Quantum cosmology: a review

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Abstract

In quantum cosmology, one applies quantum physics to the whole universe. While no unique version and no completely well-defined theory is available yet, the framework gives rise to interesting conceptual, mathematical and physical questions. This review presents quantum cosmology in a new picture that tries to incorporate the importance of inhomogeneity: De-emphasizing the traditional minisuperspace view, the dynamics is rather formulated in terms of the interplay of many interacting "microscopic" degrees of freedom that describe the space-time geometry. There is thus a close relationship with more-established systems in condensed-matter and particle physics even while the large set of space-time symmetries (general covariance) requires some adaptations and new developments. These extensions of standard methods are needed both at the fundamental level and at the stage of evaluating the theory by effective descriptions.

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1 Introduction

Quantum cosmology is based on the idea that quantum physics should apply to anything in nature, including the whole universe.[1] Quantum descriptions of all kinds of matter fields and their interactions are well known and can easily be combined into one theory — leaving aside the more complicated question of unification, which asks for a *unique* combination of all fields based on some fundamental principles or symmetries. Nevertheless, quantizing the whole universe is far from being straightforward because, according to general relativity, not just matter but also space and time are physical objects. They are subject to dynamical laws and have excitations (gravitational waves) that interact with each other and with matter. Quantum cosmology is therefore closely related to quantum gravity, the quantum theory of the gravitational force and space-time. Since quantum gravity remains unfinished, the theoretical basis of quantum cosmology is unclear. And to make things worse, there are several difficult conceptual problems to be overcome.

For a theory that stubbornly retains its highly speculative and controversial nature, quantum cosmology has a rather long history. Soon after the basic ingredients of general relativity and quantum mechanics had become known, adventurous theorists began to apply these new-found laws to the cosmos. Lemaître's "primordial atom" [1, 2] combines insights from general relativity (what would later be called the big-bang singularity) with key concepts of quantum mechanics. Tolman's attempts [3, 4] to use quantum physics in order to solve the singularity issue of general relativity were in many ways prescient and brought up problems still relevant today, especially the issue of entropy growth in an eternal universe. But further progress was hindered not only by the incomplete status of quantum mechanics at the time or a lack of interest from mainstream theorists. (After all, atomic physics provided much more practical and pressing problems.) More importantly, the mathematical issues to be solved when one tried to go beyond the simplest models were daunting. Even today, more than 80 years later, the setting remains incompletely realized in several different, sometimes competing ways in which one might be able to quantize the gravitational force: string theory, canonical quantum gravity, and non-commutative geometry to name the most popular ones. And when it comes to concrete early-universe scenarios "derived" from any one of these frameworks, the number of different and often mutually contradictory attempts rapidly multiplies.

Moreover, unlike other areas of physics that were largely reformed or newly created with the arrival of quantum mechanics, the application of quantum physics to the whole universe has not been able to make contact with observations, nor has it been able to resolve conceptual issues to a degree which would allow one to say that it works for all practical purposes. Then why would anyone be interested in quantum cosmology? In spite of all misgivings, quantum cosmology offers not only challenges but also interesting insights about a variety of questions, most importantly the fundamental nature of space-time. There are also stimulating mathematical and conceptual questions, related for instance

[1] The definition of "quantum cosmology" has changed over the years from quantization restricted to the degrees of freedom of highly symmetric classical cosmological spacetimes to the broader viewpoint taken here.

to the interplay of symmetries and discrete structures or the extraction of semiclassical features from a generally covariant and highly interacting quantum theory. This review attempts to give a modern viewpoint that takes into account previous roadblocks and points out promising activities.

1.1 The premise of quantum cosmology

The main part of this review is the presentation of quantum cosmology based on a physical and microscopic picture of quantum space-time, somewhat similar to atomic systems studied in condensed-matter physics. It turns out that this condensed-matter analogy may not only serve as a pedagogical tool to make a somewhat exotic field more broadly accessible; it will also allow one to transfer useful methods and potentially lead to cross-fertilization between different fields. One could, moreover, hope that the down-to-earth nature of systems studied in condensed-matter physics will lead to a more humble view on quantum cosmology, which tends to be attracted to grandiose questions regarding the whole universe, everything in or about it, and the origin.

We first set up the problems to be solved by quantum cosmology. General relativity has shown that space-time is physical and dynamical, and that it interacts with matter. Since matter is described by quantum physics, a consistent coupling requires space-time to be quantized as well. In the absence of any complete theory of quantum gravity (let alone experiments), however, the structure of microscopic degrees of freedom of space-time remains unknown. Even the classical nature of space-time, which will likely play the role of the continuum limit of some more fundamental quantum space-time, had not been uncovered until special relativity was developed. As in this well-known case, conceptual arguments play an important role in the approach to the quantum nature of space-time, providing guidelines in the absence of direct observations.

As shown by special relativity, and even more so by general relativity, space-time is characterized by the symmetries it enjoys. These symmetries of general covariance are not accidental but ensure that the theory is meaningful, in the sense that predictions do not depend on the choice of observers or mathematical descriptions and coordinates. They are not to be broken by quantum effects or else the theory will be inconsistent. Unlike, say, crystals that may break the rotational symmetry of the vacuum, space-time must be treated much more delicately if it is to be described in microscopic terms. This problem is a difficult one: the various approaches to quantum gravity and quantum cosmology are still looking for a consistent treatment. But recent progress has shed some light on possible outcomes.

After introductory discussions in the present section, in Sec. 2 we enter details provided by different approaches. We begin with the analog of one-particle Hamiltonians in order to understand the building blocks we are dealing with. Just as quantum mechanics provides the laws for a quantum description of idealized point-like particles, quantum cosmology should start with a quantization of what might be considered the most elementary form of space: a single point or a small uniform region around it. Clearly, without direct observations, it is difficult to guess what the elementary constituents should be. But the strong and difficult consistency conditions imposed by the symmetries of space-time provide an advantage: they can fully be implemented only by tightly controlled conceptual reasoning, which in many cases is so restrictive that fundamental properties can be inferred theoretically. As always, of course, theoretical derivations are based on some principles which may very well be wrong as far as Nature is concerned. Nevertheless, consistent sets of suitable laws for different forms of classical or quantum space-time are rare, a feature which, once consistency has been fully implemented, should strongly reduce the arbitrariness in possible forms of constituent laws.

1.2 Problems to be faced

Quantum cosmology starts with a quantization of a structureless, homogeneous chunk of space as a first approximation to a "space-time atom." Traditionally [5], the resulting systems have, under the name of minisuperspace models, been thought of as quantizations of a whole, spatially homogeneous universe that might approximate the actual one quite well at least at early times. However, to anticipate a little bit, there are several problems[2] with this picture, not just because the sense in which an actual approximation is achieved is rather uncontrolled. For this reason, we take what one may think of as the opposite view, understanding these systems not as models for the whole universe but rather as quantizations of the smallest possible and structureless contributions to space.

Another conceptual problem can already be glimpsed at this stage: Quantum cosmology attempts to find the correct quantum theory of *space-time*. But our brief sketch of a quantum-mechanical model of the elementary constituents states that we should quantize a point (or small region) in *space*. If we were to follow standard quantum mechanics, time would be provided as an external, un-quantized parameter, playing a role very different from the quantized chunk of space to be described. Such a theory could hardly be covariant. The nature of time, and not just the covariance symmetries it is crucially involved in, must therefore be rethought for a fundamental theory of quantum cosmology.

Given a consistent quantum model of building blocks of space (perhaps evolving in time), a general example of quantum space is then a collection of these chunks, patched together to form a large system which over long distances must look like the curved continuum of general relativity. The collection of chunks of space may be viewed as an approximation to inhomogeneous space by simpler homogeneous pieces as it may be of some use for a classical evaluation. But in a more fundamental view, it could also correspond to an actual physical decomposition of space if quantum gravity leads to a discrete spatial structure, as some approaches do. Two tasks are then to be faced: (i) Find the correct quantum theory

[2]This review discusses a number of key problems, introduced in due course. A problem in this class does not necessarily indicate a failure of the approach in which it appears, but it highlights important issues to be understood and therefore helps to focus research. Some problems may be considered more or less severe by different researchers. As always, the viewpoints taken in this review are partially affected by the author's opinion and degree of familiarity with other approaches, but an attempt has been made to focus on problems that are widely recognized in the general field of quantum gravity and cosmology (but not always in a single approach).

for a single patch, or the single-patch theory analogous to one-particle dynamics, and (ii) determine the rules to combine patches into a many-patch system, analogous to interacting many-body systems. When these tasks have been completed in a consistent form, quantum cosmology has been constructed. It then remains the task of finding manageable ways of evaluating the theory and comparing its predictions with possible observations.

The derivation of single-patch theories starts with the classical dynamics, given by the Friedmann equation of cosmology. At this point, the somewhat problematic role of time can already be discussed because the system is reparameterization invariant (the time coordinate can be changed at will), giving a first look on issues involved with covariance at a quantum level. The main approach used here is canonical quantum cosmology, which directly quantizes the Friedmann equation. (However, other approaches could be interpreted as taking comparable, although technically quite different, single-patch views, for instance string theory with a single string worldsheet as a test particle.) A canonical quantization of the Friedmann equation will therefore present candidates for single-patch theories. We emphasize again that the single-patch viewpoint does not amount to a traditional minisuperspace quantization, although they both quantize the Friedmann equation and are formally equivalent in some approaches. In a minisuperspace model, one typically interprets the patch as the whole universe, approximately homogeneous early on before structures had time to form. This traditional viewpoint has led to several problems, especially in discrete theories which give rise to quantum corrections that depend on the discrete patch size. In a minisuperspace model, the magnitude of these corrections is not captured properly because one is led to insert the whole size of the visible universe for the microscopic patch size.

Moreover, owing to the complicated symmetry structure in a covariant theory, there is a big difference between an exactly homogeneous model and one slightly perturbed by inhomogeneous structure. Even if classical back-reaction of inhomogeneous modes on the homogeneous background can be ignored, inhomogeneity leads to much tighter constraints on the dynamics, even of the background, than an exactly homogeneous minisuperspace model would suggest. It is much more difficult to ensure covariance of (inhomogeneous) partial differential equations compared to (exactly homogeneous) ordinary ones. Compared with the symmetry of matter distributions, which as an approximation often simplifies but still approximates realistic physical systems, space-time covariance is more subtle. It removes redundant degrees of freedom such as choices of coordinates, not directions along which structure may, at leading order, be unresolved. For these reasons, in this review we try to avoid the somewhat misleading minisuperspace view, which makes use of large exactly homogeneous spaces, and instead refer to single patches. (Minisuperspace models are discussed briefly in a historical context.)

After a detailed discussion of single-patch theories, qualitative patching models are described in Sec. 3. There are two main examples, the first one derived from the classical Belinski–Khalatnikov–Lifshitz scenario [6], according to which even highly inhomogeneous space behaves near a singularity, such as the big bang, like a collection of disconnected homogeneous pieces. This scenario leads to a patch model in which constituents are non-interacting initially (near the singularity), but slowly build up more complicated dynamics as the universe expands and dilutes to smaller densities. The second example of qualitative patching models is more recent and uses well-studied techniques from condensed-matter physics, in particular mathematical descriptions of Bose–Einstein condensation. Different patches of space (unlike individual particles in a condensate) cannot be required to occupy exactly the same state, but in a nearly homogeneous universe all the individual states should at least be close to one another. Mathematical methods of Bose–Einstein condensation then map the interacting many-patch dynamics to a non-linear single-patch dynamics of states, described by some version of the Gross–Pitaevski equation.

The exact patching of single contributions in all its details is supposed to be determined by quantum gravity, which remains incomplete. Nevertheless, several interesting indications already exist and can be exploited in models of quantum cosmology. Section 4 is devoted to these consequences, but it is brief.

Any potential candidate for quantum cosmology or quantum gravity must be evaluated and scrutinized. Another main section of this review, Sec. 5, introduces and deals with effective theories as the method of choice. Here, again, there is some overlap with condensed-matter techniques, at least at a technical level. Many-patch dynamics is usually too complicated to be dealt with directly, and in quantum gravity even numerical methods are still in their infancy and do not yet help much. Low-energy and semiclassical methods are therefore essential, by which one can approximate complicated quantum theories in interesting and observationally accessible regimes. There is a large set of techniques from different approaches, such as double-sheeted universe models (not unlike double layers of graphene) or deformed coordinate structures in non-commutative geometry, brane-world models in string theory, and different types of effective equations in canonical quantum cosmology.

Canonical effective methods are perhaps the most systematic. They begin with the well-known Ehrenfest equations of quantum mechanics but can be amended to take into account new structures required for cosmology, including a more complicated dynamics and symmetry issues such as covariance and independence of one's choice of time. The important and recurring problem of time in quantum gravity is discussed more fully in this context.

Also the problem of states plays a role: Most effective techniques make use of a simple ground, vacuum or other state around which one can expand, a reasonable choice for near-stationary and lowly-excited properties of systems in condensed-matter or particle physics. But if one tries to extend these methods to quantum cosmology, one encounters the problem that it remains unclear what the vacuum of quantum space should be, a state in which, presumably, no space-time and no geometry are excited. Moreover, even if a vacuum state existed, one would need high excitations of a large number of patches to deal with macroscopic, even cosmic and potentially high-density regions of space. One must find suitable many-body states, or perhaps an analog of finite-temperature states in which not the distribution of particle velocities but the arrangement of spatial patches is close to the classical continuum. Such states are difficult to construct and remain unknown, but some of their properties can be implemented by effective methods, also shedding light on the form of quantum space-time symmetries.

1.3 Opportunities

The challenges encountered by quantum cosmology provide us with an opportunity to revisit general methods to construct and evaluate physical quantum systems. Especially the theory of effective descriptions, but also general quantization procedures, must be extended considerably if one tries to address the problems of time, covariance, and states. In such an analysis, one may find new insights that could be useful also in other systems. Certainly from a mathematical perspective there are already several interesting examples.

A notion that should be expected in the context of many-patch systems, considered at vastly different scales that bridge fundamental candidates with potential observations, is renormalization. It is not discussed in this review because its formulation for quantum cosmology is still to be attempted. Quantum space-time does not provide an energy scale by which one could control the renormalization flow, and therefore even the basic set-up remains unclear. (See [7] for a model of possible implementations.) But the question is certainly important and presents an interesting challenge.

Models of quantum cosmology help to derive and understand possible candidates for the quantum structure of space-time. This issue plays an important role for quantum gravity as well, but is difficult to analyze at such a general level that includes all possible solutions. The restriction to cosmological space-times with some approximate symmetries, in addition to the potential for observations at high density, provides additional tools and motivations.

Physically, the ultimate pay-off of theoretical developments is the confrontation with experimental data. The final section of this review gives an outlook on potential observational tests of the theory. In the context of quantum gravity one usually thinks of the Planck density $\rho_{\rm P} = c^7/\hbar G^2$, formed by the fundamental constants of gravity, relativity and quantum physics, which is much larger by many orders of magnitude than the average density encountered in any observationally accessible regime. (It amounts to more than a trillion solar masses in a region the size of a single proton. Similarly, the distance measure given by the Planck length $\ell_{\rm P} = \sqrt{G\hbar/c^3} \approx 10^{-35}$m is much tinier than anything that can be probed by particle accelerators.) Moreover, while quantum gravity remains incomplete and usually subject to a large number of quantization ambiguities, predictions in the Planck regime are uncontrolled. But indirect effects at lower density — in which small individual quantum corrections of many patches taken together conspire to form a more sizeable contribution — are possible, much like Brownian motion which helped to detect atomic and molecular properties. There are two main examples: large extra dimensions as postulated by string theory (which have been reviewed extensively elsewhere), and different types of modifications from discrete theories such as loop quantum gravity.

2 Single-patch theories

In quantum cosmology, the analog of a single particle as an ingredient of more-complicated interacting theories is a small and uniform chunk of space. It is subject to simple laws in general relativity, which one can try to quantize by familiar methods. The result, which is formally equivalent to traditional minisuperspace models of quantum cosmology, is called here a single-patch theory. In several conceptual and practical aspects the single-patch viewpoint differs crucially from a minisuperspace picture. It turns out to be better suited for going beyond the highly simplified setting of exactly homogeneous cosmology in a realistic description of the evolution of structure in the universe.

In spite of such advantages, it is far from clear whether a single-patch theory (or a minisuperspace model) captures the correct fundamental degrees of freedom of quantum space-time. After all, a water molecule is not a water droplet. The whole process of quantizing general relativity or constructing quantum gravity has occasionally been questioned based on an analogy with hydrodynamics, in which applying quantum rules to the Navier–Stokes equation would not give the right fundamental theory (see for instance [8, 9]). Only observations can, in this case, show what the fundamental constituents of water are. Trying a quantization strategy on space-time without any observations of fundamental constituents may, from this perspective, seem preposterous.

However, there is one indication why an attempted quantization of a chunk of space should not be meaningless, although it would certainly be highly simplified compared to general space-time configurations. A chunk of space evolving in time can be seen as a local version of general relativity. Locally, the most important and basic space-time structure of this classical theory is given by the form of Lorentz or Poincaré symmetries. The same symmetries are known to play a fundamental role in high-energy physics, where they determine mathematical representations of elementary particles and other properties. The intimate relation to fundamental symmetries makes a chunk of space play a more reliable role for quantum gravity than a water droplet does for molecular physics of liquids. There is no guarantee that this strategy will lead to the right physical result, which ultimately can be tested only by experiments. But as long as the theoretical infrastructure needed to start probing the theory observationally is still being developed, the approach to quantum cosmology sketched here is promising.

2.1 Classical dynamics

The geometrical property of isotropic space, which becomes physical and dynamical in general relativity, is its volume. There are two ways in which a volume parameter can be assigned to a given region $\mathcal{V}$ in space. First, assuming some set of coordinates x^a, $a = 1, 2, 3$, covering the region, we can simply integrate and define the coordinate volume $V_0 = \int_{\mathcal{V}} \mathrm{d}^3x$. However, the result depends not just on the region but also on the choice of coordinates. To make the volume coordinate independent, we use a metric tensor h_{ab} and define $V = \int_{\mathcal{V}} \sqrt{\det h}\, \mathrm{d}^3x$. The result now depends on the choice of metric rather than coordinates. But the metric is considered a physical field in general relativity, and the volume is a measurable quantity by which properties of the metric can be determined.

For a homogeneous, isotropic and flat chunk of space, the metric $h_{ab} = a(t)^2\delta_{ab}$ in Cartesian coordinates has only one free function, the scale factor $a(t)$ depending on time.

General relativity requires that this function obeys the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho \tag{1}$$

with the energy density ρ of matter. (On the left-hand side, there could be additional geometry contributions depending on the scale factor, kc^2/a^2 with $k = \pm 1$ if space has constant positive or negative curvature, and $-\Lambda$ if there is a non-zero cosmological constant Λ.)

In order to derive the Friedmann equation, one extends the spatial metric to a space-time metric $g_{\mu\nu}$ and inserts it in the general Einstein equation. The extension, still respecting isotropy, can be done with a second free function $N(t)$, so that, written as a line element,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -N(t)^2\mathrm{d}t^2 + a(t)^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \tag{2}$$

summing over four values of μ and ν. Unlike $a(t)$, $N(t)$ has no relation whatsoever with observations; it can simply be eliminated by introducing a new (proper) time coordinate τ, integrating $\mathrm{d}\tau = \int N(t)\mathrm{d}t$. In the Friedmann equation (1), we have already assumed that the time derivative is taken with respect to proper time; otherwise the fraction would be $\dot{a}/(Na)$. The whole equation is then independent of the choice of time coordinate. In this way, the time-reparameterization part of general covariance survives even in this highly restricted setting of homogeneous cosmology.

With its energy contribution and a "kinetic" term quadratic in $\dot{a}$, the Friedmann equation looks similar to an energy-balance law. However, there are several differences to the standard form, which turn out to foreshadow the problems to be faced by quantum cosmology. We can see this more clearly if we rewrite the equation as

$$-\frac{3c^2}{8\pi G}a\dot{a}^2 + a^3\rho = 0\,. \tag{3}$$

The matter energy density ρ now appears in the combination $a^3\rho$, which is the total energy $\int_{\mathcal{V}} \rho\sqrt{\det h}\,\mathrm{d}^3x$ contained in a region of coordinate volume $V_0 = 1$. Space-time geometry contributes a term $-(3c^2/8\pi G)a\dot{a}^2$ with the right units of energy, but one that is negative. A direct application of quantum mechanics with such a Hamiltonian would not result in a stable ground state, implying the state problem of quantum cosmology.

Another difference to classical mechanics is the fact that (3) does not define the energy as a free parameter or an observable, but instead imposes a constraint because the sum of space-time and matter contributions must always vanish. In quantum mechanics, with such a Hamiltonian one would not obtain an evolution equation: the $i\hbar\partial/\partial t$-part of the Schrödinger equation would be missing. Classically, this feature implies that one can freely choose the time coordinate in which the equation is written and solved. In quantum cosmology, the missing evolution picture constitutes the problem of time.

The third problem, the one of covariance, is more difficult to see at this stage because it shows its full force only when inhomogeneity is included. Cartesian coordinates x, y and z are special and adapted to the symmetry. (They are convenient but not required: in a generally covariant theory, one may perform any transformation, not just the Euclidean ones that would preserve the form of the metric.) A transformed metric $h_{ab} \neq a(t)^2\delta_{ab}$, with position-dependent components, would not look homogeneous, even though it would enjoy the same translational and rotational symmetries and would predict the same physics. For covariance of any quantization of the theory, one needs to make sure that a quantization of the more complicated inhomogeneous-looking metric still provides the same results. One can expect that quantum corrections must have a very specific form for the function $a(t)$ solving a quantization (or an effective equation with quantum corrections) of the simple isotropic (3) to be part of a proper space-time metric that fulfills the covariance condition. We will come back to these important but somewhat technical issues at a later stage, and for now continue with exactly homogeneous models and their symmetry of time reparameterization invariance.

Reparameterization invariance implies a symmetry with a generator that should generalize the familiar energy functional, for time translation — generated by the energy — is a simple version of time reparameterization. Also this generator is provided by general relativity, most easily when one uses its canonical formulation. (See [10] for an introduction to the latter.) In the cosmological setting, by taking derivatives of the action by a time derivative of the degree of freedom a, we first obtain the momentum $p_a = -(3c^2/4\pi G)V_0 a\dot{a}$ conjugate to the scale factor, and the reparameterization generator

$$C = -\frac{2\pi G}{3c^2}\frac{p_a^2}{V_0 a} + V_0 a^3 \rho \tag{4}$$

(which is (3) times V_0, expressed using p_a instead of $\dot{a}$). The Friedmann equation amounts to the constraint equation $C = 0$. The zero, instead of an observable energy, indicates that reparameterization invariance is not a symmetry that relates different viable solutions but a gauge transformation which eliminates unphysical degrees of freedom, such as any possible effect of the choice of time coordinates. In whatever way one quantizes the theory, this gauge invariance must be respected: There must be an operator $\hat{C}$ with C as its classical limit, so that physical states ψ obey $\hat{C}\psi = 0$.[3] The classical invariance is then not destroyed, but there may still be interesting quantum corrections to the explicit expression of the constraint and the transformations it generates.

2.2 Canonical quantization

The canonical form (4) makes the first step of quantization clear: we introduce a wave function $\psi(a)$ and replace p_a with a derivative operator $\hat{p}_a = -i\hbar\partial/\partial a$. Depending on how matter is realized, by some field ϕ or a phenomenological perfect fluid, ρ might turn into an

[3] This prescription is the so-called Dirac approach to quantizing constraints. Alternatively, one may solve $C = 0$ completely, factor out the classical gauge flow, and then quantize the resulting reduced phase space. Owing to non-trivial phase-space structures, it is often more complicated to perform this procedure. Its results are not necessarily equivalent to those of the Dirac approach because the quantum gauge flow implemented in the latter procedure may differ from the classical gauge flow. Here, we focus on the Dirac approach in order to quantize the full gauge system.

operator as well. In this way, we derive from the Friedmann equation the Wheeler–DeWitt equation [5]

$$\frac{2\pi G\hbar^2}{3V_0}\frac{1}{a}\frac{\partial^2\psi}{\partial a^2}+V_0a^3\hat{\rho}\psi=0\,. \tag{5}$$

Unlike standard quantum-mechanical Hamiltonians, the Friedmann constraint does not lead to a unique ordering of the non-commuting operators $\hat{a}$ and $\hat{p}_a$. Another problem arises, the factor-ordering problem of quantum cosmology. (Presumably, one may want to choose a symmetric ordering, but this choice would be neither obvious nor unique.) More generally, the Wheeler-DeWitt equation or its extensions in different approaches usually suffer from a large dose of quantization ambiguity.

One does not expect the Wheeler–DeWitt equation to be the final form of a state equation in quantum cosmology, even when all its possible quantization ambiguities are taken into account. In addition to quantum corrections that a state obeying (5) would imply compared to classical variables subject to (4), there may be modifications because the classical picture of continuum space-time, with some line element such as (2), no longer applies when large energy density or curvature is reached. Some popular ones of these modifications are described below and in Sec. 4. Nevertheless, at reasonably small curvature, which in cosmology is realized for most, if not all, of the space-time of the accessible universe, the Wheeler–DeWitt equation should be a useful first approximation. It can shed light on general conceptual issues of quantum cosmology, show how the semiclassical limit of the state equation can be compatible with the classical dynamics, and indicate how potential quantum corrections could influence the physics of the cosmos. For instance, the equation has been used in several detailed studies of cosmic decoherence [11, 12, 13] and for potential effects on structure formation [14, 15, 16, 17].

2.3 Test structures

In order to probe candidates of quantum gravity, one can introduce their most characteristic ingredients as modifications to the Wheeler–DeWitt equation (5) and explore their consequences. Examples include new types of matter fields or condensates [18], (super)symmetries [19, 20] or extra dimensions [21] motivated by string theory, non-standard space-time structures as they may arise in non-commutative [22, 23] or fractal geometry [24], or discretization effects as studied in loop quantum gravity, causal dynamical triangulations, or related approaches.

Staying in the canonical setting, one may, as in the rather widely-studied example of loop quantum cosmology [25, 26], try to implement discrete geometrical structures by formulating the canonical pair (a, p_a) not as belonging to a continuum of spatial geometries but rather to a variable a that will, upon quantization, acquire a discrete spectrum. The derivative operator $\hat{p}_a$ is then replaced by a finite shift operator that changes a (or some function of it, such as the volume) by a smallest discrete increment. The differential equation (5), accordingly, turns into a difference equation. Since finite differences can always be Taylor expanded when variations of the wave function are small, (5) is obtained as a continuum limit, but discreteness corrections are present at high curvature where p_a (or some function of a and p_a) is large.

For details, one must use additional ingredients suggested more or less by some full theory of quantum gravity, such as loop quantum gravity [27, 28, 29] which motivated loop quantum cosmology. One must know which function of a has an equidistant spectrum after quantization, so that a finite shift operator can act on it. One must know how large or small the smallest (Planckian?) increment should be. And one must know how exactly the finite shift operator has to replace $\hat{p}_a$ in (5). All these questions are far from being answered uniquely, so that a vast set of new quantization and discretization ambiguities arises. But still, most of these discreteness modifications are quite characteristic and can show what new phenomena could be implied in addition to the quantum effects already present in the Wheeler–DeWitt equation.

One problem that all test structures, including discretization effects, must face is the "problem of re-quantization." In most cases, we do not know to what extent these ingredients are truly effective, in the sense that they could be derived in some low-energy or semiclassical limit of a general theory of quantum gravity. Some of the test structures themselves, such as the discreteness of loop quantum cosmology or extra dimensions in string theory, may be effective or coarse-grained descriptions of a more fundamental quantum theory. If they are introduced in the Friedmann equation before or while it is being quantized to a modified Wheeler–DeWitt equation, one would make the mistake of quantizing a theory that already contains quantum corrections. (Indeed, the modifications leading to finite shifts in loop quantum cosmology are often assumed to depend on $\hbar$ via the Planck length.) That such a procedure might give wrong results can already be seen for the well-understood systems of anharmonic oscillators. For a quantum-mechanical particle of mass m in a potential $V(x)$, $\hat{x}$-expectation values in semiclassical states evolve according to "classical" motion in an effective potential

$$V_{\rm eff}(x) = V(x) + \hbar\sqrt{\frac{V''(x)}{m}} + O(\hbar^2) \tag{6}$$

to first order in the $\hbar$-expansion and in an adiabatic approximation [30, 31]. (See Sec. 5.3.) If motion in this potential is re-quantized, one obtains, by inserting $V_{\rm eff}$ for V in (6) and truncating to the same order in $\hbar$, motion in a potential (6) with twice the square-root term. An iteration of the procedure can produce arbitrarily large (but wrong) quantum corrections. For this reason, combined with the problem of quantization ambiguities, one cannot make quantitative predictions in quantum cosmology until its equations have been derived from some full theory of quantum gravity or the re-quantization problem is solved in another way. But again, some effects are characteristic enough to allow interesting qualitative investigations.

2.4 Minisuperspace vs. single-patch

If space is exactly homogeneous, a quantization of some region of coordinate volume V_0 or geometrical volume V_0a^3 can be applied to any region with non-zero volume. In fact, no prediction of physical effects or observables made with such a model can depend on the value of the single parameter V_0. Homogeneity implies that it does not matter where in space the region is located, and by coordinate invariance the magnitude of V_0 cannot play a role: The value could be changed in two ways, by enlarging or shrinking the region, or by rescaling the coordinates used; since general covariance implies that coordinate choices have no effect on predictions, the value of V_0 does not affect the homogeneous dynamics and can appear only in auxiliary constructs. Making sure that it cancels in the final expressions for predictions is an important consistency condition for models of quantum cosmology.

The Wheeler–DeWitt equation (5) does not depend on V_0 or the non-invariant scale factor a separately, but only on the combination V_0a^3. It is therefore rescaling invariant for a fixed homogeneous region $\mathcal{V}$ of volume $V = V_0a^3$. Different choices of homogeneous regions, one with coordinate volume V_0 and one with λV_0, lead to different wave functions: $\psi(V)$ compared with $\psi(\lambda V)$. But since there is no absolute scale to compare V with, the λ-factor is not problematic. This conclusion might no longer be realized when one extends the quantization by other quantum-gravity effects. In particular the discreteness corrections of some models bring in a new scale, such as the Planck length. If one implements discreteness by a difference equation of constant Planckian step-size in the volume $V = V_0a^3$, for instance, terms such as $\psi(V + n\ell_{\rm P}^3)$ would not simply scale under changes of V_0.

As in this example, quantum space-time effects may make it difficult to ensure covariance under changing V_0. It may then be tempting to turn this volume into a physical parameter,[4] by requiring it to be the actual size of some distinguished region in the universe, for instance all that is accessible within the Hubble distance. Unfortunately, there are several problems with this view. First, the Hubble region is not an actual spatial region in the universe. It is rather part of the past light cone because it is defined by electromagnetic (or perhaps gravitational-wave) signals used for observations. Secondly, the classical Friedmann equation does not require one to make a distinguished choice for the region $\mathcal{V}$, and so it would seem puzzling if some quantum models would force one to select a classical region. Since these problems are usually caused by bringing in a new scale, for instance by discreteness, the only distinguished region would be one of just this size, for instance a region of the discreteness size — a true quantum effect — or a smallest possible chunk of space that cannot be further subdivided. No such limit exists in the classical continuum, and therefore it is not surprising that no distinguished region appears in the classical equation, while it shows up after bringing in quantum effects.

Traditionally, the Wheeler–DeWitt equation has been formulated as an equation quantizing the behavior of all of space at once. The wave function $\psi(a)$ or $\psi(V)$, accordingly, has been identified as the "wave function of the universe." Within Wheeler–DeWitt quantum cosmology, this view is formally consistent because the Wheeler–DeWitt equation is compatible with the classical rescaling covariance. New quantum effects, however, lead to a more complicated view and require one to change the picture. In discrete theories it is

[4]Alternatively, it has been suggested that V_0 should be taken to infinity after quantization, interpreting it as some kind of infrared regulator. However, since the value of V_0 does not affect classical observables at all, it is not a regulator. Moreover, it is then difficult to reconcile models with compact and non-compact spaces within the same setting.

more meaningful to view the quantized region as a smallest discrete chunk of space, or a spatial atom. Not only problems with any scaling dependence can be solved, but the picture also gets closer to familiar physical systems in which one first quantizes a single atomic building block and then asks how different ones interact. We take this new viewpoint in the present review, and explore it further in the next section.

Before doing so, it is useful to note additional features of this change in viewpoint. If one interprets the Wheeler–DeWitt equation as one quantizing a whole universe, there is a big leap to take when exact homogeneity is relaxed to even just a tiny bit of perturbative inhomogeneity. Many such attempts have been made and are still being made, thanks to the important role of cosmological perturbation theory for structure formation in the early universe. Most of these constructions, however, run into some kind of inconsistency related to the problem of covariance. In the isotropic context, by taking a time derivative one can combine the Friedmann equation with the continuity equation $\dot{\rho} + 3(\dot{a}/a)(\rho + p) = 0$ of matter with pressure p to obtain the second-order Raychaudhuri equation. The latter also follows from the spatial part of Einstein's equation, which is then automatically consistent with the physical requirement of energy conservation. (In general, this is a consequence of the Bianchi identity of the Einstein tensor.) The combination of evolution equations with energy conservation is much more non-trivial when not only time but also space derivatives must be taken into account for an inhomogeneous geometry. While the classical field equations are still consistent thanks to the Bianchi identity, this feature is not guaranteed when the equations are naively modified by some suspected quantum effects. Instead, one must work hard to show internal consistency and covariance of the quantum theory. No internal consistency condition need be imposed for the Wheeler–DeWitt equation of homogeneous models to be meaningful, but any kind of inhomogeneity requires delicate constructions to ensure that the resulting field equations are consistent and covariant.

A varying, inhomogeneous field behaves in a much more complicated way under spatial coordinate transformations than a constant function which would not change at all. Homogeneous models are blind to spatial-covariance requirements and only show invariance under reparameterizations of time. Time is delicate too, as shown by the problem of time which is present even for homogeneous models, but its 1-parameter transformations can be dealt with in a more direct way than covariance under changes of all space-time coordinates. The traditional view on quantum cosmology hides these problems, unlike the atomic patch-picture which always makes it clear that at all levels there is some structure and no exact homogeneity.

3 Patching models

The concept of a nearly homogeneous region of space splitting up into multiple parts has appeared in several different forms. It not only assists in simplifying the complicated nonlinear dynamics of general relativity in a more local setting in which interactions of the geometry on different regions play only a secondary role, it also prepares the ground for an application of notions and methods of condensed-matter physics in classical or quantum cosmology.

Classical versions of this concept usually take the form of fractionalization of an evolving spatial geometry into independent parts, and can be used in different regimes. At low curvature, the separate-universe approach has been useful in the study of structure formation in the early universe. At high curvature, the Belinski–Khalatnikov–Lifshitz scenario has been influential in understanding the approach to a spacelike singularity such as the one at the big bang.

Quantum versions are more recent and are technically related to Bose–Einstein condensation. Although there is no physical condensation of constituents occupying the same state, near-homogeneity of a spatial geometry allows one to think of a collection of patches being in almost the same state. The multi-variable dynamics of inhomogeneity (or an interacting many-patch system) is then modeled by a non-linear wave equation for a single patch, akin to the Gross–Pitaevski equation.

3.1 Fractionalization

As a useful approximation in cosmology, the separate-universe approach [32, 33, 34] decomposes space-time into small patches, depending on the wave length of modes considered. To lowest order in the approximation, each patch can be treated as an independent isotropic model with simple Friedmann dynamics. While this picture resembles features of the view espoused here for quantum cosmology, the scales are different: they are purely classical in the separate-universe approach while they are quantum and microscopic in quantum cosmology. Nevertheless, this approach is amenable to a quantum-gravity formulation, as given for instance in [32].

The Belinski–Khalatnikov–Lifshitz scenario [6], by contrast, works in regimes of high curvature and strong inhomogeneity, but is still able to use nearly-homogeneous dynamics at a very local level. It has been shown in several models that time derivatives in Einstein's equation dominate spatial derivatives when curvature is large [35, 36, 37, 38]. The latter can then be ignored, leaving an evolution equation according to which each point changes its geometry independently of its neighbors. The scenario is asymptotic, so that there is no general estimate on how close to a singularity it becomes a good approximation. Nor is there an estimate on the size of regions that can, for a given average curvature, be considered homogeneous. Also here, since the scenario is classical, the sizes are in general unrelated to the Planck scale. Nevertheless, the scenario supports the idea of patching of homogeneous chunks of space even in regimes far from actual homogeneity.

The original Belinski–Khalatnikov–Lifshitz scenario is purely classical, based on an analysis of Einstein's equation. But it can sometimes be strengthened or made more general by including certain quantum effects. One example is the introduction of supersymmetry and the powerful mathematical tools accompanying it [39]. And also some modifications of space-time dynamics suggested by spatial discreteness help to enhance time derivatives over spatial ones [40].

3.2 Condensation

If a nearly homogeneous collection of n patches is uncorrelated and has all constituents in the same state ψ, the total state is $\Psi = \psi^{\otimes n}$. Approximately, this quantum system with interaction potential $\hat{W}$ can then be described by an effective potential $\langle\Psi|\hat{W}|\Psi\rangle$. For delta-function interactions $W_\alpha(x_1, \ldots, x_n) = \frac{1}{2}\alpha \sum_{i\neq j} \delta(x_i - x_j)$ of pointlike particles, for instance, one obtains

$$\langle\Psi|\hat{W}_\alpha|\Psi\rangle = \frac{1}{2}\alpha(n-1)\sum_{i=1}^{n}\int |\psi(x_i)|^4 \mathrm{d}x_i \,. \tag{7}$$

The effective many-body potential therefore takes the form of the expectation value of a sum of non-interacting but non-linear, wave-function dependent "potentials" $\frac{1}{2}\alpha(n-1)|\psi(x)|^2$. If this potential is formally inserted in a 1-particle Schrödinger equation, one obtains the non-linear wave equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}\alpha(n-1)|\psi(x)|^2\psi(x) \tag{8}$$

for each particle. A rigorous derivation has been given [41, 42].

For patches of a spacelike geometry, the interaction potential, obtained from Einstein's equation, is not pointlike. It is a polynomial in the metric and its spatial derivatives, which latter can be approximated as finite differences of neighboring patch geometries. An effective condensate potential can still be derived, and it remains non-linear. However, owing to the non-pointlike nature of interactions, it is non-local [43]. It can be expressed in terms of quantum fluctuations or higher moments of a state. For instance, for quadratic interaction potentials of patch volumes V_i, $\langle\Psi|\hat{W}|\Psi\rangle$ is a sum of terms $\int \mathrm{d}V_1\mathrm{d}V_2|\psi(V_1)|^2|\psi(V_2)|^2(V_1-V_2)^2 = \int \mathrm{d}V|\psi(V)|^2\int \mathrm{d}v|\psi(V+v)|^2v^2$ and can for sharply peaked states be approximated as the volume fluctuation $(\Delta V)^2$. The corresponding Gross–Pitaevski-type equation is then non-local (and non-linear) because it has a potential $(\Delta V)^2$ that depends on the values of the wave fuction everywhere on configuration space. Related equations have also been obtained with more input from a full proposal for quantum gravity [44, 45, 46]. Such non-local equations are more difficult to analyze, but there are still some simplifications compared to any general many-patch equation as it may follow from a theory of quantum gravity.

As a general paradigm, we can see quantum cosmology as an approximate condensate of quantum gravity. A detailed realization of this picture can give access to the homogeneous background dynamics as well as inhomogeneous perturbations for structure formation. Several deep problems must, however, be solved. We must know what states Ψ and what form of operators $\hat{W}$ are meaningful, and what consistency conditions they must obey. As we will describe in the next section, different proposals can be extracted from existing candidates for quantum gravity, but there is no firm or even unique answer.

4 Quantum gravity: Fundamental theories for many-patch systems

One of the difficulties associated with quantum gravity is the fact that the candidates suggested for fundamental mathematical structures are rather far removed from the familiar classical space-time picture or even from simple constituent models such as patch systems. It is therefore hard to compute physical phenomena or predictions from these "first" principles. All the physical intuition, necessary to tell which approaches are promising and to develop them further, is extracted after some extrapolations and heuristic steps. Although the resulting models share characteristic features with the fundamental formulations, a precise relationship is often lacking. In a few cases, there are simple and solvable toy models which exhibit the expected effects. But it remains unclear what happens under more general (and more realistic) circumstances that do not provide complete mathematical control. The situation is similar to complicated condensed-matter effects which cannot yet be derived from an interacting many-body Hamiltonian. In quantum gravity one uses a similar strategy, trying to find good effective theories that may not be firmly derived but still capture interesting phenomena. The crucial difference to condensed-matter physics is that observations do not help much to pin-point the right effective equations.

In this section we briefly review the main features of the most popular candidates for quantum gravity, followed in the next section with a discussion of possible consequences at an effective level. The general description must necessarily be incomplete, but hopefully gives an idea of the diversity of different proposals.

4.1 String theory

String theory [47] begins with the dynamics of a test string moving and vibrating in classical space-time. After quantization, one obtains an interacting tower of oscillator modes that may produce properties of known elementary particles (and more). The modes include excitations suitable for a description of quantized gravitational waves, and therefore the theory promises to capture not just the quantum physics of matter but also of space-time. In order to understand characteristic properties of quantum space-time in this setting, it is important to control the rich and complicated structure of non-perturbative phenomena. If this can be achieved, full quantum space-time rather than just small excitations about it can be analyzed.

Much is known about non-perturbative effects in string theory. However, in order to derive them, one makes ample use of dualities among the different regimes and descriptions of the theory, most prominently of the AdS/CFT correspondence. At this stage, it becomes more difficult to derive a direct space-time picture of fundamental effects, although models that resemble for instance some types of topological defects [21] or different kinds of condensates of fundamental excitations [18, 48, 49] are available.

4.2 Non-commutative and fractal geometry

Much of the shape geometry of a 2-dimensional surface can be extracted from its vibration spectrum. By analogy, one may try to describe space-time geometry by the spectrum of suitable differential operators that describe waves in space-time. A convenient choice is the Dirac operator $D = \not{\partial} = i\hbar^{-1}\gamma^\mu\partial_\mu$ that determines how fermions or spinors, on whose components the matrices γ^μ act, propagate. An analog system can be seen in a graphene layer, on which phonons propagate by comparable laws.

Rather surprisingly, the abstract Dirac operator is related to gravity because the "spectral action," defined as the sum of all its eigenvalues up to some (Planckian) maximum, turns out to produce the Einstein–Hilbert action in the low-curvature limit [50, 51]. One may therefore hope that higher-order corrections in the curvature expansion, or perhaps the full non-perturbative spectral action, can bring one closer to a quantum theory of gravity.

An interesting feature is that the Dirac operators can be generalized to non-classical manifolds, in particular to those on which functions do not commute under multiplication [22]. An indirect description of a possible version of quantum, or non-commutative, space-time is then obtained. However, an outstanding problem in non-commutative geometry is a proper formulation for space-time, as opposed to 4-dimensional timeless (Euclidean) spaces as commonly used. The problem of time therefore does not play a role, but only because in this setting time remains poorly understood at an even more basic level.

Fractal or fractional [24] spaces have been proposed as an alternative deviation from the classical continuum. These effects change the cosmological dynamics, in which context the proposal has mainly been explored [52].

4.3 Wheeler–DeWitt quantum gravity

Like string theory, canonical quantum gravity starts with a controlled space-time picture. The central idea is to apply standard quantization methods to quantities that describe the geometry and curvature of space-time. As already seen in the Wheeler–DeWitt quantization of cosmological models, a possible starting point is the metric h_{ab} that determines the geometry of space. Now, however, one allows for arbitrary position-dependent symmetric matrices h_{ab}, rather than just isotropic ones that are determined by a single time-dependent function $a(t)$. The role of the momenta p^{ab} of these variables is then played by a linear combination of matrix elements $\dot{h}_{ab}$, geometrically identified as the extrinsic curvature of spatial hypersurfaces in space-time [53, 54]. (The necessity of taking a time derivative again suggests subtleties related to covariance since the choice of coordinates should not matter for physics but would affect the form of $\dot{h}_{ab}$. We come back to this issue in the next section.)

Having fields and their momenta, one can attempt a canonical quantization by postulating wave functionals $\psi[h_{ab}]$ on the space of all metrics, on which p^{ab} would become some functional derivative [5]. One would have to make mathematical sense of these operations, using methods from quantum field theory. With a well-defined Hilbert-space setting of

Wheeler–DeWitt quantum gravity one would be close to deriving an effective picture of quantum space: An effective metric could be obtained from expectation values of the metric operator in suitable semiclassical states. Unfortunately, however, it has proven difficult to define, in analogy with quantum mechanics, an inner product by some kind of integration over the unwieldy and infinite-dimensional space of all metrics.

Quantum field theory routinely deals with infinite-dimensional configuration spaces, but its standard techniques are not applicable in this context: if one tries to construct well-defined operators, one is led to integrations of the basic fields, for instance when one turns classical modes into ladder operators. Such integrations or the definition of modes require a metric for them to be well-defined, but the metric is what we are trying to quantize here. There are two obvious ways out, none of which is promising: One could use an auxiliary metric for the integrations, but its choice will then affect properties of the physical metric operators. Or one could use the same physical metric h_{ab} to define integrations, but then the resulting operators will obey a complicated non-linear algebra which makes it difficult to analyze excited states. Wheeler–DeWitt quantum gravity has therefore stayed at a formal level, but it has led to some results about possible semiclassical behaviors.

4.4 Loop quantum gravity

Loop quantum gravity [27, 28, 29] suggests a solution to the Hilbert-space problem of Wheeler–DeWitt quantum gravity by using different fields to describe the geometry. Instead of the metric one uses a densitized triad E^a_i (so that $E^a_i E^{bi} = h^{ab} \det(h_{cd})$, summed over the repeated index i), a triple of orthonormal vector fields at each point in space. Its momentum can be defined in the form of a connection A^i_a [55, 56], or a non-Abelian version of the familar electromagnetic vector potential. Like the vector potential, A^i_a can be integrated over curves in space (the eponymous loops) without requiring an auxiliary metric, and similarly the fields E^a_i can be integrated over surfaces in space to obtain an analog of electric flux. These integrated objects can be quantized in a well-defined and manageable way [57], providing powerful methods to construct the quantum state space [58].

In quantum field theory, one follows the harmonic-oscillator model of constructing state spaces by ladder operators. An operator $\hat{a}^\dagger$ that commutes with the energy operator $\hat{E}$ via $[\hat{E}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger$ raises the energy level when it acts on states, by the familiar argument: $\hat{E}(\hat{a}^\dagger|n\rangle) = (\hat{a}^\dagger\hat{E}+[\hat{E}, \hat{a}^\dagger])|n\rangle = (E_n+\hbar\omega)\hat{a}^\dagger|n\rangle$. The observer-dependent notion of energy is not fundamental in quantum gravity, and one rather refers to geometrical quantities such as areas and volumes. The values of the triad E^a_i are closely related to areas. Ladder operators should therefore be constructed from expressions h that enjoy commutator relationships of the form $[\hat{E}^a_i, \hat{h}] \propto \hat{h}$. Since A^i_a is canonically conjugate to E^a_i, a suitable expression for h takes the form $h_\ell(A) \sim \exp(i\ell A)$, or, being more careful with indices, equals the holonomy

$$h_\ell(A) = \mathcal{P}\exp(i\int_\ell t^a A^j_a \tau_j \mathrm{d}\lambda) \tag{9}$$

where $2i\tau_j$ are Pauli matrices, ℓ is now a curve in space with tangent vector t^a, and $\mathcal{P}$ denotes a suitable ordering of non-commuting SU(2)-elements along the curve.

Using these ladder operators, one is directly led to a discrete picture of space, in which the loops of the integrated A_a^i create geometrical excitations only along 1-dimensional curves or within a graph. Correspondingly, one can show that the integrated E_i^a are quantized to operators with discrete spectra [59, 60, 61]. Since E_i^a is the substitute for h_{ab} as a measure of spatial geometry, space acquires a discrete structure. It is possible to turn the full Hamiltonian constraint into a well-defined operator [62, 63], an important result which has remained elusive in the more formal Wheeler–DeWitt approach.

The theory is non-perturbative from the outset, and yet, unlike in string theory, there is an appealing picture of quantum space. Unfortunately, however, the transition to quantum space-*time* is much less tidy. Not only is the dynamics of these discrete spaces forbiddingly complicated, resembling an interacting spin system on an irregular and changing lattice. Also the problem of time plays a key but still poorly understood role, for one has to ensure that the choice of time coordinate in the definition of momenta does not affect the physics. The underlying classical symmetry can easily be broken by quantum anomalies, which would render meaningless any given proposal for the dynamics which turns out to be anomalous.

4.5 Discrete covariant approaches

There are different versions of discrete theories which, unlike loop quantum gravity, introduce their structures directly for space-time. The most advanced among them is the theory of causal dynamical triangulations [64], in which one introduces discretized space-time as a fundamental theory and looks for a second-order phase transition whose long-range correlations would give rise to a corresponding continuum theory. An influential result in this framework has been the demonstration that its quantum space-time structure differs from the classical one on small distances, expressed by a reduction of the dimension to values smaller than four [65].

As a different approach, spin-foam models [66, 67, 68, 69] have been suggested as an "evolving-graph" (or path-integral) version of loop quantum gravity. The 1-dimensional structures that describe quantum space should then evolve by moving about and branching out when new curves are created. The discrete space-time picture is therefore 2-dimensional or foam-like, with sheets swept out by moving loops connected along lines of branching events. Although there is no obvious place where a time coordinate is chosen, in contrast to canonical theories, the dynamics of such models and the question whether space-time symmetries are properly realized remain poorly understood. The intuitive space-time picture provided by these models is hard to control, with most calculations so far restricted to toy systems with a small number of links.

As a possible way to manage spin foams at a more general level, it has been suggested to use field theories on a group manifold whose Feynman diagrams in a perturbative treatment are identical to spin foams. These group-field theories [70] have evolved into an independent approach, viewed as a possible second quantization of quantum gravity.

4.6 Asymptotic safety

One could forego complicated fundamental structures if one could show that the non-renormalizable perturbative quantization of gravity becomes meaningful when expanded around a non-trivial vacuum. The classically suggested vacuum may turn out to be unsuitable for an expansion of the theory, for instance when the classical coupling constants are unstable under the renormalization flow. This flow may nevertheless have fixed points for which some or all coupling constants are large, so that an expansion around these values would be better behaved.

In order to test this proposal of asymptotic safety [71], the renormalization-group flow of classical general relativity has been studied in some detail, providing promising indications that a non-Gaussian fixed point may indeed exist. This problem is complicated because the flow away from small couplings progressively magnifies the importance of higher-curvature terms in an effective action for gravity. Coefficients of the latter play the role of new coupling constants, leading to a large parameter space to be probed. Current problems of this approach are related to the fact that one must considerably truncate the large theory space of all curvature-dependent actions in order to produce manageable models, as well as the fact that different formulations and choices of fields do not always seem to produce the same results.

5 Effective theories

Actual derivations of physical effects remain challenging in all approaches to quantum gravity, at conceptual and computational levels. But it is usually possible to parameterize characteristic effects, such as the topological defects of string theory or the discreteness of loop quantum gravity, and evaluate them in simple models in order to put those theories to the test. Such results can hardly be justified as predictions of the underlying theories, but they may still serve as a means to falsify some radical proposals and to reduce the number of choices and ambiguities.

5.1 Branes and other worlds

There are several different classes of effective models of string effects, which are usually obtained by including topological structures or additional fields in extended versions of Einstein's equation for the space-time metric (see for instance [72, 21, 73]). These modifications are suggested either by excitations of a string or from non-perturbative effects. A full derivation as an effective theory descending from the fundamental one is still lacking. Such models are rarely formulated as systems of quantum cosmology but rather as models of the classical type, with a continuous space-time structure subject to laws with quantum corrections. Therefore, the re-quantization problem is avoided, and the problems of time, covariance and states do not play a role.

5.2 Non-commutative geometry

A simple model of non-commutative geometry is obtained if one replaces the commutative algebra of functions on space-time by non-commuting 2×2-matrices. Schematically, one can then write the new Dirac operator as $\hat{D} = \begin{pmatrix} \not{\partial} & S \\ S & \not{\partial} \end{pmatrix}$ with another operator S. (The single graphene layer of an analog model for the Dirac operator then becomes a bi-layer of interacting sheets of graphene.) Since S may be position-dependent, it provides new fields of potential interest for particle or cosmological phenomenology [74]. Another effective approach is to extend the curvature expansion of the spectral action to higher orders, so that possible physical consequences can be studied at the level of higher-curvature corrections [75].

5.3 Effective canonical dynamics

The most systematic approach to effective theories exists in canonical quantum gravity. The formalism starts with the familiar Ehrenfest theorem of quantum mechanics, but includes several new features to take into account the requirements of space-time theories, such as general covariance or allowing for the freedom to change the time coordinate. Canonical methods are best suited for these extensions of familiar effective descriptions.

By Ehrenfest's theorem, expectation values of position q and momentum p of a single particle of mass m in a potential $V(q)$ change in time by

$$\frac{\mathrm{d}\langle\hat{q}\rangle}{\mathrm{d}t} = \frac{\langle[\hat{q},\hat{H}]\rangle}{i\hbar} = \frac{\langle\hat{p}\rangle}{m} \quad , \quad \frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} = -\langle V'(\hat{q})\rangle \,. \tag{10}$$

These equations closely resemble the classical ones, except that they couple to each other by the expectation value $-\langle V'(\hat{q})\rangle$ of the force, rather than the force function $-V'(\langle\hat{q}\rangle)$ evaluated in the q-expectation value. The difference of these two expressions gives rise to quantum corrections.

5.3.1 Moment dynamics.

In a more systematic treatment, one expands the quantum force $-\langle V'(\hat{q})\rangle = -V'(\langle\hat{q}\rangle) - \frac{1}{2}V'''(\langle\hat{q}\rangle)\Delta(q^2) + \cdots$ in moments $\Delta(q^a) = \langle(\hat{q} - \langle\hat{q}\rangle)^a\rangle$ of the state considered. These moments generally change in time as the state spreads, so that the system (10) must be extended by equations of motion for $\Delta(q^a)$. Given the Hamiltonian operator $\hat{H} = \hat{p}^2/2m + V(\hat{q})$, one can compute these equations in the same way in which one proves Ehrenfest's theorem: Starting with the first non-trivial choice $a = 2$ (for which $\Delta(q^2) = (\Delta q)^2$ is the q-fluctuation squared), one writes

$$\frac{\mathrm{d}\Delta(q^2)}{\mathrm{d}t} = \frac{\mathrm{d}(\langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2)}{\mathrm{d}t} = \frac{\langle[\hat{q}^2,\hat{H}]\rangle}{i\hbar} - 2\langle\hat{q}\rangle\frac{\mathrm{d}\langle\hat{q}\rangle}{\mathrm{d}t} = 2\frac{\Delta(qp)}{m}\,, \tag{11}$$

introducing the covariance $\Delta(qp)$ as an example of the more-general moments

$$\Delta(q^a p^b) = \langle(\hat{q} - \langle\hat{q}\rangle)^a(\hat{p} - \langle\hat{p}\rangle)^b\rangle_{\mathrm{Weyl}} \tag{12}$$

with totally symmetric (or Weyl) ordering.

Proceeding in this way, the whole set of infinitely many moments $\Delta(q^a p^b)$ is in general coupled to the expectation values. One can approximate this version of quantum evolution by truncating the set of moments so that only those with $a+b\leq n$ for some fixed n are considered. A finite set of coupled equations is then obtained. The truncation is suitable for a semiclassical approximation because semiclassical states obey $\Delta(q^a p^b)\sim O(\hbar^{(a+b)/2})$, but some other regimes are accessible as well. For instance, near-ground states of perturbed harmonic oscillators (or perturbed free theories) can be accessible because the harmonic-oscillator ground state is Gaussian (but not considered semiclassical). Another example is given by properties of generalized uncertainty principles derived for phase spaces with non-trivial topology [76].

The coupled dynamics of all moments can be solved partially if one assumes that moments change slowly. An adiabatic approximation can then be combined with the semiclassical one [31], with results equal to those derived from the low-energy effective action known from particle physics [30]. At lowest order, an expansion around the harmonic oscillator with frequency ω produces the equations $\mathrm{d}\langle\hat{q}\rangle/\mathrm{d}t=\langle\hat{p}\rangle/m$ and

$$\frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t}=-m\omega^2\langle\hat{q}\rangle-U'(\langle\hat{q}\rangle)-\frac{1}{2}U'''(\langle\hat{q}\rangle)\Delta(q^2)+\cdots \tag{13}$$

for expectation values, we well as (11) together with

$$\frac{\mathrm{d}\Delta(qp)}{\mathrm{d}t} = \frac{1}{m}\Delta(p^2)-m\omega^2\left(1+\frac{U''(\langle\hat{q}\rangle)}{m\omega^2}\right)\Delta(q^2)+\cdots \tag{14}$$

$$\frac{\mathrm{d}\Delta(p^2)}{\mathrm{d}t} = -2m\omega^2\left(1+\frac{U''(\langle\hat{q}\rangle)}{m\omega^2}\right)\Delta(qp)+\cdots \tag{15}$$

with anharmonicity $U(q)=V(q)-\frac{1}{2}m\omega^2q^2$. To leading adiabatic order (indicated by a subscript zero), (11), (14) and (15) are zero and one finds $\Delta_0(qp)=0$ and $\Delta_0(p^2)=m^2\omega^2(1+U''(\langle\hat{q}\rangle)/m\omega^2)\Delta_0(q^2)$. The first-order equations (such as $\mathrm{d}\Delta_0(q^2)/\mathrm{d}t=2\Delta_1(qp)/m$) then produce a consistency condition solved by $\Delta_0(q^2)=C_2/\sqrt{1+U''(\langle\hat{q}\rangle)/m\omega^2}$ with a constant C_2. If the harmonic ground state is to be obtained for $U(q)=0$, $C_2=\frac{1}{2}\hbar/m\omega$. Equation (13), formulated for $V(q)$, then shows that motion happens in the effective potential (6). To higher orders in the adiabatic approximation, higher-derivative terms are added to the classical equations [77].

5.3.2 Deparameterization.

An application of these methods to quantum cosmology must take into account the fact that the Friedmann equation is a constraint rather than an evolution generator. A popular but problematic method often used in this context, called deparameterization, consists in coupling a special choice of matter field to space-time, so that its homogeneous value can formally play the role of time. Its momentum is then the analog of the usual Hamiltonian, and standard methods can be used to analyze quantum or effective evolution. Although the method of deparameterization and the popular models obtained from it are problematic, as we will see in more detail, we describe an example in order to demonstrate some current technical developments in quantum cosmology but also their pitfalls. For different versions, see [78, 79, 80, 81].

The most common example of this kind is a free, massless scalar field ϕ with momentum p_ϕ, whose energy density $\rho = p_\phi^2/2a^6$ contributes to the Friedmann equation (4). Solving $C = 0$ for the momentum p_ϕ, we have

$$p_\phi(a, p_a) = \sqrt{\frac{4\pi G}{3c^2}}\,|ap_a|\,. \tag{16}$$

(We set $V_0 = 1$ from now on.) With this momentum interpreted as a Hamiltonian for changes with respect to ϕ (assuming positive ap_a), we obtain Hamiltonian equations of motion $\mathrm{d}a/\mathrm{d}\phi = \partial p_\phi/\partial p_a = \sqrt{4\pi G/3c^2}\,a$ and $\mathrm{d}p_a/\mathrm{d}\phi = -\partial p_\phi/\partial a = -\sqrt{4\pi G/3c^2}\,p_a$. Both variables therefore change exponentially with ϕ.

Transforming to proper time by multiplying with $\mathrm{d}\phi/\mathrm{d}\tau = \partial C/\partial p_\phi = p_\phi/a^3$ with constant p_ϕ (because $\mathrm{d}p_\phi/\mathrm{d}\tau = -\partial C/\partial\phi = 0$), one can see that the usual equations of motion follow: $a^{-1}\mathrm{d}a/\mathrm{d}\tau = a^{-1}(\mathrm{d}a/\mathrm{d}\phi)(\mathrm{d}\phi/\mathrm{d}\tau) = \sqrt{4\pi G/3c^2}\,p_\phi/a^3$ is equivalent to the Friedmann equation, and from $\mathrm{d}p_a/\mathrm{d}\phi$ one finds the Raychaudhuri equation. Deparameterization therefore allows one to reformulate constrained dynamics as a standard evolution picture, at the expense that one must provide a specific form of matter field.

5.3.3 Solvable model.

The deparameterized model has a simple ϕ-Hamiltonian (16) which shares with the harmonic oscillator the fact that it is quadratic. The Hamiltonian is not of the standard form with a kinetic term and a potential, but one may still use the commutator version of Ehrenfest's equations (10). As with the harmonic oscillator, one then finds that expectation values of a and p_a obey exactly the classical equations after quantization. There are no quantum corrections or effective potentials. The model is very special, owing to the symmetry, the specific matter content, and other properties such as the absence of spatial curvature or a cosmological constant. Nevertheless, it is useful for studying the potential of quantum space-time effects that might be implied by quantum gravity in addition to the usual quantum back-reaction of moments.

Loop quantum cosmology, for instance, suggests that the quadratic curvature dependence p_a^2 of the Friedmann equation is replaced by a periodic function, as motivated by finite shifts in quantum geometry or matrix elements of holonomies (9). After such a replacement of p_a by $\delta^{-1}\sin(\delta p_a)$ with some constant or perhaps a-dependent δ, the ϕ-Hamiltonian

$$p_\phi(a, p_a) = \sqrt{\frac{4\pi G}{3c^2}}\delta^{-1}|a\sin(\delta p_a)| \tag{17}$$

is no longer quadratic. Nevertheless, there is a hidden form of solvability [82] which one can realize after changing to variables a and $J := \delta^{-1}a\exp(i\delta p_a)$. (In this transformation, δ is not required to be constant but may depend on a by a power law [83]. This generalization is of interest in the discussion of scaling behaviors and the magnitude of quantum corrections [84].)

After quantization, these new variables obey the linear algebra $[\hat{a}, \hat{J}] = -\hbar\hat{J}$, $[\hat{a}, \hat{J}^\dagger] = \hbar\hat{J}^\dagger$ and $[\hat{J}, \hat{J}^\dagger] = 2\hbar\hat{a}$. Moreover, the ϕ-Hamiltonian is linear in $\hat{J}$, being proportional to $i(\hat{J} - \hat{J}^\dagger)$. With these features, Ehrenfest's equations for expectation values of $\hat{a}$ and $\hat{J}$ still close, and no quantum corrections other than the modification appear. One can easily solve these equations and find that the modified $a(\phi)$ does not reach zero (it behaves cosh-like), avoiding the big-bang singularity: For non-zero a, the energy density never diverges. (This solution is an example for bounce cosmology. For other versions, see [85, 15, 86].)

These formulations are of mathematical interest and they show some quantum effects, but they do not suffice to provide a reliable picture of the Planck regime or the big bang. First, they suffer from the re-quantization problem: One implements a modification of the Friedmann equation in the form (17) as motivated by properties of loop quantum gravity and its geometry. The parameter δ, referring to the step-size of discrete geometry, then likely depends on $\hbar$, and is indeed often assumed to be related to the Planck length. But then one quantizes this system which already contains some quantum corrections, and the cautionary remark of Sec. 2.3 applies.[5]

A modification such as (17) may still provide reliable qualitative information about quantum corrections even if the re-quantization problem is unsolved. However, the lack of quantitative control means that it is impossible to tell whether the modification in (17) is the dominant quantum correction under all circumstances. The main other source of corrections is, in general, quantum back-reaction to which we turn now.

There are no further quantum back-reaction terms that would couple moments to expectation values in this particular model, but it is just as special as the harmonic oscillator. With any additions to make the model more realistic, quantum back-reaction results and corrections added to the modifications appear, analogously to (13). Owing to their rather involved derivations, however, they are only incompletely known. The second problem with such models can then be seen: Based on general principles of effective theory [87, 88], one does indeed expect quantum corrections of the form of higher time derivatives, which in gravitational theories are usually expressed as higher-curvature effective actions. These corrections, on dimensional grounds, are expected to be of a tiny size given by the average density divided by the Planck density. Corrections implied by an expanded (17) for small δp_a can be estimated to be of about the same order of magnitude if $c\delta a \sim \ell_{\rm P}$, so that one cannot consistently use the whole series of the expanded $\sin(\delta p_a)$ while ignoring quantum back-reaction terms. For these reasons, the high-density regime of modified equations in

[5] Loop quantum gravity has a well-defined regularization scheme which allows one to express the dynamics in terms of holonomies. The modification of loop quantum cosmology, as in (17), however, is not a proper regularization because the limit $\delta \to 0$ cannot be taken after quantization. One can think of this crucial difference between the two frameworks as a consequence of some kind of averaging required to implement homogeneity. Since no complete derivation of cosmological models from the full theory is known one cannot pinpoint exactly where the first quantization step happens which leads to an $\hbar$-dependent δ.

loop quantum cosmology remains unclear.[6] This is an example for the incompleteness problem of quantum cosmology: Without knowing all terms of a general effective description of quantum gravity, some of which are often difficult to derive, one cannot make definitive predictions. Although such a statement may seem close to a tautology, it is worth remembering in this field.

5.4 The role of symmetries and space-time structure

Space-time enjoys an infinitely large number of important symmetries. One of the key problems of quantum gravity is to find consistent quantum realizations of these transformations, much larger than the Euclidean group of rotations and translations or even the Poincaré group that also contains Lorentz boosts and time translations.

Poincaré transformations play an important role in particle-physics models of quantum-field theory. They are generated by elements P_μ (space translations for $\mu = 1, 2, 3$ and time translations for $\mu = 0$) and $M_{\mu\nu}$ (rotations and boosts) with algebra relations

$$[P_\mu, P_\nu] = 0 \quad , \quad [M_{\mu\nu}, P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu \tag{18}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\,. \tag{19}$$

These relations depend on the metric $\eta_{\mu\nu}$ of Minkowski space-time. Since these matrix elements are just constants in Cartesian coordinates, (18) and (19) define a Lie algebra.

5.4.1 Deformations.

The covariance symmetries of general space-times are an extension from Poincaré transformations to arbitrary non-linear coordinate changes. Even though space-time is one 4-dimensional entity, it is useful to split the transformations in spatial and temporal changes, just as the Poincaré algebra contains space and time translations as well as spatial rotations and space-time boosts. One can then represent them geometrically as deformations of spatial hypersurfaces within space-time, some tangential to hypersurfaces and the rest normal.

For every vector field $\vec{w}$ on a spatial hypersurface there is a spatial deformation $S(\vec{w})$, and for every function N on a spatial hypersurface a normal (or timelike) deformation

[6] Starting with [84], there have been several activities in minisuperspace-based loop quantum cosmology, aiming for a detailed understanding of wave functions subject to a dynamics quantizing (17). Even formulating such a dynamical law is non-trivial because of certain conditions required for a so-called physical Hilbert space to be used. Moreover, some generalizations of (17), especially those with a cosmological constant, do not automatically lead to self-adjoint quantum Hamiltonians [89]. In spite of much progress in these investigations, including also the numerical front [90, 91], the high-density behavior is uncertain. Most details, analytic or numerical, are available in solvable models quantizing (17), and in fact most intuition in this field is based on the dynamics of kinetic-dominated models with vanishing or small quantum back-reaction. Numerical investigations sometimes attempt to go beyond kinetic domination, but they require specific choices of initial states and therefore lead straight to the problem of states to be discussed in more detail in Section 5.6. Nevertheless, these different methods may eventually help to shed light on the generic behavior in the Planck regime of loop quantum cosmology.

$T(N)$ which at each point x shifts the hypersurface by a distance $N(x)\vec{n}(x)$ with the unit normal vector $\vec{n}(x)$ at x. By composing different such deformations, one obtains algebraic relations that generalize the Poincaré algebra:

$$
\begin{aligned}
[S(\vec{w}_1), S(\vec{w}_2)] &= S((\vec{w}_1 \cdot \vec{\nabla})\vec{w}_2 - (\vec{w}_2 \cdot \vec{\nabla})\vec{w}_1) && (20)\\
[T(N), S(\vec{w})] &= -T(\vec{w} \cdot \vec{\nabla} N) && (21)\\
[T(N_1), T(N_2)] &= S(N_1 \vec{\nabla} N_2 - N_2 \vec{\nabla} N_1)\,, && (22)
\end{aligned}
$$

called the hypersurface-deformation algebra [92]. One can indeed check that linear N and $\vec{w}$ reproduce the Poincaré algebra.

The large number of hypersurface-deformation symmetries is a powerful input in fundamental formulations of space-time theories. Any theory which is formulated for the space-time metric, is invariant under these transformations, and has the usual classical structure of second-order partial differential equations must equal general relativity [93, 94]. Quantum effects could be realized only in higher-order derivatives, by extra ingredients such as new fields, or by a modification of the symmetries themselves. Studying hypersurface deformations and their fate in any quantum theory of gravity therefore provides access not just to the elementary space-time structure but also to its dynamics.

In contrast to the Poincaré algebra, the hypersurface-deformation algebra is infinite-dimensional. Finding well-defined representations that could transfer these important symmetries to a quantum theory of gravity is therefore difficult. Even more importantly, the hypersurface-deformation "algebra" is not an algebra in the strict sense. The gradient in the relation (22) requires one to use a metric on spatial hypersurfaces. Unlike the Minkowski metric that appears in the Poincaré algebra, the spatial metric is not constant under the general circumstances in which the hypersurface-deformation symmetries are relevant. The spatial metric is therefore an external field, independent of the symmetry generators and their coefficients N and $\vec{w}$, which must be included for a well-defined and closed formulation of the symmetries. It turns out that Lie algebroids provide the right generalizations of Lie algebras to take into account this extra ingredient [95]. However, while Lie algebroids have been studied intensively in the mathematical literature of the recent decades, they are harder to classify and to represent. Standard representation theory therefore does not provide a simple road to quantum gravity.

5.4.2 Representations.

In physics, the first approach is often a brute-force construction rather than elegant representation theory. Applied to the hypersurface-deformation algebroid, this has been the attempt of canonical quantum gravity for several decades, without much success. One of the key problems has been noticed already in the late 1970s [96, 97]: The presence of the metric in the algebroid relations leads to apparent contradictions between different features that are usually taken for granted in quantum physics.

We are looking for a representation of $S(\vec{w})$ and $T(N)$ and their relations by operators $\hat{S}(\vec{w})$ and $\hat{T}(N)$ acting on some states ψ. States that obey the required symmetries and are invariant under hypersurface deformations then solve the equations $\hat{S}(\vec{w})\psi = 0$ and $\hat{T}(N)\psi = 0$ for all $\vec{w}$ and N. We also expect that these symmetry generators are anti-Hermitian. (Their exponentials should be unitary.) The left-hand side of (22) is then anti-Hermitian, and so must be the right-hand side. The spatial generator $\hat{S}(\vec{w})$, by our assumptions, is anti-Hermitian, but in quantum gravity there is another operator for the metric h_{ab} in the gradients. We must order the right-hand side to something of the form $\frac{1}{2}(\hat{S}\hat{h} + \hat{h}\hat{S})$. This ordering does not annihilate an invariant state ψ because the metric factor in $\hat{S}\hat{h}\psi$ is not invariant under spatial deformations, while the commutator on the left-hand side annihilates invariant states. The assumption of anti-Hermition generators is therefore incompatible with a representation of the algebroid and the existence of invariant states. (In path-integral quantizations or the discrete version of spin-foam models, one does not directly represent the generators as operators. Nonetheless, the representation problem resurfaces as the anomaly problem of the path-integral measure, which remains unresolved [98, 99].)

One may relax the assumptions. There is no observable associated with the generators. (For instance, on general space-times there is no invariant notion of energy, as the usual observable related to time translations.) The requirement of anti-Hermitian generators could therefore be weakened. However, under these generalized circumstances some standard quantum notions no longer apply, and one must carefully re-assess constructions of quantum field theory. And even in this relaxed setting, it remains difficult to find representations. An encouraging development in the last few years, using the loop methods of [62, 63], has been the construction of consistent realizations in models with only two spatial dimensions [100, 101, 102, 103]. An extension to 4-dimensional space-time is more complicated but may be feasible with these new methods.

5.4.3 Signature change.

Effective methods have shown several unexpected implications of quantum effects. In this setting, one does not directly compute commutators of operators that quantize the generators in (21), but instead includes quantum terms in the classical expressions of the generators and obtains their commutator as a Poisson bracket [104]. Semiclassical features of quantum space-time can then be uncovered. Before presenting some of these consequences, it is useful to make two remarks. First, owing to the complexity of the covariance or anomaly problem, no complete picture of space-time structures in canonical quantum gravity is available yet. The following discussion is based on a set of models which appear to give a uniform and rather generic view on possible quantum effects. Nevertheless, not all quantum corrections have been included in these calculations. Secondly, since controlling inhomogeneity is an important pre-requisite for a reliable analysis of potentially observable effects of quantum gravity, several attempts have been made which circumvent an analysis of the complicated anomaly problem. In loop quantum cosmology, for instance, there are several such versions [105, 106] which *assume* that the classical space-time structure (but not the classical dynamics) is still realized in cosmological models of quantum gravity. These constructions therefore amount to quantum-field theory on a (modified) background but do not analyze full quantum space-time effects. The results described in the rest of this subsection are incomplete but nonetheless indicate that classical space-time structures may not give the full picture of the Planck regime in quantum gravity.

In particular, the characteristic features of loop quantum gravity lead to quantum corrections not just in the dynamics of gravity but also in the structure of space-time: The relation (22) is not realized in the classical form but rather as

$$[T(N_1), T(N_2)] = S(\beta(N_1\vec{\nabla}N_2 - N_2\vec{\nabla}N_1))\,, \tag{23}$$

with a function β that depends on the metric or curvature [104, 107, 108, 109, 110]. The functional form of β can be determined from the form of quantum corrections: Like effective potentials or Hamiltonians, the effective generators are defined via expectation values of the corresponding operators. The same kind of corrections in the algebra can be seen by the mentioned operator calculations [100, 101, 102, 103] in models with only two spatial dimensions. (One could expect more general versions of the algebra in which also (20) and (21) are modified, or even modified terms involving the Gauss constraint in models based on triad variables. No concrete examples have been found, and although they may be possible, they do not seem to be likely because the diffeomorphism and Gauss constraint do not receive strong corrections in loop quantum gravity.)

For $\beta \neq 1$, the space-time structure is not classical: The symmetries do not correspond to coordinate changes, and for linear N and $\vec{w}$, which classically produces the Poincaré algebra, modified transformations are obtained [111, 112]. Nevertheless, the theory is well-defined and physical observables can be computed from the generators and the dynamics they produce. (In some cases, a canonical transformation allows one to absorb β in a re-defined spatial metric [113], but the space-time geometry remains non-classical: An effective space-time obtained after the transformation would not be defined for a direct quantization of the metric.)

The most interesting aspect of these non-classical space-time models is the fact that β, with holonomy corrections from loop quantum gravity, turns negative near the Planck density [107, 108]. One can show that this sign change, in the linear limit, amounts to replacing boosts by rotations, so that transformations of 4-dimensional Euclidean space are obtained instead of those for space-time [114, 115]. In other words, at high density the time dimension turns into a spatial one, and equations of motion generated by a system subject to (23) are not deterministic. Indeed, field equations computed in this regime take the form of elliptic differential equations, not hyperbolic ones [108]. The usual initial-value problem is to be replaced by a boundary-value problem in four dimensions. (A possible analog model is a phase transition of nano-wires [116]: There is no electric conductivity in the unordered phase, but electricity — just like time in quantum cosmology — starts flowing after the transition to an ordered phase.) These consequences are an example of the drastic changes to our intuitive notion of classical space-time that can be implied by quantum effects. Any intuitive input necessary to formulate quantum gravity or cosmological models should therefore be taken with a major chunk of salt.

5.5 Problem of time

Time in general relativity is observer-dependent and not absolute. Changing time becomes part of the transformations in the hypersurface-deformation algebroid as the fundamental symmetries of space-time. Accordingly, time translations are not generated by an observable such as the energy, but by an expression which on physical states always takes the value zero. As already seen for the Friedmann equation in isotropic models, in terms of fundamental fields and moments one obtains a constraint instead of an evolution equation. The problem of time [117, 118, 119] stems from the fact that the evolution we naturally experience is hidden in the formalism of a canonical theory and must be uncovered.

Deparameterization (Sec. 5.3.2) is a simple way of re-introducing something that resembles evolution and time. However, the method is applicable only in very special circumstances of specific matter contributions. Moreover, there is a covariance problem in this context because different choices of time variables are possible but difficult to compare. Usually, the resulting quantum theories are so different that one cannot even find a unitary transformation to formulate the question of how much their predictions vary [120, 121, 122]. The most severe problem of deparameterization, however, comes from the possibility of signature change in consistent representations of the hypersurface-deformation algebroid: If one deparameterizes, one merely picks a variable that classically depends monotonically on time, and then views it as time itself. But quantization of space-time can lead to the absence of time, which remains unseen if deparameterization is used.

In milder regimes, one can at least be sure that time exists. But for time to be part of space-time, there must be strict transformations of covariance. In particular, one must be able to change one's choice of time function without affecting physical results. (Different observers must experience the same physics.) If one uses deparameterization, one must therefore show that the choice of time, such as ϕ, does not matter. However, this property has never been fully demonstrated in any cosmological (or other) model. Instead, one can easily foresee obstructions to such a result: In order to formulate deparameterized evolution, one manipulates the original constraint C, usually involving a square root as in (16). These are simple manipulations at the classical level, but different such versions require non-trivial quantization choices. Since the quantum dynamics of cosmological models can be rather complicated, one usually picks a quantization that looks most simple. But a simple choice for one time may not relate to a simple choice in another time. In this way, deparameterized models may break covariance.

More realistic models are not deparameterizable by a global time. If one solves for a momentum such as p_ϕ, it typically is dependent on "time" ϕ, unlike (16). The momentum is no longer preserved under the evolution it generates, just like a time-dependent Hamiltonian. Turning points of ϕ then become possible, where p_ϕ vanishes and the classical ϕ starts running backwards. In a quantum model, ϕ-evolution then cannot be unitary, and is questionable even before the turning point is reached.

Time coordinates are usually local. Similarly, one should expect any ϕ-time to be valid only for a finite range. For longer evolution, one would have to find a way of transforming to a different time that remains valid beyond the next turning point of ϕ. Such transformations are possible semiclassically [123, 124, 125], although it remains to be seen whether general quantum states enjoy the same feature.

As a shadow of non-unitary evolution, independence of physical results from the choice of time requires time (as an expectation value of $\hat{\phi}$) to be complex. One can see this feature, as well as general properties of the effective constraint formalism, by solving some of the moment equations implied by a constraint operator $\hat{C} = \hat{p}_\phi^2 - \hat{p}^2 + V(\hat{\phi})$ for a relativistic particle in an arbitrary ϕ-dependent potential $V(\phi)$ [123]. If we try to use ϕ as time, we have to deal with a time-dependent Hamiltonian.

We compute the effective constraint $C_Q := \langle\hat{C}\rangle$ up to second order in moments:

$$0 = C_Q = \langle\hat{p}_\phi\rangle^2 - \langle\hat{p}\rangle^2 + \Delta(p_\phi^2) - \Delta(p^2) + V(\langle\hat{\phi}\rangle) + \tfrac{1}{2}V''(\langle\hat{\phi}\rangle)\Delta(\phi^2)\,. \tag{24}$$

However, this equation is not enough for a constrained system. The moments are further restriced because $C_{\mathrm{pol}} = \langle\widehat{\mathrm{pol}}\hat{C}\rangle$ vanishes for any polynomial $\widehat{\mathrm{pol}}$ in basic operators when an invariant state with $\hat{C}\psi = 0$ is used. We obtain additional equations, including

$$0 = C_\phi = 2\langle\hat{p}_\phi\rangle\Delta(\phi p_\phi) + i\hbar\langle\hat{p}_\phi\rangle - 2p\Delta(\phi p) + V'(\langle\hat{\phi}\rangle)\Delta(\phi^2) \tag{25}$$

The ordering of $\widehat{\mathrm{pol}}\hat{C}$ can, in general, not be chosen to be symmetric because $\hat{C}$ must stay on the right for the product to annihilate invariant states. Accordingly, we see some imaginary contributions to the moment constraints.

Choosing a time variable ϕ by deparameterization means that it is no longer treated as fluctuating, just like time in quantum mechanics. We can implement this condition via moments by requiring $\Delta(\phi^2) = \Delta(\phi q) = \Delta(\phi p) = 0$ (a valid gauge fixing of the effective constrained system). Then, $\Delta(\phi p_\phi) = -\frac{1}{2}i\hbar$ from (25). Other constraints of the type (25) can be used to solve for $\Delta(p_\phi^2)$ in (24), which latter acquires a non-trivial imaginary part. For this to vanish, $\mathrm{Im}\,\langle\hat{\phi}\rangle = -\frac{1}{2}\hbar/\langle\hat{p}_\phi\rangle \neq 0$ to first order in $\hbar$ [123, 124].

Covariance, in the presence of complex time, would then suggest that also space is complex. While a covariant semiclassical picture of inhomogeneous geometries remains to be worked out, there is a clear indication that complex space-time models must be used. The physics of such systems is, however, clear: In the formalism just described, observables, such as densities in cosmology, are all real.

5.6 Problem of states

Cosmological observations do not tell us much about quantum states. Since details of fundamental and effective descriptions often depend on the class of states considered, they must be sufficiently general so as to be insensitive to ambiguities related to undetermined states. This question plays a role a two levels: for semiclassical as well as ground states.

5.6.1 Semiclassical states.

One's first idea of a semiclassical state is usually a sharply peaked Gaussian. But such a state is very special: There is only one parameter (or at most two if the state has correlations) which determines all infinitely many moments. For a general semiclassical state, by contrast, any moment of order n is a free parameter as long as it is $O(\hbar^{n/2})$. Moreover, a sharply peaked Gaussian is not a good late-time limit even in the case of a free massive particle in quantum mechanics. Assuming a sharply-peaked Gaussian at low curvature is therefore unjustified for several independent reasons.

The example of the free massive particle in quantum mechanics suggests a more refined approach. Massive objects do not become more quantum if one just waits long enough, for reasons which can be explained in a satisfactory way by decoherence [126]: The objects we usually deal with are not completely isolated but are subject to weak interactions with a large number of degrees of freedom, such as the molecules in surrounding air or the photons, neutrinos and gravitational waves of cosmic backgrounds. When these interactions, weak but acting over a long time, are integrated out, a state different from the one of an isolated object is obtained. The resulting state is no longer pure but mixed, and it approaches a near-diagonal density matrix which can be interpreted in terms of classical probabilities. When decoherence is taken into account, states need not be sharply-peaked Gaussians in order to be in agreement with semiclassical behavior. In cosmology, decoherence has been discussed in the context of inflationary structure formation [127, 11, 128, 13] and for quantum-cosmological questions [129].

5.6.2 Ground states.

As seen for C in isotropic models, Eq. (4), the time generator of gravity is unbounded from below. After quantization there is then no ground state. The powerful low-energy method of perturbing around the ground state is not available, and other distinguished states around which moments would behave adiabatically are hard to find [130]. In other words, the derivative expansion of an effective action or Hamiltonian does not exist under these circumstances. One must instead formulate effective systems in which moments are kept as independent degrees of freedom, like the coupled system (13) with (11), (14) and (15). Such systems can be analyzed and provide good information about the quantum theory. But they cannot be brought to the form of an effective action with higher-order time derivatives, or higher-curvature terms in gravity. Some of the expectations of perturbative quantum gravity are therefore not realized in general.

The absence of a ground state has important consequences for symmetries. An effective theory is covariant under certain transformations only if the zeroth-order state used to expand the quantum theory is invariant. (For instance, a Poincaré-covariant effective action is obtained when one expands around the invariant Minkowski vacuum.) It is, however, questionable whether one can find exactly invariant states in a discrete space(-time) setting. Any non-invariant choice would lead to an effective theory that is not covariant under hypersurface deformations, and therefore is not generally covariant.

In order to find an invariant state, one would have to go to high excitation levels of a many-patch theory, so that all possible patches are equally excited. One would have to find some kind of "finite-geometry" state, in analogy with finite-temperature states in quantum-field theory. Instead of the temperature (or constituent velocities), geometrical quantities would have non-zero expectation values of a uniform distribution. Candidates for such states are being considered [131, 132]. The resulting effective theories are likely to require quantum-corrected hypersurface-deformation algebroids, as a reflection of the underlying discreteness.

6 Outlook: potential observations

It is easier to break symmetries than to preserve them. In many examples of condensed-matter physics, structures that violate the Euclidean transformations of translations and rotations imply interesting effects which can be used to test the theory or its effective descriptions, and even to develop practical applications. The symmetries of space-time, however, are more fragile when one considers quantum gravity. They can easily be broken too (and have been broken many times in theoretical constructions). But if they are violated, they do not give rise to physical effects but rather to an inconsistent theory. Derivations of potentially observable effects in quantum gravity must therefore proceed much more carefully.

6.1 The problem of the Planck scale

The Planck scale is extreme and cannot be reached by any experiments in the foreseeable future. Dimensional arguments that only consider the magnitude of this scale in relation to typical observables are often taken to indicate that quantum gravity is unobservable. Moreover, since dimensional arguments work only if there is a small number of relevant parameters with the same dimension, such as the fundamentally defined Planck energy and the contingent average density of the universe, different types of corrections are often estimated to be of the same size. A careful analysis of their interplay is then required to derive reliable consequences.

In the Planck regime, however, any such detailed analysis suffers from the ambiguity problem of quantum gravity. Even if one restricts attention to just one proposal of a general theory, several quantization and other choices still have to be made for a concrete model of the early universe. Most of the time, there is not much more than one's preference or prejudice that selects crucial ingredients. (See for instance the discussion of different times in deparameterized models, Sec. 5.5.) The form of a single quantum correction, and even more so the combined effects of different corrections, can depend sensitively on such choices, and widely varying scenarios can be obtained from the same theory, making the Planck regime highly uncontrolled. Since no direct observations exist to limit the choices, with such models it is hard to improve our understanding of quantum gravity. This ambiguity problem prevents reliable physics to be drawn from direct considerations of the Planck era, where quantum-gravity corrections would be large.

For reliable input, one must look at tamer regimes with at least some indirect observational access. Such regimes have small density, compared to the Planck density, and direct quantum-gravity effects are expected to be tiny. However, the description of such regimes is more involved than the comparison of two parameters as used in dimensional arguments. At lower densities, the universe has grown out of its Planckian compactness, and developed some structure as envisioned for instance in the inflationary scenario. Not just the average density but also parameters that describe finer details are then relevant. They should only have a weak influence on quantum-gravity effects, but they can act for a long, cosmic time and eventually make themselves noticeable. Such features would have to be derived from the theory and could not be guessed by dimensional arguments.

Dimensional arguments typically fail in realistic settings. For instance, while it is easy to find the (Bohr) radius of a hydrogen atom based only on the relevant constants of nature, the radius of heavier atoms, with many constituents in their nuclei and shells, can only be computed from a detailed theory. Similarly, quantum gravity may well give rise to effects that violate expectations based on dimensional arguments. Discrete theories, for instance, have a large number of spatial or space-time constituents which make dimensional arguments fail just like heavy atoms do. In addition to the Planck density and the average matter density, for instance, a third relevant parameter could be the density of discrete patches that make up an expanding spatial region.

In order to show convincingly that additional parameters not only render dimensional arguments inapplicable but also lead to more-sizeable effects, one must find concrete mechanisms by which magnifications can occur, perhaps by the concerted action of many constituents. There is at least one indication from quantum-corrected algebroids of hypersurface-deformation: By taking into account discrete or other features, they provide non-classical space-time structures and therefore lead to effective theories more general than standard gravitational effective actions with their tiny higher-curvature terms.

6.2 Indirect effects

We just list here some candidates that have been considered for possible enlargements of tiny quantum effects.

The most well-known example is perhaps large extra dimensions in which only gravity but not the other forces can penetrate [133]. The latter requirement ensures that such models are not in conflict with observations in particle physics. The former, of large extra dimensions, provides a parameter that could lift corrections to observable magnitudes.

Another example is new fields that may serve as candidates for an inflaton. (See for instance [134, 135].) In addition to the advantage of having an explanation of inflation, such models can make their underlying theories testable by using the precise measurements of the cosmic microwave background.

Finally, discrete theories provide a clear candidate for constituents of large number. If their independent impacts on space-time physics, for instance of a propagating wave, add up, the result may be observable. Such an effect would be analogous to Brownian motion, which made it possible to prove the existence of atoms and molecules without having atomic resolution. In quantum gravity, one needs long distance or time scales to have a large number of constituents, which can be realized in signals from gamma-ray bursts or by structure building up slowly as the early universe expands. Some reviews of concrete realizations can be found in [136, 137].

Summary

Space-time presents a challenging object for quantum and effective descriptions, posing interesting questions. We could here discuss only a small part of the large body of work done on quantum cosmology, focusing instead on general problems, principles, and procedures. In most cases of discussions in the literature, at least one of the relevant requirements on quantum cosmology is violated, indicating how difficult progress in this field still is. For instance, the problem of time is circumvented in approaches that make use of a background metric, while in background independent ones it is often evaded by choosing one internal time but not asking whether results are independent of the choice. The problem of states is similarly avoided (but not solved) by assuming specific states, such as Gaussians, or by using low-energy expansions around vacuum states of free theories even in strong quantum regimes. The covariance problem is solved in some approaches, such as string theory, but presents one of the most crucial and stubborn issues for canonical and discrete space-time formulations.

We remain far from a proper understanding of quantum cosmology, especially when physics at the Planck scale is involved. At the same time, research on quantum cosmology has led to progress in our understanding of generally covariant quantum systems and often showed unexpected effects of quantum space-time.

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