Cosmological constant and vacuum energy: old and new ideas

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Abstract. The cosmological constant (CC) term in Einstein's equations, Λ, was first associated to the idea of vacuum energy density. Notwithstanding, it is well-known that there is a huge, in fact appalling, discrepancy between the theoretical prediction and the observed value picked from the modern cosmological data. This is the famous, and extremely difficult, "CC problem". Paradoxically, the recent observation at the CERN Large Hadron Collider of a Higgs-like particle, should actually be considered ambivalent: on the one hand it appears as a likely great triumph of particle physics, but on the other hand it wide opens Pandora's box of the cosmological uproar, for it may provide (alas!) the experimental certification of the existence of the electroweak (EW) vacuum energy, and thus of the intriguing reality of the CC problem. Even if only counting on this contribution to the inventory of vacuum energies in the universe, the discrepancy with the cosmologically observed value is already of 55 orders of magnitude. This is the (hitherto) "real" magnitude of the CC problem, rather than the (too often) brandished 123 ones from the upper (but fully unexplored!) ultrahigh energy scales. Such is the baffling situation after 96 years of introducing the Λ-term by Einstein. In the following I will briefly (and hopefully pedagogically) fly over some of the old and new ideas on the CC problem. Since, however, the Higgs boson just knocked our door and recalled us that the vacuum energy may be a fully tangible concept in real phenomenology, I will exclusively address the CC problem from the original notion of vacuum energy, and its possible "running" with the expansion of the universe, rather than venturing into the numberless attempts to replace the CC by the multifarious concept of dark energy.

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1 Introduction

Undoubtedly the most prominent performance of modern cosmology has been to provide observational evidence for the accelerated expansion of the universe [1, 2, 3] and for the existence of (other) large scale dynamical phenomena possibly caused by forms of matter beyond the usual baryonic component. To "explain" the accelerated evolution the name "dark energy" (DE) was coined; it refers to some mysterious form of diffuse (i.e. non-clustering) energy presumably permeating all corners of the universe, possessing negative pressure and thus being capable of boosting the expansion of the universe as a whole. Similarly, to "explain" the anomalous dynamics of galaxies and of galaxy clusters, we have imagined that there is a large deficit of matter at different astronomical scales in the form of unknown stable particles, which are neither electrons nor protons, not even neutrinos, but some form of electrically neutral heavy stuff beyond the spectrum of the Standard Model (SM) of the strong and electroweak interactions, and referred to as "dark matter" (DM) particles. We do not yet know if any of these hypotheses is true at all, although new hints (not realities, yet) might be around recently; the only thing we know for sure is the reality of the observed physical phenomena that we are trying to explain. In the following I will not elaborate on the whereabouts of the hypothetical DM particles, I will rather focus on a few aspects of the DE problem, or more specifically the cosmological constant (CC) problem [4, 5], which is perhaps the most intriguing of all cosmological puzzles – and is not obviously unrelated to the DM one.

It is often stated in the literature that the CC term, and its association with the notion of vacuum energy, cannot be a valid theoretical explanation for the accelerated expansion of the universe, and that we necessarily have to "go beyond Λ". The adduced reasons are manyfold, but perhaps the most brandished one is that the various contributions to the vacuum energy cannot possibly be successfully fine tuned to the measured value by any known mechanism, and therefore the idea of the vacuum energy and its connection with the CC is viewed as completely unnatural. Barring the fact that this need not be true, since there are dynamical mechanisms within modified gravity that could efficiently help here, starting from an arbitrarily large Λ [6], it is nevertheless a curious "reason" to wield, as usually nothing more fundamental is offered as an alternative, except a defense (tooth and nail) of some particular form of DE, say from quintessence to string landscape. Unfortunately, as we know, none of these alternatives seem to improve in any practical way the fine-tuning illness [4] – a very serious matter by the way, which for some (no less) mysterious reason is unjustly blamed to the CC option almost exclusively. In fact, the fine tuning problem in such new frameworks not only does not become milder but it gets even worse than in the CC case, simply because the traditional vacuum energy of the SM is still there, and so one has to cope with its fine tuning, plus the (no less severe) one associated to the field or string object (usually linked to some form of high energy physics) replacing the CC term. As a result the two fine tunings make the overall job even more bizarre. Let alone that, quite often, in these frameworks an extremely light new particle is predicted in the ballpark of $\sim 10^{-33}$ eV. However, we should seriously worry about the fact that such (incommensurably tiny!) mass scale is some 30 orders of magnitude smaller than the mass scale which these models aim to explain – namely the millielectronvolt ($\sim 10^{-3}$ eV) mass scale associated to the CC term! Why such strategy is not perceived as trying to solve a big problem by creating an even major one? The answer is perhaps another profound mystery of Nature; quite likely it must be that the CC problem is such a disproportionately big problem that we are – too soon – ready to redefine dramatically the scope and limits of our physical perceptions.

The "instinctive" option of replacing the vacuum energy by alternative theoretical constructs can be counterproductive, though, because in doing so many people (consciously or unconsciously) may give up the duty of explaining why the usual vacuum energy of QFT (say the SM of Particle Physics) does not participate at all in accounting for the value of the CC. Please notice that after the likely discovery of a Higgs boson at the LHC collider [7], the reality of the vacuum energy associated to the spontaneous symmetry breaking mechanism of the electroweak theory starts to acquire a very palpable reality. If we wish to face the CC problem in earnest, we should somehow move on and stop leaving the vacuum energy of the SM in the most complete oblivion, literally as if the mere fact of not thinking or talking about it would make it completely disappear from our world! If we think seriously about it, wouldn't this attitude be more typical of an inhabitant of some "Ostrichland"?

In the following I shall dwell on the properties of the CC term in Einstein's equations and generalizations thereof. Our main aim here is to consider models where the prime driving force accelerating the universe is dynamical vacuum energy and hence a time variable CC. I will also discuss some intriguing phenomenological implications of the dynamical vacuum framework as a potential source for a mild variability of the fundamental "constants" of Nature, which could help in effectively testing these ideas. For a more detailed exposition, see e.g. [8]; and for a summarized introduction to time evolving vacuum models along these lines, see [9].

2 DE and Einstein's original "constant cosmological constant"

Historically, the Λ-term in the field equations was introduced by A. Einstein 96 years ago [10], but the "CC problem" as such was formulated 50 years later by Y. B. Zeldovich [11]. The latter is the realization that the quantum theory applied to the world of the elementary particles seems to predict an effective value for Λ which is much larger than the critical density of the universe (to which the vacuum energy density is found to be comparable) – see the reviews [4, 5]. The first models trying to circumvent this tough difficulty from a fundamental quantum field theory (QFT) point of view – confer e.g. [12, 13, 14, 15, 16] – tried to use dynamical scalar fields; they tried to explain the small value of the CC density $\rho_\Lambda = \Lambda/(8\pi\, G)$, assuming that the latter was actually the energy density value of a cosmic scalar field (called "cosmon" in one of the formulations [13]) starting with a huge value in the early universe and then eventually settling it down dynamically (hence without fine tuning) to the present current value [3] $\rho_\Lambda^0 \simeq 2.5\times 10^{-47}$ GeV^4. No model of this kind ever succeeded in achieving that main aim. Later on the first quintessence models bearing this name appeared in the market [17], but with a much more modest aim: they did not try to explain the value of ρ_Λ^0, but just the cosmic coincidence problem, i.e. the reason why the current value of the CC density is so close to the present matter density – see [18] for a comprehensive exposition and more references. However, not even this lower target was ever reached satisfactorily. Nor could be reached either by the many phenomenological proposals for a time dependent cosmological term or other alternative notions [19] – see e.g. the reviews [20] and [21]. Subsequently (in the years after the discovery of the accelerated expansion of the universe [1]) a new and powerful wave of proposals invaded all corners of the cosmological literature: the multifarious notion of dark energy (DE) was born, its main purpose being to replace (in infinitely many different disguises) the function made by the Λ-term in Einstein's equations. And here we are, still fighting with the phenomenal cosmological problems triggered by Einstein's first idea to introduce the Λ term in his almost centennial General Relativity framework! Somehow Einstein knew he was going to raise such a pandemonium, as he felt that by introducing Λ in the field equations he was in danger of being interned in a madhouse: *"Ich habe wieder etwas verbrochen in der Gravitationstheorie, was mich ein wenig in Gefahr bringt, in ein Tollhaus interniert zu werden"* (A. Einstein, Letter to P. Ehrenfest, February 4th 1917).

Almost a century ago the gravitational field equations (*Die Feldgleichungen der Gravitation*) were first introduced by A. Einstein in 1915, with no cosmological term at all [22]:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 8\pi G\, T_{\mu\nu}\,. \tag{2.1}$$

It was only two years later when Einstein, in an attempt to describe the largely accepted idea, at that time, of a finite, static and closed universe hypothetical fulfilling Mach's principle, introduced the Λ term [10] and modified his field equations in the form we still write them today:

$$G_{\mu\nu} - g_{\mu\nu}\,\Lambda = 8\pi G\, T_{\mu\nu}\,. \tag{2.2}$$

The reason why he was able to introduce that term is because it is perfectly consistent with the idea of general covariance, which was the building principle of GR. This is mathematically expressed by the fact that the covariant derivative on both sides of the original field equations (2.1) gives zero: indeed $\nabla^{\mu}G_{\mu\nu} = 0$ is always (automatically) satisfied on account of the Bianchi identity of the Riemann tensor; whereas the covariant derivative on the right-hand-side can be satisfied in different ways, the simplest one being perhaps to assume that G (Newton's gravitational coupling) is a fundamental constant, and that matter is covariantly conserved (i.e. $\nabla^{\mu}T_{\mu\nu} = 0$). Under these conditions, if we compute once more the covariant derivative on both sides, but this time on the modified field equations (2.2), and under the same set of assumptions (on fixed G and matter conservation), we are immediately led to

$$\nabla_{\mu}\Lambda = \partial_{\mu}\Lambda = 0 \qquad \Rightarrow \qquad \Lambda = \text{const}\,. \tag{2.3}$$

This is of course what justifies the name "cosmological constant" to the parameter Λ introduced by Einstein in Eq. (2.2). But the justification is only partial, as it should be clear from our carefully stating the conditions under which it has been derived. If G is not constant and/or matter is not covariantly conserved (both of them being assumptions which should not be rejected too fast) then the canonical conclusion (2.3) is not guaranteed at all. A time dependent Λ, or more precisely, a spatially homogeneous function of the cosmic time, $\Lambda = \Lambda(t)$, would still be perfectly compatible with the Cosmological Principle. However, in order to still fulfill the Bianchi identity we would need either a time dependent gravitational coupling, $G = G(t)$, or to admit the possibility that matter exchanges energy with vacuum (hence that matter is not self-conserved, in a locally covariant sense), or a combination of the two possibilities. While some of these possibilities may look bizarre at first sight, it is no less bizarre than the possibility that the fundamental "constants" of Nature might not be constant after all. There are actually some hints that this could be the case – and also that the two "bizarre stories" could in fact be deeply related [24] – see Sect. 9.

The Cosmological Principle is based on the homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric [23]

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-K\,r^2} + r^2\,d\Omega^2\right]\,, \tag{2.4}$$

with $d\,\Omega^2 = r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2$. The basic cosmological equations emerging from Einstein's field equations with Λ-term (2.2) in the FLRW metric are well-known. Adopting for matter the energy-momentum tensor for a perfect cosmic fluid,

$$T_{\mu\nu} = (\rho_m + p_m)U_\mu U_\nu - p_m\,g_{\mu\nu}\,, \tag{2.5}$$

and computing the various components of the geometric tensors in (2.2), we find the desired result, which is summarized in two fundamental equations. On the one hand we have

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_m + \frac{\Lambda}{3} - \frac{K}{a^2}\,, \tag{2.6}$$

where the constant K is the spatial curvature parameter appearing in (2.4), and on the other

$$\frac{\ddot{a}}{a} = -\frac{4\pi\,G}{3}\,(\rho_m + 3p_m) + \frac{\Lambda}{3}\,. \tag{2.7}$$

Here $a = a(t)$ is the scale factor of the FLRW metric (2.4). The first equation (2.6) is called the Friedmann-Lemaître equation, whereas equation (2.7) is the acceleration equation.

If we would take the "simplest" possibility conceived by Einstein, namely a strictly constant Λ and spherical symmetry of the three-dimensional space – entailing $K = +1$ in Eq. (2.6) – one can easily derive the explicit form of the field equations in terms of just the constant matter density ρ_m, the newly introduced Λ-term and the scale factor a of the metric. We can, of course, recover Einstein's universe as a very particular case of the above dynamical equations (2.6)-(2.7). Indeed, assuming a universe made of dust (hence zero pressure, $p_m = 0$) and imposing equilibrium (static universe), i.e. $H = 0$ and $\ddot{a} = 0$, a simple relation ensues immediately between these three quantities: $4\pi G\rho_m = 1/a^2 = \Lambda$, or equivalently

$$\rho_m = \frac{1}{4\pi G\,a^2} = 2\,\rho_\Lambda\,, \tag{2.8}$$

where we recall that $\Lambda = 8\pi G\rho_\Lambda$. Clearly, it is thanks to the assumed nonvanishing, and positive, Λ-term that such relation is possible. This made Einstein happy, but his happiness was ephemeral. It is not only that E. Hubble soon provided evidence that our universe is actually expanding, but the fact (immediately noticed by A. Eddington) that even in the absence of this information the relation (2.8) is completely unrealistic, for it corresponds to an unstable position of equilibrium! It means that the slightest perturbation in the value of ρ_m around the one satisfying (2.8) triggers a runaway solution that kicks forever the "marble off the hill's top"! This result is perfectly intuitive. Indeed, the gravitational law corrected with the Λ-term reads

$$\mathbf{g}(r) = -G\,\frac{M}{r^2}\mathbf{u}_r + \frac{1}{3}\,\Lambda\, r\, \mathbf{u}_r\,, \tag{2.9}$$

where $\mathbf{u}_r$ is a unit vector directed radially outwards with respect to the position of the body of mass M creating the field. Thus, if for some reason this universe would expand slightly ($r \to r + \delta r$), this would diminish the gravitational attraction but at the same time would enhance the repulsive Λ-force because the latter is larger the larger is the separation between particles, if $\Lambda > 0$ – as implicit in Eq. (2.8). So there is no possible compensation between the two. As a result the original seed expansion, no matter how small it is, would destabilize the universe into a runaway expansion. Similarly, an initial seed contraction ($r \to r - \delta r$) would cause the universe to shrink indefinitely into the "Big Crunch".

Paradoxically, ten years later after Λ was introduced by Einstein to insure a static non-evolving universe, G. Lemaître [25] used the form (2.2) with nonvanishing Λ to discuss his dynamical models of the expansion of the universe, strongly motivated by E. Hubble's observations prior their publication in 1929. The same models had already been independently discussed (without Λ) a few years before by A. Friedmann [26] on pure mathematical grounds and with no connection whatsoever with observations. In 1931, fourteen years after Einstein had introduced the Λ-term, he finally rejected it [27], as if the observational evidence collected by Hubble against his original static model of the universe should have any impact at all on the theoretical status of the Λ term. As we know, 96 years after its introduction in the gravitational field equations, Λ is still there, "alive and kicking", we like it or not!

3 The electroweak Higgs vacuum in classical field theory

In the following we summarize the old CC problem and perform a preliminary discussion of the associated fine-tuning problem, leaving the more sophisticated effects for the subsequent sections. We wish to illustrate the problem within the context of the standard model (SM) of particle physics, and more specifically within the Glashow-Weinberg-Salam model of electroweak interactions [28]. This is the most successful QFT we have at present (together with the QCD theory of strong interactions [29]), both theoretically and phenomenologically, and therefore it is the ideal scenario where to formulate the origin of the problem. As is well-known, the unification of weak and electromagnetic interactions into a renormalizable theory requires to use the principle of local gauge symmetry in combination with the phenomenon of spontaneous symmetry breaking (SSB) [30]. It is indeed the only known way to generate all the particle masses by preserving the underlying gauge symmetry. In the SM, one must introduce a fundamental complex doublet of scalar fields. However, in order to simplify the discussion, let us just consider a field theory with a real single scalar field ϕ, as this does not alter at all the nature of the problem under discussion. To trigger SSB, one must introduce a potential for the field ϕ, which in renormalizable QFT takes the form (the tree-level Higgs potential):

$$V(\phi) = \frac{1}{2}\,m^2\,\phi^2 + \frac{1}{4!}\,\lambda\,\phi^4 \qquad (\lambda > 0)\,. \tag{3.1}$$

Since we are dealing with a problem related with the CC, we must inexcusably consider the influence of gravity. To this effect, we shall conduct our investigation of the CC problem within the semiclassical context, i.e. from the point of view of quantum field theory (QFT) in curved space-time [31, 32]. It means that we address the CC problem in a framework where gravity is an external gravitational field and we quantize matter fields only [33]. The potential in equation (3.1) is given at the moment only at the classical level, but it will eventually acquire quantum effects generated by the matter fields themselves (cf. Sect.6). In this context, we need to study what impact the presence of such potential may have on Einstein's equations both at the classical and at the quantum level.

Einstein's field equations for the classical metric in vacuo are derived from the Einstein-Hilbert (EH) action with a cosmological term $\Lambda^{(b)}$ (hereafter the CC *vacuum term*). The EH action in vacuo reads [23]:

$$S_{\rm EH} = \frac{-1}{16\pi\, G}\int d^4x\sqrt{-g}\,\left(\,R+2\,\Lambda\,\right) = -\int d^4x\sqrt{-g}\,\left(\frac{1}{16\pi\, G}\,R+\rho_{\Lambda{\rm vac}}\right). \tag{3.2}$$

Here we have defined $\rho_{\Lambda{\rm vac}}$, the energy density associated to the CC vacuum term:

$$\rho_{\Lambda{\rm vac}} = \frac{\Lambda}{8\pi\, G_N}\,. \tag{3.3}$$

The classical action including the scalar field ϕ with its potential (3.1) is

$$S = S_{EH} + \int d^4x\,\sqrt{|g|}\,\left[\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right]\,. \tag{3.4}$$

Due to the usual interpretation of Einstein's equations as an equality between geometry and a matter-energy source, it is convenient to place the $\rho_{\Lambda{\rm vac}}$ term as a part of the matter action, $S[\phi]$. Then the total action (3.4) can be reorganized as

$$S = \frac{1}{16\pi\, G_N}\int d^4x\sqrt{|g|}\,R + S[\phi]\,, \tag{3.5}$$

with

$$S[\phi] = \int d^4x\,\sqrt{|g|}\,\left[\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \rho_{\Lambda{\rm vac}} - V(\phi)\right] \equiv \int d^4x\,\sqrt{|g|}\,\mathcal{L}_\phi\,, \tag{3.6}$$

where $\mathcal{L}_\phi$ is the matter Lagrangian for ϕ. For the moment, we will treat the matter fields contained in $\mathcal{L}_\phi$ as classical fields, and in particular the potential V is supposed to take the classical form (3.1) with no quantum corrections. If we compute the energy-momentum tensor of the scalar field ϕ in the presence of the vacuum term $\rho_{\Lambda{\rm vac}}$, let us call it $\tilde{T}^{\phi}_{\mu\nu}$, we obtain

$$\tilde{T}^{\phi}_{\mu\nu} = \frac{2}{\sqrt{|g|}}\,\frac{\delta\, S[\phi]}{\delta\, g^{\mu\nu}} = 2\,\frac{\partial\mathcal{L}_\phi}{\partial\, g^{\mu\nu}} - g_{\mu\nu}\,\mathcal{L}_\phi = g_{\mu\nu}\,\rho_{\Lambda{\rm vac}} + T^{\phi}_{\mu\nu}\,, \tag{3.7}$$

where we have used $\partial\sqrt{|g|}/\partial g^{\mu\nu} = -(1/2)\sqrt{|g|}\,g_{\mu\nu}$. Here

$$T^{\phi}_{\mu\nu} = \left[\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\,g_{\mu\nu}\;\partial_\sigma\phi\,\partial^\sigma\phi\right] + g_{\mu\nu}\,V(\phi) \tag{3.8}$$

is the ordinary energy-momentum tensor of the scalar field ϕ.

In the vacuum (i.e. in the ground state of ϕ) there is no kinetic energy, so that the first term on the *r.h.s* of (3.8) does not contribute in that state. Only the potential can take a non-vanishing vacuum expectation value, which we may call $\langle V(\phi)\rangle$. Thus, the ground state value of (3.7) is

$$\langle\tilde{T}^{\phi}_{\mu\nu}\rangle = g_{\mu\nu}\,\rho_{\Lambda\mathrm{vac}} + \langle T^{\phi}_{\mu\nu}\rangle = g_{\mu\nu}\left(\rho_{\Lambda\mathrm{vac}} + \langle V(\phi)\rangle\right) \equiv \rho^{\mathrm{cl}}_{\mathrm{vac}}\, g_{\mu\nu} \qquad (m^2<0)\,, \tag{3.9}$$

where $\rho^{\mathrm{cl}}_{\mathrm{vac}}$ is the *classical vacuum energy* in the presence of the field ϕ.

If $m^2>0$ in equation (3.1), then $\langle\phi\rangle = 0 \;\Rightarrow \langle V(\phi)\rangle = 0$, and there is no SSB. The classical vacuum energy is just the original $\rho_{\Lambda\mathrm{vac}}$ term,

$$\langle\tilde{T}^{\phi}_{\mu\nu}\rangle = g_{\mu\nu}\,\rho_{\Lambda\mathrm{vac}} \qquad (m^2>0)\,. \tag{3.10}$$

This result also applies in the free field theory case. However, if the phenomenon of SSB is active, which precisely occurs when $m^2<0$, we have a non-trivial ground-state value for ϕ, or vacuum expectation value (VEV):

$$v \equiv \langle\phi\rangle = \sqrt{\frac{-6\,m^2}{\lambda}}\,. \tag{3.11}$$

In this case, there is an *induced part* of the vacuum energy at the classical level owing to the electroweak phase transition generated by the Higgs potential. This transition induces a non-vanishing contribution to the cosmological term which is usually called the "induced CC". At the classical level, it is given by

$$\rho_{\Lambda\mathrm{ind}} \equiv \langle V(\phi)\rangle = -\frac{3\,m^4}{2\lambda} = \frac{1}{4}\,m^2\,v^2 = -\frac{1}{8}\,M^2_{\mathcal{H}}\,v^2 = -\frac{1}{8\sqrt{2}}\,M^2_{\mathcal{H}}\,M^2_F\,, \tag{3.12}$$

In the last equation we have used the physical Higgs mass squared:

$$M^2_{\mathcal{H}} = \left.\frac{\partial^2\,V(\phi)}{\partial\phi^2}\right|_{\phi=v} = m^2 + \frac{1}{2}\,\lambda\,v^2 = -2m^2 > 0\,. \tag{3.13}$$

Indeed, if we redefine the Higgs field as $\mathcal{H} = \phi - v$ then its value at the minimum will obviously be zero. This is the standard position for the ground state of the field before doing perturbation theory. The physical mass is just determined by the oscillations of $\mathcal{H}$ around this minimum, i.e. it follows from the second derivative of V at $\phi = v$, as in Eq. (3.13). In the last equality of that equation we have also introduced the so-called Fermi's scale $M_F \equiv G_F^{-1/2} \simeq 293$ GeV, which is defined from Fermi's constant obtained from muon decay, $G_F \simeq 1.166\times 10^{-5}$ GeV^{-2}. The relation of G_F with the $W^{\pm}$ gauge boson mass and the $SU(2)_L$ weak gauge coupling, g, reads (at the lowest order):

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M^2_W} = \frac{1}{2\,v^2}\,, \tag{3.14}$$

where in the second equality we have used the formula for the $W^{\pm}$ mass in the SM, namely $M_W = (1/2)\,g\,v$. In this way, Eq. (3.14) provides a direct determination of the Higgs VEV in terms of Fermi's constant:

$$v = 2^{-1/4}\,G_F^{-1/2} \simeq 246\,\mathrm{GeV}\,, \tag{3.15}$$

and this relation has been used in the last equality of Eq. (3.12).

In view of the SSB phenomenon, it is clear that we must replace $g_{\mu\nu}\,\rho_{\Lambda\rm vac} \to g_{\mu\nu}\,\rho_{\rm vac}^{\rm cl}$, given by Eq. (3.9), in the expression of Einstein's equations in vacuo. This modifies the effective cosmological constant contribution in Einstein's equations. Furthermore, in the presence of incoherent matter contributions (e.g. from dust and radiation) described by a perfect fluid we have the additional contribution (2.5). Therefore, the final Einstein's equations in terms of coherent and incoherent contributions of matter, plus the vacuum energy of the fields, finally read

$$R_{\mu\nu} - \frac{1}{2}g_{ab}R = 8\pi\, G_N\,\left(\langle\tilde{T}^{\phi}_{\mu\nu}\rangle + T_{\mu\nu}\right) = 8\pi\, G_N\,\left[g_{\mu\nu}\left(\rho_{\Lambda\rm vac}+\rho_{\Lambda\rm ind}\right)+T_{\mu\nu}\right]\,. \tag{3.16}$$

We conclude that the "physical value" of the CC, at this stage, is not just the original term $\rho_{\Lambda\rm vac}$, but

$$\rho_{\Lambda\rm ph} = \rho_{\Lambda\rm vac} + \rho_{\Lambda\rm ind}\,, \tag{3.17}$$

where the induced part is given by (3.12). On the face of this result, it is pretty obvious that when we compare theory and experiment a severe fine tuning problem is conjured in equation (3.17). Indeed, the lowest order contribution from the Higgs potential, as given by equation (3.12), is already much larger than the observational value of the CC. Using the recent LHC measurement of the mass of the Higgs-like particle, suggesting the value $M_{\mathcal{H}} \simeq 125$ GeV [7], equation (3.12) yields $\rho_{\Lambda\rm ind} \simeq -1.2\times 10^{8}$ GeV^4. Thus, being the CC observed value of order $\rho_\Lambda^0 \sim 10^{-47}$ GeV^4, the electroweak vacuum energy density is predicted to be 55 orders of magnitude larger than the currently measured ρ_Λ^0:

$$\left|\frac{\rho_{\Lambda\rm ind}^{\rm EW}}{\rho_\Lambda^0}\right| = \mathcal{O}(10^{55})\,. \tag{3.18}$$

Suppose that the induced result would exactly be $\rho_{\Lambda\rm ind} = -10^8$ GeV^4 and that the vacuum density would exactly be $\rho_\Lambda^0 = +10^{-47}$ GeV^4. In such case one would have to choose the vacuum term $\rho_{\Lambda\rm vac}$ in equation (3.17) with the rather bizarre precision of 55 decimal places in order to fulfill the equation

$$10^{-47}\,{\rm GeV}^4 = \rho_{\Lambda\rm vac} + \rho_{\Lambda\rm ind} = \rho_{\Lambda\rm vac} - 10^8\,{\rm GeV}^4\,. \tag{3.19}$$

This is of course the famous fine-tuning problem.

Let us note that this problem is in no way privative of the cosmological constant approach to the DE, but is virtually present in *any* known model of the DE, in particular also in the quintessence approach [18]. Indeed, the quintessence scalar field potential $V(\varphi)$ is supposed to precisely match the value of the measured DE density at present, ρ_Λ^0, starting from a high energy scale, usually some grand unified theory (GUT) scale $\varphi = M_X$ between $\sim 10^{16}$ GeV and $M_P \sim 10^{19}$ GeV. Even in the simplest case $V(\varphi) \sim m_\varphi^2\,\varphi^2$, one finds $m_\varphi^2 \sim \rho_\Lambda^0/M_X^2$. Defining the mass scale associated to the current CC value, $m_\Lambda \equiv (\rho_\Lambda^0)^{1/4} = \mathcal{O}(10^{-3})$ eV, we have the ratio

$$\frac{m_\varphi}{m_\Lambda} \sim \frac{m_\Lambda}{M_X} \sim 10^{-30}\,, \tag{3.20}$$

which tells us that the mass of the quintessence field should be some thirty orders of magnitude smaller than the CC mass scale that one tries to explain! Apart from the numerical mass value that this implies ($m_\varphi \sim 10^{-33}$ eV), such situation is of course preposterous. Therefore, the quintessence approaches, in addition from being plagued with fine-tuning problems in no lesser degree than the original CC problem, they introduce extremely unnatural small mass scales.

However, this is not quite the end of the story yet. In QFT the induced value of the vacuum energy is much more complicated than just the simple result (3.12), and the fine-tuning problem is much more cumbersome than the one expressed in equation (3.19), see Sect.6 for some clues about this additional complication.

4 Vacuum energy: ZPE and some cosmic numerology

As we know, a nonvanishing Λ leads to a nonvanishing value of the vacuum energy density, or CC density, $\rho_\Lambda = \Lambda/(8\pi G)$. In the previous section we have seen that even at the classical level we can get a large contribution to ρ_Λ from the spontaneous symmetry breaking (SSB) of the electroweak symmetry, i.e. from the ground state of the Higgs potential. Furthermore, quantum corrections to the this potential can be quite significant, as we will discuss later. The possible confirmation of the Higgs finding at the LHC collider at CERN certainly strengthens the case for the vacuum energy in QFT, but the quantum effects which we are going to refer now do not depend on the existence of the Higgs potential as they have a very generic character. They were noted much before the Higgs potential and the SSB phenomenon entered the stage. We pay now some preliminary attention to this important (and generic) phenomenon of the quantum vacuum, only to come back to it in subsequent sections from a more rigorous point of view.

The quantum effects we are now referring to are those emerging from the vacuum-to-vacuum fluctuations of the quantum matter fields. They occur already in the free field theory, in contrast to the SSB phenomenon. These quantum fluctuations correspond to closed loop diagrams without external tails ("blobs"). They describe the infinite number of oscillators with all possible frequencies $\omega_{\mathbf{k}}$ to which we attach to any free quantum matter field. The sum of all the (nonvanishing) ground state energies of these oscillators constitutes the zero-point energy (ZPE) of the corresponding quantum field: $E_0 = (1/2)\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}$. Let us recall that the historical origin of the ZPE emerges from Planck's theory for the black-body radiation in 1900. For the average energy of an oscillator of frequency ω in equilibrium with radiation at temperature T, Planck obtained

$$E_\omega = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} + \frac{1}{2}\,\hbar\omega\,. \tag{4.1}$$

He realized immediately that $E_\omega \to (1/2)\,\hbar\omega$ for $T \to 0$. This is the *Nullpunktsenergie* or zero-point energy of the oscillator, therefore a nonvanishing one. Despite he soon tried to normalize it away treating it as an unphysical effect, thirteen years later Einstein and Stern paid clever attention to the fact that if one expands the formula (4.1) in the classical limit $kT \gg \hbar\omega$ one finds $E_\omega \simeq kT - (1/2)\,\hbar\omega + (1/2)\,\hbar\omega = kT$, as easily checked. In other words, thanks to the isolated ZPE term on the *r.h.s.* of (4.1) one can recover the expected classical limit for the average energy of an oscillator in thermal equilibrium at temperature T. This firmly convinced Einstein that the ZPE could have some physical meaning after all.

But things are not so easy. The electromagnetic field oscillates with all frequencies, and therefore one has to compute the infinite sum $E_0 = (1/2)\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}$. The unrenormalized result is of course infinite, but the renormalized quantity is perfectly finite (cf. Sect. 5) and its value may be of concern. The first considerations on the ZPE of the quantum vacuum dates back to the theoretical discussions made by W. Nernst as early as 1916 and by W. Pauli in the 1920s (and again in a more formal way in 1933, see Sect. 5). However, all these discussions were mainly focused on the electromagnetic field. As, however, the renormalized form of the ZPE is proportional to the quartic power of the renormalized mass of the corresponding field quantum ($\rho_\Lambda \propto m^4$, see below), in the electromagnetic case we have no such physical contribution to the ZPE (at least in the way it was originally conceived by the first pioneers discussing these matters) owing to the massless nature of the photon.

It was probably Y.B. Zeldovich who first raised a serious concern on the contribution to the ZPE from a "typical" massive particle [11] and expressed the necessity to subtract its leading – or "first order"– effect in the computation of the physical value of the CC [1]. In particular he noticed that the leading proton contribution to the ZPE ($\rho_\Lambda \propto m_p^4 \sim 1\text{GeV}^4$) is overwhelmingly large as compared to any cosmic density, say the current critical density ρ_c^0 expressed in typical particle physics units:

$$\rho_c^0 = \frac{3\,H_0^2}{8\pi\,G} = \frac{3}{8\pi}\,H_0^2\,M_P^2 \sim 10^{-47}\,\text{GeV}^4\,, \tag{4.2}$$

where $H_0 \sim 10^{-42}$ GeV is the current value of the Hubble rate and $M_P = 1/\sqrt{G} \sim 10^{19}$ GeV is Planck's mass (both expressed in natural units). The current matter and vacuum energy densities (ρ_m^0 and ρ_Λ^0) are proportional to (4.2) up to factors of order one, i.e. $\rho_m^0 = \Omega_M^0\,\rho_c^0$ and $\rho_\Lambda^0 = \Omega_\Lambda^0\,\rho_c^0$, with $\Omega_M^0 \sim \Omega_\Lambda^0 = \mathcal{O}(1)$. After subtracting the exceedingly large leading term in the theoretical ZPE estimate, Zeldovich realized that starting once more from m_p as the "typical" mass scale of particle physics one could produce a much more reasonable order of magnitude estimate of the CC density by means of a "second order" (or "next-to-leading") formula involving the natural presence of the gravitational coupling, G. To this purpose he concocted the dimensionally consistent "construct"

$$\rho_\Lambda \simeq G\,m_p^6 = \frac{m_p^6}{M_P^2} \sim 10^{-38}\,\text{GeV}^4\,. \tag{4.3}$$

The previous estimate still strays off the modern value, but by "only" nine orders of magnitude – in contrast to the demolishing 47 orders of magnitude that one has to face if keeping the leading term $\rho_\Lambda \sim m_p^4$. Needless to say at the time of Zeldovich there was no real measurement of Λ, although of course an upper bound estimate of the order of the critical density was in force. Therefore the order of magnitude discrepancy was not that different from the present one. Even so, formula (4.3) could somehow still be considered as a respectable estimate. An even more intriguing result obtains if one replaces the proton by the pion (whose mass $m_\pi \simeq 0.1$ GeV is roughly ten times smaller than that of the proton). Then one gets a better approximation which differs now by "only" three orders of magnitude from the correct order of magnitude result (4.2). It turns out that a kind of "cosmic prediction" of the pion mass was proposed by Weinberg in 1972 [37] through

[1] It is not my intention to make full justice, in this summarized account, to the extensive historical literature on the CC problem – see e.g. [4, 5] and [34, 35], and references therein. It will suffice to say that the connection between vacuum energy and CC had already been glimpsed by Lemaître in 1934 a few years after he found the cosmological expanding solution in the presence of a Λ-term [36].

the curious numerical relation

$$m_\pi^3 \sim \frac{H_0}{G} = H_0\, M_P^2 \sim 10^{-4}\,\mathrm{GeV}^3\,, \tag{4.4}$$

which indeed leads to $m_\pi \simeq 0.1$ GeV. Amusingly, if we now substitute Eq. (4.4) into (4.3) — after first replacing $m_p \to m_\pi$, according to our prescription – we find

$$\rho_\Lambda \sim H_0^2\, M_P^2 \sim 10^{-46}\,\mathrm{GeV}^4\,, \tag{4.5}$$

which, according to (4.2), is very close to the correct order of magnitude of the current CC density, since $\rho_\Lambda^0 = \Omega_\Lambda^0\,\rho_c^0$, with $\Omega_\Lambda^0 \simeq 0.7$. From (4.4) and (4.5) it also ensues the (no less "cabalistic") relation $\rho_\Lambda \sim m_\pi^3\, H_0$, somehow suggesting a possible link between the meson world of particle physics and cosmology. Unfortunately such relation is untenable within GR, and therefore such link is impossible if following that pathway. Equally untenable is a (subtlety disguised) form of the previous relation, which has iteratively appeared in the literature in more recent times, to wit:

$$\rho_\Lambda \sim H_0\, \Lambda_{\rm QCD}^3\,, \tag{4.6}$$

where $\Lambda_{\rm QCD} \simeq 200$ MeV is the QCD scale of the strong interactions (not far away from the pion mass). Some people has tried hard to seek for a fundamental reason behind a formula like (4.6) [38]. Numerically it is much worse than (4.5), as it yields $\rho_\Lambda \sim 10^{-44}$ GeV^4, thus failing by "only" three orders of magnitude. But it is not this numerical failure which is most disturbing (as numbers are comparable to the situation with the previous pion formula – thought of as an improved form of Zeldovich's one); the problem here is of theoretical nature. As has been emphasized in [33], an equation like (4.6) – or, for that matter, any other relation where ρ_Λ is extracted from an odd power of the Hubble rate – is incompatible with the general covariance of the effective action in QFT in curved spacetime[2]. One possibility would be to resort to fractional powers of the invariants, but are we ready for such an eccentric possibility while other, more amenable ones, are still there to be fully exploited? For example, the numerically successful relation (4.5) depends quadratically on H and therefore is compatible with the aforementioned covariance. It follows that the argument that led to Eq. (4.5) cannot be (without invoking contrived assumptions) the one that we followed above through $\rho_\Lambda \sim m_\pi^3\, H_0$ or (4.6) since the latter cannot be accepted in a natural way. It means that if there is any truth around Eq, (4.5) there must exist some completely independent pathway leading to it. We will see that there are indeed quite different paths pointing to (4.5), or in general to the more general "affine" quadratic relation

$$\rho_\Lambda(H) = c_0 + \beta\, M_P^2\, H^2\,, \tag{4.7}$$

with a non-vanishing c_0 term, β being here a dimensionless coefficient. The presence of $c_0 \neq 0$ is crucial for a realistic implementation of the model, as it enables the transition from deceleration to acceleration in this kind of models. The strict model (4.5), understood as a model of the

[2] In Sect. 9 we will see that the $\Lambda_{\rm QCD}$ scale could play a relevant cosmological role, but for quite a different reason, namely on account of its potential cosmic time dependence linked to that of the vacuum energy, and within a fully covariant formulation [24].

kind (4.7) with $c_0 = 0$, is ruled out [39, 40] as it cannot satisfy the aforementioned transition condition. This is the situation with entropic force models [41], for example, and many other models previously presented on purely phenomenological grounds. In contrast, the class of affine models (4.7) is perfectly safe in this respect, and in fact it has been successfully tested against the recent cosmological data [39, 40, 42].

In principle, we have no fundamental reason behind any of these numerical games. The only "fundamental" thing to be done here is to make sure that we subtract the leading ("first order") contribution from the ZPE, which for all kinds of known elementary particles (except perhaps for a light neutrino) is much larger than the measured ρ_Λ[3]. For a hypothetical neutrino of a few meV$= 10^{-3}$ eV, we have the suggestive result [43]

$$\rho_\Lambda \sim m_\nu^4 \sim 10^{-11}\,\mathrm{meV}^4 \sim 10^{-47}\,\mathrm{GeV}^4\,, \tag{4.8}$$

that falls in the right ballpark of the cosmic densities (4.2). In the next sections we will go in a summarized way thorough models that could justify some of the more successful evolution laws for the cosmological term, say of the form (4.7), on more fundamental grounds.

5 Zero-point energy in quantum field theory in flat spacetime

If we wish to go beyond the previous numerical games, things can get a bit harder, even if we still try to keep them as simple as possible. Let us therefore first follow a very naive formulation. Formally the ZPE of a given quantum field, say a scalar field ϕ, is obtained by selecting that part of the effective potential which does not depend on the external tails of ϕ (i.e. that part which is *not* a function of ϕ). For example, the Higgs potential (cf. Section 3) is in general *not* a part of the ZPE because it has a classical part (one which does *not* vanish when we set $\hbar = 0$). It means that it consists of all the bubble-type (vacuum-to-vacuum) diagrams at all orders in perturbation theory. The ZPE is thus a pure quantum effect: it vanishes if there is no quantum theory, $\hbar = 0$. Indeed, vacuum-to-vacuum diagrams can only exist in a field theory with vacuum fluctuations: QFT. The final result therefore can only depend on a list of parameters P (masses and coupling constants), but not on ϕ itself, which can only enter virtually in the loop propagators. If we count the loop order of perturbation theory with the corresponding power of $\hbar$, the loopwise expansion can be presented as a power series in $\hbar$:

$$V_{\rm ZPE}(P) = \hbar\, V_P^{(1)} + \hbar^2\, V_P^{(2)} + \hbar^3\, V_P^{(3)} + ... \tag{5.1}$$

(A contribution to the 21th term of this pure vacuum-to-vacuum series within the SM can be seen in Fig. 1.) From Eq. (5.1) it is obvious that even the first term depends on $\hbar$, just linearly. This is the one-loop approximation. Since we promised to keep things simple, let us evaluate the ZPE

[3] The maximum wildness (or should I say madness) of the $\sim m^4$ contributions to ρ_Λ is achieved for the Planck mass, for which the discrepancy with the current value is $M_P^4/\rho_\Lambda^0 = (M_P/m_\Lambda)^4 \sim \left(\mathcal{O}(10^{19}\mathrm{GeV})/\mathcal{O}(10^{-3}\mathrm{eV})\right)^4 \sim 10^{123}$. This is the ultimate state of "paroxysm" of the CC problem. But, as we have seen in Sect. 3 (and as we shall further emphasize later on), those 123 orders of magnitude should not be considered as the most obvious and worrisome aspect of the "real" CC problem!

of a QFT for a free scalar field at one loop only. For fermion field of spin $s_f = 1/2$ an additional factor of $4\,(= \mathrm{Tr}\,\hat{\mathbf{1}})$ and an overall minus sign should both be inserted.

A naive calculation of the coefficient $V_P^{(1)}$ in (5.1) is obtained by ignoring for the moment gravity and regularizing the infinite sum $(1/2)\sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} = (1/2)\sum_{\mathbf{k}} \hbar\sqrt{\mathbf{k}^2+m^2}$ by means of a cutoff $\mathcal{M}$. Moving to the continuum and trivially integrating the solid angle, it gives

$$V_P^{(1)} = \frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{\mathbf{k}^2+m^2} = \frac{1}{4\pi^2}\int_0^{\mathcal{M}} \mathrm{d}k\,k^2\sqrt{k^2+m^2} = \frac{\mathcal{M}^4}{16\pi^2}\left(1+\frac{m^2}{\mathcal{M}^2}-\frac{1}{4}\frac{m^4}{\mathcal{M}^4}\ln\frac{M^2}{m^2}+\cdots\right), \tag{5.2}$$

where in the expansion in powers of $m/\mathcal{M}$ we have explicitly kept the important $\sim m^4\ln m^2$ term because this one does not depend on any power of the cutoff and therefore is the term that should remain after we attempt to remove the cutoff by some renormalization procedure. In the 1930's Pauli applied this naïf approach to the electromagnetic field ($m = 0$ for the photon) and choosing the inverse of the classical electron radius for the cutoff, i.e. $\mathcal{M} = 2\pi\, m_e/\alpha$ (here α standing for the fine structure constant). Then he plugged the result into Einstein's universe formula (2.8), i.e. he replaced ρ_Λ there by the previous one-loop estimate of the ZPE, and obtained the "radius" of the ensuing universe:

$$a = \sqrt{\frac{2\pi}{G}}\,\frac{1}{\mathcal{M}^2} = (2\pi)^{-3/2}\left(\frac{M_P}{m_e}\right)\frac{\alpha^2}{m_e}\,. \tag{5.3}$$

It would not even reach the Moon! (as a matter of fact the result is appallingly small, some twenty six kilometers only!)[4]. As we can see, the larger is the vacuum density the smaller is the equilibrium "radius". This is intuitively obvious from the fact that the negative pressure associated to the CC value is very big even for a cutoff as small as the electron mass, so that if gravity has to compensate such outwards vacuum pressure the universe must be small enough to produce a gravitational (inwards) effect of the same size. He was dismayed, but the argument is itself actually non-rigorous. Indeed, the results (5.2)-(5.3) explicitly depend on the value we can arbitrarily assign to the regulator $\mathcal{M}$; in other words, it is merely an unphysical result obtained in the "bare theory". Sometimes a particular value can better approximate the final result, but in actual fact the physical result should be completely independent of $\mathcal{M}$. It means that one must first renormalize the theory before jumping to conclusions, as without renormalization the result (5.3) means very little. This implies, among other things, to get rid of the regulator before extracting meaningful conclusions on the ZPE contribution from the electromagnetic field. However, after doing so we are left with nothing since the photons have no mass and hence no contribution to the ZPE could remain from them. Indeed, upon removing the cutoff we expect that the renormalized result should be of order

$$V_P^{(1)\mathrm{renorm}} = -\frac{m^4}{64\pi^2}\ln\frac{\mu^2}{m^2}+\cdots\,. \tag{5.4}$$

where μ is some subtraction scale that must remain after renormalization. From the above formula it is clear that the ZPE becomes zero if we set $m = 0$ in it. If we would instead compute the contribution, not from the electromagnetic field, but from a scalar field of mass m, the renormalized contribution should be of the order (5.2), thus proportional to m^4. Unfortunately, we find that

[4]Strictly speaking one has to include a factor of 2 in Eq. (5.2) to account for the two helicities of the photon, so the result for the "radius" is $1/\sqrt{2}$ smaller , i.e. some ~ 18 kilometers, but of course this nicety is irrelevant here.

even after trying to make some sense out of the ZPE within QFT we are back to the huge quartic contributions to the CC which we tried to avoid in the previous section. They are back here and this becomes of course disquieting. As for Pauli's quantitative result, despite its roughness and severe limitations it transmits the correct qualitative idea that any typical choice for the cutoff scale within the particle physics domain leads to a very unsatisfactory value for the cosmological constant as compared to the observational measurements. We thus realize that the ZPE calculation becomes conflictive with observations even after renormalization. Despite we did not use a concrete renormalization scheme to guess at the renormalized form (5.4) from the regularized expression (5.2), it turns out that the latter is indeed what is obtained e.g. using dimensional renormalization (see below).

Although every scheme is in principle valid, not every scheme has the property that the renormalized quantities are in good correspondence with the physical quantities in the particular framework of the calculation. For example, it makes no much sense to renormalize QCD (the gauge theory of strong interactions) in the on-shell scheme because we never find quarks and gluons on mass shell! One is forced to use an off-shell renormalization scheme. This can complicate the physical interpretation, for there can indeed be a nontrivial gap between the renormalized parameters and the physical ones. And if all that is not enough, the CC problem [4] is of course a problem where gravity should play a role somewhere in the calculation of the ZPE, shouldn't it? So, at the end of the day, we realize that the result (5.2) is in itself far from having any physical meaning!

In an attempt to smooth out some of these problems, suppose we adopt the MS-scheme (or Minimal Subtraction scheme) in n-dimensional regularization. Then, the one-loop approximation to the ZPE renders

$$V_P^{(1)} = \frac{1}{2}\,\mu^{4-n}\int\frac{d^{n-1}k}{(2\pi)^{n-1}}\,\sqrt{\mathbf{k}^2+m^2} = \frac{1}{2}\,\beta_\Lambda^{(1)}\left(-\frac{2}{4-n}-\ln\frac{4\pi\mu^2}{m^2}+\gamma_E-\frac{3}{2}\right)\,, \tag{5.5}$$

where γ_E is Euler's constant, and

$$\beta_\Lambda^{(1)} = \frac{m^4}{2\,(4\pi)^2} \tag{5.6}$$

is the one-loop β-function for the vacuum term (see below). Notice that μ is the characteristic 't Hooft mass unit of dimensional regularization, and $n \to 4$ is understood in the final result. So obviously the ZPE is UV-divergent once more, as it could not be otherwise without renormalization. In the MS scheme one introduces, as usual, a counterterm killing the "bare bone" UV-part (the pole at $n \to 4$). In the slightly modified $\overline{\rm MS}$ scheme one collects also some additive constants. Specifically, the counterterm reads:

$$\delta\rho_\Lambda^{\overline{\rm MS}} = \frac{m^4\,\hbar}{4\,(4\pi)^2}\left(\frac{2}{4-n}+\ln 4\pi-\gamma_E\right)\,. \tag{5.7}$$

How to get now a finite (if still not physical) value of the ZPE? We just have to follow the renormalization program. Recall that the Einstein-Hilbert action from which the field equations (2.2) are derived reads as in Eq. (3.2) [23]:

$$S_{\rm EH} \;=\; \frac{-1}{16\pi\,G^{(\rm b)}}\int d^4x\sqrt{-g}\,\left(R+2\,\Lambda^{(\rm b)}\right) = -\int d^4x\sqrt{-g}\,\left(\frac{1}{16\pi\,G^{(\rm b)}}\,R+\rho_\Lambda^{(\rm b)}\right). \tag{5.8}$$

Here (and hereafter) we have rewritten the parameters $\Lambda \to \Lambda^{(b)}$, $\rho_{\Lambda \rm vac} \to \rho_\Lambda^{(b)}$ and $G \to G^{(b)}$ so as to emphasize it is the bare action, i.e. the action before any renormalization program is applied to account for the UV-divergences related to the quantum matter contributions. In particular, the CC term and the gravitational coupling are the bare ones – and so have been denoted with the superindex "b" . Even though this action is not renormalizable, we know that by including the higher order invariant terms (i.e. R^2, $R_{\mu\nu}R^{\mu\nu}$..., only relevant for the short-distance behavior of the theory) the extended action is renormalizable in the context of QFT in curved spacetime (where gravity is not quantized but the matter fields yes), provided (!) we keep the Λ-term in it. So let us maintain this term in the low-energy part of the full renormalizable action, i.e. the long-distance EH action (5.8), as this part is both necessary for the renormalization program and is accessible to observation (as we indeed know from the observed accelerated expansion).

The key point now, at the theoretical level, is that the CC-term in (5.8) is *not* yet the physical quantity, it is only the bare parameter of the bare EH action. When we include the ZPE as a part of the full action, the overall additive term is no longer $\rho_\Lambda^{(b)}$ but the sum $\rho_\Lambda^{(b)} + V_{\rm ZPE}^{(\rm bare)}$, where $V_{\rm ZPE}^{(\rm bare)}$ is given by (5.1). Next we split $\rho_\Lambda^{(b)}$ into a renormalized part plus a counterterm, $\rho_\Lambda^{(b)} = \rho_\Lambda(\mu) + \delta\rho_\Lambda$, where the renormalized part depends on the arbitrary renormalization scale μ and the counterterm depends on the regularization and renormalization scheme. Adopting $\overline{\rm MS}$ in dimensional regularization we are led to use the explicit form (5.7) for $\delta\rho_\Lambda$. Finally, since the (effective action of the) bare theory must equal (that of) the renormalized theory (in whatever renormalization scheme), we must have $\rho_\Lambda^{(b)} + V_{\rm ZPE}^{(\rm bare)} = \rho_\Lambda(\mu) + V_{\rm ZPE}(\mu)$. This condition defines the $\overline{\rm MS}$-renormalized one-loop value of the ZPE. Using the above equations, at one-loop it reads:

$$V_{\rm ZPE}^{(1)}(\mu) = \hbar\, V_P^{(1)} + \delta\rho_\Lambda^{\overline{\rm MS}} = \frac{m^4\,\hbar}{4\,(4\,\pi)^2}\left(\ln\frac{m^2}{\mu^2} - \frac{3}{2}\right)\,, \tag{5.9}$$

which is perfectly finite. Incidentally, with this result we have checked that the one-loop correction in dimensional regularization with minimal subtraction indeed provides the kind of finite correction that we guessed in Eq. (5.4) from the bare result (5.2), up to additive parts related with the subtraction procedure. This does not mean that the obtained expression is the physical result, but at least is a renormalized and hence finite one. The price for the finiteness of the renormalized result is its dependence on the arbitrary mass scale μ. However, while both pieces (5.9) and $\rho_\Lambda(\mu)$ separately depend on the scale μ, the sum

$$\rho_{\rm vac}^{(1)} = \rho_\Lambda(\mu) + V_{\rm ZPE}^{(1)}(\mu) = \rho_\Lambda(\mu) + \frac{m^4\,\hbar}{4\,(4\,\pi)^2}\left(\ln\frac{m^2}{\mu^2} - \frac{3}{2}\right)\,, \tag{5.10}$$

does *not* since by construction we started from the bare theory, which is of course μ-independent. The sum (5.10) represents the $\overline{\rm MS}$-renormalized vacuum energy of the free field at one loop. As this quantity is the same starting bare expression, just rewritten in terms of renormalized parameters, it is overall μ-independent. This is actually the main message of the renormalization group (RG): the sum of the various μ-dependencies must cancel in the renormalized effective action, and also in the renormalized S-matrix elements (when they can be defined). We can now check explicitly that (5.6) is indeed the one-loop β-function for the running of the renormalized CC. Computing the logarithmic derivative $d/d\ln\mu = \mu d/d\mu$ on both sides of (5.10) and taking into account that this gives $d\rho_{\rm vac}^{(1)}/d\ln\mu=0$ on the *l.h.s* (for the reasons just mentioned above), we immediately find the desired result:

$$\mu\frac{d\rho_\Lambda(\mu)}{d\mu}=\frac{\hbar\, m^4}{2\,(4\pi)^2}=\beta_\Lambda^{(1)}\,. \tag{5.11}$$

The RG tells us something useful about the explicit μ-dependence affecting incomplete structures of the effective action. Indeed, while we may not know the full structure of the effective action in a particular complicated situation, our educated guess in associating μ with some relevant dynamical variable of the system can give relevant information on physics, similarly as we do in particle physics. This is e.g. the case with an effective charge (or "running coupling constant"), say the QED or QCD renormalized gauge coupling $g=g(\mu)$, which is explicitly μ-dependent even though the full effective action or S-matrix element is not.

For a more rigorous connection with the curved space-time case discussed in the next section, it is convenient to approach Eq. (5.5) from a more formal point of view, namely from the notion of effective action [44]. The desired form for the effective action at one-loop reads:

$$\Gamma_{\rm eff}[\phi_c]=S[\phi_c]+\frac{i\,\hbar}{2}\,Tr\,\ln\mathcal{K}(x,x')\,. \tag{5.12}$$

where $S[\phi_c]$ is the classical action

$$S[\phi_c]=\int d^4x\,\mathcal{L}=\int d^4x\,\left[\frac{1}{2}\,g^{\mu\nu}\,\partial_\mu\phi_c\,\partial_\nu\phi_c-V_c(\phi_c)\right]\,. \tag{5.13}$$

and

$$\mathcal{K}(x,x')=\left[\Box_x+V''(\phi_c)\right]\,\delta(x-x')\,, \tag{5.14}$$

is essentially the inverse propagator in the presence of the background matter field ϕ_c. Here V_c is the tree-level or classical part of the potential for that field, typically of the form (3.1). For a free field ($\lambda=0$) it contains only the mass term: $V(\phi)=(1/2)\,m^2\,\phi^2$ and Eq. (5.14) simplifies to

$$\mathcal{K}(x,x')=\left[\Box_x+m^2\right]\,\delta(x-x')\qquad\text{(free QFT)}\,. \tag{5.15}$$

The result (5.12) was expected, namely the effective action at this order is the sum of the classical action (5.13) and the one-loop term (5.14) – notice the presence of $\hbar$ in front of it in Eq. (5.12). The quantum correction term, call it $\Gamma^{(1)}$, is generated exclusively by the vacuum diagrams, and therefore represents the ZPE. We can understand that indeed $\Gamma^{(1)}$ is associated to vacuum-to-vacuum diagrams (i.e. closed loop diagrams without external tails of quantum matter) by the fact that we have to compute a trace over all indices, including the spacetime ones:

$$\Gamma^{(1)}=\frac{i\,\hbar}{2}\,Tr\,\ln\mathcal{K}(x,x')=\frac{i\,\hbar}{2}\,\int d^4x\,\lim_{x\to x'}\,\ln[\mathcal{K}(x,x')]\,. \tag{5.16}$$

Diagrammatically we can interpret we are following a closed line starting at one point and ending at the same point, i.e. a vacuum-to-vacuum diagram or "blob". Furthermore, we have to sum these blobs over all of the spacetime points, as indicated by the above integral. Setting $\phi_c=$const. the classical action (5.13) just boils down to (minus) the classical potential times the spacetime volume Ω, i.e. $S[\phi_c]=-\int d^4xV(\phi_c)=-\Omega V(\phi_c)$. The *l.h.s.* of (5.12) can then be written as $-V_{\rm eff}\,\Omega$, where $V_{\rm eff}$ is the so-called effective potential. Clearly $V_{\rm eff} = V_c + \hbar V^{(1)}$, where the one-loop correction reads

$$V^{(1)} = -\frac{i}{2}\,\Omega^{-1}\,Tr\ln\mathcal{K}\,. \tag{5.17}$$

Next the integral can be conveniently worked out in momentum space. For the free field case (5.15) it immediately leads to

$$Tr\,\ln\mathcal{K} = \Omega\int\frac{d^nk}{(2\pi)^n}\,\ln\left(-k^2+m^2\right)\,, \tag{5.18}$$

where we moved to n dimensions to regularize the result. The spacetime volume finally cancels in (5.17) and after a simple calculation one finally retrieves the formula (5.5) and the corresponding $\overline{\rm MS}$-renormalized final result(5.10).

6 The ZPE and the full effective Higgs potential for QFT in flat spacetime

The ZPE calculation for free fields considered in the previous section is the simplest kind of quantum vacuum effect one can deal with. However, the general renormalized effective potential extending to the quantum domain the classical potential, $V_c \to V_{\rm eff}$, takes on the form

$$V_{\rm eff} = V_c + \hbar\,V_1 + \hbar^2\,V_2 + \hbar^3\,V_3 + ... \tag{6.1}$$

The quantum effects to all orders of perturbation theory arrange themselves in the form of a loopwise expansion where the number of loops is tracked by the powers of $\hbar$. Thus, at one loop we have only one power of $\hbar$, at two loops we have two powers of $\hbar$ etc. For $\hbar = 0$, however, there are no loops and the effective potential just reduces to the classical potential, V_c, given by equation (3.1) in the electroweak standard model. On the other hand, each of the loop terms in (6.1) can be split into two independent contributions, one consisting of loops with no external legs (vacuum-to-vacuum parts $V_P^{(i)}$, i.e. the ZPE contribution at ith-order) and the other involving loops with external legs of the Higgs field ϕ (i.e. the ith-loop correction $V_{\rm scal}^{(i)}(\phi)$ to the classical Higgs potential):

$$V_1 = V_P^{(1)} + V_{\rm scal}^{(1)}(\phi)\,,\qquad V_2 = V_P^{(2)} + V_{\rm scal}^{(2)}(\phi)\,,\qquad V_3 = V_P^{(3)} + V_{\rm scal}^{(3)}(\phi)....\,. \tag{6.2}$$

As a result, the effective potential (6.1) at the quantum level splits naturally into two parts, one which is ϕ-independent and another that is ϕ-dependent:

$$V_{\rm eff}(\phi) = V_{\rm ZPE} + V_{\rm scal}(\phi)\,, \tag{6.3}$$

where

$$V_{\rm ZPE} = \hbar\,V_P^{(1)} + \hbar^2\,V_P^{(2)} + \hbar^3\,V_P^{(3)} + ... \tag{6.4}$$

is the full *zero-point energy* (ZPE) contribution. It is a number, it only depends on the set of parameters $P = m, \lambda, ...$ of the classical potential, not at all on the fields. As mentioned, the latter consists in the sum of all the vacuum-to-vacuum parts of the effective potential. The ZPE part is sourced exclusively from closed loops of matter fields (i.e. vacuum loops without external ϕ-legs). In the previous section we have computed the one-loop contribution to the ZPE and for free fields, i.e. just the term $V_P^{(1)}$. The ZPE receives in general contributions to all orders of perturbation theory, except at zero loop level since $V_{\rm ZPE}$ is a pure quantum effect that vanishes for $\hbar = 0$. Now, besides the ZPE there is the scalar field dependent part of the effective potential:

$$V_{\rm scal}(\phi) = V_c(\phi) + \hbar\, V_{\rm scal}^{(1)}(\phi) + \hbar^2\, V_{\rm scal}^{(2)}(\phi) + \hbar^3\, V_{\rm scal}^{(3)}(\phi) + ... \tag{6.5}$$

This one is not purely quantum (i.e. it does not vanish for $\hbar = 0$) as the first term (the classical potential) is, of course, *n*ot proportional to $\hbar$. The above ϕ-dependent part of $V_{\rm eff}$ receives in general also contributions to all orders of perturbation theory, and vanishes for $\phi = 0$ since in this case all the loops have external ϕ legs, including the tree-level part. Thus, in the absence of SSB the effective potential boils down to just the ZPE,

$$V_{\rm eff}(\phi = 0) = V_{\rm ZPE} \tag{6.6}$$

which is a number entirely constructed from a power series of $\hbar$. On the other hand, the expression (6.5) is the effective potential excluding that ZPE number. The full effective potential (6.3) contains both contributions.

In the above formulae, all the field theoretical ingredients (m, λ, ϕ and $V_{\rm eff}$) are in fact bare quantities (m_0, λ_0, ϕ_0 and $V_{\rm eff\,0}$) that require renormalization, as the loopwise expansion is UV-divergent order by order. Renormalization means that we replace all the bare quantities with renormalized ones (in some given renormalization scheme with a specific set of renormalization conditions) plus counterterms (which are also scheme dependent and are partially fixed by the condition of canceling the UV-divergences): $m_0 = m + \delta m$, $\lambda_0 = \lambda + \delta\lambda$, $\phi_0 = Z_\phi^{1/2}\,\phi = (1 + \delta Z_\phi/2)\,\phi$... Of course, a similar splitting occurs with the vacuum term, which was originally a bare term $\rho_\Lambda^{(\rm b)}$. We must also split it into a renormalized piece plus a counterterm: $\rho_\Lambda^{(\rm b)} = \rho_\Lambda(\mu) + \delta\rho_\Lambda$. The full set of counterterms is essential to enable the loop expansion to be finite order by order in perturbation theory. For instance, if we would renormalize the theory in the $\overline{\rm MS}$ scheme in dimensional regularization, the suitable counterterm for the vacuum parameter was given in Sect. 5, see Eq. (5.7).

For a practical calculation of the first (one-loop) quantum correction to the Higgs potential (3.1), we have to compute the one-loop term (5.16) in the presence of the $\lambda \neq 0$ term in the potential. In this case,

$$\mathcal{K}(x, x') = \left[\Box_x + V''(\phi_c)\right]\, \delta(x - x') = \left[\Box_x + m^2 + \frac{1}{2}\,\lambda\,\phi_c^2\right]\, \delta(x - x')\,. \tag{6.7}$$

For ϕ_c =const. this will lead us to determine the explicit form for $V_{\rm scal}^{(1)}(\phi)$ in the above language. This introduces some complications but the calculation can still be carried out without much problems. In the constant mean field limit we may equate the bare and renormalized effective action and we obtain

$$\rho_\Lambda^{(\rm b)} + V_{\rm eff}(\phi_0, m_0, \lambda_0) = \rho_\Lambda(\mu) + V_{\rm eff}(\phi(\mu), m(\mu), \lambda(\mu); \mu)\,. \tag{6.8}$$

As always the renormalized result depends on an arbitrary mass scale μ The overall μ-dependence, however, must eventually cancel. In fact, the renormalized parameters are finite quantities which are also functions of μ: $\phi = \phi(\mu)$, $m = m(\mu)$, $\lambda = \lambda(\mu)$, $\rho_\Lambda = \rho_\Lambda(\mu)$, and since the vacuum energy cannot depend on the arbitrary scale μ, the sum of the renormalized vacuum term and the renormalized potential must be globally scale-independent (i.e. μ-independent). This is obviously so because the bare vacuum term and bare effective potential were scale-independent to start with. Thus, from (6.8) we have

$$\mu\frac{d}{d\mu}\left[\rho_\Lambda(m(\mu),\lambda(\mu);\mu)) + V_{\rm eff}(\phi(\mu);m(\mu),\lambda(\mu);\mu)\right] = 0\,. \tag{6.9}$$

This relation implies that the full effective potential is actually *not* renormalization group (RG) invariant (contrary to some inaccurate statements in the literature), but it becomes so only after we add up to it the renormalized CC vacuum part ρ_Λ. In reality, the structure of the effective potential (6.3) is such that the previous relation splits into two independent RG equations:

$$\mu\frac{d}{d\mu}\left[\rho_\Lambda(m(\mu),\lambda(\mu);\mu)) + V_{\rm ZPE}(m(\mu),\lambda(\mu);\mu)\right] = 0 \tag{6.10}$$

and

$$\mu\frac{d}{d\mu}V_{\rm scal}(\phi(\mu);m(\mu),\lambda(\mu);\mu) = 0\,. \tag{6.11}$$

Equation (6.10) shows that it is only the strict vacuum-to-vacuum part (i.e. the ZPE) the one that needs the renormalized vacuum term ρ_Λ to form a finite and RG-invariant expression, whereas the renormalized ϕ-dependent part of the potential (i.e. the tree-level plus the loop expansion with external ϕ-tails) is finite and RG-invariant by itself. This is of course the essential message from the renormalization group. Explicitly, equation (6.11) reads

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta_P\,\frac{\partial}{\partial P} - \gamma_\phi\,\phi\,\frac{\partial}{\partial\phi}\right\} V_{\rm scal}\left[P(\mu),\phi(\mu);\mu\right] = 0\,, \tag{6.12}$$

where as usual $\beta_P = \mu\partial P/\partial\mu$ ($P = m, \lambda, ...$) and $\gamma_\phi = \mu\partial\ln Z_\phi^{1/2}/\partial\mu$. Similarly, equation (6.10) can be put in the form (6.12), except that the ϕ term is absent.

Plugging equation (5.9) in the general RG equation (6.10), we find immediately that the renormalized vacuum term $\rho_\Lambda(\mu)$ obeys the one-loop RG-equation which we have found previously, i.e. Eq. (5.11).

The next step is the one-loop renormalization of the effective potential. This is standard [44], of course, although the usual discussions on this subject rarely pay much attention to disentangle the ZPE part from it. Let us do it. Once more we have to compute the one-loop expression (5.16), but in this case for the operator (6.7), so we have

$$Tr\,\ln\mathcal{K} = \Omega\int\frac{d^nk}{(2\pi)^4}\,\ln\left(-k^2+V''(\phi_c)\right)\,. \tag{6.13}$$

Substituting this expression in (5.17), we may split the result in the suggestive form

$$V_1(\phi_c) = -\frac{i}{2}\int\frac{d^nk}{(2\pi)^n}\,\ln\left[-k^2+m^2\right] - \frac{i}{2}\int\frac{d^nk}{(2\pi)^n}\,\ln\left(\frac{k^2-V''(\phi_c)}{k^2-m^2}\right)\,. \tag{6.14}$$

Notice that the first term is independent of ϕ, i.e. it only depends on the parameters of the potential. In the one-loop case, it depends only on m, but at higher loops it would also depend on λ (in the interactive theory). Diagrammatically, it corresponds to a closed one-loop diagram without external ϕ-tails, i.e. to a vacuum-to-vacuum one-loop diagram. As could be expected, this term can be identified with the one-loop contribution for the $V_{\rm ZPE}$ in Eq. (6.4), previously addressed in Sect. 5. Similarly, the second integral on the *r.h.s.* of Eq. (6.14) just gives the one-loop correction to the ϕ-dependent part of the potential, i.e. the second term on the *r.h.s.* of Eq. (6.5). Therefore Eq. (6.14) contains the full one-loop correction, which includes the ZPE part and the ϕ-dependent contribution to the potential. To it we still have to add the classical potential, cf. Eq. (6.5).

Sticking to the $\overline{MS}$ scheme in dimensional regularization to fix the counterterms, one finds the final result for the renormalized full effective potential (6.3) up to one-loop:

$$V_{\rm eff}(\phi)=\frac{1}{2}\,m^2(\mu)\,\phi^2+\frac{1}{4!}\,\lambda(\mu)\,\phi^4+\hbar\,\frac{\left(m^2+\frac{1}{2}\,\lambda\,\phi^2\right)^2}{4(4\pi)^2}\left(\ln\frac{m^2+\frac{1}{2}\,\lambda\,\phi^2}{\mu^2}-\frac{3}{2}\right)\,, \tag{6.15}$$

Notice the implicit μ-dependence of the masses and couplings. Together with the explicit μ-dependent parts of the effective potential, this insures the full RG-invariance of the effective action in the constant mean field limit, i.e. the fulfilment of Eq. (6.9).

As shown in Eq. (6.14), the two kind of one-loop effects are built in the calculation. Therefore, the expression (6.15) must boil down to the renormalized ZPE for $\phi = 0$, and indeed we verify that in this limit we recover the one-loop term on the *r.h.s.* of Eq. (5.10):

$$V_{\rm eff}(\phi=0)=\frac{\hbar\,m^4}{4(4\pi)^2}\left(\ln\frac{m^2}{\mu^2}-\frac{3}{2}\right)\,. \tag{6.16}$$

This is accordance with the expectation in Eq. (6.6). We are now ready for addressing the CC fine-tuning problem in the context of a well-defined, renormalized, and RG-invariant vacuum energy density in flat space.

Once the full effective potential has been renormalized, the two loopwise expansions (6.4) and (6.5) become finite to all orders of perturbation theory. Furthermore, the basic equation (3.17) remains formally the same in the quantum theory, i.e. the physical energy density associated to the CC is the sum of the vacuum part plus the induced part. The only difference is that the induced part now contains all the quantum effects, i.e. it reads $\rho_{\Lambda\rm ind}=\langle V_{\rm eff}^{\rm ren}(\phi)\rangle$, where $V_{\rm eff}^{\rm ren}(\phi)\equiv V_{\rm eff}(\phi(\mu);m(\mu),\lambda(\mu);\mu)$ is the renormalized effective potential. Notice that the latter includes the (renormalized) ZPE part, which was absent in the classical theory. Thus, the physical CC emerging from the renormalization program (in any given subtraction scheme) reads

$$\rho_{\rm vac}=\rho_\Lambda^{\rm ren}+\langle V_{\rm eff}^{\rm ren}(\phi)\rangle=\rho_\Lambda^{\rm ren}+V_{\rm ZPE}^{\rm ren}+\langle V_{\rm scal}^{\rm ren}(\phi)\rangle\,. \tag{6.17}$$

The formal structure of this renormalized result is valid to all orders of perturbation theory. For simplicity we have obviated the μ-dependence, which appears implicitly in all couplings and fields, and explicitly in the structure of the terms beyond the tree-level. RG-invariance of physical quantities, initially formulated in the bare theory, tells us that such μ-dependence must finally cancel among all terms in the renormalized theory. Thus $d\rho_{\rm vac}/d\ln\mu = 0$. This cancelation, however, does not mean that the role played by the scale μ is necessarily irrelevant. In fact, since the entire structure of the renormalized $\rho_{\rm vac}$ is parameterized through μ, this fact can help us to unveil the form of the quantum effects in $\rho_{\rm vac}$ and their dependence on relevant physical scales of the problem. In the cosmological case this is only possible if the theory is formulated in curved spacetime and when we consider the renormalization effects on the effective action, not just the effective potential. We will come back to this point later on.

We are now ready to formulate the extreme severity of the fine-tuning problem, which we address here only in QFT in flat spacetime. Indeed, being the expression (6.17) the precise QFT prediction of the physical value of the vacuum energy density to all orders of perturbation theory, it must be equal to the observational measured value [3], i.e. $\rho_{\rm vac} = \rho_\Lambda^0 \simeq 2.5\times 10^{-47}$ GeV^4. We have already seen in section 3 that the lowest order contribution from the Higgs potential is 55 orders of magnitude larger than ρ_Λ^0, and that this enforces us to choose the vacuum term ρ_Λ with a precision of 55 decimal places such that the sum $\rho_\Lambda + \rho_{\Lambda{\rm ind}}$ gives a number of order 10^{-47} GeV^4. The problem is that the fine-tuning game, ugly enough already at the classical level, becomes devastating at the quantum level. Indeed, recall that we have the all order expansions (6.4) and (6.5). Therefore, the quantity that must be equated to ρ_Λ^0 is not just (3.12) but the full *r.h.s.* of (6.17), which as we said is a well-defined (finite and RG-invariant) expression. It means that, instead of the "simple" equation (3.19), we must now fulfill the much more gruesome one:

$$\begin{aligned}10^{-47}\,{\rm GeV}^4 = \rho_\Lambda - 10^8\,{\rm GeV}^4 \quad &+ \quad \hbar\, V_P^{(1)} + \hbar^2\, V_P^{(2)} + \hbar^3\, V_P^{(3)}....\\ &+ \quad \hbar\, V_{\rm scal}^{(1)}(\phi) + \hbar^2\, V_{\rm scal}^{(2)}(\phi) + \hbar^3 V_{\rm scal}^{(3)}(\phi)... \end{aligned} \tag{6.18}$$

As compared to Eq. (3.19), on the *r.h.s.* of the equation we now have, in addition, two independent perturbatively renormalized series contributing to the observed value of the vacuum energy density to one-loop, two-loops, three-loops etc... up to some nth-loop order (both for the ZPE series and the series associated to the quantum corrected Higgs potential). It follows that the numerical value for the "renormalized vacuum counterterm" ρ_Λ must be changed accordingly order by order in perturbation theory. Specifically, the number ρ_Λ on the *r.h.s.* of equation (6.18) must be re-tuned with 55 digits of precision as many times as the number of diagrams (typically thousands) that contribute to the highest nth-loop still providing a contribution to the CC that is at least of the order of the experimental number placed on the *l.h.s.* of that equation. For example, let us roughly assume that each electroweak loop contributes on average a factor $g^2/(16\pi^2)$ times the fourth power of the electroweak scale $v\equiv\langle\phi\rangle\sim 250$ GeV (see section 3), where g is either the $SU(2)$ gauge coupling constant or the Higgs self-coupling, or a combination of both. It follows that the order, n, of the highest loop diagram that may contribute to the measured value of the vacuum energy density, and that therefore could still be subject to fine-tuning, can approximately be derived from

$$\left(\frac{g^2}{16\,\pi^2}\right)^n v^4 = 10^{-47}\,GeV^4\,. \tag{6.19}$$

Take e.g. g equal to just the $SU(2)$ gauge coupling constant of the electroweak SM, which satisfies $g^2/(16\pi^2) = \alpha_{\rm em}/(4\pi\sin^2\theta_W) \simeq 2.5\times 10^{-3}$. This is actually a conservative assumption because in

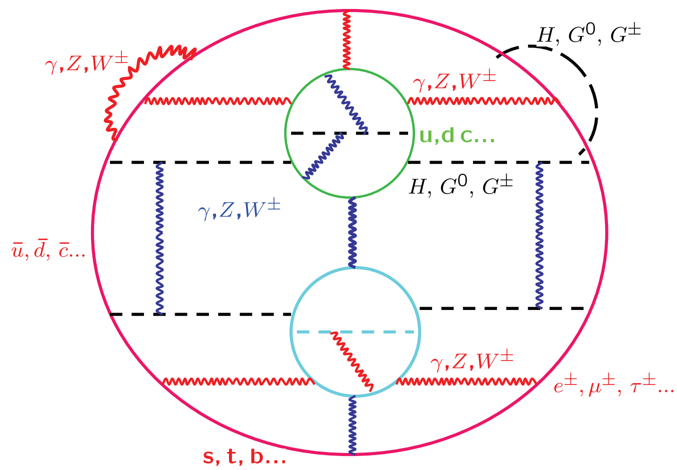

Figure 1: One of the many thousand 21th-loop vacuum-to-vacuum diagram still contributing to the value of the cosmological constant in the Standard Model of Electroweak interactions.

practice there are larger contributions in the SM associated to the big top quark Yukawa coupling, but it will suffice to illustrate the situation. Since $v \equiv \langle\phi\rangle \sim 250$ GeV, we find $n \simeq 21$. Therefore all of the (many thousand) loop diagrams pertaining to the 21th electroweak order (see Fig. 1 for a typical example) are still contributing sizeably to the value of the CC, and must therefore be readjusted by an appropriate choice of the renormalized value of the vacuum term ρ_Λ. This is of course preposterous and completely unacceptable. Even though this is nonsense, it is implicitly accepted by everyone that admits that such "technical trick" is a viable solution to the CC problem. It goes without saying that this situation worsens even more for higher energy extensions of the SM, such as in GUT's. Furthermore, replacing the cosmological term by a cosmic scalar field with some peculiar potential just iterates the same kind of fine tuning problem, let alone that it does not explain why e.g. the electroweak vacuum energy can be hidden under the rug with no relation to the CC value.

Special symmetries (such as Supersymmetry (SUSY), for example) are of little help to solve the CC problem, despite some early hopes [45], since SUSY is necessarily broken, and hence all the above problems just replicate very similarly to the SM case. Only dynamical mechanisms could really help here, namely mechanisms capable of automatically adjusting the CC to the present tiny value even starting from an arbitrarily large one in the early universe. As we know, they were first attempted using scalar fields, but these suffer from a "no-go theorem" [4]. However, these mechanisms are technically possible within generalized dynamical vacuum models with modified gravity (cf. the intriguing proposal in Ref. [6]). Unfortunately, we still don't know which fundamental theory could naturally accommodate them.

7 Zero-point energy in curved space-time

So far so good, but where does the curvature of spacetime enter the previous discussion on the ZPE in Sect. 5 or the Higgs potential in Sect. 6? Nowhere! However, it should be natural to discuss the vacuum energy in a curved background if we aim at elucidating its possible connection to the value of the cosmological term, shouldn't it? In this section, we are going to summarize the effects of curvature for the calculation of the ZPE. The discussion of the Higgs potential part in curved spacetime does not change in any fundamental way the kind of problems that we will meet for the discussion of the ZPE in a curved background, so we shall limit here to briefly consider this last aspect of the vacuum problem and leave the (somewhat bulky) details for a more comprehensive exposition elsewhere [8].

The first and most important observation is that in the presence of vacuum energy (and hence with a nonvanishing value of the CC in Einstein's equations) the constant Minkowski metric $\eta_{\mu\nu} = (+,-,-,-)$ obviously is not a solution of the field equations (2.2). It means that, with $\Lambda \neq 0$, the metric $g_{\mu\nu}$ solving them is a nontrivial one and therefore the spacetime is intrinsically curved. However, this is not what we have assumed in the calculation leading to the result (5.10). Therefore, that calculation is, in principle, not appropriate to reflect any dynamical aspect of the expanding spacetime. In particular, the arbitrary mass scale μ has no immediate connection with any physical quantity of relevant interest, in contrast to the situation of a cross-section calculation in, say, QCD, where one tries to make an association with some characteristic energy scale of the colliding or decaying process in order to estimate the quantum corrections with the help of the RG.

This is certainly a relevant issue for QFT in curved spacetime [31, 32], i.e. the theory of quantized fields on a curved classical background, where the gravitational field itself does not participate of the quantization process. In the functional formulation this means that the generating functional of the theory (from which we can derive the Green's functions from the functional derivatives of it with respect to a classical source $J(x)$) can be written as a path integral over only the quantum matter fields (collectively represented here by ϕ):

$$Z[J] = \int [D\phi] \, \exp\left\{ \frac{i}{\hbar} \left[S\,[\phi, g_{\mu\nu}] + \int d^4x \sqrt{-g(x)}\, J(x)\, \phi(x) \right] \right\} . \tag{7.1}$$

Once more $S[\phi_c]$ in that formula is the classical action for the scalar field, but adapted to the curved spacetime:

$$S[\phi, g_{\mu\nu}] = \int d^4x \, \sqrt{-g(x)} \, \left[\frac{1}{2}\, g^{\mu\nu}\, \partial_\mu\phi\, \partial_\nu\phi + \frac{1}{2}\, \xi\, \phi_c^2\, R - V_c(\phi) \right] . \tag{7.2}$$

A first complication is that this matter action generalizes the original form (5.13) in that we have included in it the generally invariant integration measure $d^4x\sqrt{-g(x)}$; and also a possible non-minimal coupling term $\xi\,\phi_c^2\,R$, where ξ is a dimensionless coefficient, which is necessary for renormalizability (see below).

In principle, the problem under discussion should also be a focus of attention for Quantum Gravity [46], the theory where the gravity field is also quantized together with the matter fields.

Unfortunately, QG is essentially non-perturbative and nonunitary [47], although some interesting perturbative results have been discussed in the literature [48]. Despite countless efforts QG does not exist as a consistent theory yet. String theory seems to be able to address some of the fundamental problems posed by QG, but it is not obvious that it can solve all of them, not even a significant part. In any case we do not wish to discuss them here. Our objective right now is much more modest, and yet not trivial. We deal with the problems of QFT in curved spacetime. In contrast to QG, the former exists as a renormalizable theory in the perturbative sense. It can be considered a fairly successful theory of the external gravitational field in interaction with quantum matter. That beautiful theory has already a long time-honored tradition, collected in several books and reviews [31, 32, 49, 50]. However, not even at present such semiclassical approach to gravity is completely understood. In fact, it is far from being so [33]. A serious hint is the fact that the ZPE contribution to the vacuum energy density in QFT in curved spacetime turns out to be exactly the same as the one we have obtained for flat space, i.e. Eq. (5.10), up to the renormalization of the higher order operators R^2, $R_{\mu\nu}R^{\mu\nu}$... in the vacuum action, including of course the low energy part represented by the Einstein-Hilbert term with cosmological constant (5.8). To see this, let us first recall that in order that the QFT theory becomes renormalizable in curved spacetime, the classical vacuum action must also contain the standard higher derivative (HD) part:

$$S_{\rm HD} = \int d^4x\sqrt{-g}\,\left\{\alpha_1^{(\rm b)}C^2 + \alpha_2^{(\rm b)}\,R^2 + \alpha_3^{(\rm b)}E + \alpha_4^{(\rm b)}\nabla^2 R\right\}\,, \tag{7.3}$$

where, $\alpha_{1,2,3,4}^{(\rm b)}$ are bare parameters, $C^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + (1/3)\,R^2$ is the square of the Weyl tensor and $E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is the topological Euler's density. The coefficients α_i of these terms are then renormalized by the quantum effects, which have the same structure, and in this way we can absorb the new infinities. The theory is thus one-loop renormalizable in curved spacetime. The HD terms are short distance effects which have no impact at low energies, i.e. they do not lead to significant corrections to the Einstein equations (2.2), valid at long distances (the situation of the present, low-energy, universe). In the FLRW metric all these terms are of order $R^2 \sim H^4$ (including $\Box R \sim H^4$), which are negligible for the entire post-inflationary history of the universe. The terms we are interested in are those which are at least of order $R \sim H^2$.

To achieve a finite renormalized theory we proceed as follows. As usual we split each bare coefficient into a renormalized term plus a counterterm:

$$\alpha_1^{(\rm b)} = \alpha_1(\mu) + \delta\alpha_1\,, \qquad\qquad \alpha_2^{(\rm b)} = \alpha_2(\mu) + \delta\alpha_2 \tag{7.4}$$

and similarly,

$$\frac{1}{G^{(\rm b)}} = \frac{1}{G(\mu)} + \delta\left(\frac{1}{G}\right)\,, \qquad\qquad \rho_\Lambda^{(\rm b)} = \rho_\Lambda(\mu) + \delta\rho_\Lambda\,, \tag{7.5}$$

We disregard here the details on the topological term and the total derivative part of the HD action. In doing these splittings the renormalized quantities are μ-dependent because they are supposed to be defined at a given renormalization point μ in RG-space, where they should hypothetically make contact with some experimental input. The point μ is arbitrary, and in fact the wisdom of the RG is that the physics should not depend on its election. However, μ can have a more or less physical meaning depending on the renormalization scheme that is used. Finally, the counterterms can be chosen to cancel the UV divergences. These have been dimensionally regularized in our approach (they appear as poles at $n = 4$) and therefore the cancelation can be easily arranged. We just require that $\delta\alpha_{1,2}$, as well as $\delta(1/G)$ and $\delta\rho_\Lambda$, are chosen so as to make finite the values of the final one-loop parameters. The use of dimensional regularization is quite standard and relatively simple, but on the other hand it is a source of ambiguity when we face the physical interpretation of the results. This ambiguity reaches a climax when we next use minimal subtraction to renormalize the theory. Again, minimal subtraction (whether in the $\overline{\rm MS}$ form or pole subtraction followed with whatever finite part we like) is simple enough, but the renormalized quantities (which become functions of the arbitrary mass scale μ) suffer from a direct physical interpretation. Let us note that we can do all this without still mentioning what is the physical system we have behind, we only know that we are renormalizing the classical vacuum action. The system can be the universe or some other gravitational framework in which we may be interested to study the quantum matter effects. It is obvious that at some point we have to put our system in context, and then assess what could be the physical quantity with which we could dream establishing a possible association with μ. This would, of course, be unnecessary if μ had been a physical subtraction point from the very beginning (say some momentum subtraction point [51]), although this has a calculational price. But if we want to still stay with the mathematical simplicity of dimensional regularization with MS-like subtraction, we are forced to look for such a physical interpretation in the last stage of the calculation. Unfortunately, not even solving this problem should mean the end of our troubles; there are still some other headaches in the list, maybe the most severe ones.

Let us however move on and see. Following the aforementioned procedure, the total effective action (classical plus one-loop corrections) can be conveniently organized as follows:

$$\Gamma = S[\phi_c] + S_{\rm HD} + S_{\rm EH} + \Gamma^{(1)}_{\rm eff} = S[\phi_c] + S^{(1)}_{\rm HD} + S^{(1)}_{\rm EH}\,, \tag{7.6}$$

where

$$S^{(1)}_{\rm HD} == \int d^4x\sqrt{-g}\,\left(\alpha_1^{(1)} C^2 + \alpha_2^{(2)}\, R^2 + ...\right) \tag{7.7}$$

and

$$S^{(1)}_{\rm EH} = -\int d^4x\sqrt{-g}\left(\frac{1}{16\pi G^{(1)}} R + \rho^{(1)}_{\rm vac}\right)\,. \tag{7.8}$$

In the above equations we have absorbed the bare terms and the quantum matter effects in the one-loop coefficients $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$ of the HD action and the one-loop Newton's constant and CC term, $G^{(1)}$ and $\rho_{\rm vac}^{(1)}$, of the low energy EH action [5]. The final result reads

$$\begin{aligned}
\alpha_1^{(1)} &= \alpha_1(\mu) - \frac{\hbar}{2(4\pi)^2}\,\frac{1}{120}\left(\ln\frac{m^2}{\mu^2} + \text{finite const.}\right) && (7.9)\\
\alpha_2^{(1)} &= \alpha_2(\mu) - \frac{\hbar}{4(4\pi)^2}\left(\frac{1}{6}-\xi\right)^2\left(\ln\frac{m^2}{\mu^2} + \text{finite const.}\right) && (7.10)\\
\frac{1}{16\pi G^{(1)}} &= \frac{1}{16\pi G(\mu)} + \frac{\hbar\, m^2}{2(4\pi)^2}\left(\frac{1}{6}-\xi\right)\left(\ln\frac{m^2}{\mu^2} + \text{finite const.}\right) && (7.11)\\
\rho_{\rm vac}^{(1)} &= \rho_\Lambda(\mu) + \frac{m^4\,\hbar}{4\,(4\,\pi)^2}\left(\ln\frac{m^2}{\mu^2} + \text{finite const.}\right). && (7.12)
\end{aligned}$$

The parameters $P_i^{(1)} = \alpha_1^{(1)},\ \alpha_2^{(1)},\ 1/G^{(1)},\ \rho_{\rm vac}^{(1)}$ on the *l.h.s.* of these expressions are the final "one-loop parameters"; they are finite at one loop because we have used the counterterms to cancel the divergences coming from the one-loop contributions to them. The very fact that such redefinition of the original coefficients of (7.3) can be made shows the practical possibility to renormalize the theory. It does not mean, however, that we can get an immediate physical interpretation of the renormalized expressions. The arbitrariness of the finite constant terms in the expressions (7.12) is only a small hint of the ambiguity and lack of direct interpretation of these formulas.

If we would continue the calculation at 2-loops, the parameters $P_i^{(1)}$ would play the role of bare parameters, in a similar way as the bare parameters (7.4)-(7.5) of the classical action, and therefore they would be UV-divergent quantities that split once more into a renormalized parameter and a corresponding counterterm. These counterterms would then be used to cancel the 2-loop divergences etc. As parameters of the bare action, all of the $P_i^{(1)}$ are of course independent of the arbitrary renormalization point μ. The explicit μ-dependence of the various terms in each $P_i^{(1)}$ must cancel in the overall expression. One source of μ-dependence comes from the corresponding "renormalized parameters", $P_i(\mu)$, which therefore "run" with the scale μ. However, as already warned before, at this point such "running" has no obvious relation with the variation of any physical quantity; whether such connection is possible cannot be unambiguously decided in the absence of a concrete physical framework. In the next section, however, we place these formulae in the cosmological context and only then some chance exists for a possible physical interpretation.

Mathematically, the μ-running is determined by the corresponding renormalization group equation (RGE) for each of the parameters, and follows from setting the total derivatives of the one-loop parameters $P_i^{(1)}$ with respect to μ to zero. For convenience we compute the logarithmic derivatives of each one of them, i.e. $dP_i^{(1)}/d\ln\mu = 0$. The explicit RGE's are immediately obtained from

[5]These are the quantum effects from the matter field ϕ encoded in the explicit computation of the one-loop part of the effective action, Eq. (5.16), in the curved space-time case. This is of course the hardest part of the calculation, that I am fully sparing to the reader here – see e.g. [31, 32]; confer also [8] for the details along the present lines.

equations (7.9)-(7.12):

$$\begin{aligned}
\frac{d\alpha_1(\mu)}{d\ln\mu} &= -\frac{\hbar}{120(4\pi)^2} \equiv \beta_1^{(1)} && (7.13)\\
\frac{d\alpha_2(\mu)}{d\ln\mu} &= -\frac{\hbar}{2(4\pi)^2}\left(\frac{1}{6}-\xi\right)^2 \equiv \beta_2^{(1)} && (7.14)\\
\frac{d}{d\ln\mu}\left(\frac{1}{16\pi G(\mu)}\right) &= \frac{\hbar\, m^2}{(4\pi)^2}\left(\frac{1}{6}-\xi\right) \equiv \beta_{G^{-1}}^{(1)} && (7.15)\\
\frac{d\rho_\Lambda(\mu)}{d\ln\mu} &= \frac{\hbar\, m^4}{2\,(4\pi)^2} \equiv \beta_\Lambda^{(1)}\,. && (7.16)
\end{aligned}$$

Notice that the obtained RGE's do not depend at all on the unspecified finite terms indicated in equations (7.9)-(7.12). The *r.h.s*'s of these expressions define the corresponding one-loop β-functions that control the running of these parameters in the renormalization scheme we have used, an MS-based one. Note, in particular, the RGE for ρ_Λ, Eq. (7.16). It coincides exactly with Eq. (5.11), the flat spacetime result. Integration of (7.16) immediately furnishes the one-loop finite result

$$\rho_{\rm vac}^{(1)} = \rho_\Lambda(\mu) + \frac{m^4\,\hbar}{4\,(4\,\pi)^2}\left(\ln\frac{m^2}{\mu^2} + \text{finite const.}\right)\,. \tag{7.17}$$

The finite additive constant in this formula is not very important, as it depends on the kind of MS-based subtraction scheme we use. For exact $\overline{\rm MS}$, the constant must of course be the same one as in Eq. (5.10), but this is quite irrelevant as it does not change at all the worrisome aspect of the result, namely the fact that we get a quantum "correction" to the vacuum energy density growing as $\sim m^4$, for a particle of mass m. So we are back to the numerological disaster first struck by Zeldovich (cf. Sect.4), which forced him to change his strategy to estimate the vacuum energy in particle physics. Somehow we have not advanced a single step since then, despite much QFT in curved spacetime! This is the result we mentioned before, and it comes as a kind of an "unexpected" surprise. It certainly does not help shed light on our poor understanding of the vacuum energy density in QFT in curved spacetime. We will come back to it in the next section.

In view of the situation, it would be unwise to rush into conclusions at this point. For example, it would make little sense to evaluate the found formula for the vacuum energy density by using some value of μ and, say, the full collection of particle masses of the SM of strong and electroweak interactions, or whatever extension of it. First, because on the face of the obtained result we feel that we are somehow in deep water and we don't know what is the next surprise we'll come across in this story; and, second, because it is obvious that the contribution from a single mass, being proportional to the quartic power of it, is completely out of range (cf. our "numerology discussion" in Sect. 4). The only possible exception is perhaps a very light neutrino mass, see Eq. (4.8), or the existence of new degrees of freedom at a similar mass scale that would determine the behavior of the cosmological term [43].

We already mentioned quantum gravity as a dreamed theory in this field, and string theory as a promising approach to it. But unfortunately neither QG nor string theory have been able to solve the vacuum energy problem either. As a matter of fact no theoretical framework at present is capable to provide a fully consistent and realistic account for the vacuum energy density in the cosmological context. Although some of the technical problems can be glimpsed in the more modest curved QFT arena [33], the situation is far from being completely (not even substantially) understood. We feel that if the problem cannot be minimally dealt with at the level of unquantized gravity, the role of QG may not be such decisive to settle this issue, especially if we limit the scope of the CC problem to the scale of the SM of strong and electroweak interactions (which is extremely far away from the Planck scale). We should judiciously expect that QFT in curved spacetime is competent enough in this energy domain so as to provide a first reliable hint of the possible solution, i.e. something that goes beyond the "cul de sac" situation we have now.

8 Dynamical vacuum energy in an expanding universe

Let us focus here on the vacuum part of the effective cosmological term, i.e. we are now mainly concerned with the ZPE in curved spacetime. With the traditional approach the metric expansion is usually made around flat space, i.e. $g_{\mu\nu} = \eta_{\mu\nu}+h_{\mu\nu}$. If one traces the details of the calculation [8], one can easily see that this is the reason why the result (7.17) is the same as in flat space, Eq. (5.9). In fact, the result is entirely dependent on the free propagator in flat spacetime, which appears as the first term of the adiabatic expansion of the Green's function in the curved case. This can be troublesome since we are dealing with the renormalization of the cosmological term, namely a term which does *not* exist in Minkowski space. Therefore, in those situations when the spacetime background is unavoidably curved, sticking to the expansion $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ misses the precise dynamical correction to the vacuum energy density that we are looking for. Although the root of the problem has been identified [33], the remedy for it is not available yet. One needs to compute not only the divergences, but also the first finite correction using a physical renormalization scheme (beyond $\overline{\rm MS}$ or the like); and, above all, one has to learn how to cope with the technical limitations of the current QFT methods for computing the renormalization corrections around a non-trivial background. For de Sitter background, for example, one expects $\rho_{\rm vac} \sim H^4$ by pure dimensional analysis, and indeed particular calculations reach that kind of conclusion [52, 53]. But no realistic attempt has ever been made for a FLRW background, where we should expect both $M_i^2 H^2$ and H^4 terms, where M_i are the particle masses. Such calculation should be a rather nontrivial one, perhaps unreachable by the presently known methods. However we expect that it should lead to a RGE of the form:

$$\frac{d\rho_\Lambda}{d\ln H^2} = \frac{1}{(4\pi)^2}\sum_i \left[a_i M_i^2\, H^2 + b_i\, H^4 + c_i \frac{H^6}{M_i^2} + ...\right] , \tag{8.1}$$

in which $\mu = H$ should sensibly act as the natural running scale in the cosmological context. Obviously the $\mathcal{O}(H^6/M_i^2)$ terms represent the decoupling contributions that should appear in a physical renormalization scheme. Notice that the scheme used in Sect. 7 was off-shell and it could not trace the decoupling effects. The above equation should describe the "change in the value of the vacuum energy density" (triggered by the quantum matter effects) associated to a "change in the curvature of the spacetime", this curvature being of order $R \sim H^2$ in the FLRW metric. It is well known that renormalization theory can only be predictive if we first input the value of the relevant parameters at a given renormalization point. It is only then when we can predict the value at another point. For instance, the e.m. fine structure constant at the scale of the electron mass, $m_e \simeq 0.5$ MeV $= 5\times 10^{-4}$ GeV, is $\alpha(m_e) \simeq 1/137$, whereas its value at the scale of the Z-boson mass, $M_Z \simeq 90$ GeV, is $\sim 7\%$ larger: $\alpha(M_Z) \simeq 1/128$. The RG approach cannot pretend to "compute" $\alpha(m_e)$ or $\alpha(M_Z)$, only the change from one value to the other when moving from $\mu = m_e$ to $\mu = M_Z$, where here we make a direct association of μ with the physical scale of the particle masses. Similarly, renormalization theory cannot aim at computing "the value" itself of the vacuum energy density and the CC, but only the evolution or running of this value after we have measured it at some cosmic energy scale. This should suffice for an effective QFT study of the running vacuum energy density, quite different from the more ambitious attempts at predicting its current value (the old CC problem [4]). These attempts are probably doomed to fail until we can first cope with the prediction, or a more fundamental understanding, of the fundamental parameters of the SM, if that is possible at all.

8.1 Running gravitational coupling and vacuum energy

While it is not possible to derive the general RGE for ρ_Λ in QFT in curved spacetime of the form given in (8.1) yet, we may be able to hint at the expected dynamical effects induced by the quantum corrections in some indirect way.

If we look at the vacuum diagrams of Fig. 2, we notice the following. The simplest diagram there is the classic "blob" with just a closed loop line of the scalar field. This one is already present in the flat spacetime case. However, the presence of the external gravitational field introduces "hair" (i.e. $h_{\mu\nu}$-tails of the classical field departing from the flat space structure). This modifies the originally "bald" quantum vacuum blob, and for this reason we have many other blob diagrams in Fig. 2, which we call the "haired" ones. The infinitely many tails are induced on all possible diagrams that are preexisting in flat space, and are generated by the non-polynomial expansion of the factor $\sqrt{-g} = 1 + \frac{1}{2}\,h + \mathcal{O}\left(h^2\right) + \mathcal{O}\left(h_{\mu\nu}\,h^{\mu\nu}\right) + ...$ in the action (7.2) of the scalar field, which we take as a free field here, i.e. $V(\phi) = (1/2)\,m^2\,\phi^2$. For the particular case of the vacuum diagrams under consideration, the expansion of $\sqrt{-g}$ is performed in the one-loop correction term of the effective action in flat space, Eq. (5.16), after we replace $d^4x \to \sqrt{-g}\,d^4x$ in it so as to account for the curvature effects. The previously computed renormalization effects from the matter field ϕ on the parameters of the purely geometric vacuum action $S_{EH} + S_{HD}$ (conf. Sect. 7) can now be viewed diagrammatically in Fig. 2, which leads more intuitively to a possible physical interpretation after we put our external gravity system in context – the FLRW cosmological one in this case.

What is important from this diagrammatic representation is to realize that the first "blob" does not interact with the curved background at all, as it has no external tails of the gravitational field. So any result that depends solely on that "bald" diagram contains the same information as in flat space. This is what happens with the unsuccessful Eq. (7.17). In contrast, the 'haired blobs" do interact with the background geometry. Let us concentrate on the first "haired blob". It contains one insertion point where infinitely many tails of the external gravitational field $h_{\mu\nu}$ can be attached. That blob, as all the other 'haired" ones, is pumping external energy-momentum from the FLRW background. The typical magnitude of the momentum should be of the order of H, which has dimension of energy in natural units. In this sense the association $\mu = H$ looks quite

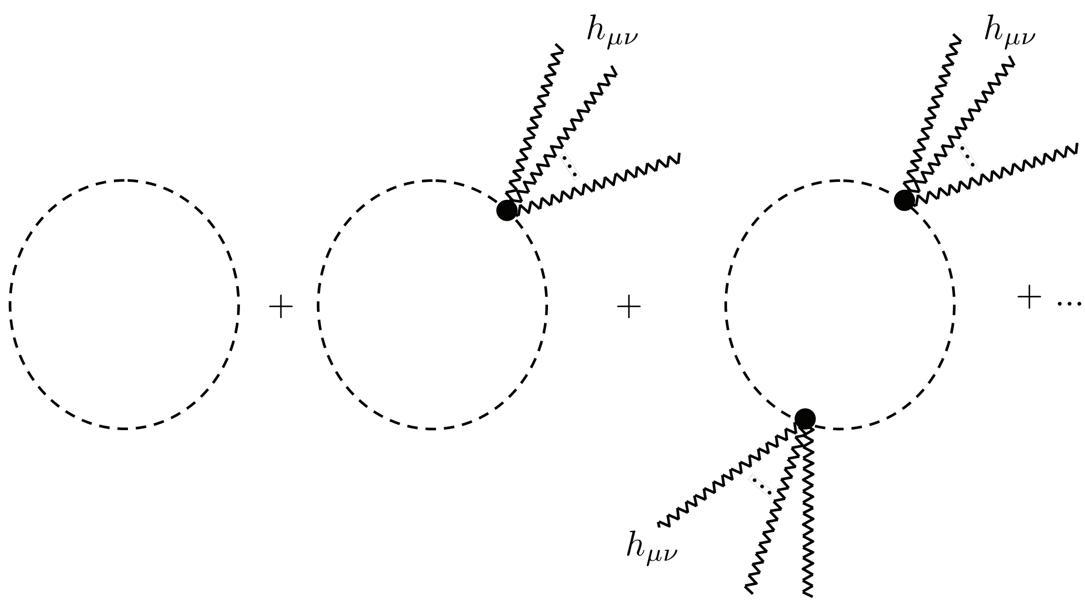

Figure 2: The one-loop vacuum-to-vacuum diagrams of the scalar matter field in the presence of an external gravitational field. The first type of contribution is the "bald blob" with no external tails, which is quartically divergent. The other contribution are the "haired blobs" with one or more external field insertion, where an arbitrary number of $h_{\mu\nu}$-tails of the background field are attached to one or more points. They appear from the expansion of the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and the corresponding determinant in the action: $\sqrt{-g} = 1 + \frac{1}{2}\,h + \mathcal{O}\left(h^2\right) + \mathcal{O}\left(h_{\mu\nu}\,h^{\mu\nu}\right) + \ldots$ with $(h \equiv g_{\mu\nu} h^{\mu\nu})$. The first haired vacuum diagram with one insertion is quadratically divergent, as there is one propagator of the scalar field. The bunch of tails organize themselves in a covariant way to generate an action term of the form $\sqrt{-g}\,m^2\,R$, and hence renormalize the inverse gravitational Newton's coupling $1/G$ in the EH action (7.8). The third type of diagram is the "doubly haired blob" and contains two insertion points. With two propagator lines, it is only logarithmically divergent; it renormalizes the coefficients $\alpha_{1,2}$ of the HD-action (7.7). The diagrams with three or more insertions of the external field are finite. In the flat spacetime case only the "bald blob" contributes, so the dynamics of the vacuum energy in an expanding universe must come from the "haired blobs", which may pump in energy from the expanding background into the vacuum matter loops.

reasonable and physically intuitive, but only after we have placed the system in the cosmological context. This is similar to the correspondence of μ with a momentum variable q in particle physics processes, although admittedly in the cosmological case the interpretation is less straightforward, specially in the low energy domain.

The bunch of $h_{\mu\nu}$-tails in the first "haired" diagram organize themselves in a covariant way to generate an action term of the form $\sqrt{-g}\,m^2\,R$, and hence renormalize the inverse gravitational Newton's coupling $1/G$ in the EH action (7.8). The precise RGE for this coupling was given before in Eq. (7.15). At this point this does not yet lead to a renormalization of the vacuum energy, but it can be related to it in an indirect way, as we shall see. Whereas in the absence of a physical context the μ-dependence would not be of much help, if we now follow the aforementioned ansatz $\mu = H$ within the cosmological context, we can immediately integrate (7.15) to find (in natural units):

$$\frac{1}{G(H)} = \frac{1}{G_0} + \frac{m^2}{2\pi}\left(\frac{1}{6} - \xi\right)\,\ln\frac{H^2}{H_0^2}\,, \tag{8.1.1}$$

where $G_0 \equiv G(H_0) = 1/M_P^2$ is the current value of the gravitational coupling. Equivalently,

$$G(H) = \frac{G_0}{1 + \nu\,\ln\left(H^2/H_0^2\right)}\,, \tag{8.1.2}$$

where $\nu = (1/2\pi)\,\left(\frac{1}{6} - \xi\right)m^2/M_P^2$ is a dimensionless coefficient which acts as the reduced β-function for the running of the gravitational coupling with the physical scale H. Since the coefficient ξ in the previous equation is not determined and the number of participating matter fields is arbitrary, we can generalize ν in the form

$$\nu = \frac{1}{2\pi}\sum_i\left(\frac{1}{6} - \xi_i\right)\frac{m_i^2}{M_P^2}\,. \tag{8.1.3}$$

It is important to realize that if the running gravitational coupling as a function of $\mu = H$ is given by Eq. (8.1.2), we cannot continue with Eq. (7.17) as a valid RGE for the running vacuum energy density, since there is a link between the running of $G(H)$ and the running of $\rho_\Lambda(H)$ which must be preserved. That link is enforced by the Bianchi identity satisfied by the Einstein tensor on the *l.h.s.* of Eq. (2.1), namely $\nabla^\mu G_{\mu\nu} = 0$. There are also the contributions provided by the HD terms in the action (7.3), which modify of course the complete field equations. These higher order effects are represented in diagrammatic form by the "double haired" diagrams in Fig. 2. But at low energies we can disregard them since they entail $\mathcal{O}(H^4)$ corrections which are negligible. In this way we obtain the following relation for the source term on the *r.h.s.* of Einstein's equations:

$$\nabla^\mu\left(G\,\tilde{T}_{\mu\nu}\right) = \nabla^\mu\,\left[G\left(T_{\mu\nu} + g_{\mu\nu}\,\rho_\Lambda\right)\right] = 0\,. \tag{8.1.4}$$

Using the FLRW metric (2.4) and the standard energy-momentum tensor for matter in the form of a perfect fluid, Eq. (2.5), a straightforward calculation from (8.1.4) provides the following "mixed" local conservation law:

$$\frac{d}{dt}\,\left[G(\rho_m + \rho_\Lambda)\right] + 3\,G\,H\,(\rho_m + p_m) = 0\,. \tag{8.1.5}$$

Therefore, if $G(t) = G(H(t))$ is evolving with the expansion rate in some particular way, the previous identity enforces $\rho_\Lambda = \rho_\Lambda(H(t))$ to evolve accordingly. Let us assume that $G(t) = G(H(t))$

runs with H as in Eq. (8.1.2). Let us also assume at this point that matter is covariantly conserved, i.e.

$$\dot{\rho}_m + 3\,H\,(\rho_m + p_m) = 0 \quad \Rightarrow \quad \rho_m(a) = \rho_m^0\,a^{3(1+\omega_m)}\,, \tag{8.1.6}$$

where $p_m = \omega_m \rho_m$ is the equation of state of matter. It is then easy to show that (8.1.5) boils down to

$$(\rho_m + \rho_\Lambda)\dot{G} + G\dot{\rho_\Lambda} = \left[(\rho_m + \rho_\Lambda)\frac{dG}{dH} + G\frac{d\rho_\Lambda}{dH}\right]\dot{H} = 0\,. \tag{8.1.7}$$

Obviously $\dot{H} \neq 0$, so we can equate to zero the expression in the parenthesis.

In the following we stick to flat space geometry, i.e. we take $K = 0$ in Eq. (2.6), and hence the metric of spacetime becomes $ds^2 = dt^2 - a^2(t)d\mathbf{x}^2$. After all, this seems to be the most plausible possibility in view of the present observational data [3] and the natural expectation from the inflationary universe. Friedmann's equation for flat space just reads

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda)\,. \tag{8.1.8}$$

Combining this equation with the Bianchi identity (8.1.7) a simple differential equation for ρ_Λ as a function of the Hubble rate emerges. Let us note that G in (8.1.8) is not constant in the present instance, but given by (8.1.2). Solving for $\rho_\Lambda = \rho_\Lambda(H)$ we find a simple "affine" quadratical law:

$$\rho_\Lambda(H) = c_0 + \frac{3\nu}{8\pi}\,M_P^2\,H^2\,, \tag{8.1.9}$$

ν being here, of course, the same coefficient defined in (8.1.3). This is a nice equation. With it we have found an explicit realization of the kind of dreamed law for the vacuum energy density suggested in Eq. (4.7), with $\beta = 3\nu/8\pi$.

Interestingly enough the obtained equation (8.1.9) is precisely of the same form as the one that would follow from solving the previously mentioned general RGE (8.1) provided we restrict the latter to the current universe, namely when the $\mathcal{O}(H^4)$ terms can be neglected, which is consistent with the approximation we used to derive (8.1.9). Integrating (8.1), and comparing with (8.1.9) it follows that ν must also be given by

$$\nu = \frac{1}{6\pi}\sum_{i=f,b} a_i \frac{M_i^2}{M_P^2}\,, \tag{8.1.10}$$

in which only the coefficients a_i of the first term on the *r.h.s* of (8.1) are involved. We shall adopt this equation as it is more general and assumes independent contributions from bosons and fermions with different multiplicities. Normalizing the obtained result (8.1.9) with respect to the current CC value $\rho_\Lambda(H_0) = \rho_\Lambda^0$, we can rewrite it as

$$\rho_\Lambda(H) = \rho_\Lambda^0 + \frac{3\nu}{8\pi}\,M_P^2\,(H^2 - H_0^2)\,. \tag{8.1.11}$$

where c_0 is related with the current value of the vacuum energy as $\rho_\Lambda^0 = c_0 + (3\nu/8\pi)\,M_P^2\,H_0^2$. The present value of the Hubble rate is $H_0 \equiv 100\,\mathrm{h}\,Km/s/Mpc = 1.0227\,h \times 10^{-10}\ \mathrm{yr}^{-1}$. The observations give $h \simeq 0.70$ (e.g. $h = 0.673 \pm 0.012$ from PLANCK [3]).

As explained around Eq. (4.7) in Sect. 4, the presence of the affine term $c_0 \neq 0$ is crucial for a realistic implementation of the model. A vacuum energy evolving only as $\sim H^2$ (with $c_0 = 0$) would be incompatible with the transition from deceleration to acceleration [40].

Let us emphasize that the affine quadratic law (8.1.11) insures a mild evolution of the vacuum energy density, specially if the parameter $|\nu| < 1$. We shall consider observational limits on this parameter in Sect. 8.5, but it is obvious from its definition (8.1.3) or (8.1.10) that the natural theoretical range expected for it is $|\nu| \ll 1$. This expectation will be confirmed by the phenomenological analysis. What is utmost important to note is that no term of the form $\sim M_i^4$ drives now the evolution of the vacuum energy. Indeed, this kind of terms are incompatible with the mild running of the gravitational coupling, as determined by Eq. (8.1.2). This equation, together with the covariant constraint imposed by the Bianchi identity (8.1.7), imply that the cosmological term must evolve in the form (8.1.11). Furthermore, from the renormalization group point of view the $\sim M_i^4$ terms are not expected on the *r.h.s.* of the general RGE (8.1) either, as otherwise this would mean that a particle with mass M_i is an active degree of freedom for the running of ρ_Λ. Since, however, the running is parameterized by the scale $\mu = H$, this would entail $H > M_i$, which is impossible for any known particle mass at any time in the matter dominated epoch (recall that $H_0 \sim 10^{-42}$ GeV) [6].

To summarize, while a direct calculation of the dynamical renormalization effects on the cosmological term is not possible at the moment without expanding around a non-flat background in the presence of massive fields and within a physical renormalization scheme, at least an indirect hint of the result should be glimpsed by requiring the consistency of the renormalization effects on the different terms of the effective action. As these terms are linked by the Bianchi identity, the possible quantum effects are tied to the general covariance of the theory. This requirement could give us an indirect clue, which can be read off on more physical grounds from the diagrams of Fig. 2. We suggested that while for flat spacetime only the "bald blob" contributes, the dynamics of the vacuum energy in an expanding universe should emerge from the "haired blobs", which may pump in energy from the expanding background into the vacuum matter loops. At low energies we found that this procedure indicates that the renormalization effects impinged upon the cosmological term of the vacuum action consists in the $\mathcal{O}(H^2)$ terms reflected in Eq. (8.1.9). These are of course the same kind of quantum effects that should be found from a direct computational approach, when it will be technically feasible.

8.2 A comment on, and an analogy with, the Casimir effect

In the meanwhile our roundabout path gives us a handle on the type of effects we are looking for. It is encouraging to see that the obtained result by the indirect procedure is free from the

[6] Even if we go to the radiation dominated epoch, at temperature T, we find from Friedman's equation (with $\rho_m \sim T^4$) that to satisfy the condition $H > M_i$ roughly means $T^2/M_P > M_i$, or equivalently $M_i^4/T^4 < M_i^2/M_P^2 \ll 1$ (for any particle). Hence, at the time when the $\sim M_i^4$ vacuum contributions start being active they are negligible as compared to the radiation contribution $\sim T^4$. We conclude that, in the RG formulation, the terms $\sim M_i^4$ remain innocuous throughout the entire cosmic history: for, at low energies, these terms are not allowed since the condition $H > M_i$ can never be fulfilled, whilst the effect becomes completely irrelevant at high energies (when that condition is possible).

unwanted $\sim m^4$ contributions that plague the traditional approach. Somehow the $\mathcal{O}(H^2)$ terms are the real effects that remain in the expanding spacetime after we subtract the flat spacetime result — quite in the same manner as when in the calculation of the Casimir effect (see e.g. [54, 55] for a review) one removes the divergence of the result upon subtracting the vacuum energy density when there are no plates. Recall that in the Casimir effect the (attractive) force between a pair of neutral, parallel conducting planes is due to the disturbance of the quantum electrodynamics (QED) vacuum caused by the presence of the boundaries. Since at zero temperature there are no real photons in between the plates, it should be the vacuum alone, i.e., the ground state of QED which causes the plates to attract each other [7]. The pressure, or force per unita area (F/A), on the plates is a pure quantum effect (proportional to $\hbar$) that goes as the inverse of the quartic power of the distance a between the plates, i.e.

$$F/A = -\frac{1}{A}\frac{dE}{da} = -\hbar\, c\, \frac{\pi^2}{240\, a^4}\,. \tag{8.2.1}$$

The Casimir effect is caused by the difference between the vibrational modes of the QED vacuum in between the plates as compared to the region outside. While the ZPE itself may not be measurable, "changes in the ZPE" are detectable. The effect would, of course, be dynamical if the distance between the plates would change in time, $a = a(t)$. Similarly, we may view the evolution of the vacuum energy in an expanding background with (dynamical) curvature $R \sim H^2(t)$, as the change that remains of the disturbed vacuum energy density after we remove the flat spacetime result – which is also contained in the curved spacetime calculation, Eq. (7.17). The mass term m^4 there is replaced here by $1/a^4$ (note that no mass contribution is possible for the Casimir effect in QED since photons are massless). Now, while the m^4 term is removed when the flat spacetime result is subtracted, the term $1/a^4$ remains in the Casimir effect because the plates, of course, stay. In the Casimir effect one regularizes the infinite mode sum $\sum_{\mathbf{k}} \frac{1}{2}\, \hbar\omega_{\mathbf{k}}$ by subtracting the infinite value of the ZPE when the plates are infinitely apart. This infinite quantity cancels against the infinite ZPE value corresponding to the two plates being at finite distance a, and in this way the final, and finite, result (8.2.1) emerges – reflecting the "differential effect" caused by the inner modes only. But notice that if letting $a \to 0$, we would strike another infinite value. This is a short distance effect or UV-divergence. The new infinity ought again to be subtracted because it corresponds to a situation where the space between the plates disappears, so no "distinctive" standing waves can form in that limit. Such situation should be the equivalent of subtracting the unphysical $\sim m^4$ term in the vacuum energy density of free fields, as it appears in taking care of the UV divergences through the MS-renormalization procedure both in Minkowski and curved spacetime cases [8].

At the end of the day only the $\mathcal{O}(H^2)$ disturbance, along with the additive c_0 term in Eq. (8.1.9), are left at low energy as the final measurable output for the curved spacetime case. Let us emphasize once more that both terms are obtained in the RG formulation, and in particular the additive term c_0 is absolutely essential for a correct phenomenological description of the transition from deceleration to acceleration. This term appears in a natural way in the present framework from the integration of the general RGE (8.1) for the vacuum energy in an expanding spacetime.

[7] For this reason the Casimir effect is usually advocated as experimental evidence for the ZPE. See [56] though.

Interestingly, a kind of equation such as (8.1.11) was suggested in [33], and previously in [57, 58, 59, 60] on more phenomenological grounds. But the first hint that an expression of that sort could be somehow derived from a specific QFT framework was provided in Ref. [61] in a conformal field theory context. More recently there have appeared alternative QFT frameworks suggesting a similar kind of evolution of the vacuum energy leading once more to dynamical terms $\sim H^2$ [62]. See also [63] for other interesting considerations along these lines.

8.3 Extension to the early universe

We expect that the low energy running vacuum model (8.1.9) can be generalized in the form (8.1) so as to include the important effects from the higher powers of H at the early inflationary times. A unified model of this kind, where the dark energy emerges at late times as a "fossil" of the early inflationary universe was presented in [61] in the framework of the generalized anomaly induced inflation [64]. These models of the early universe, combined with the low energy theory, suggest that the effective expression of the vacuum energy density should be a combination of even powers of H, essentially H^2 and H^4. Recently, in Ref. [65, 66, 67] a detailed analysis of the complete cosmic history of the universe has been presented by considering the class of models of the form

$$\rho_\Lambda(H) = c_0 + \frac{3\nu}{8\pi} M_P^2 H^2 + \frac{3\alpha}{8\pi} M_P^2 \frac{H^{2n}}{H_I^{(2n-2)}}\,, \tag{8.3.1}$$

in which $n \geqslant 2$. Clearly, this expression is an extension of Eq. (8.1.9) along the lines of the general RGE (8.1). The higher power of H should obviously be operative only for large values of H near the inflationary scale H_I (presumably a GUT scale not very far from the Planck scale) [8]. Typically we expect $n = 2$, i.e. a high energy behavior $\sim H^4$ [65]. At present, the H^4 term is of course negligible and we are effectively left with the low energy theory (8.1.9). Notice that the dimensionless coefficient α (enabling the running of the vacuum energy near the GUT scale) can be related to the coefficients of the general RGE (8.1) as follows:

$$\alpha = \frac{1}{12\pi} \frac{H_I^2}{M_P^2} \sum_{i=f,b} b_i\,. \tag{8.3.2}$$

We point out that while we use only even powers of the Hubble rate, odd powers have also been considered phenomenologically in the old literature for the treatment of the inflationary stages [69]. More recently one can also find models where odd powers are used for describing the CC evolution for the present universe, or in combination with even powers for the early universe [38, 70]. Finally, let us mention that powers of H, including linear ones, have also been used to describe the late time evolution in terms of the so-called bulk viscosity parameters, but again this is a purely phenomenological treatment of the purported DE fluid [71]. Let us insist that the general form that we propose in (8.3.1) involves only even powers of the expansion rate, at all stages of the cosmological evolution, as a most *fundamental requirement* from the general covariance of the effective action of QFT in curved spacetime [33].

[8] See Ref. [68] for interesting thermodynamical considerations on this kind of $\Lambda(H)$ dynamical vacuum models.

We shall not dwell here on the phenomenological application of the class of unified vacuum models (8.3.2), see [65, 66, 67] for details. It suffices to say that all these models start from an inflationary phase and, for all $n > 2$, they automatically reach "graceful exit" from the inflationary phase into a standard radiation dominated epoch, which is already an achievement. In general they provide an effective framework containing most of the basic ingredients that should probably be desirable for a future fundamental theory of the cosmic evolution, namely a theory capable of tackling efficiently the important cosmological problems which are still pending. And, most important, the late time cosmic expansion history is very close to the standard ΛCDM model.

8.4 Different scenarios for running cosmological parameters at low energies

We discuss now some possible scenarios for running cosmological parameters. We shall focus here on the implications for the low energy regime of the cosmological evolution and therefore we do not consider the highest powers of H introduced in Eq. (8.3.1), it will be enough to consider the H^2 dynamical effects. Once more we adopt the spatially flat FLRW metric, i.e. $K = 0$ in Eq. (2.4), for all models to be discussed henceforth. In such conditions, the relevant Friedmann's equation providing the Hubble rate reads as in Eq. (8.1.8). As stated, we assume that $\rho_\Lambda = \rho_\Lambda(t)$ and $G = G(t)$ can be functions of the cosmic time t. Furthermore, the dynamical equation for the acceleration of the universe is given by the expression (2.7), or equivalently

$$\frac{\ddot{a}}{a} = -\frac{4\pi\, G}{3}\left(\rho_m + 3p_m - 2\rho_\Lambda\right) = -\frac{4\pi\, G}{3}\left(1 + 3\omega_m\right)\rho_m + \frac{8\pi\, G}{3}\,\rho_\Lambda\,. \tag{8.4.1}$$

In the late universe ($\rho_m \to 0$) the vacuum energy density ρ_Λ dominates. It accelerates the cosmos for $\rho_\Lambda > 0$. This may occur either, because ρ_Λ is constant, and for a sufficiently old universe one finally has $\rho_m(t) < 2\,\rho_\Lambda$, or because $\rho_\Lambda(t)$ evolves with time, and the situation $\rho_\Lambda(t) > \rho_m(t)/2$ is eventually reached sooner or later than expected.

Let us come back to the general covariant conservation law, Eq. (8.1.5), and consider various possibilities [9]:

- **Model I**: G =const. and ρ_Λ =const.:

  Under these conditions and in the absence of other components in the cosmic fluid, apart from matter and a strictly constant Λ-term, the local covariant conservation law of matter-radiation is strictly fulfilled, i.e. Eq. (8.1.6). If, in addition, we have zero spatial curvature, $K = 0$, then Model I becomes the almost thirty years old flat ΛCDM, or "concordance" cosmological model [72], viz. the currently reigning standard cosmological model.

- **Model II**: G =const and $\dot{\rho}_\Lambda \neq 0$:

  Here Eq.(8.1.5) leads to the mixed conservation law:

$$\dot{\rho}_\Lambda + \dot{\rho}_m + 3\,H\left(\rho_m + p_m\right) = 0\,. \tag{8.4.2}$$

[9]The local covariant conservation equation (8.1.5) is not independent from equations (2.6) and (2.7), as it is actually a first integral of the system formed by the last two. For example, it is easy to check from equations (2.6) and (8.1.5) that the acceleration equation for the scale factor takes the usual form (2.7) – or equivalently (8.4.1) – even for time evolving G and ρ_Λ (and for nonvanishing spatial curvature K).

An exchange of energy between matter and vacuum takes place. The model can be solved only if more information is provided on e.g. the cosmic evolution of ρ_Λ, see the next section.

- **Model III**: $\dot{G} \neq 0$ and ρ_Λ =const.:

$$\dot{G}(\rho_m + \rho_\Lambda) + G[\dot{\rho}_m + 3H(\rho_m + p_m)] = 0\,. \tag{8.4.3}$$

Since G does not stay constant here, this equation implies matter non-conservation. It could be solved e.g. for G, if the (anomalous) cosmic evolution of ρ_m would be given by some particular ansatz.

- **Model IV**: $\dot{G} \neq 0$ and $\dot{\rho}_\Lambda \neq 0$:

Although several possibilities are available here, the simplest one is of course the framework that has motivated our analysis in the previous section, where matter is covariantly conserved and the dynamical interplay occurs between G and ρ_Λ through Eq. (8.1.7). The cosmological equations for this model were solved in Sect. 8.1. In the next section we solve also the "running" Models II and III and compare with the present one.

8.5 Solving the cosmological equations for Models II and III

The class of Models II and III is quite general and we cannot solve them unless we provide some more information, similarly to the situation with Model IV. Let us first concentrate on solving the class of scenarios denoted as Model II. Let ρ_M^0 be the total matter density of the present universe, which is essentially non-relativistic ($\omega_m \simeq 0$). The corresponding normalized density is $\Omega_M^0 = \rho_M^0/\rho_c^0 \simeq 0.3$, where ρ_c^0 is the current critical density. Similarly, $\Omega_\Lambda^0 = \rho_\Lambda^0/\rho_c^0 \simeq 0.7$ is the current normalized vacuum energy density, for flat space. If ρ_Λ evolves with the Hubble rate in the form indicated in Eq. (8.1.11), the non-relativistic matter density and vacuum energy density evolve with the redshift as follows [39, 58]:

$$\rho_M(z;\nu) = \rho_M^0\,(1+z)^{3(1-\nu)}\,, \tag{8.5.1}$$

and

$$\rho_\Lambda(z) = \rho_\Lambda^0 + \frac{\nu\,\rho_M^0}{1-\nu}\left[(1+z)^{3(1-\nu)} - 1\right]. \tag{8.5.2}$$

The corresponding Hubble function reads

$$H^2(z) = \frac{8\pi\,G}{3\,(1-\nu)}\left[\rho_\Lambda^0 - \nu\,\rho_c^0 + \rho_m^0\,(1+z)^{3(1-\nu)}\right]. \tag{8.5.3}$$

The crucial parameter is ν, which we have introduced in sect. 8.1. It is responsible for the time evolution of the vacuum energy. From Eq. (8.5.1) we confirm, that it accounts also for the non-conservation of matter. For $\nu = 0$ it leads to the exact local covariant conservation, which for non-relativistic matter reads

$$\rho_M(z) = \rho_M^0\,(1+z)^3\,. \tag{8.5.4}$$

Next we note that $\delta\rho_M \equiv \rho_M(z;\nu) - \rho_M(z)$ is the net amount of non-conservation of matter per unit volume at a given redshift. This expression must be proportional to ν, since we subtract the conserved part. At this order we have $\delta\rho_M = -3\,\nu\,\rho_M^0(1+z)^3\ln(1+z)$. We differentiate it with respect to time and expand in ν, and finally divide the final result by ρ_M. This provides the relative time variation:

$$\frac{\delta\dot{\rho}_M}{\rho_M} = 3\nu\,\left(1+3\ln(1+z)\right)\,H + \mathcal{O}(\nu^2)\,. \tag{8.5.5}$$

Here we have used $\dot{z} = (dz/da)\dot{a} = (dz/da)aH = -(1+z)H$. Assuming relatively small values of the redshift, we may neglect the log term and are left with:

$$\frac{\delta\dot{\rho}_M}{\rho_M} \simeq 3\nu\;H\,. \tag{8.5.6}$$

From (8.5.2) we find:

$$\frac{\dot{\rho}_\Lambda}{\rho_\Lambda} \simeq -3\nu\,\frac{\Omega_M^0}{\Omega_\Lambda^0}\,(1+z)^3\,H + \mathcal{O}(\nu^2)\,. \tag{8.5.7}$$

It is of the same order of magnitude as (8.5.6) and has the opposite sign. From a detailed analysis of the combined data on type Ia supernovae, the Cosmic Microwave Background (CMB), the Baryonic Acoustic Oscillations (BAO) and the structure formation data a direct cosmological bound on ν has been obtained in the literature [39]:

$$|\nu|^{\text{cosm.}} \lesssim \mathcal{O}(10^{-3})\,, \qquad \text{(Model II)}\,. \tag{8.5.8}$$

It is consistent with the theoretical expectations [61].

Let us now analyze Model III, which can also accommodate matter non-conservation in the form (8.5.1), but at the expense of a time varying G. We compare it with a similar model where G is also running, Model IV, but where matter is conserved.

Within the class of scenarios indicated as Model III the parameter ρ_Λ remains constant ($\rho_\Lambda = \rho_\Lambda^0$) and G is variable. This is possible due to the presence of the *non* self-conserved matter density (8.5.1). Trading the time variable by the scale factor, we can rewrite Eq. (8.4.3) as follows:

$$G'(a)\left[\rho_M(a) + \rho_\Lambda^0\right] + G(a)\left[\rho'_M(a) + \frac{3}{a}\,\rho_M(a)\right] = 0\,. \tag{8.5.9}$$

The primes indicate differentiation with respect to the scale factor. We insert equation (8.5.1) in (8.5.9), integrate the resulting differential equation for $G(a)$ and express the final result in terms of the redshift:

$$G(z) = G_0\,\left[\Omega_M^0\,(1+z)^{3(1-\nu)} + \Omega_\Lambda^0\right]^{\nu/(1-\nu)}\,. \tag{8.5.10}$$

Here $G_0 = 1/M_P^2$ is the current value of the gravitational coupling. The previous equation is correctly normalized: $G(z=0) = G_0$, due to the cosmic sum rule in flat space: $\Omega_M^0 + \Omega_\Lambda^0 = 1$. For $\nu = 0$ the gravitational coupling G remains constant: $G = G_0$. Since ρ_Λ is constant in the current scenario, the small variation of G is entirely due to the non-vanishing value of the ν-parameter in the matter non-conservation law (8.5.1). This leads to the dynamical feedback of G with matter [10]. For the present model Friedmann's equation (8.1.8) becomes:

$$H^2(z) = \frac{8\pi G(z)}{3}\left[\rho_M^0\,(1+z)^{3(1-\nu)} + \rho_\Lambda^0\right] = H_0^2\frac{G(z)}{G_0}\,\left[\Omega_M^0\,(1+z)^{3(1-\nu)} + \Omega_\Lambda^0\right]. \tag{8.5.11}$$

[10] The matter non-conservation law (8.5.1) was first suggested and analyzed in [58], and later on in [73].

Combining (8.5.10) and (8.5.11), we find the Hubble function of this model in terms of z:

$$H^2(z) = H_0^2\,\left[\Omega_M^0\,(1+z)^{3(1-\nu)} + \Omega_\Lambda^0\right]^{1/(1-\nu)}\,, \tag{8.5.12}$$

and we obtain:

$$\frac{G(z)}{G_0} = \left[\frac{H^2(z)}{H_0^2}\right]^{\nu}\,. \tag{8.5.13}$$

Since ν is presumably small in absolute value (as in the previous section), we can expand (8.5.13) in this parameter:

$$G(H) \simeq G_0\,\left(1 + \nu\,\ln\frac{H^2}{H_0^2} + \mathcal{O}(\nu^2)\right)\,. \tag{8.5.14}$$

At leading order in ν this expression for the variation of G is identical to the one found for Model IV, see Eq. (8.1.2), except for the sign of ν. The equation(8.5.14) allows us to estimate the value of the parameter ν by confronting the model with the experimental data on the time variation of G. Differentiating (8.5.14) with respect to the cosmic time, we find in leading order in ν:

$$\frac{\dot{G}}{G} = 2\nu\,\frac{\dot{H}}{H} = -2\,(1+q)\,\nu\,H\,, \tag{8.5.15}$$

where we have used the relation $\dot{H} = -(1+q)H^2$, in which $q = -\ddot{a}/(aH^2)$ is the deceleration parameter. From the known data on the relative time variation of G the bounds indicate that $|\dot{G}/G| \lesssim 10^{-12}\,\mathrm{yr}^{-1}$ [74]. If we take the present value of the deceleration parameter, we have $q_0 = 3\Omega_M^0/2 - 1 = -0.595 \simeq -0.6$ for a flat universe with $\Omega_M^0 = 0.27$. It follows:

$$\left|\frac{\dot{G}}{G_0}\right| \lesssim 0.8|\nu|\,H\,. \tag{8.5.16}$$

Taking the current value of the Hubble parameter: $H_0 \simeq 7\times 10^{-11}\,\mathrm{yr}^{-1}$ (for $h \simeq 0.70$), we obtain $|\nu| \lesssim 10^{-2}$. The real value of $|\nu|$ can be smaller, but to compare the upper bound that we have obtained with observations makes sense in view of the usual interpretation of ν in sect. 8.1 and the theoretical estimates indicated there. The constraints from Big Bang nucleosynthesis (BBN) for the time variation of G are more stringent and lead to the improved bound. Since Models III and IV share a similar kind of running law for the gravitational coupling (except for the sign of ν) we can extract the same bound for $|\nu|$ in the two models following the method of Ref. [75] and references therein, particularly [76]. The final result is

$$|\nu|^{\mathrm{BBN}} \lesssim 10^{-3} \qquad \text{(Models III and IV)}\,. \tag{8.5.17}$$

The cosmological data from different sources furnish about the same upper bound on $|\nu|$ for the two running models where matter is non-conserved, i.e. Models II and III. In both cases the upper bound on $|\nu|$ is $\sim 10^{-3}$, as shown by equations (8.5.8) and (8.5.17).

Although the order of magnitude of the bounds on $|\nu|$ are sometimes coincident for different models, the interpretation can be quite different. For example, Model IV cannot – in contrast to Models II and III – be used to explain the possible time variation of the fundamental constants of the strong interactions and the particle masses (see next section). It can only be used to explain the time variation of the cosmological parameters ρ_Λ and G in a way which is independent, in principle, from the microphysical phenomena in particle physics and nuclear physics.

9 Dynamical vacuum energy and the time variation of the fundamental constants

As we have seen in Sect. 8.1, in an expanding universe the vacuum energy density ρ_Λ is expected to be a dynamical quantity, and should exhibit a slow evolution determined by the expansion rate of the universe H. While a formal and completely rigorous proof of this contention is at the moment not feasible, the hints that it must be so are sufficiently encouraging to further spur these investigations at least from the phenomenological side [9], let alone the theoretical studies pointing to that possibility in particular QFT frameworks [61, 77]. If so, the dynamical vacuum in QFT in curved spacetime could provide an alternative scenario (beyond the usual quintessence kind of approaches) for implementing dynamical dark energy (DE) as a general paradigm for curing or alleviating the old CC problem, as well as the so-called cosmic coincidence problem. Interestingly enough, this possibility has been phenomenologically tested and profusely confronted with the latest accurate data on type Ia supernovae (SNIa), the Baryonic Acoustic Oscillations (BAOs), and the anisotropies of the Cosmic Microwave Background (CMB), see particularly [39, 40, 78, 42] for the most recent and comprehensive studies. Furthermore, a unified vacuum framework capable of describing the complete cosmic history as of the early inflationary times to the present dark energy epoch has also recently been proposed within this same kind of approach [65, 66].

If that is not enough, quite intriguingly the idea could also be tested from an entirely different vein, namely one capable of providing a new and completely independent insight into the whole subject. This novel and recent approach was first suggested in [24], and it is related with the precise measurements of the so-called Fundamental Constants of Nature and their possible time variation, see e.g. [74] (and the long list of references therein). I would like to devote some time in this review to this particular phenomenological approach to dynamical vacuum energy.

Recent measurements on the time variation of the fine structure constant and of the proton-electron mass ratio suggest that basic quantities of the Standard Model, such as the QCD scale parameter $\Lambda_{\rm QCD}$, may not be conserved in the course of the cosmological evolution [79]. The masses of the nucleons m_N and of the atomic nuclei would also be affected. Matter is not conserved in such a universe. In the framework of ideas that we have developed in Sect. 8.4, these measurements could be interpreted as a leakage of matter into vacuum or vice versa. In the following we wish to show that the amount of leakage necessary to explain the measured value of the time variation $\dot{m}_N/m_N$ could be of the same order of magnitude as the observationally allowed value of $\dot{\rho}_\Lambda/\rho_\Lambda$, with a possible contribution from the dark matter particles [24]. Therefore, the dark energy in our universe could be the dynamical vacuum energy in interaction with ordinary baryonic matter as well as with dark matter.

The QCD scale parameter is related to the strong coupling constant $\alpha_s = g_s^2/(4\pi)$. To lowest (1-loop) order one finds:

$$\alpha_s(\mu_R) = \frac{1}{\beta_0 \ln\left(\Lambda_{\rm QCD}^2/\mu_R^2\right)} = \frac{4\pi}{(11-2\,n_f/3)\,\ln\left(\mu_R^2/\Lambda_{\rm QCD}^2\right)}\,, \tag{9.1}$$

where μ_R is the renormalization point and $\beta_0 \equiv -b_0 = -(33-2\,n_f)/(12\,\pi)$ (n_f being the number of quark flavors) is the lowest order coefficient of the β-function. If we include all the know flavors, we obviously have $\beta_0 < 0$ and therefore when the renormalization scale μ_R increases the strong coupling constant diminishes (this is the famous asymptotic freedom property of QCD). However, for lower scales, namely for μ_R near $\Lambda_{\rm QCD} \sim \mathcal{O}(100)$ MeV, the strong coupling constant increases and in practice it becomes arbitrarily large. This is just the regime which we cannot explore with perturbation theory, and as we see the two regimes are roughly separated by the $\Lambda_{\rm QCD}$ scale. Not surprisingly this scale is the one that rules the calculation of the strongly interacting hadronic bound states like the proton, and in fact it dominates the calculation of its mass, see Eq. (9.5) below. The QCD scale parameter $\Lambda_{\rm QCD}$ has actually been experimentally measured with $\sim 10\%$ precision: $\Lambda_{\rm QCD} = 217 \pm 25$ MeV.

The next observation is crucial for our discussion [24]. When we embed QCD in the FLRW expanding background, the value of $\Lambda_{\rm QCD}$ need not remain rigid anymore. The value of $\Lambda_{\rm QCD}$ could change with H, and this would mean a change in the cosmic time. If $\Lambda_{\rm QCD} = \Lambda_{\rm QCD}(H)$ is a function of H, the coupling constant $\alpha_s = \alpha_s(\mu_R; H)$ is also a function of H (apart from a function of μ_R). The relative cosmic variations of the two QCD quantities are related (at one-loop) by:

$$\frac{1}{\alpha_s}\frac{d\alpha_s(\mu_R;H)}{dH} = \frac{1}{\ln\left(\mu_R/\Lambda_{\rm QCD}\right)}\left[\frac{1}{\Lambda_{\rm QCD}}\,\frac{d\Lambda_{\rm QCD}(H)}{dH}\right]. \tag{9.2}$$

The potential significance of this relation is out of discussion: if the QCD coupling constant α_s or the QCD scale parameter $\Lambda_{\rm QCD}$ undergo a small cosmological time shift, the nucleon mass and the masses of the atomic nuclei of the universe would also change in proportion to $\Lambda_{\rm QCD}$.

The cosmic dependence of the strong coupling $\alpha_s(\mu_R; H)$ can be generalized to the other couplings $\alpha_i = \alpha_i(\mu_R; H)$ [80], in particular the electromagnetic fine structure constant $\alpha_{\rm em}$. In a grand unified theory (GUT) these couplings converge at the unification point. Let $d\alpha_i$ be the cosmic variation of α_i with H. Each of the α_i is a function of μ_R, but the expression $\alpha_i^{-1}\,(d\alpha_i/\alpha_i)$ is easily seen to be independent of μ_R. As a result we can see, using Eq. (9.2), that the running of $\alpha_{\rm em}$ is related to the corresponding cosmic running of $\Lambda_{\rm QCD}$ as follows:

$$\frac{1}{\alpha_{\rm em}}\frac{d\alpha_{\rm em}(\mu_R;H)}{dH} = \frac{8}{3}\,\frac{\alpha_{\rm em}(\mu_R;H)/\alpha_s(\mu_R;H)}{\ln\left(\mu_R/\Lambda_{\rm QCD}\right)}\left[\frac{1}{\Lambda_{\rm QCD}}\,\frac{d\Lambda_{\rm QCD}(H)}{dH}\right]. \tag{9.3}$$

At the renormalization point $\mu_R = M_Z$, where both $\alpha_{\rm em}$ and α_s are well-known, one finds:

$$\frac{1}{\alpha_{\rm em}}\frac{d\alpha_{\rm em}(\mu_R;H)}{dH} \simeq 0.03\left[\frac{1}{\Lambda_{\rm QCD}}\,\frac{d\Lambda_{\rm QCD}(H)}{dH}\right]. \tag{9.4}$$

Thus the QCD scale $\Lambda_{\rm QCD}$ runs more than 30 times faster with the cosmic expansion than the electromagnetic fine structure constant. Searching for a cosmic evolution of $\Lambda_{\rm QCD}$ is thereby much easier than searching for the time variation of $\alpha_{\rm em}$. It is important to note that the cosmic running properties of the QCD scale parameter could have nontrivial implications on the physics of the fundamental constants of Nature. This is sooner said than done: how do we search for a possible cosmic running of the QCD scale? The answer is: searching for a possible running of the proton mass, a parameter which has been (and is being) monitored by many astrophysical and laboratory experiments (or more specifically, what is traced is the ratio of the proton mass to the electron mass, where the latter is of course independent of QCD) [74]. Let us take for instance the proton mass $m_p \simeq 938$ MeV, supposed to be a fundamental constant. It can be computed from $\Lambda_{\rm QCD}$, the quarks masses and the electromagnetic contribution. The precise formula reads

$$\begin{aligned} m_p &= c_{\rm QCD}\Lambda_{\rm QCD} + c_u\, m_u + c_d\, m_d + c_s\, m_s + c_{\rm em}\Lambda_{\rm QCD} \\ &= (860 + 21 + 19 + 36 + 2)\ {\rm MeV}\,, \end{aligned} \tag{9.5}$$

from which we learn that it is largely dominated by the QCD scale, namely the bulk ($\sim 92\%$) of the proton mass is given by $m_p \simeq c_{\rm QCD}\Lambda_{\rm QCD}$. We will take advantage of this fact in what follows.

Let us focus on the impact of the cosmological Models II and III of Sect. 8.4 on the non-conservation of matter in the universe. Recall that we have considered bounds on the "leakage parameter" ν within the class of these models based on the non-conservation matter density law (8.5.1). We must be careful in interpreting such a non-conservation law. For example, if we take the baryonic density in the universe, which is essentially the mass density of protons, we can write $\rho_M^B = n_p\, m_p$, where n_p is the number density of protons and $m_p^0 = 938.272013(23)$ MeV is the current proton mass. If this mass density is non-conserved, either n_p does not exactly follow the normal dilution law with the expansion, i.e. $n_p \sim a^{-3} = (1+z)^3$, but the anomalous law:

$$n_p(z) = n_p^0\,(1+z)^{3(1-\nu)} \qquad (\text{at fixed proton mass } m_p = m_p^0)\,, \tag{9.6}$$

and/or the proton mass m_p does not stay constant with time and redshifts with the cosmic evolution:

$$m_p(z) = m_p^0\,(1+z)^{-3\nu} \qquad (\text{with normal dilution } n_p(z) = n_p^0\,(1+z)^3)\,. \tag{9.7}$$

In all cases it is assumed that the vacuum absorbs the difference (i.e. $\rho_\Lambda = \rho_\Lambda(z)$ "runs with the expansion"). The first possibility implies that during the expansion a certain number of particles (protons in this case) are lost into the vacuum (if $\nu < 0$; or ejected from it, if $\nu > 0$), whereas in the second case the number of particles is strictly conserved. The number density follows the normal dilution law with the expansion, but the mass of each particle slightly changes (decreases for $\nu < 0$, or increases for $\nu > 0$) with the cosmic evolution.

For the following considerations we adopt the second point of view, i.e. Eq. (9.7). Being the matter content of the universe dominated by the dark matter (DM), we cannot exclude that the particle masses of which is made also vary with cosmic time. Let us denote the mass of the average DM particle m_X, and let ρ_X and n_X be its mass density and number density, respectively. The overall matter density of the universe can be written as follows:

$$\begin{aligned} \rho_M &= \rho_B + \rho_L + \rho_R + \rho_X = (n_p\, m_p + n_n\, m_n) + n_e\, m_e + \rho_R + n_X\, m_X \\ &\simeq n_p\, m_p + n_n\, m_n + n_X\, m_X\,. \end{aligned} \tag{9.8}$$

Here n_p, n_n, n_e, n_X (m_p, m_n, m_e, m_X) are the number densities (and masses) of protons, neutrons, electrons and DM particles. The baryonic and leptonic parts are $\rho_B = n_p\, m_p + n_n\, m_n$ and $\rho_L = n_e\, m_e$ respectively. The small ratio $m_e/m_p \simeq 5\times 10^{-4}$ implies that the leptonic contribution to the total mass density is negligible: $\rho_L \ll \rho_B$. We have also neglected the relativistic component ρ_R (photons and neutrinos).

If we assume that the mass change through the cosmic evolution is due to the time change of m_p, m_n and m_X, we can compute the mass density anomaly per unit time, i.e. the deficit or surplus with respect to the conservation law, by differentiating (9.8) with respect to time and subtracting the ordinary (i.e. fixed mass) time dilution of the number densities. The result is:

$$\delta\dot{\rho}_M = n_p\,\dot{m}_p + n_n\,\dot{m}_n + n_X\,\dot{m}_X\,. \tag{9.9}$$

The relative time variation of the mass density anomaly can be estimated as follows:

$$\frac{\delta\dot{\rho}_M}{\rho_M} = \frac{n_p\,\dot{m}_p + n_n\,\dot{m}_n + n_X\,\dot{m}_X}{n_p\,m_p + n_p\,m_p + n_X\,m_X} \simeq \frac{n_p\,\dot{m}_p + n_n\,\dot{m}_n + n_X\,\dot{m}_X}{n_X\,m_X}\left(1 - \frac{n_p\,m_p + n_n\,m_n}{n_X\,m_X}\right)\,. \tag{9.10}$$

The current normalized DM density $\Omega^0_{\rm DM} = \rho_X/\rho_c \simeq 0.23$ is significantly larger than the corresponding normalized baryon density $\Omega^0_B = \rho_B/\rho_c \simeq 0.04$. Therefore $n_X\,m_X$ is larger than $n_p\,m_p + n_n\,m_n$ by the same amount. If we assume $\dot{m}_n = \dot{m}_p$, we find approximately:

$$\frac{\delta\dot{\rho}_M}{\rho_M} = \frac{n_p\,\dot{m}_p}{n_X\,m_X}\left(1 + \frac{n_n}{n_p} - \frac{\Omega_B}{\Omega_{\rm DM}}\right) + \frac{\dot{m}_X}{m_X}\left(1 - \frac{\Omega_B}{\Omega_{\rm DM}}\right)\,. \tag{9.11}$$

In the approximation $m_n = m_p$ we can rewrite the prefactor on the *r.h.s* of Eq. (9.11) as follows:

$$\frac{n_p\,\dot{m}_p}{n_X\,m_X} = \frac{\Omega_B}{\Omega_{\rm DM}}\,\frac{\dot{m}_p}{m_p}\left(1 - \frac{n_n/n_p}{1 + n_n/n_p}\right) \simeq \frac{\Omega_B}{\Omega_{\rm DM}}\,\frac{\dot{m}_p}{m_p}\left(1 - \frac{n_n}{n_p}\right)\,. \tag{9.12}$$

The ratio n_n/n_p is of order 10% after the primordial nucleosynthesis. Since $\Omega_B/\Omega_{\rm DM}$ is also of order 10%, we can neglect the product of this term with n_n/n_p. When we insert the previous equation into (9.11), the two n_n/n_p contributions cancel each other. The expression $1 - \Omega_B/\Omega_{\rm DM}$ factorizes in the two terms on the *r.h.s* of Eq. (9.11). The final result is:

$$\left(1 - \frac{\Omega_B}{\Omega_{\rm DM}}\right)^{-1}\frac{\delta\dot{\rho}_M}{\rho_M} = \frac{\Omega_B}{\Omega_{\rm DM}}\,\frac{\dot{m}_p}{m_p} + \frac{\dot{m}_X}{m_X} = \frac{\Omega_B}{\Omega_{\rm DM}}\,\frac{\dot{\Lambda}_{\rm QCD}}{\Lambda_{\rm QCD}} + \frac{\dot{m}_X}{m_X}\,. \tag{9.13}$$

As mentioned, we have approximated $m_p \simeq c_{\rm QCD}\,\Lambda_{\rm QCD}$.

The expression $\delta\dot{\rho}_M/\rho_M$ in Eq. (9.13) must be the same as the one we have computed in (8.5.5), if we consider the models based on the generic matter non-conservation law (8.5.1). Therefore the two expressions should be equal, and we obtain approximately:

$$3\nu_{\rm eff}\,H = \frac{\Omega_B}{\Omega_{\rm DM}}\,\frac{\dot{\Lambda}_{\rm QCD}}{\Lambda_{\rm QCD}} + \frac{\dot{m}_X}{m_X}\,, \tag{9.14}$$

where we have defined

$$\nu_{\rm eff} = \frac{\nu}{1 - \Omega_B/\Omega_{\rm DM}}\,, \tag{9.15}$$

and numerically $\nu_{\rm eff} \simeq 1.2\,\nu$. The differential equation (9.14) describes approximately the relationship between the matter non-conservation law (8.5.1), the evolution of the vacuum energy density ρ_Λ (and/or G) and the time variation of the nuclear and particle physics quantities.

Various scenarios are possible. Suppose that the dark matter particles do not vary with time, i.e. $\dot{m}_X = 0$, and only the cosmic evolution of $\Lambda_{\rm QCD}$ accounts for the non-conservation of matter.

Trading the cosmic time for the scale factor through $\dot{\Lambda}_{\rm QCD} = (d\Lambda_{\rm QCD}/da)\; a\, H$ and integrating the resulting equation, we can express the final result in terms of the redshift:

$$\Lambda_{\rm QCD}(z) = \Lambda^0_{\rm QCD}\ (1+z)^{-3\,(\Omega^0_{\rm DM}/\Omega^0_B)\,\nu_{\rm eff}}\ . \tag{9.16}$$

Since the contribution of the quark masses m_u,m_d and m_s to the proton mass is small – cf. Eq. (9.5) – we can approximate the proton mass by $m_p \simeq c_{\rm QCD}\,\Lambda_{\rm QCD}$. Therefore, for the protons we have

$$m_p(z) = m^0_p\ (1+z)^{-3\,(\Omega^0_{\rm DM}/\Omega^0_B)\,\nu_{\rm eff}}\ . \tag{9.17}$$

Here $\Lambda^0_{\rm QCD}$ and m^0_p are the QCD scale and proton mass at present ($z=0$); $\Omega^0_{\rm DM}$ and Ω^0_B being the current values of these cosmological parameters.

The presence of the factor $\Omega^0_B/\Omega^0_{\rm DM}$ in the power law makes eq. (9.17) more realistic than eq. (9.7). In the case $\nu = 0$ the QCD scale and the proton mass would not vary with the expansion of the universe, but for non-vanishing ν it describes the cosmic running of $\Lambda_{\rm QCD} = \Lambda_{\rm QCD}(z)$ and $m_p = m_p(z)$. For $\nu > 0$ ($\nu < 0$) the QCD scale and proton mass decrease (increase) with the redshift. This is consistent, since for $\nu > 0$ ($\nu < 0$) the vacuum energy density is increasing (decreasing) with the redshift – cf. Eq. (8.5.2) –, and it is smaller (larger) now than in the past.

We can write down the variation of the QCD scale in terms of the Hubble rate H. With the help of Eq. (8.5.3) it is easy to see that Eq. (9.16) can be turned into an expression for $\Lambda_{\rm QCD}$ given explicitly in terms of the primary cosmic variable H:

$$\Lambda_{\rm QCD}(H) = \Lambda^0_{\rm QCD}\left[\frac{1-\nu}{\Omega^0_M}\frac{H^2}{H_0^2} - \frac{\Omega^0_\Lambda - \nu}{\Omega^0_M}\right]^{-(\Omega^0_{\rm DM}/\Omega^0_B)\,\nu_{\rm eff}/(1-\nu)}\ , \tag{9.18}$$

with $\Omega^0_M = \Omega^0_B + \Omega^0_{\rm DM}$. ν and $\nu_{\rm eff}$ are involved in (9.18), since they come from different sources. This equation satisfies the normalization condition $\Lambda_{\rm QCD}(H_0) = \Lambda^0_{\rm QCD}$ due to the cosmic sum rule for flat space: $\Omega^0_M + \Omega^0_\Lambda = 1$.

Obviously the cosmic time variation of the $\Lambda_{\rm QCD}$ scale is very small in our framework. This can be more easily assessed if we use Eqs. (9.16) and (9.18) to compute the relative time variation of the QCD scale with respect to the present value. Since ν is small it it easy to show that

$$\frac{\Lambda_{\rm QCD}(z) - \Lambda^0_{\rm QCD}}{\Lambda^0_{\rm QCD}} = -\frac{\Omega^0_{\rm DM}}{\Omega^0_B}\,\frac{\nu_{\rm eff}}{1-\nu}\,\ln\left[\frac{1-\nu}{\Omega^0_M}\,\frac{H^2(z)}{H_0^2} - \frac{\Omega^0_\Lambda - \nu}{\Omega^0_M}\right]\ . \tag{9.19}$$

As a concrete example, let us consider the studies made in Ref. [81] on comparing the H_2 spectral Lyman and Werner lines observed in the Q 0347-383 and Q 0405-443 quasar absorption systems. The comparison with the corresponding spectral lines at present may be sensitive to a possible evolution of these lines in the last twelve billion years and involves redshifts in the range $z \simeq 2.6 - 3.0$. A positive result could be interpreted as a small variation of the proton to electron mass ratio between two widely separated epochs of the cosmological evolution [81] . Assuming that $|\nu| = \mathcal{O}(10^{-3})$, as suggested by Eq. (8.5.8), it follows from the previous formulae that the relative variation of $\Lambda_{\rm QCD}$ in this lengthy time interval is only at the few percent level with respect to its present day value. From Eq. (8.5.2) we can then easily check that the corresponding variation of $\rho_\Lambda(z)$ with respect to the current value ρ^0_Λ is also of a few percent. As expected, the two scales undergo tiny variations over very long periods of time, in fact cosmological periods, and therefore the large hierarchy between them at present – namely $\Lambda_{\rm QCD} = \mathcal{O}(100)$ MeV= $\mathcal{O}(10^8)$ eV and $\rho_\Lambda^{1/4} = \mathcal{O}(10^{-3})$ eV – is essentially preserved over the cosmological evolution. However, even this small crosstalk between these two widely separated scales could be sufficient for being eventually detected by the aforementioned high precision experiments aiming at measuring very tiny variation of the proton to electron mass ratio.

Indeed, this is suggested by the fact that the expected range of values of ν is within the scope of the precision of these experiments. Consider the state of the art in the current laboratory tests, using atomic clocks. According to our estimate (9.4), the largest effect is expected to be a cosmological redshift (hence time variation) of the nucleon mass, which can be observed by monitoring molecular frequencies. These are precise experiments in quantum optics, e.g. obtained by comparing a cesium clock with 1S-2S hydrogen transitions. In a cesium clock the time is measured by using a hyperfine transition [11]. Since the frequency of the clock depends on the magnetic moment of the cesium nucleus, a possible variation of the latter is proportional to a possible variation of $\Lambda_{\rm QCD}$. A hyperfine splitting is a function of $Z\,\alpha_{\rm em}$ (Z being the atomic number) and is proportional to $Z\,\alpha_{\rm em}^2(\mu_N/\mu_B)(m_e/m_p)\,R_\infty$, where R_∞ is the Rydberg constant, μ_N is the nuclear magnetic moment and $\mu_B = e\hbar/2m_pc$ is the nuclear magneton. We have $\dot{\mu}_N/\mu_N \propto -\dot{\Lambda}_{\rm QCD}/\Lambda_{\rm QCD}$. The hydrogen transitions are only dependent on the electron mass, which we assume to be constant. The comparison over a period of time between the cesium clock with hydrogen transitions provides an atomic laboratory measurement of the ratio $\mu_{\rm pe} \equiv m_p/m_e$. The most sophisticated atomic clock experiments aim soon to reach a sensitivity limit of $\left|\dot{\Lambda}_{\rm QCD}/\Lambda_{\rm QCD}\right| < 10^{-14}\,{\rm yr}^{-1}$ [24]. Since the proton mass is given essentially by $\Lambda_{\rm QCD}$, as indicated by Eq. (9.5), we have $\dot{m}_p \simeq c_{\Lambda_{\rm QCD}}\,\dot{\Lambda}_{\rm QCD}$ and the corresponding time variation of the ratio of the proton mass would be:

$$\left|\frac{\dot{\mu}_{pe}}{\mu_{\rm pe}}\right| = \left|\frac{\dot{m}_p}{m_p}\right| \simeq \left|\frac{\dot{\Lambda}_{\rm QCD}}{\Lambda_{\rm QCD}}\right| < 10^{-14}\,{\rm yr}^{-1}\,. \tag{9.20}$$

The atomic clock result (9.20) would indicate a time variation of the ratio $\mu_{\rm pe}$, which is consistent (in absolute value) with the astrophysical measurements [81].

Using the above equations and Eq. (9.1), we can obtain the corresponding evolution of the strong coupling constant α_s with the redshift:

$$\frac{1}{\alpha_s(\mu_R;z)} = \frac{1}{\alpha_s(\mu_R;0)} + 6\,b_0\,\frac{\Omega_{\rm DM}^0}{\Omega_B^0}\,\nu_{\rm eff}\,\ln(1+z)\,, \tag{9.21}$$

where $\alpha_s(\mu_R;0)$ is the value of $\alpha_s(\mu_R;z)$ today ($z=0$). Since $b_0 > 0$, we observe that for $\nu > 0$ ($\nu < 0$) the strong interaction $\alpha_s(\mu_R;z)$ decreases (increases) with z, i.e. with the cosmic evolution. The corresponding variation of the strong coupling with the Hubble rate can also be determined [12]:

$$\frac{1}{\alpha_s(\mu_R;H)} = \frac{1}{\alpha_s(\mu_R;H_0)} + 2\,b_0\,\frac{\Omega_{\rm DM}^0}{\Omega_B^0}\,\frac{\nu_{\rm eff}}{1-\nu}\,\ln\left[\frac{1-\nu}{\Omega_M^0}\,\frac{H^2}{H_0^2} - \frac{\Omega_\Lambda^0-\nu}{\Omega_M^0}\right]. \tag{9.22}$$

[11] Recall that the cesium hyperfine clock provides the modern definition of time. In SI units, the second is defined to be the duration of 9.192631770×10^9 periods of the transition between the two hyperfine levels of the ground state of the ^{133}Cs atom

[12] We point out that a similar running of the strong coupling constant with the cosmic expansion was pointed out in a different context by J.D. Bjorken in [82].

Here $\alpha_s(\mu_R; H_0)$ is the current value of $\alpha_s(\mu_R; H)$.

Remarkably, the above expression displays the logarithmic running of the strong coupling as a function of *two* energy scales: one is the ordinary QCD running scale μ_R, the other is the cosmic scale defined by the Hubble rate H, which has dimension of energy in natural units. The second term on the *r.h.s.* depends on the product of the two β-function coefficients, the one for the ordinary QCD running (b_0) and the one for the cosmic running ($\nu \propto \nu_{\rm eff}$).

The following remarks are in order:

i) for $\nu = 0$ there is no cosmic running of the strong interaction,

ii) for $\nu > 0$ the strong coupling $\alpha_s(\mu_R; H)$ is "doubly asymptotically free". It decreases for large μ_R and also for large H, whereas for $\nu < 0$ the cosmic evolution drives the running of α_s opposite to the normal QCD running,

iii) the velocity of the two runnings is very different, because H is slowly varying with time and $|\nu| \ll 1$ and $|\nu| \ll b_0 \lesssim 1$. The cosmic running only operates in the cosmic history and is weighed with a very small β-function. But it may soon be measured in the experiments with atomic clocks and through astrophysical observations.

We should not overlook the fact that the previous equations describe not only the leading cosmic evolution of the QCD scale and the proton mass with the redshift and the expansion rate H of the universe, but they can account for the redshift evolution of the nuclear masses. For the neutron we can write approximately: $m_n \simeq c_{\rm QCD}\,\Lambda_{\rm QCD}$. For an atomic nucleus of current mass M_A and atomic number A we have $M_A = Z\,m_p + (A-Z)\,m_n - B_A$, where Z is the number of protons and $A-Z$ the number of neutrons, and B_A is the binding energy. Although B_A may also change with the cosmic evolution, the shift should be less significant, since at leading order the binding energy relies on pion exchange among the nucleons. The pion mass has a softer dependence on $\Lambda_{\rm QCD}$: $m_\pi \sim \sqrt{m_q\,\Lambda_{\rm QCD}}$, due to the chiral symmetry.

In the previous approximations we have neglected the light quark masses m_q. We can assume that the binding energy has a negligible cosmic shift as compared to the masses of the nucleons. In the limit where we neglect the proton-neutron mass difference and assume a common nucleon mass m_N^0 at present, the corresponding mass of the atomic nucleus at redshift z is given at leading order by:

$$M_A(z) \simeq A\,m_N^0\;(1+z)^{-3\,(\Omega_{\rm DM}^0/\Omega_B^0)\,\nu_{\rm eff}} - B_A\,. \tag{9.23}$$

Although the chemical elements redshift their masses, a disappearance or overproduction of nuclear mass (depending on the sign of ν) is compensated by a running of the vacuum energy ρ_Λ, which is of opposite in sign, see (8.5.7).

Above we have described a simplified case, in which the nuclear matter evolves with the cosmic evolution as a result of the evolution of the fundamental QCD scale. In this scenario the light quark masses are neglected, and the DM does not participate in the cosmic time evolution.

Alternatively we can assume that the nuclear matter does not vary with time, i.e. $\dot{\Lambda}_{\rm QCD} = 0$, and only the DM particles account for the non-conservation of matter. In general we expect a mixed situation, in which the temporal rates of change for nuclear matter and for DM particles are different:

$$\frac{\dot{\Lambda}_{\rm QCD}}{\Lambda_{\rm QCD}} = 3\,\nu_{\rm QCD}\,H\,, \qquad\qquad \frac{\dot{m}_X}{m_X} = 3\,\nu_X\,H\,. \tag{9.24}$$

We have defined the QCD time variation index, which is characteristic of the redshift rate of the QCD scale, while ν_X is the corresponding one for the DM. In this more general case we find:

$$\Lambda_{\rm QCD}(z) = \Lambda^0_{\rm QCD}\,(1+z)^{-3\,\nu_{\rm QCD}}\,, \qquad\qquad m_X(z) = m^0_X\,(1+z)^{-3\,\nu_X}\,. \tag{9.25}$$

We introduce the effective baryonic redshift index ν_B:

$$\nu_B = \frac{\Omega_B}{\Omega_{\rm DM}}\,\nu_{\rm QCD}\,. \tag{9.26}$$

The equations (9.25) satisfy the relation (9.14), provided the coefficients ν_B and ν_X are related by

$$\nu_{\rm eff} = \nu_B + \nu_X\,. \tag{9.27}$$

$\nu_{\rm QCD}$ is the intrinsic cosmic rate of variation of the strongly interacting particles. The effective index ν_B weighs the redshift rate of these particles taking into account their relative abundance with respect to the DM particles. Even if the intrinsic cosmic rate of variation of $\Lambda_{\rm QCD}$ would be similar to the DM index (i.e. if $\nu_{\rm QCD} \gtrsim \nu_X$), the baryonic index (9.26) would still be suppressed with respect to ν_X, because the total amount of baryon matter in the universe is much smaller than the total amount of DM.

In this mixed scenario the mass redshift of the dark matter particles follows a similar law as in the case of protons (9.17), except now we have $\nu_{\rm eff} \to \nu_B$. The proton would have the index $\nu_{\rm QCD}$ characteristic of the free (and bound) stable strongly interacting matter:

$$m_p(z) = m^0_p\,(1+z)^{-3\,(\Omega_{\rm DM}/\Omega_B)\,\nu_B} = m^0_p\,(1+z)^{-3\nu_{\rm QCD}}\,. \tag{9.28}$$

The DM particles have another independent index ν_X. The sum (9.27) must reproduce the original index $\nu_{\rm eff} \propto \nu$, which we associated with the non-conservation of matter.

Finally we consider the possible quantitative contribution to the matter density anomaly from the dark matter. The global mass defect (or surplus) is regulated by the value of the ν parameter, but the contribution of each part (baryonic matter and DM) depends on the values of the individual components ν_B and ν_X. We can obtain a numerical estimate of these parameters by setting the expression (9.13) equal to (8.5.6). The latter refers to the time variation of the matter density ρ_M without tracking the particular way in which the cosmic evolution can generate an anomaly in the matter conservation. The former does assume that this anomaly is entirely due to a cosmic shift in the masses of the stable particles. Taking the absolute values, we obtain:

$$3|\nu_{\rm eff}|\,H \simeq \left|\frac{4}{23}\,\frac{\dot{\Lambda}_{\rm QCD}}{\Lambda_{\rm QCD}} + \frac{\dot{m}_X}{m_X}\right| < \frac{4}{23}\times 10^{-14}\,{\rm yr}^{-1} + \left|\frac{\dot{m}_X}{m_X}\right|\,. \tag{9.29}$$

Here we have used the experimental bound (9.20) on the time variation of $\Lambda_{\rm QCD}$.

Several cases can be considered, depending on the relation between the intrinsic cosmic rates of variation of the strongly interacting particles and DM particles, $\nu_{\rm QCD}$ and ν_X. Since these indices can have either sign, we shall compare their absolute values:

- 1) $|\nu_X| \ll |\nu_B|$:

  This condition implies $|\nu_X| \ll |\nu_{\rm QCD}|$. By demanding the stronger condition $|\nu_X| \ll |\nu_B|$, we insure that the intrinsic QCD cosmic rate $|\nu_{\rm QCD}|$ is much larger than the corresponding DM rate $|\nu_X|$. We can neglect the $\dot{m}_X/m_X$ term on the *r.h.s.* of (9.29), and we recover the equations (9.16)-(9.22) with $\nu_{\rm eff} \simeq \nu_B$. Using $H_0 \simeq 7\times 10^{-11}\,{\rm yr}^{-1}$, we find:

$$|\nu_X| \simeq 0\,, \qquad |\nu_{\rm eff}| \simeq |\nu_B| < 10^{-5}\,, \qquad |\nu_{\rm QCD}| < 5\times 10^{-5}\,. \tag{9.30}$$

- 2) $|\nu_X| \simeq |\nu_B|$:

  Here we still have $|\nu_X|$ smaller than $|\nu_{\rm QCD}|$, but the requirement is weaker. It follows: $|\nu_{\rm eff}| \simeq 2|\nu_X| \simeq 2|\nu_B| = 2(\Omega_B/\Omega_{\rm DM})\,|\nu_{\rm QCD}|$, and we find

$$|\nu_{\rm eff}| < 2\times 10^{-5}\,, \qquad |\nu_X| \simeq |\nu_B| < 10^{-5}\,, \qquad |\nu_{\rm QCD}| < 5\times 10^{-5}\,. \tag{9.31}$$

- 3) $|\nu_X| \simeq |\nu_{\rm QCD}|$:

  The two intrinsic cosmic rates for strongly interacting and DM particles are similar, i.e. $\dot{\Lambda}_{\rm QCD}/\Lambda_{\rm QCD}$ and $\dot{m}_X/m_X$ do not differ significantly. In this case Eq. (9.29) leads to

$$3|\nu_{\rm eff}|\,H < \left(\frac{4}{23}+1\right)\times 10^{-14}\,{\rm yr}^{-1}\,. \tag{9.32}$$

  There are two sign possibilities ($\nu_{\rm QCD} = \pm\nu_X$), and we take the absolute value:

$$|\nu_{\rm eff}| \lesssim \left(\frac{\Omega_B}{\Omega_{\rm DM}}+1\right)\,|\nu_{\rm QCD}| \simeq |\nu_{\rm QCD}|\,. \tag{9.33}$$

  We find:

$$|\nu_{\rm eff}| \lesssim |\nu_{\rm QCD}| \simeq |\nu_X| < 5\times 10^{-5}\,. \tag{9.34}$$

- 4) $|\nu_{\rm QCD}| \ll |\nu_X|$:

  Here the nuclear part is frozen. The non-conservation of matter is entirely due to the time variation of the DM particles. Eq. (9.29) gives:

$$3\nu\,H \simeq \frac{\dot{m}_X}{m_X}\left(1-\frac{\Omega_B}{\Omega_{\rm DM}}\right)\,. \tag{9.35}$$

  We have written this expression directly in terms of the original ν parameter. In this case we cannot get information from any laboratory experiment on $\dot{m}_X/m_X$, but we do have independent experimental information on the original ν value (irrespective of the particular contributions form the nuclear and DM components). It comes from the cosmological data on type Ia supernovae, BAO, CMB and structure formation. The analysis of this data [39, 40] leads to the bound (8.5.8), which applies to all models, in which matter follows the generic non-conservation law (8.5.1) and the running vacuum law (8.1.11) — or the same matter non-conservation law and the running gravitational coupling law (8.5.14), as shown in Eq. (8.5.17).

	$\|\nu\|^{\rm cosm}$	$\|\nu\|^{\rm lab}=\|\nu_B\|$	$\|\nu_X\|^{\rm cosm}$
Model II	10^{-3} (SNIa+BAO+CMB)	10^{-5} (Atomic clocks+Astrophys.)	10^{-3}
Model III	10^{-3} (BBN)	10^{-5} (Atomic clocks+Astrophys.)	10^{-3}
Model IV	10^{-3} (SNIa+BAO+CMB)+BBN	0	0

Table 1: Upper bounds on the running index $|\nu|$ for the various models defined in sect. 8.4 – cf. [24]. Only for Models II and III a non-vanishing value of $|\nu|$ is related to non-conservation of matter and a corresponding time evolution of ρ_Λ and G, respectively. For these models, the baryonic part of ν (denoted ν_B) can be accessible to accurate lab experiments – cf. Eq. (9.20) – whereas the DM part (ν_X) can only be bound indirectly from cosmological observations (same cosmological bound as for the overall ν). For Model IV matter is conserved, and a non-vanishing value of $|\nu|$ (only accessible from pure cosmological observations) is associated to a simultaneous time evolution of ρ_Λ and G – with no microphysical implications.

Since it depends on the cosmological effects from all forms of matter, it applies to the DM particles in particular. We find:

$$|\nu_X|^{\rm cosm} \lesssim 10^{-3}\,. \tag{9.36}$$

This bound is significantly weaker than any of the bounds found for the previous scenarios in which the nuclear matter participated of the cosmic time variation. It cannot be excluded that the matter non-conservation and corresponding running of the vacuum energy in the universe is mainly caused by the general redshift of the DM particles. In this case only cosmological experiments could be used to check this possibility. If the nuclear matter also participates in a significant way, it could be analyzed with the help of experiments in the laboratory. For a summary of the bounds, see Table 1.

If in the future we could obtain a tight cosmological bound on the effective $\nu_{\rm eff}$-parameter (9.27), using the astrophysical data, and an accurate laboratory (and/or astrophysical) bound on the baryonic matter part ν_B, we could compare them and derive the value of the DM component ν_X. If $\nu_{\rm eff}$ and ν_B would be about equal, we should conclude that the DM particles do not appreciably shift their masses with the cosmic evolution, or that they do not exist. If, in contrast, the fractional difference $|\,(\nu_{\rm eff}-\nu_B)/\nu_{\rm eff}\,|$ would be significant, the DM particles should exist to compensate for it.

10 Discussion and conclusions

We have discussed a few old and new aspects of the Λ-term in Einstein's equations and its relation with the vacuum energy. After some brief historical remarks, we have dealt with the cosmological constant (CC) problem and assessed its significance and possible phenomenological implications. In doing so we have focused on the notion of vacuum energy in quantum field theory (QFT) both in flat and curved spacetime, and we have dwelled on the fact that the term which is usually interpreted as the vacuum energy density (originating from the quantum vacuum fluctuations of the matter fields) is the same both in flat and curved spacetime. This is a well-known fact, but this does not make it less disquieting, since particles with mass m provide huge quartic contributions to the zero-point energy (ZPE), $V_{\rm ZPE} \sim m^4$, and these do not seem to be acceptable from the phenomenological point of view.

In the SM of particle physics there are other, no less preoccupying, contributions to the vacuum energy that come from the spontaneous symmetry breaking (SSB) of the electroweak gauge symmetry, specifically from the Higgs mechanism. These contributions are associated to the vacuum expectation value of the Higgs potential and increase as $\langle V\rangle \sim v^2\,M_H^2 \sim 10^8$ GeV^4, where $v = \mathcal{O}(100)$ GeV is the vacuum expectation value that defines the electroweak scale, and $M_H \simeq 125$ GeV is the presumed physical mass of the Higgs boson [7]. While every single vacuum energy source alone is already vastly worrisome, the combination of them all amounts to a devastating fine tuning problem, which is further aggravated at the quantum level when we consider the many higher loop effects involved. We have illustrated quite vividly this fact in Sect. 6 by considering what are the highest loop diagrams still contributing to the CC value (and to the awful fine tuning process).

Of the two main sources of difficulties with the vacuum energy in QFT, namely the ZPE and the SSB, the reality of the former has been disputed sometimes by the inconclusive interpretations about the origin of the Casimir effect as being a pure QFT vacuum effect or something else; whereas the latter also remained in the limbo of the theoretical ideas as long as the Higgs mechanism could not be fully substantiated on the experimental side. Because of this the CC problem could remain dormant for a long time in the ethereal world of the theoretical conundrums, which are that kind of problems whose dangerousness and threatening power on the physical world of mortals is only potential, not yet factual. Such situation, however, may have given signs of changing quite dramatically in recent times. Needless to say, not because the physical world changes an inch every time the human knowledge gives a jerk in its perception of the reality, but because we might now be reaching a situation where the two giant paradigms of the fundamental physics knowledge, viz. the SM of particle physics and the SM of cosmology, have finally been put furiously face to face for a very serious parley. Indeed, we have recently heard of the exciting news from CERN about the $\sim 5\sigma$ evidence on the discovery of a bosonic Higgs-like resonance at the LHC collider [7]. If fully confirmed, this can be considered as one of the greatest triumphs of particle physics ever. But, at the same time, we should not overlook that such discovery could be certifying the very existence of the electroweak vacuum energy, with all its potential consequences for (theoretical) cosmology.

On the other hand, we should not forget that apart from the classical and quantum Higgs vacuum energy we have other sorts of vacuum fluctuations in the electroweak domain, which are perfectly "alive and kicking" since long ago. Recall that quantum corrections to high precision electroweak observables have reached a level of certainty that is beyond any possible doubt. A simple example should suffice. Take the famous Δr parameter from electroweak theory [13]. This is the famous parameter that allows to compute the $W^{\pm}$ gauge boson mass, M_W, in the on-shell renormalization scheme with quantum precision, i.e. including the quantum output from radiative

[13] See e.g. Ref.[83], and references therein, for contextual explanations on the Δr parameter, and for a comprehensive and updated study of that important electroweak parameter within the SM and beyond.

corrections. We may ask ourselves, to which extent we can attest that the "genuine" electroweak quantum effects (i.e. those beyond the pure QED running of $\alpha_{\rm em}$) have been measured when we compare the theoretical value of M_W and the experimentally measured one. To ask (and answer!) this question is important since the electroweak radiative corrections are a manifestation of the properties of the electroweak vacuum at the quantum level, namely the same vacuum that is supported by the likely existence of the Higgs particle. The answer is known, and is astounding: we know unmistakably that they are there with a confidence level of $\sim 25\sigma$! The formerly proclaimed $\sim 5\sigma$ evidence of the Higgs-like particle now pales in comparison! But the latter kind evidence is direct whereas the former is indirect, and when we put both the direct and indirect signatures together they dramatically reinforce the case for the electroweak vacuum energy; and, overall, the overwhelming evidence of the electroweak vacuum becomes even more defiant and challenging for cosmology. Somehow time has come to try to find a final solution which comes to grips with the basic notion of vacuum energy in QFT, rather than constantly eschewing the issue in an almost non-denumerable number of ways.

Although the problem is huge, and far from being solved, some avenues for its eventual solution might be looming in the horizon. We have discussed a possible reinterpretation of the results obtained in the calculation of the vacuum energy density ρ_Λ in QFT in curved spacetime, and suggested that although the resolution of the CC problem cannot be addressed at present from a rigorous computation of ρ_Λ in an expanding FLRW universe, at least some consistency relations seem to hint at the possible form of the correct dynamical dependence of that important quantity as a function of the Hubble rate. For example, if both the vacuum energy and the gravitational coupling are evolving with the Hubble rate, H, the Bianchi identity leads to the possible form for the running of the vacuum density $\rho_\Lambda = \rho_\Lambda(H)$, given the logarithmic running suggested for $G = G(H)$ in the expanding spacetime. One is led to the following behavior for the low energy regime: $\rho_\Lambda(H) = c_0 + \beta\, M_P^2\, H^2$, where M_P is the Planck mass, c_0 a constant (close to the current CC density value ρ_Λ^0) and β is a dimensionless coefficient parameterizing the CC running. This form for the evolving vacuum energy density is perfectly tenable and has been profusely tested against the latest cosmological data [39, 40]. At the same time, a nonvanishing value of β leads to a dynamical vacuum behavior which may effectively appear as quintessence, or as phantom energy, without need of invoking the existence of fundamental quintessence and phantom fields [42, 60]. This could eventually provide a strong phenomenological evidence in favor of the vacuum energy being a serious candidate for dynamical dark energy (DE).

We believe that the CC problem can only be solved as a physical problem, not just as a theoretical conundrum. This means that only through phenomenological tests it should be possible to disentangle the most difficult theoretical aspects of the CC problem. In this respect, another potentially interesting aspect of the dynamical vacuum models is the fact that they could provide an explanation for a possible variation of the so-called fundamental constants of Nature [24]. There is currently plentiful of experimental activity, both in the lab and from observations in the astrophysical domain, that will provide sooner or later interesting news on this field [74, 81]. Future data from these experiments should be very helpful for effectively testing the proposed vacuum ideas in the near future. For this reason we have discussed this issue at length in Sect. 9.

The idea of a running vacuum energy in an expanding universe, i.e. $\rho_\Lambda = \rho_\Lambda(H)$, appears as a most natural one. It would be very difficult to admit the existence of a tiny and immutable constant from the early times to the present days, maybe even ruling the entire future evolution of the universe. Nonetheless, the ultimate value that $\rho_\Lambda(H)$ takes at present, i.e. ρ_Λ^0, cannot be predicted within these models and hence can only be extracted from observations. Notice that if we would have the ability to predict this value it would be tantamount to solve the old CC problem [4]. This is of course the toughest part of the job. In our discussion, however, the running vacuum paradigm ascribes a new look to the problem, one that could perhaps make it more amenable for an eventual solution; namely it conceives the cosmological term as a time evolving variable that underwent a dramatic reduction from the inflationary time till the present days. A generic model that implements this idea is the following: $\rho_\Lambda(H) = c_0 + c_2\, H^2 + c_4\, H^4$, in which the highest power of the Hubble rate, H^4, would only be relevant for the early universe, i.e. for values of H of order of the inflationary expansion rate H_I. Models of this kind can be motivated by the general running vacuum framework that we have described, and they should have a real chance to provide a complete description of the cosmic history [65, 66, 67]. In contrast to the standard ΛCDM model, in which the two opposite poles of the cosmic history (inflation and DE) are completely disconnected, the running vacuum models offer a clue for interconnecting them and let the present DE appear as a "quantum fossil" from the inflationary universe [61]. In this way the two de Sitter epochs, viz. the primordial inflationary one and the late DE epoch, can be thought of as two vacuum dominated stages of the cosmic evolution smoothly interpolated (within a single unified model) by the radiation and matter dominated epochs.

In the kind of decaying vacuum framework that we have proposed, we can better understand why the present vacuum energy density has to be small as compared to its primordial value at the inflationary scale, i.e. $\rho_\Lambda^0 \ll \rho_\Lambda(H_I)$. Therefore, the main task that remains should be to understand why ρ_Λ^0 has the concrete value we have measured. In other words, we are left with the question of why the mass scale $m_\Lambda \equiv (\rho_\Lambda^0)^{1/4}$ associated to the cosmological term is of order of $m_\Lambda = \mathcal{O}(10^{-3})$ eV rather than, say, ten times or a hundred times bigger. While such values cannot be admitted, as they would obviously be incompatible with the observations, here we are addressing a matter of principle, i.e. we are asking: is there a fundamental physical theory which can explain the value of the vacuum energy that we measure at present? Put another way: should the millielectronvolt energy scale m_Λ be ultimately predictable, for example from the value of the Planck scale M_P? This is a very interesting question, but is not at all an obvious one, cf. [84],[85].

When we face the possibility to explain the mass scale m_Λ in our universe, we should note that in particle physics essentially all physical scales remain still unexplained. For example, we cannot explain why the value of the electron mass is $m_e \simeq 0.511$ MeV in our universe, since we do not understand why its Yukawa coupling, λ_e, takes the value it takes in order to yield the precise value of the electron mass from its product with the vacuum expectation value of the Higgs doublet: $m_e = \lambda_e\, v$. Similarly, we do not understand why we have so many fermion flavors and with widely different mass scales (or family of Yukawa couplings). In particular, why the masses of the neutrinos are much lighter, and why one species of neutrinos is possibly not far away from the same $\sim$ meV scale m_Λ associated to the cosmological constant.

Whether we can ultimately predict or not the scale of the vacuum energy in the present universe can be a debatable question, but what is perhaps less debatable is that there is, in principle, no reason why we should be able to bypass all the foreseeable difficulties well before we can minimally understand the origin of the scales of the fermion and boson masses in the SM of particle physics, and of course the values of the two basic "order parameters" that set the fundamental scales of the model, to wit: i) the electroweak scale $v \simeq G_F^{-1/2}$ (associated to the Higgs mechanism and linked to Fermi's constant); and ii) the value of the QCD scale $\Lambda_{\rm QCD}$ (associated to the strong interactions and determined by the non-perturbative confinement dynamics of QCD). In both cases an experimental input is needed to make contact with physics. For the electroweak sector (which provides the dominant contribution to the vacuum energy of the SM) we need to perform a precise measurement of Fermi's constant from muon decay. Similarly, in the domain of strong interactions we have to measure $\Lambda_{\rm QCD}$ to account for the QCD vacuum contribution. It is fascinating to entertain that the possible cosmic time dependence of $\Lambda_{\rm QCD}$ could also play a decisive role in elucidating the nature of the CC problem, as we have amply emphasized in Sect. 9.

Owing to its especial position in the realm of the physical quantities, and because of the general covariance of Einstein's equations, we should expect a tight connection of the CC with the remaining scales of the universe. For this reason the help of the phenomenological input appears as indispensable. Is this not a most fundamental physical requirement, even for the glory of the CC problem? In this sense, if the vacuum energy density is treated as a running quantity in an expanding universe, it would be natural to input its value at a given cosmic time (say, now) and then focus our efforts on finding its past and future evolution. After all it is not granted that we can reach a purely theoretical solution, unless e.g. all scales of the universe should come from a single one, say the Planck mass M_P, and all the others be referred to it through dimensionless (and computable) ratios. But it is far from obvious that we can happily make such an aprioristic assumption. For the time being quantum gravity is not a consistent theory, and we *cannot* know for certain if M_P – which is nothing but a shorthand for $(\hbar\, c/G)^{1/2}$ (G being Newton's constant) – is truly a physical scale!

The bare truth is that we still need the help from the phenomenological input so as to set the scale for the vacuum energy in our low energy universe, and there is no foreseeable change in this situation. But this does not mean we cannot hope for some progress. Take for example the RG approach we have discussed here; after we input m_Λ (or, equivalently, c_0) what remains of the CC problem is still a hard enough challenge for our intellect! The CC problem is then formulated as the problem of explaining why the dynamical term in the low energy expression $\rho_\Lambda(H) = c_0 + \beta M_P^2\, H^2$ is just the soft (and completely harmless) $\sim H^2$ contribution in the context of QFT in an expanding spacetime. Such term endows the vacuum energy density of a very mild (albeit non-negligible) time evolution, which causes no tension with the observations. It is therefore a very attractive option, in which the time effect could eventually surface in the form of a smooth dynamical DE. To check this possibility is a task reserved for the future observations. On the theoretical side, while we have given some indirect clues on how to effectively achieve an appealing scenario for $\rho_\Lambda(H)$ within QFT in curved spacetime, much more work (and thought!) is of course needed to tackle the most challenging and fierce angles of the CC problem, face to face.

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References

[1] R. Knop et al., Astrophys. J. **598** (2003) 102; A. Riess *et al.* Astrophys. J. **607** (2004) 665.

[2] D.N. Spergel, et al., Astrophys. J. Suplem., **170**, (2007) 377; E. Komatsu et al. Astrophys. J. Suppl. **192** (2011) 18 [arXiv:1001.4538].

[3] P.A.R. Ade *et al.*, PLANCK Collab. *Cosmological parameters*, arXiv:1303.5076.

[4] S. Weinberg, Rev. Mod. Phys. **61** (1989) 1.

[5] V. Sahni, A. Starobinsky, Int. J. of Mod. Phys. **A9** (2000) 373; S.M. Carroll, Living Rev. Rel. **4** (2001) 1; T. Padmanabhan, Phys. Rept. **380** (2003) 235; S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4 (2007) 115; M. Li, X.-D. Li, S. Wang, Y. Wang, Commun.Theor. Phys. **56** (2011) 525; J. Martin, Comptes Rendus Physique 13 (2012) 566.

[6] F. Bauer, J. Solà, H. Stefancic, JCAP 1012 (2010) 029 [arXiv:1006.3944]; S. Basilakos, F. Bauer, J. Solà, JCAP 1201 (2012) 050 [arXiv:1109.4739].

[7] G. Aad *et al.* (ATLAS Collab.), Phys.Lett. B 716 (2012) 1; S. Chatrchyan *et al.*. (CMS Collab.), Phys. Lett. B 716 (2012) 30.

[8] J. Solà, *The cosmological constant and the vacuum energy in quantum field theory: a review with reflections* (to appear).

[9] J. Solà, *Cosmologies with a time dependent vacuum*, J. Phys. Conf. Ser. 283 (2011) 012033 [arXiv:1102.1815].

[10] A. Einstein, Sitzungsber. Konigl. Preuss. Akad. Wiss. (1917) 142.

[11] Y. B. Zeldovich, *Cosmological constant and elementary particles*, Sov. Phys. JETP Lett **6** (1967) 3167.

[12] A.D. Dolgov, in: *The very Early Universe*, Ed. G. Gibbons, S.W. Hawking, S.T. Tiklos (Cambridge U., 1982); Y. Fujii, Phys. Rev. **D26** (1982) 2580; L. Abbott, *Phys. Lett.* **B150** (1985) 427.

[13] R.D. Peccei, J. Solà and C. Wetterich, Phys. Lett. **B195** (1987) 183.

[14] S. M. Barr, Phys. Rev. **D36** (1987) 1691; L.H. Ford, Phys. Rev. **D 35** (1987) 2339; E.T. Tomboulis, Nucl.Phys. **B329** (1990) 410.

[15] C. Wetterich, Nucl. Phys. **B302** (1988) 668; B. Ratra, P.J.E. Peebles, Phys. Rev. **D37** (1988) 3406.

[16] J. Solà, Phys.Lett. B228 (1989) 317; Int.J. Mod. Phys. **A 5** (1990) 4225.

[17] R.R. Caldwell, R.Dave, P.J. Steinhardt, Phys. Rev. Lett. **80** (1998) 1582.

[18] P.J. Peebles, B. Ratra, Rev. Mod. Phys. **75** (2003) 559; E.J. Copeland, M. Sami, S. Tsujikawa, Int. J. Mod. Phys. **D 15** 2006 1753.

[19] O. Bertolami, Nuovo Cimento, **93B**, 36, (1986); M. Ozer M. and O. Taha, Nucl. Phys., **B287**, 776, (1987); O. K. Freese K., et al., Nucl. Phys., **287**, 797, (1987); J. C. Carvalho, J. A. S. Lima and I. Waga, Phys. Rev. **D46**, 2404, (1992).

[20] J. M. Overduin and F. I. Cooperstock, Phys. Rev. D., **58**, 043506, (1998) [arXiv: astro-ph/9805260]; R.G. Vishwakarma, Class. Quant. Grav. 18 (2001) 1159 [arXiv: astro-ph/0012492]; and references therein.

[21] J. A. S. Lima, Braz. J. Phys. 34 (2004) 194 [arXiv:astro-ph/0402109].

[22] A. Einstein, *Grundgedanken der allgemeinen Relativitätstheorie und Anwendung diese Theorie in der Astronomie*, *Sitzber. Preuss. Akad. Wiss. Berlin,* (1915) 315; *Zur allgemeinen Relativitätstheorie*, *Sitzber. Preuss. Akad. Wiss. Berlin* (1915) 778; *Zur allgemeinen Relativitätstheorie (Nachtrag) Sitzber. Preuss. Akad. Wiss. Berlin,* (1915) 799.

[23] P.J.E. Peebles, *Principles of Physical Cosmology* (Princeton Univ. Press, 1993); T. Padmanabhan, *Structure Formation in the Universe* (Cambridge Univ. Press, 1993).

[24] H. Fritzsch, J. Solà, Class. Quant. Grav., **29** (2012) 215002 [arXiv:1202.5097].

[25] G. Lemaître, Ann. Soc. Sci. Brux. **A47** (1927) 49.

[26] A. Friedmann, Z. Phys. **10** (1922) 377; **21** (1924) 326.

[27] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. (1931) 235.

[28] S. Weinberg, Phys. Rev. Lett. **19** (1967) 1264; A. Salam, Elementary particle Theory; Nobel Symposium, No.8, ed. N. Svartholm (Almqvist and Wiksell, Stockholm 1969); S.L. Glashow, Nucl. Phys. **22** (1961) 579.

[29] H. Fritzsch, M. Gell-Mann, proc. of the XVI Int. Conf. on High Energy Physics, Chicago, 1972, vol 2, 135; H. Fritzsch, Murray Gell-Mann, H. Leutwyler, Phys.Lett. B47 (1973) 365.

[30] P.W. Higgs, Phys. Lett. **12** (1964) 132; Phys. Rev. Lett. **13** (1964) 508; G.S. Guralnik, C.R. Hagen and T.W.B. Kibble, Phys. Rev. Lett. **13** (1964) 585; F. Englert and R. Brout, Phys. Rev. Lett. **13** (1964) 321; T. W. B. Kibble, Phys. Rev. 155 (1967) 1554.

[31] L.E. Parker and D.J. Toms, *Quantum Field Theory in Curved Spacetime: quantized fields and gravity* (Cambridge U. Press, 2009).

[32] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge U. Press, 1982).

[33] I. L. Shapiro and J. Solà, Phys.Lett. B **682** (2009) 105, arXiv:0910.4925; arXiv:0808.0315.

[34] S. E. Rugh, H. Zinkernagel, Studies in History and Philosophy of Modern Physics, **33** (2002) 663.

[35] N. Straumann, proc. of the 3rd Int. Conf. on Dark Matter in Astro and Particle Physics (Dark 2000) [arXiv:astro-ph/0009386].

[36] G. Lemaître, *Evolution of the Expanding Universe*, Proc. of the National Academy of Sciences of the USA **20** (1934) 12-17.

[37] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[38] R. Schutzhold, Phys. Rev. Lett. **89** (2002) 081302, gr-qc/0204018.

[39] S. Basilakos, M. Plionis and J. Solà, Phys. Rev. **D 80**, (2009) 3511, arXiv:0907.4555 [astro-ph.CO]; J. Grande, J. Solà, S. Basilakos, and M. Plionis, JCAP **08** (2011) 007 [arXiv:1103.4632].

[40] S. Basilakos, D. Polarski, and J. Solà, Phys. Rev. **D 86** (043010) 2012 [arXiv:1204.4806].

[41] D. A. Easson, P. H. Frampton, G. F. Smoot, Phys. Lett. **B 696** (2011) 273.

[42] S. Basilakos, and J. Solà, *Effective equation of state for running vacuum: "mirage" quintessence and phantom dark energy* (to appear).

[43] I. L. Shapiro and J. Solà, Phys. Lett. B **475** (2000) 236, hep-ph/9910462.

[44] P. Ramond, *Field Theory. A Modern Primer* (Benjamin Cummings , Publish. Company, 1981); T.P. Cheng, L. Fong Li, *Gauge Theory of Elementary particle Physics* (Clarendon Press, Oxford, 1986); L.H. Ryder, *Quantum Field Theory* (Cambridge U. Press, 1985); S. Coleman, *Aspects of Symmetry* (Cambridge U. Press, 1985); L.S. Brown, *Quantum Field Theory* (Cambridge U. Press, 1992).

[45] J. Wess, B. Zumino, Nucl.Phys. **B78** (1974) 1.

[46] B. S. DeWitt, G. Esposito, Int. J. Geom. Meth. Mod. Phys. **5** (2008) 101 [arXiv:0711.2445].

[47] L. Smolin, *How far are we from the quantum theory of gravity?*, hep-th/0303185; S. B. Giddings, *Is string theory a theory of quantum gravity?*, arXiv:1105.6359.

[48] N.C. Tsamis, R.P. Woodard, Int. J. Mod. Phys. **D20** (2011) 2847 [arXiv:1103.5134]; M. G. Romania, N.C. Tsamis, R.P. Woodard, Lect. Notes Phys. **863** (2013) 375 [arXiv:1204.6558].

[49] B.S. DeWitt, *Quantum Field Theory in Curved Space-Time*, Phys. Rept. 19 (1975) 295; L. Parker, *Aspects of quantum field theory in curved space-time: Effective action and energy-momentum tensor*, in: Recent Developments in Gravitation, Cargèse 1978, ed. M. Lévy and S. Deser (Plenum Press, New York, 1978) 219-273; T. S. Bunch, L. Parker, Phys. Rev. **D 20** (1979) 2499.

[50] I.L. Buchbinder, S.D. Odintsov, I.L. Shapiro, *Effective Action in Quantum Gravity* (IOP, Bristol, 1992).

[51] E.V. Gorbar, I.L. Shapiro, JHEP 0302 (2003) 021.

[52] R.P. Woodard, Phys.Rev.Lett. 101 (2008) 081301.

[53] A. Kaya e-Print: arXiv:1303.5459.

[54] M. Bordag, U. Mohideen, V.M. Mostepanenko Phys.Rept. **353** (2001) 1.

[55] E. Elizalde, *Ten physical applications of spectral zeta functions* (Springer, 1995).

[56] R. L. Jaffe, Phys.Rev. D72 (2005) 021301.

[57] I.L. Shapiro, J. Solà, JHEP 02(006), 2002 [arXiv:hep-th/0012227]; I. L. Shapiro and J. Solà, Gravit. Cosmol. **9** (2003), proc. of SUSY 2002, DESY, Hamburg, hep-ph/0210329.

[58] I.L. Shapiro, J. Solà, Nucl. Phys. Proc. Suppl. **127** (2004) 71 [hep-ph/0305279]; I. L. Shapiro, J. Solà, C. España-Bonet, P. Ruiz-Lapuente, Phys. Lett. B 574 (2003) 149 [astro-ph/0303306]; JCAP 0402 (2004) 006 [hep-ph/0311171]; I.L. Shapiro, J. Solà, JHEP proc. AHEP2003/013 [astro-ph/0401015].

[59] A. Babić, B. Guberina, R. Horvat, H. Štefančić, Phys. Rev. **D 65** (085002) 2002; Phys. Rev. **D 71** (124041) 2005.

[60] J. Solà, H. Štefančić, Phys. Lett. B **624**, 147 (2005) [arXiv:astro-ph/0505133]; Mod. Phys. Lett. A **21**, 479 (2006) [arXiv:astro-ph/0507110].

[61] J. Solà, J. of Phys. **A 41** (2008) 164066 [arXiv:0710.4151].

[62] M. Maggiore, Phys. Rev. D83 (2011) 063514, arXiv:1004.1782; L. Hollenstein, M. Jaccard, M. Maggiore, E. Mitsou, Phys. Rev. D85 (2012), arXiv:1111.5575; N. Bilic, Phys.Rev. D83 (2011) 105003, arXiv:1104.1349; Phys.Lett. B707 (2012) 221, arXiv:1110.2393; N. Bilic, S. Domazet, B. Guberina, Phys.Lett. B707 (2012) 221 arXiv:1110.2393.

[63] T. Padmanabhan, Class. Quant. Grav. **22** (2005) L107-L110 [arXiv:hep-th/0406060].

[64] I. L. Shapiro and J. Solà, Phys.Lett. B **530** (2002) 10 [arXiv:hep-ph/0104182].

[65] J. A. S. Lima, S. Basilakos, and J. Solà, Mon. Not. Roy. Astron. Soc. **431** (2013) 923, [arXiv:1209.2802].

[66] S. Basilakos, J. A. S. Lima, and J. Solà, Int. J. of Mod. Phys. D (to appear).(Work awarded Honorable Mention by the Gravity Research Foundation, 2013).

[67] E. L. D. Perico, J.A.S. Lima, S. Basilakos, and J. Solà, *Complete cosmic history with a dynamical* $\Lambda = \Lambda(H)$ *term*, [arXiv:1306.0591].

[68] J. P. Mimoso, D. Pavón, Phys. Rev. **D 87**, 047302 (2013) [arXiv:1302.1972].

[69] J.A.S. Lima, J.M.F. Maia, Phys.Rev. **D49** (1994) 5597; J.A.S. Lima, M. Trodden, Phys. Rev. **D53** (1996) 4280 [astro-ph/9508049].

[70] F.R. Klinkhamer, G.E. Volovik, Phys.Rev. D79 (2009) 063527; E. C. Thomas, F. R. Urban, A. R. Zhitnitsky, JHEP 0908 (2009) 043; S. Carneiro, R. Tavakol, Phys.Rev. D80 (2009) 043528; N. Ohta, Phys. Lett. **B 695** (2011) 41.

[71] J. Ren, X-H. Meng, Phys. Lett. **B 633** (2006) 1; Int. J. Mod.Phys. **D 16** (2007) 1341.

[72] P.J.E. Peebles, Astrophys. J. 284 (1984) 439.

[73] P. Wang, X.-H. Meng, Class.Quant.Grav. **22** (2005) 283 [arXiv:astro-ph/0408495]; J.S. Alcaniz, J.A.S. Lima Phys.Rev. D72 (2005) 063516 [arXiv: astro-ph/0507372].

[74] J-P. Uzan, Living Rev. Rel. **14** (2011) 2 [arXiv:1009.5514]; Rev. Mod. Phys. **75** (2003) 403 [arXiv: hep-ph/0205340]; T. Chiba, Prog. Theor. Phys. 126 (2011) 993 [arXiv:1111.0092]; and references therein.

[75] J. Grande, J. Solà, J.C. Fabris, I.L.Shapiro, Class. Quant. Grav. **27** (2010) 105004 [arXiv:1001.0259]; I.L. Shapiro, J. Solà, H. Štefančić, JCAP **0501** (2005) 012, [arXiv:hep-ph/0410095]; J. C. Fabris, I. L. Shapiro, J. Solà, JCAP 0702 (2007) 016 [arXiv:gr-qc/0609017].

[76] F. Iocco, G. Mangano, G. Miele, O. Pisanti and P. D. Serpico, Phys. Rept. **472** (2009) 1.

[77] J. Grande, and J. Solà (in preparation).

[78] J. Grande, A. Pelinson, and J. Solà, Phys. Rev. **D79**, (2009) 043006 [arXiv:0809.3462]; J. Grande, R. Opher, A. Pelinson, and J. Solà, JCAP **0712** (2007) 007 [arXiv:0709.2130]; arXiv:0904.3293; J. Grande, J. Solà and H. Štefančić, JCAP **08** (2006) 011 [arXiv:gr-qc/0604057].

[79] H. Fritzsch, Prog. Part. Nucl. Phys. **66** (2011) 193;, Nucl.Phys. Proc. Suppl. **203-204** (2010) 3; Nucl.Phys.Proc.Suppl. 186 (2009) 221; Phys. Usp. 52 (2009) 359 [arXiv:0902.2989].

[80] X. Calmet, H. Fritzsch, Europhys. Lett. **76** (2006) 1064; Phys. Lett. **B540** (2002) 173; Eur. Phys. J **C24** (2002) 639.

[81] E. Reinhold, R. Buning, U. Hollenstein, A. Ivanchik, P. Petitjean, W. Ubachs, Phys. Rev. Lett. **96** (2006) 151101; W. Ubachs, E. Reinhold, Phys. Rev. Lett. **92** (101302) 2004; A. Ivanchik, P. Petitjean, D. Varshalovich, B. Aracil, R. Srianand, H. Chand, C. Ledoux, P. Boisse, Astron. Astrophys. **440** (2005) 45 [arXiv:astro-ph/0507174].

[82] J. D. Bjorken, Phys. Rev. **D67** (2003) 043508 [arXiv:hep-th/0210202].

[83] D. López-Val, J. Solà, Eur. Phys. J. **C73** (2013) 2393 [arXiv:1211.0311].

[84] T. Padmanabhan, *The Physical Principle that determines the Value of he Cosmological Constant*, arXiv:1210.4174; T. Padmanabhan, H. Padmanabhan [arXiv:1302.3226].

[85] B.F.L. Ward, *An Estimate of* Λ *in Resummed Quantum Gravity in the Context of Asymptotic Safety*, arXiv:1008.1046; B.F.L. Ward, arXiv:1302.0783, and references therein.