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# On leaky-box approximation to GALPROP

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ABSTRACT

The Galactic Propagation (GALPROP) numerical code is now accepted as an advanced tool for simulations of cosmic-ray diffusion and interaction in the Galaxy. The code is used for the interpretation of a large body of cosmic-ray data. In some cases, including in particular the case of stable primary and secondary nuclei, one can use a simple leaky-box model for handling of data on cosmic-ray energy spectra and composition. We find an adequate leaky-box approximation to the basic GALPROP model and estimate its accuracy.



## 1. Introduction

Cosmic-ray propagation in the Galaxy is commonly described in the diffusion approximation [1,2]. The galactic model with a flat extended cosmic-ray halo and the distribution of cosmic-ray sources (supernova remnants) in the galactic disk is accepted as the most adequate. The comprehensive numerical realization of this model is the GALPROP code [3–6] (see the Web site http://galprop.stanford.edu) that allows calculating the transport and interactions of relativistic protons, nuclei, electrons and positrons. The galactic non-thermal radio, X-ray, and gamma-ray emission produced by these energetic particles is also computed. The code incorporates detailed distributions of the interstellar gas, magnetic field and background radiation needed for such calculations.

More simple approximate models are also used in the investigations of cosmic-ray propagation as a substitute of complicated diffusion model. The most popular is the so called leaky-box model where the transport of energetic particles is described by introducing the mean escape time of cosmic rays from the Galaxy $T_e$ that is function of particle energy. The cosmic-ray density, the source density, and the gas density are characterized by some average values and do not depend on coordinates. The leaky-box approximation for the galactic diffusion model was obtained in [7] in a limiting case of fast particle diffusion in a galaxy with partly reflecting boundaries so that the life of the energetic particles is so long that their density has the time to even out over the whole volume. The observations of diffuse synchrotron radio emission and the gamma-ray emission strongly evidence against the homogeneous distribution of cosmic rays in the Galaxy. It seems that the particles with energies $10^8$–$10^{15}$ eV relatively slow diffuse in galactic magnetic fields and freely escape into the extragalactic space. Thus the leaky-box is rejected as an adequate pattern of cosmic-ray propagation. It is most clear when the rapidly decaying radioactive isotopes and the high energy electrons with severe synchrotron and inverse Compton energy losses are considered. The spatial distribution of these species produced in the galactic disk is very inhomogeneous in the cosmic-ray halo and depends on energy if the leakage time from the Galaxy is larger than the corresponding loss time. The same is true for all cosmic rays in the diffusion models with galactic wind where particles experience significant adiabatic energy losses in the non-uniform gas flow. The detailed consideration of this topic can be found in [1].

The conclusion about the absence of physical background for the use of the leaky-box model has a very important exception [8]. It is the study of diffusion and nuclear spallation of stable nuclei attended by the production of secondary relativistic nuclei in the interstellar gas. The spatial distribution in the low-density halo is the same for different stable nuclei because of the negligible nuclear spallation. The calculated intensities of stable nuclei for an observer at the galactic disk looks as corresponding leaky-box expressions even for nuclei with large cross sections and all propagation is described by some escape length of cosmic rays from the Galaxy $X_e$ (measured in g/cm$^2$) that is a function of particle energy.

It is useful to have a simple leaky-box approximation to the diffusion model that provides an adequate description of the set of data on spectra and composition of stable nuclei. It allows to determine the transport properties of cosmic rays in the Galaxy from the

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secondary-to-primary ratios B/C, sub-Fe/Fe and some other, and to find the source spectra of primary nuclei. Up to a hundred of isotopes of elements from H to Ni and even more heavy should be considered in the full-scale calculations in a wide energy range. The purpose of the present work is to find the leaky-box model that most closely reproduces the GALPROP results on the calculations of primary and secondary stable nuclei at the solar system location in the Galaxy. We also determine the precision of the leaky-box approximation that depends on the value of spallation cross sections for nuclei included in the analysis.

## 2. Cosmic-ray propagation equations

We consider the transport of cosmic-ray protons and stable nuclei in the interstellar medium. In the diffusion approximation, the isotropic part of the cosmic-ray distribution function $f(t,\mathbf{r},p)$ normalized as $N = 4\pi \int dpp^2 f$, where $N$ is the total cosmic-ray number density, obeys the equation of the form (see [1] for discussion):

$$\frac{\partial f_i}{\partial t} - \nabla(D_i \nabla f_i) + nv\sigma_i f_i + \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\left(\frac{dp}{dt}\right)_i f_i\right] - \frac{1}{p^2}\left(\frac{\partial}{\partial p}p^2\kappa_i\frac{\partial}{\partial p}f_i\right) = q_i + n\sum_{j<i} 4\pi\int_p^\infty dp' p'^2 v' \frac{d\sigma_{ij}(p,p')}{dp'} f_j(t,\mathbf{r},p'). \quad (1)$$

Here the subscript $i$ = 1, 2,... characterizes the type of nucleus (the isotope) with the charge number $Z_i$ and the atomic number $A_i$ starting with the heaviest one (no summation on $i$ is implied), $D_i(\mathbf{r},p)$ is the cosmic-ray diffusion coefficient (it may be a tensor), $n(\mathbf{r})$ is the number density of interstellar gas nuclei, $(dp/dt)_i < 0$ describes ionization energy loss, $\sigma_i(p)$ is the total cross section of nuclear spallation in the interstellar gas, $\sigma_{ij}(p,p')$ is the production cross section of nuclei $i$ by a more heavy nucleus $j$, $\kappa_i(\mathbf{r},p)$ is the diffusion coefficient on momentum (it describes possible distributed stochastic acceleration by interstellar turbulence), $q_i(t,\mathbf{r},p)$ is the source distribution. In practical work the structure of spallation term of the type $n\sigma_i$ should include nuclear interactions with interstellar hydrogen and helium nuclei that is written as $n_H\sigma_{i,H} + n_{He}\sigma_{i,He}$, where the corresponding gas number densities and cross sections are introduced. The same is true for the production cross sections. In the following, we shall use simple notations as in Eq. (1), but the real gas composition will be taken into account in the calculations. The ratio $n_{He}/n_H = 0.11$ is assumed for the interstellar gas; the ratio of cross sections $\sigma_{He}/\sigma_H$ is equal to 1.68 and 2.1 for energetic iron nuclei and protons, respectively.

The propagation model of cosmic rays in the Galaxy serves to classify and correlate experimental facts on composition, spectra, and anisotropy of various components of cosmic rays, and to find the composition of the cosmic rays at source. The appropriate value of the diffusion coefficient is determined in the process. This semiempirical approach can be complemented by the "microscopic" theory of transport of charged relativistic particles (the cosmic rays) in the turbulent interstellar medium. Unfortunately, the observational and theoretical uncertainties in the properties of the interstellar MHD turbulence do not allow yet rigorous calculation of cosmic-ray diffusion coefficient from the first principles although some important characteristics of the empirical diffusion coefficient including the estimate of its absolute value, the power law dependence on momentum and some others are supported by the theory, see [6] and references therein. The presence of magnetic inhomogeneities with the scales up to $L \sim 100$ pc suggests the applicability of the diffusion approximation up to the energies about $10^{17}$ $Z$ eV where the particle Larmor radius $r_g \approx L$($Z$ is the particle charge).

In the basic galactic model [1,2] the region of cosmic-ray diffusion in the Galaxy is a cylinder of radius $R$ and total width $2H$, see Fig. 1. The values $R$ = 30 kpc and $H$ = 4 kpc, and the independence on position of cosmic-ray diffusion coefficient will be accepted in the numerical calculations below. The spatial boundary condition at the boundaries of cosmic-ray halo is $f_i|_{\Sigma} = 0$ that implies the free exit of cosmic rays from the Galaxy to intergalactic space, where their density is negligible. These assumptions are used in the GALPROP code. (It is worth to note that the model with no boundaries but with strong increase of diffusion coefficient at large distance from the galactic disc was developed in [9].)

It is convenient to use the cosmic-ray intensity $I(E)dE = vf(p)p^2dp$as a function of kinetic energy per nucleon $E$ instead of $f(p)$. The intensity is measured in units of particles per cm$^2$ s sr (eV/nucleon), and $E$ is approximately conserved in the course of nuclear fragmentation. We shall consider the steady state problem $\partial/\partial t = 0$ and assume that the diffusion coefficient on momentum has the same spatial dependence as the gas density, i.e.

$$\kappa_i(\mathbf{r},p) = b_i(E)n(\mathbf{r}). \quad (2)$$

With these assumptions Eq. (1) transforms into the following equation for $I_i(\mathbf{r},E)$:

$$-\frac{1}{\rho(\mathbf{r})v}\nabla(D_i(\mathbf{r},E)\nabla I_i) + \frac{\sigma_i(E)}{m}I_i + \frac{\partial}{\partial E}\left[\left(w_i(E) + \frac{2b_i(E)}{A_i^2 m\sqrt{E(E+2E_0)}}\right)I_i\right] - \frac{\partial}{\partial E}\left(\frac{vb_i(E)}{A_i^2 m}\frac{\partial}{\partial E}I_i\right) = \frac{Q_i(\mathbf{r},E)}{\rho(\mathbf{r})} + \sum_{j<i}\frac{\sigma_{ij}(E)}{m}I_i, \quad (3)$$

where $\rho(r) = mn(r)$, $m$ is the average atomic mass in the interstellar gas, $w_i(E) = (dE/dx)_i$, $x$ is the matter thickness measured in g/cm$^2$, $E_0 = m_N c^2$ is the nucleon rest mass energy. The source term $Q_i = A_i p^2 q_i(\mathbf{r},p)/v$ describes the cosmic-ray production in units of particles per cm$^2$ s sr (eV/nucleon).

We shall assume that the source term is a separable function of $\mathbf{r}$, so that $Q_i = g_i(E)s(\mathbf{r})$, and the diffusion mean free path has a separable dependence on position:

$$D_i(\mathbf{r},p) = vd_i(E)l(\mathbf{r})/3, \quad (4)$$

where $d_i(E)$ is a dimensionless function.

As it was shown in [10], the solution of the system of equations (3) for all $i$ can be presented now in the form

$$I_i(\mathbf{r},E) = \int_0^\infty dyG(\mathbf{r},y)F_i(y,E). \quad (5)$$

Here the variable matter thickness is $y$, and the path length distribution function $G$ obeys the following equation that contains infor-

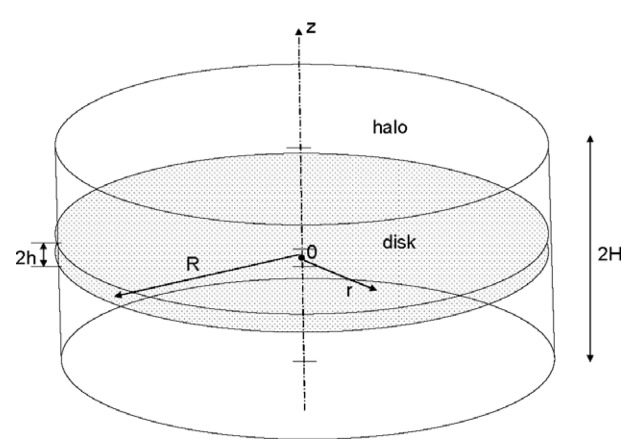


**Fig. 1.** The region of cosmic ray propagation in the Galaxy. The cosmic ray sources and the interstellar gas are distributed in the thin disk with characteristic half thickness much smaller than the height of cosmic-ray halo, $h \ll H$.

mation on the source and gas distributions and on the properties of particle wandering in the Galaxy but does not contain the type of nucleus and particle energy:

$$\rho(\mathbf{r})\frac{\partial G}{\partial y} - \frac{1}{3}\nabla l(\mathbf{r})\nabla G = s(\mathbf{r})\delta(y), \tag{6}$$

whereas function $F_i$ describes particle fragmentation and energy changes in a course of motion through the slab of material irrelevant to the position in the Galaxy where these processes occur:

$$d_i\frac{\partial F_i}{\partial y} + \frac{\partial}{\partial E}\left[\left(w_i + \frac{2b_i(E)}{A_i^2 m\sqrt{E(E+E_0)}}\right)F_i\right] - \frac{\partial}{\partial E}\left(\frac{vb_i}{A_i^2 m}\frac{\partial}{\partial E}F_i\right) + \frac{\sigma_i}{m}F_i - \sum_{j<i}\frac{\sigma_{ij}}{m}F_i = 0. \tag{7}$$

The initial condition for the last equation is $F_i(y=0) = g_i/d_i$ and no summation on $i$ is implied.

The decoupling procedure described by Eqs. (5)–(7) provides a basis for solution of a set of transport equations for stable nuclei by the exact weighted slab method [10]. It separates the process of particle diffusion, described by Eq. (6), from the processes of nuclear spallation and energy change in the interstellar medium, Eq. (7). In the absence of energy change ($w_i = b_i = 0$), Eq. (7) gives $F_1 = \frac{g_1}{d_1}\exp\left(-\frac{\sigma_1}{d_1 m}y\right)$ for the most heavy nuclei or more precisely for the primary nuclei without contribution of spallation from heavier nuclei and their intensity is

$$I_1 = g_1\int_0^\infty dx G(\mathbf{r}, d_1 x)\exp\left(-\frac{\sigma_1}{m}x\right). \tag{8}$$

The last equation clarifies the meaning of $G$ as a path length distribution function – the observed cosmic-ray intensity is proportional to the probability to traverse a matter thickness $x$ multiplied by the exponential attenuation factor, see also [10,11].

The GALPROP code numerically solves the transport Eq. (1) rewritten for the function $\psi = 4\pi p^2 f$ for all cosmic-ray species using a Crank–Nicholson implicit second-order scheme. The transport coefficients are determined by the data on $B/C$ ratio. The spatial diffusion coefficient is taken as $D = k\beta(R/R_0)^a$, $k$ = const, where $R = pc/Z$ is the particle magnetic rigidity and $\beta = v/c$, if necessary with a break ($a = a_1$ below rigidity $R_0$ and $a = a_2$ above rigidity $R_0$). The source spectrum is assumed to be a power law in momentum, $q(p) \propto p^{-\gamma_s - 2}$ and may also have breaks. In the models with particle reacceleration by the interstellar MHD turbulence, the diffusion coefficient on momentum obeys the scaling $\kappa \propto p^2V_a^2/D$, where $V_a$ is the Alfven velocity.

The prime version of the propagation model, the GALPROP plain diffusion model, has no reacceleration and is characterized by the following set of parameters [6]:

$$\begin{aligned}
&D = 2.2\times10^{28}\beta^{-2}\,\mathrm{cm^2/s} \text{ at } R \leqslant 3\,\mathrm{GV},\\
&D = 2.2\times10^{28}\beta^{-2}(R/3\,\mathrm{GV})^{0.6}\,\mathrm{cm^2/s} \text{ at } R > 3\,\mathrm{GV},\\
&p^2q \propto (R/40\,\mathrm{GV})^{-2.3} \text{ at } R \leqslant 40\,\mathrm{GV},\\
&p^2q \propto (R/40\,\mathrm{GV})^{-2.15} \text{ at } R > 40\,\mathrm{GV}.
\end{aligned} \tag{9}$$

The GALPROP diffusion model with distributed reacceleration has the following parameters:

$$\begin{aligned}
&D = 5.2\times10^{28}\beta^{-2}(R/3\,\mathrm{GV})^{0.34}\,\mathrm{cm^2/s} \text{ at all rigidities},\\
&V_a = 36\,\mathrm{km/s};\\
&p^2q \propto (R/4\,\mathrm{GV})^{-1.8} \text{ at } R \leqslant 4\,\mathrm{GV},\\
&p^2q \propto (R/4\,\mathrm{GV})^{-2.4} \text{ at } R > 4\,\mathrm{GV}.
\end{aligned} \tag{10}$$

It should be noted that the diffusion coefficient on momentum is independent on position in this version of the reacceleration model, so that condition (2) is not fulfilled and the transport Eq. (1) can not be reduced to Eqs. (5)–(7). The exception is the high energy limit $E > 40$ GeV/nucleon where the reacceleration effect on particle spectra is negligibly small and one can simply set $V_a = 0$, $\kappa = 0$, see also discussion in Section 5. Notice that the time cosmic rays spends in the Galaxy $H^2/D$ is decreasing with energy whereas the characteristic time of distributed acceleration $p^2/\kappa \propto D/V_a^2$ is increasing with energy that makes the reacceleration not efficient at high enough energies. The empirical values of the Alfven velocity $V_a$ found from the fit to cosmic-ray data are different in the models (9) and (10) but this difference lies within the limits of uncertainty of corresponding astronomical data.

More sophisticated is the diffusion model with the back reaction of cosmic rays on the interstellar turbulence which determines the cosmic ray diffusion coefficient [6]. Eq. (1) is supplemented in this case with equation for the non-linear Kraichnan-type cascade of MHD waves. The cascade is terminated below wavelengths $\sim10^{13}$ cm because of the resonant interactions of waves with cosmic-ray particles. The derived diffusion coefficient has a power law shape at high rigidities and rises sharply below a few GV. Incorporated in the GALPROP code, this model with damping is characterized by the following set of parameters:

$$\begin{aligned}
&D = 2.9\times10^{28}\beta(R/3\,\mathrm{GV})^{0.5}\,\mathrm{cm^2/s} \text{ at } R > 10\,\mathrm{GV},\\
&V_a = 22\,\mathrm{km/s};\\
&p^2q \propto (R/4\,\mathrm{GV})^{-2.4} \text{ at } R \leqslant 40\,\mathrm{GV},\\
&p^2q \propto (R/4\,\mathrm{GV})^{-2.24} \text{ at } R > 40\,\mathrm{GV}.
\end{aligned} \tag{11}$$

Here only the high-energy asymptotic of the diffusion coefficient is given and the rigidities $R > 10$ GV with negligible energy change in a course of cosmic-ray propagation are included in our present analysis.

Let us turn now to the leaky-box equations. If cosmic-ray transport is described in the leaky-box approximation, the diffusion term in Eq. (1) is substituted with the simple expression $f_i/T_{e,i}(E)$. $T_{e,i}(E)$ has the meaning of the leakage time from the Galaxy. In the steady state ($\partial/\partial t = 0$) we have the following leaky-box transport equation instead of the diffusion equation (3):

$$\frac{I_i}{X_{e,i}(E)} + \frac{\sigma_i(E)}{m}I_i + \frac{\partial}{\partial E}\left[\left(w_i(E) + \frac{2b_i(E)}{A_i^2 m\sqrt{E(E+2E_0)}}\right)I_i\right] - \frac{\partial}{\partial E}\left(\frac{vb_i(E)}{A_i^2 m}\frac{\partial}{\partial E}I_i\right) = \frac{Q_i(E)}{\rho} + \sum_{j<i}\frac{\sigma_{ij}(E)}{m}I_j, \tag{12}$$

where $X_{e,i}(E) = v\rho T_{e,i}(E)$ is the escape length and all quantities including the cosmic-ray intensity $I_i$, the source density $Q_i$, and the gas density $\rho$ do not depend on position $\mathbf{r}$.

Writing the escape length as

$$X_{e,i}(E) = \frac{X_0}{d_i(E)}, \tag{13}$$

where $X_0$ is some normalization constant, the solution of the set of equations (12) can be presented similarly to Eq. (5):

$$I_i(E) = \int_0^\infty dyG(y)F_i(y,E). \tag{14}$$

Here $G(y)$ obeys the equation

$$\rho\frac{\partial G}{\partial y} - \frac{\rho}{X_0}G = s\delta(y). \tag{15}$$

The function $F_i$ satisfies Eq. (7) as before.

The solution of Eq. (15) is

$$G(y) = \frac{s}{\rho}\exp\left(-\frac{y}{X_0}\right). \tag{16}$$

This exponential form of $G(y)$ is a distinctive characteristic of the leaky-box model.

The intensity of the most heavy nuclei $i = 1$ in the leaky-box model in the absence of energy change is determined by the same Eq. (8) as for the diffusion model but with the distribution function $G$ given by Eqs. (15) and (16). It results in the following explicit leaky-box formula for $I_1$:

$$I_1 = \frac{Q_1}{\rho}\frac{1}{\frac{1}{X_{e,1}} + \frac{\sigma_1}{m}}. \tag{17}$$

## 3. Leaky-box approximation to the GALPROP diffusion model

It is evident from the above presentation that two propagation models give approximately the same energy spectra and composition for the whole set of stable nuclei if their distribution functions $G$ are close enough. Instead of function $G$ one can work with function $I_1$, the intensity of primary nuclei calculated without contribution of spallation products of heavier nuclei and in the absence of energy change. Eq. (8) means that $I_1$ is the Laplace transform of $G$. In the following, we shall omit subscript 1.

It is instructive to introduce the following function of cross section:

$$X_{ef} = \frac{m}{\sigma}\left(\frac{I(\sigma = 0)}{I(\sigma)} - 1\right), \tag{18}$$

where the intensity $I(\sigma = 0)$ is taken at zero cross section. (It is worth noting that the ratio $\frac{m}{\sigma}$ in Eq. (18) stands for $\frac{n_H m_H + n_{He} m_{He}}{n_H \sigma_H + n_{He}\sigma_{He}}$ in the interstellar gas composed of hydrogen and helium; see comment after Eq. (1).)

Evidently $X_{ef}$ coincides with the escape length $X_e$ in the case of the leaky-box model. The intrinsic property of the leaky-box model is the independence of $X_e$ on cross section $\sigma$ which is now considered as a variable. To find how close is the diffusion model to the leaky-box model, we calculated the effective escape length $X_{ef}$ defined by Eq. (18) in a wide range of cross sections with the help of the GALPROP code. The calculations of intensity $I(\sigma)$ were made for primary nuclei without contribution of spallation from other nuclei and without account of any energy change and radioactive decays.

The results of calculations for the set of parameters (9) are presented in Fig. 2. The spallation of energetic nuclei on interstellar hydrogen and helium were taken into account, but $X_{ef}$ is shown in Fig. 2 as a function of cross section on hydrogen. It is evident that $X_{ef}(\sigma)$is nearly constant and slightly rises starting from very large cross sections $\sigma \sim 1000$mb. It is significant that the typical value of the total cross section for iron group nuclei is of the order of 700 mb and therefore the leaky-box approximation works well

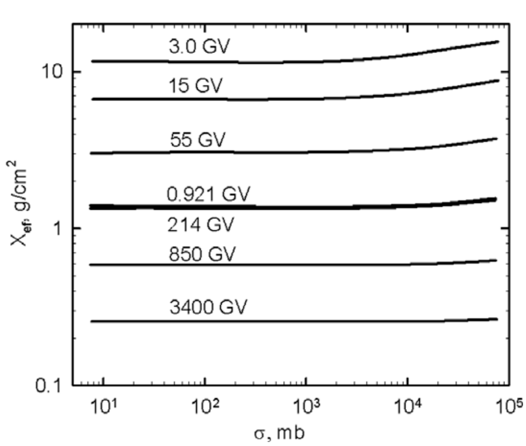


**Fig. 2.** Effective escape length determined from Eq. (18) where $I(\sigma)$ is calculated with GALPROP code in the plain diffusion model at different rigidities.

for all nuclei included in the GALPROP code. The deviation of $X_{ef}$ from its asymptotic constant value at small cross sections does not exceed ~2% for Fe nuclei at a few GeV/nucleon.

The found effective escape length that fits the GALPROP plain diffusion model is

$$X_{ef} = 19\beta^3\,\mathrm{g/cm^2} \text{ at } R \leqslant 3\,\mathrm{GV}, X_{ef} = 19\beta^3(R/3\,\mathrm{GV})^{-0.6}\,\mathrm{g/cm^2} \text{ at } R > 3\,\mathrm{GV}. \tag{19}$$

Here the dependence of $X_{ef}$ on $\beta$and$R$reflects the corresponding dependence of function $v/D_i(p)$, see Section 4.

In a similar way one can find

$$X_{ef} = 7.2(R/3\,\mathrm{GV})^{-0.34}\,\mathrm{g/cm^2} \tag{20}$$

applied at $R > 40$ GV for the reacceleration model, and

$$X_{ef} = 13(R/3\,\mathrm{GV})^{-0.5}\,\mathrm{g/cm^2} \tag{21}$$

applied at $R > 10$ GV for the model with damping.

The limits $R > 40$ GV or $R > 10$ GV after Eqs. (20) and (21) are due to the fact that below these rigidities reacceleration and damping can not be neglected, while beyond they can.

## 4. Interpretation of obtained results

The efficiency of leaky-box approximation to the diffusion model can be explained by the concentration of cosmic rays sources and the interstellar gas in a relatively thin galactic disk immersed in the flat but fat cosmic-ray halo [1,11]. We demonstrate it in this section by simple analytical solutions of the diffusion equation.

### *4.1. One-dimensional flat–halo model*

To understand why the leaky-box is a good approximation to the solution of the set of diffusion equations that describe propagation and spallation of stable nuclei in a flat–halo galaxy, let us consider a simple one-dimension case with the $z$-axis directed perpendicular to the central galactic plain. The diffusion equation for one type of primary nuclei with no energy loss is

$$-D\frac{dI}{dz} + n(z)v\sigma I = q(z). \tag{22}$$

Let us assume that the cosmic-ray sources are uniformly distributed with the density $q$ in the disk $-z- \leqslant h_s$; the interstellar gas with number density $n$ is uniformly distributed in the disk $-z- \leqslant h_g$; the cosmic rays fill the halo $-z- \leqslant H$ and it is assumed that $H > h_s > h_g$ (the corresponding numerical values are $H = 4$ kpc, $h_g \sim h_s \sim 100$pc,$n \sim 1\ \mathrm{cm^{-3}}$).

Solving Eq. (22) under the continuity conditions for intensity $I$ and the diffusion flux $-D\frac{dI}{dz}$ at the boundaries of the source and gas disks, and under the condition $I = 0$ at the halo boundaries, one can get at $z = 0$:

$$I = \frac{q}{D\lambda^2}\left(1 - \frac{1 - \lambda^2(h_s - h_g)\left(H - \frac{h_s + h_g}{2}\right)}{ch(\lambda h_g) + \lambda(H - h_g)sh(\lambda h_g)}\right), \quad \lambda = \sqrt{\frac{nv\sigma}{D}}. \tag{23}$$

Eq. (23) gives $I(\sigma = 0) = \frac{q}{D}h_s(H - \frac{h_s}{2})$. Using (18) and (23) one can find

$$X_{ef} = \frac{m}{\sigma}\left(\lambda^2 h_s\left(H - \frac{h_s}{2}\right)\left(1 - \frac{1 - \lambda^2(h_s - h_g)\left(H - \frac{h_s + h_g}{2}\right)}{ch(\lambda h_g) + \lambda(H - h_g)sh(\lambda h_g)}\right)^{-1} - 1\right) \tag{24}$$

The dependence of $X_{ef}$ on cross section (24) is illustrated in Fig. 3a. The value of $X_{ef}$ is approximately constant at small cross sections and increases at large cross sections. Thus the probable reason for increase of $X_{ef}$ with $\sigma$ in Fig. 3a lies in the difference in the widths of gas and source distributions in the GALPROP models. In the case, when the gas disk is wider than the source disk, $X_{ef}$ decreases at large cross sections, see Fig. 3a.

Eq. (24) allows finding the analytical expression for the constant part of effective escape length

$$X_{ef}(\sigma=0)=\frac{\rho v h_g H}{D}\left[\left(1-\frac{h_g}{2H}\right)-\frac{h_g^2}{2h_sH}\left(\frac{1}{3}-\frac{h_g}{4H}\right)\left(1-\frac{h_s}{2H}\right)^{-1}\right] \tag{25}$$

and to estimate the dependent on $\sigma$ relative deviation of $X_{ef}(\sigma)$ from $X_{ef}(\sigma=0)$:

$$\frac{\delta X_{ef}}{X_{ef}(\sigma=0)}\approx\frac{\sigma n v h_g^2}{6D}\left(1-\frac{h_g}{h_s}\right)\approx 0.05\beta\left(\frac{\sigma}{1000\,\mathrm{mb}}\right)\left(\frac{10^{28}\,\mathrm{cm}^2/\mathrm{c}}{D}\right)\left(1-\frac{h_g}{h_s}\right), \tag{26}$$

the sign of this deviation has an evident dependence on the ratio $h_g/h_s$ (we neglect small values $h_s/H$, $h_g/H \ll 1$ in the last equation for $\delta X_{ef}$, and assume that $\delta X_{ef} \ll X_{ef}(\sigma=0)$).

### *4.2. Galaxy with finite radius*

We show in this subsection that the account of radial galactic structure also leads to the weak increase of the efficient escape length $X_{ef}(\sigma)$ with cross section. Let us consider cosmic-ray diffusion of one type of primary stable nuclei without energy change in a galaxy with cylindrical halo of radius $R$ and the total height $2H$. The cosmic ray sources and the interstellar gas fill the infinitely thin galactic disk (i.e. $h_s/H$, $h_g/H \to 0$) with dependent on radial distance surface source density $\eta(r)$ and the constant on distance surface gas density $\mu$. Below $R = 20$ kpc, $H = 4$ kpc, and $\mu \approx 2.4\times10^{-3}$ g/cm$^2$ are accepted. The distribution of cosmic-ray sources on radius in the galactic disk $\eta(r)\propto(r/r_\odot)^{0.5}\exp(-1.0(r-r_\odot)/r_\odot)$ is taken from [3] (here $r_\odot = 8.5$ kpc is the distance of the Sun from the galactic centre).

The cosmic-ray intensity can be presented as

$$I(r,z)=2\pi\int_0^R dr_0 r_0\eta(r_0)G(r,z;r_0), \tag{27}$$

where $G(r,z;r_0)$ is the Green function for a point source located in the galactic disk at $r = r_0$, $z = 0$; $G$ satisfies the following diffusion equation in the cylindrical coordinates:

$$-\frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right)-D\frac{\partial^2 G}{\partial z^2}+\frac{\mu}{m}\delta(z)v\sigma G=\frac{\delta(r-r_0)\delta(z)}{2\pi r_0}. \tag{28}$$

The solution of Eq. (28) can be found in a form of a Fourier-Bessel series which automatically satisfies the boundary conditions $G|_\Sigma = 0$ on the sides of the halo

$$G(r,z;r_0)=\sum_{s=1}^{\infty}a_s(z;r_0)J_0(v_s r/R), \tag{29}$$

where $J_0$ is the Bessel function of zeroth order, and the $v_s$ are the roots of the equation $J_0(v) = 0$. After some routine mathematics one can obtain finally the intensity in galactic disk

$$I(r,z=0)=\sum_{s=1}^{\infty}\frac{HJ_0\left(\frac{v_s r}{R}\right)\int_0^R dr_0 r_0^2 J_0\left(\frac{v_s r_0}{R}\right)\eta(r_0)}{DR^2(J_0'(v_s))^2\left(\frac{v_s H}{R\tanh\left(\frac{v_s H}{R}\right)}+\frac{\mu v H}{2mD}\sigma\right)}. \tag{30}$$

Each term in this infinite sum can be brought to the form $\sim\frac{1}{C_s+\sigma}$ with the dependence on cross section typical for the leaky-box model, see (17). The dominant first term

$$I_d=\frac{2HJ_0\left(\frac{v_1 r}{R}\right)\int_0^R dr_0 r_0^2 J_0\left(\frac{v_1 r_0}{R}\right)\eta(r_0)}{vR^2\mu(J_0'(v_1))^2\left(\frac{2Dv_1}{\mu v R\tanh\left(\frac{v_1 H}{R}\right)}+\frac{\sigma}{m}\right)},\quad v_1\approx 2.406 \tag{31}$$

can be considered as an operative leaky-box approximation to the considered diffusion model (the sum of all other terms in (30) contribute ~10%). It is characterized by the escape length

$$X_{e,d}=\frac{\mu v R}{2Dv_1}\tanh\left(\frac{v_1 H}{R}\right). \tag{32}$$

The expression (32) transforms into the escape length $\frac{\mu v H}{2D}$ at small $v_1H/R \ll 1$ i.e. when the galactic radius $R$ is large that is in agreement with the expression for the one-dimensional model $X_{ef}(\sigma=0)$ derived from Eq. (24) in the limit $h_s/H \ll 1$, $h_g/H \ll 1$.

Fig. 3b shows the effective escape length $X_{ef}(\sigma)$ calculated at rigidities 3 GV and 100 GV with the use of Eqs. (18) and (30) and the independent on cross section escape length $X_{e,d}$. The maximum relative deviation of $X_{ef}(\sigma)$ from $X_{e,d}$ is ~0.2%. This characterizes the deviation of the leaky-box approximation from the considered diffusion model with a finite galactic radius. The deviation is about order of magnitude less than in the one-dimensional diffusion model considered in Section 4.1 at $h_s \sim 2h_g$.

We emphasize that the gas distribution accepted in the GALPROP code is more complicated function of $\boldsymbol{r}$ and $z$ than was assumed in Sections 4.1 and 4.2. The purpose of our consideration of simple diffusion models is to clarify the nature of dependence of the effective escape length $X_{ef}$ on cross sections and to explain why this dependence is so remarkably weak.

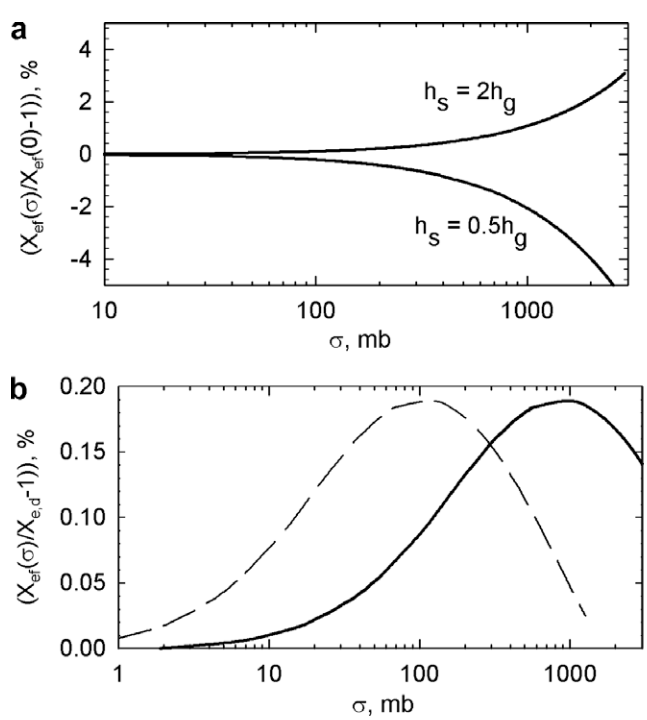


**Fig. 3.** Deviation (in percents) of effective escape length from its value at zero cross section: (a) for two versions of the one-dimensional diffusion model – the source disk is more thick than the gas disk, $h_s = 2h_g$, and the source disk is more thin than the gas disk, $h_s = h_g/2$; (b) in the model with infinitely thin galactic disk and with account of radial distribution of cosmic-ray sources in the Galaxy. Calculations were made at rigidity 3 GV (solid line) and 100 GV (dash line) assuming that $D \propto R^{0.6}$.

## 5. Discussion

The GALPROP code became a prime instrument for the study of cosmic-ray diffusion and interactions in the Galaxy [2–6].

The leaky-box model is a simple and efficient approximation to the diffusion model for a limited class of problems related to the interpretation of data on stable nuclei. In the present paper we found the value of the leaky-box escape length $X_{ef}$ (19) that reproduce the GALPROP results for the plain diffusion model and estimated the accuracy of the leaky-box approximation. The leaky-box approximation to the plain diffusion model is applicable at all particle rigidities where the concept of cosmic-ray diffusion is valid i.e. up to $\sim 10^8$ GV. The high rigidity asymptotic behavior of $X_{ef}$ were determined for the diffusion reacceleration model (20) and the diffusion model with damping (21). The leaky-box approximation is applicable in these cases at rigidities larger than 40 GV and 10 GV respectively. When working with cosmic-ray data, these results allow one to use simple leaky-box formalism described by Eq. (12) instead of making the GALPROP calculations based on more complicated Eq. (3). The use of the leaky-box approximation is convenient when one needs to find the source spectrum from the data on cosmic-ray intensity or to find the energy dependent diffusion coefficient from the observed secondary to primary ratio. With Eqs. (19)–(21) it becomes easier to compare the GALPROP results with a large body of leaky-box calculations made by different authors, see e.g. [12–18] and references therein.

The calculation of the flux of primary iron nuclei at 3 GeV/nucleon fulfilled with the leaky-box escape length (19) differs from the calculations in the GALPROP plain diffusion model by $\sim$0.5%. This difference declines with energy and is smaller for lighter nuclei.

The high accuracy of the leaky-box approximation for investigation of diffusion and spallation of stable nuclei is characteristic of the diffusion galactic models with relatively thin galactic disk surrounded by extended flat cosmic-ray halo and for nuclei with cross sections $\sigma \ll mH/X_{ef}h_g \sim 3 \times 10^3$ mb; the last condition is satisfied for all cosmic-ray nuclei. This known before result [8] is supported by the analytical calculations presented in Section 4. Eq. (30) gives useful approximate relation between the diffusion coefficient and the escape length valid at $v_1H/R < 1$.

Essential for our consideration is the decoupling procedure described in Section 2 that reduces the analysis of the entire chain of successive breakdowns of cosmic-ray nuclei to the study of only one type of primary nuclei with nuclear fragmentation and with no change of particle energy. The decoupling is possible because of the proportionality of the energy change rate (and, in particular, the ionization loss rate) to the interstellar gas density exactly as does the spallation rate. Under these conditions the steady state transport equation contains the matter thickness only (see Eqs. (5)–(7) for the diffusion model, and Eq. (12) for the leaky-box model) but not the matter thickness and the mean gas density separately (or, alternatively, the matter thickness and the time separately). The further consideration in Section 2 was reduced then to the calculation of cosmic-ray intensity as a function of the spallation cross section and the establishing how strongly $X_{ef}$ depends on cross section.

The possible presence of reacceleration uniformly distributed in the whole Galaxy (as it is assumed in the reacceleration model implemented in the GALPROP code now in use) makes the leaky-box approximation not applicable, see Introduction. It is reflected in particular in the impossibility to present the steady-state transport equation in this case in terms of the matter thickness only. The nuclear spallation and the ionization energy losses proceed as a function of passed matter thickness whereas the reacceleration proceeds in time. As a result, our approximations for the leaky-box escape length (20) and (21) for diffusion with reacceleration and the self-consistent model with damping are rigorously valid only at high enough rigidities where one can neglect reacceleration and the propagation is actually described in terms of the plain diffusion model.

It is worth noting that as it was done in [19], the reacceleration can be rigorously described in the leaky-box approximation by equation like (12) at all rigidities if the reacceleration exists only in the region adjacent to the galactic disk with the characteristic height $h_a \ll H$ to satisfy at least approximately the condition (2) needed for having Eq. (12). Even if reacceleration is distributed over the whole volume of cosmic-ray propagation in the Galaxy but the rate of reacceleration is relatively small, it can be taken into account by some correction of the escape length that extends the applicability of the leaky-box approximation to smaller rigidities $\sim$5 GV as it was done in [20]; see also the application of this procedure in [21]. The adiabatic energy loss in the diffusion–convection models (models with galactic wind) can be rigorously described in the leaky-box approximation in the special case of constant wind velocity in the halo when the adiabatic loss term works only in the thin galactic disk, see [22] for detail.


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