory; ii) this quantity admits a vertex expansion a la SFMs in which the  $M$ -th term refers just to  $M$  volume transitions, without any reference to the time at which the transition takes place; iii) the exact physical inner product is obtained by summing over just the discrete geometries; no ‘continuum limit’ is involved; and, iv) the vertex expansion can be interpreted as a perturbative expansion in the spirit of group field theory. This sum over histories reformulation of LQC also addresses certain other issues which are briefly summarized.

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## I. INTRODUCTION

In his celebrated Ph.D. work Feynman [1] began with the Hilbert space formulation of non-relativistic quantum mechanics and reformulated the unitary evolution generated by the Hamiltonian as a sum over histories. By mimicking the spirit of that procedure, Reisenberger and Rovelli [2] obtained a sum over histories formulation of loop quantum gravity (LQG). Through contributions of many researchers it has become an active research area that goes under the name ‘spin foam models’ (SFMs) (see, e.g., [3]). In LQG, spin network states can be used to construct a convenient orthonormal basis in the kinematical Hilbert space. A key challenge is to extract physical states from them by imposing constraints. Formally this can be accomplished by the group averaging procedure which also provides the physical inner product between the resulting states. A primary goal of SFMs is to provide a path integral to realize this idea.

Heuristically, the main idea behind this construction can be summarized as follows. Consider a 4-manifold  $M$  bounded by two 3-surfaces,  $S_1$  and  $S_2$ , and a simplicial decomposition thereof. One can think of  $S_1$  as an ‘initial’ surface and  $S_2$  as a ‘final’ surface. Fixing a spin network on each of these surfaces specifies an ‘initial’ and a ‘final’ quantum geometry. A quantum 4-geometry interpolating between the two can be constructed by considering the dual triangulation of  $M$  and coloring its surfaces with half integers  $j$  and edges with suitable intertwiners. The idea is to obtain the physical inner product between the two states

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by summing first over all the colorings for a given triangulation, and then over triangulations keeping the boundary states fixed. (The second sum is often referred to as ‘vertex expansion’ because the  $M$ -th term in the series corresponds to a dual triangulation with  $M$  vertices.) Since each triangulation with a coloring specifies a quantum geometry, the sum is regarded as a path integral over physically appropriate 4-geometries. In ordinary quantum mechanics and Minkowskian field theories where we have a fixed background geometry, such a path integral provides the (dynamically determined) transition amplitude for the first state, specified at initial time, to evolve to the second state at the final time. In the background independent context of quantum gravity, a priori one does not have access to a time variable and dynamics is encoded in constraints. Therefore the notion of a transition in a pre-specified time interval fails to be meaningful unless the constraints are solved via deparametrization, using a relational time variable. Rather, the sum over histories now gives the physical inner product between solutions to quantum constraints.

Over the last two years there have been significant advances in SFMs. In particular, relation to the kinematics underlying LQG has now been clearly established [4–7]. However, while the path integral is well-motivated by exploiting an interplay between general relativity and the well-understood BF theory, it requires a new key ingredient —the vertex amplitude. While this amplitude *is* severely constrained by several requirements, it has not been systematically derived following procedures used in well-understood field theories, or, starting from a well-understood Hamiltonian dynamics. Therefore a number of issues still remain. For example, the final ‘vertex expansion’ is a discrete sum in which each term is a sum over colorings, for a fixed triangulation. A priori it is somewhat surprising that the answer can be written as a *discrete* sum. Would one not have to take a continuum limit at the end as in Minkowskian field theories? The physical inner product resembles the transition amplitude used in standard quantum field theories. Is there a deeper meaning in this resemblance? Loop quantum cosmology (LQC) provides a physically interesting yet technically simple context to explore such open issues because one can arrive at a sum over histories starting from a fully controlled Hamiltonian theory [8]. The goal of this communication is to show that a detailed analysis of this example provides strong support for the general paradigm that underlies SFMs and also sharpens the discussion of some open issues. We will sketch the general ideas and summarize results, leaving the detailed derivations and discussions to a longer article [9].

## II. LQC DYNAMICS AS A SUM OVER HISTORIES

In this communication we will focus on the simplest LQC model that has been analyzed in detail [10–13]: the  $k=0$ ,  $\Lambda=0$  Friedmann model with a massless scalar field as a source. However, it is rather straightforward to extend this analysis to allow for a non-zero cosmological constant or anisotropies or to the spatially compact  $k=1$  case.

In the Hamiltonian theory one begins by fixing a (spatial) manifold  $S$ , topologically  $\mathbb{R}^3$ , cartesian coordinates  $x^i$  thereon, and a fiducial metric  $q_{ab}^o$  given by  $q_{ab}^o dx^a dx^b = dx_1^2 + dx_2^2 + dx_3^2$ . The physical 3-metric  $q_{ab}$  is then determined by a scale factor  $a$ ;  $q_{ab} = a^2 q_{ab}^o$ . For the Hamiltonian analysis one fixes a cubical fiducial cell  $\mathcal{V}$  whose volume with respect to  $q_{ab}$  is given by  $V = a^3 V_o$ . The quantity  $\nu$  defined by  $V = 2\pi\gamma\ell_{\text{Pl}}^2 |\nu|$  turns out to be a convenient

configuration variable, where  $\gamma$  is the Barbero-Immirzi parameter [13].<sup>1</sup>

The kinematical quantum states are functions  $\Psi(\nu, \phi)$  with finite norm  $\|\Psi\|^2 := \sum_{\nu} \int d\phi |\Psi(\nu, \phi)|^2$ . Thus,  $\Psi$  can have support only on a countable number of points on the  $\nu$ -axis and a (generalized) orthonormal basis in the kinematical Hilbert space  $\mathcal{H}_{\text{kin}}$  is given by  $|\nu, \phi\rangle$  with

$$\langle \nu', \phi' | \nu, \phi \rangle_{\text{kin}} = \delta_{\nu', \nu} \delta(\phi', \phi). \quad (2.1)$$

To obtain the physical Hilbert space, one first notes that the quantum constraint can be written as  $\partial_{\phi}^2 \Psi(\nu, \phi) = -\Theta \Psi(\nu, \phi)$  where  $\Theta$  is a second order difference operator

$$\begin{aligned} (\Theta \Psi)(\nu) := & -\frac{3\pi G}{4\ell_o^2} \left[ \sqrt{|\nu(\nu + 4\ell_o)|} (\nu + 2\ell_o) \Psi(\nu + 4\ell_o) - 2\nu^2 \Psi(\nu) \right. \\ & \left. + \sqrt{|\nu(\nu - 4\ell_o)|} (\nu - 2\ell_o) \Psi(\nu - 4\ell_o) \right] \end{aligned} \quad (2.2)$$

where  $\ell_o$  is related to the ‘area gap’  $\Delta = 4\sqrt{3}\pi\gamma\ell_{\text{Pl}}^2$  via  $\ell_o^2 = \Delta$ .  $\Theta$  turns out to be a positive and self-adjoint operator on the gravitational part of the kinematic Hilbert space. The form of  $\Theta$  shows that the space of solutions to the quantum constraint can be naturally decomposed into sectors in which the wave functions have support on specific ‘ $\nu$ -lattices’ [12]. For definiteness, we will restrict ourselves to the lattice  $\nu = 4n\ell_o$  where  $n$  is an integer. Details of the expression of  $\Theta$  will not be needed in this analysis.

The similarity of the form of the quantum constraint to the Klein-Gordon equation suggests that we use  $\phi$  as relational time to deparameterize the quantum theory. As in the Klein-Gordon theory, one can perform a group averaging procedure to arrive at the physical Hilbert space  $\mathcal{H}_{\text{phy}}$  [11]. Elements  $\Psi(\nu, \phi)$  of  $\mathcal{H}_{\text{phy}}$  are solutions to the ‘positive frequency’ quantum constraint equation

$$-i\partial_{\phi} \Psi(\nu, \phi) = \sqrt{\Theta} \Psi(\nu, \phi) \equiv H \Psi(\nu, \phi) \quad (2.3)$$

with a finite norm

$$\|\Psi\|_{\text{phys}}^2 = \sum_{\nu=4n\ell_o} |\Psi(\nu, \phi = \phi_0)|^2. \quad (2.4)$$

Thanks to (2.3) the norm is independent of the ‘time instant’  $\phi_0$  at which it is evaluated. Note that, because of deparameterization, the quantum constraint can be regarded as ‘evolving the state in relational time  $\phi$ ’.

Because of the close similarity of (2.3) with the Schrödinger equation, we can now pass to a sum over histories formulation of quantum dynamics a la Feynman. The object of interest is the transition amplitude  $A(\nu_f, \phi_f; \nu_i, \phi_i)$  to go from the configuration  $\nu_i$  at an initial time  $\phi_i$  to  $\nu_f$  at the final time  $\phi_f$ . It is immediate from the form of (2.3) that this amplitude depends only on the difference  $\phi_f - \phi_i$ . Therefore, without loss of generality we will set  $\phi_i = 0$  and  $\phi_f = \varphi$  and refer to this time interval as  $\mathcal{I}$ . Let us divide  $\mathcal{I}$  into  $N$  parts each of

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<sup>1</sup> In LQG the basic geometric variable is an orthonormal triad and the physical metric  $q_{ab}$  is constructed from it. If the triad has the same orientation as the fiducial one, given by the coordinates  $x^i$ , the configuration variable  $\nu$  is positive and if the orientations are opposite,  $\nu$  is negative. Physics of the model is insensitive to the triad orientation and hence to the sign of  $\nu$ . In particular quantum states satisfy  $\Psi(\nu, \phi) = \Psi(-\nu, \phi)$ .

length  $\epsilon = \varphi/N$  and write the transition amplitude as

$$\begin{aligned} A(\nu_f, \varphi; \nu_i, 0) &:= \langle \nu_f | e^{iH\varphi} | \nu_i \rangle = \sum_{\bar{\nu}_{N-1}, \dots, \bar{\nu}_1} \langle \bar{\nu}_N | e^{i\epsilon H} | \bar{\nu}_{N-1} \rangle \langle \bar{\nu}_{N-1} | e^{i\epsilon H} | \bar{\nu}_{N-2} \rangle \dots \langle \bar{\nu}_1 | e^{i\epsilon H} | \bar{\nu}_0 \rangle \\ &= \sum_{\bar{\nu}_{N-1}, \dots, \bar{\nu}_1} U_{\bar{\nu}_N \bar{\nu}_{N-1}} U_{\bar{\nu}_{N-1} \bar{\nu}_{N-2}} \dots U_{\bar{\nu}_1 \bar{\nu}_0}, \end{aligned} \quad (2.5)$$

where we have introduced a decomposition of the identity operator at each intermediate time  $\phi = n\epsilon$ ,  $n = 1, 2, \dots, N-1$ ; denoted the matrix element  $\langle \bar{\nu}_n | \exp iH\epsilon | \bar{\nu}_{n-1} \rangle$  by  $U_{\bar{\nu}_n \bar{\nu}_{n-1}}$ ; and, for notational simplicity, set  $\nu_f = \bar{\nu}_N$  and  $\nu_i = \bar{\nu}_0$ . The division of  $\mathcal{I}$  provides a skeletonization of the time interval. An assignment  $\sigma_N = (\bar{\nu}_N, \dots, \bar{\nu}_0)$  of volumes to the  $N+1$  time instants  $\phi = \epsilon n$  can be regarded as a discrete history associated with this skeletonization since one can envision the system as hopping from  $\bar{\nu}_{n-1}$  to  $\bar{\nu}_n$ . The transition amplitude is thus given by a sum of amplitudes over these discrete histories, a la Feynman:

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{\sigma_N} A(\sigma_N) \quad \text{with} \quad A(\sigma_N) = U_{\bar{\nu}_N \bar{\nu}_{N-1}} U_{\bar{\nu}_{N-1} \bar{\nu}_{N-2}} \dots U_{\bar{\nu}_2 \bar{\nu}_1} U_{\bar{\nu}_1 \bar{\nu}_0}. \quad (2.6)$$

### III. VERTEX EXPANSION: REORGANIZING THE SUM OVER HISTORIES

In this section we will reorganize the sum (2.6) in the spirit of SFMs and cast it as a ‘vertex expansion’. First note that along a path  $\sigma_N$ , the volume  $\bar{\nu}$  is allowed to remain constant along a number of time steps, then jump to another value, where it could again remain constant for a certain number of time steps, and so on. The first key idea is to group paths according to the number of *volume transitions* rather than time steps. Let us then consider a path  $\sigma_N^M$  which involves  $M$  volume transitions (clearly,  $M \leq N$ ):

$$\sigma_N^M = (\nu_M, \dots, \nu_M; \nu_{M-1}, \dots, \nu_{M-1}; \dots, \nu_2; \overbrace{\nu_1, \dots, \nu_1}^{N_2}; \underbrace{\nu_0, \dots, \nu_0}_{N_1}). \quad (3.1)$$

Thus, the volume changes from  $\nu_{m-1}$  to  $\nu_m$  at time  $\phi = N_m\epsilon$  and remains  $\nu_m$  till time  $\phi = N_{m+1}\epsilon$ . (Note that  $\nu_m$  is distinct from  $\bar{\nu}_m$  used in section II: while  $\nu_m$  is the volume after the  $m$ -th *volume transition* along the given discrete path,  $\bar{\nu}_m$  is the volume at the end of the  $m$ -th *time interval*, i.e., at  $\phi = m\epsilon$ .)

These discrete histories can be labelled more transparently by two ordered sequences

$$\sigma_N^M = \{(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0); (N_M, N_{M-1}, \dots, N_2, N_1)\}, \quad \nu_m \neq \nu_{m-1}, N_m > N_{m-1}. \quad (3.2)$$

Note that while no two *consecutive* volume values can be equal, a given volume value can repeat in the sequence;  $\nu_m$  can equal some  $\nu_n$  if  $n \neq m \pm 1$ . The probability amplitude for such a history  $\sigma_N^M$  is given by:

$$A(\sigma_N^M) = [U_{\nu_M \nu_M}]^{N-N_M-1} U_{\nu_M \nu_{M-1}} \dots [U_{\nu_1 \nu_1}]^{N_2-N_1-1} U_{\nu_1 \nu_0} [U_{\nu_0 \nu_0}]^{N_1-1} \quad (3.3)$$

The second key idea is to carry out the sum over all these amplitudes in three steps. First we keep the ordered set of volumes  $(\nu_M, \dots, \nu_0)$  fixed, but allow the volume transitions to occur at *any* value  $\phi = n\epsilon$  in the interval  $\mathcal{I}$ , subject only to the constraint that the  $m$ -th

transition occurs before the  $(m+1)$ -th for all  $m$ . The sum of amplitudes over this group of histories is given by

$$A_N(\nu_M, \dots, \nu_0) = \sum_{N_M=M}^{N-1} \sum_{N_{M-1}=M-1}^{N_M-1} \dots \sum_{N_1=1}^{N_2-1} A(\sigma_N^M). \quad (3.4)$$

Next we sum over all possible intermediate values of  $\nu_m$  such that  $\nu_m \neq \nu_{m-1}$ , keeping  $\nu_0 = \nu_i$ ,  $\nu_M = \nu_f$  to obtain the amplitude  $A(M)$  associated with the set of all paths in which there are precisely  $M$  volume transitions:

$$A_N(M) = \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \dots, \nu_0) \quad (3.5)$$

Finally the total amplitude  $A(\nu_f, \phi; \nu_i, 0)$  is obtained by summing over all volume transitions that are permissible within our initially fixed skeletonization with  $N$  time steps:

$$A(\nu_f, \phi; \nu_i, 0) = \sum_{M=0}^N A_N(M) \equiv \sum_{M=0}^N \left[ \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \dots, \nu_0) \right]. \quad (3.6)$$

Recall, however, that the Hamiltonian theory implies  $A(\nu_f, \phi; \nu_i, 0) = \langle \nu_f | e^{iH\phi} | \nu_i \rangle$ . Hence the value of the amplitude (3.6) does not depend on  $N$  at all; the skeletonization was introduced just to express this well-defined amplitude as a sum over histories. Thus, while the range of  $M$  in the sum and the amplitude  $A_N(M)$  in (3.6) both depend on  $N$ , the sum does not.

The third key idea is to get rid of the skeletonization altogether by taking the limit as  $N$  goes to infinity, to express the total transition amplitude as a vertex expansion in the spirit of the timeless framework of spin-foams. To carry out this step, we first note that a straightforward but non-trivial calculation [9] shows that  $\lim_{N \rightarrow \infty} A_N(\nu_M, \dots, \nu_0)$  exists and is given by:

$$\begin{aligned} A(\nu_M, \dots, \nu_0) &:= \lim_{N \rightarrow \infty} A_N(\nu_M, \dots, \nu_0) \\ &= \int_0^\varphi d\phi_M \int_0^{\phi_M} d\phi_{M-1} \dots \int_0^{\phi_2} d\phi_1 A(\nu_M, \dots, \nu_0; \phi_M, \dots, \phi_1), \end{aligned} \quad (3.7)$$

where,

$$\begin{aligned} A(\nu_M, \dots, \nu_0; \phi_M, \dots, \phi_1) &:= e^{i(\varphi - \phi_M)H_{\nu_M\nu_M}} (iH_{\nu_M\nu_{M-1}}) e^{i(\phi_M - \phi_{M-1})H_{\nu_{M-1}\nu_{M-1}}} \times \\ &\dots e^{i(\phi_2 - \phi_1)H_{\nu_1\nu_1}} (iH_{\nu_1\nu_0}) e^{i\phi_1 H_{\nu_0\nu_0}} \end{aligned} \quad (3.8)$$

The structure of these equations can be understood as follows. In the limit  $N \rightarrow \infty$ , the length  $\epsilon = \varphi/N$  of the elementary time intervals goes to zero and discrete sums in (3.4) are replaced by continuous integrals. Eq (3.7) says that the final, i.e.,  $M$ -th volume transition can occur anywhere in the interval  $\mathcal{I}$ , the  $(M-1)$ -th transition can occur anywhere before the  $M$ -th, and so on. In passing from (3.4) to (3.8), factors like  $[U_{\nu_M\nu_M}]^{N_M - N_{M-1}} = [1 + i((N_M - N_{M-1})\epsilon)H_{\nu_M\nu_M} + \dots]$  go over to  $e^{i(\phi_M - \phi_{M-1})H_{\nu_M\nu_M}}$  while factors like  $U_{\nu_M\nu_{M-1}} =$

$i\epsilon H_{\nu_M\nu_{M-1}} + O(\epsilon^2)$  go over to  $iH_{\nu_M\nu_{M-1}}$ .

It is trivial to carry out the integrations over  $\phi_m$  in (3.7) and express  $A(\nu_M, \dots, \nu_0)$  just in terms of matrix elements of  $H$ . For simplicity, let us consider the case when all of  $(\nu_M, \dots, \nu_0)$  are distinct. Then, we have:

$$A(\nu_M, \dots, \nu_0) = H_{\nu_M\nu_{M-1}} \cdots H_{\nu_1\nu_0} \sum_{m=0}^M \frac{e^{i\varphi H_{\nu_m\nu_m}}}{\prod_{\substack{j=0 \\ j \neq m}}^M (H_{\nu_m\nu_m} - H_{\nu_j\nu_j})}. \quad (3.9)$$

All matrix elements  $H_{\nu_m\nu_n}$  can be computed explicitly [9]. Finally, since  $\lim_{N \rightarrow \infty} A_N(\nu_M, \dots, \nu_0)$  exists, in the limit (3.6) becomes simply

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{M=0}^{\infty} A(M) \quad \text{where,} \quad A(M) = \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \dots, \nu_0). \quad (3.10)$$

Eq (3.10) mimics the vertex expansion of SFMs. More precisely, the parallels are as follows. The analog of the manifold  $M$  with boundaries  $S_i, S_f$  in SFMs is the manifold  $\mathcal{V} \times \mathcal{I}$ , where  $\mathcal{V}$  is the elementary cell in LQC and  $\mathcal{I}$  the closed interval bounded by  $\phi = 0$  and  $\phi = \varphi$ . The analog of the dual-triangulation in SFMs is just a ‘vertical’ line in  $\mathcal{V} \times \mathcal{I}$  with  $M$  marked points or ‘vertices’ (not including the two end-points of  $\mathcal{I}$ ). What matters is the number  $M$ ; the precise location of vertices is irrelevant. Coloring of the dual-triangulation in SFMs corresponds to an ordered assignment  $(\nu_M, \nu_{M-1}, \dots, \nu_1, \nu_0)$  of volumes to edges bounded by these marked points (subject only to the constraints  $\nu_M = \nu_f$ ,  $\nu_0 = \nu_i$  and  $\nu_m \neq \nu_{m-1}$ ). Each vertex signals a change in the physical volume along the quantum history. The probability amplitude associated with the given coloring is  $A(\nu_M, \dots, \nu_0)$  and a sum over colorings yields the amplitude  $A(M)$  associated with the triangulation with  $M$  ‘vertices’. The total amplitude  $A(\nu_f, \varphi; \nu_i, 0)$  is given by a sum (3.10) over these  $M$ -vertex amplitudes.

To conclude this section, we emphasize that the result was *derived* from a Hamiltonian theory. We did not postulate that the left side of (3.10) is given by a formal path integral. Rather, a rigorously developed Hamiltonian theory guaranteed that the left side is well-defined and provided the expression (2.5) for it. We simply recast this expression as a vertex expansion.

#### IV. VERTEX EXPANSION AS A PERTURBATION SERIES

We will now show that the expression (3.10) of the transition amplitude can also be obtained using a specific perturbative expansion. Structurally, this derivation of the vertex expansion is reminiscent of the perturbative strategy used in group field theory (see, e.g., [14, 15]).

Let us begin by considering the diagonal and off-diagonal parts  $D$  and  $K$  of the ‘Hamiltonian’  $H$  in the basis  $|\nu = 4n\ell_o\rangle$ , defined by their matrix elements:

$$D_{\nu'\nu} = H_{\nu\nu} \delta_{\nu'\nu}, \quad K_{\nu'\nu} = \begin{cases} H_{\nu'\nu} & \nu' \neq \nu \\ 0 & \nu' = \nu \end{cases} \quad (4.1)$$

Clearly  $H = D + K$ . The idea is to think of  $D$  as the main part of  $H$  and  $K$  as a perturbation. To implement it, introduce a 1-parameter family of operators  $H_\lambda = D + \lambda K$  as an intermediate mathematical step. The parameter  $\lambda$  will simply serve as a marker to keep track of powers of  $K$  in the perturbative expansion; we will have to set  $\lambda = 1$  at the end of the calculation.

Following the textbook procedure, let us define the ‘interaction Hamiltonian’ as

$$H_I(\phi) = e^{-iD\phi} \lambda K e^{iD\phi}. \quad (4.2)$$

Then the evolution in the interaction picture is dictated by the operator

$$\tilde{U}_\lambda(\phi) = e^{-iD\phi} e^{iH_\lambda\phi}, \quad \text{satisfying} \quad \frac{d\tilde{U}_\lambda(\phi)}{d\phi} = iH_I(\phi)\tilde{U}_\lambda(\phi), \quad (4.3)$$

whose solution is given by a time-ordered exponential:

$$\begin{aligned} \tilde{U}_\lambda(\varphi) &= \mathcal{T} e^{i \int_0^\varphi H_I(\phi) d\phi} \\ &= \sum_{M=0}^{\infty} \int_0^\varphi d\phi_M \int_0^{\phi_M} d\phi_{M-1} \dots \int_0^{\phi_2} d\phi_1 [iH_I(\phi_M)] \dots [iH_I(\phi_1)]. \end{aligned} \quad (4.4)$$

Next, let us express the evolution operator as  $U_\lambda(\varphi) = e^{iD\varphi} \tilde{U}_\lambda(\varphi)$ , with  $\tilde{U}_\lambda(\varphi)$  given by (4.4), take matrix element between initial and final states,  $|\nu_i \equiv \nu_0\rangle$  and  $|\nu_f \equiv \nu_M\rangle$ , and write out explicitly the product of the  $H_I$ 's. The result is

$$\begin{aligned} A_\lambda(\nu_f, \varphi; \nu_i, 0) &= \sum_{M=0}^{\infty} \int_0^\varphi d\phi_M \dots \int_0^{\phi_2} d\phi_1 \sum_{\nu_{M-1}, \dots, \nu_1} [e^{i(\varphi-\phi_M)D_{\nu_M\nu_M}}] (i\lambda K_{\nu_M\nu_{M-1}}) \times \\ &\quad [e^{i(\phi_M-\phi_{M-1})D_{\nu_{M-1}\nu_{M-1}}}] \dots (i\lambda K_{\nu_1\nu_0}) [e^{i\phi_1 D_{\nu_0\nu_0}}]. \end{aligned} \quad (4.5)$$

We can now replace  $D$  and  $K$  by their definition (4.1). Because  $K$  has no diagonal matrix elements, only the terms with  $\nu_m \neq \nu_{m+1}$  contribute to the sum and the sum becomes

$$A_\lambda(\nu_f, \varphi; \nu_i, 0) = \sum_{M=0}^{\infty} \lambda^M \left[ \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \dots, \nu_0) \right], \quad (4.6)$$

where  $A(\nu_M, \dots, \nu_0)$  is defined in (3.9). If we now set  $\lambda = 1$ , Eq. (4.6) reduces to Eq. (3.10) obtained independently in section III.

Thus, by formally regarding the off-diagonal piece of the Hamiltonian as a perturbation of the diagonal piece we have obtained an independent derivation of the vertex expansion of the amplitude  $A_\lambda(\nu_f, \varphi; \nu_i, 0)$  as a power series expansion in  $\lambda$ , the power of  $\lambda$  serving as a book-keeping device to keep track of the order in the vertex expansion. In this sense this alternate derivation is analogous to the vertex expansion obtained using group field theory.

## V. GROUP AVERAGING

We began our discussion by carrying out a deparametrization (using  $\phi$  as the relational time variable) because this is the procedure used in LQC to extract physics from the quantum theory. Spin foams on the other hand are based on a timeless framework. Furthermore, a convenient deparametrization is not always available even in cosmology if we allow the scalar field to have general potentials or the gravitational field to admit inhomogeneities. In this case we have to return to the full constraint and construct the physical Hilbert space differently. Then the basic object of interest is not a transition amplitude but the physical inner product. In LQC, it is given by the well-known group averaging procedure:

$$\langle \nu_f, \phi_f | \nu_i, \phi_i \rangle_{\text{phys}} = [(\nu_f, \phi_f | \int_{-\infty}^{\infty} d\alpha e^{i\alpha C} | \nu_i, \phi_i \rangle] \quad (5.1)$$

where  $C = p_\phi^2/\hbar^2 - \Theta$  is the constraint operator and the round bracket on the right side of the equation denotes a ‘generalized bra’, an element of the algebraic dual (called  $\text{Cyl}^*$  in the literature) of a suitable dense subspace of  $\mathcal{H}_{\text{kin}}$ .<sup>2</sup> Formally, integration over the ‘lapse’  $\alpha$  introduces the factor  $\delta(C)$  that is necessary to extract physical states from kinematical ones and also yields the physical inner product between the resulting physical states. This procedure can be carried out in detail [11]. If we restrict ourselves to the positive part of the spectrum of  $\hat{p}_\phi$  —or, to ‘positive frequency’ physical states— as in LQC, *the physical inner product is given precisely by the transition amplitude*  $A(\nu_f, \phi_f; \nu_i, \phi_i)$  we focused on in the last three sections (which, in turn, reproduces (2.4)). On the ‘negative frequency’ solutions it is given by the complex conjugate,  $[A(\nu_f, \phi_f; \nu_i, \phi_i)]^*$  (because the matrix elements  $H_{\nu_m, \nu_n}$  and  $\Theta_{\nu_m, \nu_n}$  are all real.) If we allow both, then the inner product is always real:  $\langle \nu_f, \phi_f | \nu_i, \phi_i \rangle_{\text{phys}} = A(\nu_f, \phi_f; \nu_i, \phi_i) + [A(\nu_f, \phi_f; \nu_i, \phi_i)]^*$ . Thus, the physical inner product, the key object in the timeless framework, can be readily constructed from the transition amplitude, the key object in the deparameterized framework.

We can also use the procedure followed in section III or IV to carry out group averaging directly, without any reference to  $\phi$  as relational time, to express the physical inner product as a vertex expansion. The two procedures yield the same result which is of course equivalent to (3.10). However, somewhat surprisingly, order by order, it is distinct from (3.10). For definiteness, let us use the perturbative method of section IV. The key object now is the operator  $e^{i\alpha C}$ . As before, let us introduce a 1-parameter family of operators  $C(\underline{\lambda}) = p_\phi^2/\hbar^2 - \Theta(\underline{\lambda})$  as follows. Set  $\Theta(\underline{\lambda}) = \underline{\mathbf{D}} + \underline{\lambda} \underline{\mathbf{K}}$  where  $\underline{\mathbf{D}}$  is the diagonal part of  $\Theta$  and  $\underline{\mathbf{K}}$  the off-diagonal part. Then, using the interaction picture one can again expand out the operator  $e^{i\alpha C(\underline{\lambda})}$  and take its matrix elements to obtain the  $\underline{\lambda}$ -physical inner product to obtain

$$\langle \nu_f, \phi_f | \nu_i, \phi_i \rangle_{\text{phy}, \underline{\lambda}} = \sum_{M=0}^{\infty} \underline{\lambda}^M \left[ \sum_{\nu_{M-1}, \dots, \nu_1} \underline{\mathbf{A}}(\nu_M, \dots, \nu_0) \right], \quad (5.2)$$

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<sup>2</sup> There is some freedom in the definition of the action of elements of  $\text{Cyl}^*$ . In LQC, this freedom is used to simplify the expression of the physical inner product [11] and the subsequent action of Dirac observables on  $\mathcal{H}_{\text{phy}}$ . We will use the same conventions here.



where

$$\underline{\Lambda}(\nu_M, \dots, \nu_0) = \Theta_{\nu_M \nu_{M-1}} \cdots \Theta_{\nu_1, \nu_0} \sum_{m=0}^M \frac{e^{i(\phi_f - \phi_i) \sqrt{\Theta_{\nu_m \nu_m}}}}{\prod_{\substack{j=0 \\ j \neq m}}^M (\Theta_{\nu_m \nu_m} - \Theta_{\nu_j \nu_j})}. \quad (5.3)$$

(As before, for simplicity we have assumed that  $(\nu_M, \dots, \nu_0)$  are distinct.) Again,  $\underline{\Lambda}$  is only a book-keeping device for the intermediate perturbative expansion and the physical inner product is obtained by setting  $\underline{\Lambda} = 1$  in the final result.

Note that Eq (5.2) is identical to Eq (4.6) (except for underbars) and Eq (5.3) has the same form as Eq (3.9). However, while Eq (3.9) contains matrix elements of  $H$ , Eq (5.3) contains matrix elements of  $\Theta = H^2$ . This fact leads to two important differences. First  $\sqrt{\Theta_{\nu_m \nu_m}}$ , the square root of the matrix element of  $\Theta$ , is distinct from  $H_{\nu_m \nu_m}$ , the matrix element of the square root of  $\Theta$ . Second, because the off diagonal elements  $\Theta_{\nu_m \nu_n}$  in Eq (5.3) are non-zero only if  $\nu_m = \nu_n \pm 4\ell_o$ , consecutive  $\nu_m$  in second sum in Eq (5.2) can differ only by  $\pm 4\ell_o$ . There is no such simplification in Eq (4.6). Because of these differences, although the physical inner product obtained by group averaging is related in a simple manner to the transition amplitude, the vertex expansion obtained in this section is completely different from that obtained in the last two sections. If we were to terminate the sum at any finite order, the results would not be simply related.

## VI. DISCUSSION

Let us start with a brief summary. In section II we began with Hamiltonian LQC, divided the time interval into  $N$  segments and expressed the transition amplitude  $A(\nu_f, \phi_f; \nu_i, \phi_i)$  as a sum (2.6) over discrete histories. In section III we reorganized this sum emphasizing volume transitions and took the  $N \rightarrow \infty$  limit to get rid of the skeletonization of the time interval. This led us to the expression (4.6) of the transition amplitude. The  $M$ -th term in this expansion corresponds to a sum over all histories in which there are precisely  $M$  volume transitions, allowed to occur at *any* time in the interval  $(\phi_f, \phi_i)$ . Therefore the expansion resembles the vertex expansion of SFMs. In section IV we showed that the same vertex expansion can be arrived at by formally splitting the Hamiltonian  $H$  into a main part  $D$  and a ‘perturbation’  $\lambda K$  and expanding the transition amplitude using standard perturbation theory in the interaction picture. (Here the ‘coupling constant’  $\lambda$  was introduced just as a mathematical label to keep track of the number of vertices in various terms and we have to set  $\lambda = 1$  at the end to recover the physical transition amplitude). This expansion in powers of  $\lambda$  resembles the vertex expansion in group field theory. Finally in section V we returned to the expression of the physical inner product that one begins with in the group averaging procedure. This features the exponential of the constraint operator  $C \equiv p_\phi^2 - \Theta$ , where  $\Theta = H^2$ . One can carry out a perturbative expansion as in section IV by formally writing  $\Theta$  into a main part  $\underline{D}$  and a perturbation  $\underline{\lambda K}$  to arrive at an expression (5.2) of the inner product as a perturbation series in  $\underline{\lambda}$ . The standard group averaging procedure implies that (for ‘positive frequency’ physical states) the physical inner product in the timeless framework equals the transition amplitude in the deparameterized framework used in LQC [11]. But whereas the perturbative expansion (4.6) involves the matrix elements of  $H$ , (5.2) involves the matrix elements of  $\Theta$  (and their square-roots). Thus, while the sum yields the same quantity, if we were to truncate the perturbation series to any finite order one would obtain distinct results. Thus, somewhat surprisingly there are two *distinct* natural vertex

expansions, one descending from the deparameterized theory and the other from the timeless framework of group averaging.

While our final result yields vertex expansions in the spirit of SFMs, we did not begin with a SFM and arrive at the vertex expansion by a symmetry reduction, e.g., by summing over the degrees of freedom other than the total volume. Our procedure is much more modest: As is usual in LQC, we carried out the symmetry reduction at the *classical* level by partial gauge fixing, constructed the Hamiltonian quantum theory and used it to obtain vertex expansions. Also, so far, vertex expansions have been discussed in SFMs only for source-free gravity, while the presence of a scalar field played a key role in LQC. Nonetheless our results provide strong support for the paradigm underlying SFMs. In addition, since we have an exactly soluble, concrete example, we can use it to analyze the status of open issues.

First, as hoped in SFMs, the physical inner product can indeed be expressed as a vertex expansion. The inner product is defined independently and this well-defined quantity is merely expanded out as a convenient series. Second, the expectation that the expansion should be derivable from a suitable Hamiltonian theory has been realized. In addition, as one would expect from [2], each vertex can be thought of as emerging from the action of the Hamiltonian operator. Finally, in our decomposition  $H = D + \lambda K$  (or  $\Theta = \underline{D} + \underline{\lambda K}$ ) one can think of  $D$  as the free part of the Hamiltonian because it does not change the volume and  $K$  as the interaction part because it does. Thus, as in group field theory, the factors of  $\lambda$  are associated with ‘interaction’ piece of the Hamiltonian (which is responsible for the volume transitions in LQC).

An issue that is often raised in the literature on SFMs is whether the physical inner product is really given by just summing over triangulations, each with a finite number of vertices, or if one should take a “continuum” limit at the end as in, e.g., lattice QCD. If one *defines* the inner product as a discretized path integral, the answer is not a priori obvious. However, since we began with a well-controlled Hamiltonian theory, in the LQC example, the answer *is* clear. We did not have to take the limit; the discrete sum provided the exact answer. It is also instructive to note that we were led to two natural vertex expansions. As we remarked in section I, in SFMs one has to introduce a vertex amplitude and, while it is constrained by several requirements, we do not have a statement of uniqueness. In LQC we were able to obtain two distinct expansions, one using the deparameterized theory and the other using the timeless framework. Are there perhaps similar inequivalent vertex expansions in more complicated models—or even the full theory—each tailored to making an aspect of the theory more transparent? Next, we saw that the inner product between physical states extracted from the kinematic basis vectors  $|\nu, \phi\rangle$  are in general complex (as is generally true for constrained systems). However, if we were to enlarge the physical Hilbert space allowing for both ‘positive *and* negative frequency’ solutions, they become real. In SFMs the obvious analog of the LQC ‘positive frequency’ restriction is a choice of an orientation. Currently, the sum involves both orientations and the inner products between the physical states extracted from spin network states are all real. The LQC analysis naturally raises a question: Should one also impose a suitable restriction allowing, e.g., only those histories with only one orientation [16]? Or, does correct quantum physics require us to have only real physical scalar products in this basis? If so, why is there a qualitative difference in LQC? Finally, if one regards group field theory as fundamental, rather than just a convenient computational tool to arrive at the spin foam vertex expansion, then one is led to take the coupling constant  $\lambda$  as a physical parameter which can run with the renormalization

group flow. However, its interpretation has been elusive. A detailed examination of the LQC example shows that it is naturally tied with the cosmological constant. If this were to hold also in the full theory, one may have a dynamical tool to analyze why the cosmological constant is so small in the low energy regime. These and several other issues will be discussed in detail in [9].

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