A note on testing pre-inflationary times with gravitational-waves

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Abstract

A simple inflationary model based on loop quantum cosmology is considered. Within this framework, we show that inflation does not necessarily erase the information prior to its onset, but that such information may leave its imprint in the energy-spectrum of the gravitational-waves generated at these earliest of times.

It is usually assumed that inflation, by its characteristics, among them a tremendous increase in the scale of the universe taking place in an extremely short time, will effectively remove any kind of information or properties already existing prior to the onset of this phenomenon (cf., e.g., [1] and references therein). In this work we use a simple example,

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from the literature on loop quantum cosmology (LQC) [2]-[12], to show that this is not necessarily the case[1]. In particular, from the model herein analysed, although the very first quantum stages in LQC are followed by inflation, still, traces of that pre-inflationary phase survive, leaving their imprint in today's energy-spectrum of the stochastic background of gravitational waves.

Before proceeding into more technical matters, let us specify several elements that distinguish our framework. To calculate the power spectrum of the gravitational waves, we apply a method based on the Bogoliubov coefficients and the differential equations obeyed by them, a method first devised by L. Parker [14] and subsequently used in other papers [15, 16, 17, 18]. Furthermore, our model includes several stages of expansion: a pre-inflationary era, followed by inflation and then the standard stage, bringing us to the present universe, whose dark energy component is, for simplicity, represented by a cosmological constant. In addition, we avoid using the sudden transition approximation [18] and, instead, with the help of Parker's differential equations, consider the transitions between the different stages as continuous.

In the model that we are considering, the potential $U = \frac{a''}{a}$, appearing in the equations for the gravitational waves (a designates the scale of the universe and the prime its derivative with respect to conformal time), shows, in the pre-inflationary phase, a specific feature, not present in the usual model based on the classical equation for the inflaton field. This feature is a small peak in the graph of U with respect to conformal time. This small barrier gives rise to a different set of initial conditions, for those gravitons whose frequencies satisfy $k^2 < U$. This, in turn, will be reflected in the shape of the energy-spectrum in the low-frequency range, for frequencies approximately below 10^{-13} rad/s. For gravitons with frequencies above this, corresponding to energies above U, the spectrum will not be changed, when compared to the classical case.

We begin with a description of our background model, closely following [11] and [19]. In LQC, a scalar field ϕ with potential $V(\phi)$, in a flat Friedmann-Robertson-Walker background, is described by the Hamiltonian [11]

$$\mathcal{H}_\phi = a^3 V(\phi) + \frac{1}{2} d_{j,l} p_\phi^2, \tag{1}$$

where $p_\phi = d_{j,l}^{-1} \dot{\phi}$ is the momentum canonically conjugate to ϕ and $d_{j,l}(a) = D_l(q)/a^3$ gives the eigenvalues of the geometrical density operator in loop quantization; j and l are the so-called ambiguity parameters, j being a half integer quantum number greater than one and where l determines the behavior of the density operator on small scales (compared to a fundamental scale a_*, introduced below) and can take any value between zero and one. $D_l(q)$ is a quantum correction factor for the density in the semiclassical regime [20, 2]. Throughout the paper we take j as a free parameter. On the other hand, we fix $l = 3/4$.

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[1]Recently, a few papers addressing this or other related issues and frameworks, have appeared in the literature [13]

The expression for $D_{3/4}(q)$ is then given by

$$D_{3/4}(q) = \left(\frac{8}{77}\right)^6 q^{3/2}\{7[(q+1)^{11/4}-|q-1|^{11/4}]-11q[(q+1)^{7/4}-sgn(q-1)|q-1|^{7/4}]\}^6, \quad (2)$$

where the variable $q \equiv (a/a_*)^2$ and $a_* \equiv \sqrt{(\gamma j)/3} l_{Pl}$ is a fundamental length scale arising in loop quantization; γ is the Barbero-Immirzi parameter, for which calculations of the black-hole entropy give $\gamma \simeq 0.274$ [20, 2].

The LQC corrections will appear in the background equations of our model through the function $D(q)$ and its derivative $\dot{D}(q)$. These effects are particularly relevant for $q \leq 1$, when the scale factor $a \leq a_*$. For $q \gg 1$ we enter the classical regime, characterized by $D(q) \to 1$ and $\dot{D}(q) \to 0$. At this point the semiclassical regime reduces to the classical case.

We assume a universe described by a flat FRW metric and going through three stages of expansion:

(a) **The pre-inflationary and the inflationary stages:** To model it, we use the chaotic inflaton potential

$$V(\phi) = \frac{1}{2}m_\phi^2\phi^2. \quad (3)$$

The background equations are the Friedmann equation, $H = \dot{a}/a$,

$$H^2 = \frac{8\pi}{3}l_{pl}^2\left[V(\phi) + \frac{1}{2D}\dot{\phi}^2\right] \quad (4)$$

and the modified Klein-Gordon equation for the inflaton, obtained from $\mathcal{H}_\phi$,

$$\ddot{\phi} + \left(3\frac{\dot{a}}{a} - \frac{\dot{D}}{D}\right)\dot{\phi} + DV_\phi(\phi) = 0, \quad (5)$$

where we take derivatives with respect to comoving time. The classical equations are obtained putting $D = 1$ and $\dot{D} = 0$.

(b) **Transition between inflation and the radiation dominated universe:** We define the end of the inflationary era, beginning of the transition era, as the point where $\ddot{a}$ becomes negative. During the transition, the energy of the inflaton will be transferred to the radiation fluid. We model this transfer through the action of a frictional term characterized by a decay constant Γ_ϕ. The equations above are now slightly changed. The Friedmann equation becomes [21]

$$H^2 = \frac{8\pi}{3}l_{pl}^2\left[V(\phi) + \frac{1}{2}\dot{\phi}^2 + \rho_R\right] \quad (6)$$

and the equation for the scalar field is

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V_\phi(\phi) = -\Gamma_\phi\dot{\phi}. \quad (7)$$

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The equation for the radiation fluid is (we use $\Gamma_\phi \simeq 2 \times 10^{-7}$):

$$\dot{\rho}_R = -4\frac{\dot{a}}{a}\rho_R + \Gamma_\phi \dot{\phi}^2 . \tag{8}$$

At the beginning of the transition era we assume $\rho_R = 0$. We end this period when when $\rho_R a^4 = const. = \rho_{R0} a_0^4$.

(c) Standard stage: This brings us to the present universe, whose dark energy component is taken, for simplicity, to be described by a cosmological constant. We, then, integrate the equations till the present time, with the content of the universe characterized by a radiation field ρ_R, dark matter+baryonic matter, with equation of state $p = 0$, and a cosmological constant representing the dark energy. The Friedmann equation is now

$$H^2 = \frac{8\pi G}{3}\left[\rho_{R0}\left(\frac{a_0}{a}\right)^4 + \rho_{M0}\left(\frac{a_0}{a}\right)^3 + \rho_{DE0}\right], \tag{9}$$

the subscript 0 denoting the values at the present time, $a(t_0) = a_0$.

Above, we wrote our expressions using comoving time (as this is the most appropriate variable when we have the fast increase in the scale factor during inflation). However, the equations for the gravitational waves take a much simpler form when we use conformal time τ, defined by writing the FRW metric in the form

$$ds^2 = s^2(\tau)\left\{-d\tau^2 + [\delta_{ij} + h_{ij}(\tau,\mathbf{x})]\,dx^i dx^j\right\}. \tag{10}$$

The tensor perturbations h_{ij} can be expanded, in the usual manner, in terms of plane waves:

$$h_{ij}(\tau,\mathbf{x}) = \sqrt{8\pi G}\sum_{p=1}^{2}\int \frac{d^3k}{(2\pi)^{3/2}a(\tau)\sqrt{2k}}\left[a_p(\tau,\mathbf{k})\varepsilon_{ij}(\mathbf{k},p)e^{i\mathbf{k}\cdot\mathbf{x}}\xi(\tau,k) + \text{h.c.}\right], \tag{11}$$

where $\mathbf{x}$ denotes the spatial coordinates, $k = |\mathbf{k}| = 2\pi/\lambda = \omega a$ is the comoving wave number; the index p runs over the two polarizations of the gravitational waves and ε_{ij} is the polarization tensor, a_p the annihilation operator and ξ the mode function. In conformal time, the mode function obeys the equation

$$\xi'' + (k^2 - a''/a)\xi = 0, \tag{12}$$

the derivatives being with respect to τ. In what follows, we call gravitational wave potential the expression

$$U = \frac{a''}{a}. \tag{13}$$

This potential is shown in figure 1, for the semiclassical case (cf. eq. (3)-(5)). The LQC main feature appears as an additional small barrier at very early times, *prior to the onset*
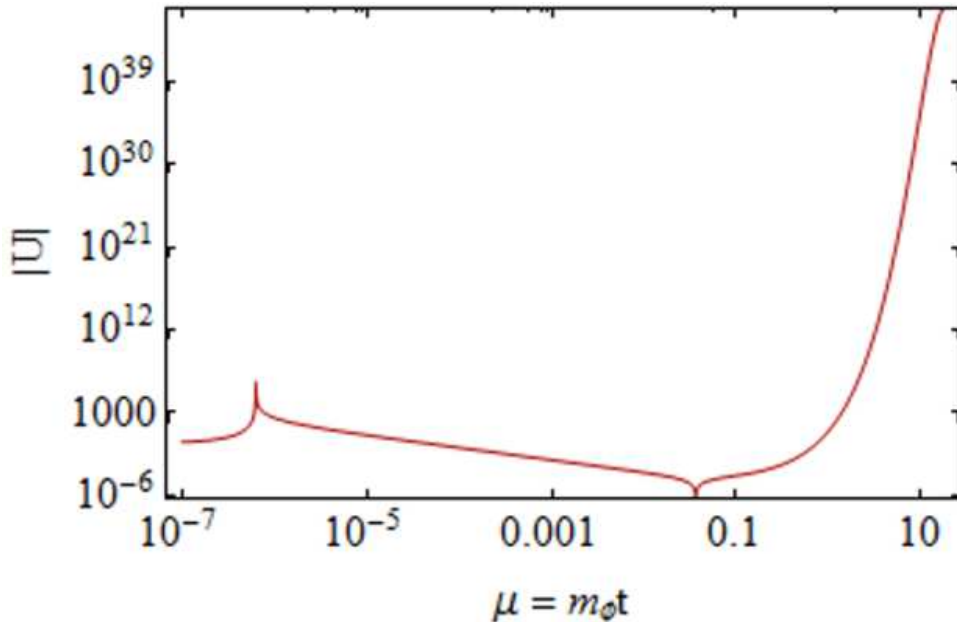
Figure 1: Gravitational wave potential $|U|$, as it appears in the LQC model herein employed.

of the inflationary stage. This small barrier gives rise to different initial conditions at the end of the pre-inflationary era, for those gravitons whose frequencies $k^2 < U$. This, in turn, will be reflected in the shape of the energy-spectrum in the low-frequency range, as will be seen.

More generically, for appropriate forms of $a''(\tau)/a(\tau)$ we may have exponentially growing solutions, with the gravitational field pumping energy into the gravitational waves. From the point of view of quantum mechanics, this corresponds to graviton production. In fact, equation (12) closely mimics a Schrödinger equation with potential barrier $U(\tau) = a''(\tau)/a(\tau)$ (with the time coordinate τ instead of the spatial coordinate). It is an important point to realize that, whenever $k^2 \lesssim U(\tau)$, the conditions exist for a significant graviton production, the equation for the gravitational waves being then of the form of a parametric oscillator, while, when $k^2 \gg U(\tau)$, the equation for the gravitational waves is that of a harmonic oscillator, with no gravitons being produced.

We now express the creation and annihilation operators in terms of the initial creation and annihilation operators $A_p^\dagger(\mathbf{k})$ and $A_p(\mathbf{k})$, through the Bogoliubov coefficients $\alpha(k,\tau)$ and $\beta(k,\tau)$ [14], [22]:

$$a_p(\tau,\mathbf{k}) = \alpha(\tau,k)A_p(\mathbf{k}) + \beta^*(\tau,k)A_p^\dagger(\mathbf{k}), \tag{14}$$

α and β satisfying the relation

$$|\alpha|^2 - |\beta|^2 = 1. \tag{15}$$

These coefficients, which in the sudden transition approximation (cf. [18, 23]) are calculated by requiring the mode functions and their derivatives to be continuous across the transition,

obey the following couple of differential equations [14]

$$\alpha' = \frac{i}{2k} \left[\alpha + \beta e^{2ik(\tau - \tau_i)} \right] \frac{a''}{a} \tag{16}$$

and

$$\beta' = -\frac{i}{2k} \left[\beta + \alpha e^{-2ik(\tau - \tau_i)} \right] \frac{a''}{a}, \tag{17}$$

which, with a change to the variables $X(k, \tau)$ and $Y(k, \tau)$

$$\alpha = \frac{1}{2}(X + Y)e^{ik(\tau - \tau_i)} \tag{18}$$

$$\beta = \frac{1}{2}(X - Y)e^{-ik(\tau - \tau_i)}, \tag{19}$$

become

$$X' = -ikY \tag{20}$$

$$Y' = -\frac{i}{k} \left(k^2 - \frac{a''}{a} \right) X, \tag{21}$$

completing our system of equations. These variables obey the constraint equation (15), which takes the form

$$X_r Y_r + X_i Y_i = 1, \tag{22}$$

the subscripts r and i denoting the real and imaginary parts. In terms of the new variables, $|\beta|^2$ is

$$|\beta|^2 = \frac{1}{4} \left[(X_r - Y_r)^2 + (X_i - Y_i)^2 \right]. \tag{23}$$

During the LQC dominated phase, we have, as mentioned, an extra barrier (cf. Fig. 1). We verified numerically that the appropriate equation of state is $p_\phi \simeq \rho_\phi$, in which case an analytical solution of equations (20) and (21) can be found and is of the form

$$X = \sqrt{\frac{\pi \theta}{2}} \left[-J_0(\theta) + iY_0(\theta) \right], \tag{24}$$

$$Y = \sqrt{\frac{\pi}{8\theta}} \left\{ -Y_0(\theta) + 2\theta Y_1(\theta) + i \left[J_0(\theta) - 2\theta J_1(\theta) \right] \right\}, \tag{25}$$

where J_n and Y_n are the Bessel functions of the first and second kind, respectively. In this case, $|\beta|^2$ has a more complicated form, but it also satisfies the limits $|\beta|^2 \to 0$, when $k \to \infty$, and $|\beta|^2 \to \infty$, when $k \to 0$. Due to the smallness of the extra barrier, only the low frequency gravitons will be affected. For the higher frequency modes, only the inflationary barrier will be important.

Initial conditions for the scale factor a_i and for $\dot{\phi}_i$ need to be set, with ϕ_i being obtained from $\dot{\phi}_i$ with the help of the uncertainty principle (see [11])

$$|\phi_i \, \dot{\phi}_i \,| \gtrsim \frac{10^3}{j^{3/2}} \left(\frac{a_i}{a_*} \right)^{12} m_{Pl}^3. \tag{26}$$

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The scale factor is $a_i \geq l_{Pl}$, otherwise the semiclassical equations would not be valid, due to the discretization of the geometry of space introduced by LQG. We therefore take $a_i = \sqrt{\gamma} l_{Pl}$.

We choose parameters and initial conditions such that we get enough inflation (number of e-folds $\simeq 70$) and satisfy the large-scale CMB anisotropies requirements [1]: $m_\phi \simeq 10^{-6} m_P$ and $\phi_{max} \gtrsim 3 m_P$. We still have the ambiguity parameter j, as a free parameter.

In the inflationary era we used, for convenience, Planck units and worked with the adimensional time $\mu = m_\phi t$. Finally, we used the same values for those parameters which are common to both models, $m_\phi = 10^{-6}$, $\Gamma_\phi = 2 \times 10^{-7}$ and $\dot{\phi}_i / m_\phi = 2$ for the semiclassical and $\phi_i = 3.3$ for the classical cases.

It is now possible to integrate the full set of equations, including the equations for X and Y, (20) and (21). The number of gravitons of gravitons created is, as is well known, given [14, 16, 18] by $|\beta_{final}|^2$. Taking into account that the density of states is $\omega^2 d\omega/(2\pi^2 c^3)$ and that each graviton contributes with two polarizations $2\hbar\omega$, then, from the definition of the energy density $dE = P(\omega)d\omega$, we have the following expression for the power-spectrum $P(\omega)$:

$$P(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3} |\beta_{final}|^2. \tag{27}$$

We choose to present our results in terms of the dimensionless relative logarithmic energy-spectrum of the gravitational waves, at present time τ_0 :

$$\Omega(\omega_0, \tau_0) = \frac{1}{\rho_{crit}(\tau_0)} \frac{d\rho_{gw}}{d \ln \omega}(\tau_0) = \frac{8\hbar G}{3\pi c^5 H^2(\tau_0)} \omega_0^4 |\beta_{final}|^2, \tag{28}$$

ρ_{crit} being the value of the present time critical density and ρ_{gw} the gravitational wave energy density

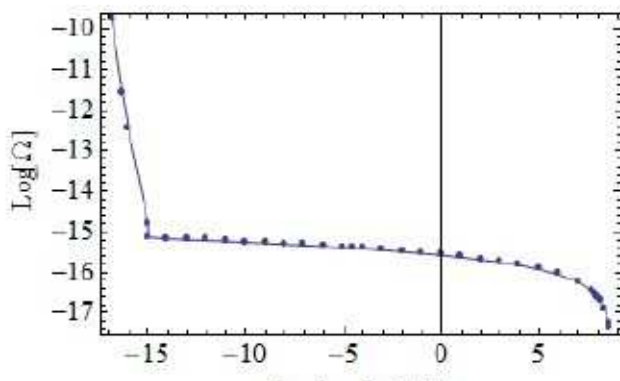$$\rho_{gw} = \int P(\omega) d\omega. \tag{29}$$

Figure 2: The energy-spectrum of the gravitational waves, for the LQC model employed in this paper, showing the sharp decrease in the low-frequency part of the spectrum.

Our results for the energy-spectrum of the gravitational waves are plotted in figure 2. For purposes of illustration, we defined values for the parameters of the classical model

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such that the results would approximately comply with the total amount of anisotropy measured by COBE which, in terms of Ω, is

$$\Omega \lesssim 1.37 \times 10^{-10}, \tag{30}$$

for frequencies corresponding to the present horizon size ω_{hor}, where we use $H = 71 \, km \, s^{-1} \, Mpc^{-1}$, giving $\omega_{hor} \simeq 10^{-17} \, rad \, s^{-1}$. We immediately notice that the gradient of the low-frequency part of the spectrum is much steeper than in the classical case, a change that is due to the extra barrier in Figure 1. This shows that physical features present in the pre-inflationary eras are not erased out by inflation, as is commonly assumed, but may leave their imprint in the spectrum of the gravitational waves[2], which was the main point of our brief note. We leave to a future paper a more thorough investigation of this topic.

Acknowledgements

References

[1] A. R. Liddle and D. H. Lyth, Cosmological inflation and large-scale structure, Cambridge University Press (2000).

[2] Martin Bojowald, *Living Rev. Relativity*, **11**, (2008), 4.

[3] Bojowald, M.; Class. Quant. Grav. 17, 1489 (2000);

[4] Bojowald, M.; Class. Quant. Grav. 18, 1071 (2001);

[5] Bojowald, M.; Class. Quant. Grav. 18, L109 (2001);

[6] Bojowald, M.; Class. Quant. Grav. 19, 2717 (2002);

[7] Bojowald, M.; Phys. Rev. Lett. 86, 5227 (2001);

[8] Bojowald, M.; Phys. Rev. Lett. 87, 121301 (2001);

[9] Bojowald, M.; Phys. Rev. Lett. 89, 261301 (2002);

[10] Velhinho, J.; Class. Quant. Grav. 21, L109 (2004);

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[2]Recently, a range of publications focusing on gravitational waves within LQC have appeared in the literature [24]-[37]. However, neither of them has pointed and addressed the issue of physical features from a pre-inflationary epoch being still extractable in observations. Most deal with inflation in LQC and the production and propagation of gravitational waves *during* inflation.

[11] Shinji Tsujikawa, Parampreet Singh, and Roy Maartens, Class. Quantum Grav. **21** (2004) 5767.

[12] M. Bojowald and G. M. Hossain, Phys. Rev. D **77** (2008) 023508 [arXiv:0709.2365 [gr-qc]].

[13] J. Mielczarek, T. Cailleteau, J. Grain, A. Barrau, arXiv:1003.4660v1; L. Krauss, S. Dodelson, S. Meyer arXiv:1004.2504v1; D. Baskaran, L. P. Grishchuk, W. Zhao, arXiv:1004.0804v1; C. G. Tsagas, Phys. Rev. **D** 81, 043501 (2010);I.A. Pedrosa, K. Bakke, C. Furtado, Phys. Lett. **B** 671 (2009) 314-317.

[14] L. Parker, Phys. Rev. **183**, 1057 (1969).

[15] L. M. E. Mendes, Density Perturbations and Gravitational Waves in Cosmology, PhD. Thesis, Instituto Superior Técnico - Universidade Técnica de Lisboa (1997).

[16] Alfredo B. Henriques, Class. Quantum Grav. **21** (2004) 3057-3065.

[17] R. G. Moorhouse, A. B. Henriques, and L. E. Mendes, Phys. Rev. D **50**, 2600 (1994).

[18] B. Allen, Phys Rev D, Vol37, 2078 (1988),

[19] M. Bojowald and H. A. Morales-Tecotl, Lect. Notes Phys. **646**, 421 (2004) [arXiv:gr-qc/0306008].

[20] Claus Kiefer, Quantum Gravity, Oxford University Press (2004).

[21] P. V. Moniz, J. M. Mourão, and P. M. Sá, Class. Quantum Grav. **10** (1993) 517-534.

[22] V. F. Mukhanov, and S. Winitzki, Introduction to Quantum Effects in Gravity, Cambridge University Press (2007).

[23] M. Giovannini, Phys. Rev. D60 (1999), 123511.

[24] E. J. Copeland, D. J. Mulryne, N. J. Nunes and M. Shaeri, Phys. Rev. D **79** (2009) 023508 [arXiv:0810.0104 [astro-ph]].

[25] J. Mielczarek, Phys. Rev. D **79** (2009) 123520 [arXiv:0902.2490 [gr-qc]].

[26] J. Mielczarek, arXiv:0908.4329 [gr-qc].

[27] D. J. Mulryne and N. J. Nunes, Phys. Rev. D **74** (2006) 083507 [arXiv:astro-ph/0607037].

[28] M. Shimano and T. Harada, Phys. Rev. D **80** (2009) 063538 [arXiv:0909.0334 [gr-qc]].

[29] J. Grain and A. Barrau, Phys. Rev. Lett. **102** (2009) 081301 [arXiv:0902.0145 [gr-qc]].

[30] J. Grain, A. Barrau and A. Gorecki, Phys. Rev. D **79** (2009) 084015 [arXiv:0902.3605 [gr-qc]].

[31] J. Grain, arXiv:0911.1625 [gr-qc].

[32] J. Grain, T. Cailleteau, A. Barrau and A. Gorecki, arXiv:0910.2892 [gr-qc].

[33] A. Barrau and J. Grain, arXiv:0805.0356 [gr-qc].

[34] J. Mielczarek and M. Szydlowski, Phys. Lett. B **657** (2007) 20 [arXiv:0705.4449 [gr-qc]].

[35] G. Calcagni and G. M. Hossain, Adv. Sci. Lett. **2** (2009) 184 [arXiv:0810.4330 [gr-qc]].

[36] J. Mielczarek, JCAP **0811** (2008) 011 [arXiv:0807.0712 [gr-qc]].

[37] J. Grain, T. Cailleteau, A. Barrau and A. Gorecki, arXiv:0910.2892 [gr-qc].