

# Quantum Cosmology <sup>\*</sup>

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## Abstract

A complete model of the universe needs at least three parts: (1) a complete set of physical variables and dynamical laws for them, (2) the correct solution of the dynamical laws, and (3) the connection with conscious experience. In quantum cosmology, item (2) is the quantum state of the cosmos. Hartle and Hawking have made the ‘no-boundary’ proposal, that the wavefunction of the universe is given by a path integral over all compact Euclidean 4-dimensional geometries and matter fields that have the 3-dimensional argument of the wavefunction on their one and only boundary. This proposal is incomplete in several ways but also has had several partial successes, mainly when one takes the zero-loop approximation of summing over a small number of complex extrema of the action. This is illustrated here by the Friedmann-Robertson-Walker-scalar model. In particular, new results are discussed when the scalar field has an exponential potential, which generically leads to an infinite number of complex extrema among which to choose.

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# 1 Motivation for a quantum state of the cosmos

A complete model of the universe needs at least three parts:

1. A complete set of physical variables (e.g., the arguments of the wavefunction) and dynamical laws (e.g., the Schrödinger equation for the wavefunction, the algebra of operators in the Hilbert space, or the action for a path integral.) Roughly speaking, these dynamical laws tell how things change with time. Typically they have the form of differential equations.
2. The correct solution of the dynamical laws (e.g., the wavefunction of the universe). This picks out the actual quantum state of the cosmos from the set of states that would obey the dynamical laws. Typically a specification of the actual state would involve initial and/or other boundary conditions for the dynamical laws.
3. The connection with conscious experience (e.g., the laws of psycho-physical experience) These might be of the form that tells what conscious experience occurs for a possible quantum state for the universe, and to what degree each such experience occurs (i.e., the measure for each set of conscious experiences [1, 2]).

Item 1 alone is called by physicists a TOE or ‘theory of everything,’ but it is not complete by itself. In this chapter I shall focus on Item 2, but even Items 1 and 2 alone are not complete, since by themselves they do not logically determine what, if any, conscious experiences occur in a universe. For example, suppose we have a unique quantum state for a TOE that consists of some completion of string/M theory. If this completion is anything like our present partial knowledge of string/M theory or of other dynamical laws in physics, it will not by itself answer the question of why most alert humans are usually consciously aware of their visual sensations but not of their heartbeats.

(Of course, one might postulate some principle whereby one would most of the time be consciously aware of visual sensations but not of one’s heartbeat, but I do not see how such a principle would be directly derivable from the quantum state and a full set of dynamical laws that are at all similar to our presently known incomplete approximations to these laws. One might propose that awareness of visual sensations would be more useful in the survival of the fittest than awareness of one’s heartbeat, but it is not obvious to me how conscious awareness contributes to survival, even though it may be correlated with some physical information processing that is useful for survival. When Fermilab Director Robert Wilson was asked by a Congressional committee what Fermilab contributes to the defense of the nation, he is reported to have responded, “Nothing. But it helps make the nation worth defending.” Similarly, consciousness may contribute nothing to the survival of an organism, but it may make the organism worth surviving and may be the selection mechanism that makes us aware of being conscious organisms.)

But even before we attack the difficult question of the relationship between conscious experience and the rest of physics, it is clear that much of what we try to describe in physics depends not only on the dynamical laws but also on some features of the quantum state of the cosmos [3]. For example, the observation that the universe has far less entropy than one might imagine, so that the entropy tends to increase (the second law of thermodynamics), cannot be purely a consequence of the familiar types of dynamical laws but must also depend on the quantum state of the cosmos.

## 2 The Hartle-Hawking proposal for the quantum state

Here I shall focus on Item 2, the quantum state of the cosmos, and in particular focus on a proposal by Hawking [4, 5] and by Hartle and Hawking [6] for this quantum state. They have proposed that the quantum state of the universe, described in canonical quantum gravity by what we now call the Hartle-Hawking wavefunction, is given by a path integral over compact four-dimensional Euclidean geometries and matter fields that each have no boundary other than the three-dimensional geometry and matter field configuration that is the argument of the wavefunction.

(Thus this proposal is sometimes called the ‘no-boundary’ proposal. However, just as in this volume my main Ph.D. advisor Kip Thorne has rebelled against his advisor John Wheeler in relabeling Wheeler’s ‘no-hair’ conjecture as the ‘two-hair’ conjecture, to count the mass and angular momentum of a Kerr black hole, so I am emboldened to rebel half as much against my other Ph.D. advisor, Stephen Hawking, by relabeling his ‘no-boundary’ proposal as the ‘one-boundary’ proposal, to count the one boundary of the path integral that is the argument of the wavefunction.)

In particular, the wavefunction for a three-geometry given by a three-metric  $g_{ij}(x^k)$ , and for a matter field configuration schematically denoted by  $\phi^A(x^k)$ , where the three-metric and the matter field configuration are functions of the three spatial coordinates  $x^k$  (with lower-case Latin letters ranging over the three values  $\{1, 2, 3\}$ ), is given by the wavefunction

$$\psi[g_{ij}(x^k), \phi^A(x^k)] = \int \mathcal{D}[g_{\mu\nu}(x^\alpha)] \mathcal{D}[\phi^\Omega(x^\alpha)] e^{-I[g_{\mu\nu}, \phi^\Omega]}, \quad (1)$$

where the path integral is over all compact Euclidean four-dimensional geometries that have the three-dimensional configuration  $[g_{ij}(x^k), \phi^A(x^k)]$  on their one and only boundary. Here a four-geometry are given by a four-metric  $g_{\mu\nu}(x^\alpha)$ , and four-dimensional matter field histories are schematically denoted by  $\phi^\Omega(x^\alpha)$ , both functions of the four Euclidean spacetime coordinates  $x^\alpha$  (with lower-case Greek letters ranging over the four values  $\{0, 1, 2, 3\}$ ).

The Hartle-Hawking ‘one-boundary’ proposal is incomplete in various ways. For example, in quantum general relativity, using the Einstein-Hilbert-matter action, the path integral is ultraviolet divergent and nonrenormalizable [7]. This nonrenormalizability also occurs for quantum supergravity [8]. String/M theory gives the

hope of being a finite theory of quantum gravity (at least for each term of a perturbation series, though the series itself is apparently only an asymptotic series that is not convergent.) However, in string/M theory it is not clear what the class of paths should be in the path integral that would be analogous to the path integral over compact four-dimensional Euclidean geometries without extra boundaries that the Hartle-Hawking proposal gives when general relativity is quantized.

Another way in which the Hartle-Hawking ‘one-boundary’ proposal is incomplete is that conformal modes make the Einstein-Hilbert action unbounded below, so the path integral seems infinite even without the ultraviolet divergence [9]. If the analogue of histories in string/M theory that can be well approximated by low-curvature geometries have actions that are similar to their general-relativistic approximations, then the string/M theory action would also be unbounded below and apparently exhibit the same infrared divergences as the Einstein-Hilbert action for general relativity. There might be a uniquely preferred way to get a finite answer by a suitable restriction of the path integral, but it is not yet clear what that might be.

A third technical problem with the Hartle-Hawking path integral is that one is supposed to sum over all four-dimensional geometries, but the sum over topologies is not computable, since there is no algorithm for deciding whether two four-dimensional manifolds have the same topology. This might conceivably be a problem that is more amenable in string/M theory, since it seems to allow generalizations of manifolds, such as orbifolds, and the generalizations may be easier to sum over than the topologies of manifolds.

A fourth problem that is likely to plague any proposal for the quantum state of the cosmos is that even if the path integral could be uniquely defined in a computable way, it would in practice be very difficult to compute. Thus one might be able to deduce only certain approximate features of the universe from such a path integral.

Despite the difficulties of precisely defining and evaluating the Hartle-Hawking ‘one-boundary’ proposal for the quantum state of the universe, it has had a certain amount of partial successes in calculating certain approximate predictions for highly simplified toy models:

1. Lorentzian-signature spacetime can emerge in a WKB limit of an analytic continuation [6, 5].
2. The universe can inflate to large size [5].
3. Models can predict near-critical energy density [5, 10].
4. Models can predict low anisotropies [11].
5. Inhomogeneities start in ground states and so can fit cosmic microwave background data [12].
6. Entropy starts low and grows with time [13, 14, 15].

### 3 Zero-loop quantum cosmology and FRW-scalar models

One can avoid many of the problems of the Hartle-Hawking path-integral, and achieve some partial successes for the ‘one-boundary’ proposal, by taking

$$\psi[g_{ij}(x^k), \phi^A(x^k)] \approx \psi_{0\text{-loop}} = \sum_{\text{some extrema}} e^{-I[g_{\mu\nu}, \phi^\Omega]}, \quad (2)$$

summing over a small set of extrema of the Euclidean action  $I$ , generally complex classical solutions of the field equations.

Even at this highly simplified approximation to the path integral, there is the question of which extrema to sum over, since typically there are infinitely many.

A simple class of models that has often been considered is the  $k = +1$  Friedmann-Robertson-Walker-scalar model, in which the three-geometry boundary is an  $S^3$  with radius  $\sqrt{2G/3\pi} a_b$  and the (real) scalar field takes the homogeneous value  $\sqrt{3/4\pi G} \phi_b$  on the boundary, where Newton’s constant is  $G$  and I have set  $\hbar$  and  $c$  equal to unity. (The numerical factors used to define the physical radius and scalar field value in terms of rescaled values  $a_b$  and  $\phi_b$  enable one to dispense with similar factors in the expressions involving  $a_b$  and  $\phi_b$ .)

Then the zero-loop approximation gives

$$\psi(a_b, \phi_b) \approx \psi_{0\text{-loop}}(a_b, \phi_b) = \sum_{\text{some extrema}} e^{-I(a_b, \phi_b)}, \quad (3)$$

where  $I(a_b, \phi_b)$  is the Euclidean action of a classical solution that is compact and has the  $S^3$  geometry and homogeneous scalar field as its one and only boundary.

‘One-boundary’ FRW-scalar histories have a time parameter  $t$  that can be taken to run from 0 (at a regular ‘center’) to 1 (at the boundary), and then to have  $\phi = \phi(t)$  and four-metric

$$ds^2 = \left(\frac{2G}{3\pi}\right) [N^2(t)dt^2 + a^2(t)d\Omega_3^2], \quad (4)$$

where  $N(t)$  is the Euclidean lapse function and  $d\Omega_3^2$  is the metric on a unit round  $S^3$ . The boundary conditions of regularity at the center are  $a(0) = 0$ ,  $\dot{a}(0)/N(0) = 1$ , and  $\dot{\phi}(0)/N(0) = 0$ , and the match to the boundary at  $t = 1$  gives  $a(1) = a_b$  and  $\phi(1) = \phi_b$ .

If the scalar field potential is  $[9/(16G^2)]V(\phi)$  (with the coefficient again chosen to simplify the formulas below in terms of the rescaled potential  $V(\phi)$ ), then the Euclidean action of the history is

$$\begin{aligned} I = -iS &= \int dt \left[ \frac{1}{2N}(-a\dot{a}^2 + a^3\dot{\phi}^2) + \frac{1}{2}N(-a + a^3V) \right] \\ &= -\frac{1}{2} \int dt \left[ \tilde{N}^{-1} \tilde{G}_{AB} \dot{X}^A \dot{X}^B + \tilde{N} \right] = - \int d\tilde{s}, \end{aligned} \quad (5)$$

where

$$\tilde{N} = a(1 - a^2V)N \equiv e^\alpha(1 - w)N, \quad (6)$$

with

$$\alpha \equiv \ln a, \quad (7)$$

$$w \equiv a^2V, \quad (8)$$

and  $d\tilde{s}$  is the infinitesimal proper distance in the auxiliary two-metric

$$\begin{aligned} d\tilde{s}^2 &= \tilde{G}_{AB}dX^A dX^B \\ &= a^4(1 - a^2V) \left( \frac{da^2}{a^2} - d\phi^2 \right) \equiv e^{4\alpha}(1 - w)(d\alpha^2 - d\phi^2) \\ &= e^{2u+2v}(1 - w)dudv = \frac{1}{4}(1 - w)dXdY. \end{aligned} \quad (9)$$

The auxiliary metric (9) has null coordinates

$$u \equiv \alpha - \phi \equiv \ln a - \phi, \quad v \equiv \alpha + \phi \equiv \ln a + \phi, \quad (10)$$

or alternate null coordinates

$$X \equiv e^{2u} \equiv e^{2\alpha-2\phi}, \quad Y \equiv e^{2v} \equiv e^{2\alpha+2\phi}. \quad (11)$$

The zero-loop or classical histories are those that extremize the Euclidean action (5). Define the rescaled proper Euclidean time (or, for short, ‘Euclidean time’)

$$\tau = \int_0^t N(t')dt' = \sqrt{\frac{3\pi}{2G}}(\text{proper radius or Euclidean ‘time’}), \quad (12)$$

which is gauge invariant, invariant under reparametrizations of the original time coordinate  $t$  when the lapse function  $N(t)$  is properly adjusted, though its value at the boundary depends upon the particular history chosen. Then extremizing the action with respect to  $N(t)$  leads to the constraint equation

$$\left( \frac{da}{d\tau} \right)^2 - a^2 \left( \frac{d\phi}{d\tau} \right)^2 = 1 - a^2V \equiv 1 - w. \quad (13)$$

Extremizing with respect to  $\phi(t)$  leads to the scalar field equation

$$\frac{d^2\phi}{d\tau^2} + \frac{3}{a} \frac{da}{d\tau} \frac{d\phi}{d\tau} = \frac{1}{2} \frac{dV}{d\phi}, \quad (14)$$

and extremizing with respect to  $a(t)$  leads to the other field equation,

$$\frac{1}{a} \frac{d^2a}{d\tau^2} + 2 \left( \frac{d\phi}{d\tau} \right)^2 = -V, \quad (15)$$

though either of these last two equations is redundant if one uses the other along with the constraint equation.

Alternatively, zero-loop or classical histories (extrema of the action) are geodesics of the auxiliary two-metric (9), with this metric giving the proper distance along an extremum as

$$d\tilde{s} = -dI = \tilde{N}dt = a(1 - a^2V) d\tau. \quad (16)$$

The Euclidean action  $I$  is then simply the negative of the proper distance from the center to the boundary along a geodesic of the auxiliary metric. If the center has  $\tau = 0$  and the boundary (where the wavefunction is being evaluated) has

$$\tau = \tau_b = \int_0^1 N(t)dt, \quad (17)$$

then the Euclidean action for the classical history is

$$I = - \int_0^1 \tilde{N}dt = - \int_0^{\tau_b} a(1 - a^2V) d\tau. \quad (18)$$

## 4 Real classical solutions for the FRW-scalar model

Therefore, a classical or extremal history  $(a(\tau), \phi(\tau))$  for the FRW-scalar model obeys the regularity conditions  $a = 0$ ,  $da/d\tau = 1$ , and  $d\phi/d\tau = 0$  at the center,  $\tau = 0$ , and so is uniquely determined, for a given rescaled potential function  $V(\phi)$ , by the value of  $\phi$  at the center,  $\phi_0 \equiv \phi(0)$ , and by the value of  $\tau$  at the boundary,  $\tau_b$ , where the wavefunction is being evaluated. (One could alternatively have  $da/d\tau = -1$  at the center, but this would give a negative value for the volume of the Euclidean geometry, and the opposite sign of the Euclidean action, so I shall reject this possibility.)

Let us for simplicity restrict attention to analytic potentials that for all real finite  $\phi$  are real finite convex functions that are bounded below by nonnegative values (which can be considered to be nonnegative cosmological constants). Since the action and extrema are invariant under replacing  $\phi$  by  $\phi' = -\phi$  if  $V(\phi)$  is replaced by  $V(\phi') = V(\phi)$ , without loss of generality we can consider the case in which the convex  $V(\phi)$  is nondecreasing as  $\phi$  is increased from  $\phi_0$ , so  $dV/d\phi \geq 0$  for  $\phi \geq \phi_0$ . (We shall not need any further the condition that  $V(\phi)$  be convex and nonnegative for real  $\phi$ , but only that  $V(\phi)$  be real, finite, positive, differentiable, and monotonically increasing for  $\phi > \phi_0$ .)

Then the scalar field equation (14), rewritten as

$$\frac{d}{d\tau} \left( a^3 \frac{d\phi}{d\tau} \right) = \frac{1}{2} a^3 \frac{dV}{d\phi}, \quad (19)$$

implies that  $\phi$  cannot decrease with  $\tau$ . Furthermore, it can stay constant only if  $\phi_0$  is at the minimum value of  $V(\phi)$ , say  $V_0$ , in which case  $\phi$  does stay at  $\phi_0$ , and one gets  $a = V_0^{-1/2} \sin(V_0^{1/2}\tau)$ , giving a 4-metric (4) that is part or all of a round

4-sphere of radius  $(1.5\pi V_0/G)^{-1/2}$ , depending on whether  $\tau_b$  is less than or equal to its maximum value of  $\pi V_0^{-1/2}$ , where  $a$  returns to zero.

But if  $dV/d\phi$  is positive at  $\phi = \phi_0$  (and hence, by assumption, remains positive for all larger  $\phi$ ), then Eq. (16) has its right hand side become positive as soon as  $a(\tau)$  becomes positive, and so  $a^3 d\phi/d\tau$  increases monotonically with real increasing  $\tau$ . It thus follows that  $\phi$  and  $V$  also increase monotonically with  $\tau$ .

The field equation (15) with real  $a$ ,  $\phi$ , and  $\tau$  and with positive  $V$  implies that  $a$  is a concave function of  $\tau$ . With  $V$  not only positive but also increasing with  $\tau$ ,  $a(\tau)$  necessarily reaches a finite maximum, say  $a_m = a_m(\phi_0)$  at the Euclidean time  $\tau_m = \tau_m(\phi_0)$ , with the functions  $a_m(\phi_0)$  and  $\tau_m(\phi_0)$  depending on the function  $V(\phi)$ , and then  $a$  returns to 0 at finite  $\tau$ , say  $\tau_s = \tau_s(\phi_0)$ , with the function  $\tau_s(\phi_0)$  also depending on the function  $V(\phi)$ .

Since  $a^3 d\phi/d\tau$  increases monotonically with  $\tau$  under the assumptions above,  $d\phi/d\tau$  becomes infinite at  $\tau = \tau_s$  as  $a$  returns to zero, giving a curvature singularity there, and one can further show that  $\phi$  goes to infinity there as well.

Given a particular fixed rescaled potential  $V(\phi)$  obeying the assumptions above for all real  $\phi$  (real, finite, differentiable, and either monotonically increasing with  $\phi$  for all  $\phi$  or else monotonically increasing in both directions away from a single minimum), we thus see that a choice of the two real parameters  $\phi_0$  and  $\tau_b$  leads to a unique classical solution, if  $\tau_b < \tau_s(\phi_0)$ , and uniquely gives the boundary values  $a_b = a(\phi_0, \tau_b)$  and  $\phi_b = \phi(\phi_0, \tau_b)$ , as well as the action  $I = I(\phi_0, \tau_b)$ .

Of course, we are really interested in evaluating  $\psi(a_b, \phi_b)$  by the zero-loop approximation and hence want the action, for each of a suitable set of extrema, as a function of  $a_b$  and  $\phi_b$  instead of as a function of  $\phi_0$  and  $\tau_b$ . To do this, we need to solve for values of the parameters  $(\phi_0, \tau_b)$  that give the desired boundary values  $(a_b, \phi_b)$ . Because the number of parameters matches the number of boundary values, we expect a discrete set of solutions, but the number of solutions may not be precisely one for each  $(a_b, \phi_b)$ .

For example, consider the case in which  $V(\phi)$  is a slowly varying function of  $\phi$ . In this case the scalar field equation (14) implies that  $\phi$  does not change much during the evolution, so as a zeroth-order approximation one can take  $\phi \approx \phi_0$  and hence also  $\phi_b \approx \phi_0$ . Then if one restricts to real values of  $\tau_b$ , one sees that there are two solutions for  $(\phi_0, \tau_b)$  given  $(a_b, \phi_b)$  if  $a_b < a_m(\phi_0) \approx a_m(\phi_b)$ , because the real classical Euclidean solution that starts at the center,  $a = 0$ , with real  $\phi_0$  there, has  $a$  increasing from 0 to its maximum  $a_m(\phi_0)$  at  $\tau = \tau_m(\phi_0)$  and then decreasing to 0 again at  $\tau = \tau_s(\phi_0)$ , so there are two solutions for  $\tau_b$ , one with  $\tau_b < \tau_m$  in which  $a$  crosses  $a_b$  while increasing with  $\tau$ , and the second with  $\tau_b > \tau_m$  in which  $a$  crosses  $a_b$  while decreasing with  $\tau$ .

On the other hand, if  $a_b > a_m(\phi_0) \approx a_m(\phi_b)$ , there are no real solutions for  $(\phi_0, \tau_b)$ , because all of the real solutions that match  $\phi_b$  have maxima for  $a$  that are smaller than  $a_b$ .

One proposal would be simply to say that the zero-loop approximation gives a wavefunction that is the sum of the  $e^{-I}$ 's when there are one or more real classical solutions matching the real boundary data that are the arguments of the wavefunc-



tion, and that the zero-loop approximation gives zero when there are no real classical solutions. However, then one would simply get zero for most large universes with nontrivial matter, such as  $\psi_{0\text{-loop}}(a_b, \phi_b) = 0$  when  $\phi_b$  is not at the minimum of the potential that is zero there (or is at the minimum if the potential is positive there) and when  $a_b$  is sufficiently large (e.g., larger than  $V_0^{-1/2}$  if the rescaled potential has a positive minimum of  $V_0$ ).

Furthermore, even for potentials allowing large classical Euclidean universes, the action for each of them would be real, and so the zero-loop approximate wavefunction would be purely real and not have the oscillatory behavior apparently necessary to describe our observations of an approximately Lorentzian universe.

Therefore, it is not adequate to restrict the zero-loop approximation to real classical Euclidean histories.

## 5 Complex classical solutions for the FRW-scalar model

To get a potentially adequate zero-loop approximation, we shall consider complex classical solutions (though still with real boundary values that are the arguments of the wavefunction). That is, we shall take the classical field equations (13)-(15) as complex analytic equations for complex quantities  $a$ ,  $\phi$ ,  $\tau$ , and  $V(\phi)$ . We shall assume that  $V(\phi)$  is a complex analytic function that for real values of  $\phi$  has the properties assumed above (real, finite, differentiable, and either monotonically increasing with  $\phi$  for all  $\phi$  or else monotonically increasing in both directions away from a single minimum). The monotonicity property of  $V(\phi)$  for real  $\phi$  is not really important but shall continue to be assumed here to simplify some of the discussion.

For a complex classical or extremal history  $(a(\tau), \phi(\tau))$  for the FRW-scalar model, we shall continue to assume the regularity conditions  $a = 0$ ,  $da/d\tau = 1$ , and  $d\phi/d\tau = 0$  at the center,  $\tau = 0$ , as complex analytic equations that are the essential input from the ‘one-boundary’ proposal when it is extended to allow complex solutions in the zero-loop approximation. Again the classical history is uniquely determined, for a given rescaled potential function  $V(\phi)$ , by the value of  $\phi$  at the center,  $\phi_0 \equiv \phi(0)$ , and by the value of  $\tau$  at the boundary,  $\tau_b$ , where the wavefunction is being evaluated. The only difference is that both  $\phi_0$  and  $\tau_b$  may be complex, though we shall only be interested in solutions that give real values for  $a_b(\phi_0, \tau_b)$  and  $\phi_b(\phi_0, \tau_b)$ .

For example, let us return to the case in which  $V(\phi)$  is slowly varying so that  $\phi$  remains close to  $\phi_0$  throughout the classical history. Then  $\phi_b \approx \phi_0$ , so for the desired real  $\phi_b$ , we can take  $\phi_0$  to be approximately real. Now let us take  $a_b$  to be much larger than  $a_m(\phi_b)$ , so there is no real classical solution matching  $(a_b, \phi_b)$  on the boundary. However, we can find a complex classical solution matching the boundary data in the following way:

First, relative to the small variation of  $V(\phi)$ , take the zeroth-order approximation that  $\phi_0 = \phi_b$ . Then the history is given by a contour in the complex  $\tau$  plane from

its value of 0 at the center to some complex value  $\tau_b$  at the boundary. Consider the contour in which  $\tau$  starts off real and increasing. The rescaled  $S^3$  size  $a$  begins increasing as  $\tau$ , but as a concave function of real  $\tau$ , it eventually reaches a maximum,  $a_m(\phi_0)$ , at  $\tau = \tau_m(\phi_0)$ , and then would decrease if  $\tau$  continued to increase along its real axis. Thus to lowest nontrivial order,  $a(\tau)$  varies quadratically with  $\tau - \tau_m$  when this quantity is small, with a negative coefficient.

Therefore, after reaching  $\tau_m$  on the real axis, make a right-angled bend in the contour for  $\tau$  so that now  $\tau - \tau_m$  takes on an imaginary value and  $a$  continues to increase. One can then follow a contour for  $\tau$  so that  $a$  stays real and increases up to the desired boundary value  $a_b$ .

If  $V(\phi)$  were positive and precisely constant, having no variation at all and thus being equivalent to a cosmological constant, the classical solution would have  $\phi = \phi_0$  everywhere and so would match the boundary condition if one chose  $\phi_0 = \phi_b$ . In this case the part of the contour with  $\tau$  increasing along the real axis from 0 to  $\tau_m$  would give the geometry of a Euclidean 4-hemisphere, and then the part of the contour with  $\tau - \tau_m$  changing in the imaginary direction would give a Lorentzian deSitter universe expanding from its time-symmetric throat at  $\tau = \tau_m$  (to be discussed more below).

If  $V(\phi)$  is not precisely constant but has a slow variation with  $\phi$ , and if  $\phi_0$  is taken to be precisely real, then along the part of the contour in which  $\tau$  increases along the real axis from 0 up to  $\tau_m$ ,  $\phi$  will develop a small positive derivative,  $d\phi/d\tau$ , and will increase slightly over its initial value of  $\phi_0$  to become, say,  $\phi_m$  at  $\tau = \tau_m$ , still real. But when one turns the corner in the contour for  $\tau$ , although  $a$  had zero time derivative there and so could remain real,  $\phi$  has a small positive time derivative and so picks up a small imaginary contribution as  $\tau - \tau_m$  increases (or decreases) in the imaginary direction, hence becoming slightly complex at  $\tau_b$  where  $a$  matches the boundary value  $a_b$ .

(The fact that  $\phi$  becomes slightly complex implies that the varying  $V(\phi)$  also becomes slightly complex, making the geometry slightly complex. This would make  $a(\tau)$  slightly complex, and hence never reaching the real boundary value  $a_b$ , if one kept  $\tau$  on a contour with  $\tau - \tau_m$  purely imaginary, though that contour would keep the time-time part of the metric, proportional to  $d\tau^2$ , purely real and negative or Lorentzian. However, one can instead, at least if  $V(\phi)$  is slowly varying, distort the contour for  $\tau$  slightly to keep  $a(\tau)$  precisely real, and hence reaching  $a_b$ , but at the cost of making  $d\tau^2$ , and hence the time-time part of the geometry, slightly complex.)

However, just as one can compensate for a slightly complex  $a(\tau)$  along the simple-minded contour by distorting the contour slightly, one can also compensate for a slightly complex value of  $\phi(\tau_b)$  by distorting the initial value  $\phi_0$  slightly into the complex, as several of us realized independently but which Lyons [16] was the first to write down; see also [17, 18] for more recent work in this area. To the lowest order in the slow variation of  $V(\phi)$ , one can see how much error there is in  $\phi(\tau_b)$  when one starts with the trial value  $\phi_0 = \phi_b$  and then evolves along the contour for  $\tau$  that keeps  $a(\tau)$  real and positive until one reaches  $a = a_b$  at  $\tau = \tau_b$ , and then correct that trial value of  $\phi_0$  by the opposite of that error. One can take this first-corrected

$\phi_0$  as a second trial value for  $\phi_0$ , follow a contour that goes from  $a = 0$  to  $a = a_b$  at a suitable  $\tau = \tau_b$ , find the error in  $\phi(\tau_b)$ , and make a second correction. In this way one can in principle iterate until one presumably finds, to sufficient accuracy, the correct complex  $\phi_0$  that leads to  $\phi = \phi_b$  at  $a = a_b$ .

For a sufficiently rapidly varying  $V(\phi)$ , this iteration procedure may not converge. For example, there may be no contour that keeps  $a$  real all the way from 0 to  $a_b$  and also allows  $\phi = \phi_b$  there. An example of this for an exponential potential will be given below. However, there can be other complex solutions that can match the boundary values  $(a_b, \phi_b)$ , even if there are none that have  $a$  stay purely real along some contour. Of course, this also highlights the possibility that even when the iterative procedure above leads to a unique complex solution (up to complex conjugation; see immediately below) matching the boundary values, there may be other complex solutions that also match the boundary values, and finding the criterion for which to include in the zero-loop approximation may be problematic.

The complex histories that lead to the desired real boundary values will have complex Euclidean action  $I$ , with the imaginary part depending on the boundary values that are the argument of the wavefunction, so the contribution that they give to the zero-loop wavefunction,  $e^{-I}$ , will be complex and have complex oscillations as a function of the boundary values. (This is true even if  $V(\phi)$  is precisely constant and positive.)

When  $V(\phi)$  is an analytic function that is real for real  $\phi$ , and when the boundary values  $a_b$  and  $\phi_b$  are real, as I am always assuming, then for any complex initial data  $(\phi_0, \tau_b)$  that leads to these real boundary values, the complex conjugate data will lead to the same real boundary values and thus also represent a history that matches the boundary values, by the analyticity of the classical equations. Therefore, complex classical histories always occur in pairs. Since the actions of the two histories in each pair will also be the complex conjugates of each other, when in the zero-loop approximation one adds up the two complex conjugate values of  $e^{-I}$ , one will always get a real sum. No matter how many pairs of complex conjugate contributions one adds (or whether one adds individual real contributions from real classical solutions), one always gets a real wavefunction (in this configuration representation), though it can be negative and hence oscillate with the boundary values in a way that the contributions of the purely real classical solutions could not.

Of course, one would expect that this feature would persist even if one multiplied the zero-loop contributions by prefactors (say to incorporate one-loop determinants) or otherwise went beyond the one-loop approximation, but this discussion shows that a real oscillating wavefunction can arise simply out of the zero-loop approximation if one allows the contributions from complex conjugate pairs of classical solutions.

## 6 FRW-scalar models with an exponential potential, $V = e^{2\beta\phi}$

To illustrate some of these ideas quantitatively, it is helpful to consider the case of an exponential potential,

$$V(\phi) = e^{2\beta\phi}, \quad (20)$$

where  $\beta$  is a real parameter that characterizes how fast the potential varies as a function of  $\phi$ .

In terms of the quantities defined by Equations (7), (8), (10), and (11), one then gets

$$w \equiv a^2 V = e^{2\alpha+2\beta\phi} = e^{(1-\beta)u+(1+\beta)v} = X^{\frac{1-\beta}{2}} Y^{\frac{1+\beta}{2}}. \quad (21)$$

The auxiliary two-metric (9) is then

$$\begin{aligned} d\tilde{s}^2 &= e^{4\alpha} (1-w) (d\alpha^2 - d\phi^2) \\ &= e^{2u+2v} (1 - e^{(1-\beta)u+(1+\beta)v}) dudv \\ &= \frac{1}{4} \left(1 - X^{\frac{1-\beta}{2}} Y^{\frac{1+\beta}{2}}\right) dXdY. \end{aligned} \quad (22)$$

This metric has a scaling symmetry, exhibited by the homothetic Killing vector

$$\mathbf{K} = (1 + \beta) \frac{\partial}{\partial u} - (1 - \beta) \frac{\partial}{\partial v}, \quad (23)$$

whose action is to multiply  $a^2 \equiv e^{2\alpha} \equiv e^{u+v}$  by a constant while keeping  $w$  (or  $\alpha + \beta\phi$ ) fixed, thereby multiplying the metric (22) by the square of this constant. This is a symmetry that maps geodesics (which represent classical solutions) onto geodesics, though multiplying the lengths of the geodesics, and hence the actions of the solutions they represent, by the same constant by which  $a^2$  is multiplied. It is this symmetry that allows one to reduce the two nontrivial parameters of a generic two-metric to one nontrivial parameter for the metric (22) and to reduce the generic second-order geodesic equation to a single first-order differential equation below.

For the exponential potential (20), the constraint equation (13) becomes

$$\left(\frac{d\alpha}{d\tau}\right)^2 - \left(\frac{d\phi}{d\tau}\right)^2 - e^{-2\alpha} + e^{2\beta\phi} = 0. \quad (24)$$

The scalar field equation (14) becomes

$$\frac{d^2\phi}{d\tau^2} + 3\frac{d\alpha}{d\tau}\frac{d\phi}{d\tau} - \beta e^{2\beta\phi} = 0, \quad (25)$$

and the field equation (15) becomes

$$\frac{d^2\alpha}{d\tau^2} + \left(\frac{d\alpha}{d\tau}\right)^2 + 2\left(\frac{d\phi}{d\tau}\right)^2 + e^{2\beta\phi} = 0. \quad (26)$$

As before, these two second-order field equations are not independent of each other if one uses the constraint (15).

The symmetry action of the homothetic Killing vector upon these classical field equations with an exponential potential is to multiply the rescaled Euclidean time  $\tau$ , the rescaled  $S^3$  size  $a = e^\alpha$  and the inverse square root of the rescaled potential,  $V^{-1/2} = e^{-\beta\phi}$ , all by the same constant.

If we take out the scaling behavior represented by the homothetic Killing vector of the auxiliary two-metric (22), we can represent the nontrivial behavior of the classical solutions by using two scale-invariant quantities. For one of them it is convenient to use  $w$  defined by Eq. (21) above. When  $\beta \neq 0$ , for the other it is convenient to use

$$u \equiv -\frac{1}{\beta} \frac{d\phi}{d\alpha}. \quad (27)$$

I shall henceforth use this definition of  $u$  rather than the previous (different) use of  $u$  for the null coordinate defined in Eq. (10). The nontrivial behavior of a classical solution is then given by the relation between  $u$  and  $w$ , say  $u(w)$  in a regime where this function is single-valued.

One can readily find that the relation between  $u$ ,  $w$ , and  $\alpha$  is given by the following two equations:

$$\frac{du}{d\alpha} = -\left(\frac{1 - \beta^2 u^2}{1 - w}\right) (2u + w - 3uw), \quad (28)$$

$$\frac{dw}{d\alpha} = -2w(1 - \beta^2 u). \quad (29)$$

The relation with the Euclidean time  $\tau$  is then given by the constraint equation

$$\left(\frac{da}{d\tau}\right)^2 = \frac{1 - w}{1 - \beta^2 u^2}. \quad (30)$$

Now we see that we can divide Eq. (28) by Eq. (29) to get a single first-order differential equation relating the two scale invariant quantities,  $u$  and  $w$ :

$$\frac{du}{dw} = -\left(\frac{1 - \beta^2 u^2}{1 - \beta^2 u}\right) \frac{2u + w - 3uw}{2w(1 - w)}. \quad (31)$$

There is a one-parameter set of solutions of Eq. (31), say labeled by the initial condition  $u(w_0)$  for some  $w_0$ . In the FRW-scalar model, the center of the FRW geometry has, if  $V$  is finite there,  $w = 0$ . This is a singular point of Eq. (31), but one can readily show that the regularity of the geometry there implies the ‘one-boundary’ condition

$$u(w) = -\frac{1}{4}w + O(w^2). \quad (32)$$

Analogous to what was done above, one should also take the positive square root of the constraint equation (30), so that  $da/d\tau = +1$  at the center and hence that the Euclidean volume is positive near there for positive real  $\tau$ .

Once one solves Eq. (31) for  $u(w)$  with the ‘one-boundary’ condition (32), one can start at the boundary value

$$w_b = a_b^2 V(\phi_b) \quad (33)$$

and choose a complex contour for  $w$  to go to the center,  $w = 0$ . Along this contour, one can start at the boundary with the boundary values  $\alpha_b \equiv \ln a_b$  and  $\phi_b$  and integrate

$$d\alpha = \frac{dw}{2w(1 - \beta^2 u)} \quad (34)$$

and

$$d\phi = -\beta u d\alpha = \frac{-\beta u dw}{2w(1 - \beta^2 u)} \quad (35)$$

to get  $\alpha(w)$  and  $\phi(w)$  along the contour. One can also evaluate the action of this classical history as

$$\begin{aligned} I(a_b, \phi_b) &= - \int a^2 \sqrt{1-w} \sqrt{d\alpha^2 - d\phi^2} \\ &= -\frac{1}{2} \int_0^{w_b} dw e^{-2\beta\phi} \sqrt{1-w} \frac{\sqrt{1-\beta^2 u^2}}{1-\beta^2 u}. \end{aligned} \quad (36)$$

However, one can also find that once  $u_b \equiv u(w_b)$  is determined, the action is given algebraically by

$$I(a_b, \phi_b) = -\frac{1}{2} a_b^2 (1 - u_b) \sqrt{\frac{1 - w_b}{1 - \beta^2 u_b^2}}. \quad (37)$$

This may be derived by considering the fact that as a function of the coordinates of the auxiliary two-metric (9) for a general FRW-scalar model, the action obeys the Hamiltonian-Jacobi equation

$$1 = (\nabla I)^2 = G^{AB} I_{,A} I_{,B} = \frac{e^{-4\alpha}}{1-w} \left[ \left( \frac{\partial I}{\partial \alpha} \right)^2 - \left( \frac{\partial I}{\partial \phi} \right)^2 \right]. \quad (38)$$

When one restricts to the exponential potential so that  $w = e^{2\alpha+2\beta\phi}$ , one gets the auxiliary two-metric (22) with its homothetic Killing vector, and one may look for solutions of the Hamilton-Jacobi equation (38) of the form

$$I = -\frac{1}{2} a^2 g(w), \quad (39)$$

where  $g(w)$  obeys the differential equation

$$\left( g + w \frac{dg}{dw} \right)^2 - \beta^2 \left( w \frac{dg}{dw} \right)^2 = 1 - w. \quad (40)$$

One can then readily check that if  $u(w)$  obeys the differential equation (31), then

$$g(w) = (1 - u) \sqrt{\frac{1 - w}{1 - \beta^2 u^2}}. \quad (41)$$

obeys the differential equation (40).

In terms of a classical solution of the field equations (24)-(26), say written as  $e^\alpha = a(\tau)$  and  $\phi = \phi(\tau)$ , the action may alternatively be written as

$$I = -\frac{1}{2}a^2 \left( \frac{da}{d\tau} + \frac{a}{\beta} \frac{d\phi}{d\tau} \right). \quad (42)$$

So far I have written as if there were a unique  $u(w)$ , obeying the differential equation (31) with the ‘one-boundary’ regularity condition (32) at the center, for each point of the complex  $w$ -plane. Indeed it is true that if one starts with the regularity condition (32) at the center and integrates (31) outward along some contour in the complex  $w$ -plane that avoids the singular points of that differential equation, one gets a unique answer for  $u(w)$  along that contour, and the analyticity of the differential equation (away from its singular points) guarantees that the result for  $u(w)$  is independent of deformations of the contour that do not cross any of the singular points. However, because the differential equation (31) does have singular points (at  $w = 0$ ,  $w = 1$ , and  $w = \infty$ , and at  $u = 1/\beta^2$  if  $\beta^2$  is neither 0 nor 1; in the latter case the zero in the denominator of the right hand side of Eq. (31) is canceled by the zero in the numerator), the result for  $u(w)$  generically depends upon the topology of the contour relative to the singular points. Thus one gets different Riemann sheets in which  $u(w)$  has different values. In particular, when one goes to the boundary value  $w_b$  and evaluates  $u_b = u(w_b)$  and then the action  $I(a_b, \phi_b)$  by Eq. (37), the action will, generically, depend upon the topology of contour in the complex  $w$ -plane from the center at  $w = 0$  to the boundary at  $w = w_b$ .

For example, if one contour leads to a complex value of  $u(w_b)$  (even though we are restricting to  $w_b$  real), there will be a complex conjugate contour that leads to the complex conjugate value of  $u(w_b)$  and hence to the complex conjugate value of the action  $I(a_b, \phi_b)$ . However, generically there will be far more than a single pair of topologically inequivalent contours leading from  $w = 0$  to  $w = w_b$ . Typically there will be an infinite number of such pairs, corresponding to winding around the various singularities arbitrarily many times. Because there are more than one singularity, one could presumably wind around one singularity an arbitrary (integer) number of times, then around another an arbitrary number of times, then around another, and so on ad infinitum, which would give an uncountably infinite number of infinite contours. However, if one restricted to finite contours, there would be merely a countable infinity of them.

One might also allow both signs for the square root in Eq. (37) for the action, but I would argue that one should not do that for a given contour in the complex  $w$ -plane. For a contour that stays at small real  $w$  in going from 0 to a small real  $w_b$  with a real Euclidean geometry, one would want the four-volume to be positive

and hence for the action to be negative,  $I \approx -a_b^2/2$ . This requires that one take the positive square root when  $1 - w$  and  $1 - \beta^2 u^2$  are both real and near unity, thus determining which branch of the square root to choose in the part of a contour when it just leaves the center,  $w = 0$ . For any contour emerging from the center (so long as it does not pass through the singular point  $w = 1$  and also avoids  $\beta^2 u^2 = 1$ ), one can follow the sign of the square root continuously as one goes from  $w = 0$  to  $w = w_b$  to get a unique answer there.

However, this consideration does show that if one deforms a contour to wrap once around a point with  $\beta^2 u^2 = 1$ , this continuity requirement on the branch choice for the square root will cause it to switch sign relative to the undeformed contour that did not wrap around the  $\beta^2 u^2 = 1$  point. Thus the value of the action will depend not only on the topology of the contour relative to the singular points  $w = 0$ ,  $w = 1$ ,  $w = \infty$ , and, generically,  $u = 1/\beta^2$  of the differential equation (31), but its sign will also depend upon its topology relative to the points  $u = 1/\beta$  and  $u = -1/\beta$ .

### 6.1 $\beta = 0$ deSitter example, $V = \text{const.}$

There are two values of  $\beta^2$  for which one can explicitly solve the differential equation (31) and get the classical solutions and their action,  $\beta^2 = 0$  and  $\beta^2 = 1$ . (Reversing the sign of  $\beta$  is equivalent to reversing the sign of the scalar field and has no effect on the differential equation (31) or on the action.)

When  $\beta = 0$ , the potential is independent of  $\phi$ . For the exponential potential  $V(\phi) = e^{2\beta\phi}$  given by Eq. (20), this would give  $V = 1$ , but one can easily generalize the result to any constant  $V$ .

It is convenient to define

$$x \equiv \sqrt{1 - w} = \sqrt{1 - Va^2} \quad (43)$$

as a useful replacement for  $w$  in certain equations. I shall choose the positive sign at the center, so there  $x = 1$ . However, for real  $a > 1/\sqrt{V}$ ,  $x$  will be purely imaginary and can have either sign, depending on which side of the singular point  $w = 1$  the contour is taken, and whether it wraps around that point an even or odd number of times.

The ‘one-boundary’ solution for constant  $V$  is

$$\phi = \phi_b, \quad (44)$$

$$a = \frac{1}{\sqrt{V}} \sin(\sqrt{V}\tau), \quad (45)$$

$$w = \sin^2(\sqrt{V}\tau), \quad (46)$$

$$x = \cos(\sqrt{V}\tau). \quad (47)$$

For real  $\tau$  ranging up to  $\pi/\sqrt{V}$ , one gets part of a round Euclidean  $S^4$  with equatorial  $S^3$  of rescaled radius

$$a_m = a(\tau_m) = \frac{1}{\sqrt{V}} \quad (48)$$



at

$$\tau_m = \frac{\pi}{2\sqrt{V}}. \quad (49)$$

If  $\tau$  ranges all the way from 0 to  $2\tau_m$ , one gets a complete Euclidean  $S^4$ .

On this  $S^4$ , define the latitudinal angle

$$\theta = \sqrt{V}(\tau_m - \tau), \quad (50)$$

so

$$a = a_m \cos \theta, \quad x = \sqrt{1 - w} = \sin \theta. \quad (51)$$

Now to get to real  $a > a_m$ , as discussed above for a general slowly varying positive potential, have  $\tau$  go along its real axis from  $\tau = 0$  to  $\tau = \tau_m$  but then make a right-angled bend in the complex  $\tau$ -plane, so that  $\tau - \tau_m$  becomes henceforth imaginary. In particular, analytically continue  $\theta$  to  $\theta = \pm i\psi$  with  $\psi$  real to get

$$a = a_m \cosh \psi, \quad x = \sqrt{1 - w} = \pm i \sinh \psi. \quad (52)$$

Then the four-metric (4) becomes

$$ds^2 = \left( \frac{2G}{3\pi V} \right) \left( -d\psi^2 + \cosh^2 \psi d\Omega_3^2 \right), \quad (53)$$

which is the real Lorentzian deSitter spacetime.

Although strictly speaking  $u$  as defined by Eq. (27) is not well defined when  $\beta = 0$ , as both the numerator and the denominator are zero, it is well defined when one starts with  $\beta \neq 0$  and then takes the limit of  $\beta$  going to zero. In particular its differential equation (31), and its ‘one-boundary’ regularity condition (32) at the center, are both well-defined when  $\beta = 0$  and lead to the solution

$$u = \frac{1}{3} \left( 1 - \frac{2}{1 - w + \sqrt{1 - w}} \right) = \frac{(x - 1)(x + 2)}{3x(x + 1)}, \quad (54)$$

one example of an equation that is simpler in terms of  $x \equiv \sqrt{1 - w}$  than in terms of  $w \equiv 1 - x^2$ .

Unlike the case of generic  $\beta$ , for  $\beta = 0$  there is only the two-fold ambiguity in  $u(w)$ , depending on the choice of the sign of the square root of  $1 - w$  for  $x = \sqrt{1 - w}$ .

When one uses this  $u(w)$  or  $u(x)$  in Eq. (37) for the Euclidean action, one gets

$$I = \frac{x_b^3 - 1}{3V} = \frac{(1 - w_b)^{3/2} - 1}{3V} = \frac{\sin^3 \theta_b - 1}{3V} = \frac{\pm i \sinh \psi_b - 1}{3V}. \quad (55)$$

Thus the Euclidean action is real (and negative) for  $w_b < 1$  (or  $a_b < 1/\sqrt{V}$ ) but is complex (but still with a negative real part) for  $w_b > 1$  (or  $a_b > 1/\sqrt{V}$ ).

In the special case of  $\beta = 0$ , there also seems to be no possibility of different topologies of the contour relative to the points that are at  $\beta^2 u^2 = 1$  for nonzero  $\beta$ , so in this special case there does not seem to be the possibility to reverse the overall

sign of the action, assuming that one always starts the contour from  $w = 0$  and the choice of the sign of the square root in Eq. (37) or (41) so that the four-volume starts off becoming positive.

For  $a_b \geq 1/\sqrt{V}$ , one gets

$$|e^{-I}|^2 = e^{+\frac{2}{3V}}, \quad (56)$$

the famous Hartle-Hawking enhancement of the relative probabilities that is greater for smaller positive  $V$  [5].

## 6.2 $\beta = 1$ example, $V = e^{2\phi}$

This special exponential potential has been discussed by [19], who give the solutions in terms of different variables, but here I shall follow my previous notation to show how they fit into my scheme for a general exponential potential.

For  $\beta = 1$ , the auxiliary two-metric (22) becomes the flat metric

$$d\tilde{s}^2 = \frac{1}{4}(1 - Y)dXdY = \frac{1}{4}(1 - w)dXdw = dXdZ, \quad (57)$$

where now  $X = a^2/V = a^4/w$  and  $Y = a^2V = w$ , and I have defined  $Z = (2w - w^2)/8$  to get the metric into the explicitly flat form with null coordinates  $X$  and  $Z$ .

One can show that only for an exponential potential, and then only for  $\beta = \pm 1$ , is the general auxiliary two-metric (9) flat for the general FRW-scalar model.

The generic geodesic of the flat auxiliary two-metric (57) has the form

$$X = CZ + D, \quad (58)$$

where  $C$  and  $D$  are arbitrary constants. The ‘one-boundary’ condition of regularity at the center,  $a = 0$ , implies that the exponential potential  $V$  must be finite and nonzero there, so both  $X$  and  $Z$  should vanish there, giving  $D = 0$  [19, 20]. Then

$$X \equiv \frac{a^2}{V} \equiv \frac{a^4}{w} \equiv \frac{w}{V^2} = CZ \equiv \frac{C}{8}(2w - w^2) = \frac{C}{8}(2a^2V - a^4V^2), \quad (59)$$

so

$$C = \frac{8a_b^4}{w_b^2(2 - w_b)} = \frac{8e^{-4\phi_b}}{2 - a_b^2e^{2\phi_b}}, \quad (60)$$

$$V^2 = \frac{8}{C(2 - w)} = \frac{V_b^2(2 - a_b^2V_b)}{2 - w}, \quad (61)$$

$$\begin{aligned} a^4 &= \frac{C}{8}w^2(2 - w) = a_b^4 \frac{w^2(2 - w)}{w_b^2(2 - w_b)} \\ &= \frac{1}{V^2} \left(2 - \frac{8}{CV^2}\right)^2 = \left(\frac{a_b^2V_b^3 - 2V_b^2}{V^3} + \frac{2}{V}\right)^2. \end{aligned} \quad (62)$$

One can further show that the differential equation (31), with the ‘one-boundary’ regularity condition (32) at the center, leads to the solution

$$u = \frac{-w}{4-3w} = \frac{1}{3} \left( 1 - \frac{4}{4-3w} \right) = \frac{x^2 - 1}{3x^2 + 1}. \quad (63)$$

In this case (only) there is no branch-cut ambiguity at all in  $u(w)$ , though there is a two-fold ambiguity in the action when it is complex, as there must be, since for any ‘one-boundary’ regular classical solution with real boundary values  $(a_b, \phi_b)$ , its complex conjugate, with a complex conjugate action, must also be a regular classical solution by the analyticity properties of the field equations and the regularity conditions.

As the negative of the geodesic path length in the the flat auxiliary two-metric (57), the Euclidean action can easily be evaluated to be

$$I = -\sqrt{X_b Z_b} = -\frac{1}{2} a_b^2 \sqrt{1 - \frac{1}{2} a_b^2 V_b}. \quad (64)$$

For  $0 < w_b < 2$ , one gets  $C > 0$  and a real Euclidean solution with real (negative) action. Unlike the case of  $\beta = 0$ , for which the action approached the negative value  $-1/(3V)$  at the boundary of the Euclidean region, there at  $w_b = 1$  or  $x_b = 0$ , here at the boundary of the Euclidean region (now at  $w_b = 2$  or  $x_b = \pm i$ ) the Euclidean action approaches zero. Along any real Euclidean classical solution, the action becomes negative and decreases as  $w$  increases from 0 to 1, and then the action increases back to zero as  $w$  further increases from 1 to 2. (It is interesting that the turning point for the action, at  $w = 1$ , does not coincide with the turning point for  $a$ , which increases from zero to a maximum as  $w$  increases from 0 to  $4/3$  and then decreases back to zero as  $w$  further increases from  $4/3$  to 2.)

For  $2 < w_b$ , one gets  $C < 0$ , and the geodesic in the  $(X, w)$  coordinates has, if one follows a contour of real  $w$ ,  $X$  starting off zero at zero  $w$  and then goes negative, decreasing as  $w$  increases from 0 to 1 and then increasing back to zero again as  $w$  increases from 1 to 2. In this region Eqs. (61) and (62) show that  $V^2$  and  $a^4$  are both negative, so the geometry is complex. Although the complex  $a$  returns to 0 at  $w = 2$ , the purely imaginary  $V$  goes to infinity there, giving a singularity in the metric, though one that could be avoided by choosing the contour in the complex  $w$ -plane to avoid the point  $w = 2$ .

It is interesting that this singular point in the geometry at  $w = 2$  is not a singular point, when  $\beta^2 = 1$ , of the differential equation (31), which then has singular points only at  $w = 0$ ,  $w = 1$ , and  $w = \infty$ . Conversely,  $w = 0$  is not a singularity of the geometry when the regularity condition (32) is imposed,  $w = 1$  is just a coordinate singularity in the flat auxiliary metric (57) that also gives no physical singularity, and  $w = \infty$  simply gives a universe that has expanded to infinite size and zero potential (if  $w_b > 2$  and if  $w$  is taken to  $\infty$  along the positive real axis).

If one continues to follow a contour of real  $w$  for  $w_b > 2$ , when  $w > 2$  one gets a precisely real Lorentzian four-metric,

$$ds^2 = \left( \frac{2G}{3\pi} \right) (-C)^{1/2} \frac{\sin 2\eta}{\cos^3 2\eta} \left( -d\eta^2 + d\Omega_3^2 \right), \quad (65)$$

for conformal time  $\eta$  in the range  $0 < \eta < \pi/4$ , and defined, up to sign and up to shifts by arbitrary multiples of  $\pi/2$ , by

$$w = 2 \sec^2 2\eta. \quad (66)$$

During the expansion of this universe from a big bang at  $\eta = 0$  to infinite size at  $\eta = \pi/4$ , the rescaled potential is given by

$$V = e^{\pm 2\phi} = 2(-C)^{-1/2} \cot 2\eta \quad (67)$$

and thus drops from an infinite value at the big bang to a value of zero at infinite expansion.

Unlike the case in which the potential  $V(\phi)$  is slowly varying, the exponential potential  $V = e^{2\beta\phi}$  with  $\beta^2 = 1$  does not allow one to follow a contour that keeps  $a$  real as one goes from the regular center to an asymptotic region where the geometry is asymptotically real and Lorentzian (in this case precisely real and Lorentzian). This one can see from Eq. (62) for  $a^4(w)$  with  $C < 0$ . If one starts from  $w = 0$  along a contour that keeps  $a$  real (initially having  $w$  depart from 0 in the purely imaginary direction), then if one follows the contour with  $a$  real, one will for large  $w$  have it asymptotically become a large (and growing) real number multiplied by one of the two complex cube roots of unity, rather than having it join the real axis where the metric is real and Euclidean.

From purely looking at this Lorentzian big-bang solution, which is singular at  $a = 0$ , one would hardly guess that it also obeys the ‘one-boundary’ regularity condition at  $a = 0$ , but from the analysis above we see that it does, having both a singular big bang singularity with  $a = 0$  and infinite  $V$  at  $w = 2$  or  $\eta = 0$ , and also having a regular center with  $a = 0$  and finite (though complex)  $V = V_b \sqrt{1 - w_b/2}$  at  $w = 0$  or  $\eta = \pm i\infty$ .

Thus we see that for a constant potential ( $\beta = 0$ ), an analytic continuation of the ‘one-boundary’ condition of regularity at  $a = 0$  leads to a precisely real and Lorentzian geometry, the deSitter four-metric (53), which is nonsingular everywhere and has no big bang or big crunch. For the exponential potential with  $\beta^2 = 1$ ,  $V = e^{\pm 2\phi}$  for the rescaled scalar field  $\phi$ , one also gets that an analytic continuation of the ‘one-boundary’ condition of regularity at  $a = 0$  leads to a precisely real and Lorentzian geometry, the four-metric (65), but this time it is a singular metric with a big bang in which both the potential energy and the kinetic energy densities of the scalar field are infinite.

### 6.3 Generic $\beta$ in the exponential potential $V = e^{2\beta\phi}$

For  $\beta^2$  different from 0 or 1, the singularities of the differential equation (31) generically lead to nontrivial branch cuts, except for the one at  $w = 1$ , which is typically accompanied by having  $u$  pass through  $\pm 1/\beta$  so that all quantities behave regularly there. The singular point at  $w = 0$  is also generically accompanied by having  $u$  pass through  $\pm 1/\beta$  (except at the beginning of a contour for which one imposes

the ‘one-boundary’ regularity condition to avoid a singularity there), but if indeed  $u = \pm 1/\beta$  at  $w = 0$ , this represents a curvature singularity, and going around it in different ways in the complex plane can lead to different results for  $u(w)$  (being on different Riemann sheets). The singular point at  $w = \infty$  cannot be accompanied by  $u = \pm 1/\beta$  but is accompanied by  $u = 1/3$  and represents a universe that has expanded to infinite size. It is also a branch-cut singularity, so that one gets different results for  $u(w)$  depending on the topology of the contour relative to that singularity, as well as to the one at  $w = 0$ .

For  $\beta^2 < 3$ , a solution for  $u(w)$  can be analytically continued to give an asymptotically real Lorentzian solution with large real  $w$  and nearly real  $a$ ,  $\phi$ , and rescaled proper Lorentzian time  $t_L = -i\tau$ , with

$$u \sim \frac{1}{3} + \frac{2}{9} \left( \frac{9 - \beta^2}{3 + \beta^2} \right) \frac{1}{w} + O\left(\frac{1}{w^3}\right) + C_1(\beta^2) \left(-\frac{1}{w}\right)^{\left(\frac{9-\beta^2}{6-2\beta^2}\right)} \left[1 + O\left(\frac{1}{w}\right)\right], \quad (68)$$

where  $C_1(\beta^2)$  is some function of  $\beta^2$ . One can then use this to find that the Euclidean action at large  $|w|$  asymptotically goes as

$$I \sim \frac{1}{\sqrt{9 - \beta^2} V} \left[ (-w)^{\frac{3}{2}} + \frac{3}{2} A (-w)^{\frac{1}{2}} - AC_2(\beta^2) (-w)^{-\frac{\beta^2}{3-\beta^2}} - \frac{3}{8} A^2 (-w)^{-\frac{1}{2}} \right], \quad (69)$$

where

$$A \equiv \left( \frac{\sqrt{9 - \beta^2}}{3 + \beta^2} \right). \quad (70)$$

For simplicity I have dropped the subscripts  $b$  on the boundary values, but what is written here as  $V$  and  $w$  should actually be  $V_b$  and  $w_b$ . Here  $C_2(\beta^2)$  is another function of  $\beta^2$ , derivable from  $C_1(\beta^2)$ .  $C_2(0) = C_2(1) = 1$ , and according to a preliminary approximate equation I have obtained for  $C_2(\beta^2)$  [18], apparently it lies between approximately 0.94 and 1 for  $0 \leq \beta^2 \leq 1$ .

Eq. (69) gives the behavior of the action ‘‘near’’ the singular point  $w = \infty$  (i.e., for  $|1/w| \ll 1$ ), with the Euclidean action being purely real (and positive) if one takes the contour in the complex  $w$ -plane to run along the negative real axis (which of course is a rather unphysical region with  $w \equiv a^2 V < 0$ ). Then to get to positive  $w$ , one can follow  $1/w$  around 0 in its complex plane from its negative axis to its positive axis. Let us define the integer  $n$  to be zero if one takes  $1/w$  counterclockwise half way around its zero, from the negative real axis to the positive real axis for  $1/w$  in the sense that goes below 0 in the complex plane. Then let nonzero  $n$  represent the number of excess times the contour is taken counterclockwise around  $1/w = 0$ . E.g., if the contour were then taken once clockwise around after reaching the positive real axis from the counterclockwise half-rotation, so that the net effect would be a clockwise half-rotation, passing once above 0 in going from the negative axis to the positive axis for  $1/w$ , then  $n = -1$ .

As a result of this, after reaching the positive  $w$  axis with some integer  $n$  representing the integer part of the winding number, one can make the replacement

$$\left(-\frac{1}{w}\right) = e^{(2n+1)\pi i} \left(\frac{1}{w}\right) \quad (71)$$

in the asymptotic formula (69) for the action, giving

$$I \sim -\frac{1}{3V} \left( \frac{\sqrt{9-\beta^2}}{3+\beta^2} \right) C_2(\beta^2) w^{-\frac{\beta^2}{3-\beta^2}} \left[ \cos \frac{(2n+1)\pi\beta^2}{3-\beta^2} + i \sin \frac{(2n+1)\pi\beta^2}{3-\beta^2} \right] - \frac{i(-1)^n}{\sqrt{9-\beta^2}V} \left[ w^{3/2} - \frac{3}{2}Aw^{1/2} + \frac{3}{8}A^2w^{-1/2} + O(w^{-3/2}) \right]. \quad (72)$$

This formula represents only a tiny class of possible contours for  $w$  in going from the regular center at  $w = 0$  to the asymptotically Lorentzian regime of a large universe with  $w \gg 1$ , with only the single integer  $n$ , representing how many times around the singularity at  $w = \infty$  the contour wraps. Thus this class of contours does not include the possibility of wrapping around the singularity at  $w = 0$  an arbitrary number of times between each possible wrapping around the singularity at  $w = \infty$ . That much larger class, though still not including the possibility of going around the singularity at  $u = 1/\beta^2$  (which seems to give merely a two-fold square-root ambiguity and so may have less effect than the other singularities), would be parametrized by an arbitrary sequence of integers and signs (with each integer in the sequence being the number of times wrapping around the singularity at  $w = 0$  between each time of wrapping once around the singularity at  $w = \infty$ , and with each sign being the sign of the wrapping around the singularity at  $w = \infty$ ). However, here we are not wrapping around the singularity at  $w = 0$  at all, so the small subclass being considered is characterized by the single integer  $n$ .

The first term in the Euclidean action above is the real part of the Euclidean action,  $I_R$ , which gives the magnitude of the zero-loop approximation to the wavefunction if only that history contributes,  $|e^{-I}|^2 = e^{-2I_R}$ . If  $n = 0$  or  $n = -1$ , the two simplest contours, then  $I_R$  is negative for  $\beta^2 < 1$  (assuming that  $C_2(\beta^2)$  remains positive) and vanishes for  $\beta^2 = 1$ , this last fact being consistent with the exact solution given above for  $\beta^2 = 1$ . Presumably these simplest contours then make  $I_R$  positive when  $\beta^2$  is increased beyond 1. If  $C_2(\beta^2)$  remains positive for all  $\beta^3$  up to 3, then the simplest contours would make  $I_R$  oscillate an infinite number of times as  $\beta^2$  is increased to 3, the maximum value it can have and still lead to an approximately Lorentzian universe that can expand to arbitrarily large size. For example, if the asymptotic formula (72) were accurate for the real part of the action,  $I_R$ , that quantity would oscillate and pass through zero at

$$\beta^2 = \frac{6N-3}{2N+1} \quad (73)$$

for each positive integer  $N$  (unrelated to  $n$ , which is being set equal to 0 or -1 to give the simplest possible contours), assuming that  $C_2(\beta^2)$  does not pass through infinity at any of those points to prevent  $I_R$  from passing through zero.

If the winding number  $n$  can take on arbitrary integer values, then for values of  $\beta^2$  other than those given by Eq. (73),  $I_R$  can take on both positive and negative values. This highlights the question of what classical histories are to be included if the zero-loop approximation is supposed to be a reasonable approximation to the ‘one-boundary’ wavefunction.

In models in which  $Re\sqrt{g}$ , the real part of the proper four-volume per coordinate four-volume, has a fixed sign, Halliwell and Hartle [21] proposed that one should include only classical histories in which the sign is positive. The most conservative interpretation of this in models such as those being considered here, in which the sign of  $Re\sqrt{g}$  can vary along a contour, is that one should choose the square root of this metric determinant  $g$  to give a real value in the part of contour near the center, which is the same as my choice of the sign of the square root in Eq. (41) to make my  $g(w)$  (not the determinant of the metric) positive for small  $w$  in the part of the contour near its beginning at the center,  $w = 0$ .

But one might also interpret the Halliwell-Hartle proposal as requiring that, for a monotonically increasing real time coordinate  $t$  along the complex contour for  $w$ , one have  $Re\sqrt{g} \propto Re(a^3 d\tau/dt)$  be positive along the entire contour. It remains to be seen whether this can always be satisfied for the models considered above. (It does have the ugliness of depending on the details of the contour even within the same topological class of how it winds around the various singularities, whereas one would prefer simply a restriction to a particular topological class.) But if it can, one might conjecture that it might be satisfied only for the contours given above with  $n = 1$  and  $n = -1$  (which are a complex-conjugate pair).

Alternatively, one might simply postulate that for the FRW-scalar model with an exponential potential, for large  $w_b$  one should just use the simplest single complex-conjugate pair of contours, those given above with  $n = 0$  and  $n = -1$ . Both of these have the same real part,  $I_R$ , and opposite imaginary parts, say  $I_I$  for  $n = 0$ , so  $n = 0$  gives  $I = I_R + iI_I$  and  $n = -1$  gives the complex conjugate  $I' = I^* = I_R - iI_I$ . (In general the action for some  $n$  is the complex conjugate of the action with a new integer  $n' = -n - 1$ , e.g.,  $n' = -1$  for  $n = 0$ .) Then if we take the zero-loop approximation to be given by those two contributions, we get

$$\psi_{0\text{-loop}}(a_b, \phi_b) = e^{-I} + e^{-I'} = 2e^{-I_R} \cos I_I. \quad (74)$$

However, it would be highly desirable to have some physically motivated principles for selecting a suitable set of complex classical solutions for the zero-loop approximation.

## 7 Summary

- The Hartle-Hawking ‘one-boundary’ proposal is one of the first attempts to say which wavefunction, out of all those satisfying the dynamical equations, correctly describes our universe.
- There are severe problems doing the path integral it calls for.
- At the zero-loop level it makes a number of remarkable predictions (large nearly flat Lorentzian universe, second law of thermodynamics, etc.)
- However, even here there are generically an infinite number of complex extrema to choose from, and it is not quite clear how to do this properly. Certainly

one important goal of this approach to quantum cosmology would be to give a specification of which extrema to use, and then of course one would like to compare the results with as many observations as possible.

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